Technical Note No. 24*
Options, Futures, and Other Derivatives, Ninth Edition
John Hull

Proof That Forward and Futures Prices Are Equal When Interest Rates Are Constant

This Note demonstrates that forward and futures prices are equal when interest rates are constant. Suppose that a futures contract lasts for n days and that F_i is the futures price at the end of day i ($0 < i < n$). Define δ as the risk-free rate per day (assumed constant). Consider the following strategy.[1]

1. Take a long futures position of e^{δ} at the end of day 0 (i.e., at the beginning of the contract).
2. Increase long position to $e^{2\delta}$ at the end of day 1.
3. Increase long position to $e^{3\delta}$ at the end of day 2.

And so on.

Table 1
The Investment Strategy to Show That Futures and Forward Prices Are Equal

Day	0	1	2	...	$n-1$	n
Futures price	F_0	F_1	F_2	...	F_{n-1}	F_n
Futures position	e^{δ}	$e^{2\delta}$	$e^{3\delta}$	...	$e^{n\delta}$	0
Gain/loss	0	$(F_1 - F_0)e^{\delta}$	$(F_2 - F_1)e^{2\delta}$	...	...	$(F_n - F_{n-1})e^{n\delta}$
Gain/loss compounded to day n	0	$(F_1 - F_0)e^{n\delta}$	$(F_2 - F_1)e^{n\delta}$	...	...	$(F_n - F_{n-1})e^{n\delta}$

This strategy is summarized in Table 1. By the beginning of day i, the investor has a long position of $e^{\delta i}$. The profit (possibly negative) from the position on day i is

$$(F_i - F_{i-1})e^{\delta i}$$

Assume that the profit is compounded at the risk-free rate until the end of day n. Its value at the end of day n is

$$(F_i - F_{i-1})e^{\delta i}e^{(n-i)\delta} = (F_i - F_{i-1})e^{n\delta}$$

The value at the end of day n of the entire investment strategy is therefore

[1] This strategy was proposed by J. C. Cox, J. E. Ingersoll, and S. A. Ross, "The Relation between Forward Prices and Futures Prices," *Journal of Financial Economics* 9 (December 1981): 321–46.

$$\sum_{i=1}^{n}(F_i - F_{i-1})e^{n\delta}$$

This is

$$\begin{aligned}&[(F_n - F_{n-1}) + (F_{n-1} - F_{n-2}) + \cdots + (F_1 - F_0)]e^{n\delta}\\ =&(F_n - F_0)e^{n\delta}\end{aligned}$$

Because F_n is the same as the terminal asset spot price, S_T, the terminal value of the investment strategy can be written

$$(S_T - F_0)e^{n\delta}$$

An investment of F_0 in a risk-free bond combined with the strategy involving futures just given yields

$$F_0e^{n\delta} + (S_T - F_0)e^{n\delta} = S_Te^{n\delta}$$

at time T. No investment is required for all the long futures positions described. It follows that an amount F_0 can be invested to give an amount $S_Te^{n\delta}$ at time T.

Suppose next that the forward price at the end of day 0 is G_0. Investing G_0 in a riskless bond and taking a long forward position of $e^{n\delta}$ forward contracts also guarantees an amount $S_Te^{n\delta}$ at time T. Thus, there are two investment strategies—one requiring an initial outlay of F_0 and the other requiring an initial outlay of G_0—both of which yield $S_Te^{n\delta}$ at time T. It follows that in the absence of arbitrage opportunities

$$F_0 = G_0$$

In other words, the futures price and the forward price are identical. Note that in this proof there is nothing special about the time period of one day. The futures price based on a contract with weekly settlements is also the same as the forward price when corresponding assumptions are made.