Supplementary Materials for Static and Dynamic Event-triggered Mechanisms for Distributed Control of Parallel Inverters in Low-Voltage islanded Microgrids

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I. PRELIMINARY

For completeness, some equations of the manuscript are relisted here.

The active and reactive power flows, P_i and Q_i, are

$$\begin{aligned} P_i &= \sum_{j=1}^{n} G_{ij} V_i V_j cos(\delta_{ij}) \\ Q_i &= \sum_{j=1}^{n} G_{ij} V_i V_j sin(\delta_{ij}) \end{aligned} \tag{1}$$

The $P-\dot{V}/Q-\omega$ droop control method of inverter $i \in \{1, \cdots m\}$ is represented as

$$\begin{aligned} D_{pi}\dot{V}_i(t) &= P_i^* - P_i(t) - p_i(t) \\ \dot{p}_i &= k_{pri}\dot{V}_i(t) \\ D_{qi}\dot{\theta}_i(t) &= -(Q_i^* - Q_i(t) - q_i(t)) \\ \dot{q}_i &= -k_{qri}\dot{\theta}_i(t) \end{aligned} \tag{2}$$

The structure-preserving model is adopted to model the constant power flows P_{Li} and Q_{Li} at load $i \in \{m+1, \cdots, n\}$.

$$\begin{aligned} D_{Lpi}\dot{\theta}_i(t) &= -P_{Li} - P_i(t) \\ D_{Lqi}\dot{V}_i(t) &= -Q_{Li} - Q_i(t) \end{aligned} \tag{3}$$

The event-triggered restoration mechanism (ETSM) is designed as:

$$\begin{aligned} k_{pi}\dot{p}_i(t) &= P_i^* - P_i(t) - p_i(t) + \lambda_{pi} \sum_{j \in N_i} \left(\frac{p_j(t^j_{g'(t)})}{D_{pj}} - \frac{p_i(t^i_g)}{D_{pi}} \right), \, t \in \left[t^i_g, t^i_{g+1}\right) \\ k_{qi}\dot{q}_i(t) &= Q_i^* - Q_i(t) - q_i(t) + \lambda_{qi} \sum_{j \in N_i} \left(\frac{q_j(\tau^j_{h'(t)})}{D_{qj}} - \frac{q_i(\tau^i_h)}{D_{qi}} \right), \, t \in \left[\tau^i_h, \tau^i_{h+1}\right) \end{aligned} \tag{4}$$

where $\lambda_{pi} = \lambda_{qi} = \lambda$.

The measurement errors with respect to $p_i(t)$ and $q_i(t)$ are defined as

$$\begin{aligned} e_{pi}(t) &= p_i(t) - p_i(t^i_g), \quad t \in \left[t^i_g, t^i_{g+1}\right) \\ e_{qi}(t) &= q_i(t) - q_i(\tau^i_h), \quad t \in \left[\tau^i_h, \tau^i_{h+1}\right) \end{aligned} \tag{5}$$

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II. PROOF OF THEOREM 1

Proof: Construct the following function

$$U=\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}V_iV_jG_{ij}\cos(\theta_i-\theta_j)\right)+\sum_{i=1}^{m}Q_i^*\theta_i-\sum_{i=m+1}^{n}Q_{Li}\theta_i-\sum_{i=1}^{m}P_i^*\ln V_i+\sum_{i=m+1}^{n}P_{Li}\ln V_i+\frac{1}{2}\sum_{i=1}^{m}p_i^2+\frac{1}{2}\sum_{i=1}^{m}q_i^2 \tag{6}$$

Define $\nu_i = lnV_i$ for $i\in\{1,...,m\}$. According to (1) and (6), we have

$$\frac{\partial U}{\partial\theta_i}=\frac{1}{2}\left(-2*\sum_{j=1}^{n}V_iV_jG_{ij}\sin(\theta_i-\theta_j)\right)+Q_i^*=-Q_i(t)+Q_i^*,\quad i\in\{1,...,m\} \tag{7}$$

$$\frac{\partial U}{\partial\theta_i}=\frac{1}{2}\left(-2*\sum_{j=1}^{n}V_iV_jG_{ij}\sin(\theta_i-\theta_j)\right)-Q_{Li}^*=-Q_i(t)-Q_{Li}^*,\quad i\in\{m+1,...,n\} \tag{8}$$

$$\begin{aligned}\frac{\partial U}{\partial\nu_i}&=\frac{1}{2}\frac{\partial}{\partial\nu_i}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\nu_i}e^{\nu_j}G_{ij}\cos(\theta_i-\theta_j)\right)-P_i^*\\&=\frac{1}{2}\left(2*\sum_{i=1}^{n}e^{\nu_i}e^{\nu_j}G_{ij}\cos(\theta_i-\theta_j)\right)-P_i^*\\&=P_i(t)-P_i^*,\quad i\in\{1,...,m\}\end{aligned} \tag{9}$$

$$\frac{\partial U}{\partial\nu_i}=P_i(t)+P_{Li},\quad i\in\{m+1,...,n\} \tag{10}$$

$$\frac{\partial U}{\partial_{pi}}=p_i(t),\qquad\frac{\partial U}{\partial_{qi}}=q_i(t) \tag{11}$$

In summary, we have

$$\begin{aligned}&\frac{\partial U}{\partial\theta_i}=\begin{cases}-Q_i(t)+Q_i^* & i\in\{1,...,m\}\\-Q_i(t)-Q_{Li} & i\in\{m+1,...,n\}\end{cases}\\&\frac{\partial U}{\partial\nu_i}=\begin{cases}P_i(t)-P_i^* & i\in\{1,...,m\}\\P_i(t)+P_{Li} & i\in\{m+1,...,n\}\end{cases}\\&\frac{\partial U}{\partial_{pi}}=p_i(t),\qquad\frac{\partial U}{\partial_{qi}}=q_i(t)\end{aligned} \tag{12}$$

According to (2,3,4,5) and (12), the closed-loop system can be written as follows.
For inverter $i\in\{1,...,m\}$

$$D_{pi}\dot{V}_i(t)=D_{pi}\dot{e^{\nu_i(t)}}=e^{\nu_i(t)}D_{pi}\dot{\nu}_i(t)=-(P_i(t)-P_i^*)-p_i(t)=-\frac{\partial U}{\partial\nu_i}-\frac{\partial U}{\partial_{pi}} \tag{13}$$

$$\begin{aligned}k_{pi}\dot{p}_i(t)&=-(P_i(t)-P_i^*)-p_i(t)+\lambda_{pi}\sum_{j\in N_i}\left(\frac{p_j(t^j_{g'(t)})}{D_{pj}}-\frac{p_i(t^i_g)}{D_{pi}}\right)\\&=-\frac{\partial U}{\partial\nu_i}-\frac{\partial U}{\partial_{pi}}-\lambda\sum_{j=1}^{m}\frac{l_{ij}}{D_{pj}}p_j\left(t^j_{g'(t)}\right)\\&=-\frac{\partial U}{\partial\nu_i}-\frac{\partial U}{\partial_{pi}}-\lambda\sum_{j=1}^{m}\frac{l_{ij}}{D_{pj}}\left(p_j(t)-e_{pj}(t)\right)\end{aligned} \tag{14}$$

$$D_{qi}\dot{\theta}_i(t)=-(-Q_i(t)+Q_i^*)+q_i(t)=-\frac{\partial U}{\partial\theta_i}+\frac{\partial U}{\partial_{qi}} \tag{15}$$

$$
\begin{aligned}
k_{qi}\dot{q}_i(t) &= -Q_i(t) + Q_i^* - q_i(t) + \lambda_{qi}\sum_{j\in N_i}\left(\frac{q_j(\tau^j_{h'(t)})}{D_{qj}} - \frac{q_i(\tau^i_h)}{D_{qi}}\right) \\
&= \frac{\partial U}{\partial \theta_i} - \frac{\partial U}{\partial_{qi}} - \lambda\sum_{j=1}^{m}\frac{l_{ij}}{D_{qj}}\left(q_j(t) - e_{qj}(t)\right)
\end{aligned} \tag{16}
$$

In summary, we have

$$
\begin{aligned}
e^{\nu_i(t)} D_{pi}\dot{\nu}_i(t) &= -\frac{\partial U}{\partial \nu_i} - \frac{\partial U}{\partial_{pi}} \\
k_{pi}\dot{p}_i(t) &= -\frac{\partial U}{\partial \nu_i} - \frac{\partial U}{\partial_{pi}} - \lambda\sum_{j=1}^{m}\frac{l_{ij}}{D_{pj}}\left(p_j(t) - e_{pj}(t)\right) \\
D_{qi}\dot{\theta}_i(t) &= -\frac{\partial U}{\partial \theta_i} + \frac{\partial U}{\partial_{qi}} \\
k_{qi}\dot{q}_i(t) &= \frac{\partial U}{\partial \theta_i} - \frac{\partial U}{\partial_{qi}} - \lambda\sum_{j=1}^{m}\frac{l_{ij}}{D_{qj}}\left(q_j(t) - e_{qj}(t)\right)
\end{aligned} \tag{17}
$$

where l_{ij} is the element in the Laplacian matrix L.

For load $i \in \{m+1, ..., n\}$

$$
\begin{aligned}
D_{Lpi}\dot{\theta}_i(t) &= -\frac{\partial U}{\partial \nu_i} \\
e^{\nu_{Li}(t)} D_{Lqi}\dot{\nu}_i(t) &= \frac{\partial U}{\partial \theta_i}
\end{aligned} \tag{18}
$$

Denote the $m \times n$ dimensional zero and identity matrix as $\mathbf{0_{m\times n}}$ and $\boldsymbol{I_{m\times n}}$, respectively. Denote the vectors $\boldsymbol{\theta(t)} \triangleq [\theta_1(t), ..., \theta_n(t)]^T$, $\boldsymbol{p(t)} \triangleq [p_1(t), ..., p_n(t)]^T$, $\boldsymbol{q(t)} \triangleq [q_1(t), ..., q_n(t)]^T$, $\boldsymbol{\nu(t)} \triangleq [\nu_1(t), ..., \nu_n(t)]^T$, $\boldsymbol{e_p(t)} \triangleq [e_{p1}(t), ..., e_{pn}(t)]^T$, $\boldsymbol{e_q(t)} \triangleq [e_{q1}(t), ..., e_{qn}(t)]^T$, $\boldsymbol{x(t)} \triangleq \left[\boldsymbol{\nu(t)}^T, \boldsymbol{p(t)}^T, \boldsymbol{\theta(t)}^T, \boldsymbol{q(t)}^T\right]^T$, $\boldsymbol{e(t)} \triangleq \left[\mathbf{0}^T_{\mathbf{1\times n}}, \boldsymbol{e_p(t)}^T, \mathbf{0}^T_{\mathbf{1\times n}}, \boldsymbol{e_q(t)}^T\right]^T$.

Besides, define the matrix $\boldsymbol{\lambda} \triangleq diag(\lambda_1, ..., \lambda_m)$, $\boldsymbol{k_p} \triangleq diag(k_{p1}, ..., k_{pm})$, $\boldsymbol{k_q} \triangleq diag(k_{q1}, ..., k_{qm})$, $\boldsymbol{D_p} \triangleq diag(D_{p1}, ..., D_{pm})$, $\boldsymbol{D_{Lp}} \triangleq diag(D_{Lp(m+1)}, ..., D_{Lpn})$, $\boldsymbol{D_q} \triangleq diag(D_{q1}, ..., D_{qm})$, $\boldsymbol{D_{Lq}} \triangleq diag(D_{Lq(m+1)}, ..., D_{Lqn})$, $\boldsymbol{e^{\nu(t)}} \triangleq diag(e^{\nu_1(t)}, ..., e^{\nu_m(t)})$, $\boldsymbol{e^{\nu_L(t)}} \triangleq diag(e^{\nu_{L(m+1)}(t)}, ..., e^{\nu_{Ln}(t)})$. Then,set $\boldsymbol{D(t)} \triangleq diag(\boldsymbol{e^{\nu(t)}D_p}, \boldsymbol{e^{\nu_L(t)}D_{Lq}}, \boldsymbol{k_p}, \boldsymbol{D_q}, \boldsymbol{D_{Lp}}, \boldsymbol{k_q})$ and $\boldsymbol{A} \triangleq \boldsymbol{A_1} + \boldsymbol{A_2}$, $\boldsymbol{A_2} \triangleq diag(\mathbf{0_{n\times n}}, \boldsymbol{\lambda L D_p^{-1}}, \mathbf{0_{n\times n}}, \boldsymbol{\lambda L D_q^{-1}})$ and $\boldsymbol{A_1}$ is shown as follows.

$$
A_1 \triangleq \begin{pmatrix}
\begin{pmatrix} I_{m\times m} & \\ & 0_{(n-m)\times(n-m)} \end{pmatrix} & \begin{pmatrix} I_{m\times m} \\ 0_{(n-m)\times m} \end{pmatrix} & \begin{pmatrix} 0_{m\times m} & \\ & -I_{(n-m)\times(n-m)} \end{pmatrix} & 0_{n\times m} \\
\begin{pmatrix} I_{m\times m} & 0_{m\times(n-m)} \end{pmatrix} & I_{m\times m} & 0_{m\times n} & 0_{m\times m} \\
\begin{pmatrix} 0_{m\times m} & \\ & I_{(n-m)\times(n-m)} \end{pmatrix} & 0_{n\times m} & \begin{pmatrix} I_{m\times m} & \\ & 0_{(n-m)\times(n-m)} \end{pmatrix} & \begin{pmatrix} -I_{m\times m} \\ 0_{(n-m)\times m} \end{pmatrix} \\
0_{m\times n} & 0_{m\times m} & \begin{pmatrix} -I_{m\times m} & 0_{m\times(n-m)} \end{pmatrix} & I_{m\times m}
\end{pmatrix} \tag{19}
$$

Based on the above vectors and matrix, the closed-loop system (17,18) can be rewritten in a compact form as

$$
\boldsymbol{D}\frac{d}{dt}\boldsymbol{x} = -\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}} + \boldsymbol{A}_2\boldsymbol{e}(t) \tag{20}
$$

The derivative of U with respect to time t along with the solution of system (20) is

$$
\begin{aligned}
\frac{d}{dt}U =& \left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \frac{d\boldsymbol{x}}{dt} \\
=& -\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}\frac{\partial U}{\partial \boldsymbol{x}} + \left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}_2 e(t) \\
=& -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}\frac{\partial U}{\partial \boldsymbol{x}} - \frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}}(\boldsymbol{A_1}+\boldsymbol{A_2})\frac{\partial U}{\partial \boldsymbol{x}} + \left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}_2 e(t) \\
=& -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}\frac{\partial U}{\partial \boldsymbol{x}} - \frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A_1}\frac{\partial U}{\partial \boldsymbol{x}} \\
& -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A_2}\frac{\partial U}{\partial \boldsymbol{x}} \\
& +\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}_2 e(t) \\
=& -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A}\frac{\partial U}{\partial \boldsymbol{x}} - \frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}A_1}\frac{\partial U}{\partial \boldsymbol{x}} \\
& -\frac{1}{2}\left(\boldsymbol{p^T\lambda k_p^{-1}LD_p^{-1}p} + \boldsymbol{q^T\lambda k_q^{-1}LD_q^{-1}}\right) \\
& +\left(\boldsymbol{p^T\lambda k_p^{-1}LD_p^{-1}e_p} + \boldsymbol{q^T\lambda k_q^{-1}LD_q^{-1}e_q}\right)
\end{aligned}
\tag{21}
$$

Defining the vectors $\boldsymbol{\hat{p}(t) \triangleq p_1(t^1_{g^{'}(t)}),...,p_m(t^m_{g^{'}(t)})}^T$ and $\boldsymbol{\hat{q}(t) \triangleq q_1(\tau^1_{h^{'}(t)}),...,q_m(\tau^m_{h^{'}(t)})}^T$, we obtain

$$
\begin{aligned}
& -\frac{1}{2}\left(\boldsymbol{p^T\lambda k_p^{-1}LD_p^{-1}p}\right) + \left(\boldsymbol{p^T\lambda k_p^{-1}LD_p^{-1}e_p}\right) \\
&= \boldsymbol{p^T\lambda k_p^{-1}LD_p^{-1}}\left(\boldsymbol{e_p} - \boldsymbol{\frac{1}{2}p}\right) \\
&= \frac{1}{2}\left(\boldsymbol{e_p}+\boldsymbol{\hat{p}}\right)\boldsymbol{\lambda k_p^{-1}LD_p^{-1}}\left(\boldsymbol{e_p}-\boldsymbol{\hat{p}}\right) \\
&= \frac{1}{2}\left(\boldsymbol{e_p\lambda k_p^{-1}LD_p^{-1}e_p} - \boldsymbol{\hat{p}\lambda k_p^{-1}LD_p^{-1}\hat{p}}\right)
\end{aligned}
\tag{22}
$$

Since the communication graph G is undirected and connected, we have

$$
\begin{aligned}
& \boldsymbol{e_p\lambda k_p^{-1}LD_p^{-1}e_p} \\
&= \sum_{i=1}^{m}\frac{\lambda k_{pr}e_{pi}}{D_{pi}}\sum_{j\in N_i}\left(\frac{e_{pi}}{D_{pi}} - \frac{e_{pj}}{D_{pj}}\right) \\
&\leq \lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} + \lambda k_{pr}\sum_{i=1}^{m}\sum_{j\in N_i}\left|\frac{e_{pi}}{D_{pi}}\right|\cdot\left|\frac{e_{pj}}{D_{pj}}\right| \\
&\leq \lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} + \lambda k_{pr}\sum_{i=1}^{m}\sum_{j\in N_i}\left(\frac{e_{pi}^2}{2D_{pi}^2} + \frac{e_{pj}^2}{2D_{pj}^2}\right) \\
&= \lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} + \frac{1}{2}\lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} + \frac{1}{2}\lambda k_{pr}\sum_{i=1}^{m}\sum_{j\in N_i}\left(\frac{e_{pj}^2}{D_{pj}^2}\right) \\
&= \lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} + \frac{1}{2}\lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} + \frac{1}{2}\lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2} \\
&= 2\lambda k_{pr}\sum_{i=1}^{m}|N_i|\frac{e_{pi}^2}{D_{pi}^2}
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\hat{\boldsymbol{p}}\boldsymbol{\lambda}\boldsymbol{k_p^{-1}}\boldsymbol{L}\boldsymbol{D_p^{-1}}\hat{\boldsymbol{p}} \\
&= \sum_{i=1}^{m} \lambda k_{pr} \frac{p_i(t^i_{g'(t)})}{D_{pi}} \sum_{j\in N_i} \left(\frac{p_i(t^i_{g'(t)})}{D_{pi}} - \frac{p_j(t^j_{g'(t)})}{D_{pj}} \right) \\
&= \lambda k_{pr} \sum_{i=1}^{m} \sum_{j\in N_i} \left(\frac{p_i^2(t^i_{g'(t)})}{D_{pi}^2} - \frac{p_i(t^i_{g'(t)}) p_j(t^j_{g'(t)})}{D_{pi}D_{pj}} \right) \\
&= \lambda k_{pr} \sum_{i=1}^{m} \sum_{j\in N_i} \left(\frac{p_i^2(t^i_{g'(t)})}{2D_{pi}^2} + \frac{p_j^2(t^j_{g'(t)})}{2D_{pj}^2} - \frac{p_i(t^i_{g'(t)}) p_j(t^j_{g'(t)})}{D_{pi}D_{pj}} \right) \\
&= \lambda k_{pr} \sum_{i=1}^{m} \sum_{j\in N_i} \frac{1}{2} \left(\frac{p_i(t^i_{g'(t)})}{D_{pi}} - \frac{p_j(t^j_{g'(t)})}{D_{pj}} \right)^2
\end{aligned}
\tag{24}
$$

Combing (22,23) and (24), we have

$$
\begin{aligned}
&-\frac{1}{2}\left(\boldsymbol{p^T}\boldsymbol{\lambda}\boldsymbol{k_p^{-1}}\boldsymbol{L}\boldsymbol{D_p^{-1}}\boldsymbol{p}\right) + \left(\boldsymbol{p^T}\boldsymbol{\lambda}\boldsymbol{k_p^{-1}}\boldsymbol{L}\boldsymbol{D_p^{-1}}\boldsymbol{e_p}\right) \\
&\leq \lambda k_{pr} \sum_{i=1}^{m} \left[|N_i| \frac{e_{pi}^2}{D_{pi}^2} - \sum_{j\in N_i} \frac{1}{4} \left(\frac{p_i(t^i_{g'(t)})}{D_{pi}} - \frac{p_j(t^j_{g'(t)})}{D_{pj}} \right)^2 \right]
\end{aligned}
\tag{25}
$$

With a similar analysis, the following equation is derived.

$$
\begin{aligned}
&-\frac{1}{2}\left(\boldsymbol{q^T}\boldsymbol{\lambda}\boldsymbol{k_q^{-1}}\boldsymbol{L}\boldsymbol{D_q^{-1}}\boldsymbol{q}\right) + \left(\boldsymbol{q^T}\boldsymbol{\lambda}\boldsymbol{k_q^{-1}}\boldsymbol{L}\boldsymbol{D_q^{-1}}\boldsymbol{e_q}\right) \\
&\leq \lambda k_{qr} \sum_{i=1}^{m} \left[|N_i| \frac{e_{qi}^2}{D_{qi}^2} - \sum_{j\in N_i} \frac{1}{4} \left(\frac{q_i(\tau^i_{h'(t)})}{D_{qi}} - \frac{q_j(\tau^j_{h'(t)})}{D_{qj}} \right)^2 \right]
\end{aligned}
\tag{26}
$$

With the definition of matrix $\boldsymbol{D}$ and $\boldsymbol{A_1}$, we have

$$
\begin{aligned}
&\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D^{-1}}\boldsymbol{A_1}\frac{\partial U}{\partial \boldsymbol{x}} \\
&= \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^*\right)^2 + \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^*\right)p_i + \sum_{i=m+1}^{n} \frac{1}{D_{Lqi}V_i}\left(P_i + P_{Li}\right)\left(Q_i + Q_{Li}\right) + \sum_{i=1}^{m} \frac{1}{k_{pi}}\left(P_i - P_i^*\right)p_i + \sum_{i=1}^{m} \frac{1}{k_{pi}}p_i^2 \\
&+ \sum_{i=1}^{m} \frac{1}{D_{qi}}\left(Q_i - Q_i^*\right)^2 + \sum_{i=1}^{m} \frac{1}{D_{qi}}\left(Q_i - Q_i^*\right)q_i + \sum_{i=m+1}^{n} -\frac{1}{D_{Lpi}}\left(P_i + P_{Li}\right)\left(Q_i + Q_{Li}\right) + \sum_{i=1}^{m} \frac{1}{k_{qi}}q_i^2 + \sum_{i=1}^{m} \frac{1}{k_{qi}}\left(Q_i - Q_i^*\right)q_i \\
&= \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^*\right)\left(P_i - P_i^* + p_i\right) + \sum_{i=1}^{m} \frac{k_{pri}}{D_{pi}}\left(P_i - P_i^* + p_i\right)p_i + \sum_{i=1}^{m} \frac{1}{D_{qi}}\left(Q_i - Q_i^* + k_{qi}q_i\right)\left(Q_i - Q_i^* + q_i\right) \\
&\geq \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^*\right)\left(P_i - P_i^* + p_i\right) + \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^* + p_i\right)k_{pri}p_i + \sum_{i=1}^{m} \frac{1}{D_{qi}}\left(Q_i - Q_i^* + q_i\right)^2 \\
&= \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^* + k_{pri}p_i\right)\left(P_i - P_i^* + p_i\right) + \sum_{i=1}^{m} \frac{1}{D_{qi}}\left(Q_i - Q_i^* + q_i\right)^2 \\
&\geq \sum_{i=1}^{m} \frac{1}{D_{pi}V_i}\left(P_i - P_i^* + p_i\right)^2 + \sum_{i=1}^{m} \frac{1}{D_{qi}}\left(Q_i - Q_i^* + q_i\right)^2
\end{aligned}
\tag{27}
$$

Substituting (22,23,25,26,27) into (21) yields

$$\begin{aligned}\frac{d}{dt}U \leq &-\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D}^{-1}\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}}\\ &+\sum_{i=1}^{m}\left[\lambda k_{pr}\left|N_i\right|\frac{e_{pi}^2(t)}{D_{pi}^2}-\sum_{j\in N_i}\frac{\lambda k_{pr}}{4}\left(\frac{p_i(t^i_{g'(t)})}{D_{pi}}-\frac{p_j(t^j_{g'(t)})}{D_{pj}}\right)^2-\frac{1}{2D_{pi}V_i}(P_i(t)-P_i^*(t)+p_i(t))^2\right]\\ &+\sum_{i=1}^{m}\left[\lambda k_{qr}\left|N_i\right|\frac{e_{qi}^2(t)}{D_{qi}^2}-\sum_{j\in N_i}\frac{\lambda k_{qr}}{4}\left(\frac{q_i(\tau^i_{h'(t)})}{D_{qi}}-\frac{q_j(\tau^j_{h'(t)})}{D_{qj}}\right)^2-\frac{1}{2D_{qi}}(Q_i^*(t)-Q_i(t)-q_i(t))^2\right]\end{aligned} \tag{28}$$

Considering the static event-triggered mechanism, we obtain

$$\begin{aligned}e_{pi}^2(t)&\leq\sum_{j\in N_i}\frac{D_{pi}^2}{4\left|N_i\right|}\left(\frac{p_j(t^i_{g'(t)})}{D_{pj}}-\frac{p_i(t^i_g)}{D_{pi}}\right)^2+\frac{D_{pi}(P_i^*(t)-P_i(t)-p_i(t))^2}{2\lambda k_{pr}V_i\left|N_i\right|}+\frac{D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\eta_{pi}\\ e_{qi}^2(t)&\leq\sum_{j\in N_i}\frac{D_{qi}^2}{4\left|N_i\right|}\left(\frac{q_j(\tau^j_{h'(t)})}{D_{qj}}-\frac{q_i(\tau^i_h)}{D_{qi}}\right)^2+\frac{D_{qi}(Q_i^*(t)-Q_i(t)-q_i(t))^2}{2\lambda k_{qr}\left|N_i\right|}+\frac{D_{qi}^2}{\lambda k_{qr}\left|N_i\right|}\eta_{qi}\end{aligned} \tag{29}$$

Thus,

$$\begin{aligned}\frac{d}{dt}U &\leq -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D}^{-1}\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}}+\sum_{i=1}^{m}(\eta_{pi}+\eta_{qi})\\ &=-\sum_{i=1}^{m}\frac{q}{2D_{pi}V_i}(P_i(t)-P_i^*(t)+p_i(t))^2-\sum_{i=1}^{m}\frac{1}{2D_{qi}}(Q_i^*(t)-Q_i(t)-q_i(t))^2\\ &\quad-\lambda k_{pr}\sum_{\mathrm{i}=1}^{m}\sum_{j\in N_i}\frac{1}{4}\left(\frac{p_i\left(t\right)}{D_{p,i}}-\frac{p_j\left(t\right)}{D_{p,j}}\right)^2-\lambda k_{qr}\sum_{\mathrm{i}=1}^{m}\sum_{j\in N_i}\frac{1}{4}\left(\frac{q_i\left(t\right)}{D_{q,i}}-\frac{q_j\left(t\right)}{D_{q,j}}\right)^2+\sum_{i=1}^{m}(\eta_{p,i}+\eta_{q,i})\end{aligned} \tag{30}$$

Since the boundedness of x can be deduced by the boundedness of $U(x)$ through a similar analysis of that in [1], [2], equation (30) indicates that the closed-loop system (20) converges into the set S and with the LaSalle's invariance principle, we have

$$\begin{aligned}S \triangleq (\theta_1,...,\theta_n,p_1,...,p_m,...,V_1,...,V_n,q_1,...,q_m)|&\sum_{i=1}^{m}\frac{q}{2D_{pi}V_i}(P_i(t)-P_i^*(t)+p_i(t))^2\\ +\sum_{i=1}^{m}\frac{1}{2D_{qi}}(Q_i^*(t)-Q_i(t)-q_i(t))^2+\lambda k_{pr}&\sum_{\mathrm{i}=1}^{m}\sum_{j\in N_i}\frac{1}{4}\left(\frac{p_i(t)}{D_{pi}}-\frac{p_j(t)}{D_{pj}}\right)^2\\ +\lambda k_{qr}\sum_{i=1}^{m}\sum_{j\in N_i}\frac{1}{4}\left(\frac{q_i\left(t\right)}{D_{qi}}-\frac{q_j\left(t\right)}{D_{qj}}\right)^2\leqslant&\sum_{i=1}^{m}(\eta_{pi}+\eta_{qi})\end{aligned} \tag{31}$$

In other words, the frequency and voltage of the microgrid staring from a neighbourhood of the equilibrium converges into an arbitrate small neighbourhood of the equilibrium since η_{pi} and η_{qi} can be defined small enough. In a mathematical statement, we have $|\omega_i-\omega^*|<C_\omega$, C_ω is an arbitrary small positive constants and $|V_i-V^*|<5\%V^*$.

With (31), we can obtain that $\left|\frac{p_i(t)}{D_{pi}}-\frac{p_j(t)}{D_{pj}}\right|\leqslant\frac{2}{\sqrt{\lambda k_{pr}}}\sqrt{\sum_{i=1}^{m}(\eta_{pi}+\eta_{qi})}$, for $j\in N_i$. Defining the variable $\xi_{pi}(t)\triangleq P_i^*-P_i(t)-p_i(t)$, we can also have that $|\xi_{pi}(t)|\leqslant\sqrt{2D_{pi}V_i\sum_{i=1}^{m}(\eta_{pi}+\eta_{qi})}$.

Since

$$
\begin{aligned}
\left|\frac{p_i(t)}{D_{pi}}-\frac{p_j(t)}{D_{pj}}\right| &= \left|\frac{P_i^*-P_i(t)-\xi_{pi}(t)}{D_{pi}}-\frac{P_j^*-P_j(t)-\xi_{pj}(t)}{D_{pj}}\right| \\
&= \left|\frac{P_i^*}{D_{pi}}\left(1-\frac{P_i(t)}{P_i^*}-\frac{\xi_{pi}(t)}{P_i^*}\right)-\frac{P_j^*}{D_{pj}}\left(1-\frac{P_j(t)}{P_j^*}-\frac{\xi_{pj}(t)}{P_j^*}\right)\right| \\
&= \frac{P_i^*}{D_{pi}}\left|-\frac{P_i(t)}{P_i^*}-\frac{\xi_{pi}}{P_i^*}+\frac{P_j(t)}{P_j^*}+\frac{\xi_{pj}(t)}{P_j^*}\right| \\
&\geqslant \frac{P_i^*}{D_{pi}}\left(\left|\frac{P_i(t)}{P_i^*}-\frac{P_j(t)}{P_j^*}\right|-\left|\frac{\xi_{pi}}{P_i^*}-\frac{\xi_{pj}}{P_j^*}\right|\right)
\end{aligned} \tag{32}
$$

we can derive

$$
\begin{aligned}
\left|\frac{P_i(t)}{P_i^*}-\frac{P_j(t)}{P_j^*}\right| &\leqslant \frac{D_{pi}}{P_i^*}\left|\frac{p_i(t)}{D_{pi}}-\frac{p_j(t)}{D_{pj}}\right|+\left|\frac{\xi_{pi}(t)}{P_i^*}-\frac{\xi_{pj}(t)}{P_j^*}\right| \\
&\leqslant \frac{2D_{pi}}{P_i^*\sqrt{\lambda k_{pr}}}\sqrt{\sum_{i=1}^m(\eta_{pi}+\eta_{qi})}+\frac{\sqrt{2D_{pi}V_i\sum_{i=1}^m(\eta_{pi}+\eta_{qi})}}{P_i^*}+\frac{\sqrt{2D_{pj}V_j\sum_{i=1}^m(\eta_{pi}+\eta_{qi})}}{P_j^*} \\
&= \left(\frac{2D_{pi}}{P_i^*\sqrt{\lambda k_{pr}}}+\frac{\sqrt{2D_{pi}V_i}}{P_i^*}+\frac{\sqrt{2D_{pj}V_j}}{P_j^*}\right)\sqrt{\sum_{i=1}^m(\eta_{pi}+\eta_{qi})}
\end{aligned} \tag{33}
$$

With the same analysis, we have

$$
\left|\frac{Q_i(t)}{Q_i^*}-\frac{Q_j(t)}{Q_j^*}\right| \leqslant \left(\frac{2D_{qi}}{Q_i^*\sqrt{\lambda k_{qr}}}+\frac{\sqrt{2D_{qi}}}{Q_i^*}+\frac{\sqrt{2D_{qj}}}{Q_j^*}\right)\sqrt{\sum_{i=1}^m(\eta_{pi}+\eta_{qi})} \tag{34}
$$

Let $\eta^{'}=\sqrt{\sum_{i=1}^m(\eta_{pi}+\eta_{qi})}$, $C_{Pi}=\left(\frac{2D_{pi}}{P_i^*\sqrt{\lambda k_{pr}}}+\frac{\sqrt{2D_{pi}V_i}}{P_i^*}+\frac{\sqrt{2D_{pj}V_j}}{P_j^*}\right)\eta^{'}$ and $C_{Qi}=\left(\frac{2D_{qi}}{Q_i^*\sqrt{\lambda k_{qr}}}+\frac{\sqrt{2D_{qi}}}{Q_i^*}+\frac{\sqrt{2D_{qj}}}{Q_j^*}\right)\eta^{'}$. This concludes the proof.

■

III. PROOF OF THEOREM 2

Proof: With equations (4) and (5), the following inequality is derived. For $t\in[t_g^i,t_{g+1}^i)$:

$$
\frac{d}{dt}|e_{pi}(t)| \leqslant |\dot{e}_{pi}(t)| = |\dot{p}_i(t)| = \frac{k_{pr}}{D_{pi}}\left|P_i^*-P_i(t)-p_i(t)+\lambda_i\sum_{j\in N_i}\left(\frac{p_j(t^j_{g^{'}(t)})}{D_{pj}}-\frac{p_i(t_g^i)}{D_{pi}}\right)\right| \leqslant M_{pi} \tag{35}
$$

where M_{pi} is positive constant and the last inequality holds due to the convergence verified by Theorem 1.

Thus, for $t\in[t_g^i,t_{g+1}^i)$, we have $|e_{pi}(t)|\leqslant M_{pi}(t-t_g^i)$.

Since the adjacent triggering event occurs when the inequality (36) holds according to the SETM,

$$
e_{pi}^2(t) > \sum_{j\in N_i}\frac{D_{pi}^2}{4\,|N_i|}\left(\frac{p_j(t^i_{g^{'}(t)})}{D_{pj}}-\frac{p_i(t_g^i)}{D_{pi}}\right)^2+\frac{D_{pi}(P_i^*(t)-P_i(t)-p_i(t))^2}{2\lambda k_{pr}V_i\,|N_i|}+\frac{D_{pi}^2}{\lambda k_{pr}\,|N_i|}\eta_{pi} \tag{36}
$$

we have

$$
M_{pi}(t_{g+1}^i-t_g^i) > \sqrt{\sum_{j\in N_i}\frac{D_{pi}^2}{4\,|N_i|}\left(\frac{p_j(t^i_{g^{'}(t)})}{D_{pj}}-\frac{p_i(t_g^i)}{D_{pi}}\right)^2+\frac{D_{pi}\left(P_i^*-P_i(t_{g+1}^i)-p_i(t_{g+1}^i)\right)^2}{2\lambda k_{pr}V_i\,|N_i|}+\frac{D_{pi}^2}{\lambda k_{pr}\,|N_i|}\eta_{pi}} \tag{37}
$$

Which also means

$$t_{g+1}^{i}-t_{g}^{i}>\frac{1}{M_{pi}}\sqrt{\sum_{j\in N_i}\frac{D_{pi}^{2}}{4\left|N_i\right|}\left(\frac{p_j(t_{g'(t)}^{i})}{D_{pj}}-\frac{p_i(t_g^i)}{D_{pi}}\right)^2+\frac{D_{pi}\left(P_i^*-P_i(t_{g+1}^i)-p_i(t_{g+1}^i)\right)^2}{2\lambda k_{pr}V_i\left|N_i\right|}+\frac{D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\eta_{pi}} \tag{38}$$

With a similar analysis, we obtain

$$\tau_{h+1}^{i}-\tau_{h}^{i}>\frac{1}{M_{qi}}\sqrt{\sum_{j\in N_i}\frac{D_{qi}^{2}}{4\left|N_i\right|}\left(\frac{q_j(\tau_{h'(\tau_{h+1}^j)}^{j})}{D_{qj}}-\frac{q_i(\tau_h^i)}{D_{qi}}\right)^2+\frac{D_{qi}(Q_i^*-Q_i(\tau_{h+1}^j)-q_i(\tau_{h+1}^j))^2}{2\lambda k_{qr}\left|N_i\right|}+\frac{D_{qi}^2}{\lambda k_{qr}\left|N_i\right|}\eta_{qi}} \tag{39}$$

This concludes the proof.

■

IV. Proof of theorem 3

Proof: Construct the following function

$$U_d=U+\sum_{i=1}^{m}\varphi_{pi}+\sum_{i=1}^{m}\varphi_{qi} \tag{40}$$

With the similar analysis as (28), we have

$$
\begin{aligned}
\frac{d}{dt}U \leq & -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D}^{-1}\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}} + \sum_{i=1}^{m}\left(\dot{\varphi}_{pi} + \dot{\varphi}_{qi}\right) \\
& + \sum_{i=1}^{m}\left[\lambda k_{pr}\left|N_i\right|\frac{e_{pi}^2(t)}{D_{pi}^2} - \sum_{j\in N_i}\frac{\lambda k_{pr}}{4}\left(\frac{p_i(t_{g'(t)}^i)}{D_{pi}} - \frac{p_j(t_{g'(t)}^j)}{D_{pj}}\right)^2 - \frac{1}{2D_{pi}V_i}\left(P_i(t) - P_i^*(t) + p_i(t)\right)^2\right] \\
& + \sum_{i=1}^{m}\left[\lambda k_{qr}\left|N_i\right|\frac{e_{qi}^2(t)}{D_{qi}^2} - \sum_{j\in N_i}\frac{\lambda k_{qr}}{4}\left(\frac{q_i(\tau_{h'(t)}^i)}{D_{qi}} - \frac{q_j(\tau_{h'(t)}^j)}{D_{qj}}\right)^2 - \frac{1}{2D_{qi}}\left(Q_i^*(t) - Q_i(t) - q_i(t)\right)^2\right] \\
\leq & -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D}^{-1}\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}} + \sum_{i=1}^{m}\left(\dot{\varphi}_{pi} + \dot{\varphi}_{qi}\right) + \sum_{i=1}^{m}\left(\alpha_{pi}\varphi_{pi} + \alpha_{qi}\varphi_{qi}\right) \\
& + \sum_{i=1}^{m}\left[-\sum_{j\in N_i}\frac{\lambda k_{pr}(1-\alpha_{pi})}{4}\left(\frac{p_i(t_{g'(t)}^i)}{D_{pi}} - \frac{p_j(t_{g'(t)}^j)}{D_{pj}}\right)^2 - \frac{1-\alpha_{pi}}{2D_{pi}V_i}\left(P_i(t) - P_i^*(t) + p_i(t)\right)^2\right] \\
& + \sum_{i=1}^{m}\left[-\sum_{j\in N_i}\frac{\lambda k_{qr}(1-\alpha_{qi})}{4}\left(\frac{q_i(\tau_{h'(t)}^i)}{D_{qi}} - \frac{q_j(\tau_{h'(t)}^j)}{D_{qj}}\right)^2 - \frac{1-\alpha_{qi}}{2D_{qi}}\left(Q_i^*(t) - Q_i(t) - q_i(t)\right)^2\right] \\
\leq & -\frac{1}{2}\left(\frac{\partial U}{\partial \boldsymbol{x}}\right)^T \boldsymbol{D}^{-1}\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}} + \sum_{i=1}^{m}\left(\left(\alpha_{pi} - \beta_{pi}\right)\varphi_{pi} + \left(\alpha_{qi} - \beta_{qi}\right)\varphi_{qi}\right) \\
& - \sum_{i=1}^{m}\sum_{j\in N_i}\frac{1}{4}\left(\lambda k_{pr}(1-\alpha_{pi}) - \frac{\alpha_{pi}\gamma_{pi}D_{pi}^2}{|N_i|}\right)\left(\frac{p_i(t_{g'(t)}^i)}{D_{pi}} - \frac{p_j(t_{g'(t)}^j)}{D_{pj}}\right)^2 \\
& - \sum_{i=1}^{m}\frac{1}{2\lambda k_{pr}D_{pi}V_i}\left(\lambda k_{pr}(1-\alpha_{pi}) - \frac{\alpha_{pi}\gamma_{pi}D_{pi}^2}{|N_i|}\right)\left(P_i^* - P_i(t) - p_i(t)\right)^2 \\
& - \sum_{i=1}^{m}\sum_{j\in N_i}\frac{1}{4}\left(\lambda k_{qr}(1-\alpha_{qi}) - \frac{\alpha_{qi}\gamma_{qi}D_{qi}^2}{|N_i|}\right)\left(\frac{q_i(\tau_{h'(t)}^i)}{D_{qi}} - \frac{q_j(\tau_{h'(t)}^j)}{D_{qj}}\right)^2 \\
& - \sum_{i=1}^{m}\frac{1}{2\lambda k_{qr}D_{qi}}\left(\lambda k_{qr}(1-\alpha_{qi}) - \frac{\alpha_{qi}\gamma_{qi}D_{qi}^2}{|N_i|}\right)\left(Q_i^* - Q_i(t) - q_i(t)\right)^2 \\
& - \gamma_{pi}e_{pi}^2(t) - \gamma_{qi}e_{qi}^2(t) \\
\leq & \; 0
\end{aligned} \tag{41}
$$

The last inequality of (41) holds if the parameters are selected such that (42) holds and φ_{pi} and φ_{qi} are both positive variables with the similar analysis as [3].

$$
\begin{aligned}
& \alpha_{pi} \leq (\lambda k_{pr}|N_i|)/(\lambda k_{pr}|N_i| + \gamma_{pi}D_{pi}^2) \\
& \alpha_{qi} \leq (\lambda k_{qr}|N_i|)/(\lambda k_{qr}|N_i| + \gamma_{qi}D_{qi}^2) \\
& \alpha_{pi} \leq \beta_{pi}, \alpha_{qi} \leq \beta_{qi}
\end{aligned} \tag{42}
$$

Equation (41) shows that the solution of the closed-loop system (20) converges into set $S^{'} \triangleq \{\boldsymbol{x}|\boldsymbol{D}^{-1}\boldsymbol{A}\frac{\partial U}{\partial \boldsymbol{x}} = 0\}$ by applying the LaSalle's invariance principle [2], which further means $\frac{\partial U}{\partial \boldsymbol{x}} = 0$. According to (20), we have

$$
\begin{aligned}
& for\, i = 1, ..., m \\
& \qquad P_i^* - P_i(t) - p_i(t) = 0 \\
& \qquad P_i^* - P_i(t) - p_i(t) + \lambda_i \sum_{j \in N_i} \left(\frac{p_j(t^j_{g'(t)})}{D_{pj}} - \frac{p_i(t^i_g)}{D_{pi}} \right) = 0,\, t \in \left[t^i_g, t^i_{g+1}\right) \\
& \qquad Q_i^* - Q_i(t) - q_i(t) = 0 \\
& \qquad Q_i^* - Q_i(t) - q_i(t) + \lambda_j \sum_{j \in N_i} \left(\frac{q_j(\tau^j_{h'(t)})}{D_{qj}} - \frac{q_i(\tau^i_h)}{D_{qi}} \right) = 0,\, t \in \left[\tau^i_h, \tau^i_{h+1}\right) \\
& for\, i = m+1, ..., n \\
& \qquad - P_{Li} - P_i(t) = 0 \\
& \qquad - Q_{Li} - Q_i(t) = 0
\end{aligned}
\tag{43}
$$

This implies that $S^{'}$ is the set of all equilibriums of the closed-loop system (20). The solution of (20) starts from a neighborhood of its initial equilibrium asymptotically converges to an equilibrium in the neighborhood.

In a mathematical statement, we have $\omega_i = \omega^*$, $P_i^* - P_i(t) - p_i(t) \equiv 0$ and $Q_i^* - Q_i(t) - q_i(t) \equiv 0$ hold for $i = 1, ..., m$.

Besides, we have $\sum\limits_{j \in N_i} \left(\frac{p_j(t)}{D_{pj}} - \frac{p_i(t)}{D_{pi}} \right) = 0$ and $\sum\limits_{j \in N_i} \left(\frac{q_j(t)}{D_{qj}} - \frac{q_i(t)}{D_{qi}} \right) = 0$ at the equilibrium point for $i = 1, ..., m$, which further means $\frac{p_j(t)}{D_{pj}} = \frac{p_i(t)}{D_{pi}}$ and $\frac{q_j(t)}{D_{qj}} = \frac{q_i(t)}{D_{qi}}$ for all $i, j = 1, ..., m$.

Since the droop coefficient are selected as $\frac{P_j^*}{D_{pj}} = \frac{P_i^*}{D_{pi}}$ and $\frac{{Q_j}^*}{D_{qj}} = \frac{{Q_i}^*}{D_{qi}}$, we have $\frac{P_j(t)}{D_{pj}} = \frac{P_i(t)}{D_{pi}}$ and $\frac{Q_j(t)}{D_{qj}} = \frac{Q_i(t)}{D_{qi}}$.

Thus, we can deduce that $\frac{P_j(t)}{P_j^*} = \frac{P_i(t)}{P_i^*}$ and $\frac{Q_j(t)}{Q_j^*} = \frac{Q_i(t)}{Q_i^*}$ at the equilibrium point for all $i, j = 1, ..., m$.

This concludes the proof. ∎

V. Proof of Theorem 4

Proof: With the dynamic distributed event-triggered mechanism of $p_i(t)$, the local error e^2_{pi} and the dynamic regulation of φ_{pi} during $t \in \left[t^i_g, t^i_{g+1}\right)$ are shown as follows.

$$
\begin{aligned}
& e^2_{pi}(t) \leqslant \alpha_{pi} \left[\sum_{j \in N_i} \frac{D^2_{pi}}{4\left|N_i\right|} \left(\frac{p_j(t^j_{g'(t)})}{D_{pj}} - \frac{p_i(t^i_g)}{D_{pi}} \right)^2 + \frac{D_{pi}}{2\lambda k_{pr} V_i \left|N_i\right|} \left(P_i^* - P_i(t) - p_i(t)\right)^2 + \frac{D^2_{pi}}{\lambda k_{pr} \left|N_i\right|} \varphi_{pi} \right] \quad t \in \left[t^i_g, t^i_{g+1}\right) \\
& \dot{\varphi}_{pi} = -\beta_{pi}\varphi_{pi} + \gamma_{pi} \left(\sum_{j \in N_i} \frac{\alpha_{pi} D^2_{pi}}{4\left|N_i\right|} \left(\frac{p_j(t^j_{g'(t)})}{D_{pj}} - \frac{p_i(t^i_g)}{D_{pi}} \right)^2 + \frac{\alpha_{pi} D_{pi}}{2\lambda k_{pr} V_i \left|N_i\right|} \left(P_i^* - P_i(t) - p_i(t)\right)^2 - e^2_{pi}(t) \right)
\end{aligned}
\tag{44}
$$

According to (44), we have

$$
\dot{\varphi}_{pi} \geq - \left(\beta_{pi} + \gamma_{pi} \frac{\alpha_{pi} D^2_{pi}}{\lambda k_{pr} \left|N_i\right|} \right) \varphi_{pi} \tag{45}
$$

Thus, we obtain

$$
\varphi_{pi} \geq \varphi_{pi}(0) e^{-\left(\beta_{pi} + \gamma_{pi} \frac{\alpha_{pi} D^2_{pi}}{\lambda k_{pri} \left|N_i\right|} \right) t} \tag{46}
$$

where $\varphi_{pi}(0) \geq 0$ [3].

Assume that there exists Zeno behaviour, then $lim_{k \to \infty} t^i_k = T_c$ for some $i \in \{1, ..., n\}$ with an accumulation time T_c. According to the definition of limitation, for a given constant

$$
\varepsilon_a = \frac{D_{pi} \sqrt{\alpha_{pi} \varphi_{pi}(0)}}{2M_{pi} \sqrt{\lambda k_{pr} \left|N_i\right|}} e^{-\frac{1}{2} \left(\beta_{pi} + \gamma_{pi} \frac{\alpha_{pi} D^2_{pi}}{\lambda k_{pr} \left|N_i\right|} \right) T_c} > 0 \tag{47}
$$

there exists a positive integer N_c, such that for $\forall k > N_c$, we have

$$
t^i_k \in \left[T_c - \varepsilon_a, T_c\right) \tag{48}
$$

Consider the following equation for $t > t_k^i$:

$$M_{pi}(t - t_k^i) \leq \frac{D_{pi}\sqrt{\alpha_{pi}\varphi_{pi}(0)}}{\sqrt{\lambda k_{pr}\left|N_i\right|}} e^{-\frac{1}{2}\left(\beta_{pi}+\gamma_{pi}\frac{\alpha_{pi}D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\right)T_c} \tag{49}$$

Since $\int_{t_k^i}^t |\dot{e}_{pi}(t)| \leq M_{pi}(t - t_k^i)$, then

$$\begin{aligned} |e_{pi}(t)| &= \left|\int_{t_k^i}^t \dot{e}_{pi}\left(t\right)\right| \leq \int_{t_k^i}^t \left|\dot{e}_{pi}\left(t\right)\right| \\ &\leq \frac{D_{pi}\sqrt{\alpha_{pi}\varphi_{pi}(0)}}{\sqrt{\lambda k_{pr}\left|N_i\right|}} e^{-\frac{1}{2}\left(\beta_{pi}+\gamma_{pi}\frac{\alpha_{pi}D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\right)T_c} \end{aligned} \tag{50}$$

According to (46) and (50), we have

$$\begin{aligned} e_{pi}^2(t) &\leq \frac{\alpha_{pi}D_{pi}^2}{\lambda k_{pr}|N_i|}\varphi_{pi} \\ &\leq \alpha_{pi}\left[\sum_{j\in N_i}\frac{D_{pi}^2}{4\left|N_i\right|}\left(\frac{p_j(t_{g'(t)}^j)}{D_{pj}} - \frac{p_i(t_g^i)}{D_{pi}}\right)^2 + \frac{D_{pi}}{2\lambda k_{pr}V_i\left|N_i\right|}(P_i^* - P_i(t) - p_i(t))^2 + \frac{D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\varphi_{pi}\right] \end{aligned} \tag{51}$$

Therefore, equation (49) is one sufficient condition of (51). Thus, we derive

$$e_{pi}^2(t) > \alpha_{pi}\left[\sum_{j\in N_i}\frac{D_{pi}^2}{4\left|N_i\right|}\left(\frac{p_j(t_{g'(t)}^j)}{D_{pj}} - \frac{p_i(t_g^i)}{D_{pi}}\right)^2 + \frac{D_{pi}}{2\lambda k_{pr}V_i\left|N_i\right|}(P_i^* - P_i(t) - p_i(t))^2 + \frac{D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\varphi_{pi}\right] \tag{52}$$

is one sufficient condition of

$$M_{pi}(t - t_k^i) > \frac{D_{pi}\sqrt{\alpha_{pi}\varphi_{pi}(0)}}{\sqrt{\lambda k_{pr}\left|N_i\right|}} e^{-\frac{1}{2}\left(\beta_{pi}+\gamma_{pi}\frac{\alpha_{pi}D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\right)T_c} \tag{53}$$

Selecting $k = N > N_c$ and $t = t_{N+1}^i$,

$$e_{pi}^2(t_{N+1}^i) > \alpha_{pi}\left[\sum_{j\in N_i}\frac{D_{pi}^2}{4\left|N_i\right|}\left(\frac{p_j(t_{g'(t_N^i)}^j)}{D_{pj}} - \frac{p_i(t_g^i)}{D_{pi}}\right)^2 + \frac{D_{pi}}{2\lambda k_{pr}V_i\left|N_i\right|}(P_i^* - P_i(t) - p_i(t))^2 + \frac{D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\varphi_{pi}\right] \tag{54}$$

where t_{N+1}^i and t_N^i are two neighbouring triggering time instants. Then

$$M_{pi}(t_{N+1}^i - t_N^i) > \frac{D_{pi}\sqrt{\alpha_{pi}\varphi_{pi}(0)}}{\sqrt{\lambda k_{pr}\left|N_i\right|}} e^{-\frac{1}{2}\left(\beta_{pi}+\gamma_{pi}\frac{\alpha_{pi}D_{pi}^2}{\lambda k_{pr}\left|N_i\right|}\right)T_c} \tag{55}$$

Combing (47) and (55), we have

$$t_{N+1}^i - t_N^i > 2\varepsilon_a \tag{56}$$

Noting that (56) contradicts (48), which means the aforementioned assumption is invalid. Thus, Zeno behavior is excluded. This concludes the proof. ■

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