INTRODUCTORY DIFFERENTIAL TOPOLOGY AND AN APPLICATION TO THE HOPF FIBRATION

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ABSTRACT. This expository paper presents fundamental ideas in differential topology that extend differentiation in Euclidean space to arbitrary manifolds. The concepts introduced culminate in an application to the Hopf fibration, a projection-like map from the 3-sphere to the 2-sphere.

CONTENTS

1. INTRODUCTION

This paper assumes a basic understanding of point-set topology (i.e. open sets, continuous maps), introductory real analysis, and linear algebra. Differential topology concerns calculus on manifolds—topological spaces that locally resemble $\mathbb{R}^n$. We rely on this resemblance to define *smooth*, or differentiable, manifolds, which lay the foundation for smooth functions. The total derivative from real analysis generalizes to the *differential*, along with its various theorems that describe the local behaviors of functions. We develop a foundation that answers questions such as, "what is the derivative of a function defined between surfaces?"

Examples and propositions in the first half of the paper prepare key facts underlying our discussion of the Hopf fibration in the latter half. The Hopf map is a *fiber bundle* that shows the four-dimensional 3-sphere is locally the product space of the 1- and 2-spheres. Fiber bundles describe local relationships between spaces such as smooth manifolds, which evokes a connection to differential topology. In this paper, we apply a basic understanding differential topology to show that the Hopf map is a fiber bundle.

2. Foundations of Differential Topology

2.1. Topological Manifolds.

Roughly speaking, a manifold is a space that resembles $\mathbb{R}^n$ locally. The most basic type is known as a *topological manifold.*

Definition 2.1. Suppose M is a topological space. We say that M is a topological n-manifold if the following are satisfied:

(1) M is *Hausdorff*: for all $p, q \in M$, there exist disjoint open sets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
(2) M is *second-countable*: there exists a countable basis for the topology of M.
(3) M is *locally Euclidean of dimension* n: for all $p \in M$, there exist
  (a) an open subset $U \subseteq M$ containing p,
  (b) some open subset $\widehat{U} \subseteq \mathbb{R}^n$,
  (c) and a homeomorphism $\varphi : U \to \widehat{U}$.

By our definition above, the homeomorphism $\varphi : U \to \widehat{U}$ between open subsets of the manifold and $\mathbb{R}^n$ expresses the exact notion of a manifold resembling Euclidean space. Note that φ is specific to U, so we can expect a different homemorphism for each open set of our manifold.

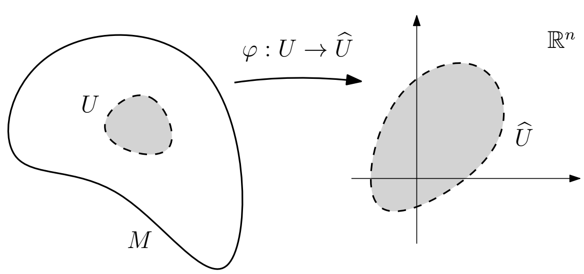

Figure 1. A coordinate chart

Definition 2.2. If M is an n-dimensional manifold, a *coordinate chart* on M is a pair (U, φ). We refer to U as the *coordinate domain*, and φ as the *local coordinate map.*

The following are examples of topological manifolds:

Example 2.3. A simple, yet important topological manifold is $\mathbb{R}^n$ itself. Note that metric spaces—which includes $\mathbb{R}^n$—are Hausdorff, and a countable basis for the metric topology on $\mathbb{R}^n$ is the collection of open balls with rational centers and radii (since $\mathbb{Q}$ is countable). Our chart can simply be $(\mathbb{R}^n, \mathrm{Id})$—a global domain and the identity function.

Example 2.4. Let U be an open subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^k$ be a continuous function. The *graph* of f, defined as the subspace

$$\Gamma(f) := \{(x, f(x)) \mid x \in U\} \subseteq \mathbb{R}^{n+k},$$

is a topological manifold. Note that since $\Gamma(f)$ is a subspace of $\mathbb{R}^{n+k}$, it is Hausdorff and second-countable.

To show $\Gamma(f)$ is locally Euclidean, let $\varphi : \Gamma(f) \to U$ be defined by $\varphi(x, y) = x$. Notice that φ is the projection $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ restricted to the domain $\Gamma(f)$, and since projections and restrictions of continuous functions are continuous, we know φ is continuous. Note that φ is also invertible; its inverse $\varphi^{-1} : U \to \Gamma(f)$ is defined by $\varphi^{-1}(x) = (x, f(x))$, and because its components

$$\pi_1 \circ \varphi^{-1}(x) = x,$$
$$\pi_2 \circ \varphi^{-1}(x) = f(x),$$

are continuous, φ^{-1} is continuous overall. Thus, we have shown that $\varphi : \Gamma(f) \to U$ is a homeomorphism, which means $\Gamma(f)$ is a topological manifold. In fact, we have shown that $(\Gamma(f), \varphi)$ is a global coordinate chart.

Example 2.5. The n-sphere $\mathbb{S}^n$ is a topological n-manifold. Once again, since $\mathbb{S}^n$ is a Euclidean subspace, it is Hausdorff and second-countable. To show $\mathbb{S}^n$ is locally Euclidean, let $p = (x^1, x^2, \ldots, x^{n+1}) \in \mathbb{S}^n$. For $1 \le i \le n + 1$, we define

$$U_i^+ := \{x \in \mathbb{R}^{n+1} \mid \pi_i(x) > 0\},$$
$$U_i^- := \{x \in \mathbb{R}^{n+1} \mid \pi_i(x) < 0\},$$

which correspond to "halves" of $\mathbb{R}^{n+1}$ where the ith component of $x \in \mathbb{R}^{n+1}$ is either positive or negative, respectively. Note that $U_i^\pm$ is open in $\mathbb{R}^{n+1}$, which means $U_i^\pm \cap \mathbb{S}^n$ is open with respect to the subspace topology on $\mathbb{S}^n$.

Without loss of generality, suppose $p \in U_i^\pm \cap \mathbb{S}^n$ for some $1 \le i \le n$. Let $f^\pm : \mathbb{B}^n \to \mathbb{R}$ be the map on the open unit ball $\mathbb{B}^n$ defined by

$$f^\pm(x) = \pm\sqrt{1 - \|x\|^2}.$$

Observe that the graph of $f^\pm$ is equal to $U_i^\pm \cap \mathbb{S}^n$; as an example, for $n = 1$ the graph of $f^+(x) = \sqrt{1 - \|x\|^2} = \sqrt{1 - x^2}$ in $\mathbb{R}^2$ yields the upper hemicircle of $\mathbb{S}^1$. By Example 2.4, we have that $U_i^\pm \cap \mathbb{S}^n$ is a topological manifold with a global domain and homeomorphism $\varphi_i^\pm : U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n$ defined by

$$\varphi_i^\pm(x) = \underbrace{(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1})}_{\text{exclusion of } x^i}.$$

Thus, we have shown that $(U_i^\pm \cap \mathbb{S}^n, \varphi_i^\pm)$ is the local chart containing p and that $\mathbb{S}^n$ is an n-dimensional manifold.

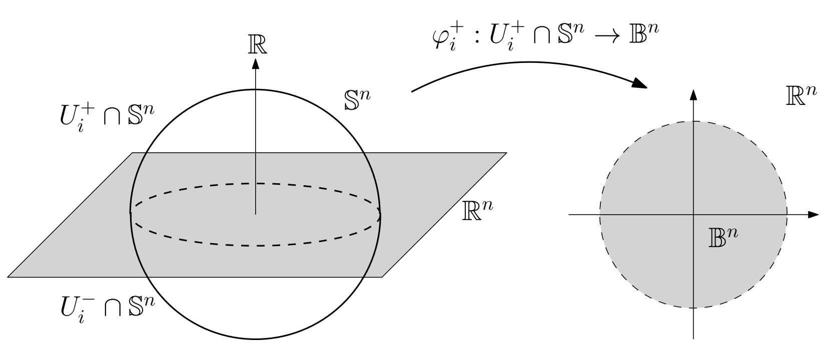

FIGURE 2. A chart for $\mathbb{S}^n$

Remark 2.6. Note that for $1 \leq i \leq n$, the union of $U_i^+ \cap \mathbb{S}^n$ and $U_i^- \cap \mathbb{S}^n$ excludes points on the n-sphere where their ith components are equal to 0 (points on the equator). An equatorial point p for the pair $U_i^\pm \cap \mathbb{S}^n$ will actually lie inside some other pair of hemispheres $U_j^\pm \cap \mathbb{S}^n$, as shown in Figure 3. For $\mathbb{S}^n$, we have $n+1$ possible equators. As such, we expect $n+1$ many hemispherical pairs of charts, or $2(n+1)$ total chart domains.

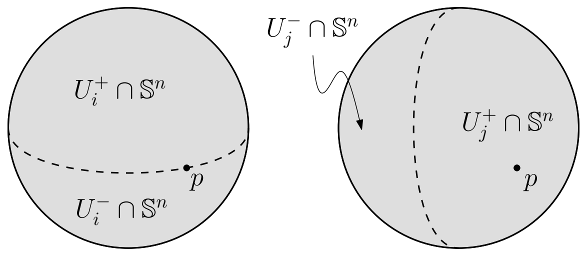

FIGURE 3. Multiple charts for $\mathbb{S}^n$

2.2. **Smooth Maps and Smooth Manifolds.**

On top of our topological manifold, we introduce a *smooth structure* that allows us to describe which functions are differentiable, or *smooth*. We begin with a definition for smooth functions between Euclidean space:

Definition 2.7. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A function $F : U \to V$ is **smooth** if each of its component functionals have continuous partial derivatives on all orders.

Remark 2.8. Roughly speaking, we are interested in an open domain U because we want to approach a point as close as possible from all directions when computing partial/directional derivatives. A single-variable analogue of this is determining the derivative of $f(x) = x^2$ at $x = 1$ on the closed interval $[0, 1]$; our domain only allows us to approach $x = 1$ from the left.

Definition 2.9. If a smooth function $F : U \to V$ is also bijective with a smooth inverse map, then F is a **diffeomorphism**.

Note that our definition for smoothness above applies only to functions between Euclidean spaces; it does not (yet) make sense to take partial derivatives of functions on manifolds. To adapt Definition 2.7 to manifolds, we "convert" the coordinates on our manifold to those in Euclidean space via some chart (U, φ) and require the newly parametrized function to be smooth. In other words, if (U, φ) is some chart on M, then we say $f : M \to \mathbb{R}$ is smooth if and only if $f \circ \varphi^{-1} : \widehat{U} \to \mathbb{R}$ is smooth.

Now consider the following example:

Example 2.10. Let $M = \mathbb{R}^2$ be our manifold, and consider the (global) homeomorphism $\varphi : M \to \mathbb{R}^2$ defined by $\varphi(u, v) = (u^3, v^3)$; this is indeed a homeomorphism, as its inverse $\varphi^{-1} : \mathbb{R}^2 \to M$ is defined by $\varphi^{-1}(x, y) = (x^{1/3}, y^{1/3})$, and both φ^{-1} and φ are continuous. Now consider the differentiable function $f : M \to \mathbb{R}$ defined by $f(x, y) = x$. The composition $f \circ \varphi^{-1} : M \to \mathbb{R}$ yields

$$(f \circ \varphi^{-1})(x, y) = x^{1/3},$$

and differentiating gives

$$(f \circ \varphi^{-1})'(x, y) = \frac{1}{3}x^{2/3}.$$

Note then that $f \circ \varphi^{-1}$ is not differentiable at any point $(0, y) \in M$. However, if we picked a different local coordinate map, say $\mathrm{Id} : M \to \mathbb{R}^2$, then $f \circ \mathrm{Id}^{-1}$ would be smooth everywhere. So is f smooth or not?

In the example above, the characterization of f as a smooth map depends on the chart used. To prevent such inconsistencies, we require our charts to be *smoothly compatible*.

Definition 2.11. Let M be a topological n-manifold. If (U, φ) and (V, ψ) are intersecting charts, then the composition $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is known as the *transition map* from φ to ψ.

Definition 2.12. Charts (U, φ) and (V, ψ) are *smoothly compatible* if either their transition maps are diffeomorphisms or $U \cap V = \varnothing$.

To illustrate how smooth compatibility eliminates a dependence on our choice of chart, let $f : M \to \mathbb{R}$ be a map that is considered smooth on (U, φ), and suppose (V, ψ) is an intersecting smoothly compatible chart. As such, $f \circ \varphi^{-1}$ is smooth and $\psi \circ \varphi^{-1}$ is a diffeomorphism. Observe then that

$$(f \circ \varphi^{-1}) \circ (\psi \circ \varphi^{-1})^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1} = f \circ \psi^{-1}.$$

Since the composition of smooth functions is smooth, $f \circ \psi^{-1}$ must be smooth as well. As a result, (V, ψ) does not alter the smoothness of f.

Keeping smooth compatibility in mind, we define a smooth structure on a topological manifold with the following:

Definition 2.13. An *atlas* $\mathcal{A}$ for M is a collection of charts where the union of coordinate domains covers M.

Definition 2.14. An atlas $\mathcal{A}$ for M is a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible.

A smooth atlas is the exact structure that we need on a manifold to consistently discern whether a map on the manifold is smooth, regardless of our choice of charts. If such a structure exists, then we have a *smooth manifold*.

Example 2.15. The family of $2(n + 1)$ charts $\{(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})\}$ from Example 2.5 is a smooth atlas of $\mathbb{S}^n$. Let $(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})$ and $(U_j^{\pm} \cap \mathbb{S}^n, \varphi_j^{\pm})$ be two charts from the family. Note that if $i = j$, the chart domains $U_i^{+} \cap \mathbb{S}^n$ and $U_i^{-} \cap \mathbb{S}^n$ are disjoint, which means the two charts are smoothly compatible. Without loss of generality, suppose $i < j$; if $x = (x^1, \ldots, x^n)$, then observe that

$$\begin{aligned}
\varphi_j^{\pm} \circ (\varphi_i^{\pm})^{-1}(x) &= \varphi_j^{\pm}(x^1, \ldots, \underbrace{\pm\sqrt{1 - \|x\|^2}}_{i\text{th position}}, \ldots, x^n), \\
&= \underbrace{(x^1, \ldots, \pm\sqrt{1 - \|x\|^2}, \ldots, x^{j-1}, x^{j+1}, \ldots, x^n)}_{\text{exclusion of } x^j}.
\end{aligned}$$

Computing the Jacobian matrix of $\varphi_j^{\pm} \circ (\varphi_i^{\pm})^{-1}(x)$ reveals that $\varphi_j^{\pm} \circ (\varphi_i^{\pm})^{-1}(x)$ is smooth, which implies $(U_i^{\pm}, \varphi_i^{\pm})$ and $(U_j^{\pm}, \varphi_j^{\pm})$ are smoothly compatible. Since the union of the chart domains covers $\mathbb{S}^n$, we have that $\{(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})\}$ is a smooth atlas, hence $\mathbb{S}^n$ is a smooth manifold.

Example 2.16. The following are atlases for $\mathbb{R}^n$.

$$\mathcal{A}_1 = \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\},$$
$$\mathcal{A}_2 = \{(B_1(x), \mathrm{Id}_{B_1(x)}) \mid x \in \mathbb{R}^n\}.$$

In the example above, both atlasses use the identity map as coordinate maps, yet $\mathcal{A}_2$ involves more chart domains. We want to work with as few charts as possible, so we require our smooth atlas to be *maximal*.

Definition 2.17. A smooth atlas $\mathcal{A}$ is *maximal* if it is not properly contained by another atlas.

In Example 2.16, each ball of $\mathcal{A}_2$ is contained in $\mathbb{R}^n$ of $\mathcal{A}_1$, hence $A_1 \subset A_2$. In fact, $\mathcal{A}_2$ is a maximal atlas among all atlasses with charts using the identity map.

2.3. **Smooth Map Between Manifolds.**

With a smooth structure in place, we define smooth functions on/between smooth manifolds by the following.

Definition 2.18. Suppose M is a smooth n-manifold and $f : M \to \mathbb{R}^k$ is a real function. We say f is *smooth* if for all $p \in M$, there exists a smooth chart (U, φ) containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth on $\varphi(U)$.

Definition 2.19. Let M, N be smooth manifolds, and let $F : M \to N$ be a map. We say F is *smooth* if for all $p \in M$, there exist smooth charts (U, φ) and (V, ψ) containing p and $F(p)$, respectively, such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

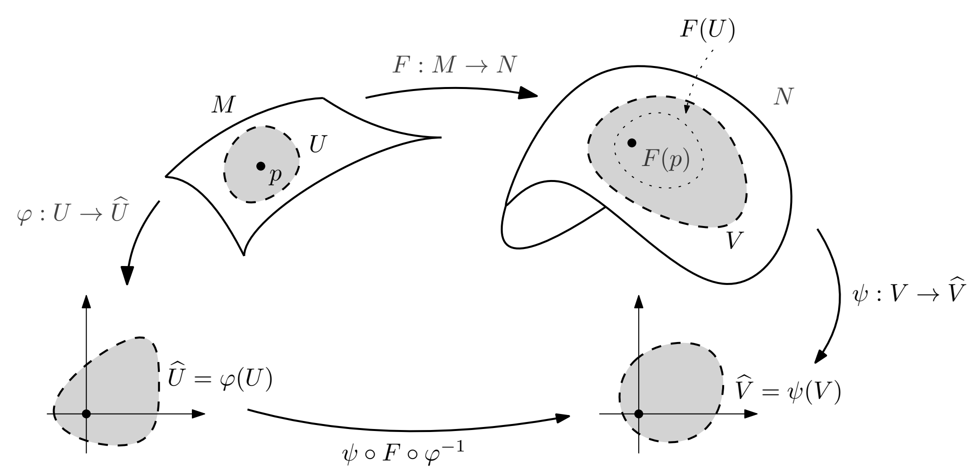

Figure 4. A smooth map between manifolds

We call $\psi \circ F \circ \varphi^{-1}$ the *coordinate representation* of F. An alternate, yet equivalent definition for smooth maps between manifolds is given by the following:

Definition 2.20. A map $F : M \to N$ between smooth manifolds is *smooth* if for all $p \in M$, there exist smooth charts (U, φ) and (V, ψ) containing p and $F(p)$, respectively, such that $U \cap F^{-1}(V)$ is open in M and the composition $\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$ is smooth.

Remark 2.21. Before continuing any further, note that a coordinate map φ for a chart (U, φ) is also a diffeomorphism. The composition $\varphi \circ \varphi^{-1} : \varphi(U) \to \varphi(U)$ is equal to the identity $\mathrm{Id} : \varphi(U) \to \varphi(U)$. Since the identity map is a diffeomorphism, we know φ and φ^{-1} are both smooth.

2.4. **Tangent Spaces and the Differential.**

Given a map $F : M \to N$ between smooth manifolds M and N, we want to define the *derivative* of F, analogous to the total derivative for functions between Euclidean spaces. Such a map is known as the *differential*, which we define as a map between the *tangent spaces* of M and N.

To begin, recall the definitions of the total and directional derivatives:

Definition 2.22. Suppose $U \subseteq \mathbb{R}^m$ is open, and let $f : U \to \mathbb{R}^n$ be a function. We say f is *differentiable* at $a \in U$ if there exists a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{u \to 0} \frac{\|f(a+u) - f(a) - Tu\|}{\|u\|} = 0.$$

If such a linear map exists, then T is known as the *total derivative of f at a*, denoted $f'(a) := T$.

Definition 2.23. Let $f : U \to \mathbb{R}^n$ be a function and let $u \in \mathbb{R}^m$ be a unit vector. Then the *directional derivative* of f at $a \in U$ in the direction u is defined as

$$D_v|_a(f) := \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}.$$

If v is a standard basis vector e_i, then the directional derivative is known as a *partial derivative*, denoted

$$D_v|_a = \left.\frac{\partial}{\partial x^i}\right|_a .$$

If $f : U \to \mathbb{R}^n$ is differentiable at $a \in U$ and $f = (f_1, f_2, \dots, f_n)$, then the total derivative $f'(a)$ can be represented by the Jacobian matrix of f,

$$f'(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(a) & \frac{\partial f_1}{\partial x^2}(a) & \cdots & \frac{\partial f_1}{\partial x^m}(a) \\ \frac{\partial f_2}{\partial x^1}(a) & \frac{\partial f_2}{\partial x^2}(a) & \cdots & \frac{\partial f_2}{\partial x^m}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x^1}(a) & \frac{\partial f_n}{\partial x^2}(a) & \cdots & \frac{\partial f_n}{\partial x^m}(a) \end{pmatrix},$$

where each column vector is the image of a standard basis vector in $\mathbb{R}^m$ under $f'(a)$ and is tangent to the image of f at $f(a)$, as depicted in Figure 5. Notice that e_1, e_2 are also tangent to the neighborhood $U \subseteq \mathbb{R}^m$ at a. As such, the total derivative is roughly a map between tangent spaces. Figure 5 also reflects a natural interpretation of tangent vectors in Euclidean space as concrete arrows radiating from a point, and this visual construction defines the *geometric tangent space*.

Definition 2.24. For $a \in \mathbb{R}^n$, the *geometric tangent space* at a is the set

$$\mathbb{R}^n_a := \{a\} \times \mathbb{R}^n.$$

An element in $\mathbb{R}^n_a$ is known as a *geometric tangent vector*, denoted v_a.

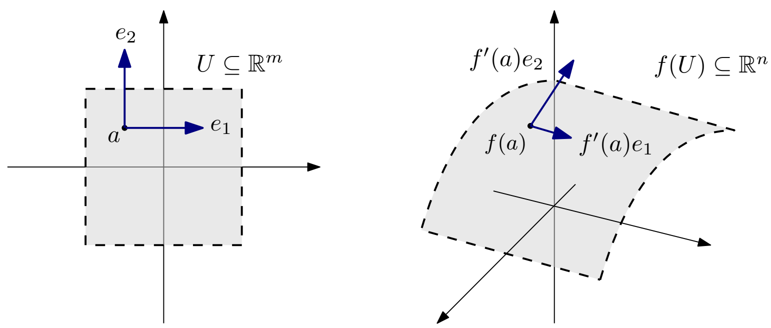

FIGURE 5. Transformation of standard basis vectors under $f'(a)$

Note that $\mathbb{R}^n_a$ is "unique" to each $a \in \mathbb{R}^n$; if $b \in \mathbb{R}^n$ and $a \neq b$, then $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ are disjoint. In addition, $\mathbb{R}^n_a$ is isomorphic to $\mathbb{R}^n$, which means $\mathbb{R}^n_a$ is like a copy of $\mathbb{R}^n$ sitting on top of a, from which geometric tangent vectors radiate.

However, the geometric tangent space does not exist for arbitrary manifolds. For one, not all manifolds are subspaces of $\mathbb{R}^n$, where geometric tangent vectors can be placed on top of a point on the manifold. In other words, we cannot assume a manifold exists within an ambient space like $\mathbb{R}^n$. Furthermore, not all manifolds are vector spaces, hence the manifold and its tangent space cannot share the same space.

Instead, consider the following definition of a tangent space:

Definition 2.25. For $p \in M$, a map $w : C^\infty(M) \to \mathbb{R}$ is a *derivation at p* if it is linear over $\mathbb{R}$ and satisfies the product rule:

$$w(fg) = f(a)wg + g(a)wf.$$

Definition 2.26. The *tangent space* of M, denoted T_pM is the set of all derivations of $C^\infty(M)$ at a. Vector addition and scalar multiplication are defined by

$$(w_1 + w_2)f = w_1 f + w_2 f,$$
$$(cw)f = c(wf).$$

For some intuition, consider $M = \mathbb{R}^n$ and $a \in \mathbb{R}^n$. Given some direction $v \in \mathbb{R}^n$, the directional derivative $D_v|_a$ is a derivation, as one-dimensional derivatives satisfy the product rule. Hence, derivations are roughly like directional derivatives. In fact, this is an accurate interpretation for $M = \mathbb{R}^n$:

Theorem 2.27. *For $a \in \mathbb{R}^n$, the geometric tangent space $\mathbb{R}^n_a$ is isomorphic to the set of derivations $T_a\mathbb{R}^n$ via the map $v_a \mapsto D_v|_a$.*

In other words, each derivation is a directional derivative given by some vector. Note that this statement holds for $\mathbb{R}^n$, not necessarily for all manifolds. Nonetheless, Theorem 2.27 should hopefully motivate our definition by derivations, as the isomorphism between $\mathbb{R}^n_a$ and $T_a\mathbb{R}^n$ shows an agreement between our concrete and abstract definitions of the tangent space. And like $\mathbb{R}^n_a$, $T_a\mathbb{R}^n$ is an n-dimensional vector space with a "standard" basis.

Corollary 2.28. *For $a \in \mathbb{R}^n$, the n derivations*

$$\left.\frac{\partial}{\partial x^1}\right|_a, \dots, \left.\frac{\partial}{\partial x^n}\right|_a \quad \text{defined by} \quad \left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $T_a\mathbb{R}^n$.

Proof. Note that the map $v_a \mapsto D_v|_a$ maps the standard basis vectors $e_1|_a, \dots, e_n|_a$ of $\mathbb{R}^n_a$ to the partial derivatives

$$\left.\frac{\partial}{\partial x^1}\right|_a, \dots, \left.\frac{\partial}{\partial x^n}\right|_a,$$

and because $v_a \mapsto D_v|_a$ is an isomorphism, the image of the standard basis vectors of $\mathbb{R}^n_a$ form a basis of $T_a\mathbb{R}^n$. □

In this paper, we will refer to $\partial/\partial x^i|_a$ as a *standard basis derivation* of $T_a\mathbb{R}^n$. Having defined the tangent space for a smooth manifold, we now define the *differential*—the "total derivative" for manifolds.

Definition 2.29. Let $F : M \to N$ be a smooth map between smooth manifolds M and N. For each $p \in M$, the *differential of F at p* is a map $dF_p : T_pM \to T_{F(p)}M$ such that for $v \in T_pM$, $dF_p(v)$ is a derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Note that $f \circ F \in C^\infty(M)$, so it makes sense for v to act on $f \circ F$. We can verify that $dF_p(v)$ is indeed a derivation; linearity holds because $v \in T_pM$ is linear, and the product rule holds by the following: for some $f, g \in C^\infty(N)$, we have

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)), \\ &= f \circ F(p) \cdot v(g \circ F) + g \circ F(p) \cdot v(f \circ F), \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)f. \end{aligned}$$

The following are some useful results about the differential from [3].

Proposition 2.30. *Let M, N, P be smooth manifolds, $F : M \to N$, $G : N \to P$ be smooth maps, and let $p \in M$.*

(1) $dF_p : T_pM \to T_{F(p)}N$ is linear.
(2) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.
(3) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
(4) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proposition 2.31. *Let U be an open subset of a smooth manifold M, and let $\iota : U \to M$ denote the inclusion map. Then for $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.*

From Proposition 2.30, we see that the differential behaves like the total derivative. In fact, the differential generalizes the total derivative, as the latter involves only Euclidean spaces. Proposition 2.31 allows us to take the tangent space at a point to be either that of the overall manifold or that of an open neighborhood containing the point.

Because the differential is a linear map, we expect a matrix representation of the differential. Determining a representation requires us to identify a basis for each tangent space, and to do so, we exploit the chart diffeomorphism between a manifold and its Euclidean look-alike.

Let M be a smooth m-dimensional manifold and $p \in M$. If U is a chart domain containing p and is diffeomorphic to $\widehat{U} \subseteq \mathbb{R}^m$ via $\varphi : U \to \widehat{U}$, then the differential $d\varphi_p : T_pU \to T_{\varphi(p)}\widehat{U}$ is an isomorphism (Proposition 2.30). Note that the differentials of inclusion functions $\iota : U \to M$ and $\widehat{\iota} : \widehat{U} \to \mathbb{R}^m$ are also isomorphisms (Proposition 2.31), so altogether the composition

$$d\left(\iota \circ \varphi^{-1} \circ \widehat{\iota}^{-1}\right)_{\varphi(p)} = d\iota_p \circ d\varphi^{-1}_{\varphi(p)} \circ d(\widehat{\iota}^{-1})_{\varphi(p)} : T_{\varphi(p)}\mathbb{R}^m \to T_pM \tag{2.32}$$

is an isomorphism. Consequently, this implies $\dim(T_pM) = \dim(T_{\varphi(p)}\mathbb{R}^m) = m$. For simplicity, we refer to (2.32) as $d(\varphi^{-1})_{\varphi(p)}$. If the directional derivatives

$$\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \left.\frac{\partial}{\partial x^2}\right|_{\varphi(p)}, \ldots, \left.\frac{\partial}{\partial x^m}\right|_{\varphi(p)}$$

form a basis of $T_{\varphi(p)}(\mathbb{R}^m)$, then their image under $d(\varphi^{-1})_{\varphi(p)}$ yields a basis of T_pM. We denote the (standard) basis derivations of T_pM by the following:

$$\left.\frac{\partial}{\partial x^i}\right|_p := d(\varphi^{-1})_{\varphi(p)} \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}.$$

By the definition of differentials, $\partial/\partial x^i|_p$ acts on $f \in C^\infty(M)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}).$$

Having constructed a basis for T_pM, we now examine how the basis transforms under the differential of a map between manifolds. But first, let us consider how the differential reduces to the total derivative for a map between Euclidean spaces.

Let $F : U \to V$ be a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, where $F = (F^1, F^2, \ldots, F^n)$. As a convention moving forward, $(x^1, x^2, \ldots, x^m)$ and $(y^1, y^2, \ldots, y^m)$ will denote coordinates in the domain and codomain, respectively. For $p \in U$, the basis of $T_p\mathbb{R}^m$—which is isomorphic to T_pU—consists of

$$\left.\frac{\partial}{\partial x^1}\right|_p, \left.\frac{\partial}{\partial x^2}\right|_p, \ldots, \left.\frac{\partial}{\partial x^m}\right|_p.$$

Similarly, the basis of $T_{F(p)}\mathbb{R}^n$ consists of

$$\left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \left.\frac{\partial}{\partial y^2}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^n}\right|_{F(p)}.$$

By the definition of differentials, $dF_p : \mathbb{R}^m \to \mathbb{R}^n$ acts on $f \in C^\infty(\mathbb{R}^n)$ by

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^1}\right|_p (f \circ F).$$

Applying the chain rule for partial derivatives, we thus have

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \sum_{j=1}^n \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \left(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f. \tag{2.33}$$

The expression above expresses the image of $\partial/\partial x^i|_p$ as a linear combination of the basis vectors in $T_{F(p)}\mathbb{R}^n$. Thus, the matrix representation of dF_p is given by

$$dF_p = \begin{pmatrix} \frac{\partial F_1}{\partial x^1}(p) & \frac{\partial F_1}{\partial x^2}(p) & \cdots & \frac{\partial F_1}{\partial x^m}(p) \\ \frac{\partial F_2}{\partial x^1}(p) & \frac{\partial F_2}{\partial x^2}(p) & \cdots & \frac{\partial F_2}{\partial x^m}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x^1}(p) & \frac{\partial F_n}{\partial x^2}(p) & \cdots & \frac{\partial F_n}{\partial x^m}(p) \end{pmatrix},$$

which accurately corresponds to the total derivative of F. This special case also manifests itself in the more general case.

Now suppose that $F : M \to N$ is a smooth function between manifolds M and N, where $\dim(M) = m$ and $\dim(N) = n$. For $p \in M$, let (U, φ) and (V, ψ) be the charts containing p and $F(p)$, respectively, and let $\widehat{p} = \varphi(p)$. As such, we have that

$$\widehat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$$

is a smooth map between $\mathbb{R}^m$ and $\mathbb{R}^n$. Observe then that $\psi^{-1} \circ \widehat{F} = F \circ \varphi^{-1}$, which implies

$$d(\psi^{-1})_{\widehat{F}(\widehat{p})} \circ d\widehat{F}_{\widehat{p}} = dF_p \circ d(\varphi^{-1})_{\widehat{p}}.$$

For some standard basis vector $\partial/\partial x^i|_{\widehat{p}} \in T_{\widehat{p}}\mathbb{R}^m$, we observe its transformation under the two composition of differentials above; first, it follows from (2.33) that

$$\begin{aligned} \left(d(\psi^{-1})_{\widehat{F}(\widehat{p})} \circ d\widehat{F}_{\widehat{p}}\right) \frac{\partial}{\partial x^i}\bigg|_{\widehat{p}} &= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(d\widehat{F}_{\widehat{p}}\left(\frac{\partial}{\partial x^i}\bigg|_{\widehat{p}}\right)\right), \\ &= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(\sum_{j=1}^{n} \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p})\ \frac{\partial}{\partial y^j}\bigg|_{\widehat{F}(\widehat{p})}\right). \end{aligned}$$

Note that $d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(\frac{\partial}{\partial y^j}\Big|_{\widehat{F}(\widehat{p})}\right) = \frac{\partial}{\partial y^j}\Big|_{F(p)}$, which means

$$\left(d(\psi^{-1})_{\widehat{F}(\widehat{p})} \circ d\widehat{F}_{\widehat{p}}\right) \frac{\partial}{\partial x^i}\bigg|_{\widehat{p}} = \sum_{j=1}^{m} \frac{\partial \widehat{F}^j}{dx^i}(\widehat{p})\ \frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

Note also that $\left(dF_p \circ d(\varphi^{-1})_{\widehat{p}}\right) \frac{\partial}{\partial x^i}\Big|_{\widehat{p}} = dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)$, so altogether, we have:

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \sum_{j=1}^{m} \frac{\partial \widehat{F}^j}{dx^i}(\widehat{p})\ \frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{2.34}$$

From (2.34), observe that the matrix representation of dF_p with respect to the standard basis derivations of T_pM and $T_{F(p)}N$ is equal to the Jacobian matrix of $\widehat{F}'(\widehat{p}) = (\psi \circ F \circ \varphi^{-1})'(\widehat{p})$. Therefore, we can naturally represent dF_p by the Jacobian matrix of its coordinate representation $\widehat{F}$.

2.5. **Submersions and Immersions.**

Since the differential is linear, the *rank* of the map—the dimension of the map's image—reveals insights about the local behavior of F.

Definition 2.35. Let $F : M \to N$ be a smooth map between smooth manifolds. The *rank of F at $p \in M$* is defined as the rank of $dF_p : T_pM \to T_{F(p)N}$.

Remark 2.36. Since dF_p and $\widehat{F}'(\widehat{p})$ share the same matrix representation (assuming standard basis), the rank of F is equal to the rank of the Jacobian matrix of $\widehat{F}$.

In particular, we are interested in when F achieves *full rank*:

Definition 2.37. A smooth map $F : M \to N$ has *full rank at a point $p \in M$* if $\operatorname{rank}(dF_p) = \min(\dim M, \dim N)$.

If $F : M \to N$ has full rank at $p \in M$ and $\operatorname{rank}(dF_p) = \dim M$, then dF_p is injective. If instead $\operatorname{rank}(dF_p) = \dim N$, then dF_p is injective. Note that the converses of these two statements hold as well. From these two cases, we define *immersions* and *submersions*:

Definition 2.38. Let $F : M \to N$ be a smooth map with full rank at $p \in M$. If dF_p is injective, then dF_p is an *immersion.* If dF_p is surjective, then dF_p is a *submersion.*

Example 2.39. Recall from Proposition 2.31 that the differential of the inclusion function $\iota : U \to M$ from an open subset to a manifold is an isomorphism between T_pU and T_pM for $p \in M$. As such, the inclusion map is both a submersion and immersion.

The following are some important results about submersions and immersions given in [3]:

Proposition 2.40. *Suppose $F : M \to N$ is a smooth map, and F has full rank at $p \in M$. Then there exists a neighborhood $U \subseteq M$ of p such that $F|_U$ has full rank.*

Theorem 2.41 (Inverse Function Theorem)**.** *Suppose $F : M \to N$ is a smooth map. If dF_p is invertible for some $p \in M$, then there exist connected neighborhoods $U_0 \subseteq M$ of p and $V_0 \subseteq N$ of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Theorem 2.42 (Rank Theorem)**.** *Suppose $F : M \to N$ is a smooth map with constant rank r, $\dim M = m$, and $\dim N = n$. Then for $p \in M$, there exist charts (U, φ) and (V, ψ) containing p and $F(p)$, respectively, such that $F(U) \subseteq V$ and*

$$\widehat{F} = (x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

The Inverse Function Theorem in $\mathbb{R}^n$ generalizes to Theorem 2.41, which is used to show Theorem 2.42. Note that combining Proposition 2.40 with Theorem 2.42 yields the following corollary.

Corollary 2.43 (Local Submersion/Immersion Theorem)**.** *Suppose $F : M \to N$ is a smooth map with full rank at $p \in M$. Then there exist neighborhoods U of p and V of $F(p)$ such that*

$$\widehat{F}(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n)$$

if F is a submersion, and

$$\widehat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$$

if F is an immersion.

In loose terms, if dF_p is surjective or injective, then $\widehat{F}$ behaves like a projection or inclusion (respectively) within a sufficiently small neighborhood of a point $p \in M$. In particular, the inclusive behavior of immersions is relevant to a submanifold that sits inside a larger manifold. We end our discussion of differential topology by returning to the idea of an ambient space when we defined tangent tangent vectors.

Definition 2.44. If M and N are smooth manifolds, a *smooth embedding of M into N* is an immersion $F : M \to N$ such that M is homeomorphic to its image $F(M) \subseteq N$ in the subspace topology.

For instance, the inclusion map is a smooth embedding; an open submanifold U of M is homeomorphic to $\iota(U) = U$ and ι is an immersion (Example 2.39).

Theorem 2.45. *Let $F : M \to N$ be a smooth map. If S is an immersed or embedded submanifold of M, then the restriction $F|_S : S \to N$ is smooth. If instead S is an immersed submanifold of N containing $F(M)$ and $F : M \to S$ is continuous, then $F : N \to S$ is smooth.*

In other words, smooth maps between manifolds are also smooth when restricted to submanifolds. We reference this fact in a later proof.

3. The Hopf Fibration

3.1. Fiber Bundles and Fibrations.

Our discussion of the Hopf fibration begins by defining a *fiber bundle*:

Definitions 3.1. Let B be a connected space, and $p : E \to B$ be a continuous map from a *total space* E to the *base space* B. We say p is a *fiber bundle with fiber F* if p is surjective and for $x \in B$,

1. $p^{-1}(\{x\})$ is homeomorphic to F, and
2. there exists an open neighborhood $U_x \subset B$ of x and a homeomorphism $\Psi_{U_x} : p^{-1}(U_x) \to U_x \times F$. In other words, the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_x) \arrow[r, "\Psi_{U_x}", "\cong"'] \arrow[d, "p"'] & U_x \times F \arrow[d, "\text{proj}"] \\ U_x \arrow[r, phantom, "="] & U_x \end{tikzcd}$$

If E, B, and F are smooth manifolds and Ψ_{U_x} and p are diffeomorphisms (restricted according to diagram above), then p is known as a *smooth fiber bundle.*

To clarify, a fiber looks like the preimage of every point on the base space. The surjectivity of $p : E \to B$ ensures that all preimages are non-empty. Our diagram above indicates that $p = \text{proj} \circ \Psi_{U_x}$ for a sufficiently small neighborhood U_x of $x \in B$, which characterizes p as a projection-like map of $p^{-1}(U_x)$ onto U_x.

Example 3.2. Consider a Möbius band as our total space E and the circle running through its center (bolded in Figure 3.2) as our base space B. Let x be a point on B and $U_x \subset B$ be some sufficiently small neighborhood of x. As depicted, the fiber of x is some vertical segment of points on E, and the preimage $p^{-1}(U_x)$ of the neighborhood closely resembles the Cartesian product $U_x \times F$. This resemblance is formally established by the homeomorphism Ψ_{U_x} between the two sets.

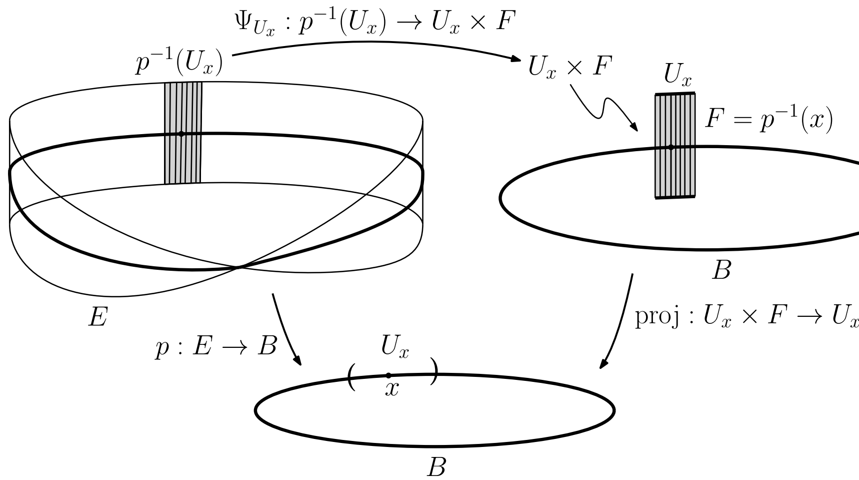

Figure 6. Fiber bundle for Möbius band

While the preimage of a sufficiently small neighborhood looks like a Cartesian product, globally the Möbius band exhibits twisting, hence it is not a Cartesian product of a fiber F and the base space B. As such, fiber bundles are also known as "twisted Cartesian products."

Remark 3.3. If the total space of a fiber bundle is the product of its fiber and base space, then such a bundle is a *trivial fiber bundle.*

Similar to how manifolds locally resemble Euclidean space, total spaces locally resemble product spaces. In the context of differential topology, a fiber bundle describes a manifold locally by referencing only two other manifolds.

3.2. Hopf Map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$.

Our definition of the Hopf map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ follows the construction via quaternions described in [1]. We summarize some notable properties about quaternions.

Definitions 3.4. Let 1, i, j, and k correspond to standard basis vectors e_1, e_2, e_3, and e_4 of $\mathbb{R}^4$, respectively. The set of *quaternions* is defined as

$$\mathbb{H} := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

A quaternion of the form $bi + cj + dk$ is known as a *pure quaternion.* We define the product of quaternions according to $i^2 = j^2 = k^2 = ijk = -1$.

Remark 3.5. The products $i^2 = j^2 = k^2 = ijk = -1$ sufficiently allows us to deduce $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, and $ik = -j$, which implies quaternion multiplication is non-commutative.

By construction, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$, and the set of all pure quaternions is isomorphic to $\mathbb{R}^3$. And like the complex numbers, quaternions have similar notions of conjugation and norm; for $r = a + bi + cj + dk$, we have

$$\overline{r} = a - bi - cj - dk,$$
$$\|r\|^2 = a^2 + b^2 + c^2 + d^2 = r \cdot \overline{r}.$$

Furthermore, there exists a multiplicative inverse for non-zero quaternions:

$$r^{-1} = \frac{\overline{r}}{\|r\|}.$$

Like complex numbers, quaternions describe rotation in space; for some quaternion $r = a + bi + cj + dk$ and point $p = xi + yj + zk \equiv (x, y, z)$ in $\mathbb{R}^3$, the map

$$R_r(p) = r \cdot p \cdot r^{-1}$$

describes the rotation of p around the axis (b, c, d) by $\theta = 2\cos^{-1}(a)$.

We now introduce the Hopf map:

Definition 3.6. For $r = a + bi + cj + dk \in \mathbb{S}^3$, the Hopf map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ is defined by

$$\eta(r) = rir^{-1} = (a^2 + b^2 - c^2 - d^2)i + 2(ad + bc)j + 2(bd - ac)k$$

Given some unit quaternion—which represents a point on $\mathbb{S}^3$—the Hopf map returns with a rotation of the point $i \equiv (1, 0, 0) \in \mathbb{R}^3$. Hence, the image of η is a subset of $\mathbb{S}^2$. The Hopf map η is a fiber bundle with fiber $\mathbb{S}^1$, base space $\mathbb{S}^2$, and total space $\mathbb{S}^3$. Our proof of the Hopf fibration relies on Lemma 3.7, which renders our task to a problem in differential topology.

Lemma 3.7 (Ehresmann's Lemma, [2])**.** *Let $F : M \to N$ be a map between manifolds M and N such that F is*

(1) A surjective submersion, and
(2) A proper map (guaranteed if M is compact).

Then F is a fiber bundle.

Note that Lemma 3.7 does not require us to explicitly identify the fiber of η. For the following propositions, we prove η is surjective and has fiber $\mathbb{S}^1$. We begin by determining the fiber of $i \equiv (1, 0, 0) \in \mathbb{S}^2$.

Proposition 3.8. $\eta^{-1}(i) = \{\cos(t) + \sin(t)i \mid t \in \mathbb{R}\}$.

Proof. Suppose $r = a + bi + cj + dk \in \mathbb{S}^3$ such that $\eta(p) = i$. And so, we have

$$a^2 + b^2 + c^2 + d^2 = 1, \qquad a^2 + b^2 - c^2 - d^2 = 1.$$

Subtracting the latter from the former yields $c^2 + d^2 = 0$, which implies $c = d = 0$. Thus, we have $a^2 + b^2 = 1$, which means

$$\eta^{-1}(i) = \{a + bi \in \mathbb{S}^3 \mid a^2 + b^2 = 1\},$$

where trigonometric parametrization gives $\eta^{-1}(i) = \{\cos(t) + \sin(t)i \mid t \in \mathbb{R}\}$. □

Proposition 3.9. *η is surjective, and the preimage of a point on $\mathbb{S}^2$ is a circle.*

Proof. To show η is surjective, we show that for any $p \in \mathbb{S}^2$, there exists a quaternion on $\mathbb{S}^3$ that maps to p. Let $p = (x, y, z) \in \mathbb{S}^2$. Observe that $p \neq (-1, 0, 0)$ can be described as the rotation of $i \equiv (1, 0, 0)$ by π radians around the vector from the origin to the midpoint of i and p. This vector has normalized coordinates

$$\frac{1}{\sqrt{x+1}}\,(x + 1, y, z)\,.$$

Since the angle of rotation is π, the real component of our quaternion is equal to 0. And so, the quaternion

$$r := \frac{1}{\sqrt{x+1}}\left((x+1)i + yj + zk\right)$$

rotates i to p, which proves η is surjective for $p \neq (-1,0,0)$. To show that the preimage of all such p are circles, suppose $s \in \mathbb{S}^3$ such that $\eta(s) = sis^{-1} = p$. Since $p = rir^{-1}$, we have $sis^{-1} = rir^{-1}$, which implies

$$(r^{-1}s)\cdot i \cdot (s^{-1}r) = i.$$

By this result, we find $r^{-1}s \in \eta^{-1}(i)$, which means $r^{-1}s = \cos(t) + i\sin(t)$ for some $t \in \mathbb{R}$. And so, we have $s = r(\cos(t) + i\sin(t))$, which means every point on $\eta^{-1}(p)$ can be parametrized by $r(\cos(t) + i\sin(t))$. In other words, $\eta^{-1}(p)$ is a rotation of $\eta^{-1}(i)$ by the quaternion r.

Our construction of the rotation quaternion does not define a preimage for $p = (-1,0,0)$. Following similar steps as shown in Proposition 3.2, we find that

$$\eta^{-1}(-i) = \{\cos(t)j + \sin(t)k \mid t \in \mathbb{R}\},$$

which completes our proof that η is surjective and all preimages are circles. $\square$

To prove η is a submersion, we require the following lemma.

Lemma 3.10. *Let $U \subseteq \mathbb{R}^{n+1}$ be an n-dimensional subspace and $T : \mathbb{R}^{n+1} \to \mathbb{R}^n$ a linear map with full rank. If $V \subseteq \mathbb{R}^n$ is an $(n-1)$-dimensional subspace containing $T(U)$, then the restriction $T|_U : U \to V$ has rank $n-1$.*

Proof. By the Rank-Nullity Theorem on T and $T|_U$, we have

$$\dim(\ker T) = \dim(\mathbb{R}^{n+1}) - \dim(\mathrm{Im}T),$$
$$\dim(\mathrm{Im}T|_U) = \dim(U) - \dim(\ker T|_U).$$

If $\dim(\mathrm{Im}T) = n$, then $\dim(\ker T) = 1$. Note that $\ker T|_U \subseteq \ker T$, which implies $\dim(\ker T|_U) \leq \dim(\ker T) = 1$. Consequently, we have $\dim(U) - \dim(\ker T|_U) \geq n-1$, which means $\dim(\mathrm{Im}T|_U) \geq n-1$. Given that $\mathrm{Im}T|_U \subseteq V$ and $\dim V = n-1$, we have $\dim(T|_U) \leq n-1$. Thus, we must have $\dim(T|_U) = n-1$. $\square$

Theorem 3.11. *$\eta : \mathbb{S}^3 \to \mathbb{S}^2$ is a fiber bundle.*

Proof. To show that $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ is a fiber bundle, we invoke Lemma 3.7; we have shown that η is surjective and because $\mathbb{S}^3$ is compact, we know η is a proper map. Thus, it suffices to show that η is smooth and its differential $d\eta_p : T_p\mathbb{S}^3 \to T_{\eta(p)}\mathbb{S}^2$ is a submersion—in other words, has full rank—for any $p \in \mathbb{S}^3$.

First, consider the extension $H : \mathbb{R}^4 \to \mathbb{R}^3$ of $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ defined by

$$H(a,b,c,d) = (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)).$$

Since H is a map between Euclidean spaces, the differential $dH_p : T_p\mathbb{R}^4 \to T_{H(p)}\mathbb{R}^3$ is given by its Jacobian matrix

$$dH_p = \begin{pmatrix} 2a & 2b & -2c & -2d \\ 2d & 2c & 2b & 2a \\ -2c & 2d & -2a & 2b \end{pmatrix}.$$

Note that $\mathbb{S}^3$ is a smooth submanifold of $\mathbb{R}^4$ (immersion by the inclusion map). Since H is smooth, its restriction $H|_{\mathbb{S}^3} : \mathbb{S}^3 \to \mathbb{R}^3$ is smooth as well by Theorem

2.45. Observe that $\mathrm{Im}(H|_{\mathbb{S}^3}) = \mathrm{Im}(\eta) \subseteq \mathbb{S}^2$, and the same theorem tells us that the codomain restriction $\eta = H|_{\mathbb{S}^3} : \mathbb{S}^3 \to \mathbb{S}^2$ is smooth.

Let $p = (a, b, c, d) \in \mathbb{S}^3$. To show $d\eta_p$ has a full rank of $\dim(T_{\eta(p)}\mathbb{S}^2) = \dim(\mathbb{S}^2) = 2$, we instead show dH_p has a full rank of 3 and apply Lemma 3.10. Thus, it suffices to verify that there are always three linearly independent column vectors of dH_p.

Note that the first three column vectors are linearly independent if $c \neq 0$; the determinant of the 3×3 minor yields

$$\det \begin{pmatrix} 2a & 2b & -2c \\ 2d & 2c & 2b \\ -2c & 2d & -2a \end{pmatrix} = -8c(a^2 + b^2 + c^2 + d^2) = -8c,$$

which is nonzero if $c \neq 0$. If $c = 0$, then consider the second, third, and fourth column vectors: the determinant of the minor is equal to

$$\det \begin{pmatrix} 2b & 0 & -2d \\ 0 & 2b & 2a \\ 2d & -2a & 2b \end{pmatrix} = 8b(a^2 + b^2 + d^2) = 8b,$$

which is nonzero for $b \neq 0$. If $b = 0$ as well, then consider the first, third, and fourth column vectors; the determinant is equal to

$$\det \begin{pmatrix} 2a & 0 & -2d \\ 2d & 0 & 2a \\ 0 & -2a & 0 \end{pmatrix} = 8a(a^2 + d^2) = 8a,$$

which is nonzero for $a \neq 0$. If $a = 0$, then we are left with

$$dH_p = \begin{pmatrix} 0 & 0 & 0 & -2d \\ 2d & 0 & 0 & 0 \\ 0 & 2d & 0 & 0 \end{pmatrix}.$$

If $a = b = c = 0$ and $a^2 + b^2 + c^2 + d^2 = 1$, then $d = 1$. As such, the first, second, and fourth column vectors are linearly independent, and altogether, we have shown $\mathrm{rank}(dH_p) = 3$.

If $\dim(T_p\mathbb{S}^3) = 3$ and $\dim(T_p\mathbb{S}^2) = 2$ and dH_p has full rank, then it follows from Lemma 3.10 that the restriction $d\eta_p : T_p\mathbb{S}^3 \to T_{\eta(p)}\mathbb{S}^2$ has rank equal to 2. Since we have shown that η is a submersion, η is a fiber bundle by Lemma 3.7. $\square$

In our proof of the Hopf fibration, Lemma 3.7 was a key ingredient that linked fiber bundles to differential topology. Rather than showing that each $x \in \mathbb{S}^2$ has a neighborhood U_x such that $\eta^{-1}(U_x)$ is homeomorphic to $U_x \times \mathbb{S}^1$, we showed that η is a smooth submersion.

Recall from Corollary 2.43 that if $F : M \to N$ is a smooth map with full rank at a point on M, then $\widehat{F}$ behaves like a projection for a neighborhood of the point. Technicalities aside, this property reflects how η "projects" a preimage $\eta^{-1}(U_x)$ on $\mathbb{S}^3$ onto the neighborhood U_x on $\mathbb{S}^2$, as depicted in the commutative diagram of Definition 3.1.

In general, Lemma 3.7 reveals that submersions naturally give rise to fibrations that describe the composition of a manifold. Our investigation shows that $\mathbb{S}^3$ locally looks like $\mathbb{S}^2 \times \mathbb{S}^1$. The fibration of $\mathbb{S}^3$ by fibers $\mathbb{S}^1$ can be visualized in 3-space via the *stereographic projection* of $\mathbb{S}^4$. This process is described in detail in [4]. In summary, circles on $\mathbb{S}^3$ containing the *projection point* are mapped to lines in $\mathbb{R}^3$, while circles that do not are mapped to circles in $\mathbb{R}^3$. Since the fibers of $\mathbb{S}^3$ are all circles, their image under stereographic projection are also circles. A dynamic visualization of the Hopf fibration can be found at [6].

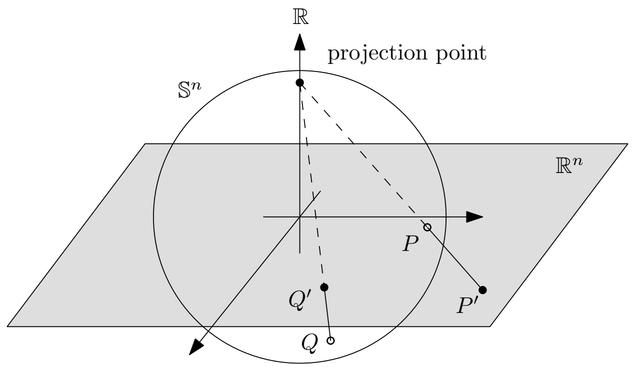

FIGURE 7. Stereographic projection of $\mathbb{S}^n$

Acknowledgments

I would like to thank my mentor Adrian Chu for advising and carefully guiding me through the fundamentals of differential topology, as well as suggesting our study of the Hopf fibration. This paper would not have been possible without his guidance and support. I would also like to thank Prof. Peter May for successfully organizing the REU virtually and for giving me this invaluable opportunity to develop my interests in mathematics. Special thanks go to my first-year math instructor Trevor Hyde for his constructive feedback and insights about the contents of this paper.

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