for $f \in I$ locally $f = \sum C_{\alpha,f}(y)\, x^{\alpha} \Rightarrow \pi_*^{-1}(f.m) = \boxed{\sum C'_{\alpha,f}(y)\, x'^{\alpha}}$

where

$$C'_{\alpha,f}(y') = y_q'^{\,|\alpha|-m}\; C_{\alpha,f}(y_1' y_q', \dots, y_q', \dots, y_m')$$

while

$$f|_S = C_{0,f}|_S \quad \text{and} \quad \pi_*^{-1}(f.m)|_{S'} = C'_{0,f}(y')|_{S'},$$

$$\pi_*^{-1}(f.m)|_{S'} = (C'_{0,f})|_{S'} = y_q'^{\,-m}\,(\underbrace{\pi^* C_{0,f}}_{\text{composition}})|_{S'} = y_q^{-m}\, \pi|_S^*(C_{0,f}|_S)$$

$$= y_q^{-m}\, \pi|_S^*(f|_S) = \pi|_{S*}^{-1}(f|_S, m)$$

$$\Rightarrow \pi_*^{-1}(I.m)|_{S'} = \pi|_{S*}^{-1}((I.m)|_S).$$

(4). Notations as in (3).

As before, $*$ $\mathrm{Supp}(\pi_*^{-1} I, m) \cap S' = \bigcap_{\substack{f \in I \\ |\alpha|<m}} \mathrm{Supp}(C'_{\alpha,f}|_{S'}, m-|\alpha|)$

Now $C'_{\alpha,f} = y_m'^{\,-m+|\alpha|}\, \pi^*(C_{\alpha,f}) \subseteq \pi_*^{-1} D^{|\alpha|}(I.m)$.

back to $*$ LHS $\supseteq \bigcap_{0 \le i < m} \mathrm{Supp}(\pi_*^{-1} D^i(I.m)|_{S'})$

$$= \bigcap_{i=0} \mathrm{Supp}(\pi|_{S*}^{-1}(D^i(I.m)|_S)).$$

$$= \mathrm{Supp}(\pi|_{S*}^{-1} C(I.m)|_S).$$

So $S' \cap \mathrm{Supp}(\pi_*^{-1} C(I.m)) \supseteq S' \cap \mathrm{Supp}(\pi_*^{-1}(I, m)) \supseteq \mathrm{Supp}(\pi|_{S*}^{-1} C(I.m)|_S)$.

?//

$\mathrm{Supp}(\pi_*^{-1} C(I.m)|_{S'})$

$\|$

$\mathrm{Supp}(\pi|_{S*}^{-1}(C(I.m)|_S))$.

$\square$

Remark: The above proposition says that, $S \subset X$ not contained in $\mathrm{supp}(I,m)$, an order reduction for $C(I,m)|_S$ on S lifts naturally to an "order reduction" on X.

To be more specific $\quad Z \subset S \subset X$

$$\begin{array}{llllll} \mathbb{B}: & \pi: & X_r \to & \cdots \to & X_0 & C(I,m). \\ \uparrow \text{lift} & & \uparrow & & \uparrow & \\ \mathbb{B}_S: & \pi_S: & S_r \to & \cdots \to & S_0 \supseteq C(I,m)|_S & \end{array}$$

(1) $Z_i \subseteq \mathrm{Supp}([C(I,m)|_S]_i) \Rightarrow Z_i \subseteq \mathrm{Supp}[C(I,m)_i] \cap S_i$

(2) $\mathrm{Supp}([C(I,m)|_S]_r) = \phi \Rightarrow \mathrm{Supp}[C(I,m)_r] \cap S_r = \phi$.

i.e. $\mathrm{supp}(I,m)_r = \mathrm{supp}(C(I),m)_r$ is disjoint with S_r.

(3) If $\mathbb{B}_S$ is funstorial resp to smooth morphisms, then the natural lifting is also functorial resp to smooth morphism.

In fact
$$\begin{tikzcd} S_Y \arrow[r] \arrow[d, "h_Y"'] & Y \arrow[d, "h \to \text{smooth}"] \\ Z \subset S \arrow[r, hook] & X \end{tikzcd}$$

$h: Y \to X$ smooth $\Rightarrow h_Y: S_Y \to S$ smooth.

Lift $\mathbb{B}_S$ to $\mathbb{B}$, blow-up center is Z.

functionality for $\mathbb{B}_S$ imply blow up center for S_Y is $h_Y^{-1}(Z)$.

lift ~~to~~ Y, blow up center is again $h_Y^{-1}(Z) \subset Y$.

$\square$

Rem: In previous case, we only consider the restriction $\to$ ord reduction $\to$ lifting that end up with $S_r \cap \mathrm{Supp}(I,m)_r = \phi$.

Key: If we can find $S \supseteq \mathrm{Supp}(I,m)$ such that each time.

maxi cont $\longrightarrow$ $S_i \supseteq \mathrm{Supp}[(I,m)_i]$ then we end up with

$\phi = S_r \cap \mathrm{Supp}(I,m)_r = \mathrm{Supp}(I,m)_r$!

Def-Prop 2.7 (Hypersurface of Maximal contact).

The maxi contact ideal sheaf of (I,m) is $(T(I),1) := D^{m-1}(I,m)$ $\quad m =$ maxord I.

For any $x \in \mathrm{Supp}(I,m) = \mathrm{Supp}(T(I))$, $\exists$ open neighbor $x \in U_x$, and a smooth element $h \in T(I)(U)$ $\quad$ ($V(h) =: H$ is sm hypersurface on U_x). with $I|_H \neq 0$, we call H a hypersurface of maximal contact.

Rem: x^2+y^3, maxord $= 2$, $\quad D((x^2+y^3)) = (x, y^2)$ $\quad x+cy^2$ is a hy surf of m.c.

Now, $\pi: \underset{\substack{\| \\ U_1}}{Bl_Z U} \longrightarrow U$ a sm blow up with $Z \subseteq Supp(I.m) \cap H$, we have
$Supp(\pi_*^{-1}(I.m)) \subset \pi^{-1}H$.

Proof: $\underset{\substack{\| \\ Supp(I \cdot m)}}{Supp(T(I),1)} \subseteq V(h) = H$.

since $\pi_*^{-1}(h,1) \subseteq \pi_*^{-1}(T(I),1) \subseteq (T(\pi_*^{-1}(I)),1)$

$\Rightarrow Supp(\pi_*^{-1}I, m) = Supp(T(\pi_*^{-1}I),1) \subseteq supp(\pi_*^{-1}h.)) = h^{-1}H$.

Rem! the maximal contact hypersurface is local and depends on choice of h.

(That is where $\mathcal{H}(I)$ plays a role).

Lem 2.8 Let $(X, I, m\ E)$ be a marked triple, $m = \max ord\ I$.

for any $u, v \in T(X,m)_x$ at $x \in Supp(I.m)$ that are smooth and snc with E. Then we have automorphism $\widehat{\phi_{uv}}$ of $\widehat{X_x} = Spec\ \widehat{\mathcal{O}_{x,X}}$

s.t.
(1) $\widehat{\phi^*_{uv}}(\mathcal{H}(I))_x = (\mathcal{H}\ I)_x$
(2) $\widehat{\phi^*_{uv}}\ E = E$
(3) $\widehat{\phi^*_{uv}}(u) = v$
(4) $Supp(\hat{I}.m) = V(T(\hat{I},m))$ is in the fixed point set of $\widehat{\phi_{uv}}$.

Proof: Step 1 construction.

Take $u = u_1, u_2, \cdots, u_n$ s.t. both u or v, $u_2, u_3 \cdots u_n$ form local coordinates and is compatible with E.

Set $\widehat{\phi_{uv}}(u) = v$ $\quad \widehat{\phi_{uv}}(u_i) = u_i$ for $i > 0$.

Step 2: Varification.

Let $h := v - u \in T(I)$. $\quad \forall f \in \hat{I}$

$$\begin{aligned}\widehat{\phi^*_{uv}} f &= f(u_1 + h, u_2, \cdots, u_n) \\ &= f(u_1, \cdots, u_n) + \frac{\partial f}{\partial u_1} h + \frac{1}{2!}\frac{\partial^2 f}{\partial u_1} h^2 + \cdots \\ &\subseteq \hat{I} + \widehat{DI} \cdot \widehat{TI} + \cdots + \underset{\substack{\psi \\ \frac{\partial^i f}{\partial u_i}}}{\widehat{D^i I}} \cdot \underset{\substack{\psi \\ h^i}}{\widehat{TI}^i}\end{aligned}$$

$\Rightarrow \widehat{\phi^*_{uv}}\ \hat{I} \subset H\hat{I}$. Similarly $\widehat{\phi^*_{uv}}(D^i\hat{I}) \subset \mathcal{H}\ D^i\hat{I}$ $\quad \widehat{\phi^*_{uv}}\ \widehat{T(I)} \subset \underset{\substack{\vee \\ \widehat{T(I)}}}{\widehat{\mathcal{H}(T(I))}}$

to sum up, $\hat{\phi}^*_{uv}(D^i\hat{I}\cdot\widehat{T(I)}^i) \subset D^i\hat{I}\cdot\widehat{T(I)}^i + \cdots + D^{m-1}\hat{I}\cdot\widehat{T(I)}^{m-i-1}\cdot\widehat{T(I)}^{m_j} \subset H(\hat{I})$.

$\Rightarrow\ \hat{\phi}^*_{uv}\mathcal{H}(\hat{I}) \subset \mathcal{H}(\hat{I})$. Noetherian properties garantees that

$\hat{\phi}^{*\,n}_{uv}(\mathcal{H}(\hat{I})) = \hat{\phi}^{*\,n-1}_{uv}(\mathcal{H}(\hat{I})) \Rightarrow$ (1) ✓.

(2) (3) ✓ by construction

(4) $h=0$ is fixed by $\hat{\phi}^*_{uv}$ $\Rightarrow$ $Supp(T(I),1)$ is fixed $\Rightarrow$ $Supp(I,m)$ fixed.

Formal local uniqueness imply étale equivalence.

Lem 2.9 Settings as in Lem 2.8.
Then there exists étale neighborhoods

$$\phi_u, \phi_v : \underset{\tilde{x}}{U} \to \underset{x}{X} \quad \text{of } x = \phi_u(\tilde{x}) = \phi_v(\tilde{x})$$

$$\begin{tikzcd} \tilde{x}\in U \arrow[r, bend left, "\phi_u"] \arrow[r, bend right, "\phi_v"'] & X \ni x \end{tikzcd}$$

s.t. (1) $\phi_u^*(X, \mathcal{H}(I), m, E) = \phi_v^*(X, \mathcal{H}(I), m, E) := (\tilde{X}, \mathcal{H}(\tilde{I}), m, \tilde{E})$

(2) $\phi_u^*(u) = \phi_v^*(v)$

(3) $\mathbb{B}: X_r \to \cdots \to X_0$ be a seq of sm blow-up with Z_i in $Supp(I, m)$

then $\phi_u^*\mathbb{B}(X, \mathcal{H}(I), m, E) = \phi_v^*\mathbb{B}(X, \mathcal{H}(I), m, E) : \tilde{X}_r \to \cdots \to \tilde{X}_0$

$\phi_{ui}, \phi_{vi}: \tilde{X}_i \to X_i$ satisfies

$\phi_{ui}^{-1}(V(u), i) = \phi_{vi}^{-1}(V(v), i)$ and

$\phi_{ui}(\tilde{y}_i) = \phi_{vi}(\tilde{y}_i)$ $\forall\, \tilde{y} \in Supp(\tilde{I}_i, m)$.

Remark: Lemma 2.9 allow us to glue restricted resolution! $\forall x \in X$.
that is, $\forall\, U_{(u,x)}$ and $U_{(v,x)}$ two open set that restrict to $V(u)$, $V(v)$
and def blow up seq and lift to Blow up Seq $B_u(U_{(u,x)})$ $B_v(U_{(v,x)})$

$$\exists \quad \begin{tikzcd} U_{(uv,x)} \arrow[r, bend left, "\phi_u"] \arrow[r, bend right, "\phi_v"'] & U_{(u,x)} \cap U_{(v,x)}. \end{tikzcd}$$

s.t. $\phi_u^* B(U_u \cap U_v) = \phi_v^* B(U_u \cap U_v)$.

$\Rightarrow$ restricted to $U_u \cap U_v$, the blow up center for $B_u(U)$, $B_v(U)$ coincide!

We can glue blow up center and globalize it as in $\underline{\underline{LL}}$.

and sm func preserved.

§ 3. Algorithm of Resolution

Now we are ready to prove

$$\text{Ord II in dim} \le n-1 \xrightarrow{T_1} \text{Ord I in dim } n \xrightarrow{T_2} \text{Ord II in dim } n.$$

div smooth !

T_1: We start from (X, I, E) $E = (E^1, \cdots, E^s)$ maxord $I \le m$

If maxord $< m$, the process is trivial. We assume maxord $I = \boxed{m}$

Step 1: Construct $\Pi_1 : (X_{r_1}, I_{r_1}, E_{r_1}) \to \cdots \to (X, I, E)$ s.t.

$\mathrm{Supp}\, \Pi_{1*}^{-1} E \cap \mathrm{Supp}\,(I_{r_1}, m) = \phi$.

Consider the equivalent ideal $\boxed{C(I)}$ now

1.1. Let Z_0 be the union of all irr comp of E^1 contained in $\mathrm{Supp}(I, m)$.

Blow up Z_0, we have (for irr comp E^{1k} in E^1 for example) $\Pi_0 : X_1 \to X$

$$\mathrm{maxord}_{E^{1k}} I = m \Rightarrow \max_{\Pi_{0*}^{-1} E^{1k}} \Pi_0^{-1} I \le m - m = 0.$$

$\Rightarrow \Pi_{0*}^{-1} E^{1k}$ and $\mathrm{supp}\,(I_{r_1}, m)$ are disjoint.

1.2. Now, set $S = E^1$, $E_S = (E - E^1)|_S = (\phi, E^2|_S, \cdots, E^s|_S)$

consider $(S, I|_S, m, E|_S)$ apply Ord II in dim $\le n-1$, we get

$BO_1 : (S_{E_1}, I_{E_1}|_{S_{E_1}}, m, E_{S_{E_1}})$ lift $(X_{r_{E_1}}, I_{r_{E_1}}, m, E_S) \to \cdots \to (\cdots)$

s.t. $\mathrm{supp}\,(I_{r_{E_1}}, m) \cap \Pi_{E_1 *}^{-1} S = \phi$

$$\Pi_{E_1 *}^{-1} S = \Pi_{E_1 *}^{-1} E_1$$

Inductively, we have $\Pi_1 : (\cdots) \to \cdots (\cdots)$ s.t. $\mathrm{Supp}\, \Pi_{1*}^{-1} E \cap \mathrm{Supp}\,(I_{r_1}, m) = \phi$

Functoriality follows from previous remark.

Step 2: Start from $J = H(C(I_{r_1}))$, $Y = X_{r_1}$, $F = (\Pi_*^{-1} E, E^{s+1}, \cdots, E^{s+r_1})$

excep.

(Y, J, m, F).

$\forall y \in \mathrm{Supp}\,(J, m)$, $\exists U_y$. $h \in T(J)$ Since in step 1, all blow-up is snc,

we can take h s.t. H_h snc with F ($\Pi_*^{-1} E$ away from $\mathrm{supp}(I, m)$). ※

locally consider $(H_h, J|_{H_h}, m, F|_{H_h})$

Apply Ord II in dim $< n$ and lift it and globalized it to

$$\Pi_2 : (X_{r_2}, I_{r_2}, E_{r_2}) \to (X_{r_1}, I_{r_1}, E_{r_1})$$

s.t. $\mathrm{Supp}\,(I_{r_2}, m) = \phi$!

(Note, all blow up seq is also for $C(I)$ and I, $TH(C(I)) = C(I)$)

☆ ↑ ✓

Functoriality follows from previous rem.

□.

T_2: We start with a marked triple (X, I, m, E) $E = (E^1, \cdots, E^s)$.

Step 0:

We may write $I = N(I) M(I)$, $M(I) = \mathcal{O}_X(-\sum c_i E_i)$ and $\operatorname{Supp} N(I)$ does not contain any of E^i!

Rem: if $E = \phi$, $M(I) = \phi$ and $I = N(I)$.

Step 1: Write $(N(I), 1) + (I, m) = (J, s)$ here s a number.

Write $m_J = \operatorname{maxord} J$, Run Ord I to (X, J, E) with m_J.

we get (X_1, J_1, E_1) s.t. $\operatorname{maxord} J_1 = m_{J_1} < m_J$.

(Note, $\operatorname{Supp}(J, m_J)_k \subseteq \operatorname{Supp}(J, s)_k = \operatorname{Supp}(N(I), 1)_k \cap \operatorname{Supp}(I, m)_k$).

Inductively we get

(X_r, J_r, E_r) s.t. $\operatorname{maxord} J_r < s$.

this implies $\phi = \operatorname{Supp}(J, s)_r = \operatorname{Supp}(N(I), 1)_r \cap \operatorname{Supp}(I, m)_r$

$$\operatorname{Supp}(N(I), 1)_r = \operatorname{Supp}(N(I)_r).$$

To sum up, we have $(X_r, I_r, m, E_r) \to \cdots \to (X, I, m, E)$

s.t. $N(I)_r \cap \operatorname{Supp}(I, m)_r = \phi$.

Rem: $N(I)_r$ $N(I_r)$ differs by some exceptional comps E^k $(k > s)$

$M(I)_r$ $M(I_r)$ and is contained in $M(I_r)$.

Step 2: $I = M(I) = \mathcal{O}_X(-\sum a_j E^j)$ $E = (E^1, E^2, \cdots, E^s, \cdots)$.

2.1 $\operatorname{Sub}\{E^1, E^2, \cdots, E^s, \cdots\}$ has a lexicographic order.

$(x \cdots$ $)$.

2.2. $\forall x \in X$, set $p(x) = (\{E^{j_1}, \cdots, E^{j_k}\})$ the maximal subset (in above order) satisfying

(1) E^{j_i} pass x $\forall$ $1 \leq i \leq k$.

(2) $\sum_k a_{j_i} \geq m$ (3) $a_{j_1} + \cdots + \hat{a}_{j_i} + \cdots + a_{j_k} < m$.

$D_{p(x)} = \bigcap_{i=1} E^{j_i}$, and it is the locus that is a maximal component of $\operatorname{Supp}(I, m)$.

$\nu = (\operatorname{maxord} I$, member of maximal comp of $\operatorname{supp}(I, m)$ attain maxord$)$.

$= (m, n)$

Each time, we blow up $D_{p(x)}$, $D_{p(x)} = \{x_{j_1} = \cdots = x_{j_K} = 0\}$.

$\forall x \in D_{p(x)}$, since $\sum_{i=1}^{K} a_{j_i} - m < a_{j_i}$ $\forall 1 \le i \le K$,

$I_x = \prod_t x_t^{a_t}$

$\parallel$

$a_{j_1} + \cdots + \widehat{a_{j_i}} + \cdots + a_{j_K} - m + a_{j_i}$

and $\pi_*^{-1} I_x \underset{\text{in coord } x_{j_i}}{=} \prod_{t \ne j_i} x_t^{a_t} \cdot x_{j_i}^{\underline{\sum a_{j_i} - m}}$ $< a_{j_i}$

$\Rightarrow \operatorname{ord}_y \pi_*^{-1}(I) < \operatorname{ord}_x I$ $\forall$ $\pi(y) = x$.

$\Rightarrow$ ν decrease strictly in the lexi-cographic order $\Rightarrow$ the procedure terminates with $\nu = (<m, *)$. $\Rightarrow$ max ord $I_r < m$ eventually.

Functoriality follows from the sm invariant property of $p(x)$ and ν.

$\square$.

OCE: Commute resp to closed embedding or Ord II $m=1$ $E = \phi$

I_S

$\tau : S \to X$

Now, $\tau_* I_S$ in X we have locally
$\tau_* I_S$ is adding some smooth element $\{u\}$.

$\operatorname{Ord}(\tau_* I_S, 1) = 1$ $T(\tau_* I_S) = \tau_* I_S \Leftarrow$ sm element.

Recall T_1: We restrict to sm hysnf in T to do induction.
so for this case, we just restrict $\tau_* I_S$ to $\{u\}$ and exactly get I, the procedure commute with closed embedding

$\square$