Title: Midterm Exam I
Course Name: Winter-2016-MAT223H1-S-LEC0301.LEC5101.LEC0201.LE
Instructor: Sean Uppal
Instructor Email: uppal@math.utoronto.ca
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Total Score: 100% (60/60)
Score: 6.0/6.0

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Midterm Exam I

#749   2 of 11

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

1. Suppose that the augmented matrix of a system of linear equations has been reduced to $\begin{bmatrix} 1 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$. Which of the

following statements describes the set of solutions to the system?

(A) infinitely many solution with three parameters
(B) infinitely many solution with two parameters
(C) infinitely many solution with one parameter
(D) unique solution
(E) no solution

$x_1 = 1 - x_2$
$x_3 = 1$
$x_4 = 0$
$x = \begin{bmatrix} 1-t \\ t \\ 1 \\ 0 \end{bmatrix}$

2. Let A be an $n \times n$ matrix and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct solutions to the system $A\mathbf{x} = \mathbf{b}$. Which of the following statements are TRUE?

(i) A cannot be carried to I_n by elementary row operations ✓

(ii) $3\mathbf{x}_1 + 2\mathbf{x}_2$ is a solution to the system $A\mathbf{x} = 5\mathbf{b}$ ✓

(iii) $3\mathbf{x}_1 + 2\mathbf{x}_2$ is a solution to the system $A\mathbf{x} = \mathbf{b}$ ✗

(A) (ii) only
(B) (iii) only
(C) (i) and (iii) only
(D) (i) and (ii) only
(E) none of (i), (ii), or (iii)

$3A x_1 = 3b$
$2A x_2 = 2b$
$A(3x_1 + 2x_2) = 5b$.

3. Let A, B, and C be $n \times n$ matrices. Which of the following statements are TRUE?

(i) $(AB)^2 = A^2 B^2$ ✗

(ii) $(ABC)^T = A^T B^T C^T$ ✗

(iii) If $A^2 = 0$, then $A = 0$. ✗

(A) (i) and (ii) only
(B) none of (i), (ii), or (iii)
(C) (ii) and (iii) only
(D) (i), (ii), and (iii)
(E) (iii) only

$A \cdot A = 0$
$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

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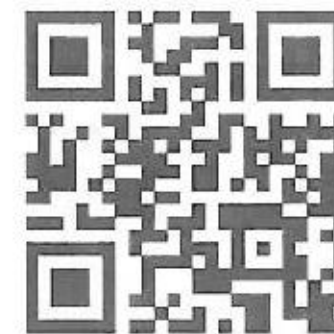
Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth <u>2 marks</u>.

For each question, choose the <u>BEST</u> option from the given options.

4. Which of the following matrices are symmetric?

   (i) $A = [a_{ij}]$ where $a_{ij} = i^2 + j^2$ $\checkmark$ $A^T = [a_{ji}]$

   (ii) $A = [a_{ij}]$ where $a_{ij} = i^2 - j^2$ $\times$

   (iii) $A = [a_{ij}]$ where $a_{ij} = 2i + 2j$ $\checkmark$

   (A) (i) only
   (B) (ii) only
   (C) (iii) only
   (D) (i) and (ii) only
   (E) (i) and (iii) only

5. Let A be an $m \times n$ matrix such that the system $A\mathbf{x} = \mathbf{b}$ has a <u>unique solution</u>. Which of the following statements are TRUE?

   (i) $\operatorname{rank}(A) = n$ $\checkmark$

   (ii) the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\checkmark$

   (iii) A is invertible

   $\begin{pmatrix} 1 & 2 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$
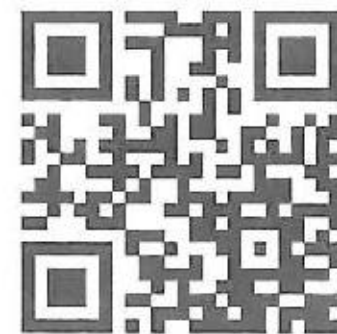
   (A) (i) and (ii) only
   (B) (i) and (iii) only
   (C) (ii) and (iii) only
   (D) (i) only
   (E) (ii) only

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Score: 10.0/10.0

Part II - Short Answer Questions. Write your solutions in the space provided below each question.

1. Find all solutions to homogeneous system of linear equations

$$2x_2 + 2x_3 + 2x_4 = 0$$
$$x_1 - x_3 - 3x_4 = 0$$
$$2x_1 + 3x_2 + x_3 + x_4 = 0$$
$$-2x_1 + x_2 + 3x_3 - 2x_4 = 0$$

[10 marks]

$$\begin{pmatrix} 0 & 2 & 2 & 2 & | & 0 \\ 1 & 0 & -1 & -3 & | & 0 \\ 2 & 3 & 1 & 1 & | & 0 \\ -2 & 1 & 3 & -2 & | & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & -1 & -3 & | & 0 \\ 0 & 2 & 2 & 2 & | & 0 \\ 2 & 3 & 1 & 1 & | & 0 \\ -2 & 1 & 3 & -2 & | & 0 \end{pmatrix} \xrightarrow[R_4 + 2R_1]{R_3 - 2R_1}$$

$$\begin{pmatrix} 1 & 0 & -1 & -3 & | & 0 \\ 0 & 2 & 2 & 2 & | & 0 \\ 0 & 3 & 3 & 7 & | & 0 \\ 0 & 1 & 1 & -8 & | & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & -1 & -3 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 3 & 3 & 7 & | & 0 \\ 0 & 1 & 1 & -8 & | & 0 \end{pmatrix} \xrightarrow[R_4 - R_2]{R_3 - 3R_2}$$

$$\begin{pmatrix} 1 & 0 & -1 & -3 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & 4 & | & 0 \\ 0 & 0 & 0 & -9 & | & 0 \end{pmatrix} \xrightarrow[R_4 + \frac{9}{4}R_3]{\frac{1}{4}R_3} \begin{pmatrix} 1 & 0 & -1 & -3 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow[R_2 - R_3]{R_1 + 3R_3}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & | & 0 \\ 0 & 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

which is $\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0 \end{cases}$ Let $x_3 = t \in \mathbb{R}$

$\Rightarrow \begin{cases} x_1 = x_3 = t \\ x_2 = -x_3 = -t \\ x_4 = 0 \end{cases} \nearrow \quad X = \begin{bmatrix} t \\ -t \\ t \\ 0 \end{bmatrix}$

Conclusion: $X = \begin{bmatrix} t \\ -t \\ t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$. $\left(\begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = t \\ x_4 = 0 \end{cases} \right)$

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2. Suppose a linear system of equations in the variables x_1, x_2, x_3 has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 1 & c & 1 & 1 \\ c & 1 & 1 & -2 \end{array}\right]$$

(a) For what values of c is the system (i) inconsistent, (ii) consistent with a unique solution, and (iii) consistent with infinitely many solutions? [10 marks]

$$A = \begin{pmatrix} 1 & 1 & C & | & 1 \\ 1 & C & 1 & | & 1 \\ C & 1 & 1 & | & -2 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & C & | & 1 \\ 0 & C-1 & 1-C & | & 0 \\ C & 1 & 1 & | & -2 \end{pmatrix}$$

if $c = 0$, then the matrix should be

$$A = \begin{pmatrix} 1 & 1 & 0 & | & 1 \\ 0 & -1 & 1 & | & 0 \\ 0 & 1 & 1 & | & -2 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1 + R_2}$$

if $C \neq 0$.

$$\xrightarrow{R_3 - cR_1} \begin{pmatrix} 1 & 1 & C & | & 1 \\ 0 & C-1 & 1-C & | & 0 \\ 0 & 1-C & 1-C^2 & | & -2-C \end{pmatrix} \xrightarrow{R_3 + R_2}$$

$$\begin{pmatrix} 1 & 1 & C & | & 1 \\ 0 & C-1 & 1-C & | & 0 \\ 0 & 0 & 2-C-C^2 & | & -2-C \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & -1 & 1 & | & 0 \\ 0 & 0 & 2 & | & -2 \end{pmatrix} \xrightarrow[\frac{1}{2}R_3]{R_2} \begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & -1 \end{pmatrix}$$

$\text{rank}(A) = n = 3$

So when $c = 0$, it's consistent with one solution.

i) When $2 - C - C^2 = 0$ and $-2 - C \neq 0$, A is inconsistent.

$C^2 + C - 2 = 0 \qquad -2 - C \neq 0$

$(C+2)(C-1) = 0 \qquad C \neq -2$.

$C_1 = -2 \quad C_2 = 1$

So when $C = 1$, A is inconsistent

because it has a row of $(0 \dots 0 | *)$

$*$ means non-zero.

iii) When $2 - C - C^2 = 0$ and $-2 - C = 0$

A is consistent with infinitely many solutions, because $\text{rank}(A) < n$

That means $C = -2$.

ii) When $2 - C - C^2 \neq 0$, A has unique solution because $\text{rank}(A) = n$ (# of variables)

$C \neq -2$ and $C \neq 1$

Conclusion: (i) $c = 1$ (ii) $C \neq -2$ and $C \neq 1$ (iii) $c = -2$

3. (a) Define what it means for a matrix A to be in row-echelon form. [2 marks]

① The row of zeros is at the bottom of the matrix. $\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$

② The non-zero rows begin with the leading 1 $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix}$

③ In the next row, entry of leading 1 should be on the right of the leading 1 of the row above it.

3. (b) Suppose $A = \begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix}$, and the reduced row-echelon form of A is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Determine a, b, and c.

[6 marks]

$A \xrightarrow[R_3 - 3R_1]{R_2 + R_1} \begin{pmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 1 & c-3a & -3 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 0 & -2a+b+c & -4 \end{pmatrix} \xrightarrow{-R_2}$
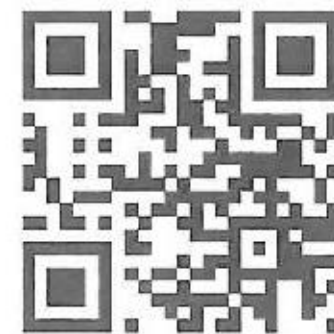
$\begin{pmatrix} 1 & 0 & a & 1 \\ 0 & 1 & -(a+b) & 1 \\ 0 & 0 & -2a+b+c & -4 \end{pmatrix}$
Set $\begin{cases} a=2 \\ -(a+b)=-5 \\ -2a+b+c=0 \end{cases} \longrightarrow \begin{cases} a=2 \\ b=3 \\ c=1 \end{cases}$

$\longrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \xrightarrow{-\frac{1}{4}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_3} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

which is exactly what we want.

So $a=2, b=3, c=1$.

4. (a) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Find the augmented matrix of the system $AX - XA = I_2$ in the variables x_1, x_2, x_3, x_4. [4 marks]

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} - \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} x_1-x_3 & x_2-x_4 \\ 2x_1 & 2x_2 \end{pmatrix} - \begin{pmatrix} x_1+2x_2 & -x_1 \\ x_3+2x_4 & -x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -2x_2-x_3 & x_1+x_2-x_4 \\ 2x_1-x_3-2x_4 & 2x_2+x_3 \end{pmatrix}$$

which is equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ This means $\longrightarrow$

$\begin{cases} -2x_2-x_3 = 1 \\ x_1+x_2-x_4 = 0 \\ 2x_1-x_3-2x_4 = 0 \\ 2x_2+x_3 = 1 \end{cases}$

So the augmented matrix is

$$\begin{pmatrix} 0 & -2 & -1 & 0 & | & 1 \\ 1 & 1 & 0 & -1 & | & 0 \\ 2 & 0 & -1 & -2 & | & 0 \\ 0 & 2 & 1 & 0 & | & 1 \end{pmatrix}$$

4. (b) A matrix B is said to be a **square root** of a matrix A if $BB = A$. Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. [4 marks]

Call the two square root matrix C and D.

$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ check: $CC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = A$.

$D = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ check $D \cdot D = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = A$.

So ① $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ② $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

5. (a) Find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}$. [4 marks] $(A|I) \longrightarrow (I|A^{-1})$.

$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array} \right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array} \right) \xrightarrow[-R_3]{R_1-R_2}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array} \right) \xrightarrow[R_2+2R_3]{R_1-3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & 3 \\ 0 & -1 & 0 & -4 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array} \right) \xrightarrow{-R_2}$

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & 3 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array} \right)$ So the $A^{-1} = \left(\begin{array}{ccc} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{array} \right)$

5. (b) Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Express the variables x_1, x_2, x_3 in terms of $z_1, z_2,$ and z_3. **Suggestion**: Use part (a). [4 marks]

$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow A^{-1} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A^{-1} \cdot A \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow A^{-1} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = I \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Recall A^{-1} in 5(a).

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \left(\begin{array}{ccc} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{array} \right) \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 & -2 & 5 \\ 2 & -1 & -1 \\ 16 & -3 & 8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12z_1 - 2z_2 + 5z_3 \\ 2z_1 - z_2 - z_3 \\ 16z_1 - 3z_2 + 8z_3 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 12z_1 - 2z_2 + 5z_3 \\ x_2 = 2z_1 - z_2 - z_3 \\ x_3 = 16z_1 - 3z_2 + 8z_3 \end{cases}$
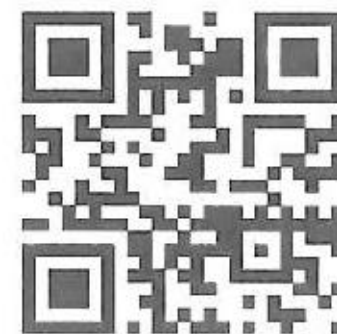
6. (a) Show that if A is a 2×2 diagonal matrix of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, then $AB = BA$ for any 2×2 matrix B. [2 marks]

Let $B = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$ where b, c, d, e are random real numbers.   Assume $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

$AB = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} ab & ac \\ ad & ae \end{bmatrix}$

$BA = \begin{bmatrix} b & c \\ d & e \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ab & ac \\ ad & ae \end{bmatrix}$

then $AB = BA = \begin{bmatrix} ab & ac \\ ad & ae \end{bmatrix}$.

Because a, b, c, d are random, so it works for any 2×2 matrix B.

6. (b) Let A be a 2×2 matrix with the property that $AB = BA$ for all 2×2 matrices B. Show that A must be a diagonal matrix of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. [4 marks]

Let $B = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$ where b, c, d, e are random real numbers

$A = \begin{bmatrix} f & g \\ h & i \end{bmatrix}$ where f, g, h, i are constants. (i.e. f, g, h, i do not change)

Assume $AB = BA$.

$AB = \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} bf + dg & cf + eg \\ bh + di & ch + ei \end{bmatrix}$

$BA = \begin{bmatrix} b & c \\ d & e \end{bmatrix} \begin{bmatrix} f & g \\ h & i \end{bmatrix} = \begin{bmatrix} bf + ch & bg + ci \\ df + eh & dg + ei \end{bmatrix}$

$AB = BA \Rightarrow \begin{cases} bf + dg = bf + ch \\ cf + eg = bg + ci \\ bh + di = df + eh \\ ch + ei = dg + ei \end{cases} \Rightarrow \begin{cases} dg = ch \\ cf + eg = bg + ci \\ bh + di = df + eh \end{cases}$

Figure out that $dg = ch$ holds for every choice of c and d.
So g and h must be 0.

So $\begin{cases} 0 = 0 \\ cf = ci \\ di = df \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ c(f - i) = 0 \\ d(i - f) = 0 \end{cases}$

because c and d are random

So $\begin{cases} f - i = 0 \\ i - f = 0 \end{cases} \Rightarrow \boxed{f = i}$

Let $f = i = a$ (a is some constant)

$A = \begin{bmatrix} f & g \\ h & i \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

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$$-6+3+3$$

$$\begin{pmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ -1 & -1 & 3 & -2 \\ 3 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 1 & -5 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

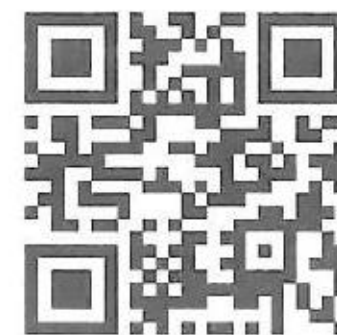$$c \cdot \frac{h(b-e)}{d} = g(b-e)$$

$$ch = gd$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$cf - ci = bg - eg$$

$$c(f-i) = g(b-e)$$

$$bh - eh = df - di$$

$$h(b-e) = d(f-i)$$

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$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} - \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & -2 \end{array} \right) = \begin{pmatrix} x_1 - x_3 & x_2 - x_4 \\ 2x_1 & 2x_2 \end{pmatrix} - \begin{pmatrix} x_1 + 2x_2 & -x_1 \\ x_3 + 2x_4 & -x_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array} \right) = \begin{pmatrix} -x_3 - 2x_2 & x_1 + x_2 - x_4 \\ 2x_1 - x_3 - 2x_4 & 2x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} -2x_2 - x_3 = 1 \\ x_1 + x_2 - x_4 = 0. \\ 2x_1 - x_3 - 2x_4 = 0. \\ 2x_2 + x_3 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$x_1 = 1 - x_2 + 2x_3 = 1 + x_3$$
$$x_2 = x_3 = t$$

$$\left(\begin{array}{cccc|c} 0 & -2 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & -1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{array} \right)$$