A ROADMAP TO THE GAUSS-BONNET THEOREM

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Abstract. This paper will outline and motivate essential topics in differential geometry leading up to the Gauss-Bonnet Theorem for regular surfaces in $\mathbb{R}^3$.

Contents

1. Introduction

Classical differential geometry is the study of curves and surfaces in $\mathbb{R}^3$, and the Gauss-Bonnet Theorem is one of its most well-known results for uniting geometric quantities such as angles and curvature with topological invariants such as the Euler characteristic and genus. One variation of the theorem is as follows: if R is a regular region on a orientable regular surface S, then

$$\int_{\partial R} k_g \; ds + \iint_R K \; dA + \sum_{i=1}^{p} (\pi - \varphi_i) = 2\pi\chi(R),$$

where k_g is the *geodesic curvature* of ∂R, K the *Gaussian curvature* of S, $\varphi_1, \dots, \varphi_p$ the *interior angles* of R, and $\chi(R)$ the *Euler characteristic* of R. This paper will attempt to explain and motivate these concepts through images and intuitive constructions whenever possible. The progression of topics leading up to a proof of the theorem in this paper will largely follow that of do Carmo's *Differential Geometry of Curves and Surfaces* [2], while introducing modern mathematical language and filling in or condensing details to existing proofs.

The author assumes a solid understanding of multivariable calculus and linear algebra, along with some familiarity with topics in point-set topology such as open and closed sets, continuity, connectedness, and compactness. The paper will focus on the geometric side of the Gauss-Bonnet Theorem, and may serve as a rough introduction to differential geometry.

2. Crash Course on Classical Differential Geometry

2.1. **Calculus on Surfaces.** Intuitively, a surface in $\mathbb{R}^3$ is a set that looks flat locally. To extend calculus from Euclidean space to surfaces, we would like them to have nice properties, namely that they are 'smooth' and do not intersect themselves.

Definition 2.1. A set $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$, there exist an open neighborhood $V \subseteq \mathbb{R}^3$ of p, an open set $U \subseteq \mathbb{R}^2$, and a map $\mathbf{x} : U \to V \cap S$ such that

(1) $\mathbf{x}$ is *smooth*; if $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, then $x, y, z : U \to \mathbb{R}$ are continuously differentiable on all orders,
(2) $\mathbf{x}$ is a *homeomorphism*; $\mathbf{x}$ is invertible, and both $\mathbf{x}$ and $\mathbf{x}^{-1}$ are continuous,
(3) the total derivative $d\mathbf{x}_p : \mathbb{R}^2 \to \mathbb{R}^3$ of $\mathbf{x}$ at p is injective.

We say $\mathbf{x}$ *locally parametrizes* S *near* p, and we refer to the pair $(U, \mathbf{x})$ as a *(local) chart containing* p.

Conditions (1) and (3) account for smoothness, while (2) prevents self-intersections.

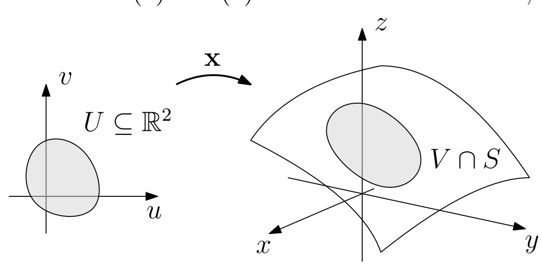

Figure 1. Schematic for charts

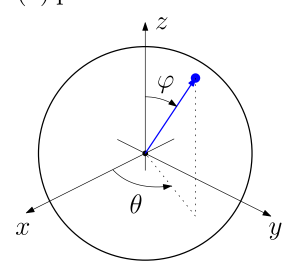

Figure 2. Sphere

Throughout this paper, we say a function $F = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ defined on an open subset $U \subset \mathbb{R}^n$ is *smooth* if its component functions $f_1, \ldots, f_m : U \to \mathbb{R}$ have continuous partial derivatives on all orders.

Example 2.2. The *graph* of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is the set

$$\Gamma(f) := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}.$$

It is parametrized by $\mathbf{x}(u, v) = (u, v, f(u, v))$ with inverse $\mathbf{x}^{-1}(x, y, z) = (x, y)$. Both $\mathbf{x}$ and $\mathbf{x}^{-1}$ are continuous, and the total derivative

$$d\mathbf{x}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_u & f_v \end{pmatrix}$$

at $p \in \Gamma(f)$ is indeed injective.

Example 2.3. A sphere S_r with radius $r > 0$ is a regular surface parametrized by

$$\mathbf{x}(\theta, \varphi) = (r \cos\theta \sin\varphi, r \sin\theta \sin\varphi, r \cos\varphi) \qquad \text{on} \qquad U = (0, 2\pi) \times (-\pi, \pi).$$

Note that $(U, \mathbf{x})$ actually fails to cover two circles on S_r, but we can resolve this by constructing similar charts. We can check that $\mathbf{x}$ is smooth and its differential

$$d\mathbf{x}_p = \begin{pmatrix} -r \sin\theta \sin\varphi & r \cos\theta \cos\varphi \\ r \cos\theta \sin\varphi & r \sin\theta \cos\varphi \\ 0 & -r \sin\varphi \end{pmatrix}$$

is injective for $p \in U$.

Condition (3) in Definition 2.1 also ensures that surfaces are smooth enough to be locally approximated by a *tangent space*, which we will realize as follows.

Definitions 2.4. A *path in* $\mathbb{R}^m$ is a continuous function $\alpha : I \to \mathbb{R}^m$ defined on an open interval $I \subseteq \mathbb{R}$. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is smooth, then the *velocity of* α *at* $t_0 \in I$ is the vector

$$\alpha'(t_0) := \frac{d}{dt}\alpha(t_0 + t)\Big|_{t=0} = \big(\alpha_1'(t_0), \ldots, \alpha_m'(t_0)\big).$$

We say α is *parametrized by arc length* if $\|\alpha'(s)\| = 1$ for all $s \in I$.

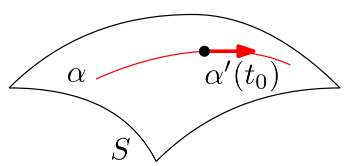

FIGURE 3. Path and velocity

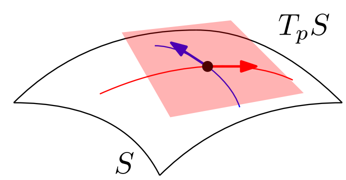

FIGURE 4. Tangent space

Definition 2.5. If S is a regular surface and $p \in S$, then the *tangent space* T_pS *at* p is the collection of velocity vectors of all smooth paths $\alpha : I \to S$ where $p \in \alpha(I)$. We say $w \in T_pS$ is a *tangent vector of* S *at* p.

The following is an equivalent way to realize the tangent space at a point.

Proposition 2.6. *If* $(U, \mathbf{x})$ *is a chart containing* $p \in S$*, then* $T_pS = d\mathbf{x}_p(\mathbb{R}^2)$.

Proof. If $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, then

$$d\mathbf{x}_p = \begin{pmatrix} \mathbf{x}_u(q) & \mathbf{x}_v(q) \end{pmatrix} := \begin{pmatrix} \dfrac{\partial x}{\partial u}(q) & \dfrac{\partial x}{\partial v}(q) \\ \dfrac{\partial y}{\partial u}(q) & \dfrac{\partial y}{\partial v}(q) \\ \dfrac{\partial z}{\partial u}(q) & \dfrac{\partial z}{\partial v}(q) \end{pmatrix}.$$

If $d\mathbf{x}_p$ is injective, then the column vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ form a basis of $d\mathbf{x}_p(\mathbb{R}^2)$. To see that $T_pS \subseteq d\mathbf{x}_p(\mathbb{R}^2)$, let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a smooth path such that $\alpha(0) = p$ and $\alpha'(0) = w$. Since α is a path in S, the composition $\mathbf{x}^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to U$ defines a path $t \mapsto (u(t), v(t))$ in U. Then $\alpha(t) = \mathbf{x}(u(t), v(t))$. Applying the chain rule yields

$$\alpha'(0) = \mathbf{x}_u(q)\ u'(0) + \mathbf{x}_v(q)\ v'(0)$$

which expresses the velocity $\alpha'(0)$ as a linear combination of $\mathbf{x}_u(q)$ and $\mathbf{x}_v(q)$. To see that $d\mathbf{x}_p(\mathbb{R}^2) \subseteq T_pS$, let $w \in \mathbb{R}^2$ and take a smooth path $\beta : (-\varepsilon, \varepsilon) \to U$ where $\beta(0) = q$. Note that $\mathbf{x} \circ \beta : (-\varepsilon, \varepsilon) \to S$ is a path in S, so if $\beta(t) = (u(t), v(t))$, then

$$(\mathbf{x} \circ \beta)'(0) = \mathbf{x}_u(q)\ u'(0) + \mathbf{x}_v(q)\ v'(0)$$

We have $\beta'(0) = (u'(0), v'(0))$ and $d\mathbf{x}_p\big(\beta'(0)\big) = d\mathbf{x}_p(w) = (\mathbf{x} \circ \beta)'(0)$, which shows $d\mathbf{x}_p(w)$ is the velocity of some path in S that passes through p. $\square$

It follows from Proposition 2.6 that if $(U, \mathbf{x})$ and $(V, \mathbf{y})$ are charts containing p, then $d\mathbf{x}_p(\mathbb{R}^2) = T_pS = d\mathbf{y}_p(\mathbb{R}^2)$.

In differential geometry, we consider functions defined on regular surfaces. However, if $f : S \to \mathbb{R}^n$ is a function on a regular surface S, we cannot merely compute its total derivative as if it were a function from $\mathbb{R}^3$ to $\mathbb{R}^n$; total derivatives only make sense for functions defined on open subsets of Euclidean space. Sadly, not all open subsets of S are open in $\mathbb{R}^3$. We will instead characterize the smoothness of f by expressing its domain in Euclidean coordinates with local charts.

Definition 2.7. Let S be a regular surface and G an open subset of S. We say $f : G \to \mathbb{R}^n$ is *smooth at* $p \in G$ if there exists a chart $(U, \mathbf{x})$ containing p such that $f \circ \mathbf{x} : \mathbf{x}^{-1}(G) \to \mathbb{R}^n$ is smooth.

While it seems the smoothness of $f : G \to \mathbb{R}^n$ depends on the choice of chart $(U, \mathbf{x})$, our definition of regular surfaces actually prevents this.

Proposition 2.8 ([2], 70)**.** *Let S be a regular surface and $p \in S$. If $(U, \mathbf{x})$ and $(V, \mathbf{y})$ are charts containing p and $W := \mathbf{x}(U) \cap \mathbf{y}(V)$, then the composition*

$$\mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$$

is a diffeomorphism. In other words, $\mathbf{x}^{-1} \circ \mathbf{y}$ is invertible and both itself and its inverse are smooth.

If $(U, \mathbf{x})$ and $(V, \mathbf{y})$ contain $p \in S$, then notice that $f \circ \mathbf{y} = (f \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \mathbf{y})$. If $f \circ \mathbf{x}$ and $\mathbf{x}^{-1} \circ \mathbf{y}$ are smooth, then their composition must also be smooth. Aside from telling whether $f : G \to \mathbb{R}^n$ is smooth, we would like to determine its *differential*, the extension of total derivatives to our context of surfaces.

Definition 2.9. If $f : G \to \mathbb{R}^n$ is smooth on an open subset $G \subseteq S$ and $p \in G$, then the *differential of f at p* is the map $df_p : T_pS \to \mathbb{R}^n$ defined by

$$df_p(w) := (f \circ \alpha)'(0) \qquad\qquad \text{for } w \in T_pS,$$

where $\alpha : (-\varepsilon, \varepsilon) \to S$ is a smooth path such that $\alpha(0) = p$ and $\alpha'(0) = w$.

Like total derivatives, differentials are linear maps between vector spaces. Notice that Definition 2.9 does not require us to choose a chart containing p, as T_pS and paths on surfaces can be described without charts. We only need to if we want a concrete matrix representation; if $(U, \mathbf{x})$ is a chart and $f = (f_1, \dots, f_n)$, then

$$df_p = \begin{pmatrix} \dfrac{\partial(f_1 \circ \mathbf{x})}{\partial u}(q) & \dfrac{\partial(f_1 \circ \mathbf{x})}{\partial v}(q) \\ \vdots & \vdots \\ \dfrac{\partial(f_n \circ \mathbf{x})}{\partial u}(q) & \dfrac{\partial(f_n \circ \mathbf{x})}{\partial v}(q) \end{pmatrix}$$

which is the total derivative of $f \circ \mathbf{x}$ at $q = \mathbf{x}^{-1}(p)$. For practical reasons, we will always choose local charts and treat df_p as the total derivative of $f \circ \mathbf{x}$. We conclude this section by extending smoothness and differentials to functions between surfaces.

Definition 2.10. Suppose V_1 and V_2 are open subsets of regular surfaces S_1 and S_2. We say $f : V_1 \to V_2$ is *smooth at* $p \in V_1$ if for any charts $(U_1, \mathbf{x}_1)$ of p and $(U_2, \mathbf{x}_2)$ of $f(p)$, the composition

$$\mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1 : \mathbf{x}_1^{-1}(V_1) \to \mathbf{x}_2^{-1}(V_2)$$

is smooth. The *differential of f at p* is the map $df_p : T_pS_1 \to T_{f(p)}S_2$ defined by

$$df_p(w) := (f \circ \alpha)'(0) \qquad \text{for } w \in T_pS_1,$$

where $\alpha : (-\varepsilon, \varepsilon) \to S_1$ is a smooth path such that $\alpha(0) = p$ and $\alpha'(0) = w$.

If $(\widehat{f}_1, \widehat{f}_2) = \mathbf{x}_2^{-1} \circ f \circ \mathbf{x}_1$, then one matrix representation of df_p is

$$df_p = \begin{pmatrix} \dfrac{\partial \widehat{f}_1}{\partial u}(q) & \dfrac{\partial \widehat{f}_1}{\partial v}(q) \\ \dfrac{\partial \widehat{f}_2}{\partial u}(q) & \dfrac{\partial \widehat{f}_2}{\partial v}(q) \end{pmatrix} \qquad \text{where } q = \mathbf{x}^{-1}(p).$$

2.2. **Measurements and the First Fundamental Form.** In geometry, we are interested in quantities such as length, angle measure, and area. We study these quantities on a regular surface S by defining an inner product $\langle \cdot, \cdot \rangle_p : T_pS \times T_pS \to \mathbb{R}$ on T_pS for all $p \in S$. For our purposes, T_pS will inherit the inner product on $\mathbb{R}^3$:

$$\langle w_1, w_2 \rangle_p := \langle w_1, w_2 \rangle \qquad \text{for } w_1, w_2 \in T_pS.$$

We will omit the subscript p in $\langle \cdot, \cdot \rangle_p$ and simply treat inner products on T_pS as inner products on $\mathbb{R}^3$. An important quantity to study for $w \in T_pS$ is its inner product with itself, which allows us to talk about its length.

Definition 2.11. If S is a regular surface and $p \in S$, then the *first fundamental form* $\mathrm{I}_p : T_pS \to \mathbb{R}$ *of S at p* is defined by

$$\mathrm{I}_p(w) := \langle w, w \rangle = \|w\|^2 \qquad \text{for } w \in T_pS.$$

If $(U, \mathbf{x})$ is a chart containing p, then we can express I_p in terms of local coordinates; let $w \in T_pS$ and $\alpha : (-\varepsilon, \varepsilon) \to S$ a smooth path such that $\alpha(0) = p$ and $\alpha'(0) = w$. If $\alpha(t) = \mathbf{x}(u(t), v(t))$, then

$$\begin{aligned} \mathrm{I}_p(w) = \langle w, w \rangle &= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle, \\ &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle (v')^2, \end{aligned}$$

where $\mathbf{x}_u$ and $\mathbf{x}_u$ are evaluated at $q = \mathbf{x}^{-1}(p)$ and u' and v' at $t = 0$. We define the functions $E, F, G : U \to \mathbb{R}$ by

$$E := \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p, \qquad F := \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p, \qquad G := \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p,$$

which will appear frequently throughout the paper.

Definition 2.12. A chart $(U, \mathbf{x})$ of a regular surface S is *orthogonal* if $F(q) = 0$ and *isothermal* if $F(q) = 0$ and $E(q) = G(q) = \lambda(q)$ for all $q \in U$, where $\lambda : U \to \mathbb{R}$ is a smooth function.

Our proof of the Gauss-Bonnet Theorem will use the nontrivial fact that all regular surfaces can be parametrized locally by isothermal charts [1]. We are ready to define arc lengths and angles on regular surfaces.

Definition 2.13. Let $(U, \mathbf{x})$ be a local chart of S and $\alpha : I \to \mathbf{x}(U)$ a path in $\mathbf{x}(U)$. The *arc length* $s : I \to \mathbb{R}$ *of α from $a \in I$* is defined by

$$s(t) := \int_a^t \|\alpha'(r)\| dr = \int_a^t \sqrt{\mathrm{I}_{\alpha(r)}(\alpha'(r))}\, dr.$$

If $\alpha(r) = \mathbf{x}(u(r), v(r))$, then

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\ dr,$$

where E, F, and G are evaluated at $\mathbf{x}^{-1} \circ \alpha(r) = (u(r), v(r))$ and u' and v' at r.

Definition 2.14. If $(U, \mathbf{x})$ is a chart containing $p \in S$ and $v, w \in T_pS$, then define the *angle between* v *and* w to be $0 \leq \theta \leq \pi$ such that

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\|w\|}.$$

To discuss area—and more generally, integration—on regular surfaces, we would like our regions of interest to be sufficently 'well-behaved' subsets of S.

Definitions 2.15. A *path segment in* $\mathbb{R}^m$ is a continuous function $\beta : [a, b] \to \mathbb{R}^m$ defined on a closed interval $[a, b] \subset \mathbb{R}$. We say β is

- *simple* if the restriction $\beta|_{(a,b)} : (a, b) \to S$ is injective,
- *closed* if $\beta(a) = \beta(b)$,
- and *piecewise-smooth* if there exists a finite subset $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$ of $[a, b]$ where $t_0 = a$, $t_{n+1} = b$, and β is smooth on (t_i, t_{i+1}) for all $0 \leq i \leq n$.

We say $\{t_0, \ldots, t_{n+1}\}$ *partitions* $[a, b]$ and $\beta(t_0), \ldots, \beta(t_n)$ are the *vertices* of β.

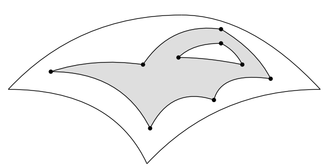

Figure 5. Regular region

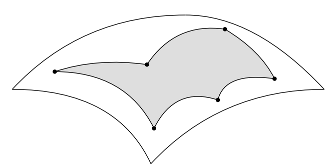

Figure 6. Simple region

Definitions 2.16. Let S be a regular surface. We say $R \subset S$ is a *regular region* if

(1) R is compact and connected,
(2) and the boundary ∂R is a finite union of the images of simple, closed, and piecewise smooth path segments. We refer to such path segements as the *boundary paths of* R.

A regular region R is *simple* if ∂R is the image of exactly one path segment. The *vertices of* R are the vertices from its boundary paths.

Definition 2.17. Let R be a regular region such that $R \subseteq \mathbf{x}(U)$ for some chart $(U, \mathbf{x})$ of a regular surface S. Define the *surface area of* R to be

$$\iint_{\mathbf{x}^{-1}(R)} \|\mathbf{x}_u \times \mathbf{x}_v\|\ du\ dv.$$

If $f : S \to \mathbb{R}$ is a continuous function on S, then the *surface integral of* f *over* R is defined as

$$\iint_R f\ dA := \iint_{\mathbf{x}^{-1}(R)} f \circ \mathbf{x}\ \|\mathbf{x}_u \times \mathbf{x}_v\|\ du\ dv$$

We can check that surface integrals do not depend on our choice of chart $(U, \mathbf{x})$ by applying the change of variables formula. Since $R \subset S$ is compact in S, we know $\mathbf{x}^{-1}(R)$ must be compact in U and thus $\iint_R f \; dA$ is finite. A useful fact is that $\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{EG - F^2}$, which follows from $\|\mathbf{x}_u \times \mathbf{x}_v\| = \|\mathbf{x}_u\|\|\mathbf{x}_v\| \sin\theta$.

In the context of regular regions, we often assign an orientation to its boundary paths. Without defining this rigorously, a boundary path is *positively-oriented* if a normal vector traversing along the path has the surface to its left.

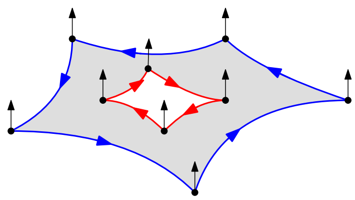

FIGURE 7. Path orientation

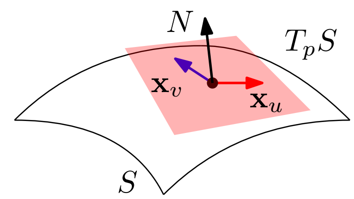

FIGURE 8. Surface normal

2.3. **The Gauss Map and the Second Fundamental Form.** We now introduce the language that geometers use to discuss the curvature of surfaces. Intuitively, curvature measures how much a surface bends along some direction at a point. More formally, we want to study how *tangent spaces* change locally.

Recall that if $(U, \mathbf{x})$ is a chart of a regular surface S, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ at some $p \in \mathbf{x}(U)$ span T_pS. If we imagine T_pS as a plane in $\mathbb{R}^3$, then

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

is a unit vector normal to T_pS. To examine how tangent spaces change, it suffices then to consider how their unit normals change accordingly. To make our work a little easier, we will assume that our regular surfaces are *orientable*—that the unit normals at each point vary 'smoothly' on the surface.

Definition 2.18. A regular surface S is *orientable* if there exists a smooth function $N : S \to \mathbb{R}^3$ such that $N(p)$ is a unit vector normal to T_pS for all $p \in S$. We refer to N as an *orientation of* S.

The following examples illustrate the distinction between orientable and non-orientable surfaces.

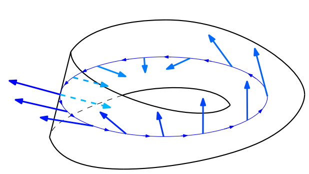

FIGURE 9. Mobius strip

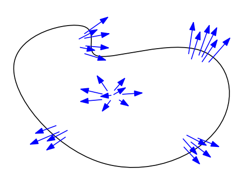

FIGURE 10. Blob

The Möbius strip is non-orientable because the unit normal at a point flips after traversing along the strip, which prevents the existence of a smooth map N. On the other hand, the 'blob' is orientable because a unit normal never flips after traversing any closed path. And so, an orientation N gives a consistent notion of 'inward' and 'outward' for a surface.

Definition 2.19. A chart $(U, \mathbf{x})$ of S is *compatible with an orientation* N if

$$N(p) = \frac{\mathbf{x}_u(q) \times \mathbf{x}_v(q)}{\|\mathbf{x}_u(q) \times \mathbf{x}_v(q)\|}$$

for all $p \in S$ and $q = \mathbf{x}^{-1}(p)$.

Since $\|N(p)\| = 1$ for all $p \in S$, the image of N is a subset of S^2. We also refer to the orientation $N : S \to S^2$ as the *Gauss map on* S. Note that $N(p)$ is normal to both T_pS and $T_{N(p)}S^2$, so the two tangent spaces are parallel (Figure 11). Hence, we treat $T_{N(p)}S^2$ as T_pS and dN_p as a linear map between T_pS and itself. We frame our study of curvature in terms of dN_p and the *second fundamental form*.

Definition 2.20. Let S be an oriented regular surface with orientation $N : S \to S^2$. We define the *second fundamental form of* S *at* p to be

$$\mathrm{II}_p(w) = -\langle dN_p(w), w\rangle.$$

To see how $\mathrm{II}_p(w)$ conveys information about the curvature of S at p, let $w \in T_pS$ be a unit vector and $\alpha : (-\varepsilon, \varepsilon) \to \mathbf{x}(U)$ a smooth path such that $\alpha(0) = p$ and $\alpha'(0) = w$. By Definition 2.10, we have $dN_p(w) = (N \circ \alpha)'(0)$.

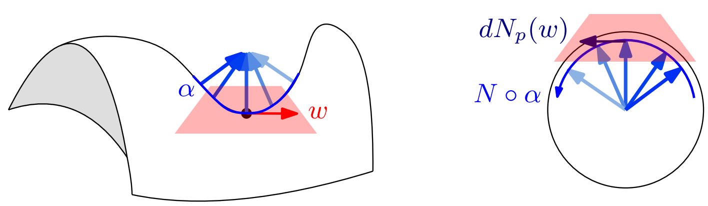

Figure 11. Gauss map

As depicted in Figure 11, $dN_p(w)$ is the velocity vector of a path on S^2. Roughly speaking, $dN_p(w)$ represents the direction the surface normals at p will 'tilt' towards while moving along α; in Figure 11, the surface normals tilt leftward at $t = 0$. The inner product $\langle dN_p(w), w\rangle$ represents the intensity of the tilt in the direction of w, and the negative of the inner product indicates whether our surface 'bends' towards or away from the surface normal at p while moving along α:

(1) If $\mathrm{II}_p(w) > 0$, then S bends towards $N(p)$.
(2) If $\mathrm{II}_p(w) < 0$, then S bends away from $N(p)$.
(3) If $\mathrm{II}_p(w) = 0$, then S does not bend at all.

As depicted in Figure 11, we have $\mathrm{II}_p(w) > 0$, which correctly corresponds to the surface bending upwards and towards $N(p)$ along α.

More concretely, if $(U, \mathbf{x})$ is a chart containing p and $\alpha(t) = \mathbf{x}(u(t), v(t))$, then

$$\begin{aligned} dN_p(w) &= \frac{d}{dt}(N \circ \mathbf{x}(u(t), v(t)))\Big|_{t=0}, \\ &= \frac{\partial(N \circ \mathbf{x})}{\partial u}(0)\ u'(0) + \frac{\partial(N \circ \mathbf{x})}{\partial v}(0)\ v'(0). \end{aligned}$$

For simplicity, let $N_u := \partial(N \circ \mathbf{x})/\partial u$ and $N_v := \partial(N \circ \mathbf{x})/\partial v$. Just as we obtained the coefficient functions E, F, and G for the first fundamental form, there are special coefficients for the second fundamental form. Observe that

$$\begin{aligned}\mathrm{II}_p(w) &= -\langle N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v'\rangle_p, \\ &= -\langle N_u, \mathbf{x}_u\rangle_p (u')^2 - \langle N_u, \mathbf{x}_v\rangle_p u'v' - \langle N_v, \mathbf{x}_u\rangle_p u'v' - \langle N_v, \mathbf{x}_v\rangle_p (v')^2.\end{aligned}$$

Note that $\langle N \circ \mathbf{x}, \mathbf{x}_u\rangle = \langle N \circ \mathbf{x}, \mathbf{x}_v\rangle = 0$, so differentiating in u and v yield

$$\begin{aligned} 0 &= \langle N_u, \mathbf{x}_u\rangle + \langle N, \mathbf{x}_{uu}\rangle, & 0 &= \langle N_{uv}, \mathbf{x}_u\rangle + \langle N, \mathbf{x}_{uv}\rangle, \\ 0 &= \langle N_u, \mathbf{x}_v\rangle + \langle N, \mathbf{x}_{vu}\rangle, & 0 &= \langle N_v, \mathbf{x}_v\rangle + \langle N, \mathbf{x}_{vv}\rangle. \end{aligned}$$

We define

$$\begin{aligned} e &= \langle N, \mathbf{x}_{uu}\rangle = -\langle N_u, \mathbf{x}_u\rangle, \\ f &= \langle N, \mathbf{x}_{vu}\rangle = \langle N, \mathbf{x}_{uv}\rangle = -\langle N_u, \mathbf{x}_v\rangle = -\langle N_v, \mathbf{x}_u\rangle, \\ g &= \langle N, \mathbf{x}_{vv}\rangle = -\langle N_v, \mathbf{x}_v\rangle, \end{aligned}$$

and refer to e, f, and g as the second fundamental form coefficients.

It follows from $\langle N_u, \mathbf{x}_v\rangle = \langle N_v, \mathbf{x}_u\rangle$ that dN_p is a symmetric operator, and as such, there exists an orthonormal basis of T_pS consisting of eigenvectors of dN_p by the Spectral Theorem.

Definitions 2.21. If $N : S \to S^2$ is an orientation of a regular surface S, then the *principal curvatures* $\kappa_1(p)$ and $\kappa_2(p)$ at $p \in S$ are defined as

$$\begin{aligned} \kappa_1(p) &= -\min\left\{\mathrm{II}_p(w) : \|w\| = 1, w \in T_pS\right\}, \\ \kappa_2(p) &= -\max\left\{\mathrm{II}_p(w) : \|w\| = 1, w \in T_pS\right\}. \end{aligned}$$

A unit vector $w \in T_pS$ is a *principal direction* if $-\mathrm{II}_p(w)$ equals $\kappa_1(p)$ or $\kappa_2(p)$. The product $K(p) := \kappa_1(p)\kappa_2(p)$ is known as the *Gaussian curvature of S at p.*

Remark 2.22. The principal curvatures $\kappa_1(p)$ and $\kappa_2(p)$ are also the eigenvalues of dN_p. Moreover, $K(p) = \kappa_1(p)\kappa_2(p) = \det(dN_p)$.

We now derive an expression for Gaussian curvature in terms of the first and second fundamental form coefficients.

Theorem 2.23. *If S is a regular surface with orientation $N : S \to S^2$, then the Gaussian curvature K is given by*

$$K = \frac{eg - f^2}{EG - F^2}.$$

Proof. First, let $dN_p = (a_{ij}) \in \mathbb{R}^{2\times 2}$ such that

$$dN_p(\mathbf{x}_u) = N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \qquad dN_p(\mathbf{x}_v) = N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v.$$

We have

$$\begin{aligned} -e &= \langle N_u, \mathbf{x}_u\rangle = a_{11}E + a_{21}F, & -f &= \langle N_u, \mathbf{x}_v\rangle = a_{11}F + a_{21}G, \\ -f &= \langle N_v, \mathbf{x}_u\rangle = a_{12}E + a_{22}F, & -g &= \langle N_v, \mathbf{x}_v\rangle = a_{12}F + a_{22}G, \end{aligned}$$

or more succinctly,

$$\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{(dN_p)^T} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

We can solve for the entries of $(dN_p)^T$ by matrix inversion, but observe that

$$\det\begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} = \det\left((dN_p)^T\right)\cdot\det\begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

so $eg - f^2 = \det(dN_p)\cdot(EG - F^2)$ and therefore $K = \dfrac{eg - f^2}{EG - F^2}$. ☐

The sign of K supplies us with information about a surface's shape around a point.

(1) If $K(p) < 0$, then the principal curvatures have opposite signs and there are two directions along which the surface bends away from or towards $N(p)$.
(2) If $K(p) > 0$, then the principal curvatures—hence all second fundamental forms—have the same sign. The surface either bends away from or towards $N(p)$ for all directions.
(3) If $K(p) = 0$, then one of the principal curvatures vanish and the surface does not bend along some direction.

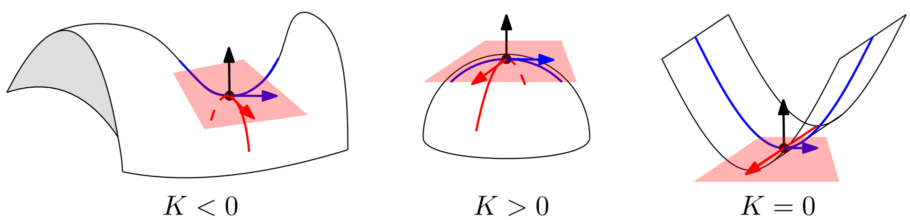

FIGURE 12. Gaussian curvature signs

Intuitively and by computation, the sphere's uniform 'roundness' implies that the principal curvatures at every point have the same sign and thus the Gaussian curvature is positive. We sketch a proof of the following—perhaps surprising—result.

Proposition 2.24. *If S is a compact, orientable regular surface, then there exists a point with positive Gaussian curvature.*

Sketch. If S is compact, then S is closed and bounded. Let Σ be a sphere that encloses and is tangent to S, and let p be their point of tangency. Assuming both S and Σ have normals pointing outward, then Σ bends away from the normal $N(p)$ at p. For Σ to enclose S, S must also bend away from $N(p)$ to stay inside Σ. ☐

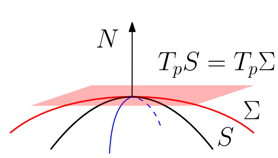

FIGURE 13. Positive Gaussian curvature for compact surfaces

2.4. **Gauss' Remarkable Formula.** In this section, we present but omit the proof of an important formula for Gaussian curvature. From now on, let S be an orientable regular surface and assume all charts are compatible with an orientation $N : S \to S^2$. To simplify computations, we also assume that all charts are orthogonal. The following is the setup of the remarkable formula.

First, recall that if $(U, \mathbf{x})$ is a chart of S and $p \in \mathbf{x}(U)$, then $\mathbf{x}_u$ and $\mathbf{x}_v$ form a basis of T_pS and are normal to N. If we consider $\mathbf{x}_u$ and $\mathbf{x}_v$ as vectors not only in T_pS but also in $\mathbb{R}^3$, then $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ is a basis of $\mathbb{R}^3$. We can express $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vu}$, and $\mathbf{x}_{vv}$ in terms of this basis, so let $\Gamma^k_{ij} \in \mathbb{R}$ for $1 \leq i, j, k \leq 2$ such that

$$\mathbf{x}_{uu} = \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + eN, \qquad \mathbf{x}_{vu} = \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + fN,$$
$$\mathbf{x}_{uv} = \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + fN, \qquad \mathbf{x}_{vv} = \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + gN.$$

Because $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, we have $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$. Taking the inner product between the four expressions with $\mathbf{x}_u$ and $\mathbf{x}_v$ gives us

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \Gamma^1_{11}E = \frac{E_u}{2}, \quad \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \Gamma^1_{12}E = \frac{E_v}{2}, \quad \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = \Gamma^1_{22}E = -\frac{G_u}{2},$$
$$\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = \Gamma^2_{11}G = -\frac{E_v}{2}, \quad \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \Gamma^2_{12}G = \frac{G_u}{2}, \quad \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \Gamma^2_{22}G = \frac{G_v}{2}.$$

As introduced in our proof of Theorem 2.23, suppose that

$$dN_p(\mathbf{x}_u) = N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \qquad dN_p(\mathbf{x}_v) = N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v.$$

Noting that $(\mathbf{x}_{uu})_v = (\mathbf{x}_{uv})_u$, our setup allows us to obtain the following.

Theorem 2.25 (Gauss Formula)**.** *If S is an orientable regular surface, then*

$$-EK = \left(\Gamma^2_{12}\right)_u - \left(\Gamma^2_{11}\right)_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}.$$

A proof of formula is available on p. 237 of [2]. Substituting the values of Γ^k_{ij} into the Gauss formula will gives us the following expression for Gaussian curvature in terms of only E, F, and G.

Lemma 2.26. *If $(U, \mathbf{x})$ is an orthogonal chart of S, then*

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

2.5. **Geodesic Curvature.** A part of the Gauss-Bonnet Theorem concerns the *geodesic curvature* of a regular region's boundary paths. In this section, we examine geodesic curvature in a general manner by studying *smooth tangent vector fields* along paths on oriented regular surfaces.

Definition 2.27. Let $(U, \mathbf{x})$ be a chart of S and $\alpha : I \to S$ a smooth path such that $\alpha(I) \subset \mathbf{x}(U)$. A *smooth tangent (vector) field w along α* assigns to each $t \in I$ a vector $w(t) \in T_{\alpha(t)}S$ such that there exist smooth functions $a, b : I \to \mathbb{R}$ where

$$w(t) = a(t)\mathbf{x}_u(q) + b(t)\mathbf{x}_v(q)$$

for all $t \in I$ and $q = \mathbf{x}^{-1} \circ \alpha(t)$.

Example 2.28. If $\alpha : I \to S$ is a smooth path such that $\alpha(I) \subset \mathbf{x}(U)$ for some chart $(U, \mathbf{x})$ of S, then the map $t \mapsto \alpha'(t)$ is a smooth tangent field along α. Since α

is smooth, there exist smooth functions $u, v : I \to \mathbb{R}$ such that $\alpha(t) = \mathbf{x}(u(t), v(t))$. Denoting $q = \mathbf{x}^{-1} \circ \alpha(t)$, we have

$$\alpha'(t) = u'(t)\mathbf{x}_u(q) + v'(t)\mathbf{x}_v(q).$$

We will denote $t \mapsto \alpha'(t)$ by a' and refer to it as the *velocity vector field of* α.

Definition 2.29. If w is a smooth tangent field along a smooth path $\alpha : I \to S$, then define the *covariant derivative* $Dw/dt(t)$ *of* w *at* $t \in I$ to be

$$\frac{Dw}{dt}(t) := \left(\text{projection of } w'(t) = \frac{dw}{dt}(t) \text{ onto } T_{\alpha(t)}S\right).$$

By construction, $Dw/dt(t)$ belongs to $T_{\alpha(t)}S$. If w is a smooth tangent field of unit vectors along α, then $\langle w(t), w(t)\rangle = 1$ for all $t \in I$. Differentiating with respect to t yields

$$\langle w'(t), w(t)\rangle + \langle w(t), w'(t)\rangle = 0$$

and therefore $\langle w(t), dw/dt(t)\rangle = 0$. If $w'(t)$ is orthogonal to $w(t)$, then its projection $Dw/dt(t)$ must also be orthogonal to $w(t)$. Notice that the surface normal $N(\alpha(t))$ is orthogonal to both $Dw/dt(t)$ and $w(t)$, so $Dw/dt(t)$ must be parallel to the unit vector $N(\alpha(t)) \times w(t)$. Let $\lambda : I \to \mathbb{R}$ be the smooth function such that

$$\frac{Dw}{dt}(t) = \lambda(t)\,(N(\alpha(t)) \times w(t))$$

for all $t \in I$. We define the *algebraic value of* Dw/dt *at* $t \in I$ to be $\lambda(t)$, which we will denote by $\left[Dw/dt(t)\right]$.

Definition 2.30. Let $\alpha : I \to S$ be a smooth path parametrized by arc length. We define the *geodesic curvature* $k_g(s)$ *of* α *at* $s \in I$ to be

$$k_g(s) := \left[\frac{D\alpha'}{ds}(s)\right].$$

If $\alpha : I \to S$ is parametrized by arc length, then notice that $D\alpha'/ds(s)$ is the projection of $\alpha''(s)$ onto $T_{\alpha(s)}S$ (Figure 14). If we follow our intuition that $T_{\alpha(s)}S$ is a first-order local approximation of S, then the geodesic curvature $[D\alpha'/ds(s)]$ measures how much α 'turns' *on* S—or equivalently, how much α turns away from the plane spanned by $\alpha'(s)$ and N. A notable class of smooth paths on a surface are those with zero geodesic curvature everywhere.

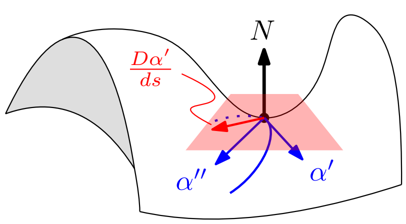

FIGURE 14. Geodesic curvature

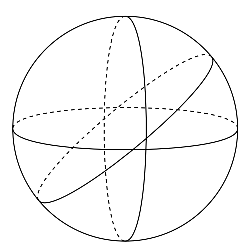

FIGURE 15. Great circles

Definition 2.31. A smooth path $\alpha : I \to S$ parametrized by arc length is *locally geodesic at* $s_0 \in I$ if $D\alpha'/ds(s_0) = 0$. We say α is a *geodesic* if $D\alpha'/ds(s) = 0$ for all $s \in I$.

Intuitively, $D\alpha'/dt \equiv 0$ means that α is like a straight line in S—yet not always a straight line in $\mathbb{R}^3$. Since straight lines minimize distances between points in $\mathbb{R}^2$, geodesics are often referred to as *locally length-minimizing curves*. Geodesics on a sphere are its *great circles*—circles with radius equal to that of the sphere.

We would like to obtain a concrete expression of the algebraic value of covariant derivatives in terms of coefficients such as E, F, and G. We will first show that if v and w are smooth unit tangent vector fields along a smooth path $\alpha : I \to S$, then $[Dw/dt] - [Dv/dt]$ is equal to the rate of change of the angle between v and w.

Lemma 2.32. *Let $a, b : I \to \mathbb{R}$ be smooth functions where $a^2 + b^2 \equiv 1$ and there exists $t_0 \in I$ and $\varphi_0 \in [0, 2\pi)$ such that*

$$a(t_0) = \cos(\varphi_0), \qquad b(t_0) = \sin(\varphi_0).$$

Then the smooth function

$$\varphi(t) := \varphi_0 + \int_{t_0}^{t} (ab' - a'b)dr$$

satisfies $\varphi(t_0) = \varphi_0$ and $\cos(\varphi(t)) = a(t)$ and $\sin(\varphi(t)) = b(t)$ for all $t \in I$.

Proof. By construction of φ, we have $\varphi(t_0) = \varphi_0$ and $\varphi' \equiv ab' - a'b$. If $a^2 + b^2 \equiv 1$, then $2aa' + 2bb' \equiv 0$ and $aa' \equiv -bb'$. We will show that $a \equiv \cos\varphi$ and $b \equiv \sin\varphi$ by showing

$$X := (a - \cos\varphi)^2 + (b - \sin\varphi)^2 \equiv 0.$$

First, we show that X is constant; differentiating X and applying $aa' \equiv -bb'$ yields

$$\begin{aligned} X' &= 2(a - \cos\varphi)(a' + \varphi' \sin\varphi) + 2(b - \sin\varphi)(b' - \varphi' \cos\varphi), \\ &= a\varphi' \sin\varphi - b\varphi' \cos\varphi - a' \cos\varphi - b' \sin\varphi. \end{aligned}$$

Since $\varphi' \equiv ab' - a'b$, substituting φ' above shows that $X' \equiv 0$ and thus X is constant. Because $a(t_0) - \cos(\varphi_0) = 0$ and $b(t_0) - \sin(\varphi_0) = 0$, we see that $a - \cos\varphi \equiv 0$ and $b - \sin\varphi \equiv 0$. ☐

Lemma 2.33. *Let v, w be smooth unit tangent vector fields along a smooth path $\alpha : I \to S$ and $\varphi : I \to \mathbb{R}$ the angle from v to w. Then*

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\varphi}{dt}.$$

Proof. Let $w_\perp := N \times w$ and $v_\perp := N \times v$. Then $\{v, v_\perp\}$ and $\{w, w_\perp\}$ are positive orthonormal bases of $T_{\alpha(t)}S$ for all $t \in I$. We have that $\langle v, v\rangle = \langle v_\perp, v_\perp\rangle = 1$ and $\langle v, v_\perp\rangle = 0$, and differentiating them implies

$$\langle v', v\rangle = \langle v'_\perp, v_\perp\rangle = 0, \qquad \langle v', v_\perp\rangle = \langle v, v'_\perp\rangle. \tag{2.34}$$

Since φ is the angle from v to w, we have

$$w = (\cos\varphi)v + (\sin\varphi)v_\perp, \qquad w_\perp = (-\sin\varphi)v + (\cos\varphi)v_\perp.$$

Differentiating w yields

$$w' = (-\varphi' \sin\varphi)v + (\cos\varphi)v' + (\varphi' \cos\varphi)v_\perp + (\sin\varphi)v'_\perp.$$

Since $w_\perp = N \times w$ is a unit vector, we have $[Dw/dt] = \langle w', w_\perp \rangle$. An expansion of $\langle w', w_\perp \rangle$ in terms of v and $v_\perp$, followed by an application of (2.34), reduces to

$$\left[\frac{Dw}{dt}\right] = \langle w', w_\perp \rangle = \frac{d\varphi}{dt} + \left[\frac{Dv}{dt}\right] \qquad \square$$

Lemma 2.35. *Let $(U, \mathbf{x})$ be an isothermal chart compatible with the orientation of a regular surface S. If w is a smooth tangent vector field along a smooth path $\alpha(t) = \mathbf{x}(u(t), v(t))$ in $\mathbf{x}(U)$ and φ is the angle from $\mathbf{x}_u$ to w, then*

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u v' - E_v u'\right) + \frac{d\varphi}{dt}.$$

Proof. If $(U, \mathbf{x})$ is isothermal and compatible with the orientation of S, then the vector fields

$$\mathbf{u}(t) := \frac{\mathbf{x}_u(u(t), v(t))}{\sqrt{E(u(t), v(t))}} \qquad\qquad \mathbf{v}(t) := \frac{\mathbf{x}_v(u(t), v(t))}{\sqrt{G(u(t), v(t))}}$$

along α form an orthonormal basis of $T_{\alpha(t)}S$ with $N = \mathbf{u} \times \mathbf{v}$. Lemma 2.33 tells us

$$\left[\frac{Dw}{dt}\right] = \left[\frac{d\mathbf{u}}{dt}\right] + \frac{d\varphi}{dt},$$

so it suffices to compute $[d\mathbf{u}/dt]$. Observe that

$$\frac{d\mathbf{u}}{dt} = \frac{1}{E}\left(\left(\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v'\right)\sqrt{E} - \mathbf{x}_u\left(\frac{E_u u' + E_v v'}{2\sqrt{E}}\right)\right),$$

so we have

$$\left[\frac{d\mathbf{u}}{dt}\right] = \left\langle \frac{d\mathbf{u}}{dt}, \mathbf{v} \right\rangle = \left\langle \frac{\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v'}{\sqrt{E}}, \frac{\mathbf{x}_v}{\sqrt{G}} \right\rangle.$$

Since $\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = -\frac{1}{2}E_v$ and $\langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \frac{1}{2}G_u$, we see that

$$\left[\frac{d\mathbf{u}}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u u' - E_v v'\right). \qquad \square$$

3. Triangulations and the Euler Characteristic

We now present the topological side of the Gauss-Bonnet Theorem. We begin by revisiting regular regions and defining *exterior angles* and *triangles* on surfaces. We will list but not prove a few important facts related to *triangulations* and the *Euler characteristic.*

To define exterior angles, let $\alpha : [0, \ell] \to S$ be a simple, closed, piecewise-smooth path on S with a partition $\{t_0, \dots, t_{n+1}\}$ of $[0, \ell]$. For each $0 \le i \le n$, define the *left and right velocities of α at t_i* to be the vectors

$$\alpha'_-(t_i) := \lim_{t \to t_i^-} \alpha'(t), \qquad\qquad \alpha'_+(t_i) := \lim_{t \to t_i^+} \alpha'(t),$$

respectively. Because α is piecewise-smooth, $\alpha'_-(t_i)$ and $\alpha'_+(t_i)$ are non-zero vectors that also lie on $T_{\alpha(t_i)}S$. Define the *ith exterior angle* to be $\theta_i \in [-\pi, \pi]$ such that $|\theta_i|$ is the angle between $\alpha'_-(t_i)$ and $\alpha'_+(t_i)$ (Definition 2.14) and θ_i is positive if α turns counterclockwise about the surface normal and negative otherwise.

We will later use the following fact without proof.

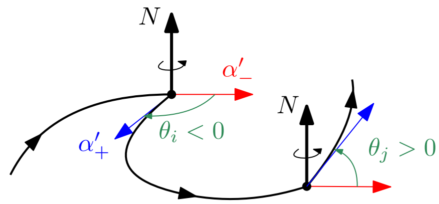

FIGURE 16. Exterior angles

Theorem 3.1. *If $\alpha : [0, \ell] \to S$ is a simple, closed, piecewise smooth path with a partition $\{t_0, t_1, \ldots, t_n, t_{n+1}\}$ of $[0, \ell]$ and exterior angles $\theta_0, \theta_1, \ldots, \theta_n$, then*

$$\sum_{i=0}^{n} \big(\varphi_i(t_{i+1}) - \varphi_i(t_i)\big) + \sum_{i=0}^{n} \theta_i = \pm 2\pi,$$

where φ_i is the angle function from $\mathbf{x}_u$ to $\alpha'(t)$ for $t \in [t_i, t_{i+1}]$. The sign of 2π is positive if α is positively-oriented and negative otherwise.

Theorem 3.1 will give us the factor of π in the Gauss-Bonnet Theorem. We now define triangles and triangulations.

Definition 3.2. A *triangle on S* is a simple region with three vertices and three non-vanishing exterior angles.

Definition 3.3. A *triangulation $\mathcal{T}$ of a regular region $R \subset S$* is a finite collection $\{T_i\}_{i=1}^n$ of triangles T_i such that

(1) $R = \bigcup_{i=1}^n T_i$
(2) and $T_i \cap T_j \neq \varnothing$ only if $T_i \cap T_j$ is either a common edge or common vertex of T_i and T_j.

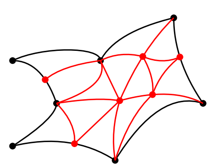

FIGURE 17. Triangulation

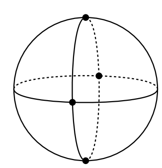

FIGURE 18. Triangulated sphere

Our proof of the Gauss-Bonnet Theorem will assume the following facts

Lemma 3.4 ([2], 272)**.** *Every regular region $R \subset S$ can be triangulated.*

Lemma 3.5 ([2], 272)**.** *Let $\{(U_\lambda, \mathbf{x}_\lambda)\}_{\lambda\in\Lambda}$ be a family of charts compatible with the orientation of S. Then there exists a triangulation of R such that each triangle is a subset of $\mathbf{x}_\lambda(U_\lambda)$ for some $\lambda \in \Lambda$.*

Definition 3.6. Suppose $\mathcal{T}$ is a triangulation of a regular region $R \subset S$. Let E, F, and V denote the total number of edges, faces, and vertices of $\mathcal{T}$, respectively. Define the Euler characteristic of R to be

$$\chi(R) := V - E + F.$$

Lemma 3.7. *The Euler characteristic of a regular region $R \subset S$ does not depend on triangulation.*

Proof. Let $\{V_i\}_{i=1}^{\ell}$ be the set of vertices of some triangulation $\mathcal{T}$ of R. Let $V_{\ell+1}$ be a point in R that does not already belong to $\{V_i\}_{i=1}^{\ell}$ and suppose it lies in a region bounded by a triangle $T \in \mathcal{T}$. Let ΔV, ΔE, and ΔF denote the change in the number of vertices, edges, and faces, respectively, upon appending $V_{\ell+1}$ to $\{V_i\}_{i=1}^{\ell}$. Now consider the following cases.

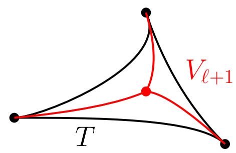

FIGURE 19. Inside T

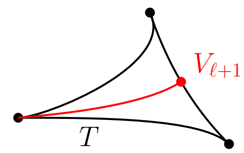

FIGURE 20. On ∂T

(1) If $V_{\ell+1}$ lies in the interior of a triangle T, then $\Delta V = 1$, $\Delta E = 3$, and $\Delta F = 2$. We have
$$\Delta V - \Delta E + \Delta F = 1 - 3 + 2 = 0,$$
so the Euler characteristic of $\{V_i\}_{i=1}^{\ell+1}$ is still $V - E + F$.

(2) If $V_{\ell+1}$ lies on the boundary of T, then $\Delta V = 1$, $\Delta E = 2$, $\Delta F = 1$. Thus,
$$\Delta V - \Delta E + \Delta F = 1 - 2 + 1 = 0.$$

The Euler Characteristic of the triangulation defined by $\{V_i\}_{i=1}^{\ell+1}$ is equal to
$$(V - E + F) + (\Delta V - \Delta E + \Delta F) = V - E + F,$$
so the Euler Characteristic is invariant under adding vertices.

If $\{V_i\}_{i=1}^{\ell}$ and $\{V_j\}_{j=1}^{k}$ are vertices of two different triangulations of R, then appending $\{V_j\}_{j=1}^{k}$ to $\{V_i\}_{i=1}^{\ell}$ shows that $\{V_i\}_{i=1}^{\ell}$ and $\{V_j\}_{j=1}^{k}$ must have the same Euler characteristics. □

Lemma 3.8 ([2], 273)**.** *If $S \subset \mathbb{R}^3$ is a compact, connected, and orientable regular surface, then $\chi(S)$ takes on values $-2, 0, 2, 4, \ldots$. Define the genus g of S to be*
$$g := \frac{2 - \chi(S)}{2}.$$

The genus is also known as the number of holes that a closed surface has. For instance, a sphere has genus 0, a torus genus 1, and adding 'handles' to a surface increases its genus by 1.

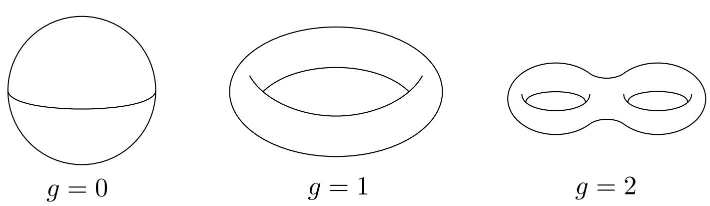

FIGURE 21. Genus

4. The Gauss-Bonnet Theorem

We are finally well-equipped with tools to prove variations of the Gauss-Bonnet Theorem. The 'local' version concerns regular regions that lie entirely within $\mathbf{x}(U)$ for some chart $(U, \mathbf{x})$ of an orientable regular surface S, whereas the 'global' version extends this result to regular regions that lie across multiple charts. The proof of the latter will draw upon the topological facts in the previous section.

Theorem 4.1 (Local Gauss-Bonnet)**.** *Let S be an orientable regular surface and $(U, \mathbf{x})$ an isothermal chart where U is homeomorphic to an open disk. Let $R \subset \mathbf{x}(U)$ be a simple region and $\beta : [0, \ell] \to S$ a positively-oriented, simple, closed, piecewise-smooth path segment parametrized by arc length such that $\beta\big([0, \ell]\big) = \partial R$. Then*

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} k_g \; ds + \int_R K \; dS + \sum_{i=0}^{n} \theta_i = 2\pi,$$

where $\{s_0, \ldots, s_{n+1}\}$ partitions $[0, \ell]$ and $\theta_0, \theta_1, \ldots, \theta_n$ are the exterior angles of β.

Proof. Let $\beta(s) = \mathbf{x}(u(s), v(s))$ and $Q := \mathbf{x}^{-1}(R)$. By Lemma 2.35, we have

$$k_g = \frac{1}{2\sqrt{EG}} \left(G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\varphi_i}{ds},$$

where φ_i measures the angle from $\mathbf{x}_u$ to $\beta'(s)$ for $s \in [s_i, s_{i+1}]$. Then

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} k_g \; ds = \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \left(\frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_u}{2\sqrt{EG}} \frac{du}{ds} \right) ds + \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} \; ds.$$

By Stokes' Theorem, we have

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \left(\frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_u}{2\sqrt{EG}} \frac{du}{ds} \right) ds = \int_Q \left(\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right) du \; dv.$$

Since $(U, \mathbf{x})$ is isothermal, we have $E(q) = G(q) = \lambda(q)$ and $F(q) = 0$ for all $q \in Q$. Notice that $\sqrt{EG - F^2} = \sqrt{EG} = \lambda$. By Lemma 2.26, we have

$$\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u = -K\lambda,$$

which means

$$\int_Q \left(\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right) du \; dv = \int_Q -K\lambda \; du \; dv = -\int_R K \; dS.$$

By Theorem 3.1, we have

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds} \; ds = \sum_{i=0}^{n} \big(\varphi_i(s_{i+1} - s_i) \big) = 2\pi - \sum_{i=0}^{n} \theta_i.$$

Altogether, we see that

$$\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} k_g \; ds = -\int_R K \; dA + 2\pi - \sum_{i=0}^{n} \theta_i,$$

which proves the local Gauss-Bonnet Theorem. ∎

Notation 4.2. Let C denote the image of a boundary path β and write

$$\int_C k_g\ ds := \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} k_g\ ds.$$

Theorem 4.3 (Global Gauss-Bonnet)**.** *Let $R \subset S$ be a regular region and $\beta_1, \ldots, \beta_n$ positively-oriented boundary paths. If $\theta_1, \ldots, \theta_p$ are the exterior angles of ∂R, then*

$$\sum_{i=1}^{n} \int_{C_i} k_g\ ds + \iint_R K\ dA + \sum_{j=1}^{p} \theta_j = 2\pi\chi(R),$$

where C_i is the image of β_i.

Proof. Let $\{(U_\lambda, \mathbf{x}_\lambda)\}_{\lambda\in\Lambda}$ be a family of charts compatible with the orientation of S. By Lemma 3.5, there exists a triangulation $\mathcal{T}$ of R such that each triangle in $\mathcal{T}$ is a subset of $\mathbf{x}_\lambda(U_\lambda)$ for some $\lambda \in \Lambda$. Suppose $\mathcal{T}$ has V vertices, E edges, and F faces, and let $\theta_{j1}, \theta_{j2}, \theta_{j3}$ denote the exterior angles of the jth triangle.

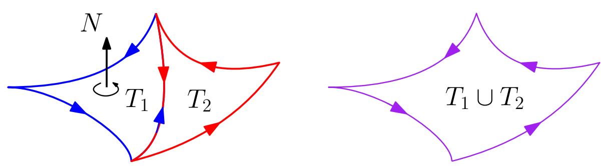

Figure 22. Boundary sum

Assuming that the boundary paths of the triangles are positively-oriented, observe that if T_i and T_j are triangles sharing an edge C, then

$$\int_{\partial T_i} k_g\ ds + \int_{\partial T_j} k_g\ ds = \int_{\partial(T_i\cup T_j)} k_g\ ds$$

(Figure 22). Thus, applying the Local Gauss-Bonnet Theorem to each triangle and summing the result across all F of them yields

$$\sum_{i=1}^{n} \int_{C_i} k_g\ ds + \iint_R K\ dA + \sum_{j=1}^{F}\sum_{k=1}^{3} \theta_{jk} = 2\pi F. \tag{4.4}$$

Now define the *interior angle* $\varphi_{jk} := \pi - \theta_{jk}$. We have

$$\sum_{j=1}^{F}\sum_{k=1}^{3} \theta_{jk} = \sum_{j=1}^{F}\sum_{k=1}^{3} \pi - \sum_{j=1}^{F}\sum_{k=1}^{3} \varphi_{jk} = 3\pi F - \sum_{j=1}^{F}\sum_{k=1}^{3} \varphi_{jk}. \tag{4.5}$$

Let E_i and V_i denote the number of edges and vertices (respectively) inside R, and E_b and V_b the number of edges and vertices (respectively) on ∂R. Observe that $3F = 2E_i + E_b$; there are a total of $3F$ edges if we disregard overlaps between triangles. Of these edges, E_b of them lie on ∂R and belong to exactly one triangle while the remaining are edges shared between triangles, and there are $2E_i$ of them. Thus, we have $3F = 2E_i + E_e$, and substituting into (4.5) yields

$$\sum_{j=1}^{F}\sum_{k=1}^{3} \theta_{jk} = 2\pi E_i + \pi E_b - \sum_{j=1}^{F}\sum_{k=1}^{3} \varphi_{jk}.$$

We will now examine the vertices of $\mathcal{T}$. The V_b boundary vertices either came from paths $\beta_1, \ldots, \beta_n$ or from triangulation. Let V_{bp} and V_{bt} denote these two quantities, respectively. There were p exterior angles from the paths, so $V_{bp} = p$. Observe that:

- If a boundary vertex is from triangulation, then the sum of the interior angles about it is π.
- If a boundary vertex is from $\beta_1, \ldots, \beta_n$, then the sum of interior angles is equal to $\sum_{j=1}^{p}(\pi - \theta_j) = \pi V_{bp} - \sum_{j=1}^{p} \theta_j$.
- The sum of angles about an interior vertex is 2π.

And so, we have

$$\begin{aligned}\sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk} &= 2\pi V_i + \pi V_{bt} + \pi V_{bp} - \sum_{j=1}^{p}\theta_j, \\ &= 2\pi V_i + \pi V_b - \sum_{j=1}^{p}\theta_j, \\ \sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} &= 2\pi E_i + \pi E_b - \left(2\pi V_i + \pi V_b - \sum_{j=1}^{p}\theta_j\right).\end{aligned}$$

Since $\beta_1, \ldots, \beta_n$ are closed, we have $E_b = V_b$. Then

$$\begin{aligned}\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} &= 2\pi E_i + \pi E_b + \overbrace{\pi(E_b - V_b)}^{0} - \left(2\pi V_i + \pi V_b - \sum_{j=1}^{p}\theta_j\right), \\ &= 2\pi E_i + 2\pi E_b - 2\pi V_i - 2\pi V_b + \sum_{j=1}^{p}\theta_j, \\ &= 2\pi(E - V) + \sum_{j=1}^{p}\theta_j.\end{aligned}$$

Substitution into (4.4) finally gives us

$$\begin{aligned}\sum_{i=1}^{n}\int_{C_i} k_g\, ds + \iint_R K\, dA + \left(2\pi(E - V) + \sum_{j=1}^{p}\theta_j\right) &= 2\pi F, \\ \sum_{i=1}^{n}\int_{C_i} k_g\, ds + \iint_R K\, dA + \sum_{j=1}^{p}\theta_j &= 2\pi(V - E + F), \\ &= 2\pi\chi(R).\end{aligned}$$

□

Corollary 4.6. *If S is a compact, orientable regular surface, then*

$$\iint_S K\, dA = 2\pi\chi(S) = 2\pi(2 - 2g). \tag{4.7}$$

Proof. Since S is compact, there are no boundary paths for the entire surface—hence no exterior angles. We obtain (4.7) by applying the Global Gauss-Bonnet Theorem and Lemma 3.8. □

As mentioned in the introduction, the Gauss-Bonnet Theorem presents a striking link between geometric quantities and topological invariants. Corollary 4.6 is one of the most well-known realizations of the theorem, whereby the integral of Gaussian curvature over a compact surface is related its number of holes. The following are some additional consequences of the Gauss-Bonnet Theorem.

Corollary 4.8. *Suppose S is an orientable surface and T a geodesic triangle—a triangle whose edges are geodesics. If K has constant sign on T and $\varphi_1, \varphi_2, \varphi_3$ are the interior angles of T, then*

$$\begin{cases} \varphi_1 + \varphi_2 + \varphi_3 = \pi & \text{if } K = 0, \\ \varphi_1 + \varphi_2 + \varphi_3 > \pi & \text{if } K > 0, \\ \varphi_1 + \varphi_2 + \varphi_3 < \pi & \text{if } K < 0. \end{cases}$$

The proof follows immediately from the Global Gauss-Bonnet Theorem. Interestingly, if S is a plane, then $K = 0$ and we see that the sum of a triangle's interior angles is π—just as we know it to be. If we construct a triangle on a sphere with three great circles, then we see the sum of its interior angles will be greater than π.

Corollary 4.9. *Suppose S is a compact, connected, and orientable regular surface with at least one hole. Then there exist points on S with positive, zero, and negative Gaussian curvature.*

Proof. If S has at least one hole, then $g \geq 1$ and $\chi(S) = 2 - 2g < 0$. Corollary 4.6 tells us $\iint_S K \, dA < 0$, which implies there exists a point with negative Gaussian curvature. If S is compact, then there exists a point with positive Gaussian curvature (Proposition 2.24). By the continuity of K and the Intermediate Value Theorem, there exists a point on S with zero Gaussian curvature. $\square$

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