HAUSDORFF AND MINKOWSKI DIMENSION OF BROWNIAN PATHS

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Contents

1. Introduction

Brownian motion is one of the important stochastic processes in modern probability theory and is used to model random continuous movement. The process is named after botanist Robert Brown, who investigated in 1827 the irregular jiggling motion of pollen grains suspended in water. Physicists later attributed this random movement to the bombardment of pollen grains by fast-moving molecules in water.

In this expository paper, we will introduce the mathematical definition of Brownian motion and motivate the Hausdorff and Minkowski notions of dimension. We will prove that Brownian paths in $\mathbb{R}^d$ for $d \geq 2$ have Hausdorff and Minkowski dimension 2 almost surely with the help of Frostman's Lemma from potential theory.

2. Brownian Motion

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *stochastic process* is a collection $\{X(t)\}_{t\in I}$ of random variables $X(t)$ indexed by $t \in I$, which are measureable functions that map from Ω to some measureable space E. For our purposes, we take I to be the closed interval $[0, 1] \subset \mathbb{R}$ and $E = \mathbb{R}^d$ for some $d \geq 1$.

For a point or "experiment" in $\omega \in \Omega$, we associate with it the function $f_\omega : [0, 1] \to \mathbb{R}^d$ defined by evaluating $X(t)$ at ω for all $t \in [0, 1]$, namely

$$f_\omega(t) := X(t)(\omega).$$

Roughly speaking, in the same way that random variables represent "random" or unknown values, we can treat a stochastic process $\{X(t)\}_{t\in[0,1]}$ as an random function $X : [0, 1] \to \mathbb{R}^d$. Consequently, the image $X\big([0, 1]\big)$ represents an random subset of $\mathbb{R}^d$. Brownian motion is a special class of stochastic processes, and we begin by defining them on the real line.

Definition 2.1. We say a real-valued stochastic process $\{B(t)\}_{t\in[0,1]}$ is a *Brownian motion in* $\mathbb{R}$ if

(1) $B(0) \equiv 0$ and a random function $B(t)$ is continuous almost surely i.e.
$$\mathbb{P}\{B(t) \text{ is continuous}\} = \mathbb{P}\{\omega \in \Omega : f_\omega(t) = B(t)(\omega) \text{ is continuous}\} = 1,$$
(2) for any $0 \le t_1 \le \cdots \le t_{n-1} \le 1$ with $t_0 = 0$ and $t_n = 1$, the increments $B(t_k) - B(t_{k-1})$ are independent for all $k = 1, 2, \ldots, n$,

(3) $B(t) - B(s)$ is normally distributed with mean 0 and variance $|t-s|$ for any $0 \le s, t \le 1$ i.e.
$$\mathbb{P}\{B(t) - B(s) \le x\} \le \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi|t-s|}} \exp\left(-\frac{u^2}{2|t-s|}\right) \mathrm{d}u.$$

The three conditions together give us a model for one-dimensional random continuous movement. Intuitively, the third condition guarantees that for large $t-s$ time increments, we can expect the difference $B(t) - B(s)$ to be large as well. It might be surprising that $B(t)$ could be continuous almost surely if the increments $B(t) - B(s)$ are random for any $0 \le s < t \le 1$. Norbert Wiener proved in 1923 that there indeed exists a stochastic process that satisfies all three properties listed above. Later in 1954, Paul Lévy gave a construction of Brownian motion by first defining $B(t)$ inductively for all t of the form $k/2^n$ where $n \in \mathbb{N}$ and $0 \le k \le 2^n$ and then expressing $B(t)$ for all $t \in [0,1]$ as a series that converges uniformly on $[0,1]$ almost surely. We refer readers to [3] for a proof of Lévy's construction. Note that while Brownian random functions are continuous almost surely, they are also nowhere differentiable almost surely.

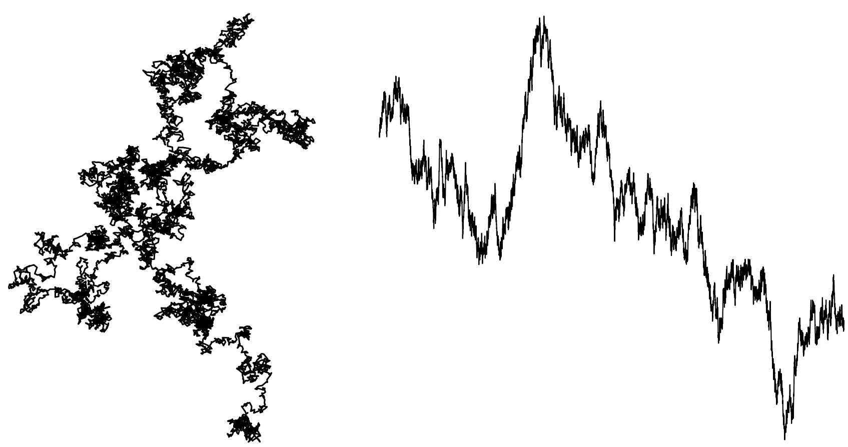

FIGURE 1. Brownian path in $\mathbb{R}^2$ (left), Brownian graph in $\mathbb{R}^{1+1}$ (right)

To extend our definition of Brownian motion in $\mathbb{R}$, we say that a vector-valued stochastic process $\{B(t)\}_{t\in[0,1]}$ with $B(t) = (B_1(t), \ldots, B_d(t))$ is a *Brownian motion in* $\mathbb{R}^d$ if its component processes $\{B_1(t)\}, \ldots, \{B_d(t)\}$ are independent Brownian motions in $\mathbb{R}$. Viewing stochastic processes as random functions, we define a *Brownian path in* $\mathbb{R}^d$ to be the image $B([0,1])$ and a *Brownian graph in* $\mathbb{R}^{d+1}$ to be
$$\Gamma(B) := \{(t, B(t)) \in \mathbb{R}^{d+1} : t \in [0,1]\}$$

for a Brownian random function $B : [0, 1] \to \mathbb{R}^d$.

It follows the definition of Brownian motion that the joint cumulative density function of $B(s) - B(t)$ is given by

$$\begin{aligned} F_d(x_1, \dots, x_d) &= \mathbb{P}\{B_i(t) - B_i(s) \le x_i \text{ for all } 1 \le i \le d\} \\ &= \prod_{i=1}^{d} \mathbb{P}\{B_i(t) - B_i(s) \le x_i\} \\ &= \prod_{i=1}^{d} \int_{-\infty}^{x_i} \frac{1}{\sqrt{2\pi|t-s|}} \exp\left(-\frac{r_i^2}{2|t-s|}\right) \mathrm{d}r_i \\ &= C_d |t-s|^{-s/2} \int_H \exp\left(-\frac{r_1^2 + \cdots + r_d^2}{2|t-s|}\right) \mathrm{d}r_1 \dots \mathrm{d}r_d \end{aligned}$$

where C_d is a constant that depends only on d and $H = (-\infty, x_1] \times \cdots \times (-\infty, x_d]$. Changing F_d to polar coordinates (using Jacobian formula from [4]), we obtain the following distribution function for the distance between $B(t)$ and $B(s)$

$$\begin{aligned} P(\rho) &:= \mathbb{P}\{\|B(t) - B(s)\| \le \rho\} \\ &= \widetilde{C_d} |t-s|^{-s/2} \int_0^{\rho} r^{d-1} \exp\left(-\frac{r^2}{2|t-s|}\right) \mathrm{d}r \end{aligned} \tag{2.2}$$

where $\widetilde{C_d}$ is a constant dependent on d that differs slightly from C_d.

3. Notions of Dimension

3.1. **Hausdorff and Minkowski dimensions.** Given a set $F \subseteq \mathbb{R}^d$, what can we say about its *dimension*? Loosely speaking, the dimension of a set is in some sense a description of its complexity. If F is a vector subspace of $\mathbb{R}^d$, then we can borrow the linear algebraic definition of dimension, namely the maximal number of (linearly) independent directions that can belong to F. However, this definition is too rigid if we want to talk about the dimension of, say, a curve, cube, or arbitrary set in $\mathbb{R}^d$. In this section, we introduce the Minkowski and Hausdorff definitions of dimension for studying the geometry of sets in Euclidean space.

Assume for now that we are in $\mathbb{R}^3$. A reasonable notion of dimension should reflect our intuition that a line segment is less "complex" than a square on the xy-plane, which is less complex than a solid cube. One way to illustrate this is to consider a *$1/n$-cube mesh of* $\mathbb{R}^3$, whereby we partition $\mathbb{R}^3$ into $1/n \times 1/n \times 1/n$ cubes with non-intersecting interiors.

Observe that only n^3 many of these cubes are necessary to cover the unit cube $[0, 1]^3$, n^2 for one of its square faces, and n for one of the square's edges. As we shrink the size of our mesh i.e. $n \to \infty$, we see that the number of mesh cubes covering each shape grows at different rates, which is quantified by the exponent on n. In a perhaps contrived way, we may take the exponents on n to be a definition of dimension, as doing so matches our intuition that lines, planes, and cubes are somehow 1, 2, and 3-dimensional objects, respectively.

As an attempt to build this "exponent of n" definition of dimension, let $N_n(F)$ be the number of $1/n$-mesh cubes that intersect a bounded subset $F \subseteq \mathbb{R}^d$. Assuming that $N_n(F)$ grows proportionally to n^s where s is the desired dimension of F, we have $N_n(F) = Cn^s$ for some constant $C > 0$ and thus

$$s = \frac{\log(N_n(F)) - \log(C)}{\log(n)}.$$

Taking the limit $n \to \infty$ (assuming it exists) gets rid of $\log(C)$ and gives us

$$s = \lim_{n\to\infty} \frac{\log(N_n(F))}{\log(n)} = \lim_{\delta\to 0^+} \frac{\log(N_\delta(F))}{\log(1/\delta)}$$

where $N_\delta(F)$ is defined as the number of δ-mesh cubes intersecting F. We now have a formula for the desired dimension s of F, though this rests on the existence of the limit. Nonetheless, the ratio $\log(N_\delta(F))/\log(1/\delta)$ is the key to defining the *lower and upper Minkowski dimensions of* F.

Definition 3.1. Let $F \subseteq \mathbb{R}^d$ be a bounded subset. We define the *lower and upper Minkowski dimensions of* F to be

$$\underline{\dim}_M(F) := \liminf_{\delta\to 0} \frac{\log(N_\delta(F))}{\log(1/\delta)} \quad \text{and} \quad \overline{\dim}_M(F) := \limsup_{\delta\to 0} \frac{\log(N_\delta(F))}{\log(1/\delta)},$$

respectively. If the lower and upper Minkowski dimensions coincide, then we refer to their common value as the *Minkowski dimension of* F, denoted $\dim_M(F)$.

By definition, it follows that $\underline{\dim}_M(F) \leq \overline{\dim}_M(F)$. Note that $\underline{\dim}_M(F)$ and $\overline{\dim}_M(F)$ are also referred to as the lower and upper *box* or *box-counting dimensions* based on our definition of $N_\delta(F)$. It is a well-known fact that $N_\delta(F)$ could alternatively be defined as

- the least number of δ-balls covering F,
- or the least number of balls with diameter δ covering F.

While the value of $N_\delta(F)$ will differ based on its definition, their differences are actually negligible when we take limits. In some of the later results about lower and upper Minkowski dimension, it may be useful to employ one of these alternate definitions.

Unfortunately, there are a few downsides that come with using Minkowski dimension as quantification of a set's size or complexity. First, the possibility that $\underline{\dim}_M(F) < \overline{\dim}_M(F)$ prevents us from always assigning a single number as the dimension of a set. Another issue is that the lower and upper Minkowski dimensions of a bounded set F coincide exactly with those of its closure (see Proposition 3.4 of [2]). This implies that the set of rationals on the closed interval $[0, 1]$ has the same Minkowski dimensions as $[0, 1]$, which seems to defy the fact that the former is countable while the latter is not. These issues are resolved with Hausdorff's notion of dimension, which we motivate as follows.

Suppose once again that we are in $\mathbb{R}^3$ and $C \subseteq \mathbb{R}^3$ is a cube with side length ℓ. Any of its square faces has area ℓ^2, and its overall volume is ℓ^3. Length, area, and volume are all quantities for describing content, and we compute them by raising ℓ to some exponent. The key insight here is that depending on the set we wish to study, there is a special exponent on length that gives a meaningful description of content. Roughly speaking, it does not make much sense to compute the "length" of a square as we might for a line segment or curve. We can try by covering the square with (disjoint) line segments and summing up their total lengths, but this sum always amounts to ∞. Likewise, it does not make much sense to compute the "volume" of a square; we can also try approximating the square's "volume" by covering the square with cubes and totaling their volumes, but we can make this sum arbitrarily small by shrinking the constituent cubes' side lengths. Hence, the "volume" of a square must be 0. We see that the exponents 1 and 3 on length—associated with computing lengths and volumes, respectively—are either too low or too high to quantify a square's content. It turns out that the most appropriate exponent on length is 2, which corresponds to computing the square's surface area (finite and nonzero). Thus, we might define the dimension of an arbitrary set in $\mathbb{R}^d$ to be its "best exponent on length."

Definition 3.2. Let $F \subseteq \mathbb{R}^d$. For $s \geq 0$ and $\delta > 0$, define

$$\mathcal{H}^s_\delta(F) := \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s : F \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } \operatorname{diam}(U_i) < \delta \text{ for all } i \geq 1\right\}.$$

We define the *s-dimensional Hausdorff measure of* F to be the limit

$$\mathcal{H}^s(F) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(F).$$

One can prove that $\mathcal{H}^s$ indeed defines a measure (in fact, on Borel subsets of $\mathbb{R}^d$). We should think of $\mathcal{H}^s(F)$ as measuring the s-dimensional content of F, and in particular, $\mathcal{H}^s$ for $s = 1$, 2, and 3 as generalizations of length, area, and volume, respectively.

A key feature of Hausdorff measures is that if $\mathcal{H}^s(F) < \infty$ for some $s \geq 0$, then

$$\mathcal{H}^t(F) = \begin{cases} \infty & 0 \leq t < s, \\ 0 & s < t. \end{cases}$$

We define the *Hausdorff dimension of* F to be the critical $s \geq 0$ where the 0-∞ jump occurs, namely

$$\dim_H(F) := \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\} = \inf\{t \geq 0 : \mathcal{H}^t(F) = 0\}.$$

The critical dimension $s = \dim_H(F)$ is in some sense the "best" because $\mathcal{H}^s(F)$ could be finite and nonzero. However, note that $\mathcal{H}^s(F)$ could still be 0 or ∞.

In most cases, it is rather difficult to prove the Hausdorff and Minkowski dimensions of a set precisely from their definitions. A better strategy is to find suitable lower and upper bounds for the dimensions. Note that the two notions of dimension do not always coincide; for instance, one can show that the set of points $\{1/k\}_{k=1}^{\infty}$ on the x-axis has Hausdorff dimension 0 and lower Minkowski dimension 1/2. It is true in general that a bounded set's Hausdorff dimension is at most its lower Minkowski dimension.

Theorem 3.3. *If $F \subseteq \mathbb{R}^d$ is a bounded subset, then* $\dim_H(F) \leq \underline{\dim}_M(F)$.

Proof. It suffices to show that for any $s \geq 0$ where $\underline{\dim}_M(F) < s$, we must have $\dim_H(F) \leq s$, or equivalently $\mathcal{H}^s(F) = 0$. First, let $\varepsilon > 0$ such that $\underline{\dim}_M(F) + \varepsilon < s$. For each $k \in \mathbb{N}$, let $\delta_k < 1/k$ so that

$$\frac{\log(N_{\delta_k}(F))}{\log(1/\delta_k)} \leq \underline{\dim}_M(F) < s - \varepsilon.$$

It follows then that $N_{\delta_k}(F) < \delta_k^{\varepsilon - s}$ and $N_{\delta_k}(F)\delta_k^s < \delta_k^{\varepsilon}$. Now let $B_1, \dots, B_n$ be a collection of $n = N_{\delta_k}(F)$ balls with diameter δ_k covering F. Observe that

$$\begin{aligned} \mathcal{H}^s_{\delta_k}(F) &\leq \sum_{i=1}^{n} \operatorname{diam}(B_i)^s \\ &= N_{\delta_k}(F) \cdot \delta_k^s \\ &< \delta_k^{\varepsilon}, \end{aligned}$$

where taking $k \to \infty$ gives us $\mathcal{H}^s(F) = 0$ as desired. □

We may also examine how a set's Hausdorff and Minkowski dimensions change under various transformations, particularly those which are Hölder continuous. We say a function $f : \mathbb{R}^d \to \mathbb{R}^n$ is *Hölder continuous* on a subset $F \subseteq \mathbb{R}^d$ if there exist constants $\lambda > 0$ and $C > 0$ such that

$$\|f(x) - f(y)\| \leq C\|x - y\|^{\lambda}$$

for all $x, y \in F$. For simplicity, we say that f is λ-Hölder (with constant C). Hölder continuity generalizes Lipschitz continuity ($\lambda = 1$) and can also be understood as a regularity condition for functions. We have the following bounds on the Hausdorff and upper Minkowski dimensions of images of Hölder continuous functions.

Proposition 3.4. *If $f : \mathbb{R}^d \to \mathbb{R}^n$ is λ-Hölder with constant C on $F \subseteq \mathbb{R}^d$, then*

$$\mathcal{H}^{s/\lambda}(f(F)) \leq C^{s/\lambda}\mathcal{H}^s(F)$$

for any $s > 0$. It follows that

$$\dim_H(f(F)) \leq \frac{1}{\lambda}\dim_H(F). \tag{3.5}$$

For proofs of the two inequalities, see Propositions 2.2 and 2.3 in [2].

Proposition 3.6. *If $f : \mathbb{R}^d \to \mathbb{R}^n$ is λ-Hölder on a compact subset $K \subseteq \mathbb{R}^d$, then*

$$\overline{\dim}_M(f(K)) \leq \frac{1}{\lambda}\overline{\dim}_M(K).$$

Proof. For any $k \in \mathbb{N}$, let $0 < \delta_k < 1/k$ such that

$$\sup_{\delta < 1/(k+1)} \frac{\log(N_\delta(K))}{\log(1/\delta)} \leq \frac{\log(N_{\delta_k}(K))}{\log(1/\delta_k)} \leq \sup_{\delta < 1/k} \frac{\log(N_\delta(K))}{\log(1/\delta)},$$

which guarantees

$$\overline{\dim}_M(K) = \lim_{k \to \infty} \frac{\log(N_{\delta_k}(K))}{\log(1/\delta_k)}.$$

Let $U_1, \dots, U_n$ be a collection of $n = N_{\delta_k}(K)$ sets with diameter at most δ_k that cover K. The images $f(U_1), \dots, f(U_n)$ cover $f(K)$, and if f is λ-Hölder continuous, then $\operatorname{diam}(f(U_i)) \leq C \operatorname{diam}(U_i)^{\lambda} \leq C\delta_k^{\lambda}$ for all $1 \leq i \leq n$. We have n sets of diameter at most $C\delta_k^{\lambda}$ covering $f(K)$, but by the definition of $N_{C\delta_{\lambda}^{k}}(f(K))$, we must have $N_{C\delta_\lambda^k}(f(K)) \leq n$. Now observe that

$$\begin{aligned}\frac{\dfrac{\log(N_{C\delta_k^\lambda}(f(K)))}{-\log(C\delta_k^\lambda)}}{\dfrac{\log(N_{\delta_k}(K))}{-\log(\delta_k)}} &= \frac{\log(N_{C\delta_k^\lambda}(f(K)))}{\log(N_{\delta_k}(K))} \cdot \frac{\log(\delta_k)}{\log(C) + \lambda \log(\delta_k)} \\ &\leq \frac{\log \delta_k}{\log(C) + \lambda \log(\delta_k)} \\ &\leq \frac{1}{\lambda} \cdot \frac{\log \delta_k}{\log(C^{1/\lambda}) + \log \delta_k}.\end{aligned}$$

Using the fact that $\limsup_{n\to\infty} a_n b_n = b \limsup_{n\to\infty} a_n$ if (a_n) and (b_n) are bounded sequences and (b_n) converges to b, we see that taking the supremum limit as $k \to \infty$ of the inequality above shows $\overline{\dim}_M(f(K)) \leq \dfrac{1}{\lambda}\overline{\dim}_M(K)$. □

We will rely on the results above in conjunction with the following to show that Brownian paths have Hausdorff and Minkowski dimensions equal to 2 almost surely.

Theorem 3.7. *For any $0 < \lambda < \frac{1}{2}$, there exist constants $C > 0$ and $R > 0$ such that*

$$\|B(t) - B(s)\| < C|t - s|^\lambda$$

almost surely for any $t, s \in [0, 1]$ where $|t - s| < H$.

Despite the seemingly irregular shape of Brownian paths, Theorem 3.7 suggests that Brownian paths are actually well-behaved with high probability. This result is proven as Proposition 16.1 in [2] and draws on Lévy's construction of Brownian motion on $[0, 1]$.

3.2. **Frostman's Lemma.** It is usually more difficult to prove a lower bound than an upper bound for the Hausdorff dimension of some set $F \subseteq \mathbb{R}^d$; for the latter, it suffices to find for each $\delta > 0$ a δ-cover $U_\delta^{(1)}, U_\delta^{(2)}, \dots$ of F that closely bounds $\mathcal{H}_\delta^s(F)$ from above and to compute the limit of $\sum_{i=1}^\infty \operatorname{diam}(U_\delta^i)^s$ as $\delta \to 0$. In contrast, bounding the infimum in the definition of $\mathcal{H}_\delta^s(F)$ from below before taking the $\delta \to 0$ limit is rather tricky. Fortunately, Frostman's Lemma reduces our task of computing lower bounds on $\dim_H(F)$ to finding a *mass distribution μ on F*—a finite measure on $\mathbb{R}^d$ with support F. This approach provides a surprising application of potential theory (a subarea of mathematical physics) to the study of fractals. To state the lemma, we introduce the following generalizations of potential and energy from physics.

Definitions 3.8. Let $F \subseteq \mathbb{R}^d$ be a bounded subset and μ a mass distribution F. For $s \geq 0$, we define the *s-potential* of μ at a point $x \in \mathbb{R}^d$ to be

$$\phi_s(x) := \int_{\mathbb{R}^d} \frac{d\mu(y)}{\|x - y\|^s}$$

and the *s-energy of* μ to be

$$I_s(\mu) := \int_{\mathbb{R}^d} \phi_s(x) \mathrm{d}\mu(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{d\mu(y) d\mu(x)}{\|x - y\|^s}.$$

For concreteness, F could be a solid rod in $\mathbb{R}^3$ and μ a distribution of the charges along F. Each point $x \in \mathbb{R}^3$ experiences a potential due to the charged rod, and summing up the potentials at each point of the rod gives the rod's total energy (sorry, but I'm no physicist).

Lemma 3.9 (Frostman 1935)**.** *Let $F \subseteq \mathbb{R}^d$ be a bounded subset and $s \geq 0$. If there exists a mass distribution μ on F such that $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and therefore $s \leq \dim_H(F)$.*

Remark 3.10. There is actually a partial converse, but we will not require it in this paper.

A full proof of the lemma is given in [2], though the main implication of finding μ such that $I_s(\mu) < \infty$ is that the "mass" of

$$F_1 = \left\{ x \in F : \limsup_{\delta \to 0} \frac{\mu(B_\delta(x))}{\delta^s} > 0 \right\}$$

is negligible i.e. $\mu(F_1) = 0$. Roughly speaking, F_1 consists of points in F where mass is "s-dimensionally dense." If $\mu(F_1) = 0$, then μ-almost every point does not carry too much mass. For this to be true, we need F to be "large enough" so that mass can be spread out. As such, we arrive at a lower bound for $\dim_H(F)$.

Example 3.11. As a simple demonstration of Frostman's Lemma, let F be the unit segment $[0,1] \times \{0\}$ in $\mathbb{R}^2$. We will show that the Hausdorff dimension of F is at least one—as one might expect—by showing that $\dim_H(F) \geq s$ for any $s < 1$.

For every $s < 1$, it suffices to consider a uniform mass distribution on F, namely

$$\mu(A) := \mathcal{L}\left(A \cap F\right)$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}$ and A is a subset of $\mathbb{R}^2$ where $A \cap F$ is Lebesgue measureable (when viewed as a subset of $\mathbb{R}$). Then

$$I_s(\mu) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{\|x-y\|^s} = \int_0^1 \int_0^1 \frac{\mathrm{d}t\mathrm{d}u}{|u-t|^s},$$

which is finite if $s < 1$ (note that this is not true if $s = 1$).

4. Dimension of Brownian Paths

We now present the main result of this paper, which was proven in 1953 by Samuel J. Taylor, a student of Abram S. Besicovitch. Our proof will draw inspiration from the versions presented in [1] and [2].

Theorem 4.1. *Brownian paths in $\mathbb{R}^d$ for $d \geq 2$ have Hausdorff and Minkowski dimension 2 almost surely.*

Proof. We will establish the almost surely Hausdorff and Minkowski dimensions of Brownian paths simultaneously by leveraging Theorem 3.3 and the definitions of lower and upper Minkowski dimension, which imply

$$\dim_H B\left([0,1]\right) \leq \underline{\dim}_M B([0,1]) \leq \overline{\dim}_M B([0,1])$$

for any Brownian path $B([0,1])$. It suffices for us to show that $2 \leq \dim_H B([0,1])$ and $\overline{\dim}_M B([0,1]) \leq 2$ almost surely.

For the latter inequality, applying Proposition 3.6 to the fact that Brownian paths are λ-Hölder for all $0 < \lambda < 1/2$ almost surely shows that

$$\overline{\dim}_M B([0,1]) \leq \frac{1}{\lambda}\overline{\dim}_M([0,1]) = \frac{1}{\lambda} \leq 2$$

almost surely.

To show that $\dim_H B([0,1]) \geq 2$ almost surely, it suffices by Frostman's Lemma for us to define for each Brownian path $B([0,1])$ a mass distribution μ_B such that $I_s(\mu_B) < \infty$ for all $s < 2$ almost surely. Showing that $I_s(\mu_B) < \infty$ almost surely is equivalent to showing

$$\mathbb{P}\left\{I_s(\mu_B) = \infty\right\} = \lim_{N\to\infty} \mathbb{P}\{I_s(\mu_B) > N\} = 0.$$

Because s-potentials are nonnegative, Markov's inequality tells us

$$\mathbb{P}\left\{I_s(\mu_B) > N\right\} \leq \frac{\mathbb{E}(I_s(\mu_B))}{N}.$$

Thus, if we show that $\mathbb{E}(I_s(\mu))$ is bounded, then taking $N \to \infty$ gives $\mathbb{P}\{I_s(\mu_B) = \infty\} = 0$ and therefore $\mathbb{P}\{I_s(\mu_B) < \infty\} = 1$.

The mass distribution μ_B that we should consider for each path $B([0,1])$ is defined by

$$\mu_B(A) := \mathcal{L}\left(B^{-1}(A)\right)$$

for all $A \subseteq \mathbb{R}^d$, where $B^{-1}(A) \subseteq [0,1]$ is the preimage of A under a Brownian random function B and $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$. In concrete terms, $\mu_B(A)$ represents the amount of time that the Brownian path B spends inside A. One can verify that μ_B is indeed a mass distribution on $B([0,1])$. It follows from our definition of μ_B that

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_B(x) = \int_0^1 f(B(t))\mathrm{d}t$$

for all $f : [0,1] \to \mathbb{R}^d$, which means

$$I_s(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu_B(x)\mathrm{d}\mu_B(y)}{\|x-y\|^s} = \int_0^1\int_0^1 \frac{\mathrm{d}t\mathrm{d}u}{\|B(t)-B(u)\|^s}.$$

Taking the expectation of $I_s(\mu)$ gives us

$$\mathbb{E}\left(I_s(\mu)\right) = \mathbb{E}\int_0^1\int_0^1 \frac{\mathrm{d}t\mathrm{d}u}{\|B(t)-B(u)\|^s} = \int_0^1\int_0^1 \mathbb{E}\left(\|B(t)-B(u)\|^{-s}\right)\mathrm{d}t\mathrm{d}u.$$

Using the cumulative density $P(\rho)$ of $\|B(t) - B(u)\|$ from (2.2), we have

$$\begin{aligned}\mathbb{E}\left(\|B(t)-B(u)\|^{-s}\right) &= \int_0^\infty \rho^{-s}\mathrm{d}P(\rho) \\ &= \int_0^\infty \rho^{-s}\left(\widetilde{C_d}|t-u|^{-d/2}\exp\left(-\frac{\rho^2}{2|t-u|}\right)\right)\mathrm{d}\rho.\end{aligned}$$

A change of variables with $v = \dfrac{\rho^2}{|t-u|}$ gives

$$\begin{aligned}\mathbb{E}\left(\|B(t)-B(u)\|^{-s}\right) &= \frac{\widetilde{C_d}|t-u|^{-s/2}}{2}\int_0^\infty v^{\frac{d-s-2}{2}}\exp\left(-\frac{v}{2}\right)\mathrm{d}v \\ &= K|t-u|^{-s/2},\end{aligned}$$

where K is a constant that does not depend on $|t-u|$. We see that

$$\mathbb{E}\left(I_s(\mu)\right) = \int_0^1 \int_0^1 K|t-u|^{-s/2} \mathrm{d}t\mathrm{d}u,$$

which is finite if $s < 2$. Thus, $I_s(\mu) < \infty$ almost surely for all $s < 2$ as desired. □

5. Additional Results

Our main result concerns the almost surely Hausdorff and Minkowski dimensions of Brownian *paths*, but a similar result holds for *graphs* of Brownian paths in $\mathbb{R}$.

Theorem 5.1. *Graphs of Brownian paths in* $\mathbb{R}$ *have Hausdorff and Minkowski dimension* $3/2$ *almost surely.*

The overall strategy of bounding the Hausdorff dimension from below with Frostman's Lemma and the upper Minkowski dimension from above with the almost surely Hölder continuity property of Brownian paths still applies. In particular, the following is true for graphs of real-valued Hölder continuous functions.

Lemma 5.2. *If* $f : I \to \mathbb{R}$ *is* λ*-Hölder on an interval* $I \subseteq \mathbb{R}$*, then*

$$\overline{\dim}_M(\Gamma(f)) \leq 2 - \lambda.$$

This result is proven as Lemma 5.1.2 in [1]. Since Brownian random functions $B : [0,1] \to \mathbb{R}$ are λ-Hölder almost surely for all $0 < \lambda < 1/2$, it follows that $\overline{\dim}_M(\Gamma(B)) \leq 3/2$ almost surely. When proving the lower bound on $\dim_H(\Gamma(B))$, the mass distribution on $\Gamma(B)$ that we should consider is defined by

$$\mu_B(A) := \mathcal{L}\left\{t \in [0,1] : (t, B(t)) \in A\right\}$$

for subsets $A \subseteq \mathbb{R}^d$ where the set $\{t \in [0,1] : (t, B(t)) \in A\}$ is Lebesgue measureable. Similar to the mass distribution we defined in Theorem 4.1, $\mu_B(A)$ here represents the total amount of time that the curve $(t, B(t))$ spends inside A. It remains then to show that $\mathbb{E}I_s(\mu_B) < \infty$ for all $s < 3/2$ to arrive at $I_s(\mu_B) < \infty$ almost surely.

For most of this paper, our primary focus has been on defining notions of dimension and computing them for Brownian paths. We conclude with the fact that 2-dimensional Hausdorff measure of a Brownian path in $\mathbb{R}^d$ for $d \geq 2$ is 0 almost surely [1]. In rough terms, this means that even at the critical Hausdorff dimension, most Brownian paths cover zero "area."

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