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John Hull

Calculation of Moments for Valuing Asian Options

We consider the problem of calculating the first two moments of the arithmetic average price of an asset in a risk-neutral world when the average is calculated from discrete observations. Suppose that the asset price is observed at times T_i ($1 \le i \le m$). We define variables as follows:

S_i : The value of the asset at time T_i
F_i : The forward price of the asset for a contract maturing at time T_i
σ_i : The implied volatility for an option on the asset with maturity T_i
ρ_{ij} : Correlation between return on asset up to time T_i and the return on the asset up to time T_j
P : Value of the arithmetic average
M_1 : First moment of P in a risk-neutral world
M_2 : Second moment of P in a risk-neutral world

With these definitions

$$M_1 = \frac{1}{m} \sum_{i=1}^{m} F_i$$

Also

$$P^2 = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} S_i S_j$$

In this case

$$\hat{E}(S_i S_j) = F_i F_j e^{\rho_{ij}\sigma_i\sigma_j\sqrt{T_i T_j}}$$

It can be shown that when $i \le j$

$$\rho_{ij} = \frac{\sigma_i \sqrt{T_i}}{\sigma_j \sqrt{T_j}}$$

so that

$$\hat{E}(S_i S_j) = F_i F_j e^{\sigma_i^2 T_i}$$

and

$$M_2 = \frac{1}{m^2} \left[\sum_{i=1}^{m} F_i^2 e^{\sigma_i^2 T_i} + 2 \sum_{j=1}^{m} \sum_{i=1}^{j-1} F_i F_j e^{\sigma_i^2 T_i} \right]$$