Convolution of two real functions using a complex FFT

Let us suppose we need to calculate the convolution between two real functions f and g on a one-dimensional lattice of dimensions n:

$$h(j) = f \star g(j) = \sum_{\ell=0}^{n-1} f(\ell) g(\ell - j) \ . \tag{1}$$

From general properties of FFT we have that

$$h(j) = (\widetilde{\tilde{f}(p)\tilde{g}(p)})(x) \ , \tag{2}$$

where the FFT of a function $f(u)$ is indicated with $\tilde{f}(v)$ and u and v are conjugate variables. Let us suppose now that the function f, g are real. Clearly, also the function h is real.

A FFT of a real function $f(p)$ defined in the region $p = 0, \cdots, n-1$ satisfy the property

$$\tilde{f}(p) = \tilde{f}(n-p)^* \ . \tag{3}$$

The calculation of the FFT of a real function can be performed in a different way. Let us define

$$f_e(j) = f(2j) \ , \qquad\qquad f_o(j) = f(2j+1) \ , \tag{4}$$

where now $j = 0, \cdots, n/2-1$. The FFT of $f(j)$ can be rewritten as

$$\begin{aligned} \tilde{f}(p) &= \sum_{\ell=0}^{n-1} f(\ell) e^{-\frac{2\pi i}{n}\ell p} \\ &= \sum_{\ell=0}^{n/2-1} \left(f_e(\ell) + f_o(\ell) e^{-\frac{2\pi i}{n} p} \right) e^{-\frac{2\pi i}{n/2}\ell p} \\ &= \tilde{f}_e(p) + e^{-\frac{2\pi i}{n} p} \tilde{f}_o(p) \ . \end{aligned} \tag{5}$$

The FFT of f_e and f_o are clearly defined only for $p = 0, \cdots, n/2-1$. Consider now the complex function

$$F(j) = f_e(j) + i f_o(j) \Longrightarrow \tilde{F}(p) = \tilde{f}_e(p) + i\tilde{f}_o(p) \ . \tag{6}$$

Given that both f_e and f_o are real functions, we have that

$$\begin{aligned}
\tilde{F}(n/2-p) &= \tilde{f}_e(n/2-p) + i\tilde{f}_o(n/2-p) \\
&= \tilde{f}_e(p)^* + i\tilde{f}_o(p)^* \\
&\Downarrow \qquad (7)\\
\tilde{f}_e(p) &= \frac{1}{2}\left(F(p) + F(n/2-p)^*\right) \ , \qquad (8)\\
\tilde{f}_o(p) &= -\frac{i}{2}\left(F(p) - F(n/2-p)^*\right) \ .
\end{aligned}$$

From these two functions we can calculate $\tilde{f}$ by simply performing a complex FFT with half the points.

This property can also be used in the opposite sense, when calculating an inverse FFT of a function that is known to be real. We know that

$$h(j) = \sum_{p=0}^{n-1} \tilde{h}(p) e^{\frac{2\pi i}{n} jp} \ . \qquad (9)$$

We can easily calculate the inverse FFT for its “even” and “odd” part

$$\begin{aligned}
h_e(j) = h(2j) &= \sum_{p=0}^{n/2-1} \left(\tilde{h}(p) + \tilde{h}(n/2+p)\right) e^{\frac{2\pi i}{n} jp} \ , \\
h_o(j) = h(2j+1) &= \sum_{p=0}^{n/2-1} e^{\frac{2\pi i}{n} p} \left(\tilde{h}(p) - \tilde{h}(n/2+p)\right) e^{\frac{2\pi i}{n} jp} \ . \qquad (10)
\end{aligned}$$

Since the function h is by hypothesis real, we can calculate its invese FFT knowing only the first half of the values of $\tilde{h}$. In fact $\tilde{h}(n/2+p) = \tilde{h}(n/2-p)^*$, thus defining

$$H(j) = h_e(j) + ih_o(j) \qquad (11)$$

we have that

$$\tilde{H}(p) = \tilde{h}_e(p) + i\tilde{h}_o(p) \ , \qquad (12)$$

where

$$\begin{aligned}
\tilde{h}_e(p) &= \tilde{h}(p) + \tilde{h}(n/2-p)^* \ , \\
\tilde{h}_o(p) &= e^{\frac{2\pi i}{n} p} \left(\tilde{h}(p) - \tilde{h}(n/2-p)^*\right) \ . \qquad (13)
\end{aligned}$$

To calculate the convolution of two real objects we can apply both the two reasonement, thus using only half of the point in each FFT step.