Technical Note No. 6*
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Differential Equation for Price of a Derivative on a Stock Providing a Known Dividend Yield

Define f as the price of a derivative dependent on a stock that provides a dividend yield at rate q. We suppose that the stock price, S, follows the process

$$dS = \mu S\,dt + \sigma S\,dz$$

where dz is a Wiener process. The variables μ and σ are the expected growth rate in the stock price and the volatility of the stock price, respectively. Because the stock price provides a dividend yield, μ is only part of the expected return on the stock.[1]

Because f is a function of S and t, it follows from Ito's lemma that

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dz$$

Similarly to the procedure of Section 15.6, we can set up a portfolio consisting of

$$\begin{array}{rl} -1: & \text{derivative} \\ +\dfrac{\partial f}{\partial S}: & \text{stock} \end{array}$$

If Π is the value of the portfolio,

$$\Pi = -f + \frac{\partial f}{\partial S}S \tag{1}$$

and the change, $\Delta\Pi$, in the value of the portfolio in a time period Δt is as given by equation (15.14):

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

In time Δt the holder of the portfolio earns capital gains equal to $\Delta\Pi$ and dividends on the stock position equal to

$$qS\frac{\partial f}{\partial S}\,\Delta t$$

Define ΔW as the change in the wealth of the portfolio holder in time Δt. It follows that

$$\Delta W = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + qS\frac{\partial f}{\partial S}\right)\Delta t \tag{2}$$

[1] In a risk-neutral world $\mu = r - q$ as indicated in equation (17.7) of the book.

Because this expression is independent of the Wiener process, the portfolio is instantaneously riskless. Hence

$$\Delta W = r\Pi\,\Delta t \tag{3}$$

Substituting from equations (1) and (2) into equation (3) gives

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + qS\frac{\partial f}{\partial S}\right)\Delta t = r\left(-f + \frac{\partial f}{\partial S}S\right)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

This is equation (17.6) in the book.