Now we state the final principalization thm.

PIII : For any triple $(X, I, \boxed{E})$ where X is sm var / k char $= 0$, E snc div on X, $I \subset \mathcal{O}_X$ ideal sheaf, then there exist a smooth blow-up seq functor $\mathcal{B}(X, I, E)$, such that all centers of blow-ups are snc with E, and

(1) $\pi^{-1} I \cdot \mathcal{O}_X$ is an ideal sheaf of snc div,

(2) $\mathcal{B}$ commutes with smooth morphisms,

(3) $\mathcal{B}$ commutes closed embeddings whenever $\boxed{E = \emptyset}$

(4) $\mathcal{B}$ commutes with field extensions (seperable)

CIII : (Functorial Strong "Embedded" Resolution of Sing)

Let Y be a subvar of sm var X / k char 0. Then there exists a seq of smooth blow ups $\pi : \widetilde{X} = X_r \to \cdots \to X_1 \to X_0 = X$ with centers snc with $\emptyset$, such that

(1) $\pi|_Y : \widetilde{Y} \to Y$ is a proj bir with $\widetilde{Y}$ main comp of $\pi^{-1}(Y)$, here π is iso near η_Y.

(2) $\pi|_Y$ is iso over $(Y \setminus \mathrm{Sing}\, Y)$, $\pi^{-1}(\mathrm{Sing}\, Y)$ snc on $\widetilde{Y}$.

(3) π commutes with sm morphisms

(4) π commutes with closed embeddings.

(5) π commutes with field extensions.

Rem: The above resolution depends on how we embed varieties, and can only resolve sings of varieties that can be embedded into sm varieties

Problem 1: Not every var has an embedding to sm var.

Problem 2: To remove the influence of embedding, we need to glue resolutions, but the glued mor may not be projective.

Problem 3: To ensure canonicity, can we find "canonical embedding"?

$P\mathrm{III} \Rightarrow C\mathrm{III}$: As in the proof of $P\mathrm{II} \Rightarrow C\mathrm{II}$, we have a seq of smooth blow-ups $\Pi: \widetilde{X} = X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$ which appears as a subseq of the "principalization blow-up seq" of (X, I_Y, ϕ). Such that Π is an iso near η_Y, and Π induce a resolution $\Pi|_Y: \widetilde{Y} \to Y$.

(3)-(5) follows directly from $P\mathrm{III}$ (2)-(3).

Now, for (2) $\forall y \in Y \subset X$ smooth point of Y, $\exists\, y \in U \subset X$ s.t.

$$(\mathbb{A}^n, I_0 = \langle x_1, x_2, \cdots, x_k \rangle) \xleftarrow[\text{étale}]{\pi} (U, I_Y|_U)$$

$$I_Y|_U = \pi^{-1} I_0 \cdot \mathcal{O}_U.$$

$\Rightarrow$ take $y' \in Y \subset X$ s.t. $\Pi|_Y$ is an iso near $y' \in Y$

Functoriality
$$\begin{tikzcd} & (U, I_Y|_U) \arrow[dl, "\pi"'] \\ (\mathbb{A}^n, I_0) & \\ & (U', I_Y|_{U'}) \arrow[ul, "\pi'"] \end{tikzcd}$$
$\Rightarrow \Pi|_Y$ is iso near $y \in Y$.

so $\Pi|_Y$ is iso $/(Y \setminus \operatorname{Sing} Y)$.

$\Pi^{-1}(\operatorname{Sing} Y)$ is snc, this follows from all blow up center is snc with ϕ.

L1 (Gluing Lemma)

Let $\mathcal{B}$ be a blow-up seq "functor" defined for affine varieties over k of char 0, such that commutes with smooth morphism, then $\mathcal{B}$ has a unique extension to all varieties.

Proof: Let X be a variety $/k$, $\{U_i\}$ an affine cover

For each i, $\mathcal{B}$ assigns a center Z_{i0} on U_i s.t. the first blow up is $Bl_{Z_{i0}} U_i \longrightarrow U_i$ for $\mathcal{B}(U_i)$.

Consider $\phi_i^\circ: U_{ij} \to U_i$ $\quad \phi_j^\circ: U_{ij} \to U_j$ $\quad (U_{ij} = U_i \cap U_j)$

$$\phi_i^{\circ *}\mathcal{B}(U_i) = \mathcal{B}(U_{ij}) = \phi_j^{\circ *}\mathcal{B}(U_j)$$

$\Rightarrow Z_{i0}|_{U_{ij}} = Z_{j0}|_{U_{ij}} = Z_{ij0}$ $\Rightarrow$ this gaurantees the projectivity. we can glue all Z_{i0} naturally to Z_0 on X, and get a "canonical" glued blow-up $Bl_{Z_0} X \to X$.

This process does not depend on choice of $\{U_i\}$, since for other cover $\{V_i\}$, then we can repeat above process to $\{U_i\} \cup \{V_i\}$, this solves the problem.

Repeat the process, we can construct a seq of blow up functor for X, and functoriality for sm morphism and field extension follows from the construction.

$\square$.

Rem: For any $h: Y \to X$, we can write it as $h|_{U_i}: h^{-1}(U_i) \to U_i$, and apply gluing argu).

L2 (Local canonicity of embedding)

Let X be an affine variety X and $i_1 X \hookrightarrow A^n$, $i_2 X \hookrightarrow A^m$ two closed embeddings then we have a further embedding

$$i_1' : X \xrightarrow{i_1} A^n \longrightarrow A^{n+m} \quad \text{and} \quad i_2' : X \xrightarrow{i_2} A^m \longrightarrow A^{n+m}$$

$$x \to i_1(x) \to (i_1(x), 0) \qquad x \to i_2(x) \to (0, i_2(x))$$

such that i_1' and i_2' are equivalent under a (nonlinear) automorphism of A^{n+m}.

Proof: We extend i_1 to $j_1: A^m \to A^n$, the extension is not unique.

i_2 to $j_2: A^n \to A^m$.

$$\begin{tikzcd} & X \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ A^n \arrow[rr, shift left, "j_2"] & & A^m \arrow[ll, shift left, "j_1"] \end{tikzcd} \Rightarrow \begin{tikzcd} R_X & k[y^m] \arrow[l, two heads, "\varphi"'] \\ k[x^n] \arrow[u, two heads, "\phi"] \arrow[ur, "j^\#"'] & \end{tikzcd}$$

: $\forall x_i$, take $y_i \in \varphi^{-1}\phi(x_i)$

def $j^\#: x_i \to y_i$..

Let $\vec{x}$ be cords on A^n, $\vec{y}$ on A^m.

$$\varphi_1: (\vec{x}, \vec{y}) \longrightarrow (\vec{x}, \vec{y} + j_2(\vec{x})) \qquad A^{n+m} \to A^{n+m}$$

$$\varphi_2: (\vec{x}, \vec{y}) \longrightarrow (\vec{x} + j_1(\vec{y}), \vec{y}).$$

$$X \xrightarrow{i_1'} A^{n\times m} \xrightarrow{\varphi_1} A^{n\times m}$$

$$\vec{x} \qquad (i_1(x), 0) \qquad (i_1(x), j_2 i_1(x) = i_2(x)).$$

$$X \xrightarrow{i_2'} A^{n\times m} \xrightarrow{\varphi_2} A^{n\times m}$$

$$\vec{x} \qquad (0, i_2(x)) \qquad (i_1(x), i_2(x)).$$

(In char 0, k is automatically an infinite field).

L3: Let $h: Y \to X$ be a smooth morphism, $y \in Y$ a closed point and $i: X \to A_X$ closed embedding to sm var. Then $\exists$ open set $f(y) \in A_X^\circ \subseteq A_X$, a smooth affine var A_Y° with a smooth morphism

$$h_A: A_Y^\circ \to A_X^\circ$$

Set $Y^\circ = h_A^{-1}(X \cap A_X^\circ) \ni y$, it has a closed embedding $Y^\circ \xrightarrow{j} A_Y^\circ$, such that the following diagram commutes and is fiber product.

$$\begin{tikzcd} Y^\circ \arrow[r, "j"] \arrow[d, "h"'] & A_Y^\circ \arrow[d, "h_A"] \\ X^\circ \arrow[r, "i"] & A_X^\circ \end{tikzcd}$$

$\text{aff} \to A^N$

$\downarrow$

Proof: The problem is local, we assume X, Y, A_X affine, $\underline{Y \subset X \times A^N}$.

If h is rel of dim d, consider the closed pt $x = h(y)$, by taking general projection $\sigma: A^N_x \to A^{d+1}_x$, we may assume $h^{-1}(x) \to A^{d+1}_x$ is finite mor and is an embedding near $y \in h^{-1}(x)$. (Need k to be infinite field).

Now, shrinking Y and X, we may assume Y is an open subset of a hypersurface $H \subset X \times A^{d+1}$, defined $\sum_I \phi_I Z^I$, ϕ_I reg func on X, Z cor for A^{d+1}.

Now, $X \hookrightarrow A_X$ closed embedding, we extend ϕ_I to Φ_I regular functions on A_X,

set $A_Y = (\sum \Phi_I Z^I = 0) \subset A_X \times A^{d+1} \to A_X$

$\Rightarrow$ $Y \subset A_Y$ and $A_Y \to A_X$ is smooth near $y \in A_Y$.

$$\Rightarrow \begin{tikzcd} y \in Y^\circ \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & A_Y^\circ \arrow[d, "h_A"] \\ X^\circ \arrow[r, "i"] & A_X^\circ \end{tikzcd}$$

Return to our Main Goal: C III $\to$ Main Goal.

By C III we construct $\mathbb{B}(X)$ for affine varieties.

We need to check

① $\mathbb{B}(X)$ is indep of choice of embedding

② $\mathbb{B}(X)$ functorial respect to smooth morphism

③ $\mathbb{B}(X)$ is functorial resp field ext.

① Follows from L2 :

$$\begin{matrix} i_1 X \to Y_1 \to A^N \\ \qquad\qquad\quad \| \to Id. \\ i_2 X \to Y_2 \to A^N \end{matrix}$$

By CIII (4). we have the uniqueness.

② Let $h: Y \to X$ smooth, by L3, $\forall y \in Y$, we fix embedding $X \to A_X$, $\exists$

$$\begin{tikzcd} Y \arrow[d, "h"'] & \supset & Y^\circ \arrow[r, hook] \arrow[d, "h^\circ"] \arrow[dr, phantom, "\square"] & A_Y^\circ \arrow[d, "h_A"] \\ X & \supset & X^\circ \arrow[r, hook] & A_X^\circ \end{tikzcd}$$

that is a fiber products.

Apply CIII (3) to $(A_Y^\circ, I_{Y^\circ}) \to (A_X^\circ, I_{X^\circ})$ we have

$h_A^* B(A_X^\circ, I_{X^\circ}, \phi)$ is an extension of $\mathbb{B}(A_Y^\circ, I_{Y^\circ}, \phi)$

$\Rightarrow$ $h^* \mathbb{B}(X^\circ) = \mathbb{B}(Y^\circ)$ (as extension)

$\Rightarrow$ by argue as in L1, we have $\mathbb{B}$ is functorial resp sm morp.

③ Follows from CIII (5).

Now, we defined $\mathbb{B}$ for affine vars, by L1, we extend uniquely to a resolution "functor" for all vars / k char k=0.

$\square$.

Cor (Log resolution) : Let Y be a closed subscheme in a variety X, then there exists a birational proj morphism $f: \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and

$$f^{-1}(Y) \sqcup Excep \text{ is a snc div on } \widetilde{X}$$

Proof: By "Main Goal", we have

$$f_1 : \widetilde{X}_1 \to X \quad s.t. \quad \widetilde{X}_1 \text{ is smooth.}$$

Now consider $(\widetilde{X}_1, f_1^{-1} I_Y \cdot O_{\widetilde{X}_1} \cdot I(-Excep f_1))$, by PII we have

$$f_2 : \widetilde{X} \to \widetilde{X}_1 \quad s.t.$$

$f_2^{-1}(f_1^{-1} I_Y \cdot O_{\widetilde{X}_1} I(-Excep f_1)) \cdot O_{\widetilde{X}}$ is an ideal sheaf of snc div.

$\square$.

§1. Marked ideals and ord reductions.

We reduce PII to an inductive order reduction process in this section, and we introdue the "marked ideals" that play an important role in the proces.

Def 1.1 (Order) Let X be a smooth variety, $0 \neq I \subset \mathcal{O}_X$ ideal sheaf, For a point $x \in X$ (not necessarily closed), we define

$$\mathrm{Ord}_x I = \max \{ r : m_x^r \mathcal{O}_{x.X} \supset I \cdot \mathcal{O}_{x.X} \}.$$

Rem: ① Ord_x is constructible, upper-semi-continuous on X.

② the maximal order of I along subvar $Z \subset X$ is

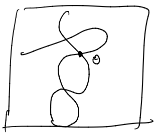
$$\text{max-ord}_Z I = \max \{ \mathrm{ord}_z I, z \in Z \}.$$

when $Z = X$, we may omit Z.

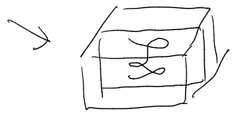
③ If $V(I)$ is contained in a smooth hyperplane, then $\mathrm{ord}_x I = 1$ $\forall x \in \mathrm{Supp}\, I$.

(In this case, $\exists u \in I$, $\mathrm{ord}_x u = 1$ $\forall x \in \mathrm{supp}\, u$).

plane



$\mathrm{ord}_0 I_C = 2$ as plane curve



$I_C' = (I_C, u)$ $\quad \mathrm{ord}_x I_C' = 1$.

Def 1.2 (Marked ideals) A marked ideal on a sm var X is a pair (I, m) where $I \subset \mathcal{O}_X$ is an ideal sheaf and m is a natural number.

The support of a marked ideal (I, m) is defined as

$$\mathrm{Supp}\,(I, m) = \{ x \in X \mid \mathrm{Ord}_x I \geq m \}.$$

Rem: $\mathrm{Supp}\,(I, 1) = \mathrm{Supp}\,(I)$, $\mathrm{Supp}\,(I, m)$ is closed.

Def 1.3 (Birational transform of Ideals and Marked ideals)

Let $0 \neq I \subset \mathcal{O}_X$ be an ideal sheaf for a smooth variety X, There is a unique largest div $\mathrm{Div}(I)$ s.t. $I \subset \mathcal{O}_X(-\mathrm{Div}(I))$. We may write

$$I = \mathcal{O}_X(-\mathrm{Div}(I)) \cdot I_{\mathrm{codim} \geq 2} \qquad I_{\mathrm{codim} \geq 2} = I \cdot \mathcal{O}_X(\mathrm{Div}(I))$$

Let $f: \tilde{X} \to X$ be a smooth blow-up with center $Z \subset X$ and exceptional div $f^{-1}(Z) := E$. We define the bir transform for I as

$$f_*^{-1} I = \mathcal{O}_{\tilde{X}}(\operatorname{Ord}_{\eta_Z}(I) \cdot E) \cdot f^{-1} I \cdot \mathcal{O}_{\tilde{X}}.$$

We define the bir trans for marked ideal as

$$f_*^{-1}(I, m) = (\mathcal{O}_{\tilde{X}}(mE) \cdot f^{-1} I \cdot \mathcal{O}_{\tilde{X}}, m)$$

Remark: ① In application, we require $Z \subseteq \operatorname{Supp}(I, m)$, and the bir trans for marked ideal is well defined in this case.

② The exceptional div here is different from the usual ones. If $f: \tilde{X} \to X$ is a trivial blow-up, then f is Id, $E = Z$. And in this case, $f_*^{-1} I_Z = \mathcal{O}_{\tilde{X}}(E) \cdot \mathcal{O}_{\tilde{X}}(-E) = \mathcal{O}_{\tilde{X}}$

③ f is empty blow-up, then $f_*^{-1} I = I$.

④ $(g \in I, m)$ is called a marked function. And in local coordinate $(x_1, \cdots x_n)$, $Z := (x_1 = \cdots = x_r = 0)$, blow up Z, the bir transform of (f, m) is

$$f_*^{-1}(g, m) = (y_r^{-m} g(y_1 y_r, \cdots, y_{r-1} y_r, y_r, x_{r+1}, \cdots, x_{r+n}), m)$$ in the local chart resp to x_r $\left(f_*^{-1} g = y_r^{-\operatorname{ord}_Z g} g(\cdots)\right)$

⑤ Let $Z \subset X$ with $\operatorname{Ord}_{\eta_Z} I = \max\operatorname{ord} I = m$, (I, m) marked ideal, $\pi: Bl_Z X \to X$ then $\max\operatorname{ord} \pi_*^{-1} I \le \max\operatorname{ord} I$.

thus $\operatorname{ord}_x f = m$ by semi-cont. $x = f(y)$.

Semi-continuity: $\forall x \in Z$, $\operatorname{ord}_x I = m$. $\forall$ $y \in \pi^{-1}(Z)$, $\exists f \in I$ s.t. $\operatorname{ord}_Z f = m$. $\pi_*^{-1} f = y_r^{-m} f(y_1 y_r, \cdots)$

$\Rightarrow \operatorname{ord}_y \pi_*^{-1} f \le \operatorname{ord}_y f(y_1 y_r, \cdots) - m \le 2m - m = m$. $\Rightarrow \operatorname{ord}_y \pi_*^{-1} I \le \operatorname{ord}_y \pi_*^{-1} f \le m \Rightarrow \max\operatorname{ord} \pi_*^{-1} I \le m$.

⑥ $Z \subsetneq H \subset X$, where H is a hypersurface, $I \subseteq \mathcal{O}_X$ $I|_H \neq 0$. $Z \subseteq \operatorname{Supp}(I, m)$.

$\pi: Bl_Z X \to X$ $\quad \pi|_H$ $Bl_Z H \to H$

$(\pi_H)_*^{-1}(I|_H) \supset (\pi_*^{-1} I)|_{Bl_Z H}$ $\quad (\pi_H)_*^{-1}(I|_H, m) = (\pi_*^{-1}(I, m))|_{Bl_Z H}$

(When we do restriction on H, ord may increase, but when we assign an order as in marked ideal, everything is fine)

⑦ $f: Y \to X$ smooth, $f(y) = x$, $I \subseteq \mathcal{O}_X$ then

$$\operatorname{ord}_y f^{-1} I \cdot \mathcal{O}_Y = \operatorname{ord}_x I \quad (\text{Check for étale, for } \times \mathbb{A}^n).$$

⑧ The bir trans for ideals and marked ideals are only defined for seq of sm blow-ups.

Def-Prop 1.4 (Arithmetic Operation on Marked ideals)

Let (I_1, m_1), (I_2, m_2) be two marked ideals on sm var X, we introduce the following

$(I_1, m_1)\cdot(I_2, m_2) = (I_1 I_2, m_1+m_2)$, $\sum_i^n (I_i, m_i) = (\sum_i I_i^{c_i}, \mathrm{lcm}(m_1, \cdots) m_n))$

here $c_i = \mathrm{lcm}(m_1, \cdots, m_n)/m_i$. ($m_i \neq 0$ above). (I_i, m_i) marked ideal on X.

We have the following basic properties.

(1) $\mathrm{Supp}(\sum_i^n (I_i, m_i)) = \bigcap_{i=1}^n \mathrm{Supp}(I_i, m_i)$.

(2) $\mathrm{Supp}(I_1, m_1) \cap \mathrm{Supp}(I_2, m_2) \subseteq \mathrm{Supp}((I_1, m_1)\cdot(I_2, m_2))$

(3) $\mathrm{Supp}(I^c, cm) = \mathrm{Supp}(I, m)$

(4) Let $\pi: Y = Bl_Z \to X$ be a smooth blow-up for $Z \subseteq \mathrm{Supp}(I_1, m_1) \cap \mathrm{Supp}(I_2, m_2)$

we have $\pi_*^{-1}[(I_1, m_1) + (I_2, m_2)] = \pi_*^{-1}(I_1, m_1) + \pi_*^{-1}(I_2, m_2)$

$\pi_*^{-1}[(I_1, m_1)\cdot(I_2, m_2)] = \pi_*^{-1}(I_1, m_1)\cdot\pi_*^{-1}(I_2, m_2)$.

Remark: Here $\sum(I_i, m_i)$ is in fact not even associate operation, it is just a formal notation!

(1) $\forall x \in \bigcap_i \mathrm{Supp}(I_i, m_i)$, $\mathrm{ord}_x I_i^{c_i} \geq \mathrm{lcm}(m_1, \cdots, m_n)$

$\Rightarrow \mathrm{ord}_x I_1^{c_1} + \cdots + I_n^{c_n} \geq \mathrm{lcm} \Rightarrow \bigcap_i \mathrm{Supp}(I_i, m_i) \subseteq \mathrm{Supp}(\sum(I_i, m_i))$

$\forall x \in \mathrm{Supp}(\sum_i (I_i, m_i))$, if $\exists f_i \in I_i$ s.t. $\mathrm{ord}_x f_i < m_i$

$\Rightarrow \mathrm{ord}_x(\sum(I_i, m_i)) < c_i m_i \Rightarrow \Leftarrow$. $\Rightarrow \mathrm{ord}_x I_i \geq m_i$ $\forall i$, $\forall x$. $\square$.

(2) follows from definition.

(3) by (2) we have $\mathrm{Supp}(I, m) \subseteq \mathrm{Supp}(I^c, cm)$.

$\forall x \in \mathrm{Supp}(I^c, cm)$, assume $\mathrm{ord}_x f < m$ for some $f \in I$

$\Rightarrow \mathrm{ord}_x f^c < cm \Rightarrow \mathrm{ord}_x I^c < cm \Rightarrow \Leftarrow$. $\square$.

(4).

$\pi_*^{-1}((I_1, m_1)(I_2, m_2)) = \pi_*^{-1}(I_1 I_2, m_1+m_2)$

$Y = Bl_Z X$.

$\downarrow$

$Z \subset X$

$Z \subseteq \bigcap_i \mathrm{Supp}(I_i, m_i)$

$= \pi^{-1}(I_1 I_2)\cdot\mathcal{O}_Y \cdot \mathcal{O}_Y((m_1+m_2)E)$

$= \pi^{-1}(I_1)\cdot\mathcal{O}_Y\cdot\mathcal{O}_Y(m_1 E)\cdot\pi^{-1}(I_2)\mathcal{O}_Y\cdot\mathcal{O}_Y(m_2 E)$

$= \pi_*^{-1}(I_1, m_1)\cdot\pi_*^{-1}(I_2, m_2)$

$\pi_*^{-1}((I_1, m_1) + (I_2, m_2)) = \pi_*^{-1}(I_1^{c_1} + I_2^{c_2}, \mathrm{lcm}(m_1, m_2))$

$= \pi^{-1} I_1^{c_1}\cdot\mathcal{O}_Y\cdot\mathcal{O}_Y(c_1 m_1) + \pi^{-1} I_2^{c_2}\cdot\mathcal{O}_Y\cdot\mathcal{O}_Y(c_2 m_2) = \pi_*^{-1}(I_1, m_1) + \pi_*^{-1}(I_2, m_2)$.

Now we introduce the main object in the order reduction process.

Def 1.5 ① Let (X, I, m, E) be the object (resp. (X, I, E))

(1) X is sm var $/k$ char $=0$

(2) $0 \neq (I, m) \subset \mathcal{D}_X$ marked ideal (resp. $0 \neq I \subset \mathcal{O}_X$ ideal)

(3) ☆ $E := (E^1, \cdots, E^s)$ an ordered set of smooth divisors, s.t. $\sum E^i$ is snc, E^i can be zero divisor.

has no order just a snc div. (X, I, E)

② And a smooth blow-up of (X, I, m, E) (resp. (X, I, E)) is a sm blow-up $\pi : Bl_Z X \to X$ such that

(1) Z is snc with $\bigcup_{i=1}^{s} E^i$ ☆ (2) $Z \subseteq \mathrm{supp}(I, m)$

③ The bir transform of (X, I, m, E) (resp. (X, I, E)) is

$$\pi_*^{-1}(X, I, m, E) = (Bl_Z X, \pi_*^{-1}(I, m), \pi_{tot}^{-1} E)$$

here $\Rightarrow \pi_{tot}^{-1} E = (\pi_*^{-1} E^1, \pi_*^{-1} E^2, \cdots, \pi_*^{-1} E^s, E^{s+1} = Excep(\pi))$

$$\pi_*^{-1}(X, I, E) = (Bl_Z X, \pi_*^{-1} I, \pi_{tot}^{-1} E).$$

Remark: In the principlization we introduced triple (X, I, E) with E just snc div. and the "transform" of (X, I, E) on $Bl_Z X$ is $(Bl_Z X, \pi^{-1} I \cdot \mathcal{O}_{Bl_Z X}, \pi_*^{-1} E)$.

Def 1.6. (Sequence of blow-ups for (X, I, m, E) and (X, I, E)).

① A smooth blow-up seq of (X, I, m, E) is $(E = (E^1, E^2, \cdots, E^s))$

$\boxed{BO}$: $\pi : (X_r, I_r, m_r, E_r) \to (X_{r-1}, I_{r-1}, m_{r-1}, E_{r-1}) \to \cdots \to (X_0, I_0, m_0, E_0)$

s.t. (1) Each blow-up is smooth with center Z_i snc with $\bigcup_{i=1}^{s} E^i$

(2) $(I_{i+1}, m) = \pi_*^{-1}(I_i, m)$, $E_{i+1} = \pi_{tot}^{-1} E_i$

☆ (3) $Z_i \subseteq \mathrm{supp}(I_i, m_i)$.

② A smooth blow-up seq of (X, I, E) of $\boxed{\text{order } m}$ is

$\boxed{B^m}$: $\pi : (X_r, I_r, E_r) \to \cdots \to (X_0, I_0, E_0)$

$(1)^* = (1)$

$(2)^*$: $I_{i+1} = \pi_*^{-1} I_i$ $\quad E_{i+1} = \pi_{tot}^{-1} E_i$

$(3)^*$: ☆ $\mathrm{Ord}_{\eta_{Z_i}} I_i = m \quad \forall 0 \le i \le r-1$.

Rem: In both definitions, empty blow-ups are not allowed, but trivial blow-ups are allowed.