

**Technical Note No. 7\***  
**Options, Futures, and Other Derivatives, Ninth Edition**  
**John Hull**

**Differential Equation for Price of a Derivative  
on a Futures Price**

Suppose that the futures price  $F$  follows the process

$$dF = \mu F dt + \sigma F dz \quad (1)$$

where  $dz$  is a Wiener process and  $\sigma$  is constant.<sup>1</sup> Because  $f$  is a function of  $F$  and  $t$ , it follows from Ito's lemma that

$$df = \left( \frac{\partial f}{\partial F} \mu F + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} \sigma^2 F^2 \right) dt + \frac{\partial f}{\partial F} \sigma F dz \quad (2)$$

Consider a portfolio consisting of

$$\begin{aligned} -1 : & \quad \text{derivative} \\ + \frac{\partial f}{\partial F} : & \quad \text{futures contracts} \end{aligned}$$

Define  $\Pi$  as the value of the portfolio and let  $\Delta\Pi$ ,  $\Delta f$ , and  $\Delta F$  be the change in  $\Pi$ ,  $f$ , and  $F$  in time  $\Delta t$ , respectively. Because it costs nothing to enter into a futures contract,

$$\Pi = -f \quad (3)$$

In a time period  $\Delta t$ , the holder of the portfolio earns capital gains equal to  $-\Delta f$  from the derivative and income of

$$\frac{\partial f}{\partial F} \Delta F$$

from the futures contract. Define  $\Delta W$  as the total change in wealth of the portfolio holder in time  $\Delta t$ . It follows that

$$\Delta W = \frac{\partial f}{\partial F} \Delta F - \Delta f$$

The discrete versions of equations (1) and (2) are

$$\Delta F = \mu F \Delta t + \sigma F \Delta z$$

and

$$\Delta f = \left( \frac{\partial f}{\partial F} \mu F + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} \sigma^2 F^2 \right) \Delta t + \frac{\partial f}{\partial F} \sigma F \Delta z$$

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<sup>1</sup> From the arguments in Section 18.7, the drift is zero in the traditional risk-neutral world.

where  $\Delta z = \epsilon \sqrt{\Delta t}$  and  $\epsilon$  is a random sample from a standardized normal distribution. It follows that

$$\Delta W = \left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial F^2} \sigma^2 F^2 \right) \Delta t \quad (4)$$

This is riskless. Hence it must also be true that

$$\Delta W = r \Pi \Delta t \quad (5)$$

If we substitute for  $\Pi$  from equation (3), equations (4) and (5) give

$$\left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial F^2} \sigma^2 F^2 \right) \Delta t = -r f \Delta t$$

Hence

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} \sigma^2 F^2 = r f$$

This is equation (18.8) in the 9th edition.