Technical Note No. 18*
Options, Futures, and Other Derivatives, Ninth Edition
John Hull

Valuation of a Compounding Swap

Consider a compounding swap where floating rate cash flows in a swap are compounded forward at LIBOR plus a spread rather than being paid. We can use forward rate agreements to show that the value of the floating side is the same as the value it would have if forward rates were realized. In other words, the swap can be valued similarly to a regular swap by assuming that future interest rates equal today's forward rates.

Suppose that t_0 is the time of the payment date immediately preceding the valuation date and that the payment dates following the valuation date are at times t_1, t_2, ..., t_n. Define $\tau_i = t_{i+1} - t_i$ $(0 \leq i \leq n-1)$ and other variables as follows[1]

L: Principal on the floating side of swap
Q_i: Value of floating side compounded forward to time t_i (Q_0 is known)
Q_i^*: Value of floating side compounded forward to time t_i if forward rate is realized
R_i: LIBOR rate from t_i to t_{i+1} for $i \geq 1$ (R_0 is known)
F_i: Forward rate applicable to period between time t_i and t_{i+1} (all known)
s_p: Spread above LIBOR at which interest is paid on the floating side of the swap (20 basis points in Business Snapshot 33.2)
s_c: Spread above LIBOR at which floating interest compounds (10 basis points in Business Snapshot 33.2)

We assume that the spread s_c is applied first to Q_i and then the result is compounded forward at R_i to produce $Q_i(1 + R_i\tau_i)(1 + s_c\tau_i)$. (This assumption is discussed at the end of the Note.)

The value of the floating side of the swap at time t_1 is known. It is:

$$Q_1 = Q_0[(1 + R_0\tau_0)(1 + s_c\tau_0)] + L(R_0 + s_p)\tau_0$$

The first term on the right hand side is the result of compounding the floating payments from time t_0 to t_1. The second term is the floating payment at time t_1.

The value of the floating side at time t_2 is not known and depends on R_1. It is

$$Q_2 = Q_1[(1 + R_1\tau_1)(1 + s_c\tau_1)] + L(R_1 + s_p)\tau_1 \tag{1}$$

However, we can costlessly enter into two FRAs today:

1. An FRA to exchange, at time t_2, R_1 for F_1 on a principal of $Q_1(1 + s_c\tau_1)$
2. An FRA to exchange, at time t_2, R_1 for F_1 on a principal of L

The first FRA shows that the first term on the right hand side of equation (1) has the same present value as a cash flow of $Q_1(1 + F_1\tau_1)(1 + s_c\tau_1)$ at time t_2. The second FRA shows that the second term on the right hand side of equation (1) has the same present value as a cash flow of $L(F_1 + s_p)\tau_1$ at time t_2. The value of the floating side of the swap at time t_2 is, therefore the same as the value of a cash flow of

$$Q_1[(1 + F_1\tau_1)(1 + s_c\tau_1)] + L(F_1 + s_p)\tau_1$$

[1] All rates are here expressed with a compounding frequency reflecting their maturity. Three-month rates are expressed with quarterly compounding; six-month rates are expressed with semi-annual compounding; etc.

at time t_2. This means that Q_2 at time t_2 can costlessly be converted to Q_2^* at time t_2.

Consider next time t_3. The compounded forward amount at time t_3 is

$$Q_3 = Q_2[(1 + R_2\tau_2)(1 + s_c\tau_2)] + L(R_2 + s_p)\tau_2 \tag{2}$$

To deal with the first term on the right hand side, we note that a cash flow of $Q_2[(1 + R_2\tau_2)(1 + s_c\tau_2)]$ at time t_3 is worth the same as $Q_2(1 + s_c\tau_2)$ at time t_2. This from our earlier result is worth the same as $Q_2^*(1 + s_c\tau_2)$ at time t_2. This in turn is worth the same as $Q_2^*[(1 + F_2\tau_2)(1 + s_c\tau_2)]$ at time t_3. To deal with the second term, we note that we can today enter into an FRA to exchange, at time t_3, R_2 for F_2 on a principal of L. These two observations show that a cash flow of Q_3 at time t_3 is worth the same as a cash flow of Q_3^* at time t_3.

Similarly, a cash flow of Q_4 at time t_4 is worth the same as a cash flow of Q_4^* at time t_4; a cash flow of Q_5 at time t_5 is worth the same as a cash flow of Q_5^* at time t_5; and so on. In particular, a cash flow of Q_n at time t_n is worth the same as a cash flow of Q_n^* at time t_n so that the result is proved.

In practice it may be the case that Q_i is compounds forward to $Q_i[1 + (R_i + s_c)\tau_i]$ rather than to $Q_i(1 + R_i\tau_i)(1 + s_c\tau_i)$. There is then an approximation. The result is only true when small terms of the form $s_cR_i\tau_i^2$ are ignored.

Example 33.1 provides an application of the result in this Technical Note. (It does make the approximation just mentioned).