## Technical Note No. 30\* Options, Futures, and Other Derivatives, Ninth Edition John Hull

## The Return for a Security Dependent on Multiple sources of Uncertainty

In this Note we prove a result relating the excess return to market prices of risk when there are multiple sources of uncertainty.

Suppose there are n stochastic variables following Wiener processes. Consider n+1 traded securities whose prices depend on some or all of the n stochastic variables. Define  $f_j$  as the price of the jth security  $(1 \le j \le n+1)$ . We assume that no dividends or other income is paid by the n+1 traded securities. It follows from Ito's lemma in Technical Note 29 that the securities follow processes of the form

$$df_j = \mu_j f_j dt + \sum_{i=1}^n \sigma_{ij} f_j dz_i \tag{1}$$

Since there are n+1 traded securities and n Wiener processes, it is possible to form an instantaneously riskless portfolio,  $\Pi$ , using the securities. Define  $k_j$  as the amount of the jth security in the portfolio, so that

$$\Pi = \sum_{j=1}^{n+1} k_j f_j \tag{2}$$

The  $k_j$  must be chosen so that the stochastic components of the returns from the securities are eliminated. From equation (1) this means that

$$\sum_{j=1}^{n+1} k_j \sigma_{ij} f_j = 0 \tag{3}$$

for  $1 \le i \le n$ . The return from the portfolio is then given by

$$d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j dt$$

The cost of setting up the portfolio is

$$\sum_{j=1}^{n+1} k_j f_j$$

<sup>\* ©</sup>Copyright John Hull. All Rights Reserved. This note may be reproduced for use in conjunction with Options, Futures, and Other Derivatives by John C. Hull.

<sup>&</sup>lt;sup>1</sup> This is not restrictive. A non-dividend-paying security can always be obtained from a dividend-paying security by reinvesting the dividends in the security.

If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

$$\sum_{j=1}^{n+1} k_j \mu_j f_j = r \sum_{j=1}^{n+1} k_j f_j \tag{4}$$

or

$$\sum_{j=1}^{n+1} k_j f_j(\mu_j - r) = 0 \tag{5}$$

Equations (3) and (5) can be regarded as n+1 homogeneous linear equations in the  $k_j$ 's. The  $k_j$ 's are not all zero. From a well-known theorem in linear algebra, equations (3) and (5) can be consistent only if for all j

$$f_j(\mu_j - r) = \sum_{i=1}^n \lambda_i \sigma_{ij} f_j \tag{6}$$

or

$$\mu_j - r = \sum_{i=1}^n \lambda_i \sigma_{ij} \tag{7}$$

for some  $\lambda_i$   $(1 \le i \le n)$  that are dependent only on the state variables and time. Dropping the j subscript, this show that for any security f dependent on the n stochastic variables

$$df = \mu f \, dt + \sum_{i=1}^{n} \sigma_i f dz_i$$

where

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

This proves the required result.