A Lower Bound for Joint TDOA-PDOA Localization

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***Abstract*—Estimating the location of a target is essential for many applications such as asset tracking, navigation, and data communications. Time-difference-of-arrival (TDOA) based localization has the main advantage that it does not require synchronization between the transmitting and the receiving sides. Phase-difference-of-arrival (PDOA) provides additional information that can be leveraged to enhance localization performance. In this paper, we derive a lower bound for joint TDOA-PDOA localization based on the Cramer-Rao lower bound (CRLB).**

I. Localization Problem Statement

Consider a 2-D space with N anchors located at $\mathbf{r}_i = [x_i, y_i]^T$, $i = 1, ..., N$, and a target located at a point $\mathbf{p} = [x, y]^T$. The difference of the distances between the target and two anchors i and j is given by

$$d_{ij}(\mathbf{p}) = d_i(\mathbf{p}) - d_j(\mathbf{p}), \tag{1}$$

where $d_i(\mathbf{p})$ is the distance between the target and anchor i, and is given by

$$d_i(\mathbf{p}) = ||\mathbf{p} - \mathbf{r}_i|| = \sqrt{(x-x_i)^2 + (y-y_i)^2}. \tag{2}$$

Here, $||\cdot||$ denotes the L_2 norm.

A noisy TDOA measurement over anchor i and anchor j ($i \neq j$) can be expressed as

$$\tau_{ij} = \frac{d_{ij}(\mathbf{p})}{v} + w_{t,ij}, (i, j = 1, ..., N), \tag{3}$$

where v is the signal propagation speed and $w_{t,ij}$ is the TDOA measurement error.

For N anchors, there are $M = N(N-1)/2$ (N choose 2) possible anchor pairs for TDOA measurement. We denote the full set of all anchor pairs as

$$I_{\mathbf{f}} = \{(i, j)|1 \leq j < i \leq N\}. \tag{4}$$

Assuming that we have $m < M$ measurements, we can collect these measurements in an $m \times 1$ vector $\boldsymbol{\tau} = [\tau_{ij}, ...]^T$, $(i, j) \in I_{\mathbf{f}}$. The signal model for the TDOA measurements becomes [1]

$$\boldsymbol{\tau} = \frac{\mathbf{d}(\mathbf{p})}{v} + \mathbf{w}_t, \tag{5}$$

where $\mathbf{d} = [d_{ij}, ...]^T$, $\mathbf{w}_t = [w_{t,ij}, ...]^T$. The problem of source localization is to estimate the target position $\mathbf{p}$ given $\mathbf{r}_i$, and $\boldsymbol{\tau}$.

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II. CRLB for TDOA-based Localization

The Cramer-Rao lower bound (CRLB) is a bound on the covariance matrix of unbiased estimators. The CRLB is given by $\mathbf{J}^{-1}$, where $\mathbf{J}$ is the Fisher information matrix (FIM) defined as

$$\mathbf{J} = \mathrm{E}\left\{[\frac{\partial}{\partial \mathbf{p}} \log f(\boldsymbol{\tau}; \mathbf{p})][\frac{\partial}{\partial \mathbf{p}} \log f(\boldsymbol{\tau}; \mathbf{p})]^T\right\}. \tag{6}$$

Here $f(\boldsymbol{\tau}; \mathbf{p})$ is the probability density function (PDF) of $\boldsymbol{\tau}$. Assuming that the measurement error vector $\mathbf{w}_t$ in (5) is Gaussian with zero mean and a full-rank covariance matrix $\boldsymbol{\Sigma}_t$ is independent of $\mathbf{p}$, we can have the PDF of $\boldsymbol{\tau}$ [1]

$$f(\boldsymbol{\tau}; \mathbf{p}) = \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det(\boldsymbol{\Sigma}_t)}} \exp\left[-\frac{1}{2}(\boldsymbol{\tau} - \frac{\mathbf{d}}{v})^T \boldsymbol{\Sigma}_t^{-1} (\boldsymbol{\tau} - \frac{\mathbf{d}}{v})\right]. \tag{7}$$

Based on [2], we can calculate the CRLB using a subset set of anchor pairs with anchor 1 as the reference given by

$$I_{\mathbf{s}} = \{(i, 1)|1 < i \leq N\}. \tag{8}$$

The CRLB can be obtained from the diagonal elements of the inverse 2×2 fisher information matrix $\mathbf{F}_t$ as [3]

$$\mathbf{F}_t = \frac{\mathbf{G}\boldsymbol{\Sigma}_t^{-1}\mathbf{G}^T}{v^2}, \tag{9}$$

where $\boldsymbol{\Sigma}_t$ is the covariance matrix of the TDOA estimation noise, and $\mathbf{G}$ is a matrix containing geometry information and is given by

$$\mathbf{G} = [\mathbf{g}_{21}\ \mathbf{g}_{31} \cdots\ \mathbf{g}_{N1}] \in \mathbf{R}^{2\times(N-1)}, \tag{10}$$

$$\mathbf{g}_{i1} = \frac{\mathbf{p} - \mathbf{r}_i}{||\mathbf{p} - \mathbf{r}_i||} - \frac{\mathbf{p} - \mathbf{r}_1}{||\mathbf{p} - \mathbf{r}_1||}. \tag{11}$$

Assuming that the estimations of the signal arrival time at different anchors are independent, we can model the estimation error at anchor i as a zero-mean Gaussian variable with a variance $\sigma_{t,i}^2$. Note that a bias may occur in the signal arrival time estimation as a result of a clock offset, which makes the arrival time estimation error non-zero-mean. Since all anchors are assumed to be synchronized to the same clock, the subtraction of the arrival times at two different anchors to form TDOA measurements eliminates this clock offset. Hence, $\boldsymbol{\Sigma}_t$ can be formed as the following $(N-1) \times (N-1)$ matrix

$$\boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_{t,1}^2 + \sigma_{t,2}^2 & \sigma_{t,1}^2 & \cdots & \sigma_{t,1}^2 \\ \sigma_{t,1}^2 & \sigma_{t,1}^2 + \sigma_{t,3}^2 & \cdots & \sigma_{t,1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{t,1}^2 & \sigma_{t,1}^2 & \cdots & \sigma_{t,1}^2 + \sigma_{t,N}^2 \end{bmatrix}. \tag{12}$$

III. Phase-difference Estimation

Given a signal frequency f, the phase-difference of arrival (PDOA) can be expressed as

$$\phi_{ij}(\mathbf{p}) = 2\pi f \tau_{ij}(\mathbf{p}) = 2\pi f(\frac{d_{ij}(p)}{v}) + w_{p,ij}. \quad (13)$$

Here, ϕ_{ij} is the PDOA between the received signals at anchors R_i and R_j, $w_{p,ij} = 2\pi f w_{p,ij}$ is the PDOA measurement error. An estimation returns only a $wrapped$ version of PDOA that is given by [4]

$$\psi_{ij}(\mathbf{p}) = \mathrm{wrap}(\phi_{ij}(\mathbf{p})) = \phi_{ij}(\mathbf{p}) - k_{ij}(\mathbf{p}). \quad (14)$$

Here $\psi_{i,j} \in [-\pi, \pi)$ is the wrapped PDOA, and

$$k_{ij}(\mathbf{p}) = \mathrm{round}(\phi_{ij}(\mathbf{p})/2\pi), \quad (15)$$

where $\mathrm{round}(.)$ is rounding a number to the closest integer.

IV. A Lower Bound on PDOA-based Localization

Based on (3), (13) and (15), we can write

$$\tau_{ij} = \frac{\phi_{ij}(\mathbf{p})}{2\pi f} = \frac{\psi_{ij}(\mathbf{p}) + k_{ij}(\mathbf{p})}{2\pi f}, (i, j = 1, ..., N), \quad (16)$$

where $k_{ij}(\mathbf{p})$ is multiple times of 2π indicating the difference (wrapped cycles) between the true phase-difference and the wrapped phase-difference as

$$K(\mathbf{p}) = \phi_{i,j}(\mathbf{p}) - \mathrm{wrap}(\phi_{i,j}(\mathbf{p})). \quad (17)$$

Deriving the CRLB requires the derivative of $k_{ij}(p)$ given in 15. This poses a difficulty since the rounding function is not continuous, and hence not differentiable, in $\mathbf{p}$. To circumvent this issue, we propose, as a lower bound (LB) on PDOA position estimation, and *optimistic* bound that coicides with the CRLB for a priori known integers $k_{ij}(\mathbf{p})$. Under the assumption that $k_{ij}(\mathbf{p})$ are known precisely, the ambiguous PDOA observations can be transformed into unambiguous measurements given by the form

$$\boldsymbol{\phi} = 2\pi f \left(\frac{\mathbf{d}(\mathbf{p})}{v} + \mathbf{w}_p \right), \quad (18)$$

which is similar to (5) except for the noise term $\mathbf{w}_p = [w_{p,ij}, ...]^T$. We use the CRLB based on (18) as a LB on PDOA position estimation. This bound is then similar to (9), we just need to replace the covariance matrix of TDOA noise $\boldsymbol{\Sigma}_t$ with the covariance matrix of phase-difference estimation noise $\boldsymbol{\Sigma}_p$. The FIM for localization based on (18) is

$$\mathbf{F}_p = \frac{\mathbf{G}\boldsymbol{\Sigma}_p^{-1}\mathbf{G}^T}{v^2}, \quad (19)$$

where

$$\boldsymbol{\Sigma}_p = \begin{bmatrix} \sigma_{p,1}^2 + \sigma_{p,2}^2 & \sigma_{p,1}^2 & \cdots & \sigma_{p,1}^2 \\ \sigma_{p,1}^2 & \sigma_{p,1}^2 + \sigma_{p,3}^2 & \cdots & \sigma_{p,1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{p,1}^2 & \sigma_{p,1}^2 & \cdots & \sigma_{p,1}^2 + \sigma_{p,N}^2 \end{bmatrix}, \quad (20)$$

$$\sigma_{p,i} = \frac{\delta_{p,i}}{2\pi f}. \quad (21)$$

Here, $\delta_{p,i}^2$ is the variance of phase-difference estimation error in radian and $\sigma_{p,i}^2$ is the corresponding variance in second. The proposed approximated lower bound can be obtained from the diagonal elements of $\mathbf{F}_p^{-1}$, i.e.,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix} \leq \mathrm{PDOA\ LB} = \mathrm{diag}(\mathbf{F}_p^{-1}), \quad (22)$$

where σ_x, σ_y are the mean squared error (MSE) of the position estimation in x and y, respectively.

V. Joint Lower Bound

Similar to PDOA LB, we rely on (18) to derive a lower bound on joint TDOA-PDOA estimation. To this end, we form the joint measurement model from (5) and (18) as

$$\begin{bmatrix} \boldsymbol{\tau} \\ \frac{\boldsymbol{\phi}}{2\pi f} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{d}(\mathbf{p})_t}{v} \\ \frac{\mathbf{d}(\mathbf{p})_p}{v} \end{bmatrix} + \begin{bmatrix} \mathbf{w}_t \\ \mathbf{w}_p \end{bmatrix}. \quad (23)$$

Based on (23), we can write the FIM for the joint model as

$$\mathbf{F}_j = \frac{\begin{bmatrix} \mathbf{G} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_t & \mathbf{O}_{N-1} \\ \mathbf{O}_{N-1} & \boldsymbol{\Sigma}_p \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{G} & \mathbf{G} \end{bmatrix}^T}{v^2}, \quad (24)$$

where $\mathbf{O}_{N-1}$ is a $(N-1) \times (N-1)$ zero matrix. The joint lower bound can be obtained from the inversed joint FIM, i.e.,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix} \leq \mathrm{Joint\ LB} = \mathrm{diag}(\mathbf{F}_j^{-1}). \quad (25)$$

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