The Gauss-Bonnet Theorem

Chen Lin

University of Chicago

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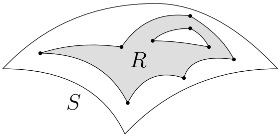

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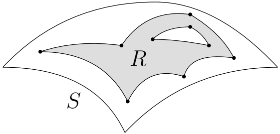

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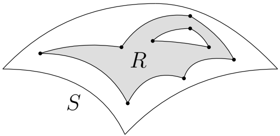

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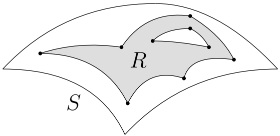

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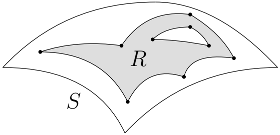

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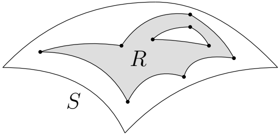

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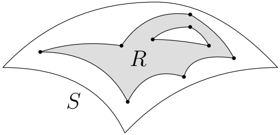

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- k_g, K, and θ_i are geometric, but $\chi(R)$ is topological!

Outline

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4. Some corollaries

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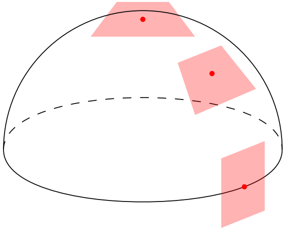

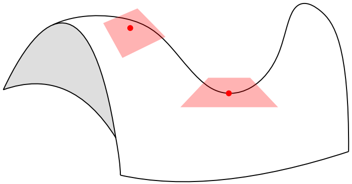

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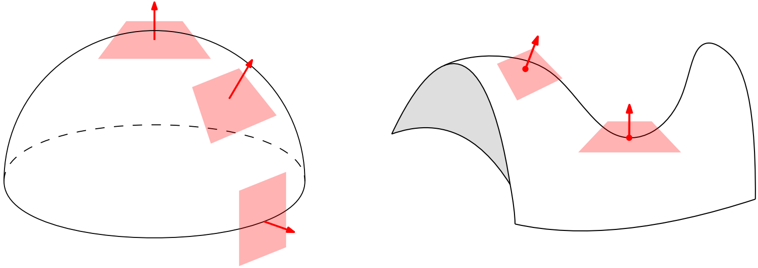
$S \subset \mathbb{R}^3$ is a **surface** if it is 'smooth' and looks flat locally.

- Tangent planes exist at each point
- Identify the tangent plane with a unit normal vector



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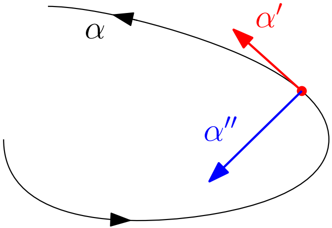

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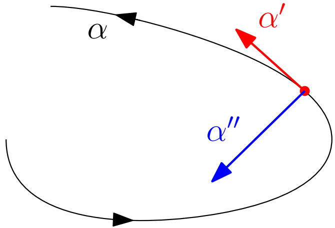

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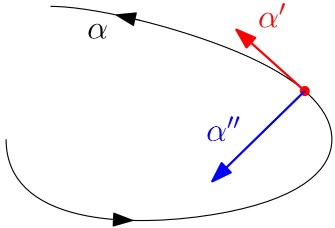

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- We'll start with paths on surfaces

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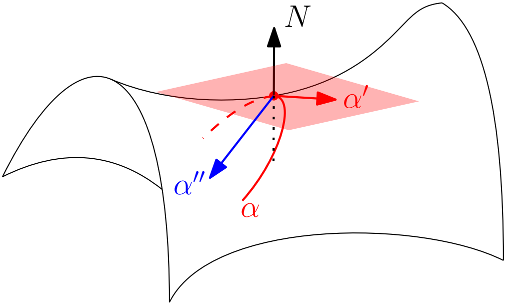

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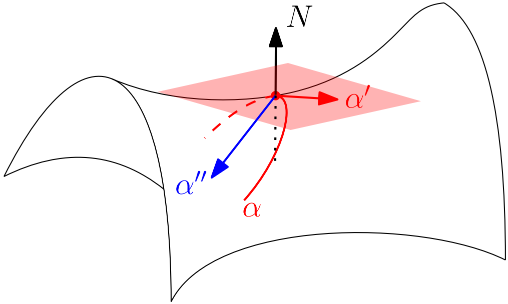

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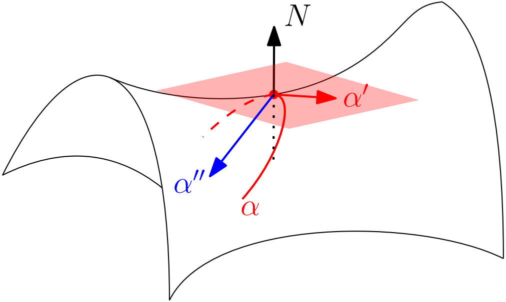

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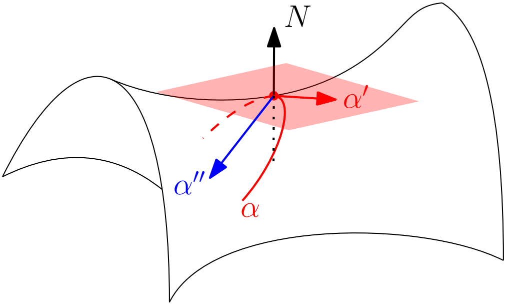

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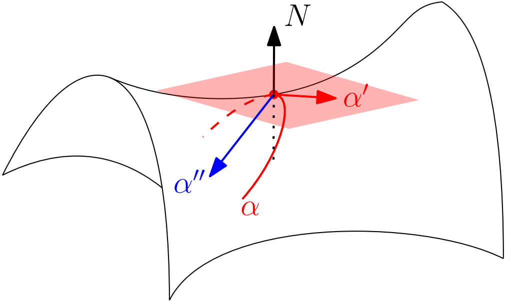

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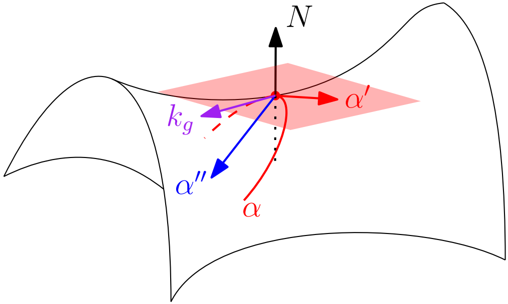

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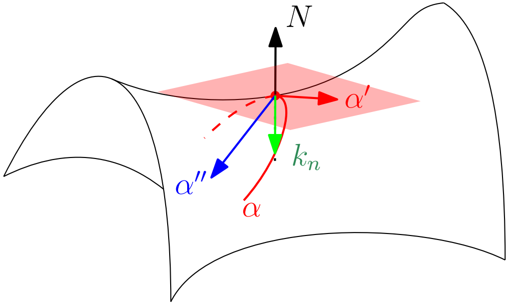

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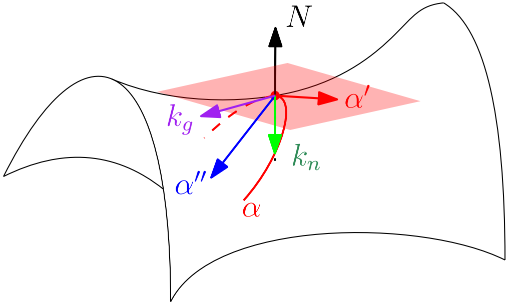

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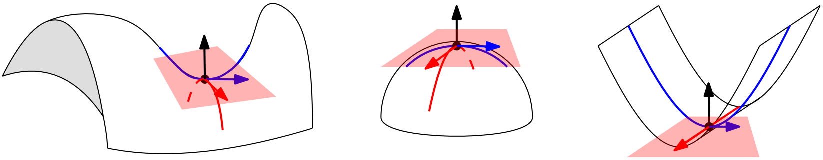

$K < 0$ $\qquad$ $K > 0$ $\qquad$ $K = 0$

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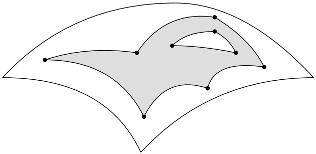
Think of a region R as a 'nice' subset of S



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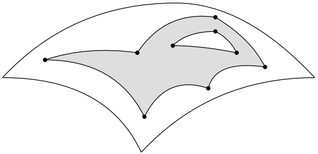

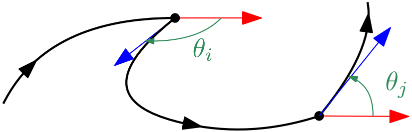

Denote exterior angles with $\theta_1, \ldots, \theta_p$ and interior as $\varphi_i := \pi - \varphi_i$.

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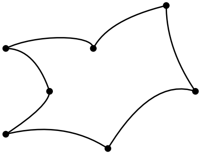

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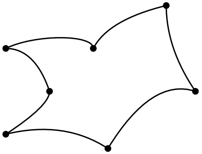

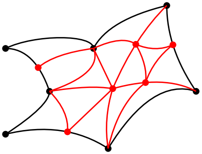

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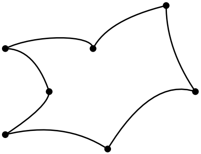

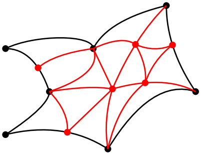

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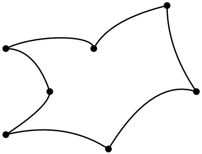

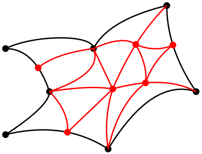

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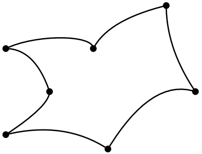

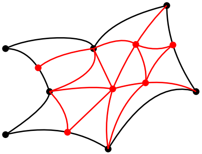

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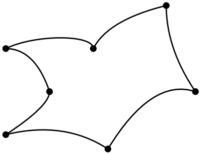

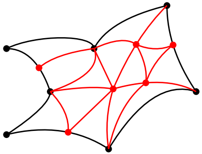

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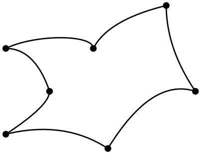

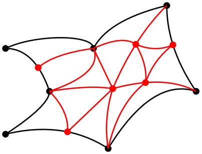

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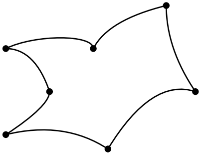

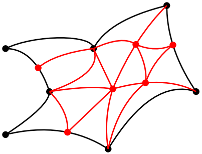

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$\chi(R)$ does not depend on triangulation!

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Uses Stokes' Theorem: $\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R \mathrm{curl}\mathbf{F} \cdot N \, dA$

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3. Sum everything up and $\chi(R)$ will pop out.

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$$\begin{aligned}\iint_S K\ dA &= 2\pi\chi(S),\\ &= 2(2-2g).\end{aligned}$$
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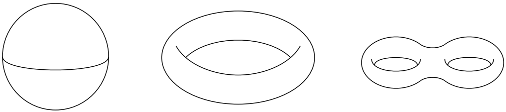

$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

Why care? (continued)

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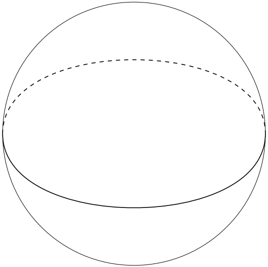
- Take some sphere.



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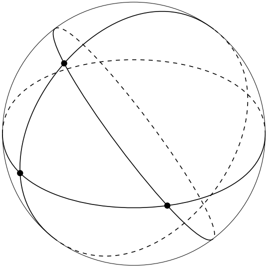

- Draw a triangle whose sides are ‘straight-lines’.

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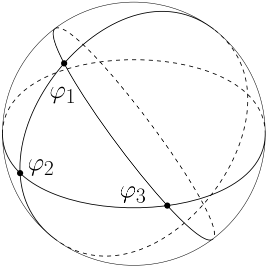
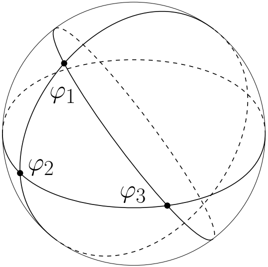


- Draw a triangle whose sides are 'straight-lines'.
- The Gauss-Bonnet Theorem will tell us

$$\varphi_1 + \varphi_2 + \varphi_3 > \pi.$$

Why care? (continued)

- Take some sphere.



- Draw a triangle whose sides are 'straight-lines'.
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- I think that's pretty epic.

THANK YOU!