Technical Note No. 11*
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The Manipulation of Credit Transition Matrices

Suppose that $\mathbf{A}$ is an $N \times N$ matrix of credit rating changes in one year. This is a matrix such as the one shown in Table 24.5. The matrix of credit rating changes in m years is $\mathbf{A}^m$. This can be readily calculated using the normal rules for matrix multiplication.

The matrix corresponding to a shorter period than one year, say six months or one month is more difficult to compute. We first use standard routines to calculate eigenvectors $\mathbf{x_i}$, $\mathbf{x_2}$, ..., $\mathbf{x_N}$ and the corresponding eigenvalues λ_1, λ_2, ..., λ_N. These have the property that

$$\mathbf{A}\mathbf{x_i} = \lambda_i \mathbf{x_i} \tag{1}$$

Define $\mathbf{X}$ as a matrix whose ith column is $\mathbf{x_i}$ and $\mathbf{\Lambda}$ as a diagonal matrix where the ith diagonal element is λ_i. From equation (1)

$$\mathbf{AX} = \mathbf{X\Lambda}$$

so that

$$\mathbf{A} = \mathbf{X\Lambda X^{-1}}$$

From this it is easy to see that the nth root of A is

$$\mathbf{X\Lambda^* X^{-1}}$$

where $\mathbf{\Lambda}^*$ is a diagonal matrix where the ith diagonal element is $\lambda_i^{1/n}$.

Some authors such as Jarrow, Lando, and Turnbull prefer to handle this problem in terms of what is termed a *generator matrix*.[1] This is a matrix Γ such that the transition matrix for a short period of time Δt is $\mathbf{1} + \mathbf{\Gamma}\Delta t$ and the transition matrix for longer period of time, t, is

$$\exp(t\mathbf{\Gamma}) = \sum_{k=0}^{\infty} \frac{(t\mathbf{\Gamma})^k}{k!}$$

[1] See R. A. Jarrow, D. Lando, and S.M. Turnbull, "A Markov model for the term structure of credit spreads" *Review of Financial Studies*, 10 (1997), 481–523.