# An Introduction to Minimal Surfaces in $\mathbb{R}^3$

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# Outline

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4. Examples of minimal surfaces
5. Remarks

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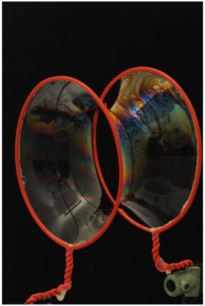
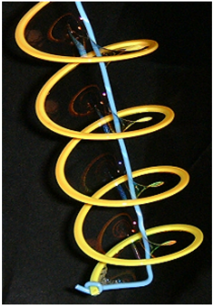
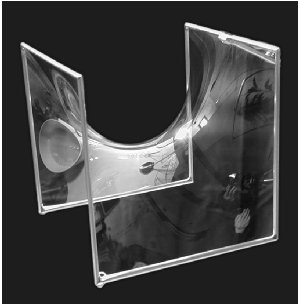

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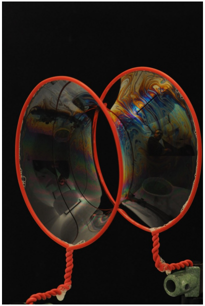

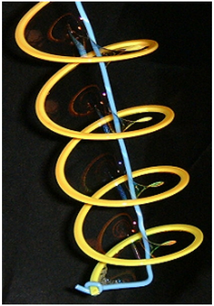

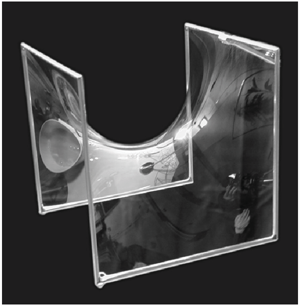

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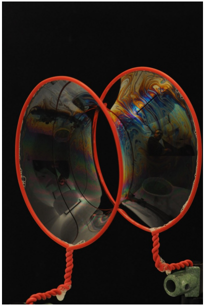
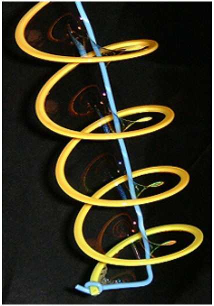
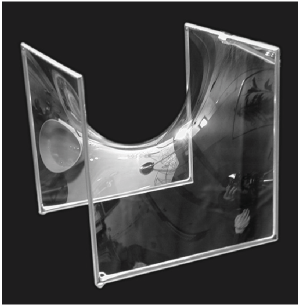

- Think of them as solutions to optimization problems

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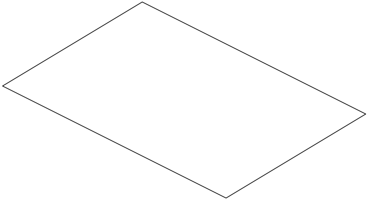
**Examples:**

- Planes



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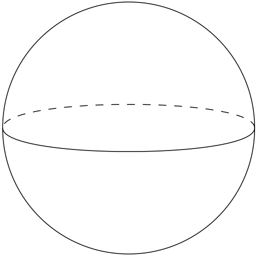
- Spheres



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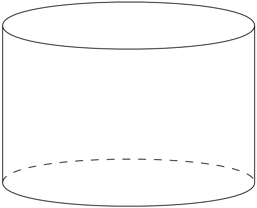
- Spheres
- Cylinders



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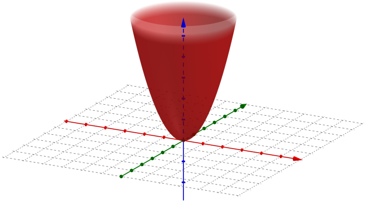
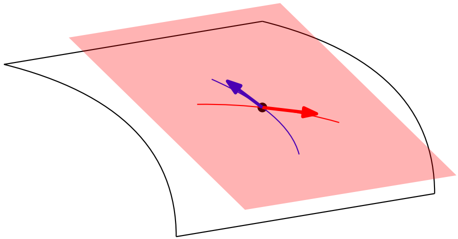
- Graphs of smooth $f : \mathbb{R}^2 \to \mathbb{R}$



# Tangent Space

The *tangent space* $T_pS$ is the collection of velocity vectors of curves passing through $p \in S$.



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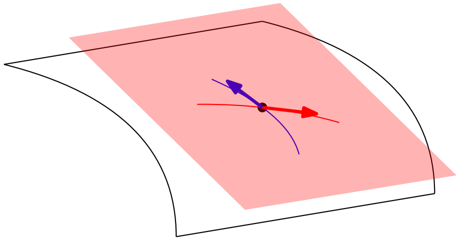

Given local parametrization $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ and $\mathbf{x}(0, 0) = p$, consider the *coordinate curves*:

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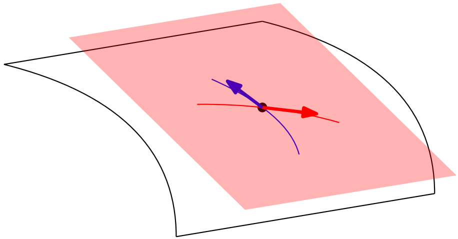

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Let $\mathbf{x}_u$ and $\mathbf{x}_v$ be the respective velocity vectors at $p$.

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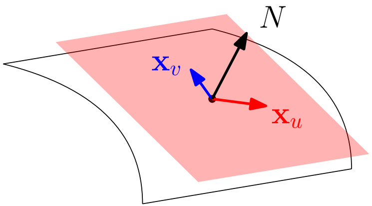

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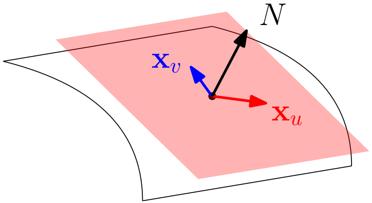
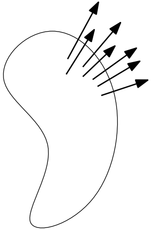
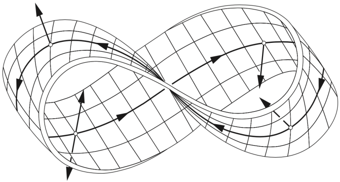


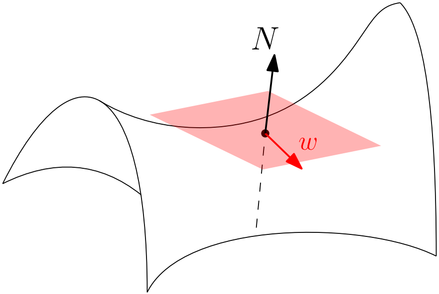

- We will assume our surfaces have a 'consistent' normal direction.





**Normal Curvature**

- Take $w \in T_pS$ with $\|w\| = 1$.

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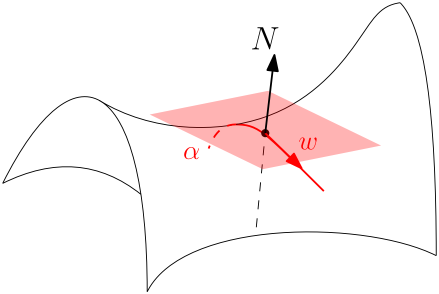


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- Take $w \in T_pS$ with $\|w\| = 1$.
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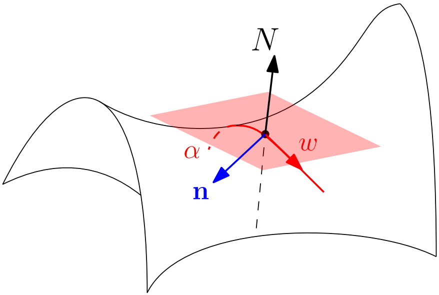
$$\mathbf{n} = \frac{\alpha''(0)}{\|\alpha''(0)\|}.$$

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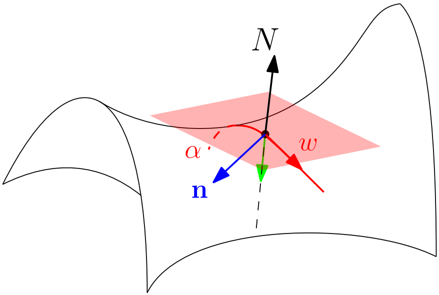


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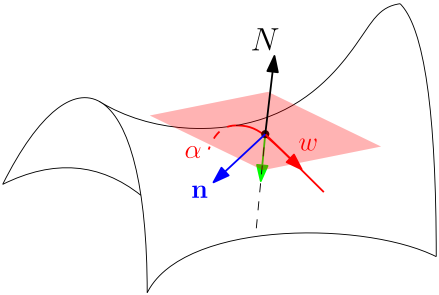



**Interpretation**

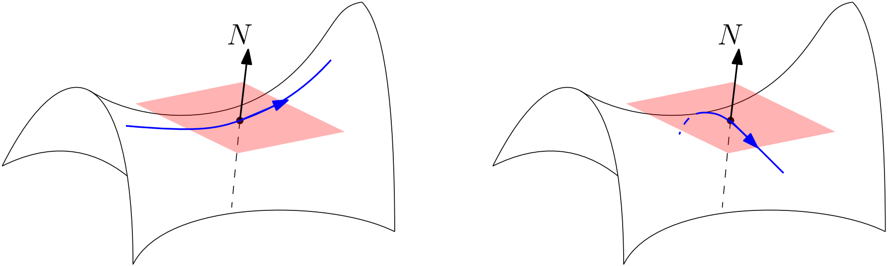
- High $(+)$ normal curvature $\implies$ curve 'bends' or 'accelerates' towards $N$

# Principle Curvatures

- Define the *principal curvatures*

$$\kappa_1 := \max_{\|w\|=1} (\text{normal curvature w.r.t. } w),$$
$$\kappa_2 := \min_{\|w\|=1} (\text{normal curvature w.r.t. } w).$$

- *Principal directions* either maximize or minimize normal curvature



# Mean Curvature and Minimal Surfaces

Define *mean curvature* $H$

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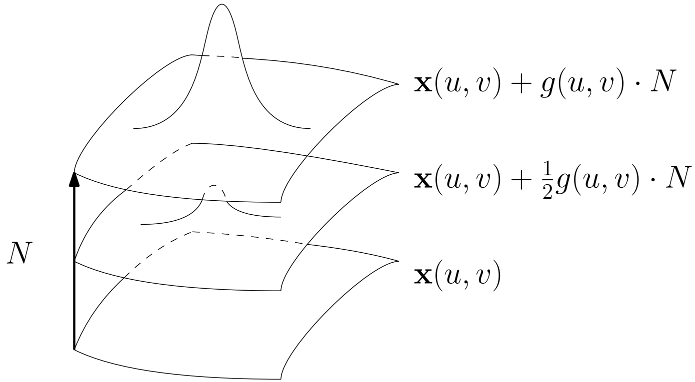
- i.e. $\kappa_1 = -\kappa_2$
- But what is minimized with 'minimal surfaces?'

# Variational Characterization of Minimal Surfaces

Assume $S$ is parametrixed by $\mathbf{x}(u, v)$ with domain $\mathbb{R}^2$.

1. Take some closed disk $D \subset \mathbb{R}^2$ and perturbation $g : D \to \mathbb{R}$
2. "Stretch" $S$ along its normal direction by $tg$ where $t \in \mathbb{R}$ small

$$\mathbf{x}^t(u, v) = \mathbf{x}(u, v) + tg(u, v) \cdot N$$



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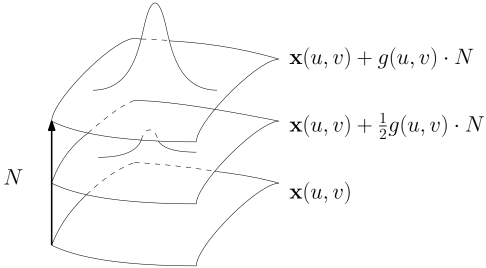


3. Compute area of stretched $S$ by factor $t$

$$A(t) = \int_D \|\mathbf{x}_u^t \times \mathbf{x}_u^t\| \, du \, dv$$

**Equivalent Formula of 'Minimal'**:

We say $S$ is *minimal* if $A'(0) = 0$ for any disk $D$ and perturbation $g$.

# Equivalence of Minimal Surface Definitions

**Theorem**

$H = 0$ everywhere $\iff A'(0) = 0$ for any $D$ and $h : \bar{D} \to \mathbb{R}$

*Sketch of Proof.*

- Some computation yields

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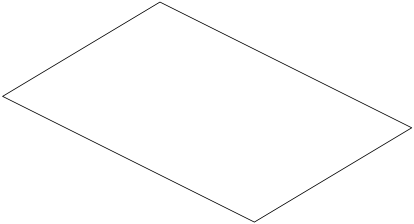
- Converse using contradiction:
    - If $H \neq 0$ at $p \in S$, then pick $g$ so that $gH > 0$.
    - Then $A'(0) < 0$ on a certain disk.

# Some Facts About Minimal Surfaces

**Quick Facts:**

- Planes are minimal.



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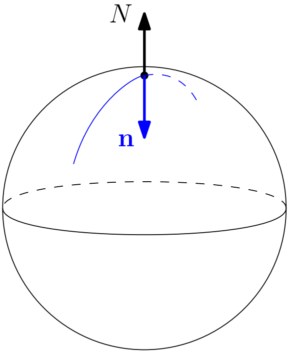

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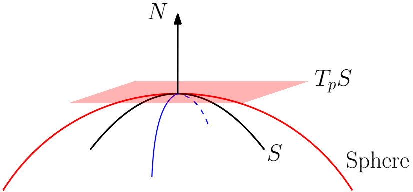
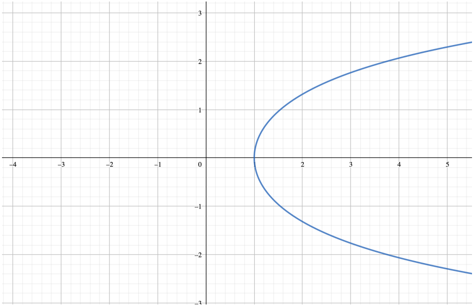
- Compact surfaces are not minimal.

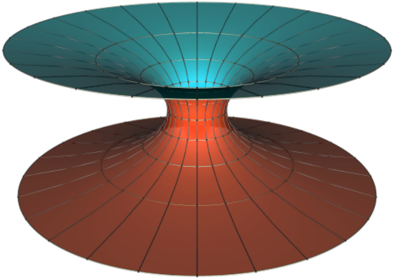


# Example: Catenoid

Parametrization given by:

$$\mathbf{x}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$



The catenoid is the only minimal surface of revolution (aside from plane)

# Example: Helicoid

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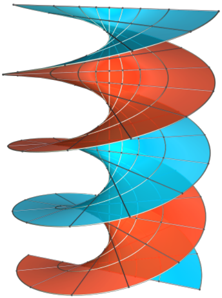

# Example: Enneper's Surface

Parametrization given by:

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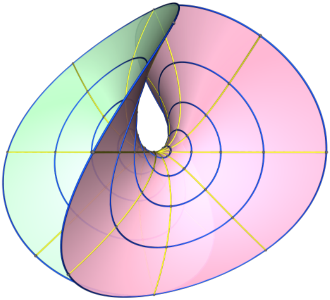

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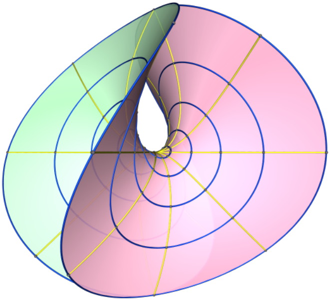


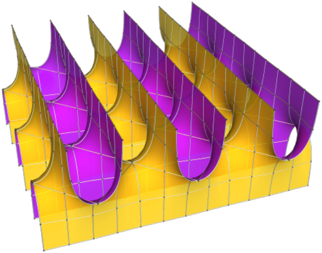
- Known for having self-intersections.
- Invariant after $\pi/2$ rotation about $z$-axis, followed by reflection over $xy$-plane

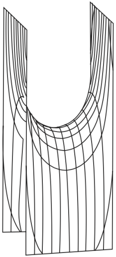
# Example: Doubly-Scherk Surface

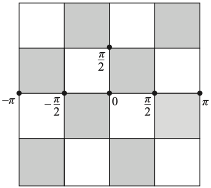
Parametrization given by:

$$\mathbf{x}(u,v) = \left( \arg\left(\frac{\zeta+i}{\zeta-i}\right), \arg\left(\frac{\zeta+1}{\zeta-1}\right), \log\left|\frac{\zeta^2+1}{\zeta^2-1}\right| \right)$$

where $\zeta = u + iv$ and $\arg\zeta$ is the angle from the real axis to $\zeta$.








Known for being periodic.

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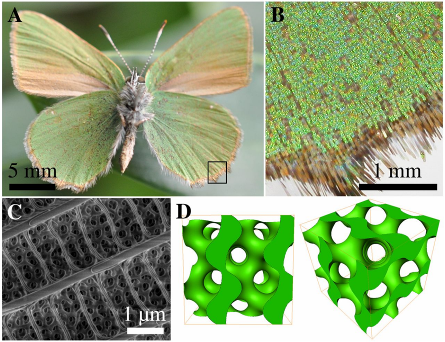
  - e.g. Butterfly wing colors



# THANK YOU!