Additional Practice Problems with Solutions

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Problem One

Recall the example from the slides. Students had preferences

- $A : S \succ M \succ T$;
- $B : S \succ T \succ M$;
- $C : M \succ T \succ S$

and universities have preferences

- $S : B \succ C \succ A$;
- $M : A \succ C \succ B$;
- $T : A \succ B \succ C$.

Suppose universities proposed to students instead of the other way around. What is the outcome of Deferred Acceptance if this were the case?

Solution:

	1	2	3
S	B	B	B
M	A	A	A
T	A	B	C

Thus, the pairing is $(B, S); (A, M); (C, T)$. As this is the same as the student-proposing result, we can conclude that this is actually the only stable match, since it is both the best and the worst for both students and universities.

Problem Two

We mentioned that the proposing side has truth-telling as its best response and would get their most preferred stable match. This problem investigates that in more detail.

Suppose we are matching circles $1, 2, 3, 4$ on one side and squares A, B, C, D on the other. Circle preferences are

- $1 : A \succ B \succ D \succ C$;
- $2 : A \succ B \succ C \succ D$;
- $3 : B \succ D \succ C \succ A$;
- $4 : A \succ D \succ B \succ C$;

while square preferences are

- $A : 3 \succ 1 \succ 4 \succ 2$;
- $B : 2 \succ 3 \succ 1 \succ 4$;
- $C : 1 \succ 2 \succ 4 \succ 3$;
- $D : 1 \succ 4 \succ 3 \succ 2$.

Part a

Fine the circle-proposing stable match and the square-proposing stable match.

Solution:

	1	2	3	4
1	A	A	A	A
2	A	B	B	B
3	B	B	D	C
4	A	D	D	D

The circle-proposing pairing is $(1, A); (2, B); (3, C); (4, D)$.

	1	2	3
A	3	3	3
B	2	2	2
C	1	2	4
D	A	1	1

The square-proposing pairing is $(1, D); (2, B); (3, A); (4, C)$.

Part b

Verify that the circles like the circle-proposing result more while the squares like the square-proposing result more.

Solution:

Circle 2 is matched with B in either case, so we do not need to worry about them. Circle 1 is matched with A when circles propose and D when squares propose, and 1 prefers $B \succ D$. Circle 3 is matched with C when circles propose and A when squares propose, and 3 prefers $C \succ A$. Circle 4 is matched with D when circles propose and D when squares propose, and 4 prefers $D \succ B \succ C$. Thus, circles $1, 3, 4$ are better off in the circle proposing outcome while circle 2 is indifferent between the two.

Part c

Suppose we implement the circle-proposing Deferred Acceptance algorithm, but allow squares to report whatever preferences they want (for instance, Square A might say that $4 \succ 2$ and no other circles are accessible). What should the squares collectively do?

Solution:

The squares should say that the only acceptable circle to them is the result of the square-proposing Deferred Acceptance algorithm. As such, A says that only 3 is acceptable, B says that only 2 is acceptable, C says that only 4 is acceptable, and D says that only 1 is acceptable. The new outcome of circle-proposing Deferred Acceptance is:

	1	2	3	4
1	A	B	D	D
2	A	B	B	B
3	B	D	C	A
4	A	D	B	C

which is exactly the square-optimal outcome. In general, the non-proposing side can always say that only their optimal match is acceptable.

Problem 3

Our final problem relaxes two of the main assumptions. First, we now suppose that each university has two spots instead of one. Second, this problem will have five students, so the number of students is not equal to the number of spots.

Suppose we have universities A, B, C and students $1, 2, 3, 4, 5$. Each university has two spots, but each student can still only enroll in one university. University preferences are:

- $A : 1 \succ 3 \succ 5 \succ 4 \succ 2$;
- $B : 5 \succ 2 \succ 4 \succ 1 \succ 3$;
- $C : 1 \succ 2 \succ 3 \succ 5 \succ 4$;

and student preferences are:

- $1 : A \succ B \succ C$;
- $2 : B \succ C \succ A$;
- $3 : B \succ A \succ C$;
- $4 : A \succ B \succ C$;
- $5 : A \succ B \succ C$.

Part a

We first modify the student proposing Deferred Acceptance Algorithm. Instead of each university tentatively accepting its top student, they now accept their top two students. Everything else is the same. What is the student-proposing outcome?

Solution:

	1	2	3	4	5
1	A	A	A	A	A
2	B	B	B	C	C
3	B	B	A	A	A
4	A	B	B	B	B
5	A	A	A	B	B

The student proposing outcome is $(1, A); (2, B); (3, A); (4, B); (5, C)$ with C's second spot remaining unfilled.

Part b

We next modify the university proposing Deferred Acceptance Algorithm. Instead of each university proposing to one student at a time, they now start by proposing to their top two students at once. In every subsequent iteration, each university proposes once for each unfilled spot: universities with two tentative acceptances do not propose, universities with one tentative acceptance propose to their top remaining student, and universities with no tentative acceptances propose to their top two remaining students. What is the university-proposing outcome?

Solution:

	1	2	3
A	1,3	1,3	1,3
B	2,5	2,5	2,5
C	1,2	3,5	4

The university proposing outcome is $(1, A); (2, B); (3, A); (4, C); (5, B)$. Note that this is different from the student proposing outcome, but it is still University C with a second unmatched spot.