1 ISiTGR extensions to ΛCDM

The Integrated Software in Testing General Relativity (ISiTGR) offers an extension to the ΛCDM model based on a phenomenological approach to Modified Gravity (MG). In other words, it is based on the modification of the perturbed field equations through the inclusion of phenomenological modified gravity parameters.

1.1 Functional Form

ISiTGR is a patch that goes at the top of CosmoMC. Therefore, it works based on line element of the perturbed FLRW metric

$$ds^2 = a(\tau)^2[-(1+2\Psi)d\tau^2 + (1-2\Phi)\gamma_{ij}dx^i dx^j], \tag{1}$$

where $\gamma_{ij} = \delta_{ij}\left[1 + \frac{K}{4}(x^2+y^2+z^2)\right]^{-2}$ and $K = -\Omega_k \mathcal{H}_0$. Now, after analyzing the first-order perturbed Einstein field equations in Fourier space, we can obtain a pair of equations describing the evolution of Ψ and Φ fields, given by

$$(k^2 - 3K)\Phi = -4\pi G a^2 \sum_i \rho_i \Delta_i, \tag{2}$$

and

$$k^2(\Psi - \Phi) = -12\pi G a^2 \sum_i \rho_i (1+w_i)\sigma_i. \tag{3}$$

Here, we can observe that in the limit of zero-shear stress, $\Psi = \Phi$.

Now, ISiTGR introduces phenomenological parameters into equations (2) and (3) that quantifies the devitions from General Relativity at large scales. In the next table we explicitely list the modified gravity parameters that we are considering, the modifications to equations (2) and (3), and the functional form of these ones, as is implemented into the current version of ISiTGR.

set of MG parameters	MG first order perturbed equations	functional form of MG parameters
(μ,γ)	$(k^2-3K)\Psi = -4\pi G a^2 \mu(a,k)\left[\rho\Delta + 3\frac{k^2-3K}{k^2}(\rho+p)\sigma\right]$ $k^2[\Phi-\gamma(a,k)\Psi] = 12\pi G a^2\mu(a,k)(\rho+p)\sigma$	$\mu = 1+E_{11}\Omega_{DE}(a)\left[\frac{1+c_1(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$ $\gamma = 1+E_{22}\Omega_{DE}(a)\left[\frac{1+c_2(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$
(μ,Σ)	$(k^2-3K)\Psi = -4\pi G a^2 \mu(a,k)\left[\rho\Delta + 3\frac{k^2-3K}{k^2}(\rho+p)\sigma\right].$ $k^2[\Phi+\Psi] = -4\pi G a^2\Sigma(a,k)\left[\frac{2}{1-3K/k^2}\rho\Delta + 3(\rho+p)\sigma\right]$	$\mu = 1+\mu_0\frac{\Omega_{DE}(a)}{\Omega_\Lambda}\left[\frac{1+c_1(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$ $\Sigma = 1+\Sigma_0\frac{\Omega_{DE}(a)}{\Omega_\Lambda}\left[\frac{1+c_2(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$
(Q,D)	$(k^2-3K)\Phi = -4\pi G a^2 Q(a,k)\rho\Delta$ $k^2[\Phi+\Psi] = -\frac{8\pi G a^2}{1-3K/k^2}D(a,k)\rho\Delta - 12\pi G a^2 Q(a,k)(\rho+p)\sigma$	$Q = 1+Q_0\frac{\Omega_{DE}(a)}{\Omega_\Lambda}\left[\frac{1+c_1(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$ $D = 1+D_0\frac{\Omega_{DE}(a)}{\Omega_\Lambda}\left[\frac{1+c_2(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$
		$Q = [Q_0 e^{-k/k_c} + Q_\infty(1-e^{-k/k_c}) - 1]a^s + 1$ $D = [D_0 e^{-k/k_c} + D_\infty(1-e^{-k/k_c}) - 1]a^s + 1$
(Q,R)	$(k^2-3K)\Phi = -4\pi G a^2 Q(a,k)\rho\Delta$ $k^2[\Psi-R(a,k)\Phi] = -12\pi G a^2 Q(a,k)(\rho+p)\sigma$	$Q = 1+Q_0\frac{\Omega_{DE}(a)}{\Omega_\Lambda}\left[\frac{1+c_1(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$ $R = 1+R_0\frac{\Omega_{DE}(a)}{\Omega_\Lambda}\left[\frac{1+c_2(\lambda H/k)^2}{1+(\lambda H/k)^2}\right]$
		$Q = [Q_0 e^{-k/k_c} + Q_\infty(1-e^{-k/k_c}) - 1]a^s + 1$ $R = [R_0 e^{-k/k_c} + R_\infty(1-e^{-k/k_c}) - 1]a^s + 1$

Here, we can notice that in this new version of ISiTGR we parameterize the MG parameters with a dark energy dependent evolution given by the parameter $\Omega_{DE}(a)$ and a scale dependent evolution given by the term between square brackets. However, we can notice that for the (Q, D) and (Q, R) parameterization we also have a monotonic evolution and the a^s dependence, these equations correspond to the original versions of ISiTGR. Therefore, in this version we add new forms for the MG parameters based in the proposals given in the last years.

1.1.1 Dark energy dependence

We model a dark energy dependence of the MG parameters by defining the parameter $\Omega_{DE}(a) \equiv \frac{\rho_v(t)}{\rho_c(t)}$, that accounts for the dark energy evolution. Then, by solving explicitely for $\rho_v(t)$ we can obtain a formula to compute Ω_{DE} for a given equation of state for the dark energy. Indeed, we get

$$\Omega_{DE}(a) = \Omega_v \left(\frac{H_0}{H}\right)^2 e^{-3\int_{a_0}^{a}[1+w(a)]\frac{da}{a}}, \qquad H = \frac{\dot{a}}{a} \tag{4}$$

We set $a_0 = 1$ and solve for this dark energy parameter for four different models: The traditional ΛCDM model, $wCDM$ model, the CPL parameterization, and the pivot equation of state for the CPL parameterization. We write (4) for each of these models in the next table:

Dark Energy model	Dark Energy density	Dark Energy evolution
$w = -1$	$\rho_v(t) = \rho_v^{(0)}$	$\Omega_{DE} = \Omega_v \left(\frac{H_0}{H}\right)^2$
$w = w_0$	$\rho_v(t) = \rho_v^{(0)} a^{-3(1+w_0)}$	$\Omega_{DE} = \Omega_v \left(\frac{H_0}{H}\right)^2 a^{-3(1+w_0)}$
$w = w_0 + (1-a)w_a$	$\rho_v(t) = \rho_v^{(0)} a^{-3(1+w_0+w_a)} e^{3w_a(a-1)}$	$\Omega_{DE} = \Omega_v \left(\frac{H_0}{H}\right)^2 a^{-3(1+w_0+w_a)} e^{3w_a(a-1)}$
$w = w_p + (a_p - a)w_a$	$\rho_v(t) = \rho_v^{(0)} a^{-3(1+w_p+a_p w_a)} e^{3w_a(a-1)}$	$\Omega_{DE} = \Omega_v \left(\frac{H_0}{H}\right)^2 a^{-3(1+w_p+a_p w_a)} e^{3w_a(a-1)}$

Table 1: Dark energy parameterizations given in ISiTGR, corresponding to ΛCDM, wCDM, CPL parameterization, and the pivot CPL equation of state. Once the user picked one of these models, automatically ISiTGR selects the corresponding dark energy density and the corresponding dark energy evolution.

1.1.2 Scale dependence

the ISiTGR code adds a scale dependence to the MG parameters as proposed by Planck collaboration in 2015. This is, we add a factor of the form

$$S_{1,i}(a,k) = \frac{1 + c_i\,(\lambda H/k)^2}{1 + (\lambda H/k)^2}, \tag{5}$$

to the MG parameters. The index $i = 1, 2$ stands for the first and the second MG parameter. Now, the purpose of the for of $S_{1,i}$ is that at small scales (large k) the factor $S_{1,i} \to 1$, while at large scales (small k) $S_{1,i} \to c_i$. Therefore, at small scales (5) will agreed with General Relativity while for large scales it will quantifies a deviation of General Relativity through the parameters c_1 and c_2 that provide information regarding of how the two MG parameters evolve at large scales. Even more, these two parameters quantify that maybe one of the two MG parameters is more affected by the scale than the other one. Finally, λ also quantifies the deviations from GR at larges scales, recovering General Relativity when for example $\lambda = 0$.

1.2 Binning methods

Finally, ISiTGR incorporates and alternative approach to evolve the MG parameters, called Binning Method. This method allows the user to evolve the MG parameters through an independent evolution based on bins of redshift and scale. Moreover, ISiTGR has two different ways to evolve the MG parameters using this binning method. The first one, the traditional binning method in which one evolves the MG parameters in two different predefined redshift and scale bins, without a functional form of the parameters. However, ISiTGR implements the traditional binning method with some control in the transition. The second way, is the hybrid method incorporated in ISiTGR, in which we still have two predefined bins, but we allow an independent monotonically functional evolution for the MG parameters in each bin.

If $X(a,k)$ represents any MG parameters, its binning form, as implemented in ISiTGR, can be written as,

$$X(a,k) = \frac{1 + X_{z_1}(k)}{2} + \frac{X_{z_2}(k) - X_{z_1}(k)}{2}\tanh\left(\frac{z - z_{div}}{z_{tw}}\right) + \frac{1 - X_{z_2}(k)}{2}\tanh\left(\frac{z - z_{TGR}}{z_{tw}}\right), \tag{6}$$

where z_{div} is the specific redshift at which the transition between the two bins happens, and z_{TGR} is the redshift from which below of it, General Relativity(GR) is tested. Also, z_{tw} acts as a transition width for the hyperbolic tangent function.

Moreover, It is worth noting that the parameter $X(a, k)$ is now in terms of two new parameters $X_{z_1}(a, k)$ and $X_{z_2}(a, k)$.
Now, as we mentioned, ISiTGR provides some control in the transition between the bins for the traditional binning method. We do this by giving to the parameters $X_{z_1}(a, k)$ and $X_{z_2}(a, k)$ the form

$$X_{z_1}(a, k) = \frac{X_2 + X_1}{2} + \frac{X_2 - X_1}{2} \tanh\left(\frac{k - k_c}{k_{tw}}\right), \tag{7}$$

and

$$X_{z_2}(a, k) = \frac{X_4 + X_3}{2} + \frac{X_4 - X_3}{2} \tanh\left(\frac{k - k_c}{k_{tw}}\right), \tag{8}$$

respectively.
Moreover, ISiTGR provides another method that gives some control in the transition between the redshift bins, by giving a functional form for the $X_{z_1}(a, k)$ and $X_{z_2}(a, k)$ parameters. This method is called the hybrid binning method, and the above functions are parameterized as

$$X_{z_1}(a, k) = X_1 e^{-k/k_c} + X_2(1 - e^{-k/k_c}), \tag{9}$$

and

$$X_{z_2}(a, k) = X_3 e^{-k/k_c} + X_4(1 - e^{-k/k_c}). \tag{10}$$

This way of parameterizing these functions provides a smoother transition between the bins.

2 Modifications to equations.f90

In this section we explicitely describe the changes made to the file *equations.f90*, the principal file that contains the background and perturbartion evolution equations in the usual CAMB program. All changes made to the original file correspond to the CosmoMC 2018 July version. Any change to each part of the code starts after the comment ! *>ISiTGR MOD START* and finish before the comment ! *<ISiTGR MOD END*. So any part of code written inside these region is a modification as a part of the ISiTGR patch. In the following, we describe the modifications given by the ISiTGR patch on the *equations.f90* file.

2.1 Module ISiTGR

This module is completely new in the *equations.f90* file. It starts with the definition of Modified Gravity variables and logical variables that allow us to turn on or turn

off a certain MG parameterization.
Then, the module contains 16 functions which encoded the information of the MG parameterizations: (μ,γ), (μ,Σ), (Q,D) and (Q,R). Also, 4 of these 16 functions are used when the user wants to use the binning methods available for the (Q,D) and (Q,R) parameterizations. These parameterizations correspond to phenomenological modifications to the field equations given in the Newtonian Gauge. Then, we can write the MG perturbed field equations in the most general case as,

$$(k^2-3K)\Psi = -4\pi G a^2 \mu \sum_i \left[\rho_i \Delta_i + 3\left(\frac{k^2-3K}{k^2}\right)\rho_i(1+w_i)\sigma_i\right], \tag{11}$$

$$k^2(\Phi-\gamma\Psi) = 12\pi G a^2 \mu \sum_i \rho_i(1+w_i)\sigma_i, \tag{12}$$

for the (μ,γ) parameterization. In the case of (μ,Σ), instead of (12), we need to consider

$$k^2(\Phi+\Psi) = -4\pi G a^2 \Sigma \sum_i \left[\frac{2\rho_i\Delta_i}{1-3K/k^2} + 3\rho_i(1+w_i)\sigma_i\right]. \tag{13}$$

Then, the equations (11) and (13) are used when working with the (μ,Σ) parameterization.
For the (Q,D) case, we have the MG perturbed field equations

$$(k^2-3K)\Phi = -4\pi G a^2 Q \sum_i \rho_i\Delta_i, \tag{14}$$

and

$$k^2(\Phi+\Psi) = \left(\frac{-8\pi G a^2}{1-3K/k^2}\right) D \sum_i \rho_i\Delta_i - 12\pi G a^2 Q \sum_i \rho_i(1+w_i)\sigma_i. \tag{15}$$

Furthermore, if we consider the (Q,R) parameterization, we need to use

$$k^2(\Psi-R\Phi) = -12\pi G a^2 Q \sum_i \rho_i(1+w_i)\sigma_i, \tag{16}$$

instead of (15). Here, it is important to notice that $\Sigma = D$ only in the zero anisotropic shear case.
Now, we can write the explicit form of the MG gravity parameters and its derivatives as are coded in the ISiTGR module, for the (μ,γ) and (μ,Σ) parameterizations as:

(μ,γ) case:

$$\mu = 1 + E_{11}\Omega_{DE}(a)S_{1,1}(a,k), \tag{17}$$

$$\mu' = E_{11}[\Omega'_{DE}(a)S_{1,1}(a,k) + \Omega_{DE}(a)S'_{1,1}(a,k)], \tag{18}$$

$$\gamma = 1 + E_{22}\Omega_{DE}(a)S_{1,2}(a,k), \tag{19}$$

$$\gamma' = E_{22}[\Omega'_{DE}(a)S_{1,2}(a,k) + \Omega_{DE}(a)S'_{1,2}(a,k)]. \tag{20}$$

(μ, Σ) case:

$$\mu = 1 + \mu_0 \frac{\Omega_{DE}(a)}{\Omega_v} S_{1,1}(a,k), \tag{21}$$

$$\mu' = \mu_0 \frac{[\Omega'_{DE}(a)S_{1,1}(a,k) + \Omega_{DE}(a)S'_{1,1}(a,k)]}{\Omega_v}, \tag{22}$$

$$\Sigma = 1 + \Sigma_0 \frac{\Omega_{DE}(a)}{\Omega_v} S_{1,2}(a,k), \tag{23}$$

$$\Sigma' = \Sigma_0 \frac{[\Omega'_{DE}(a)S_{1,2}(a,k) + \Omega_{DE}(a)S'_{1,2}(a,k)]}{\Omega_v}. \tag{24}$$

Here, the user can choose a dark energy parameterization as given in table (1). Then ISiTGR will compute the corresponding Ω_{DE}.
Furthermore, the ISiTGR code add the scale dependent contributions by calculating the quantities

$$S_{1,i}(a,k) = \frac{1 + c_i S_2(a,k)}{1 + S_2(a,k)}, \quad (i = 1,2), \qquad S_2(a,k) = \left(\frac{\lambda H}{k}\right)^2 \tag{25}$$

and the corresponding derivative

$$S'_{1,i}(a,k) = \frac{S'_2(a,k)(c_i - 1)}{[1 + S_2(a,k)]^2}, \qquad S'_2(a,k) = \frac{2S_2(a,k)(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}}, \tag{26}$$

where the parameters λ, c_1 and c_2 quantifies the scale dependent evolution of the MG parameters. Thus, we represent the scale dependent part with the functions $S_{1,i}(k)$ and $S_2(k)$. Moreover, the cases (Q, D) and (Q, R) are similarly implemented as the (μ, Σ) case, following the normalized dark energy dependence.

2.2 Subroutine MassiveNuVarsOut

The purpose of this subroutine is to calculate quantities that are relevant when you are considering massive neutrinos, in case the user turn on this possibility. Basically, in this subroutine we are calculating the contribution of massive neutrinos to the quantities $8\pi G a^2 \sum_i \rho_i \Pi_i (3w_i + 1)$, $8\pi G a^2 \sum_i \rho_i \Pi_i (3w_i + 2)$ and $8\pi G a^2 \sum_i \rho_i \Pi_i (3w_i + 1 + \beta_k)$, where the anisotropic shear Π_i is defined as $\Pi_i = \frac{3}{2}(1 + w_i)\sigma$. Thefore, the modification made by ISiTGR to this subroutine is simple, we are computing the contribution of massive neutrinos to these quantities that will become relevant in other parts of the code when summing over all species with non-zero stress shear. Then, we define the quantities

$$\text{dgpi\_3wplus1} = 8\pi G a^2 \sum_i \rho_i \Pi_i (3w_i + 1), \tag{27}$$

$$\text{dgpi\_3wplus2} = 8\pi G a^2 \sum_i \rho_i \Pi_i (3w_i + 2), \tag{28}$$

and

$$\text{dgpi\_3wplus1plusbetak} = 8\pi G a^2 \sum_i \rho_i \Pi_i (3w_i + 1 + \beta_k), \tag{29}$$

as are used in the ISiTGR patch. Now, our modification to the *MassiveNuVarsOut* subroutine is to compute the contributions of massive neutrinos to the above quantities. Then, in this subroutine just compute the contributions $8\pi G a^2 \rho_\nu \Pi_\nu (3w_\nu + 1)$, $8\pi G a^2 \rho_\nu \Pi_\nu (3w_\nu + 2)$ and $8\pi G a^2 \rho_\nu \Pi_\nu (3w_\nu + 1 + \beta_k)$.

2.3 Subroutine derivs

Most of the modifications to the ISiTGR patch are in this subroutine. It is worth mentioning that, older versions of CosmoMC used to compute the derivatives of relevant quantities in this subroutine and the outputs in another subroutine called *output*. Now, for recent versions, all is done in this single subroutine, called *derivs*.

In order to get the MG equations with the contributions of massive neutrinos, considering the flat case and cases with spatial curvature different from zero, we wrote a patch in this subroutine consisting in five major steps:

ISiTGR patch for MG	
(I)	Compute σ_{CAMB} (called just σ in the code). For this part, we also obtain Φ, Ψ
(II)	Get the contributions of massive neutrinos for $\sum_i \rho_i \dot{\Pi}_i$. This contributions are computed up to a certain order, so we did not use z_{CAMB}.
(III)	Calculate $\dot{\sigma}_{\text{CAMB}}$.
(IV)	Compute $k\dot{\eta}$. Then, it is straigthforward to get z_{CAMB}, $\dot{\Phi}$, $\dot{\Psi}$.
(V)	Obtain $\Phi + \Psi$ for the Weyl potential and $\dot{\Phi} + \dot{\Psi}$ for the Integrated Sachs-Wolfe effect.

Table 2: five major steps computed into the ISiTGR patch in the *derivs* subroutine inside CAMB.

Now, we explain more in detail each of these steps and derive the equations used in the code.

(I) Get $\sigma_{\mathbf{CAMB}}$.

First of all, we call the different MG functions computed into the Module ISiTGR to this subroutine *derivs*. Then, we need to add the contributions for radiation, like photons and massless neutrinos, and the contributions of massive neutrinos to quantities like dgpi= $\sum_i \hat{\rho}_i \Pi_i$. The code uses the definition$\sum_i \hat{\rho}_i \equiv 8\pi G a^2 \sum_i \rho_i$ and works directly with $\hat{\rho}_i$.
Next, we obtain σ_{CAMB} by

$$\sigma_{CAMB} = \frac{k(\eta - \Phi)}{\mathcal{H}}, \tag{30}$$

where in the code $\mathcal{H} = \frac{a'}{a} = \dot{a} =$ adotoa. Therefore, we calculate σ_{CAMB} through Φ, but in turn, for some parameterizations we need to get Φ in order to compute Ψ. So in order to get (30) in the code, we first compute these quantities given by (11), (12) and (13). However, the code works with the anisotropic shear which is defined as $\Pi_i = \frac{3}{2}(1 + w_i)\sigma$. Then we can solve for the MG potentials in order to get

$$\Psi = -\frac{\mu}{2k^2} \sum_i [\beta_k \Delta_i \hat{\rho}_i + 2\Pi_i \hat{\rho}_i] \tag{31}$$

and

$$\Phi = \frac{\mu}{k^2} \sum_i \Pi_i \hat{\rho}_i + \gamma \Psi, \tag{32}$$

where Φ is given for the (μ, γ) parameterization and $\beta_k \equiv (1 - \frac{3K}{k^2})^{-1}$. While, for the (μ, Σ) parameterization, we have

$$\Phi = -\frac{\Sigma}{k^2} \sum_i [\beta_k \Delta_i \hat{\rho}_i + \Pi_i \hat{\rho}_i] - \Psi. \tag{33}$$

Thus, in the code we first compute Ψ, then depending on the parameterization we are using, we compute Φ through (32) or (33). Afterwards, we simply compute σ_{CAMB} by using (30). Finally, for the (Q, D) and (Q, R) parameterizations, we simply use

$$\Phi = -\frac{Q}{2k^2} \sum_i \beta_k \Delta_i \hat{\rho}_i \tag{34}$$

and

$$\Psi = -\frac{1}{k^2} \sum_i [\beta_k D \Delta_i \hat{\rho}_i + Q \Pi_i \hat{\rho}_i] - \Phi, \tag{35}$$

where we use that $D \equiv \frac{Q}{2}(1 + R)$ to go from (Q, D) parameterization to the (Q, R) parameterization. It is important to mention that, in the case of non-zero stress shear $D \equiv \frac{Q}{2}(1 + R)$ is a definition while $\Sigma = \frac{\mu}{2}(1 + \gamma)$ is a condition obtained when you have $\Pi_i \to 0$.

(II) Contributions of massive neutrinos to $\sum_i \rho_i \dot{\Pi}_i$.

In this part of the ISiTGR patch given in the *derivs subroutine*, we compute the contributions of radiation and massive neutrinos to the quantity $\sum_i \rho_i \dot{\Pi}_i$, by calling some parts of the code that are already written, but taking care of the quantities that contains the term z_{CAMB}.
First, we get the contributions by photons and then the contributions by massless

neutrinos. Later, we need to compute the contributions of massive neutrinos, but here we comment the ones that contains z_{CAMB}.

(III) Get $\dot{\sigma}_{\mathbf{CAMB}}$.

Here, we compute $\dot{\sigma}_{\rm CAMB}$ for each parameterization. There exist two gauge transformations for the MG potentials between the Newtonian and Synchronous gauges, given by

$$\Phi = \eta - \mathcal{H}\alpha \tag{36}$$

and

$$\Psi = \dot{\alpha} + \mathcal{H}\alpha, \tag{37}$$

where $\alpha = \frac{1}{k^2}\left(\frac{\dot{h}}{2} + 3\dot{\eta}\right)$. However, since $\sigma_{CAMB} = k\alpha$ we can use (37) to get

$$\dot{\sigma}_{CAMB} = k\Psi - \mathcal{H}\sigma_{CAMB}. \tag{38}$$

Then, since we got σ_{CAMB} in part (I), it is straightforward to compute (38).

(IV) Calculate $\dot{\eta}$.

Our duty in this part is to compute the derivative of η with respect to the conformal time. Since we want to include the massive neutrinos in our analysis, we need to compute $\dot{\eta}$ without neglecting the contributions to the anisotropic shear and its derivative. Then, when we consider a non-zero shear, the transformations between the MG parameters do not get a simple form, so it is better to compute $\dot{\eta}$ for each set of MG parameters.
In the ISiTGR code we use $k\dot{\eta}$ but here we will calculate $\dot{\eta}$ analitically. Then, if we want to calculate this quantity for the (μ, γ) parameterization we need to take the derivative of (36), to obtain,

$$\dot{\eta} = \dot{\mathcal{H}}\alpha + \mathcal{H}\dot{\alpha} + \dot{\Phi}. \tag{39}$$

Now, since we are doing this for the (μ, γ) parameterization, we need to take the derivative of (32) and plug it into (39),

$$\begin{aligned}\dot{\eta} = \dot{\mathcal{H}}\alpha + \mathcal{H}\dot{\alpha} + \frac{\dot{\mu}}{2k^2}\sum_i \left[2\Pi_i\hat{\rho}_i(1-\gamma) - \gamma\beta_k\Delta_i\hat{\rho}_i\right] + \frac{\mu}{2k^2}\sum_i \Big\{2\dot{\Pi}_i\hat{\rho}_i(1-\gamma) + \\ \dot{\hat{\rho}}_i\left[2\Pi_i(1-\gamma) - \gamma\beta_k\Delta_i\right] - \dot{\gamma}\hat{\rho}_i(\beta_k\Delta_i + 2\Pi_i) - \gamma\beta_k\hat{\rho}_i\dot{\Delta}_i\Big\}.\end{aligned} \tag{40}$$

Here, we need to observe that there are three quantities that we need to compute, $\dot{\alpha}$, $\dot{\hat{\rho}}_i$ and $\dot{\Delta}_i$. The first quantity can be computing by the relation (37) combined with (31),

$$\dot{\alpha} = -\frac{\mu}{2k^2}\sum_i \left[\beta_k \Delta_i \hat{\rho}_i + 2\Pi_i \hat{\rho}_i\right] - \mathcal{H}\alpha. \tag{41}$$

Also, it is know that $\dot{\Delta}_i$ is given by

$$\dot{\Delta}_i = 3(1+w_i)(\dot{\Phi} + \mathcal{H}\Psi) + 3\mathcal{H}w_i\Delta_i - 2\mathcal{H}\Pi_i - kq_i^{(N)} f_1, \tag{42}$$

where $f_1 \equiv 1 + \frac{3(\mathcal{H}^2 - \dot{\mathcal{H}})}{k^2}$ and the superscript (N) denote that in this relation, the heat flux q_i is given in the Newtonian gauge.
Plugging the equations (41) and (42) into (40), we obtain

$$\begin{aligned}\dot{\eta} = {}& \alpha(\dot{\mathcal{H}} - \mathcal{H}^2) + \frac{\dot{\mu}}{2k^2}\sum_i \left[2(1-\gamma)\Pi_i\hat{\rho}_i - \gamma\beta_k\Delta_i\hat{\rho}_i\right] + \frac{\mu}{2k^2}\sum_i \hat{\rho}_i \Big\{ -\mathcal{H}[\beta_k\Delta_i + 2\Pi_i] + \\ & 2\dot{\Pi}_i(1-\gamma) - \dot{\gamma}(\beta_k\Delta_i + 2\Pi_i) - \gamma\beta_k[3(\dot{\Phi} + \mathcal{H}\Psi)(1+w_i) + 3\mathcal{H}\Delta_i w_i - 2\mathcal{H}\Pi_i - kq_i^{(N)} f_1] \Big\} \\ & + \frac{\mu}{2k^2}\sum_i \dot{\hat{\rho}}_i[2\Pi_i(1-\gamma) - \gamma\beta_k\Delta_i].\end{aligned} \tag{43}$$

Now, we need to recall that $\hat{\rho}_i$ is defined as $\hat{\rho}_i = 8\pi G a^2 \rho_i$. Then, by using the relationship provided by the continuity equation of the FLRW metric, $\dot{\rho}_i = -3\mathcal{H}(1+w_i)\rho_i$, we can obtain that

$$\dot{\hat{\rho}}_i = -\mathcal{H}\hat{\rho}_i(1+3w_i). \tag{44}$$

Moreover, we can combined (36) and (37) to obtain

$$\dot{\Phi} + \mathcal{H}\Psi = \dot{\eta} + \alpha(\mathcal{H}^2 - \dot{\mathcal{H}}). \tag{45}$$

With (44) and (45) we can rewrite (43) as

$$\begin{aligned}\dot{\eta} = {}& \frac{1}{2f_{\mu,\gamma}}\Big\{ -2k^2\alpha(\mathcal{H}^2 - \dot{\mathcal{H}}) + \dot{\mu}\sum_i \left[2(1-\gamma)\Pi_i\hat{\rho}_i - \gamma\beta_k\Delta_i\hat{\rho}_i\right] \\ & + \mu\sum_i \Big[-\mathcal{H}\beta_k\Delta_i\hat{\rho}_i - 2\mathcal{H}\Pi_i\hat{\rho}_i + 2(1-\gamma)\dot{\Pi}_i\hat{\rho}_i - \mathcal{H}\hat{\rho}_i(1+3w_i)[2(1-\gamma)\Pi_i - \gamma\beta_k\Delta_i] \\ & - \dot{\gamma}(\beta_k\Delta_i + 2\Pi_i)\hat{\rho}_i - 3\gamma\beta_k\mathcal{H}\Delta_i\hat{\rho}_i w_i + 2\gamma\beta_k\mathcal{H}\Pi_i\hat{\rho}_i + k\gamma\beta_k f_1 q_i^{(N)}\hat{\rho}_i - 3\alpha\gamma\beta_k(\mathcal{H}^2 - \dot{\mathcal{H}})(1+w_i)\hat{\rho}_i \Big] \Big\},\end{aligned} \tag{46}$$

where we defined $f_{\mu,\gamma} = k^2 + \frac{3}{2}\beta_k\mu\gamma\sum_i(1+w_i)\hat{\rho}_i$.
Finally, we can notice that in the previous equation, the heat flux is computed in the Newtonian gauge. However, CAMB calculates the quantities in the Synchronous gauge. Therefore, we need to conver the heat flux into the Synchronous gauge by applying the transformation

$$q_i^{(N)} = q_i^{(S)} + k\alpha(1+w_i). \tag{47}$$

Therefore, using (47) we obtain

$$\begin{aligned}
\dot{\eta}_{(\mu,\gamma)} = \frac{1}{2f_{\mu,\gamma}}\Bigg\{ & k\mu\gamma\beta_k f_1 \sum_i q_i^{(S)}\hat{\rho}_i + \sum_i \beta_k\Delta_i\hat{\rho}_i[\mathcal{H}\mu(\gamma-1) - \dot{\mu}\gamma - \mu\dot{\gamma}] \\
& + 2\mu(1-\gamma)\sum_i \dot{\Pi}_i\hat{\rho}_i + k^2\alpha[-2(\mathcal{H}^2 - \dot{\mathcal{H}}) + \mu\gamma\beta_k\sum_i\hat{\rho}_i(1+w_i)] \\
& - 2[\mu\dot{\gamma} + \dot{\mu}(\gamma-1)]\sum_i \Pi_i\hat{\rho}_i - 2\mathcal{H}\mu\sum_i\Pi_i\hat{\rho}_i(3w_i+2) \\
& + 2\mathcal{H}\mu\gamma\sum_i\hat{\rho}_i\Pi_i(3w_i+1+\beta_k)\Bigg\}.
\end{aligned} \tag{48}$$

Here, the subscript in $\dot{\eta}$ means that this quantity is given in the (μ,γ) parameterization. However, it is worth mentioning that in the actual patch of ISiTGR, we computed instead $\dot{\eta}k$ and we used that $\sigma_{CAMB} = k\alpha$. Finally, it is important to mention that, for the contributions $\sum_i\hat{\rho}_i(1+w_i)$ the code uses that $w_\nu = 0$ for massive neutrinos as an approximation, since for early times the equation of state of massive neutrinos behave as radiation.

For the (μ,Σ) parameterization, we can find that the analogous of (40) is

$$\begin{aligned}
\dot{\eta} = \dot{\mathcal{H}}\alpha + \mathcal{H}\dot{\alpha} + \frac{1}{2k^2}\sum_i \Bigg\{ & \beta_k(\dot{\mu} - 2\dot{\Sigma})\sum_i\Delta_i\hat{\rho}_i + \beta_k(\mu - 2\Sigma)\sum_i(\Delta_i\dot{\hat{\rho}}_i + \dot{\Delta}_i\hat{\rho}_i) \\
& + 2(\dot{\mu} - \dot{\Sigma})\sum_i\Pi_i\hat{\rho}_i + 2(\mu-\Sigma)\sum_i(\Pi_i\dot{\hat{\rho}}_i + \dot{\Pi}_i\hat{\rho}_i)\Bigg\}.
\end{aligned} \tag{49}$$

after computing the derivative of (33) instead of the derivative of (32), since we are now in the (μ,Σ) parameterization instead of the (μ,γ) parameterization.
Now, since $\dot{\alpha} = \Psi - \mathcal{H}\alpha$ does not depend on Φ and (31) is valid for both parameterizations, then we can still use (41) in this case. Furthermore, it is clear that we can

still use the results given in (42) and (44). Then, after substituting these equations into (49) and using (45) and (47) we obtain

$$\dot{\eta}_{(\mu,\Sigma)} = \frac{1}{2f_{\mu,\Sigma}}\Bigg\{k\beta_k(2\Sigma-\mu)f_1\sum_i q_i^{(S)}\hat{\rho}_i + \beta_k[(\dot{\mu}-2\dot{\Sigma})+2\mathcal{H}(\Sigma-\mu)]\sum_i \Delta_i\hat{\rho}_i$$
$$+2(\mu-\Sigma)\sum_i \dot{\Pi}_i\hat{\rho}_i + 2[(\dot{\mu}-\dot{\Sigma})+\mathcal{H}\beta_k(2\Sigma-\mu)-\mathcal{H}\mu]\sum_i \Pi_i\hat{\rho}_i$$
$$+2\mathcal{H}(\Sigma-\mu)\sum_i \Pi_i(1+3w_i)\hat{\rho}_i + k^2\alpha\left[\beta_k(2\Sigma-\mu)\sum_i(1+w_i)\hat{\rho}_i - 2(\mathcal{H}^2-\dot{\mathcal{H}})\right]\Bigg\}, \tag{50}$$

where $f_{\mu,\Sigma} = k^2 + \frac{3}{2}\beta_k(2\Sigma-\mu)\sum_i \hat{\rho}_i(1+w_i)$ and f_1 is still defined as before.
On the other hand, it is important to mention that, we cannot go from (50) to (48) just by using the usual relation $\Sigma = \frac{\mu}{2}(1+\gamma)$, because this is only valid in the case of zero shear. Therefore, we need to get a more general relation including a non-zero shear term in order to do this.
Finally, we can also write an expression for $\dot{\eta}$ for the (Q,D) case, given by

$$\dot{\eta}_{(Q,D)} = -\frac{1}{2f_Q}\Bigg\{2k^2\alpha(\mathcal{H}^2-\dot{\mathcal{H}}) + [2\mathcal{H}(D-Q)+\dot{Q}]\beta_k\sum_i \Delta_i\hat{\rho}_i$$
$$-k^2\alpha\sum_i \beta_k Q(1+w_i)\hat{\rho}_i - k\beta_k Q f_1\sum_i q_i^{(S)}\hat{\rho}_i - 2\mathcal{H}Q(\beta_k-1)\sum_i \hat{\rho}_i\Pi_i\Bigg\}. \tag{51}$$

Here, $f_Q = k^2 + \frac{3}{2}\beta_k Q\sum_i(1+w_i)\hat{\rho}_i$. Moreover, we can compute this quantity also for the (Q,R) case by using that $R = \frac{2D}{Q} - 1$, which is a general relation even for the non-zero shear case.

(V) Weyl potential and Integrated Sachs-Wolfe (ISW) effect.

The last part of the derivs subroutine is what it was the output subroutine, in the previous versions of CAMB. At this point, we only coded the contributions of these MG parameterizations to the Weyl potential and ISW. The simple case is the Weyl potential, which is given by $\frac{k^2}{2}(\Phi+\Psi)$ and we can calculated through the set of equations (31)-(33).
In order to get the ISW effect contribution we need to know the derivatives of the MG potentials. We can easily code $\dot{\Phi}$ by recalling (39), so at this stage we have already computed $\dot{\eta}$ and $\alpha = \frac{\sigma_{CAMB}}{k}$, so we can get $\dot{\Phi}$ by

$$\dot{\Phi} = \dot{\eta} - \dot{\mathcal{H}}\alpha - \mathcal{H}\dot{\alpha}. \tag{52}$$

Then, we can obtain $\dot{\Psi}$ by directly deriving (31) which only depends on μ, so it will be valid for both (μ, γ) and (μ, Σ) parameterizations. Doing this, we obtain

$$\dot{\Psi} = -\frac{\dot{\mu}}{2k^2}\sum_i \left[\beta_k \Delta_i \hat{\rho}_i + 2\Pi_i \hat{\rho}_i\right] + \frac{\mu}{2k^2}\sum_i \left[(\beta_k \Delta_i + 2\Pi_i)\dot{\hat{\rho}}_i + (\beta_k \dot{\Delta}_i + 2\dot{\Pi}_i)\hat{\rho}_i\right]. \quad (53)$$

Here we can substitute (42), (44) and (47) into the above equation to get

$$\begin{aligned} \dot{\Psi} = -\frac{\dot{\mu}}{2k^2}\sum_i \left[\beta_k \Delta_i \hat{\rho}_i + 2\Pi_i \hat{\rho}_i\right] + \frac{\mu}{2k^2}\sum_i \Big\{ &\mathcal{H}\beta_k \Delta_i \hat{\rho}_i - 2\dot{\Pi}_i \hat{\rho}_i + k\beta_k f_1 q_i^{(S)} \hat{\rho}_i \\ + \beta_k (1+w_i)\hat{\rho}_i [k^2 \alpha f_1 - 3(\dot{\Phi} + \mathcal{H}\Psi)] + 2\mathcal{H}\Pi_i \hat{\rho}_i (1+3w_i) &+ 2\beta_k \mathcal{H}\hat{\rho}_i \Pi_i \Big\}. \end{aligned} \quad (54)$$

Thus, we can compute the contribution to the ISW effect through (52) and (54).

On the other hand, if we want to compute the ISW effect for the (Q, D) and (Q, R) we need to derive another expression for $\dot{\Psi}$. Following similar steps we can obtain

$$\begin{aligned} \dot{\Psi} = -\dot{\Phi} - \frac{1}{k^2}\sum_i \Big\{ &\beta_k \dot{D}\Delta_i \hat{\rho}_i + \dot{Q}\Pi_i \hat{\rho}_i - \beta_k D\mathcal{H}\Delta_i \hat{\rho}_i - Q\mathcal{H}\Pi_i (3w_i+1)\hat{\rho}_i + Q\dot{\Pi}_i \hat{\rho}_i \\ &- 2\beta_k D\mathcal{H}\Pi_i \hat{\rho}_i - k\beta_k D f_1 q_i^{(S)} \hat{\rho}_i + \beta_k (1+w_i)\hat{\rho}_i [3D(\dot{\Phi} + \mathcal{H}\Psi) - k^2 \alpha f_1 D] \Big\}. \end{aligned} \quad (55)$$

Finally, here you can find the name of some important quantities as are written in the code of ISiTGR or CAMB:

$$\text{gqpi} = \sum_i \Pi_i \hat{\rho}_i, \quad (56)$$

$$\text{gqpi\_3wplus2} = \sum_i \Pi_i \hat{\rho}_i (2+3w_i), \quad (57)$$

$$\text{gqpi\_wplus1} = \sum_i \Pi_i \hat{\rho}_i (2+3w_i), \quad (58)$$

$$\text{dgq} = \sum_i q_i^{(S)} \hat{\rho}_i, \quad (59)$$

$$\text{TGR\_rhoDelta} = \sum_i \Delta_i \hat{\rho}_i, \quad (60)$$

$$\text{pidot\_sum} = \sum_i \dot{\Pi}_i \hat{\rho}_i, \tag{61}$$

$$\text{gpres} = 8\pi G a^2 p_i \equiv \hat{p}_i, \tag{62}$$

$$\text{dgrho} = \sum_i \delta_i \hat{\rho}_i, \tag{63}$$

also in the code grhoc$\rightarrow$ ρ_c^0, grhob$\rightarrow$ ρ_b^0, grhor$\rightarrow$ $\rho^0_{masslessnu}$, grhog$\rightarrow$ $\rho^0_{photons}$, grhonu$\rightarrow$ $\rho^0_{massivenu}$, grhov$\rightarrow$ ρ^0_Λ, while grhoc_t$\rightarrow$ $\rho_c = \rho_c^0 a^{-(1+0)}$ for example.