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John Hull

Hermite Polynomials and Their Use for Integration

As explained in Section 25.10, the Gaussian copula model requires functions to be integrated over a normal distribution between $-\infty$ and $+\infty$. Gaussian quadrature approximates the integral as

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-F^2/2} g(F) dF \approx \sum_{k=1}^{M} w_k g(F_k) \tag{1}$$

The approximation gets better as M increases. It has the property that it is exact when $g(F)$ is a polynomial of order M.

The determination the w_k and F_k involves Hermite polynomials. If you want to avoid getting into the details of this, values of w_k and F_k for different values of M can be downloaded from a spread sheet on the author's web site.

The first few Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

A recurrence relationship for calculating higher order polynomials is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and an equation for the derivative with respect to x is

$$H_n'(x) = 2nH_{n-1}(x)$$

Define x_k $(1 \leq k \leq n)$ as the n roots of $H_n(x)$ (that is, the n values of x for which $H_n(x) = 0$) and

$$w_k^* = \frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H_{n-1}(x_k)]^2}$$

A key result is

$$\int_{-\infty}^{\infty} f(x) dx \approx \sum_{k=1}^{n} w_k^* e^{x_k^2} f(x_k) \tag{2}$$

Setting $x = F/\sqrt{2}$ and

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} g(\sqrt{2}x)$$

equation (2) gives

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-F^2/2} g(F) dF \approx \sum_{k=1}^{n} \frac{1}{\pi} w_k^* g(F_k)$$

or alternatively

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-F^2/2} g(F) dy \approx \sum_{k=1}^{n} w_k g(F_k)$$

where

$$w_k = \frac{w_k^*}{\sqrt{\pi}} \qquad F_k = \sqrt{2} x_k$$

This is the result in equation (1), with $n = M$.

This leaves the problem of calculating the n roots of a Hermite polynomial. A program for doing this is 'gauher' in "Numerical Recipes for C: The Art of Scientific Computing" by Press, Flanery, Teukolsky, and Vetterling, Cambridge University Press.