

Technical Note No. 31*
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Properties of Ho–Lee and Hull–White Interest Rate Models

This note presents some of the math underlying the Ho–Lee and Hull–White one-factor models of the term structure. It follows the approach in Hull and White (1993).¹

In a one-factor term structure, model the process for a zero-coupon bond price in the traditional risk-neutral world must have a return equal to the short rate r . Suppose that $v(t, T)$ is the volatility. Then:

$$dP(t, T) = rP(t, T)dt + v(t, T)P(t, T)dz \quad (1)$$

In this note, we will assume that v is a function only of t and T . Because the bond's price volatility declines to zero at maturity $v(t, t) = 0$.

From Ito's lemma, for any times T_1 and T_2 with $T_2 > T_1$

$$d \ln P(t, T_1) = \left[r - \frac{v(t, T_1)^2}{2} \right] dt + v(t, T_1)dz(t) \quad (2)$$

$$d \ln P(t, T_2) = \left[r - \frac{v(t, T_2)^2}{2} \right] dt + v(t, T_2)dz(t) \quad (3)$$

Define $f(t, T_1, T_2)$ as the forward rate for the period between time T_1 and T_2 as seen at time t

$$f(t, T_1, T_2) = -\frac{\ln P(t, T_2) - \ln P(t, T_1)}{T_2 - T_1}$$

From equations (2) and (3)

$$df(t, T_1, T_2) = \left[\frac{v(t, T_2)^2 - v(t, T_1)^2}{2(T_2 - T_1)} \right] dt - \left[\frac{v(t, T_2) - v(t, T_1)}{T_2 - T_1} \right] dz(t)$$

Define $R(t, T)$ as the zero rate for the period between t and T .

$$R(t, T) = f(0, t, T) + \int_0^t df(\tau, t, T)$$

so that

$$R(t, T) = f(0, t, T) + \int_0^t \left[\frac{v(\tau, T)^2 - v(\tau, t)^2}{2(T - t)} \right] d\tau - \int_0^t \left[\frac{v(\tau, T) - v(\tau, t)}{T - t} \right] dz(\tau) \quad (4)$$

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¹ See J. Hull and A. White, "Bond Option Pricing Based on a Model for the Evolution of Bond Prices," *Advances in Futures and Options Research*, 6 (1993), 1–13.

As T approaches t , $R(t, T)$ becomes $r(t)$ and $f(0, t, T)$ becomes the instantaneous forward rate, $F(0, t)$ so that

$$r(t) = F(0, t) + \int_0^t \frac{\partial}{\partial t} \frac{v(\tau, t)^2}{2} d\tau - \int_0^t \frac{\partial}{\partial t} v(\tau, t) dz(\tau)$$

or

$$r(t) = F(0, t) + \int_0^t v(\tau, t) v_{tt}(\tau, t) d\tau - \int_0^t v_t(\tau, t) dz(\tau) \quad (5)$$

where subscripts denote partial derivatives. To calculate the process for r we differentiate with respect to t . Because $v(t, t) = 0$, this gives

$$dr = \left\{ F_t(0, t) + \int_0^t [v(\tau, t) v_{tt}(\tau, t) + v_t(\tau, t)^2] d\tau - \int_0^t v_{tt}(\tau, t) dz(\tau) \right\} dt - v_t(\tau, t)|_{\tau=t} dz(t) \quad (6)$$

Case 1: Ho-Lee; $v(t, T) = \sigma(T - t)$

In the case, equation (5) gives

$$r(t) = F(0, t) + \sigma^2 t^2 / 2 - \int_0^t \sigma dz(\tau) \quad (7)$$

and equation (6) gives

$$dr(t) = [F_t(0, t) + \sigma^2 t] dt + \sigma dz$$

This is the Ho-Lee model

$$dr = \theta(t) dt + \sigma dz$$

We have proved equation (31.11)

$$\theta(t) = F_t(0, t) + \sigma^2 t$$

Also from equation (4)

$$R(t, T) = f(0, t, T) + \sigma^2 tT / 2 - \int_0^t \sigma dz(\tau) \quad (8)$$

From equations (7) and (8)

$$R(t, T) = f(0, t, T) + \sigma^2 tT / 2 + r(t) - F(0, t) - \sigma^2 t^2 / 2 = f(0, t, T) - F(0, t) + \sigma^2 t(T - t) / 2 + r(t)$$

Because

$$\ln P(t, T) = -R(t, T)(T - t)$$

It follows that

$$\ln P(t, T) = -f(0, t, T)(T - t) + F(0, t)(T - t) - \sigma^2 t(T - t)^2 / 2 - r(t)(T - t)$$

The forward bond price $P(0, T)/P(0, t)$ equals $e^{-f(0, t, T)(T-t)}$ so that this becomes

$$\ln P(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)(T - t) - \sigma^2 t(T - t)^2/2 - r(t)(T - t)$$

This proves equation (31.12):

$$P(t, T) = A(t, T)e^{-r(t)(T-t)}$$

where

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)(T - t) - \frac{1}{2}\sigma^2 t(T - t)^2$$

Case 2: Hull–White; $v(t, T) = \sigma(1 - e^{-a(T-t)})/a$

In this case, equation (5) gives

$$r(t) = F(0, t) + \frac{\sigma^2}{a^2}(1 - e^{-at}) - \frac{\sigma^2}{2a^2}(1 - e^{-2at}) - \int_0^t \sigma e^{-a(t-\tau)} dz(\tau) \quad (9)$$

Equation (6) gives

$$dr(t) = \left\{ F_t(0, t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) + \int_0^t \sigma a e^{-a(t-\tau)} dz(\tau) \right\} dt - \sigma dz(t) \quad (10)$$

Substituting for

$$\int_0^t \sigma e^{-a(t-\tau)} dz(\tau)$$

from equation (9) into equation (10) we obtain

$$dr(t) = \left\{ F_t(0, t) + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) - ar(t) + aF(0, t) + \frac{\sigma^2}{a}(1 - e^{-at}) - \frac{\sigma^2}{2a}(1 - e^{-2at}) \right\} dt - \sigma dz(t)$$

or

$$dr(t) = \left\{ F_t(0, t) + aF(0, t) - ar(t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \right\} dt - \sigma dz(t)$$

This is the Hull–White model

$$dr(t) = (\theta(t) - ar) dt + \sigma dz$$

with

$$\theta(t) = F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

This proves equation (31.14). From equation (4)

$$R(t, T) = f(0, t, T) + \frac{\sigma^2[e^{-2a(T-t)} - e^{-2aT} - 1 + e^{-2at} - 4e^{-a(T-t)} + 4e^{-aT} + 4 - 4e^{-at}]}{4a^3(T-t)}$$

$$+ \frac{\sigma(e^{-aT} - e^{-at})}{a(T-t)} \int_0^t e^{a\tau} dz(\tau) \quad (11)$$

From equation (9)

$$\sigma \int_0^t e^{a\tau} dz(\tau) = -r(t)e^{at} + F(0,t)e^{at} + \frac{\sigma^2}{a^2}(e^{at} - 1) - \frac{\sigma^2}{2a^2}(e^{at} - e^{-at})$$

so that

$$R(t, T) = f(0, t, T) + \frac{\sigma^2[e^{-2a(T-t)} - e^{-2aT} - 1 + e^{-2at} - 4e^{-a(T-t)} + 4e^{-aT} + 4 - 4e^{-at}]}{4a^3(T-t)} \\ + \frac{(e^{-aT} - e^{-at})}{a(T-t)} \left[-r(t)e^{at} + F(0,t)e^{at} + \frac{\sigma^2}{a^2}(e^{at} - 1) - \frac{\sigma^2}{2a^2}(e^{at} - e^{-at}) \right]$$

Now

$$\ln P(t, T) = -R(t, T)(T-t)$$

and the forward bond price $P(0, T)/P(0, t)$ equals $e^{-f(0, t, T)(T-t)}$. After some tedious algebra we get

$$\ln P(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)B(t, T) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1) - r(t)B(t, T)$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

showing that

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

where

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + F(0, t)B(t, T) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1)$$

This proves equation (31.17).

Bond Options

Consider a European option with strike price K and maturity T on a zero-coupon bond where the maturity of the bond is s . The forward price of the bond underlying the option as seen at time t , $F_B(t, T, s)$, is

$$F_B(t, T, s) = \frac{P(t, s)}{P(t, T)}$$

Using the results in equations (2) and (3) we get

$$d \ln F_B(t, T, s) = \frac{v(t, T)^2 - v(t, s)^2}{2} dt + [v(t, s) - v(t, T)] dz$$

This shows that the $P(T, s) = f_B(T, T, s)$ is lognormal when $v(t, T)$ is function only of t and T . The variance $\ln P(T, s)$ is then

$$\sigma_P^2 = \int_0^T [v(t, s) - v(t, T)]^2 dt$$

In the case of Ho-Lee $v(t, T) = \sigma(T - t)$ and $\sigma_P^2 = \sigma^2(s - T)^2 T$. In Hull-White $v(t, T) = \sigma B(t, T)$ so that

$$\sigma_P^2 = \sigma^2 \int_0^T [B(t, s) - B(t, T)]^2 dt = \frac{\sigma^2}{2a^3} [1 - e^{-a(s-T)}]^2 (1 - e^{-2aT})$$

In both cases bond options can be valued using Black's model. The average variance rate of the forward bond price is σ_P^2/T . This leads to the results in Section 31.4.