

Technical Note No. 8*
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John Hull

Analytic Approximation for Valuing American Options

Consider an option on a stock providing a dividend yield equal to q . We will denote the difference between the American and European option price by v . Because both the American and the European option prices satisfy the Black–Scholes differential equation, v also does so. Hence,

$$\frac{\partial v}{\partial t} + (r - q)S \frac{\partial v}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} = rv$$

For convenience, we define

$$\begin{aligned}\tau &= T - t \\ h(\tau) &= 1 - e^{-r\tau} \\ \alpha &= \frac{2r}{\sigma^2} \\ \beta &= \frac{2(r - q)}{\sigma^2}\end{aligned}$$

We also write, without loss of generality,

$$v = h(\tau)g(S, h)$$

With appropriate substitutions and variable changes, this gives

$$S^2 \frac{\partial^2 g}{\partial S^2} + \beta S \frac{\partial g}{\partial S} - \frac{\alpha}{h}g - (1 - h)\alpha \frac{\partial g}{\partial h} = 0$$

The approximation involves assuming that the final term on the left-hand side is zero, so that

$$S^2 \frac{\partial^2 g}{\partial S^2} + \beta S \frac{\partial g}{\partial S} - \frac{\alpha}{h}g = 0 \tag{1}$$

The ignored term is generally fairly small. When τ is large, $1 - h$ is close to zero; when τ is small, $\partial g / \partial h$ is close to zero.

The American call and put prices at time t will be denoted by $C(S, t)$ and $P(S, t)$, where S is the stock price, and the corresponding European call and put prices will be denoted by $c(S, t)$ and $p(S, t)$. Equation (1) can be solved using standard techniques. After boundary conditions have been applied, it is found that

$$C(S, t) = \begin{cases} c(S, t) + A_2 \left(\frac{S}{S^*}\right)^{\gamma_2} & \text{when } S < S^* \\ S - K & \text{when } S \geq S^* \end{cases}$$

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The variable S^* is the critical price of the stock above which the option should be exercised. It is estimated by solving the equation

$$S^* - K = c(S^*, t) + \left\{ 1 - e^{-q(T-t)} N[d_1(S^*)] \right\} \frac{S^*}{\gamma_2}$$

iteratively. For a put option, the valuation formula is

$$P(S, t) = \begin{cases} p(S, t) + A_1 \left(\frac{S}{S^{**}} \right)^{\gamma_1} & \text{when } S > S^{**} \\ K - S & \text{when } S \leq S^{**} \end{cases}$$

The variable S^{**} is the critical price of the stock below which the option should be exercised. It is estimated by solving the equation

$$K - S^{**} = p(S^{**}, t) - \left\{ 1 - e^{-q(T-t)} N[-d_1(S^{**})] \right\} \frac{S^{**}}{\gamma_1}$$

iteratively. The other variables that have been used here are

$$\begin{aligned} \gamma_1 &= \left[-(\beta - 1) - \sqrt{(\beta - 1)^2 + \frac{4\alpha}{h}} \right] / 2 \\ \gamma_2 &= \left[-(\beta - 1) + \sqrt{(\beta - 1)^2 + \frac{4\alpha}{h}} \right] / 2 \\ A_1 &= - \left(\frac{S^{**}}{\gamma_1} \right) \left\{ 1 - e^{-q(T-t)} N[-d_1(S^{**})] \right\} \\ A_2 &= \left(\frac{S^*}{\gamma_2} \right) \left\{ 1 - e^{-q(T-t)} N[d_1(S^*)] \right\} \\ d_1(S) &= \frac{\ln(S/K) + (r - q + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \end{aligned}$$

Options on stock indices, currencies, and futures contracts are analogous to options on a stock providing a constant dividend yield. Hence the quadratic approximation approach can easily be applied to all of these types of options.