An Introduction to Minimal Surfaces in $\mathbb{R}^3$

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December 24, 2021

Outline

1. Motivation

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2. Surfaces, tangent spaces, and curvature

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3. Variational characterization of minimal surfaces

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2. Surfaces, tangent spaces, and curvature
3. Variational characterization of minimal surfaces
4. Examples of minimal surfaces
5. Remarks

Motivation for Minimal Surfaces

- Soap films

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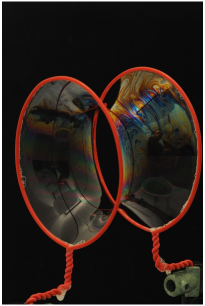
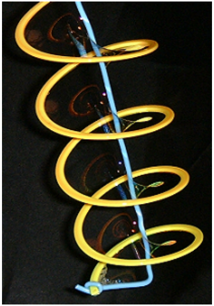
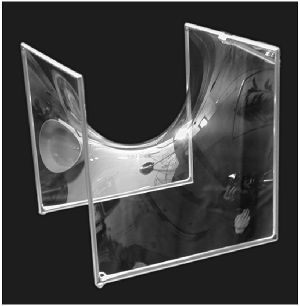
  - Given any non-self-intersecting, closed curve Γ, is there a minimal surface with boundary Γ? (Plateau's Problem)

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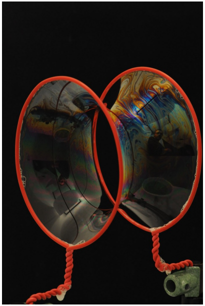
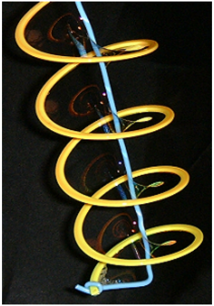
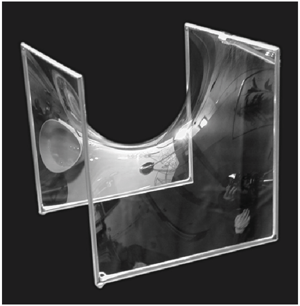
  - Yes! Solved independently by T. Radó (1930) and J. Douglas (1931)

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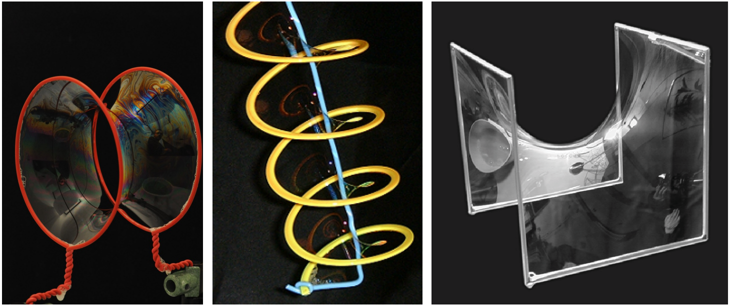

- Think of them as solutions to optimization problems

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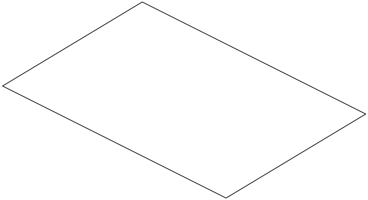
- Planes



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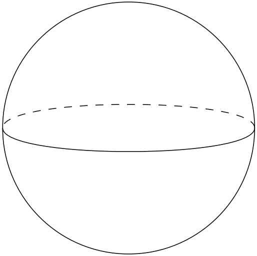

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Examples:

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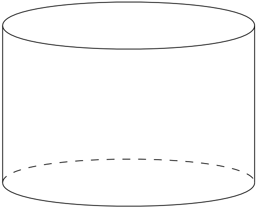
- Spheres
- Cylinders



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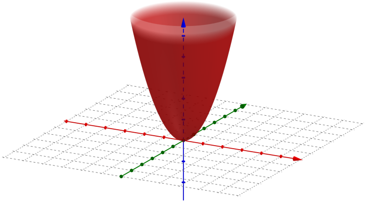
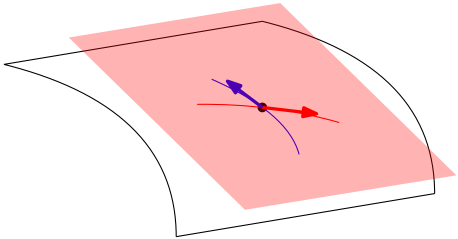
- Spheres
- Cylinders
- Graphs of smooth $f : \mathbb{R}^2 \to \mathbb{R}$



Tangent Space

The *tangent space* T_pS is the collection of velocity vectors of curves passing through $p \in S$.



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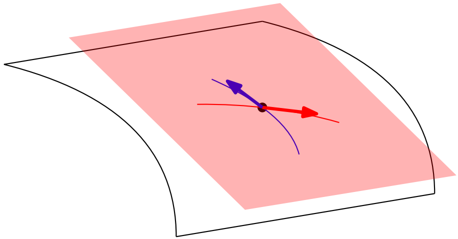

Given local parametrization $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ and $\mathbf{x}(0,0) = p$, consider the *coordinate curves*:

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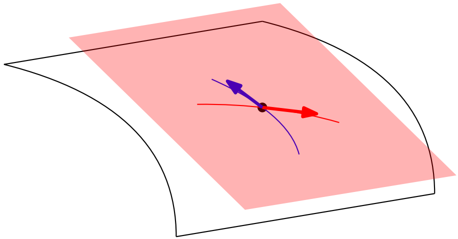

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Let $\mathbf{x}_u$ and $\mathbf{x}_v$ be the respective velocity vectors at p.

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- $\{\mathbf{x}_u, \mathbf{x}_v\}$ is a basis of T_pS

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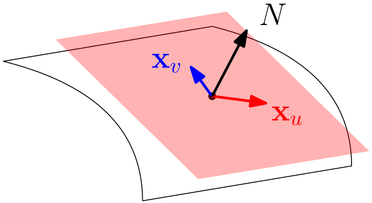

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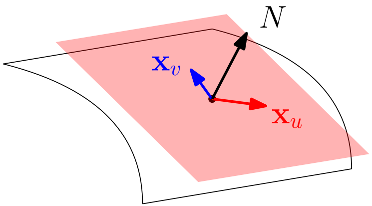
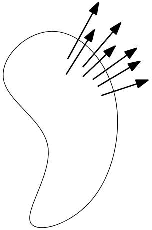
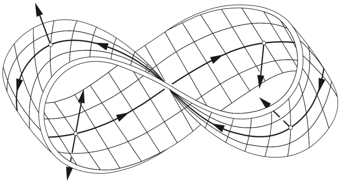


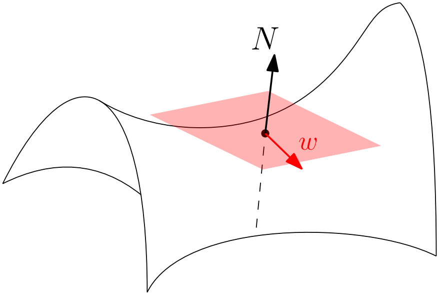
- We will assume our surfaces have a ‘consistent’ normal direction.





Normal Curvature

- Take $w \in T_pS$ with $\|w\| = 1$.



Interpretation

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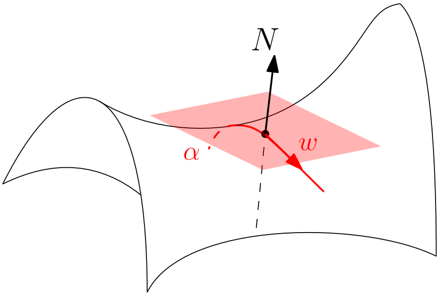

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- Take $w \in T_pS$ with $\|w\| = 1$.
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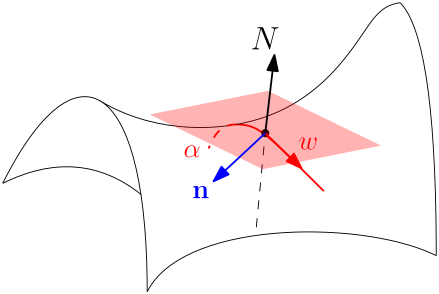
- Compute *curvature of* α at p:

$$\mathbf{n} = \frac{\alpha''(0)}{\|\alpha''(0)\|}.$$



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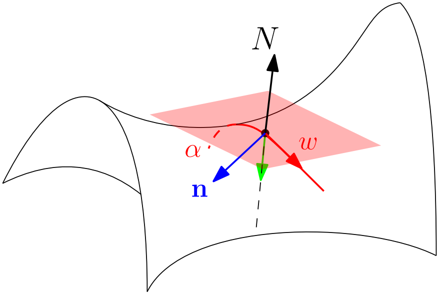

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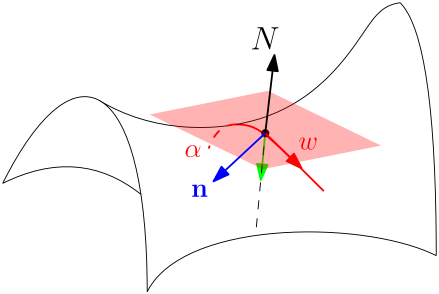


Interpretation

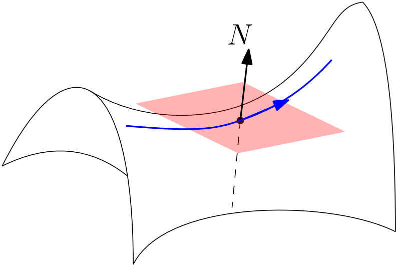
- High $(+)$ normal curvature $\implies$ curve 'bends' or 'accelerates' towards N

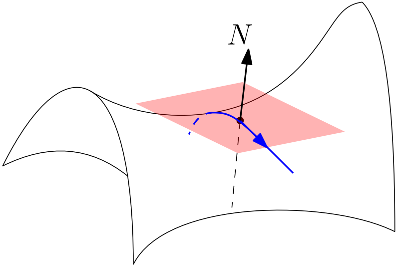
Principle Curvatures

- Define the *principal curvatures*

$$\kappa_1 := \max_{\|w\|=1} (\text{normal curvature w.r.t. } w),$$
$$\kappa_2 := \min_{\|w\|=1} (\text{normal curvature w.r.t. } w).$$

- *Principal directions* either maximize or minimize normal curvature





Mean Curvature and Minimal Surfaces

Define *mean curvature* H

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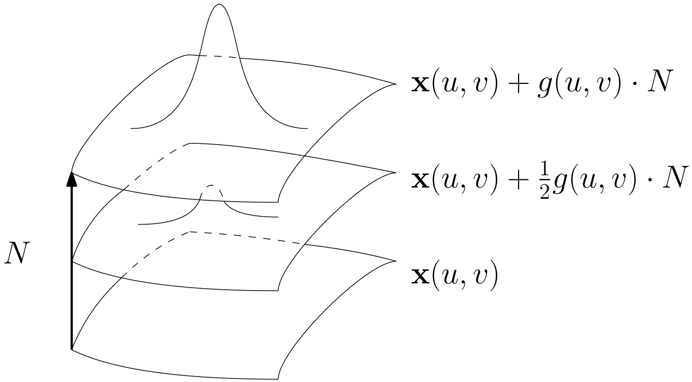
- i.e. $\kappa_1 = -\kappa_2$
- But what is minimized with 'minimal surfaces?'

Variational Characterization of Minimal Surfaces

Assume S is parametrixed by $\mathbf{x}(u,v)$ with domain $\mathbb{R}^2$.

1. Take some closed disk $D \subset \mathbb{R}^2$ and perturbation $g : D \to \mathbb{R}$
2. "Stretch" S along its normal direction by tg where $t \in \mathbb{R}$ small

$$\mathbf{x}^t(u,v) = \mathbf{x}(u,v) + tg(u,v) \cdot N$$



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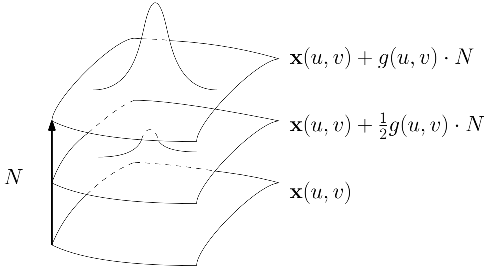

3. Compute area of stretched S by factor t

$$A(t) = \int_D \|\mathbf{x}_u^t \times \mathbf{x}_u^t\| \, du \, dv$$

Equivalent Formula of 'Minimal':

We say S is *minimal* if $A'(0) = 0$ for any disk D and perturbation g.

Equivalence of Minimal Surface Definitions

Theorem

$H = 0$ everywhere $\iff$ $A'(0) = 0$ for any D and $h : \bar{D} \to \mathbb{R}$

Sketch of Proof.

- Some computation yields

$$A'(0) = -\int_D 2gH\|\mathbf{x}_u \times \mathbf{x}_v\| \, du \, dv$$

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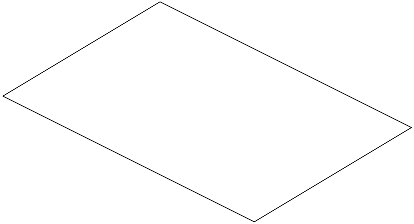
  - Then $A'(0) < 0$ on a certain disk.

Some Facts About Minimal Surfaces

Quick Facts:

- Planes are minimal.



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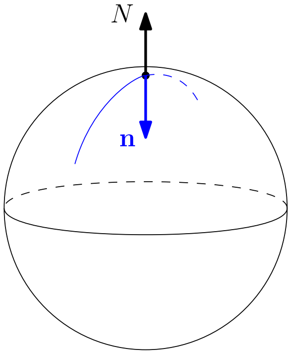

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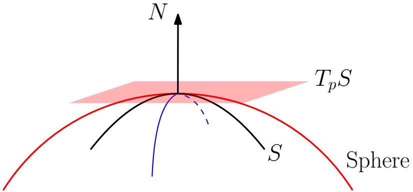
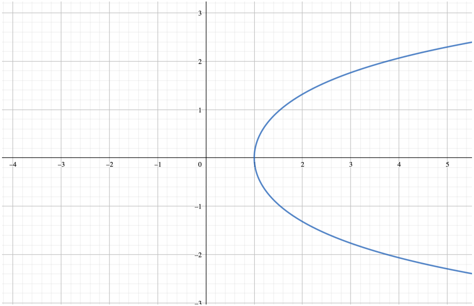
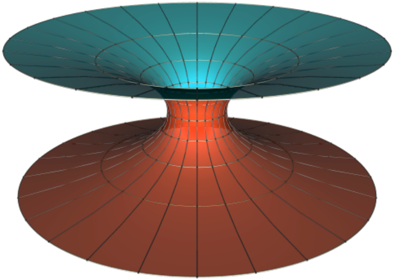


Example: Catenoid

Parametrization given by:

$$\mathbf{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$





The catenoid is the only minimal surface of revolution (aside from plane)

Example: Helicoid

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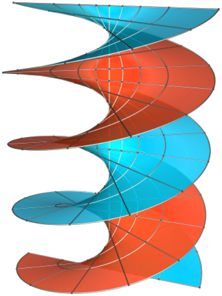

Example: Enneper's Surface

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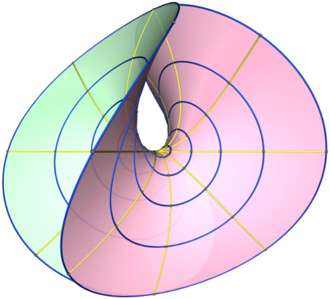

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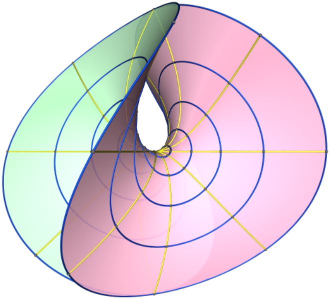


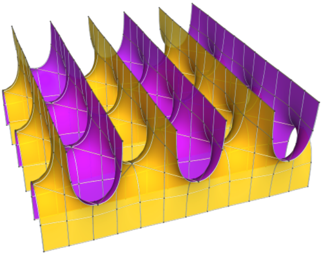
- Known for having self-intersections.
- Invariant after $\pi/2$ rotation about z-axis, followed by reflection over xy-plane

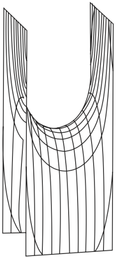
Example: Doubly-Scherk Surface

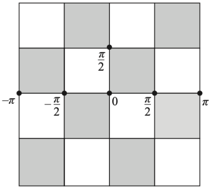
Parametrization given by:

$$\mathbf{x}(u,v) = \left(\arg\left(\frac{\zeta+i}{\zeta-i}\right), \arg\left(\frac{\zeta+1}{\zeta-1}\right), \log\left|\frac{\zeta^2+1}{\zeta^2-1}\right|\right)$$

where $\zeta = u + iv$ and $\arg\zeta$ is the angle from the real axis to ζ.







Known for being periodic.

Remarks

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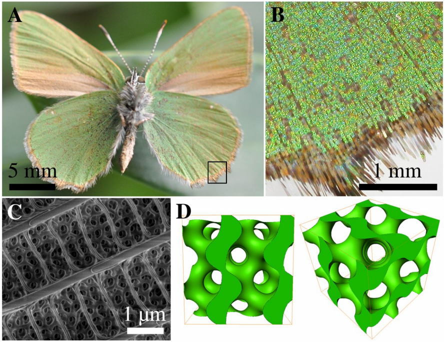
  - e.g. Butterfly wing colors



THANK YOU!