Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

E

4. Which of the following matrices are symmetric?

(i) $A = [a_{ij}]$ where $a_{ij} = i^2 + j^2$ ✓ $A^T = [a_{ji}]$

(ii) $A = [a_{ij}]$ where $a_{ij} = i^2 - j^2$ X

(iii) $A = [a_{ij}]$ where $a_{ij} = 2i + 2j$ ✓

(A) (i) only
(B) (ii) only
(C) (iii) only
(D) (i) and (ii) only
(E) (i) and (iii) only

A

5. Let A be an $m \times n$ matrix such that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Which of the following statements are TRUE?

(i) $\text{rank}(A) = n$ ✓

(ii) the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution ✓

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

(iii) A is invertible

(A) (i) and (ii) only
(B) (i) and (iii) only
(C) (ii) and (iii) only
(D) (i) only
(E) (ii) only

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Score: 10.0/10.0

Part II - Short Answer Questions. Write your solutions in the space provided below each question.

1. Find all solutions to homogeneous system of linear equations

$$\begin{aligned} 2x_2 + 2x_3 + 2x_4 &= 0 \\ x_1 - x_3 - 3x_4 &= 0 \\ 2x_1 + 3x_2 + x_3 + x_4 &= 0 \\ -2x_1 + x_2 + 3x_3 - 2x_4 &= 0 \end{aligned}$$

[10 marks]

$$\left(\begin{array}{cccc|c} 0 & 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right) \xrightarrow[R_4+2R_1]{R_3-2R_1}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right) \xrightarrow[R_4-R_2]{R_3-3R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -9 & 0 \end{array}\right) \xrightarrow[R_4+\frac{9}{4}R_3]{\frac{1}{4}R_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2-R_3]{R_1+3R_3}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{ which is } \begin{cases} X_1 - X_3 = 0 \\ X_2 + X_3 = 0 \\ X_4 = 0 \end{cases}$$

Let $X_3 = t \in \mathbb{R}$

$$\Rightarrow \begin{cases} X_1 = X_3 = t \\ X_2 = -X_3 = -t \\ X_4 = 0 \end{cases} \qquad X = \begin{bmatrix} t \\ -t \\ t \\ 0 \end{bmatrix}$$

Conclusion: $X = \begin{bmatrix} t \\ -t \\ t \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$ $\left(\begin{cases} X_1 = t \\ X_2 = -t \\ X_3 = t \\ X_4 = 0 \end{cases} \right)$

Score: 10.0/10.0

2. Suppose a linear system of equations in the variables x_1, x_2, x_3 has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 1 & c & 1 & 1 \\ c & 1 & 1 & -2 \end{array}\right]$$

(a) For what values of c is the system (i) inconsistent, (ii) consistent with a unique solution, and (iii) consistent with infinitely many solutions? [10 marks]

$$A = \left(\begin{array}{ccc|c} 1 & 1 & C & 1 \\ 1 & C & 1 & 1 \\ C & 1 & 1 & -2 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & C & 1 \\ 0 & C-1 & 1-C & 0 \\ C & 1 & 1 & -2 \end{array}\right)$$

if $C \neq 0$:

$$\xrightarrow{R_3 - CR_1} \left(\begin{array}{ccc|c} 1 & 1 & C & 1 \\ 0 & C-1 & 1-C & 0 \\ 0 & 1-C & 1-C^2 & -2-C \end{array}\right) \xrightarrow{R_3 + R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & C & 1 \\ 0 & C-1 & 1-C & 0 \\ 0 & 0 & 2-C-C^2 & -2-C \end{array}\right)$$

if $c=0$, then the matrix should be

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right) \xrightarrow[R_3 + R_2]{R_1 + R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right) \xrightarrow[\frac{1}{2}R_3]{-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$\text{rank}(A) = n = 3$

So when $C=0$, it's consistent with one solution.

i) When $2-C-C^2=0$ and $-2-C \neq 0$, A is inconsistent.

$C^2+C-2=0$ $\qquad -2-C \neq 0$

$(C+2)(C-1)=0$ $\qquad C \neq -2$.

$C_1=-2$ $C_2=1$

So when $C=1$, A is inconsistent because it has a row of $(0 \ldots 0 \mid *)$

$*$ means non-zero

iii) When $2-C-C^2=0$ and $-2-C=0$

A is consistent with infinitely many solutions, because $\text{rank}(A) < n$

That means $C=-2$.

ii) when $2-C-C^2 \neq 0$, A has unique solution because $\text{rank}(A) = n$ (# of variables)

$C \neq -2$ and $C \neq 1$

Conclusion: (i) $C=1$ (ii) $C \neq -2$ and $C \neq 1$ (iii) $C=-2$.

Score: 8.0/8.0

3. (a) Define what it means for a matrix A to be in row-echelon form. [2 marks]

① The row of zeros is at the bottom of the matrix.

② The non-zero rows begin with the leading 1

③ In the next row, entry of leading 1 should be on the right of the leading 1 of the row above it.

3. (b) Suppose $A = \begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix}$, and the reduced row-echelon form of A is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Determine a, b, and c.

[6 marks]

$$A \xrightarrow[R_3-3R_1]{R_2+R_1} \begin{pmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 1 & c-3a & -3 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 0 & -2a+b+c & -4 \end{pmatrix} \xrightarrow{-R_2}$$

$$\begin{pmatrix} 1 & 0 & a & 1 \\ 0 & 1 & -(a+b) & 1 \\ 0 & 0 & -2a+b+c & -4 \end{pmatrix}$$

Set $\begin{cases} a=2 \\ -(a+b)=-5 \\ -2a+b+c=0 \end{cases} \longrightarrow \begin{cases} a=2 \\ b=3 \\ c=1 \end{cases}$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \xrightarrow{-\frac{1}{4}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2-R_3]{R_1-R_3} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is exactly what we want.

So $a=2, b=3, c=1$.

Score: 8.0/8.0

4. (a) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Find the augmented matrix of the system $AX - XA = I_2$ in the variables x_1, x_2, x_3, x_4. [4 marks]

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} - \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} X_1 - X_3 & X_2 - X_4 \\ 2X_1 & 2X_2 \end{pmatrix} - \begin{pmatrix} X_1 + 2X_2 & -X_1 \\ X_3 + 2X_4 & -X_3 \end{pmatrix}$$

$$= \begin{pmatrix} -2X_2 - X_3 & X_1 + X_2 - X_4 \\ 2X_1 - X_3 - 2X_4 & 2X_2 + X_3 \end{pmatrix}$$ which is equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ This means $\Rightarrow$

$$\begin{cases} -2X_2 - X_3 = 1 \\ X_1 + X_2 - X_4 = 0 \\ 2X_1 - X_3 - 2X_4 = 0 \\ 2X_2 + X_3 = 1 \end{cases}$$

So the augmented matrix is

$$\left(\begin{array}{cccc|c} 0 & -2 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & -1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

4. (b) A matrix B is said to be a **square root** of a matrix A if $BB = A$. Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. [4 marks]

Call the two square root matrix C and D.

$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ check: $CC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = A$.

$D = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ check $D \cdot D = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = A$.

So ① $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ② $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

Score: 8.0/8.0

5. (a) Find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}$. [4 marks] $(A|I) \longrightarrow (I|A^{-1})$.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \xrightarrow[-R_3]{R_1-R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right) \xrightarrow[R_2+2R_3]{R_1-3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & 3 \\ 0 & -1 & 0 & -4 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right) \xrightarrow{-R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & 3 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right) \quad \text{So the } A^{-1} = \begin{pmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

5. (b) Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Express the variables x_1, x_2, x_3 in terms of z_1, z_2, and z_3. **Suggestion**: Use part (a). [4 marks]

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow A^{-1} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A^{-1} \cdot A \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow A^{-1} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = I \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Recall A^{-1} in 5(a).

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{pmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 & -2 & 5 \\ 2 & -1 & -1 \\ 16 & -3 & 8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12z_1 - 2z_2 + 5z_3 \\ 2z_1 - z_2 - z_3 \\ 16z_1 - 3z_2 + 8z_3 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 12z_1 - 2z_2 + 5z_3 \\ x_2 = 2z_1 - z_2 - z_3 \\ x_3 = 16z_1 - 3z_2 + 8z_3 \end{cases}$$

Score: 6.0/6.0

6. (a) Show that if A is a 2×2 diagonal matrix of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, then $AB = BA$ for any 2×2 matrix B. [2 marks]

Let $B = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$ where b, c, d, e are random real numbers. Assume $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

$AB = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} ab & ac \\ ad & ae \end{bmatrix}$

$BA = \begin{bmatrix} b & c \\ d & e \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ab & ac \\ ad & ae \end{bmatrix}$

then $AB = BA = \begin{bmatrix} ab & ac \\ ad & ae \end{bmatrix}$.

Because a, b, c, d are random, so it works for any 2x2 matrix B.

6. (b) Let A be a 2×2 matrix with the property that $AB = BA$ for all 2×2 matrices B. Show that A must be a diagonal matrix of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. [4 marks]

Let $B = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$ where b, c, d, e are random real numbers

$A = \begin{bmatrix} f & g \\ h & i \end{bmatrix}$ where f, g, h, i are constants (i.e. f, g, h, i do not change)

Assume $AB = BA$.

$AB = \begin{bmatrix} f & g \\ h & i \end{bmatrix} \begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} bf+dg & cf+eg \\ bh+di & ch+ei \end{bmatrix}$

$BA = \begin{bmatrix} b & c \\ d & e \end{bmatrix} \begin{bmatrix} f & g \\ h & i \end{bmatrix} = \begin{bmatrix} bf+ch & bg+ci \\ df+eh & dg+ei \end{bmatrix}$

$AB = BA \Rightarrow \begin{cases} \cancel{bf}+dg = \cancel{bf}+ch \\ cf+eg = bg+ci \\ bh+di = df+eh \\ ch+\cancel{ei} = dg+\cancel{ei} \end{cases} \Rightarrow \begin{cases} dg = ch \\ cf+eg = bg+ci \\ bh+di = df+eh \end{cases}$

Figure out that $dg = ch$ holds for every choice of c and d.

So g and h must be 0.

So $\begin{cases} 0 = 0 \\ cf = ci \\ di = df \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ c(f-i) = 0 \\ d(i-f) = 0 \end{cases}$

because c and d are random

So $\begin{cases} f-i = 0 \\ i-f = 0 \end{cases} \Rightarrow \boxed{f = i}$

Let $f = i = a$ (a is some constant)

$A = \begin{bmatrix} f & g \\ h & i \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

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$-6+3+3$

$$\begin{pmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ -1 & -1 & 3 & -2 \\ 3 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 1 & -5 & -3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c \cdot \frac{h\cancel{(b-e)}}{d} = g\cancel{(b-e)}$$

$$ch = gd$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$cf - ci = bg - eg$$

$$c(f-i) = g(b-e)$$

$$bh - eh = df - di$$

$$h(b-e) = d(f-i)$$

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$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} - \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & -2 \end{array}\right)$$

$$= \begin{pmatrix} X_1-X_3 & X_2-X_4 \\ 2X_1 & 2X_2 \end{pmatrix} - \begin{pmatrix} X_1+2X_2 & -X_1 \\ X_3+2X_4 & -X_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$= \begin{pmatrix} -X_3-2X_2 & X_1+X_2-X_4 \\ 2X_1-X_3-2X_4 & 2X_2+X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} -2X_2-X_3 = 1 \\ X_1+X_2-X_4 = 0. \\ 2X_1-X_3-2X_4 = 0. \\ 2X_2+X_3 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$X_1 = 1 - X_2 + 2X_3 = 1 + X_3$

$X_2 = X_3 = t$

$$\left(\begin{array}{cccc|c} 0 & -2 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & -1 & -2 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

C

1. Suppose that the augmented matrix of a system of linear equations has been reduced to $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. Which of the following statements describes the set of solutions to the system?

$x_1 = 1 - x_2$
$x_3 = 1$
$x_4 = 0$
$x = \begin{bmatrix} 1-t \\ t \\ 1 \\ 0 \end{bmatrix}$

(A) infinitely many solution with three parameters
(B) infinitely many solution with two parameters
(C) infinitely many solution with one parameter
(D) unique solution
(E) no solution

D

2. Let A be an $n \times n$ matrix and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct solutions to the system $A\mathbf{x} = \mathrm{b}$. Which of the following statements are TRUE?

(i) A cannot be carried to I_n by elementary row operations ✓

(ii) $3\mathbf{x}_1 + 2\mathbf{x}_2$ is a solution to the system $A\mathbf{x} = 5\mathbf{b}$ ✓

(iii) $3\mathbf{x}_1 + 2\mathbf{x}_2$ is a solution to the system $A\mathbf{x} = \mathrm{b}$ X.

$3Ax_1 = 3b$
$2Ax_2 = 2b$.
$A(3x_1+2x_2) = 5b$.

(A) (ii) only
(B) (iii) only
(C) (i) and (iii) only
(D) (i) and (ii) only
(E) none of (i), (ii), or (iii)

B

3. Let A, B, and C be $n \times n$ matrices. Which of the following statements are TRUE?

(i) $(AB)^2 = A^2B^2$ X

(ii) $(ABC)^T = A^TB^TC^T$ X.

(iii) If $A^2 = 0$, then $A = 0$. X

$A \cdot A = 0$.
$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(A) (i) and (ii) only
(B) none of (i), (ii), or (iii)
(C) (ii) and (iii) only
(D) (i), (ii), and (iii)
(E) (iii) only

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