Algebraic representations of the circle

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August 15, 2024

0.0.1. We are interested in the following examples of groups and their representation theory.

- $G \subset GL_n$ a classical affine algebraic group scheme,
- G a coaffine group stack, such as $G = B\mathbb{G}_a$,
- G an animated group, such as $G = S^1$,
- G an abelian variety such as $G = E$ an elliptic curve.

0.0.2. Very roughly speaking, representation theory cannot see non-affineness of groups, but higher representation theory can. We will explain the following table.[1]

	G 0-affine? affinization?	BG 0-affine?	G 1-affine?	BG 1-affine?
G affine sch.	yes	yes if G unipotent no otherwise	yes	yes
G coaffine st.	yes	yes	yes	yes
G animated	no $\operatorname{Spec}\mathcal{O}(G)$ coaffine	no	yes	no
G abelian var.	no $\operatorname{Spec}\mathcal{O}(G)$ coaffine	no	yes	???
$G = B\mathbb{G}_a$	yes	yes	yes	yes
$G = S^1$	no $\operatorname{Spec}\mathcal{O}(G) = B\mathbb{G}_a$	no	yes	no
$G = E$	no $\operatorname{Spec}\mathcal{O}(G) = B\mathbb{G}_a$	no	yes	???
	$\mathrm{Rep}(G)$ only knows $\mathrm{Aff}(G)$	"Koszul duality"	$2\mathrm{Rep}(G)$ only knows $1\mathrm{Aff}(G)$	"equiv'n corresp."

0.0.3. The notion of 1-affineness was initially studied by Gaitsgory.

1 0-categorical

1.0.1. We define the category of representations in the usual way.

Definition 1.0.2. Let G be a group algebra in prestacks over k. Then, $\mathcal{O}(G)$ is a coalgebra object in $\mathbf{Vect}_k$ and we define

$$\mathrm{Rep}(G) := \mathrm{Mod}_{\mathbf{Vect}_k}(\mathcal{O}(G)).$$

It is a general fact that

$$\mathrm{Rep}(G) \simeq \mathrm{QCoh}(BG).$$

[1] Often, up to renormalization. E.g. $B^4\mathbb{G}_a$ is not 1-affine, but I suspect this is just a renormalization issue?

1.0.3. Our first observation is that $\mathrm{Rep}(G)$ cannot distinguish between G and its (0-)affinization $\operatorname{Spec}\mathcal{O}(G)$. So the study of $\mathrm{Rep}(G)$ is the same as the study of $\mathrm{Aff}(G)$.

1.0.4. Our next observation is that sometimes, the category $\mathrm{Rep}(G)$ is "affine." Let k denote the trivial representation, and note that $\mathrm{End}_G(k,k) \simeq \mathcal{O}(BG)$. We have commuting adjoint functors

$$\begin{tikzcd}
\mathrm{Rep}(G) \arrow[rr, "\simeq"] \arrow[d, equal] & & \mathrm{QCoh}(BG) \arrow[d, shift left, "\Gamma"] \\
\mathrm{Comod}(\mathcal{O}(G)) \arrow[rr, shift right, "(-)^G := \mathrm{Hom}_G(k,-)"'] & & \mathrm{Mod}(\mathcal{O}(BG)) \arrow[ll, shift right, "-\otimes_{\mathcal{O}(BG)} k"'] \arrow[u, shift left, "-\otimes_{\mathcal{O}(BG)}\mathcal{O}_{BG}"]
\end{tikzcd}$$

We say a group G has a 0-equivariantization correspondence if the adjunction above defines inverse equivalences, possibly up to renormalization.[2] This is closely related to the (0-)affineness of BG.

1.0.5. Let us check this in examples.

- If G is a reductive affine algebraic group, then obviously $(-)^G : \mathrm{Rep}(G) \to \mathrm{Mod}(k)$ is not an equivalence. If G is unipotent, say $G = B\mathbb{G}_a$, then $\mathrm{Rep}(G)$ is generated by the trivial representation, and $\mathrm{End}_{\mathbb{G}_a}(k,k) \simeq k[\eta]$ where $\eta \in \mathrm{Ext}^1_{\mathbb{G}_a}(k,k)$, so we have equivalences

$$\mathrm{Rep}(\mathbb{G}_a) \underset{-\otimes_{k[\eta]} k}{\overset{(-)^{\mathbb{G}_a}}{\rightleftarrows}} \mathrm{Mod}(k[\eta])$$

  Note that the augmentation module $k \in \mathrm{Mod}(k[\eta])$ is non-compact and goes to the infinite-dimensional cofree $\mathbb{G}_a$-representation.

- If G is coaffine, then it is affine by definition.[3] For example, for $G = B\mathbb{G}_a$ we have the usual Koszul duality, first identifying $\mathrm{Rep}(B\mathbb{G}_a)$ with comodules for $\mathcal{O}(B\mathbb{G}_a)$, then with modules for its k-linear dual $\mathcal{O}(B\mathbb{G}_a)^* \simeq k[\lambda]$:

$$\mathrm{Rep}(B\mathbb{G}_a) \simeq \mathrm{Mod}(k[\lambda]) \underset{-\otimes_{k[u]} k}{\overset{(-)^{B\mathbb{G}_a}}{\rightleftarrows}} \mathrm{Mod}(k[u])$$

  where $k[u] \simeq \mathcal{O}(B^2\mathbb{G}_a)$. Note the above isn't quite right, we have to renormalize.

- If $G = S^1$, obviously this is not affine. Its affinization is $B\mathbb{G}_a$.

- If $G = E$, it also not affine, and its affinization is again $B\mathbb{G}_a$. Indeed, if V is a representation of E, then the action map must have proper image, therefore 0-dimensional image. But it must also be connected, so it is the identity.

2 1-categorical

2.0.1. We now discuss higher representations and sheaves.

Definition 2.0.2. For any group object G, the category $\mathrm{QCoh}(G)$ is a comonoidal category. We define the 2-category of *2-representations*:

$$2\mathrm{Rep}(G) := \mathbf{Comod}_{\mathbf{dgCat}_k}(\mathrm{QCoh}(G)).$$

We also define for any prestack X the 2-category of *2-quasicoherent sheaves* on X to be the category of sheaves of categories on X.

$$2\mathrm{QCoh}(X) := \mathrm{ShCat}(X).$$

[2]This isn't precisely defined, of course. We leave it open to interpretation.

[3]We allow ourselves some flexibility with what this means, e.g. the Spec vs. cSpec.

We say that X is *1-affine* if $2\mathrm{QCoh}(X) \simeq \mathbf{Mod}(\mathrm{QCoh}(X))$. The *1-affinization*[4] $1\mathrm{Aff}(X)$ of X is a 1-affine Y with a map $X \to Y$ defining an equivalence $2\mathrm{QCoh}(X) \simeq \mathbf{Mod}(\mathrm{QCoh}(Y))$.

2.0.3. Assuming that G is 1-affine,[5] we have

$$2\mathrm{Rep}(G) \simeq 2\mathrm{QCoh}(BG).$$

Assuming that BG is 1-affine, we then have an equivariantization correspondence:

$$2\mathrm{Rep}(G) \simeq \mathbf{Mod}(\mathrm{QCoh}(BG)).$$

2.0.4. Examples.

1. For G affine algebraic, the correspondence is well-known. For example, one can recover via descent that

$$\mathrm{QCoh}(X/G) \otimes_{\mathrm{QCoh}(BG)} \mathbf{Vect}_k \simeq \mathrm{QCoh}(X)$$

   with the usual $\mathrm{QCoh}(G)$-action.

2. As discussed earlier, $B^2\mathbb{G}_a$ is 0-affine, so it is 1-affine, so it $B\mathbb{G}_a$ satisfies 1-equivariantization.

3. On the other hand, S^1 satisfied 0-equivariantization for the dumb reason that $\mathrm{Rep}(S^1)$ is basically $\mathrm{Rep}(B\mathbb{G}_a)$, even though BS^1 was not 0-affine. However, S^1 does not satisfy 1-equivariantization. For example, consider the regular representation $\mathrm{QCoh}(S^1) \simeq \mathrm{QCoh}(\mathbb{G}_m)$. We have

$$(\mathrm{QCoh}(S^1)^{\mathrm{QCoh}(S^1)} \otimes_{\mathrm{QCoh}(BS^1)} \mathbf{Vect}_k \simeq \mathbf{Vect}_k \otimes_{\mathrm{QCoh}(BS^1)} \mathbf{Vect}_k \simeq \mathbf{Vect}_k \otimes_{\mathrm{QCoh}(B^2\mathbb{G}_a)} \mathbf{Vect}_k \simeq \mathrm{QCoh}(B\mathbb{G}_a).$$

   This is the full subcategory of $\mathrm{QCoh}(S^1)$ where the automorphism is unipotent.

4. I have nothing intelligent to say about the elliptic situation.

3 Examples for S^1 and $B\mathbb{G}_a$ actions

3.0.1. Let's try to see the phenomenon in these examples. First, Cartier duality.

Theorem 3.0.2. *There is an equivalence of monoidal categories*

$$(\mathrm{QCoh}(S^1), \circ) \simeq (\mathrm{QCoh}(\mathbb{G}_m), \otimes).$$

$$(\mathrm{QCoh}(B\mathbb{G}_a), \circ) \simeq (\mathrm{QCoh}(\widehat{\mathbb{G}}_a), \otimes).$$

Invariants is identified with the $!$-fiber at 1, and coinvariants with the $*$-fiber.

3.0.3. So, a category with an S^1-action sheafifies over $\mathbb{G}_m$, and a category with a $B\mathbb{G}_a$-action is one supported at $1 \in \mathbb{G}_m$.

3.0.4. But we can do even more. Let us take the following general set-up. Let $\mathbf{X}$ be an integer lattice, $T = B\mathbf{X}$ the corresponding topological torus, $\check{T} = \mathrm{Spec}\, k\mathbf{X}$ the dual algebraic torus, $\mathfrak{t} = \mathbf{X} \otimes_{\mathbb{Z}} k$ the Lie algebra, and $\check{\mathfrak{t}}$ its dual. In fact, a category with an S^1-action, or a category over $\check{T}$, naturally sheafifies over "2-shifted" version of $\mathbb{T}^*_{\check{T}}$, namely $\check{T} \times \mathfrak{t}[2]$, and with a $B\mathfrak{t}$-action sheafifies over $\mathfrak{t}[2]$.

[4] I am not sure about existence nor uniqueness.
[5] I think this can be relaxed, but I'm not entirely sure how.

3.0.5. An important example: We can define an S^1-action on $\mathrm{Coh}(X)$ by a map $X \to \mathbb{G}_m$. Then, $\mathrm{Coh}(X)$ is a sheaf of categories over $\mathbb{G}_m \times \mathbb{A}^1[2]$. Given a $t \in \mathbb{G}_m$, we have:

$$\mathrm{Coh}(X)|_{\{z\}\times\mathbb{A}^1[2]} = \mathrm{Coh}(f^{-1}(z))$$

$$\mathrm{Coh}(X)|_{\{z\}\times\mathbb{G}_m[2]} = \mathrm{MF}(X, f-1)$$

$$\mathrm{Coh}(X)|_{\{z\}\times\widehat{\{0\}}} = \mathrm{Perf}(f^{-1}(z))$$

$$\mathrm{Coh}(X)|_{\widehat{\{z\}}\times\{0\}} = \mathrm{Coh}_{f^{-1}(z)}(X)$$

Let's justify some of these, assuming X is smooth for simplicity. The first one is "obvious", i.e. $\mathrm{Coh}(X) \otimes_{\mathrm{Perf}(\mathbb{G}_m)} \mathrm{Perf}(\{z\}) = \mathrm{Coh}(f^{-1}(z))$. There is a $k[u]$-module structure on this category. The u-torions are just the perfect complexes, and that tells us the second and third. For the fourth, we want to set $u = 0$, and you get the orthogonal Lagrangian.

3.0.6. Not every S^1-action on $\mathrm{Coh}(X)$ comes from viewing X over $\mathbb{G}_m$ when X is a stack or derived scheme. For example, there is an S^1-action on $\mathrm{Coh}(\mathcal{L}X)$. When X is a scheme, this is the de Rham differential (in fact, $\mathrm{Coh}(\mathcal{L}X)$ lives over $\widehat{\{1\}}$, i.e. gets a $B\mathbb{G}_a$-action). When $X = BG$, it is the tautological automorphism on adjoint-equivariant bundles on G.

3.0.7. Question: what is $\mathrm{Coh}(\mathcal{L}(BG)) \otimes_{\mathcal{O}(\mathbb{G}_m)} \{1\}$? Is it $\mathrm{Coh}(\widehat{\mathcal{L}}(BG))$?

4 Elliptic curves E

4.0.1. I know basically nothing about this. See recent work of Sibilla–Tomasini, which mentions work of Grojnowski.