Resolution of Singularities.

§ 0. Preliminarlies and Main Goal.

A variety is an integral seperate scheme of finite type over a field k.

Main Goal: Let X be a variety over a field of char zero. Then there exists a canonical desingularization of X, that is a smooth variety $\widetilde{X}$ and a proj bir morphism

$$res_X : \widetilde{X} \longrightarrow X$$

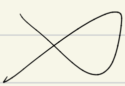
such that

(1) $res_X^{-1}(\mathrm{Sing}\, X)$ is a divisor with simple normal crossings.

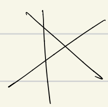
(2) res_X is functorial respect to smooth morphisms and field extension.

Remark: $E = \sum E^i$, E^i irr is called snc (on smooth var)

if E^i is smooth, and for each closed pt, $\exists$ local cors $z_1, \cdots z_n$ s.t. $E^i = (z_{k_i}^{d_i} = 0)$, $E = (\prod_i z_{k_i}^{d_i} = 0)$.



normal crossing



snc.

Remark: Res "functor", associate each object $X \in Var$ an $\widetilde{X} \in Var$ and a bir proj $res_X : \widetilde{X} \to X$.

we say it is functorial resp to sm mor if

$\forall$ $h : Y \to X$ smooth mor, $\exists$ $\tilde{h} : \widetilde{Y} \to \widetilde{X}$ s.t.

the diagram
$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\tilde{h}"'] \arrow[d] & \widetilde{X} \arrow[d, "res_X"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$
is a fiber product.

Prop 1: Let Res be a resolution "functor" that is functorial resp to sm mor, then $\forall$ var X, $res_X : \widetilde{X} \to X$ is an isomorphism over $X \setminus Sing(X)$.

Proof: Let $x \in X$ be a smooth point.

$$\begin{tikzcd} X \ni x \arrow[d] & U \arrow[l, hook', "i"', "\text{open}"] \arrow[r, "\pi"] & \mathbb{A}_k^{\dim X} \arrow[dll] \\ k & & \end{tikzcd}$$

π is étale

Claim 1^*: G is an algebraic group$/k$ char 0, then $res_G : \widetilde{G} \to G$ is an isomorphism.

Assuming claim 1^*:

$$\begin{tikzcd} U \arrow[r, "\pi"] & \mathbb{A}_k^{\dim X} \\ \widetilde{U} \arrow[u, "res_U"] \arrow[r, "\widetilde{\pi}"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{A}_k^{\dim X} \arrow[u, "res = id"'] \end{tikzcd}$$

since [the diagram] is a fiber product

$\Rightarrow$ res_U is an isomorphism.

Now,

$$\begin{tikzcd} x \in X & U \arrow[l, hook'] \\ \widetilde{X} \arrow[u, "res_X"] & U \arrow[l] \arrow[u, "id"'] \arrow[ul, phantom, "\circlearrowright"] \end{tikzcd}$$

$\Rightarrow$ $res_X|_U$ is an iso

$\square$

Proof of claim 1^*: suppose $res_G : \widetilde{G} \to G \ni g_1$ is not iso on g_1.

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{\phi}"] \arrow[d, "res_G"'] & \widetilde{G} \arrow[d, "res_G"] \\ G \arrow[r] & G \\ g_2 \arrow[r, "g_1 \cdot g_2^{-1}"', "= \phi"] & g_1 \end{tikzcd}$$

here g_2 is a general pt s.t. res_G is iso.

$$\Rightarrow 1 \le \dim \widetilde{\phi}^{-1} res_G^{-1}(g_1) = \dim res_G^{-1} \phi^{-1}(g_1) = \dim res_G^{-1}(g_2) = 0$$

$\Rightarrow\Leftarrow$

$\square$

Prop 2: Let Res be a resolution "functor" as above.
Let $X \in Var/k$ char $= 0$, G is an algebraic group acting on X, then the G-action lifts to a G action on $\widetilde{X}$ s.t.

$res_X : \widetilde{X} \to X$ is G-equivariant. i.e.

$res_X(g(\widetilde{x})) = g\, res_X(\widetilde{x})$.

Proof: Note that the G action is a smooth morphism.

i.e. $G \times X \xrightarrow{\phi_G} X$ is smooth.

① Consider $\pi_1 : G \times X \longrightarrow X$ projection.

$$\begin{tikzcd}
\widetilde{(G\times X)} \arrow[r, "\widetilde{\pi}_1"] \arrow[d, "res"'] \arrow[dr, phantom, "\circlearrowright"] & \widetilde{X} \arrow[d] \\
G \times X \arrow[r, "\pi_1"'] & X
\end{tikzcd}$$

is fiber product $\Rightarrow$ $\widetilde{(G\times X)} \cong G \times \widetilde{X}$.

Consider $\phi_G : G\times X \to X$.

$$\begin{tikzcd}
G\times\widetilde{X} \arrow[r, "\widetilde{\phi}_G"] \arrow[d] & \widetilde{X} \arrow[d] \\
G \times X \arrow[r, "\phi_G"'] & X
\end{tikzcd}$$

we now show that $\widetilde{\phi}_G$ gives a group action on $\widetilde{X}$.

②.

$$\begin{tikzcd}[row sep=small, column sep=small]
G\times G\times X \arrow[rr, "Id\times\phi_G"] \arrow[dd, "m_G\times Id"'] & & G\times X \arrow[dd, "\phi_G"] & \\
& G\times G\times\widetilde{X} \arrow[ul] \arrow[rr] \arrow[dd] & & G\times\widetilde{X} \arrow[ul] \arrow[dd] \\
G\times X \arrow[rr] & & X & \\
& G\times\widetilde{X} \arrow[ul] \arrow[rr, "\phi_G"] & & \widetilde{X} \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
G\times G\times\widetilde{X} \arrow[r] \arrow[d] & G\times\widetilde{X} \arrow[d] \\
G\times\widetilde{X} \arrow[r] & \widetilde{X}
\end{tikzcd}$$

We know, the diagram commutes over $X \setminus Sing\, X$.

$$\begin{tikzcd}
g_1\times g_2\times \underset{\widetilde{x}\in}{\widetilde{X}} \arrow[r] \arrow[d] \arrow[dr, bend left, "\phi_1"] \arrow[dr, bend right, "\phi_2"'] & g_1\times g_2\widetilde{X} \arrow[d] \\
g_1g_2\times\widetilde{X} \arrow[r] & \widetilde{X}
\end{tikzcd}$$

$\forall\, \widetilde{x} \in \widetilde{X}$, $\exists$ a general smooth curve not in $res^{-1}(Reg\, X)$.
denoted as $\widetilde{C} \ni \widetilde{x}$,

$\phi_1(C) = \phi_2(C) \,/\, (X \setminus Sing\, X) \overset{\text{by valuative criteria}}{\Longrightarrow} \phi_1(C) = \phi_2(C)$

$\Rightarrow \phi_1(\widetilde{x}) = \phi_2(\widetilde{x})$

$\square$.

One of the key idea is, instead of considering the resolution problem, we consider the "Principalization" problem.

Easy version, weak.

PI: Let X be a smooth variety over $/k$ char $=0$, $I \subset \mathcal{O}_X$ non-zero ideal sheaf.
Then $\exists\ f: \tilde{X} \to X$ proj bir, $\tilde{X}$ sm, such that

$$f^{-1}I \cdot \mathcal{O}_{\tilde{X}} \subset \mathcal{O}_{\tilde{X}} \text{ is an invertible ideal sheaf.}$$

Rem: $f^{-1}I \cdot \mathcal{O}_{\tilde{X}}$ is the image of f^*I under $f^*\mathcal{O}_X \to \mathcal{O}_{\tilde{X}}$.

Cor I: (Elimination of indeterminacies).
Let X be a smooth variety $/k$ char $=0$, $g: X \dashrightarrow \mathbb{P}$ a rational map to some proj space. Then $\exists$ sm var $\tilde{X}$ and proj bir mor $f: \tilde{X} \to X$ such that $g \circ f: \tilde{X} \to \mathbb{P}$ is a morphism.

$$\begin{tikzcd} \tilde{X} \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ X \arrow[r, dashed, "g"'] & \mathbb{P} \end{tikzcd}$$

Proof: Since $\mathbb{P}$ is projective, $\exists\ Z \subseteq X$ with codim ≥ 2 s.t. $g: X \setminus Z \to \mathbb{P}$ is a morphism. (Valuative criterion for properness)
By algebraic Hartogs thm, $g^*\mathcal{O}(1)|_{X\setminus Z}$ extends uniquely to a linebundle on X, denoted as L.
Let $J \subset L$ be a subsheaf generated by $g^*H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, $I = J \otimes L^{-1} \subset \mathcal{O}_X$
Take $f: \tilde{X} \to X$ s.t. $f^{-1}I \cdot \mathcal{O}_X$ is invertible ideal sheaf.
Since $f^*I = f^*J \otimes (f^*L)^{-1} \Rightarrow f^*J = f^*I \otimes f^*L$.

$\tau: f^*I \hookrightarrow \mathcal{O}_{\tilde{X}}$ $f^{-1}I \cdot \mathcal{O}_{\tilde{X}}$ that defines $f^{-1}I \cdot \mathcal{O}_{\tilde{X}}$

$\tau \otimes f^*L$: $f^*I \otimes f^*L = f^*J \twoheadrightarrow \underline{f^{-1}I \cdot \mathcal{O}_{\tilde{X}} \otimes f^*L}$. invertible.

$\Rightarrow \mathrm{Im}(f^*J)$ is a subsheaf of L' generated by $(g\circ f)^* H(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \Rightarrow$ No base locus on $\tilde{X} \Rightarrow$ defines a mor.

Before we state a stronger version of "principalization", let's recall some notations.

Smooth blow-up: $Z \overset{\text{closed}}{\subset} X$, $\pi : Bl_Z : \widetilde{X} \to X$

We say π is a smooth blow-up if X, Z are both smooth, Z sm in X.

We say π is trivial if Z is Cartier, π is iso.

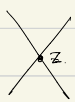
We say π is empty if $Z = \phi$, π is iso.

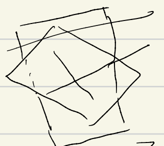
snc — center: E is a snc divisor, $E = \sum E_i$, $Z \subset X$

We say Z is snc with E if $\exists$ local cor system $\{z_1, \dots, z_n\}$

s.t. $Z = (z_{j_1} = \cdots = z_{j_s} = 0)$ $\quad E_i = (z_{c(i)} = 0)$

If $Z \not\subset \operatorname{Supp} E$, then $E|_Z$ is snc.





$$\begin{array}{ccccccccc} & & Z_{n-1} & & & & Z_1 & & Z_0 \\ & & \cap & & & & \cap & & \cap \\ \text{Let } \pi : X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X_0 = X \end{array}$$

be a seq of blow-ups of sm variety X, E snc div on X, Z_i is snc with E if

$Z_i \subset X_i$ is snc with $\pi_{i*}^{-1} E + \sum_{j<i} \pi_{ij}^{-1}(Z_j)$ (snc).

Here $\pi_i : X_i \to X$ $\quad \pi_{ij} : X_i \to X_j$.

When $E = \phi$, Z_i is snc with exceptional set.

Now we state a stronger version.

PII: Let X be a sm variety $/k$ char $=0$, $I \subset \mathcal{O}_X$ a nonzero ideal sheaf, E snc div on X, $\exists$ seq of sm blow-ups

$$\pi : \tilde{X} = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

whose center has snc with E, such that (center smooth)

☆ (1) $\pi^{-1} I \cdot \mathcal{O}_{\tilde{X}}$ is the id' sheaf of a snc div, and (snc with excep).

(2) π is functorial resp to sm morphism.

Rem: (2) gaurantees that $\pi|_{X \setminus \mathrm{Supp}\, I}$ is an iso.

CorII: (Non functorial "weak" "Embedded" Resolution of Sing)

Let Y be a closed subvariety of a sm variety X/k char 0. Then there is a bir proj mor $\pi : \tilde{X} \to X$ such that π is iso near η_Y i.e.

$\pi|_Y : \tilde{Y} \to Y$ proj bir, and.

$\tilde{Y}$ has snc with $\sqcup$ excep divs on $\tilde{X}$.

$\downarrow$ smooth. $\longrightarrow (\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$

(Not sure $\pi|_Y$ is iso over $Y \setminus \mathrm{Sing}\, Y$, $(\pi|_Y)^{-1}(\mathrm{Sing}\, Y)$ snc on $\tilde{Y}$).

Proof: (of CII assuming PII)

Let I_Y be the ideal sheaf of $Y \subseteq X$. Let

$\pi_r : X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ be a seq of sm blow-ups whose centers are snc with ϕ as in PII such that $\pi_r^{-1} I_Y \cdot \mathcal{O}_{X_r}$ is principal.

① If π_r is an iso over $\eta_Y \subseteq X$, since $\pi^{-1} I_Y \cdot \mathcal{O}_{X_r}$ is an snc div, we can find an irr comp of $\mathrm{Supp}\, \pi^{-1} I_Y \cdot \mathcal{O}_{X_r}$, denoted as $\tilde{Y}$ and $\tilde{Y} \to Y$ is bir, $\tilde{Y}$ is smooth.

② If π_r is not an iso ~~over~~ η_Y. $\exists j$ s.t $\eta_Y \subseteq Z_j$ and $\eta_Y \nsubseteq Z_i$ for $i < j$. Since π is iso over $X \setminus Y$,

and $\pi_j(Z_j) \subseteq \mathrm{Supp}(I_Y) = Y \Rightarrow \exists$ irr comp of Z_j, denoted as $\tilde{Y}$

s.t. $\tilde{Y} \to Y$ bir.

Remark: ① In PII, only $\pi : \tilde{X} \to X$ is "functorial" resp to sm mor, but we we choose middle blow up $\pi_j : X_j \to X$, "func" fails.

Let's introduce some notations and definitions.

Def 1: (Blow-up sequence) Let X be a var, a blow-up seq of length r starting with X is a chain of morphisms

$$\begin{tikzcd} \mathbb{B}: \quad \pi: X_r \arrow[r, "\pi_{r-1}"] & X_{r-1} \arrow[r] & \cdots \arrow[r, "\pi_1"] & X_1 \arrow[r, "\pi_0"] & X_0 = X \\ & Z_{r-1} \arrow[u, phantom, "\cup"] & & Z_1 \arrow[u, phantom, "\cup"] & Z_0 \arrow[u, phantom, "\cup"] & Y \arrow[ul] \end{tikzcd}$$

where each π_i is a blow up of X_i with center Z_i.

$\pi_{ij} : X_i \to X_j$ $\qquad \pi_i : X_i \to X$

We say a blow-up seq is smooth if X_i, Z_i are all smooth

We allow trivial and empty blow-ups in the seq X.

Def 2 (Pull back blow-up seq by sm morphism)

$\mathbb{B}$ be a blow-up seq as above. $h: Y \to X$ smooth morphism.

$$\begin{tikzcd} h^*\mathbb{B} \quad h^*\pi: X_r \times_X Y \arrow[r] & X_{r-1} \times_X Y \arrow[r] & \cdots \arrow[r] & X_1 \times_X Y \arrow[r] & X_0 \times_X Y = Y \\ & Z_{r-1} \times_X Y \arrow[u, phantom, "\cup"] & & Z_1 \times_X Y \arrow[u, phantom, "\cup"] & Z_0 \times_X Y \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$h^*\mathbb{B}$ is called the pull back of $\mathbb{B}$ by h.

Rem: ① $h^*\mathbb{B}$ defines a blow up seq. i.e.

$$\begin{tikzcd} Bl_Z X \times_X Y \arrow[r] \arrow[d, "\pi_Y"'] & Bl_Z X \arrow[d, "\pi_X"] \\ Y \arrow[r, "h"] & X \end{tikzcd}$$

π_Y is a blow-up of $h^{-1}Z$.

In general, h not smooth, $h^*\mathbb{B}$, see examp

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Bl_0\mathbb{P}^2 \arrow[d] \\ Bl_0\mathbb{P}^2 \arrow[r, "h"] & \mathbb{P}^2 \end{tikzcd}$$

$S \to Bl_0\mathbb{P}^2$ obviously not a blow up $\qquad S = Bl_0\mathbb{P}^2 \times_{\mathbb{P}^2} Bl_0\mathbb{P}^2$.

② $\mathbb{B}$ is smooth, then $h^*\mathbb{B}$ is smooth blow-up
③ h is not surjective, $h^*\mathbb{B}$ may contain extra empty blow-up

Def 3 (Restriction to closed subvariety)
Let $\mathbb{B}$ as above, $j: S \to X$ is a closed emb. def

$$\mathbb{B}|_S = \begin{array}{ccccccccc} S_r & \to & S_{r-1} & \to & \cdots & \to & S_1 & \to & S_0 = S. \\ & & \cup & & & & \cup & & \cup \\ & & Z_{r-1} \cap S_{r-1} & & & & Z_1 \cap S_1 & & Z_0 \cap S_0 \end{array}$$

here we need $\eta_S \notin Z_j$, (In fact we require all Z_i has image strictly contained in S in application)

Def 4 (Push forward resp to closed embedding)
$j: S \to X$ closed embedding.

$$\mathcal{B}_S : \Pi: \begin{array}{ccccccc} S_r \to & S_{r-1} & \to \cdots \to & S_1 & \to & S_0 = S \\ & \cup & & \cup & & \cup \\ & Z^S_{r-1} & & Z^S_1 & & Z^S_0 \end{array}$$ blow-up seq for S.

define
$j_* \mathcal{B}_S$ as $j_* \Pi$:

$$\begin{array}{ccccccccc} X_r \to & X_{r-1} & \to \cdots \to & X_2 & \to & X_1 & \to & X_0 = X \\ & \cup & & \cup & & \uparrow & & \cup \\ & Z^S_{r-1} & & Z^S_2 & & S_1 & & Z^S_0 \\ & & & & & \cup & & \\ & & & & & Z^S_1 & & \end{array}$$

Remark: if $\mathbb{B}(S)$ is smooth, then $j_* \mathcal{B}(S)$ is smooth.

Now we consider a triple (X, I, E), where
X is smooth var, I ideal sheaf $\subseteq \mathcal{O}_X$, div $E \subseteq X$

$\mathbb{B}(X, I, E)$ $\quad \Pi: X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$

$I_i = \Pi_i^{-1} I \cdot \mathcal{O}_{X_i} \qquad E_i = \Pi_{i*}^{-1} E$ on X_i

$X_i \times_X Y \quad h^*(I_i) = h_i^{-1} I_i \cdot O_{X_i \times_X Y}$

$$\begin{array}{ccc} & & E_i \times_X Y \\ \downarrow & & \downarrow \\ \underline{X_i} & I_i & E_i \end{array}$$

We extend push forward, pull back for triple.

① $h: Y \to X$ smooth, $\boxed{h^* B(X, I, E)}$: as $h^*(I_i) = h_i^{-1} I_i \cdot O_{X_i \times_X Y}$

$h^*(E_i) = E_i \times_X Y$

② $j: S \to X$ is a closed embedding. $j_* B_S(X, \underset{I_S}{I_S}, E_S)$

$E_S \subset S \subset X \qquad E_{S_i} \subset S_i \subset X_i$ natural.

def : $j_* I_S \cdot O_X = (j^\#)^{-1}(j_* I_S) \qquad j^\# : O_X \to j_* O_S \xleftarrow{} j_* I_S$

$O_X / j_* I_S \cdot O_X = j_*(O_S / I_S)$

locally : $j: \operatorname{Spec} R/A \to \operatorname{Spec} R. \qquad \varphi: R \to R/A$, $\bar{I}$

$j_* \bar{I} \cdot O_{\operatorname{Spec} R} = \varphi^{-1}(\bar{I})$

define $j_*(I_{S_i}) = j_{i*} I_{S_i} \cdot O_{X_i} \qquad j_i : S_i \to X_i$

Def 5 : (Functorial Package)

① $B(X, I, E)$ commute with smooth mor $h: Y \to X$ if

$\underline{h^* B(X, I, E)}$ is an extension $\underline{B(Y, h^{-1} I \cdot O_Y, h^{-1} E)}$.

extension : $h^* B$ is $B(Y, h^{-1} I \cdot O_Y, h^{-1} E)$ by adding some empty blow-ups.

② $B(X, I, E)$ commutes with closed embedding if

$I_S \quad S \xrightarrow{j} X \supset E$

$$B(X, j_* I_S \cdot O_X, E) = j_* B(S, I_S, E|_S)$$

$$\Rightarrow j^*(X, j_* I_S \cdot O_X, E) = B(S, I_S, E|_S)$$

Now we state the final principalization thm.

$P\mathrm{III}$: For any triple (X, I, E) where X is sm var / k char $= 0$, E snc div on X, $I \subset \mathcal{O}_X$ ideal sheaf, then there exist a smooth blow-up seq functor $\mathcal{B}(X, I, E)$, such that all centers of blow-ups are snc with E, and

(1) $\pi^{-1} I \cdot \mathcal{O}_X$ is an ideal sheaf of snc div,

(2) $\mathcal{B}$ commutes with smooth morphisms,

(3) $\mathcal{B}$ commutes closed embeddings whenever $E = \phi$.

(4) $\mathcal{B}$ commutes with field extensions (seperable)

$C\mathrm{III}$: (Functorial Strong "Embedded" Resolution of Sing)

Let Y be a subvar of sm var X/k char 0. Then there exists a seq of smooth blow ups $\pi : \tilde{X} = X_r \to \cdots \to X_1 \to X_0 = X$ with centers snc with ϕ, such that

(1) $\pi|_{\tilde{Y}} : \tilde{Y} \to Y$ is a proj bir with $\tilde{Y}$ main comp of $\pi^{-1}(Y)$, here π is iso near η_Y.

(2) $\pi|_{\tilde{Y}}$ is iso over $(Y \setminus \mathrm{Sing}\, Y)$, $\pi^{-1}(\mathrm{Sing}\, Y)$ snc on $\tilde{Y}$.

(3) π commutes with sm morphisms

(4) π commutes with closed embeddings.

(5) π commutes with field extensions.

Rem: The above resolution depends on how we embed varieties, and can only resolve sings of varieties that can be embedded into sm varieties

Problem 1: Not every var has an embedding to sm var.

Problem 2: To remove the influence of embedding, we need to glue resolutions, but the glued mor may not be projective.

Problem 3: To ensure canonicity, can we find "canonical embedding"?