Prop 1.7. Let $h: Y \to X$ be a smooth morphism, then for any $BO(X, I, m, E)$ (resp. $B^m(X, I, E)$), $h^* BO(X, I, m, E)$ (resp. $h^* B^m(X, I, E)$) is well defined.

Proof:

$$\boxed{\begin{matrix} J_r = h_r^{-1} I_r \cdot O_{Y_r} \\ \vdots \\ J_2 = h_2^{-1} I_2 O_{Y_2} \\ J_1 = h_1^{-1} I_1 \cdot O_{Y_1} \\ J = h^{-1} I \cdot O_Y \end{matrix}}
\quad
\begin{tikzcd}
Y_0 \times X_r \arrow[r, "h_r"] \arrow[d, dashed] & X_r \arrow[d, dashed] & I_r \\
Y_0 \times X_2 \arrow[r, "h_2"] \arrow[d] & X_2 \arrow[d, "\pi_1"] & I_2 \\
Y_0 \times X_1 \arrow[r, "h_1"] \arrow[d] & X_1 \arrow[d, "\pi_0"] & I_1 \\
Y_0 \arrow[r, "h"] & X_0 & I_0
\end{tikzcd}$$

Fact: $\forall y \in Y$, $x = h(y)$, we have

$\operatorname{ord}_y h^{-1} I \cdot O_Y = \operatorname{ord}_x I$. — Def 1.3 Rem ⑦.

$\tilde{\pi}_i : Y_{i+1} \to Y_i$

We need to check: $(J_{i+1}, m) = \tilde{\pi}_{i*}^{-1}(J_i, m)$.

$$\Leftrightarrow \ (h_{i+1}^{-1} I_{i+1} \cdot O_{Y_{i+1}}, m) = \tilde{\pi}_{i*}^{-1}(h_i^{-1} I_i \cdot O_{Y_i}, m)$$

$$\Leftrightarrow \ (h_{i+1}^{-1} I_{i+1} \cdot O_{Y_{i+1}}, m) = \tilde{\pi}_i^{-1}(h_i^{-1} I_i \cdot O_{Y_i}, m) \cdot O_{Y_{i+1}}(m\, h_{i+1}^{-1}(E_{i+1})).$$

E_{i+1} is the excep div of $\pi_i : X_{i+1} \to X_i$.

$$// \quad h_{i+1}^{-1}(\pi_i^{-1} I_i \cdot O_{X_{i+1}} \cdot O_{X_i}(m E_i)) \cdot O_{Y_{i+1}}, m).$$

$$\begin{tikzcd}
Y_{i+1} \arrow[r, "h_{i+1}"] \arrow[d, "\tilde{\pi}_i"'] \arrow[dr, phantom, "\circlearrowright"] & X_{i+1} \arrow[d, "\pi_i"] \\
Y_i \arrow[r, "h_i"] & X_i
\end{tikzcd}$$

Rem: We say BO (resp B^m) commutes with sm mor if

may not be surjective. ordered $=$.

$\underset{\sim\sim\sim}{h^* BO(X, I, m, E)}$ is an extension of $\underset{\sim\sim\sim}{BO(Y, h^{-1} I \cdot O_Y, m, h^{-1} E)}$ (resp. B^m). ☆(Just as blow-up sequence)☆

Rem: for closed embeddings, as we mentioned before

$j: x \in X \hookrightarrow A$, it can happen that

$\operatorname{ord}_x j_* I \cdot O_A = 1$!!!

so functoriality resp to closed embedding in general does not make sense.

I_x in O_A contain some smooth element.

But, for $(I, 1)$, functoriality resp to closed embedding still make sense.

Now, we introduce two order reduction theorem

Ord I (ord reduction for ideals)

$\boxed{\text{For every } m \in \mathbb{N}}$, there is a smooth blow-up sequence functor

Necessary.

$\mathcal{B}^m$ $\boxed{\text{of order } m}$ defined on $(X, I, E := (E^1, \cdots, E^s))$, $\boxed{\text{maxord } I \le m}$

$$\mathcal{B}^m: (X_r, I_r, E_r) \to \cdots \to (X_1, I_1, E_1) \to (X_0, I_0, E_0)$$

$$Z_i \qquad\qquad Z_1 \subset X_1 \qquad Z_0 \subset X_0$$

i.e.,

(each center Z_i snc with E, $\mathrm{Ord}_{\eta_{Z_i}} I_i = m$)

(1) $\text{max-ord } I_r < m$

(2) $\mathcal{B}^m$ commutes with smooth morphism and change of fields.

Ord II (Order reduction theorem for marked ideals)

$\boxed{\text{For every } m \in \mathbb{N}}$, there is a smooth blow-up seq functor BO defined on (X, I, m, E) such that

E: ordered set of sm div.

$$BO: \Pi: (X_r, I_r, m, E_r) \to \cdots \to (X, I, m, E)$$

i.e.,

($Z_i \subset \mathrm{Supp}(I_i, m)$, Z_i snc with E) $\to$ $\mathrm{ord}_{\eta_{Z_i}} I_i \ge m$ ($m = \text{maxord}$) $\Downarrow$ $\mathrm{ord}_{\eta_{Z_i}} I_i = m$.

(1) $\mathrm{Supp}(I_r, m) = \phi$

(2) commutes with sm mor and change of fields.

Remark: If $m = \text{max-ord } I$, $\mathcal{B}^m(X, I, E) = BO(X, I, m, E)$; ☆

In this case, $\mathrm{ord}_{\eta_{Z_i}} I_i = m$.

$\pi^{-1}I$. $\quad R \xrightarrow{\pi} R/A$. $\quad I$

$j^{\#} : \mathcal{O}_X \to j_*\mathcal{O}_S$.

$j_*\mathcal{I}_S \cdot \mathcal{O}_X = j^{\#-1}(j_*\mathcal{I}_S)$.

$\boxed{\text{OCE}}$: $j : S \xrightarrow{} X$ closed embedding ($\mathcal{I}_S$)

$$BO(X, j_*\mathcal{I}_S \cdot \mathcal{O}_X, 1, \emptyset) = j_* BO(S, \mathcal{I}_S, 1, \phi)$$

The main inductive steps of the proof is

$$\text{Ord II in dim} \le n-1$$

→ the induction has nothing to do with "m". ☆

$$\Downarrow T_1 \quad \to \boxed{\text{use restriction.}} \ ☆.$$

$$\text{Ord I in dim } n$$

$$\Downarrow T_2$$

$$\text{Ord II in dim } n.$$

$\boxed{\text{We only use } m=1}$

By

$$\text{Ord II} + \text{OCE} \boxed{\xRightarrow{T_3}} P\,\text{III} \Rightarrow \text{Main Goal}.$$

Now we prove T_3.

Proof of T_3: We start from a triple (X, I, E) ← just snc divisor.

Step 1: Write $E = \bigsqcup_{i=1}^{k} D_i$, set $\tilde{E} = (D_1, D_2, \cdots, D_k)$ D_i sm div.

For any point $x \in X$, set $S_{X,E}(x) = \{\text{number of div in } \{D_i\} \text{ passing } x\}$.

$$S(X,E) = \max\{S_{X,E}(x) \mid x \in X\}.$$

Set $\underline{H^{S(X,E)}} = \bigsqcup_{\substack{A \in \{1,\cdots,k\} \\ |A| = S(X,E)}} \bigcap_{i \in A} D_i$ we blow up $H^{S(X,E)}$.

Here $H^{S(X,E)}$ is a smooth center.

$\pi_0 : X_1 \to X_0 = X$ is a sm blow-up of center $H^{S(X,E)}$

Consider the corresponding $S_{X_1, \pi_{0*}^{-1}E}(x_1)$ $\quad S(X_1, \pi_{0*}^{-1}E)$.

$$\downarrow$$

$$(\pi_*^{-1}D_1, \cdots, \pi_*^{-1}D_k).$$

$$S(X_1, \pi_{0*}^{-1}E) < S(X, E).$$

Repeat this procedure, we construct $\widetilde{B}: X_r \to X_{r-1} \to \cdots \to X_0 = X$ s.t. $(\pi_i : X_i \to X)$

$$0 = S(X_r, \pi_{r*}^{-1} E) < S(X_{r-1}, \pi_{r-1*}^{-1} E) < \cdots < S(X, E)$$

$\Rightarrow$ $\pi_{r*}^{-1} E$ is a disjoint union of irr sm components,

We now show $\widetilde{B}(X, I, E)$ is functorial resp to sm morp.

$\forall\, h : Y \longrightarrow X$ $y \in Y$, $x = h(y)$, we have $\underbrace{S_{X,E}(x) = S_{Y, h^{-1}E}(y)}$

thus $S(X,E) = S(Y, h^{-1}E)$. Since each time we blow up the maximal locus of $S_{X,E}$ $(S_{Y, h^{-1}E})$ $\Rightarrow$ $H^{S(Y, h^{-1}E)} = h^{-1} H^{S(X,E)}$

thus the functoriality resp to sm morp follows.

Rem: This idea is used in [Wło05], where we use a function to control the blow up process, and this function is inv resp to sm mor.

Step 2: Now consider $\left(X_r, \underbrace{\pi_r^{-1} I \cdot \mathcal{O}_{X_r}}_{J}, (\underbrace{\pi_{r*}^{-1} E}_{F_0}, F_1, F_2, \cdots, F_r)\right)$

here $\pi_{r*}^{-1} E$ is smooth.

$m = 1$.

Apply Ord II to $(X_r, J, 1, F)$, we get

$BO(X_r, J, 1, F) : \pi : X_n \to \cdots \to X_r$

s.t. $\underline{\operatorname{Supp} \pi_*^{-1}(J, 1) = \emptyset}$.

$\Rightarrow$ $\underline{\pi_1^{-1} J \cdot \mathcal{O}_{X_n} = \mathcal{O}_{X_n}(\operatorname{Excep} \pi)}$ here $\operatorname{Excep} \pi$ is snc, π exceptional

And $X_n \longrightarrow X_r \longrightarrow X_0$ gives the principalization.

Sm functoriality follows from Ord II and construction.

$\underline{\text{functoriality resp to closed embedding follows from OCE}}$.

P III $E = \emptyset$

in this case, step 1 is trivial.

§ 2. Derivative of ideal sheaves.

In this section, we introduce the derivative of ideal sheaves. (Major problem appear in char $p>0$).

Def 2.1 X sm var$/k$ $\boxed{\text{char } 0}$. Let $Der_X : \mathcal{O}_X \to \mathcal{O}_X$ denote the sheaf of k-derivatives, it gives a k-bilinear map

$$Der_X \times \mathcal{O}_X \to \mathcal{O}_X.$$

$D(I) := Im(Der_X \times I)$

In local coordinates near $p: (x_1, \cdots, x_n)$, I generated by $f_1 \cdots f_s$

$$D(I)_p = \left\{ \frac{\partial g}{\partial x_i} \mid g \in I \right\} \overset{\star}{=} \left(\boxed{\frac{\partial f_i}{\partial x_j}}, \boxed{f_i} \mid 1 \le i \le s,\ 1 \le j \le n \right)$$

Rem: ($f = \frac{\partial (xf)}{\partial x} - x\frac{\partial f}{\partial x}$).

and we def

$D^{r+1}(I) = D(D^r(I))$. $\quad I \subset D(I) \subset \cdots \subset D^{m-1}(I) \subset D^m(I) = \mathcal{O}_X$ $\quad m = \text{maxord } I$,

Obviously $D^r(D^s(I)) = D^{r+s}(I)$.

For marked ideals (I, m), $D^r(I, m) = (D^r(I), m-r)$.

Rem: ① In char $p>0$, the correct derivative is $\frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial u^{|\alpha|}} = D^\alpha$.

and (try to def:

$$D^\alpha(I) = \left(\frac{\partial^{|\beta|}}{\partial u^{|\beta|}} f_j \quad 0 \le |\beta| \le |\alpha|,\ f_j \text{ generator of } I,\ u \text{ local cor} \right).$$

② In this case, $D^i(D^j(I)) \nsubseteq D^{i+j}(I)$ might happen.

for: char $p=2$ $\quad I = (x^3)$.

$$D^1(D^1(I)) = D^1(x^3, 3x^2) = (x^3, 3x^2) \not\supseteq$$

$$D^2(I) = \left(x^3, \frac{\partial x^3}{\partial x}, \frac{1}{2}\frac{\partial^2 x^3}{\partial x^2}\right) = (x^3, 3x^2, 3x)$$

Prop 2.2 Notations as above.

① $D^r(I \cdot J) \subset \sum_{i \geq 0}^{r} D^i(I)\, D^{r-i}(J)$

② $\mathrm{Supp}(I, m) = \mathrm{Supp}(D^r(I), m-r)$ for $r < m$ char 0.

③ $h: Y \to X$ sm, then $D(h^{-1}I \cdot \mathcal{O}_Y) = h^{-1}D(I) \cdot \mathcal{O}_Y$

Proof: ① follows from chain rule.

② local set $I \in (f_1, \cdots f_s)$ $D(I) = (f_i, \frac{\partial f_i}{\partial x_j})$ locally. near x

if $x \in \mathrm{Supp}(I, m) \Rightarrow \mathrm{ord}_x f_i \geq m \Rightarrow \mathrm{ord}_x \frac{\partial f_i}{\partial x_j} \geq m-1$.

if $x \in \mathrm{Supp}(D(I), m-1) \Rightarrow \mathrm{ord}_x \frac{\partial f_i}{\partial x_j} \geq m-1 \Rightarrow$ ✓.

$\mathrm{Supp}(I, m) = \mathrm{Supp}(D(I), m-1)$, inductively we are done.

③
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X \times A^n \ \ I \arrow[dl, "\pi"] \\ X \ \ J & \end{tikzcd}$$

g etale, π proj.

$D(\pi^{-1}J \cdot \mathcal{O}_{X \times A^n}) = \pi^{-1} D(J) \cdot \mathcal{O}_{X \times A^n}$. ✓.

now we check etale.

Now we consider $I_p \otimes \hat{\mathcal{O}}_p$, we have

$\forall\ y \in Y,\ x = g(y)$, $\widehat{\mathcal{O}_{Y,y}} = \widehat{\mathcal{O}_{X \times A^n, x}}$

$\Rightarrow$ Commutative follows.

□.

Remark: for (1), Set $I = (f)$, $J = (g)$ $IJ = (fg)$

$D(IJ) = (fg, \partial(fg))$

$\subsetneq$

$D(I)J + ID(J) = (fg, f\partial g, (\partial f) \cdot g)$.

Lemma 2.3 (Bir transform and derivative ideal)

Let (I, m) be a marked ideal, $\pi: Y \to X \supset Z$ a smooth blow up with center $Z \subseteq \mathrm{Supp}(I, m)$

Then $\pi_*^{-1}(D^j(I, m)) \subset D^j(\pi_*^{-1}(I, m))$ for $j \geq 0$.

Proof. This is a local problem, take $y \in Y$, $x \in Z \subset X$. choose local chart $(x_1, \cdots x_n)$ near x s.t. $Z = (x_1 = \cdots = x_r = 0)$

and the local chart resp to x_r on $Bl_Z X$:

$$y_1 = x_1/x_r, \cdots, y_{r-1} = x_{r-1}/x_r, \quad y_r = x_r, \cdots, y_n = x_n.$$

$\forall f \in I$ $\pi_*^{-1}(f, m) = y_r^{-m} f(y_1 y_r, \cdots, y_{r-1} y_r, y_r, \cdots y_n)$

$$\begin{cases} \pi_*^{-1}(\frac{\partial f}{\partial x_r}, m-1) = y_r \frac{\partial}{\partial y_r} \pi_*^{-1}(f, m) - y_r \sum_{i<r} \frac{\partial}{\partial y_i} \pi_*^{-1}(f, m) + \boxed{(m, 1) \cdot \pi_*^{-1}(f, m)} & j = r \\ \pi_*^{-1}(\frac{\partial}{\partial x_j} f, m-1) = \frac{\partial}{\partial y_j} \pi_*^{-1}(f, m) \cdot y_r & j > r \\ \pi_*^{-1}(\frac{\partial}{\partial x_j} f, m-1) = \frac{\partial}{\partial y_j} \pi_*^{-1}(f, m) & j < r. \end{cases}$$

product of marked ideal

$\Rightarrow$ $\pi_*^{-1}(D(I, m)) \subset D(\pi_*^{-1}(I, m))$

Inductively we are done.

A major idea of Hironaka is that, instead of dealing with I, we deal with some "equivalent ideal" that enrich I, and the enriched ideal behaves well under centern restriction.

Def 2.4 (Coefficient ideal and Homogenized ideal)

Let (I, m) be a marked ideal such that $m = \text{max-ord}\, I$ on sm var X/char$p=0$ k.

We def D-Balanced. $(D^i I)^m \subset I^{m-i}$ $\forall i < m$ $W(I, n)$.

$$\boxed{C(I, m)} = (I, m) + D(I, m) + \cdots + D^{m-1}(I, m). \quad (+ \cdots + D^\infty(I, m))$$

and MC-Invariant: $T(I) \cdot D(I) \subset I$

$$\boxed{H(I, m)} = (H(I), m) = (I, m) + D(I, m) \cdot (T(I), 1) + \cdots D^i(I, m) \cdot (T(I), 1)^i + \cdots + D^{m-1}(I, m) \cdot (T(I), 1)^{m-1}$$

$$= (I + DI \cdot TI + \cdots + D^i I \cdot (TI)^i + \cdots + \underbrace{D^{m-1} I \cdot (TI)^{m-1}}, m).$$

Here $\underbrace{T(I) = D^{m-1} I}$. ☆

$x^2 + y^3$ $\quad C(I) = (x^2 + y^3, 2) + ((x, y^2), 1)^2$

$= \underbrace{(x^2, xy^2, y^3, 2)}.$

[Wot 05]

Prop 2.5 (1) $\mathrm{Supp}(\mathcal{H}(I,m)) = \mathrm{Supp}(C(I,m)) = \mathrm{Supp}(I,m)$.

(2) $\forall Z \subseteq \mathrm{Supp}((I,m))$ smooth on X, $\pi: Bl_Z X \to X$, we have

$$\mathrm{Supp}(\pi_*^{-1} \mathcal{H}(I,m)) = \mathrm{Supp}(\pi_*^{-1} C(I,m)) = \mathrm{Supp}(\pi_*^{-1}(I,m))$$

(3) $h: Y \to X$ smooth, then

$$\mathcal{H}(h^{-1}I \cdot \mathcal{O}_Y) = h^{-1}\mathcal{H}(I)\cdot \mathcal{O}_Y$$

$$C(h^{-1}I\cdot\mathcal{O}_Y, m) = h^{-1}C(I,m)\cdot\mathcal{O}_Y.$$

Proof:

(1) By Def-Prop 1.4 (1)-(3) Prop 2.2 (2)

$$\underset{\supseteq\ \mathrm{Supp}(I,m)}{\mathrm{Supp}(\mathcal{H}(I,m))} = \bigcap_{i=0}^{m-1} \mathrm{Supp}\left(D^i(I,m)\cdot (T(I),1)^i\right) \supseteq \bigcap_{i=0}^{m-1} \underbrace{\mathrm{Supp}(D^i(I,m)) \cap \mathrm{Supp}(T(I),1)}_{= \mathrm{Supp}(I,m)}.$$

Similar for $C(I,m)$.

(2) $$\underset{\supseteq\ \mathrm{Supp}(\pi_*^{-1}I,m)}{\mathrm{Supp}(\pi_*^{-1}\mathcal{H}(I,m))} = \bigcap_{i=0}^{m-1} \left(\pi_*^{-1} D(I,m) \cdot \pi_*^{-1}(T(I)^i, i)\right) \supseteq \bigcap_{i=0}^{m-1} \underbrace{\mathrm{Supp}\left(D^i(\pi_*^{-1}I, m-i)\cdot T(\pi_*^{-1}I)^i, i\right)}_{= \mathrm{Supp}(\pi_*^{-1}I,m)}.$$

Lem 2.3

Similar for $C(I,m)$.

(3) Follows from Prop 2.2 (3).

Rem: Above propersition says that, any order reduction process

To be more speific

$$\pi: X_r \to X_{r-1} \to \cdots \to X_0 = X \quad \text{a seq of blow-up}$$
$$Z_{r-1} \subseteq X_{r-1}, \quad Z_0 \subseteq X_0$$

(1) $Z_i \subseteq \mathrm{Supp}(I_i, m)$ iff $Z_i \subseteq \mathrm{Supp}\,\mathcal{H}(I_i,m)$, $(C(I_i,m))$

(2) $\mathrm{Supp}((I,m)_r) = \emptyset$ iff $\mathrm{Supp}(\mathcal{H}(I,m))_r = \emptyset$ $(C(I,m)_r)$.

$\ \ $ $\uparrow$ bir trans on X_r.

(3) Prop 2.5 (3) gaurantees sm functoriality for $\mathcal{H}(I)$, $C(I)$, vice versa.

Now we consider the restriction problem

Prop 2.6 Let (X, I, m) be triple s.t. (I, m) marked ideal on sm $X/k=0$
S smooth subvariety on X not contained in $\mathrm{Supp}(I, m)$, $Z \subseteq S \cap \mathrm{Supp}(I, m)$
$\pi: Bl_Z X \to X$ the smooth blow up, $\pi|_S : Bl_Z S \to S$, $Bl_Z S = S'$

Then (1) $\mathrm{Supp}(I, m) \cap S \subseteq \mathrm{Supp}(I|_S, m)$

(2) $\mathrm{Supp}(C(I, m)) \cap S = \mathrm{Supp}(C(I, m)|_S)$

(3) $\pi|_{S*}^{-1}((I, m)|_S) = (\pi_*^{-1}(I, m))|_{S'}$

(4) $\mathrm{Supp}(\pi_*^{-1} C(I, m)) \cap S' = \mathrm{Supp}(\pi|_{S*}^{-1}(C(I, m)|_S))$

Proof: (1) follows from the fact that when we do restriction, ord will not decrease.

(2) Let $x_1 \cdots x_k, y_1 \cdots y_{n-k}$ be local parameters at x s.t. $x \in S$
$S := (x_1 = \cdots = x_k = 0)$ $\forall f \in I$, $f = \sum C_{\alpha,f} x^\alpha = \sum C_{\alpha,f}(y) x^\alpha$

Now, $x \in \mathrm{Supp}(I, m) \cap S$ iff $\mathrm{ord}_x(C_{\alpha,f}(y))|_S \geq m - |\alpha|$ for all $f \in I$, $1 \le |\alpha| \le m-1$

$$C_{\alpha,f}|_S = \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big|_S \in D^\alpha(I)|_S.$$

thus

$$\mathrm{Supp}(C(I, m)) = \mathrm{Supp}(I, m) \cap S = \bigcap_{\substack{f \in I \\ |\alpha| \le m}} \mathrm{Supp}(C_{\alpha,f}|_S, m - |\alpha|) \supseteq \bigcap_{0 \le i \le m-1} \mathrm{Supp}(D^i I|_S, m-i)$$

$$\supseteq \mathrm{Supp}(C(I, m)|_S). \qquad = \mathrm{Supp}(C(I, m)|_S).$$

(3). Notations as in (2), $Z \subset S \subset X$, $\pi|_S : S' \to S$.
$Z := (x_1 = \cdots = x_k = y_1 = \cdots = y_q = 0)$.

for $x \in Z \subset S \subset X$, locally blow up can write as

$$x_1' = x_1/y_q \;\cdots\; x_k' = x_k/y_q, \; y_1' = y_1/y_q \cdots \; y_q' = y_q \;\; y_{q+1}' = y_{q+1} \cdots$$

then strict transform of S (denoted as S') is locally defined by
$x_1' = x_2' = \cdots = x_k' = 0 \subset X' = Bl_Z X$.