Technical Note No. 29*
Options, Futures, and Other Derivatives, Ninth Edition
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Proof of Extensions to Ito's Lemma

Options, Futures and Other Derivatives proves Ito's lemma for a function of a single stochastic variable. Here we present a generalized version of Ito's lemma for the situation where there are several sources of uncertainty.

Suppose that a function, f, depends on the n variables $x_1, x_2, \ldots, x_n$ and time, t. Suppose further that x_i follows an Ito process with instantaneous drift a_i and instantaneous variance b_i^2 $(1 \leq i \leq n)$, that is,

$$dx_i = a_i\,dt + b_i\,dz_i \tag{1}$$

where dz_i is a Wiener process $(1 \leq i \leq n)$. Each a_i and b_i may be any function of all the x_i's and t. A Taylor series expansion of Δf gives

$$\Delta f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial x_j}\Delta x_i\,\Delta x_j + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial t}\Delta x_i \Delta t + \cdots \tag{2}$$

Equation (1) can be discretized as

$$\Delta x_i = a_i\,\Delta t + b_i \epsilon_i \sqrt{\Delta t}$$

where ϵ_i is a random sample from a standardized normal distribution. The correlation, ρ_{ij}, between dz_i and dz_j is defined as the correlation between ϵ_i and ϵ_j. In the book's proof of Ito's lemma when there is only one stochastic variable it was argued that

$$\lim_{\Delta t \to 0} \Delta x_i^2 = b_i^2\,dt$$

Similarly,

$$\lim_{\Delta t \to 0} \Delta x_i\,\Delta x_j = b_i b_j \rho_{ij}\,dt$$

As $\Delta t \to 0$, the first three terms in the expansion of Δf in equation (2) are of order Δt. All other terms are of higher order. Hence

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial x_j} b_i b_j \rho_{ij}\,dt$$

This is the generalized version of Ito's lemma. Substituting for dx_i from equation (1) gives

$$df = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i\,\partial x_j} b_i b_j \rho_{ij} \right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} b_i\,dz_i \tag{3}$$

For an alternative generalization of Ito's lemma suppose that f depends on a single variable x and that the process for x involves more than one Wiener process:

$$dx = a\,dt + \sum_{i=1}^{m} b_i\,dz_i$$

In this case

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x\,\partial t}\Delta x \Delta t + \cdots$$

$$\Delta x = a\,\Delta t + \sum_{i=1}^{m} b_i \epsilon_i \sqrt{\Delta t}$$

and

$$\lim_{\Delta t \to 0} \Delta x_i^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j \rho_{ij}\,dt$$

where as before ρ_{ij} is the correlation between dz_i and dz_j This leads to

$$df = \left(\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sum_{i=1}^{m}\sum_{j=1}^{m} b_i b_j \rho_{ij}\right) dt + \frac{\partial f}{\partial x}\sum_{i=1}^{m} b_i\,dz_i \tag{4}$$

Finally consider the more general case where f depends on variables x_i $(1 \leq i \leq n)$ and

$$dx_i = a_i\,dt + \sum_{k=1}^{m} b_{ik} dz_k$$

A similar analysis shows that

$$df = \left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}a_i + \frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}\sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik} b_{jl} \rho_{kl}\right) dt + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\sum_{k=1}^{m} b_{ik}\,dz_k \tag{5}$$