Orthosymplectic enumerative geometry

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Abstract

This thesis studies enumerative invariants counting orthogonal and symplectic objects in linear categories arising from algebraic geometry, as a first step towards generalizing known results and methods in linear enumerative geometry to general non-linear moduli problems.

The main focus of the thesis is the construction of orthosymplectic Donaldson–Thomas invariants and the study of their properties. Examples include invariants counting self-dual representations of self-dual quivers with potential, invariants counting orthosymplectic complexes of coherent sheaves on Calabi–Yau threefolds, a motivic version of Vafa–Witten type invariants counting orthosymplectic Higgs complexes on surfaces, and so on. We prove wall-crossing formulae relating these invariants for different stability conditions, and we carry out explicit computations in some cases.

Statement of authorship

This thesis primarily consists of the author's own work, based on [26; 28; 29], except for parts of §3.3, §3.7, and §5.2, which are also based on the author's collaboration with Davison, Halpern-Leistner, Ibáñez Núñez, Kinjo, and Pădurariu [30–32].

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Contents

Conventions

We assume familiarity with basic concepts in algebraic geometry and higher category theory. Standard textbooks include Hartshorne [71] and *the Stacks project* [144] for the former, and Lurie [109; 110; 112] for the latter.

We write $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the set of natural numbers. Write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ for the cyclic group of order $n \in \mathbb{Z}_{>0}$. Denote by $\mathrm{Sp}(2n)$ (rather than $\mathrm{Sp}(n)$) the n-th symplectic group for $n \in \mathbb{N}$, as an algebraic group over a given base field.

For an integer $n > 0$, an n- or ∞-*category* means an $(n, 1)$- or $(\infty, 1)$-category. For a category (or a higher category) $\mathscr{C}$, and objects $x, y \in \mathscr{C}$, write $\mathscr{C}(x, y)$ for the set (or space) of morphisms from x to y in $\mathscr{C}$. We freely use the language of higher category theory. For example, functors into higher categories always mean higher functors; limits and colimits in higher categories always mean homotopy limits and colimits; algebra objects in higher categories are always homotopy coherent; and so on.

We deal with set-theoretic size issues following Lurie [109, §1.2.15], by assuming the following *axiom of universes*: For every set x, there exists a Grothendieck universe U such that $x \in U$. We use these universes implicitly. For example, 'the category of sets' refers to sets living in a fixed but unspecified universe, whereas the category itself lives in a bigger universe.

Chapter 1
Introduction

1.1 Overview

1.1.1. Enumerative geometry is the geometric study of moduli spaces arising in algebraic geometry and other areas of mathematics. From its early origins in ancient times up to the present day, it has served as a rich source of inspiration and problems, contributing to the development of many new geometric theories and methods.

The modern study of enumerative geometry began with, among others, the study of the moduli space of holomorphic vector bundles on a compact Riemann surface. Foundational works on this moduli space include the seminal works of Mumford [120], Harder and Narasimhan [70], Atiyah and Bott [7], and others, spanning several decades.

The ideas and methods developed in these works are powerful and far-reaching, and continue to inspire new developments today in understanding more complicated moduli spaces, such as the moduli of coherent sheaves on higher dimensional varieties. However, it has remained true that moduli spaces of *linear* nature, such as those of vector bundles or sheaves, are better understood than the non-linear ones, such as those of principal G-bundles where G is an algebraic group that is not $\mathrm{GL}(n)$, or $\mathrm{SL}(n)$, etc.

1.1.2. One major difficulty in enumerative geometry is that moduli spaces are often *stacky*, or that they are more naturally seen as *moduli stacks*, meaning that their points can acquire non-trivial automorphism groups. This often causes technical complexity in studying such moduli spaces, and this is the main issue that this thesis addresses.

In this thesis, we take a first step in the enumerative study of non-linear moduli stacks by studying *orthosymplectic* moduli stacks, such as those of principal G-bundles when $G = \mathrm{O}(n)$ or $\mathrm{Sp}(n)$, drawing ideas from existing methods in linear enumerative geometry.

Orthosymplectic moduli stacks are closely related to linear ones, making them easier among the non-linear ones, while also allowing us to get a first glimpse at the behaviour of non-linear moduli stacks, which will eventually lead us to an *intrinsic* theory of enumerative geometry, which we describe in §1.5.1.

We note that although some non-linear moduli spaces are well-understood, such as those in Gromov–Witten theory, as in Li and Tian [105], the automorphism groups there are usually finite, and do not cause the issues that we discuss here.

1.1.3. A central topic in enumerative geometry is the study of *enumerative invariants*, which are numbers or other types of data constructed from the geometry of moduli spaces, usually having an interpretation as counting points in such moduli spaces in a certain sense. See, for example, Kiem [95] for a survey of the subject.

For example, when the moduli space is discrete, the number of points in the moduli space is an enumerative invariant. As another example, when the moduli space is smooth and compact, its fundamental class in homology is another enumerative invariant. More interesting cases are those where the moduli space can be stacky and singular, where invariants are constructed from different types of *virtual geometry* of the moduli space.

1.1.4. Many different flavours of enumerative invariants have been developed in the linear case in the past decades. Notable examples include the following:

(i) Intersection pairings on moduli spaces of semistable vector bundles on Riemann surfaces. These were first computed by Witten [161] using physical methods, and rigorously proved by Jeffrey and Kirwan [80] in the case of smooth moduli spaces, later generalized by Jeffrey, Kiem, Kirwan, and Woolf [79] to the singular case. Another computation was recently done by the author [25] using the framework of Joyce [88], giving an equivalent (in the smooth case) but more concise and more conceptual formula.

(ii) Invariants counting coherent sheaves on algebraic surfaces, sometimes called *algebraic Donaldson invariants*. These can be seen as a generalization of the *virtual fundamental* *classes* of Behrend and Fantechi [12] from Deligne–Mumford stacks to Artin stacks, and were studied by Mochizuki [118], with recent developments by Joyce [88].

(iii) *Donaldson–Thomas invariants* counting coherent sheaves on Calabi–Yau threefolds. This theory was initiated by Donaldson and Thomas [55] and Thomas [151], and later developed by Behrend [10], Joyce [81–85], Joyce and Song [89], and Kontsevich and Soibelman [99], using motivic methods.

(iv) *Cohomological Donaldson–Thomas theory*, which can be seen as a categorification of Donaldson–Thomas theory, where the invariants are vector spaces rather than numbers, and are sometimes called *BPS cohomology* after the physicists Bogomol'nyi, Prasad, and Sommerfield. This was conjecturally proposed by Kontsevich and Soibelman [100], and rigorously understood through the works of Efimov [57], Meinhardt and Reineke [117], and Davison and Meinhardt [46] in the linear case, and understood very recently in the non-linear case by Hennecart [73; 74] and the author et al. [30].

(v) *Categorical Donaldson–Thomas theory*, which is a further categorification of cohomological Donaldson–Thomas theory, where the invariants are categories rather than vector spaces. This theory was recently developed in a series of works of Toda [154; 155] and Pădurariu and Toda [125–133].

(vi) *Vafa–Witten invariants* counting Higgs sheaves on algebraic surfaces, arising from the work of Vafa and Witten [160], and developed by Tanaka and Thomas [148; 149] and Thomas [152].

(vii) *Donaldson–Thomas theory for Calabi–Yau fourfolds*, or the theory of *DT4 invariants* counting coherent sheaves on Calabi–Yau fourfolds. With its foundations developed by Cao and Leung [38], Borisov and Joyce [18], and Oh and Thomas [122; 123], the theory has seen increasing interest recently.

However, until very recently, it had not been clear how to generalize any of these theories outside the linear case, except the case (i), which was discussed in Teleman and Woodward [150].

1.1.5. A main reason that many of the above theories are restricted to the linear case is the difficulty of dealing with *strictly semistable points*, or points in the semistable locus in the moduli stack that have positive-dimensional automorphism groups, making the semistable locus a genuine Artin stack, rather than a scheme or a Deligne–Mumford stack. When this happens, defining invariants usually requires heavy extra work. For many theories listed above, techniques to deal with this are only available in the linear case. See §1.2.6 below for more detailed examples of technical issues involved here.

Note that when there are no strictly semistable points, that is, when the moduli stack is *Deligne–Mumford*, we often do not need to restrict to the linear case, since the technical issues mentioned above are not present.

1.1.6. This thesis will mainly focus on developing an orthosymplectic version of the theory (iii) above, that is, an orthosymplectic version of Donaldson–Thomas theory, as a first step towards understanding general non-linear moduli stacks and their enumerative invariants, especially how to deal with strictly semistable points in these cases.

During the preparation of this thesis, the author and his collaborators [30–33] have been working on a more general framework of enumerative geometry, called *intrinsic enumerative geometry*, which allows the generalization of many results in this thesis to more general moduli stacks. We will discuss this framework in slightly more detail in §1.5 below.

1.2 Linear enumerative geometry

1.2.1. In this section, we sketch through the main ideas in existing theories of linear enumerative geometry, such as those listed in §1.1.4. We will discuss generalizations of these ideas outside the linear case in §§1.3–1.5 below.

We describe a general process that applies to most variants of linear enumerative geometry, and we outline the main technical difficulties in them, which we will be facing again when trying to generalize them outside the linear case.

1.2.2. The setting. In linear enumerative geometry, we usually start with a linear category $\mathscr{A}$, together with a moduli stack $\mathscr{X}$ of objects in $\mathscr{A}$. Thus, points in $\mathscr{X}$ correspond to objects in $\mathscr{A}$, and automorphism groups of points in $\mathscr{X}$ correspond to automorphism groups of objects in $\mathscr{A}$.

Typical examples of such linear categories $\mathscr{A}$ include:

- The category $\mathsf{Coh}(Y)$ of coherent sheaves on a smooth projective $\mathbb{C}$-variety Y.
- The category $\mathsf{Mod}(\mathbb{C}Q)$ of representations of a quiver Q over $\mathbb{C}$, or its variants, such as the category $\mathsf{Mod}(\mathbb{C}Q, W)$ of representations of Q with potential W (see §4.1.7).

In all these cases, we have natural choices of moduli stacks, which are algebraic stacks over $\mathbb{C}$.

Such a moduli stack $\mathcal{X}$ usually has infinitely many connected components, and we consider its decomposition

$$\mathcal{X} = \coprod_{\alpha \in \pi_0(\mathcal{X})} \mathcal{X}_\alpha \tag{1.2.2.1}$$

into connected components, where $\pi_0(\mathcal{X})$ is the set of connected components of $\mathcal{X}$, and then study invariants counting points in each component $\mathcal{X}_\alpha$.

The set $\pi_0(\mathcal{X})$ has the structure of a commutative monoid, induced by the direct sum in $\mathcal{A}$. We denote the monoid operation by $+$, and its unit by 0. The component $\mathcal{X}_0 \subset \mathcal{X}$ is usually a single point $\{0\}$, and we have direct sum morphisms

$$\oplus \colon \mathcal{X}_\alpha \times \mathcal{X}_\beta \longrightarrow \mathcal{X}_{\alpha+\beta} \tag{1.2.2.2}$$

for $\alpha, \beta \in \pi_0(\mathcal{X})$.

1.2.3. Stability conditions. A next ingredient in constructing enumerative invariants is a *stability condition.* This is needed because it can often happen that the components $\mathcal{X}_\alpha$ are not quasi-compact, so that counting all points in them will not be meaningful. Instead, we decompose $\mathcal{X}_\alpha$ into smaller quasi-compact pieces which we count points in.

A *stability condition* in this case can be defined as a map

$$\tau \colon \pi_0(\mathcal{X}) \setminus \{0\} \longrightarrow T$$

to some totally ordered set T, satisfying certain conditions. For a point in $\mathcal{X}_\alpha$ representing an object $x \in \mathcal{A}$, the value $\tau(\alpha)$ is called the *slope* of x, also denoted by $\tau(x)$. Such an object is called *τ-semistable* if the slopes of its non-zero subobjects do not exceed its own slope, that is, we have $\tau(x') \leqslant \tau(x)$ for all $0 \neq x' \subset x$.

This condition determines a *semistable locus*

$$\mathcal{X}_\alpha^{\mathrm{ss}}(\tau) \subset \mathcal{X}_\alpha \ ,$$

which we usually require to be a quasi-compact open substack. We also require that each object $x \in \mathscr{A}$ has a unique *Harder–Narasimhan filtration*, that is a filtration of the form

$$\begin{tikzcd} 0 = x_0 \arrow[r, hook] & x_1 \arrow[r, hook] \arrow[d, two heads] & x_2 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & x_k = x \arrow[d, two heads] \\ & y_1 & y_2 & & y_k\,, \end{tikzcd} \tag{1.2.3.1}$$

where each quotient $y_i = x_i / x_{i-1}$ is non-zero and τ-semistable, with $\tau(y_1) > \cdots > \tau(y_k)$.

1.2.4. Stratifications. Geometrically, on the moduli stack, the existence and uniqueness of Harder–Narasimhan filtrations correspond to a stratification

$$\mathcal{X}_\alpha = \bigcup_{\substack{\alpha = \alpha_1 + \cdots + \alpha_n : \\ \tau(\alpha_1) > \cdots > \tau(\alpha_n)}} \mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau) * \cdots * \mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)\,, \tag{1.2.4.1}$$

where $\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau) * \cdots * \mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)$ denotes the stack of all filtrations in $\mathscr{A}$ with stepwise quotients lying in $\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau), \ldots, \mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)$, in that order, and we run over all ways to write α as a sum of non-zero classes α_i with strictly decreasing slopes. This is often a Θ-*stratification* in the sense of Halpern-Leistner [67].

In light of the stratification (1.2.4.1), one might argue that to make sense of counting points in $\mathcal{X}_\alpha$, we may instead count points in the semistable loci $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)$. In other words, we construct enumerative invariants depending on α and τ, based on the geometry of $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)$, and this will be a satisfactory answer to the question of counting points in $\mathcal{X}_\alpha$.

1.2.5. Motivic relations. We can often extract from the decomposition (1.2.4.1) precise relations between the enumerative information of $\mathcal{X}_\alpha$ and the $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)$. This process is straightforward in the motivic setting, and will be useful in this thesis.

More precisely, we consider *rings of motives*, which are roughly rings generated by classes $[\mathcal{Z}]$ of algebraic stacks $\mathcal{Z}$, with the *cut-and-paste relations*

$$[\mathcal{Z}] = [\mathcal{Z}'] + [\mathcal{Z} \setminus \mathcal{Z}'] \tag{1.2.5.1}$$

for closed substacks $\mathcal{Z}' \subset \mathcal{Z}$. The stratification (1.2.4.1) directly leads to the relation

$$[\mathcal{X}_\alpha] = \sum_{\substack{\alpha = \alpha_1 + \cdots + \alpha_n : \\ \tau(\alpha_1) > \cdots > \tau(\alpha_n)}} [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] * \cdots * [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \tag{1.2.5.2}$$

in the *motivic Hall algebra*, which we discuss in detail in Chapter 5, where $*$ denotes the multiplication in the motivic Hall algebra, defined by taking the stack of filtrations with given stepwise quotients, so that the product is equal to $[\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau) * \cdots * \mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)]$, the motive of the stack of filtrations appearing in (1.2.5.2).

Relations like (1.2.5.2) are important tools in studying enumerative invariants, and can be used to obtain *wall-crossing formulae*, which relate invariants defined for different stability conditions τ. We will discuss this more in §1.4.7 below.

1.2.6. Enumerative invariants. As we mentioned in §1.1.5, a major difficulty in constructing invariants is that the semistable loci $\mathcal{X}^{\mathrm{ss}}_{\alpha}(\tau)$ might contain *strictly semistable points*, or points with positive-dimensional automorphism groups, even if we ignore (i.e., quotient out) the copy of $\mathbb{G}_{\mathrm{m}}$ present in the automorphism group of every non-zero object as scalar automorphisms.

The presence of strictly semistable points means that the moduli stack is genuinely stacky, and is an Artin stack instead of a Deligne–Mumford stack. In many cases, this causes technical difficulty that requires heavy extra work to deal with, often involving further decompositions of the enumerative information of $\mathcal{X}^{\mathrm{ss}}_{\alpha}(\tau)$ into even simpler pieces which we finally call *enumerative invariants*. We demonstrate this in the following examples:

Firstly, in the motivic setting, we would like to make sense of the Euler characteristic of the moduli stack. Naïvely, we would like to define the Euler characteristic of a quotient stack U/G to be $\chi(U/G) = \chi(U)/\chi(G)$, where U is a scheme acted on by an algebraic group G. However, this almost never works, since we have $\chi(G) = 0$ for any algebraic group G of positive rank, so that we would have to define $\chi(*/G) = \infty$. To obtain a finite number requires a machinery developed by Joyce [81–85]. Roughly, we consider a further decomposition

$$[\mathcal{X}^{\mathrm{ss}}_{\alpha}(\tau)] = \sum_{\substack{\alpha=\alpha_1+\cdots+\alpha_n: \\ \tau(\alpha_1)=\cdots=\tau(\alpha_n)}} \frac{1}{n!} \cdot \epsilon_{\alpha_1}(\tau) * \cdots * \epsilon_{\alpha_n}(\tau) \tag{1.2.6.1}$$

in the motivic Hall algebra, where the sum is over ordered partitions of α into non-zero classes α_i of equal slope, and the *epsilon motives* $\epsilon_{\alpha}(\tau)$ are uniquely determined by the requirement that (1.2.6.1) holds for all classes α. The choice of the coefficients $1/n!$ ensures that $\epsilon_{\alpha}(\tau)$, which is the leading term in (1.2.6.1), has a well-defined Euler characteristic (after multiplying by the motive $[\mathbb{G}_{\mathrm{m}}]$ to correct for the scalar automorphisms). We can read from (1.2.6.1) that the epsilon motive $\epsilon_{\alpha}(\tau)$ is the motive $[\mathcal{X}^{\mathrm{ss}}_{\alpha}(\tau)]$ with certain parts removed on the strictly semistable locus, and we regard it as a good representative of $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)$. These epsilon motives are used to define Donaldson–Thomas invariants by Joyce and Song [89].

Secondly, in the cohomological setting, this problem corresponds to the phenomenon that the cohomology of $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)$ can be infinite-dimensional. For example, we have $\mathrm{H}^\bullet(*/\mathbb{G}_\mathrm{m}; \mathbb{Q}) \simeq \mathbb{Q}[t]$, a free polynomial ring. To obtain finite-dimensional invariants, one considers decompositions of the form

$$\mathrm{H}(\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)) = \bigoplus_{\substack{\alpha=\alpha_1+\cdots+\alpha_n: \\ \tau(\alpha_1)=\cdots=\tau(\alpha_n)}} \Bigl(\mathrm{BPS}_{\alpha_1}(\tau) \otimes \cdots \otimes \mathrm{BPS}_{\alpha_n}(\tau) \otimes \mathbb{Q}[t_1, \ldots, t_n]\Bigr)^{\mathrm{Aut}(\alpha_1,\ldots,\alpha_n)}, \tag{1.2.6.2}$$

where H denotes either rational cohomology or its certain variants, the sum is over unordered partitions of α into non-zero classes α_i of equal slope, and $\mathrm{Aut}(\alpha_1, \ldots, \alpha_n)$ is the group of permutations of $\{1, \ldots, n\}$ that preserve the sequence $(\alpha_1, \ldots, \alpha_n)$. The spaces $\mathrm{BPS}_\alpha(\tau)$ are usually finite-dimensional, sometimes called *BPS cohomology*, and we regard them as good cohomological representations of $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)$. This type of decompositions were conjectured by Kontsevich and Soibelman [100], and proved in different generalities by Efimov [57], Davison and Meinhardt [46], Hennecart [73; 74], and the author et al. [30].

Thirdly, in the homological setting, we would like to define a generalized version of virtual fundamental classes of $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)$. Such a construction is not directly available in the presence of strictly semistable points. The works of Mochizuki [118] and Joyce [88] deal with this by constructing an auxiliary moduli space of *Bradlow pairs* after Bradlow [20], which contains no strictly semistable points, then transporting the virtual fundamental classes there to the original moduli space. It would be interesting to explore whether this approach has an interpretation as a decomposition-type construction similar to the previous cases.

1.3 Orthosymplectic enumerative geometry

1.3.1. The main subject of this thesis is *orthosymplectic enumerative geometry*, extending the techniques and constructions in linear enumerative geometry discussed above to the case of moduli stacks of orthogonal or symplectic objects. We see this as a first step towards a further generalization to more general algebraic stacks, which we will discuss in §1.5.1.

1.3.2. Self-dual linear categories. We now introduce the basic set-up of orthosymplectic enumerative geometry. We start with a linear category $\mathscr{A}$ as in §1.2.2, equipped with a *contravariant involution*, that is the following data:

(i) An equivalence $(-)^\vee \colon \mathscr{A} \xrightarrow{\sim} \mathscr{A}^{\mathrm{op}}$.

(ii) A natural isomorphism $\eta \colon \mathrm{id}_{\mathscr{A}} \xRightarrow{\sim} (-)^{\vee\vee}$, satisfying compatibility conditions.

For example, $\mathscr{A}$ could be the category of vector bundles of finite rank on a smooth projective curve, where the involution $(-)^\vee$ is given by taking the dual bundle, and the natural isomorphism η can be chosen to be $\varepsilon = \pm 1$ times the usual identification.

We have the notion of a *self-dual object* of $\mathscr{A}$, defined as a pair (x, ϕ), where $x \in \mathscr{A}$ is an object, and $\phi \colon x \xrightarrow{\sim} x^\vee$ is an isomorphism satisfying $\phi = \phi^\vee$. We have a groupoid $\mathscr{A}^{\mathrm{sd}}$ of such self-dual objects. For example, in the case of vector bundles, $\mathscr{A}^{\mathrm{sd}}$ consists of either orthogonal or symplectic vector bundles, depending on the choice of the sign $\varepsilon = \pm 1$ mentioned above, which is a part of the data of the involution.

Note that $\mathscr{A}$ cannot be taken to be the category of coherent sheaves on a smooth projective variety of positive dimension, as it is not equivalent to its dual category. See §2.1.7 for explanations, and see §1.4.3 for a modification that fits into our framework.

1.3.3. Moduli stacks. In the setting above, suppose that we are given a moduli stack $\mathcal{X}$ of objects in $\mathscr{A}$, as in §1.2.2, such that the involution on $\mathscr{A}$ induces a $\mathbb{Z}_2$-action on $\mathcal{X}$. The fixed locus $\mathcal{X}^{\mathrm{sd}} = \mathcal{X}^{\mathbb{Z}_2}$, defined as a (2-categorical) limit in the 2-category of algebraic stacks, is the moduli stack of self-dual objects in the sense above.

Again, we consider the decomposition

$$\mathcal{X}^{\mathrm{sd}} = \coprod_{\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})} \mathcal{X}^{\mathrm{sd}}_\theta \tag{1.3.3.1}$$

into connected components. We also often consider the monoid action

$$\begin{aligned} \oplus^{\mathrm{sd}} \colon \mathcal{X} \times \mathcal{X}^{\mathrm{sd}} &\longrightarrow \mathcal{X}^{\mathrm{sd}} , \\ (x, y) &\longmapsto x \oplus y \oplus x^\vee , \end{aligned} \tag{1.3.3.2}$$

where we equip $x \oplus y \oplus x^\vee$ with the self-dual structure induced by that of y. This type of operations often give rise to modules for various algebras defined from $\mathcal{X}$. For example, in

§5.4, we will use a variant of this action to define the *motivic Hall module* for the *motivic Hall algebra* associated to $\mathcal{X}$.

In particular, the set $\pi_0(\mathcal{X}^{\mathrm{sd}})$ also acquires an action by the monoid $\pi_0(\mathcal{X})$, which we denote simply by $(\alpha, \theta) \mapsto \alpha + \theta + \alpha^\vee$ for $\alpha \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$.

We remark that although the stack $\mathcal{X}^{\mathrm{sd}}$ and the set $\pi_0(\mathcal{X}^{\mathrm{sd}})$ also carry commutative monoid structures given by the direct sum, these structures are less often used in this thesis.

1.3.4. Stability conditions. Now, suppose that we have a stability condition τ on $\mathcal{A}$, as in §1.2.3. We also assume that τ is compatible with the self-dual structure, in that $\tau(x^\vee) = -\tau(x)$ for all non-zero objects $x \in \mathcal{A}$, where $t \mapsto -t$ is an order-reversing involution of the totally ordered set T that τ is valued in. Assume that there is a unique element $0 \in T$ fixed by this involution.

Then, for any non-zero self-dual object $(x, \phi) \in \mathcal{A}^{\mathrm{sd}}$, we necessarily have $\tau(x) = 0$, and in the Harder–Narasimhan filtration

$$\begin{tikzcd} 0 = x_0 \arrow[r, hook] & x_1 \arrow[r, hook] \arrow[d, two heads] & x_2 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & x_k = x \arrow[d, two heads] \\ & y_1 & y_2 & & y_k \end{tikzcd} \tag{1.3.4.1}$$

of x, the self-dual structure ϕ induces isomorphisms $y_i \xrightarrow{\sim} y_{k+1-i}^\vee$ of the factors (see §2.2.4). In particular, if k is odd, the middle factor $y_{(k+1)/2}$ admits an induced self-dual structure. For convenience, when k is even, we sometimes think of it as having the zero self-dual object as the middle factor.

Therefore, heuristically speaking, if we think of objects of $\mathcal{A}$ as composed of semistable objects via Harder–Narasimhan filtrations as in §1.2.3, we should think of an object of $\mathcal{A}^{\mathrm{sd}}$ as composed of a series of semistable objects of $\mathcal{A}$, which are those in the left half of the Harder–Narasimhan filtration, together with a single semistable self-dual object in $\mathcal{A}^{\mathrm{sd}}$ in the middle, allowed to be zero; the factors in the right half are dual to those on the left, and do not contain new information.

1.3.5. Stratifications. Geometrically, similarly to §1.2.4, the existence and uniqueness of

Harder–Narasimhan filtrations correspond to a stratification

$$\mathcal{X}_\theta^{\mathrm{sd}} = \bigcup_{\substack{\theta=\alpha_1+\alpha_1^\vee+\cdots+\alpha_n+\alpha_n^\vee+\rho: \\ \tau(\alpha_1)>\cdots>\tau(\alpha_n)>0}} \mathcal{X}_{\alpha_1}^{\mathrm{ss}}(\tau) \diamond \cdots \diamond \mathcal{X}_{\alpha_n}^{\mathrm{ss}}(\tau) \diamond \mathcal{X}_\rho^{\mathrm{sd,ss}}(\tau) , \tag{1.3.5.1}$$

where we run over classes $\alpha_i \in \pi_0(\mathcal{X}) \setminus \{0\}$ and $\rho \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, and we allow $n = 0$, giving the leading term, the semistable locus $\mathcal{X}_\theta^{\mathrm{sd,ss}}(\tau)$. We denote by $\mathcal{X}_{\alpha_1}^{\mathrm{ss}}(\tau) \diamond \cdots \diamond \mathcal{X}_{\alpha_n}^{\mathrm{ss}}(\tau) \diamond \mathcal{X}_\rho^{\mathrm{sd,ss}}(\tau)$ the stack of *self-dual filtrations*, or filtrations of the form in §1.3.4, with stepwise quotients lying in $\mathcal{X}_{\alpha_1}^{\mathrm{ss}}(\tau), \ldots, \mathcal{X}_{\alpha_n}^{\mathrm{ss}}(\tau), \mathcal{X}_\rho^{\mathrm{sd,ss}}(\tau), \mathcal{X}_{\alpha_n^\vee}^{\mathrm{ss}}(\tau), \ldots, \mathcal{X}_{\alpha_1^\vee}^{\mathrm{ss}}(\tau)$, in that order, where we allow the middle term of the filtration to be zero. As in the linear case, this is also usually a Θ-stratification.

Again, similarly to §1.2.5, the stratification (1.3.5.1) implies the motivic relation

$$[\mathcal{X}_\theta^{\mathrm{sd}}] = \sum_{\substack{\theta=\alpha_1+\alpha_1^\vee+\cdots+\alpha_n+\alpha_n^\vee+\rho: \\ \tau(\alpha_1)>\cdots>\tau(\alpha_n)>0}} [\mathcal{X}_{\alpha_1}^{\mathrm{ss}}(\tau)] \diamond \cdots \diamond [\mathcal{X}_{\alpha_n}^{\mathrm{ss}}(\tau)] \diamond [\mathcal{X}_\rho^{\mathrm{sd,ss}}(\tau)] \tag{1.3.5.2}$$

in the *motivic Hall module* for the motivic Hall algebra, which we introduce in Chapter 5, and this will be important for studying orthosymplectic enumerative invariants.

1.3.6. Enumerative invariants. We now explain how to construct enumerative invariants in the orthosymplectic case, in the presence of strictly semistable points, parallel to the descriptions in §1.2.6.

Firstly, in the motivic setting, which is the main focus of this thesis, we roughly consider a further decomposition

$$[\mathcal{X}_\theta^{\mathrm{sd,ss}}(\tau)] = \sum_{\substack{\theta=\alpha_1+\alpha_1^\vee+\cdots+\alpha_n+\alpha_n^\vee+\rho: \\ \tau(\alpha_1)=\cdots=\tau(\alpha_n)=0}} \frac{1}{2^n n!} \cdot \epsilon_{\alpha_1}(\tau) \diamond \cdots \diamond \epsilon_{\alpha_n}(\tau) \diamond \epsilon_\rho^{\mathrm{sd}}(\tau) \tag{1.3.6.1}$$

in the motivic Hall module, which is parallel to the decomposition (1.2.6.1) in the linear case. The epsilon motives $\epsilon_{\alpha_i}(\tau)$ are the ones defined in the linear case, and the new epsilon motives $\epsilon_\theta^{\mathrm{sd}}(\tau)$, defined by the relation (1.3.6.1), are one of the main constructions of this thesis, which we will use to define orthosymplectic Donaldson–Thomas invariants, and the choice of the coefficients $1/2^n n!$ ensure that they have well-defined Euler characteristics. See Chapter 5 for details.

Secondly, in the cohomological setting, we hope to obtain decompositions of the form

$$\mathrm{H}(\mathscr{X}_\theta^{\mathrm{sd,ss}}(\tau)) = \bigoplus_{\substack{\alpha=\alpha_1+\alpha_1^\vee+\cdots+\alpha_n+\alpha_n^\vee+\rho: \\ \tau(\alpha_1)=\cdots=\tau(\alpha_n)=0}} \Bigl(\mathrm{BPS}_{\alpha_1}(\tau) \otimes \cdots \otimes \mathrm{BPS}_{\alpha_n}(\tau) \otimes \mathrm{BPS}_\rho^{\mathrm{sd}}(\tau) \otimes \mathbb{Q}[t_1, \ldots, t_n]\Bigr)^{\mathrm{Aut}^{\mathrm{sd}}(\alpha_1,\ldots,\alpha_n)}, \tag{1.3.6.2}$$

where $\mathrm{Aut}^{\mathrm{sd}}(\alpha_1, \ldots, \alpha_n)$ is the group of $\mathbb{Z}_2$-equivariant permutations of the set $\{1, 1^\vee, \ldots, n, n^\vee\}$, where the $\mathbb{Z}_2$-action exchanges each pair $(i, i^\vee)$, such that the sequence $(\alpha_1, \alpha_1^\vee, \ldots, \alpha_n, \alpha_n^\vee)$ remains unchanged after the permutation. The spaces $\mathrm{BPS}_\theta^{\mathrm{sd}}(\tau)$ are finite-dimensional, sometimes called *BPS cohomology*, and are regarded as the cohomological enumerative invariants in this setting. Such decompositions were first conjectured by Young [165], and then partially proved in different generalities as special cases of the results of Hennecart [73; 74] and the author et al. [30], and we refer to these works for details.

Thirdly, in the homological setting, the author's preprint [27] attempts to formulate a precise statement about how the invariants should behave, and constructs these invariants in the case of self-dual quivers, although we are not yet able to verify that these invariants satisfy all the expected properties.

1.3.7. Graded and filtered points. A helpful framework for studying enumerative geometry outside the linear case, and for finding the correct generalizations of notions from the linear case, is the theory of *stacks of graded and filtered points* due to Halpern-Leistner [67].

For an algebraic stack $\mathscr{X}$, its stacks of graded and filtered points are defined as mapping stacks

$$\mathscr{G}rad(\mathscr{X}) = \mathscr{M}ap(*/\mathbb{G}_\mathrm{m}, \mathscr{X}), \tag{1.3.7.1}$$

$$\mathscr{F}ilt(\mathscr{X}) = \mathscr{M}ap(\mathbb{A}^1/\mathbb{G}_\mathrm{m}, \mathscr{X}), \tag{1.3.7.2}$$

where $\mathbb{G}_\mathrm{m}$ acts on $\mathbb{A}^1$ by scaling. See §3.2 for details.

For example, in the linear case, if $\mathscr{X}$ is the moduli stack of objects in a linear category $\mathscr{A}$, then $\mathscr{G}rad(\mathscr{X})$ and $\mathscr{F}ilt(\mathscr{X})$ are usually the stack of $\mathbb{Z}$-graded objects and the stack of $\mathbb{Z}$-indexed filtrations in $\mathscr{A}$, respectively.

In the orthosymplectic case, we consider the moduli stack $\mathscr{X}^{\mathrm{sd}}$ of self-dual objects in a self-dual linear category $\mathscr{A}$. In this case, $\mathscr{G}rad(\mathcal{X}^{\mathrm{sd}})$ is usually the stack of self-dual objects $(E, \phi) \in \mathscr{A}^{\mathrm{sd}}$, equipped with $\mathbb{Z}$-gradings $E = \bigoplus_{i \in \mathbb{Z}} E_i$, such that $\phi(E_i) = E_{-i}^{\vee}$ for all i. Equivalently, this is the information of a series of objects $(E_i)_{i>0}$, together with a single self-dual object (E_0, ϕ_0). The stack $\mathscr{F}ilt(\mathcal{X}^{\mathrm{sd}})$ can also be described as the stack of *self-dual filtrations* indexed by $\mathbb{Z}$, similar to those appearing in §1.3.4. See §3.4 for details.

1.4 Main results

1.4.1. Orthosymplectic Donaldson–Thomas invariants. As mentioned above, the main constructions of this thesis are those of *orthosymplectic Donaldson–Thomas invariants*, including a numeric version and an enhanced, motivic version, which we discuss in Chapters 5 and 6, respectively.

More precisely, recall from §1.2.6 and §1.3.6 the epsilon motives $\epsilon_\alpha(\tau)$ and $\epsilon_\theta^{\mathrm{sd}}(\tau)$. The linear and orthosymplectic Donaldson–Thomas invariants are rational numbers defined by

$$\mathrm{DT}_\alpha(\tau) = \int_{\mathcal{X}_\alpha} (1 - \mathbb{L}) \cdot \epsilon_\alpha(\tau) \cdot \nu_{\mathcal{X}} \, d\chi \,, \tag{1.4.1.1}$$

$$\mathrm{DT}_\theta^{\mathrm{sd}}(\tau) = \int_{\mathcal{X}_\theta^{\mathrm{sd}}} \epsilon_\theta^{\mathrm{sd}}(\tau) \cdot \nu_{\mathcal{X}^{\mathrm{sd}}} \, d\chi \,, \tag{1.4.1.2}$$

where $\int (-) \, d\chi$ denotes taking the weighted Euler characteristic, and $\nu_{(-)}$ denotes the *Behrend function* of a stack. The extra factor $(1 - \mathbb{L})$ roughly accounts for the fact that in the linear case, every non-zero object has a copy of $\mathbb{G}_{\mathrm{m}}$ in its automorphism group, given by scalar automorphisms, whereas this is not the case for orthosymplectic objects. See §5.1 for more explanations of these definitions.

The linear invariants $\mathrm{DT}_\alpha(\tau)$ have seen many applications and connections with other fields of mathematics. The invariants themselves admit rich structures, such as wall-crossing structures as in Kontsevich and Soibelman [101], and a geometric structure on the stability space as in Bridgeland [24]. They have interesting relations to Gromov–Witten invariants that are not yet well-understood, as conjectured by Maulik, Nekrasov, Okounkov, and Pandharipande [114; 115]. They are also related to different aspects of representation theory, such as those studied by Kontsevich and Soibelman [99], Nagao [121], Córdova and Shao [42], and many others. We hope that many of the above constructions and applications will have analogous versions for the orthosymplectic invariants.

1.4.2. Invariants for quivers. A basic example of our theory is the construction of orthosymplectic Donaldson–Thomas invariants for *self-dual quivers with potential*, which we discuss in §4.1 and §8.1. These invariants are an orthosymplectic analogue of the usual Donaldson–Thomas theory for quivers with potential, studied in Joyce and Song [89, Ch. 7] and Kontsevich and Soibelman [99, §8]. Self-dual quivers were first introduced by Derksen and Weyman [51], and studied by Young [163–165] in the context of enumerative geometry.

Roughly speaking, a *self-dual quiver* is a quiver Q equipped with a contravariant involution $(-)^\vee \colon Q \simeq Q^{\mathrm{op}}$, where Q^{op} is the opposite quiver of Q, obtained from Q by reversing the direction of arrows. For example, we could take the quiver

$$Q = \left(\begin{tikzcd}[sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd} \right),$$

with the involution $(-)^\vee$ given by horizontal flipping. The involution induces a self-dual structure on the abelian category of representations of Q, which, in the above example, is roughly given by

$$\begin{tikzcd}[sep=small] & E_2 \arrow[dr, "e_{24}"] & & & & E_2^\vee \arrow[dr, "e_{12}^\vee"] & \\ E_1 \arrow[ur, "e_{12}"] \arrow[dr, "e_{13}"'] & & E_4 & \arrow[r, leftrightsquigarrow, "(-)^\vee"] & E_4^\vee \arrow[ur, "e_{24}^\vee"] \arrow[dr, "e_{34}^\vee"'] & & E_1^\vee \\ & E_3 \arrow[ur, "e_{34}"'] & & & & E_3^\vee \arrow[ur, "e_{13}^\vee"'] & \end{tikzcd},$$

where E_i are vector spaces and e_{ij} are linear maps. See §4.1 for the precise set-up. We are then interested in representations that are *self-dual*, meaning in the above example that $E_4 \simeq E_1^\vee$, and that there are self-dual structures on the vector spaces E_2 and E_3, together with conditions on the morphisms e_{ij}. The theory also works for quivers with potential.

This can be regarded as a local model for counting orthosymplectic sheaves on varieties, which we will discuss below.

We also provide an algorithm for computing Donaldson–Thomas invariants for self-dual quivers when the potential is zero, and present some numerical results. We mention a relation between self-dual quivers and orthosymplectic coherent sheaves in Example 8.2.5.

1.4.3. Invariants for threefolds. Another main example of our theory is the construction of

Donaldson–Thomas invariants counting *orthosymplectic complexes* on Calabi–Yau threefolds, which are perfect complexes of coherent sheaves equipped with self-dual structures. These are an orthosymplectic version of the usual Donaldson–Thomas theory counting coherent sheaves on Calabi–Yau threefolds, studied by Thomas [151], Joyce and Song [89], and Kontsevich and Soibelman [99].

As mentioned in §1.3.2, the category of coherent sheaves on a Calabi–Yau threefold does not fit into our framework, and we use an alternative approach involving the derived category of coherent sheaves, which we describe below.

For a smooth projective Calabi–Yau threefold Y over $\mathbb{C}$, we consider a Bridgeland stability condition $\tau = (Z, \mathscr{P})$ on Y in the sense of Bridgeland [22], such that it is compatible with a chosen self-dual structure on the derived category $\mathsf{D}^{\mathsf{b}}\mathsf{Coh}(Y)$. Then there is an abelian subcategory

$$\mathscr{P}(0) \subset \mathsf{D}^{\mathsf{b}}\mathsf{Coh}(Y)$$

of semistable objects of slope 0, which inherits a self-dual structure. We then define invariants counting self-dual objects in this category, or *τ-semistable orthosymplectic complexes*. See §4.2 and §8.3 for more details.

We expect that these invariants are related to counting D-branes in string theories on Calabi–Yau 3-*orientifolds*, discussed in, for example, Witten [162, §5.2], Diaconescu, Garcia-Raboso, Karp, and Sinha [54], and Hori and Walcher [76].

1.4.4. Invariants for curves and surfaces. We also introduce the following invariants similar to the above construction for threefolds.

For an algebraic curve C over a field, we define Donaldson–Thomas invariants counting semistable orthogonal or symplectic bundles on C, analogous to the motivic invariants counting semistable vector bundles considered by Joyce [85, §6.3]. We discuss this in §8.2.

For an algebraic surface S over $\mathbb{C}$ which is either a del Pezzo, K3, or an abelian surface, we also define motivic Vafa–Witten type invariants counting *orthosymplectic Higgs complexes* on S, which is similar to the Vafa–Witten invariants of Tanaka and Thomas [148; 149], although we work in the motivic setting, which is different from their approach using equivariant localization. We discuss this in §4.3 and §8.4. The main reason for restricting to this class of surfaces is that we do not know how to construct Bridgeland stability conditions on the derived category of Higgs sheaves on S if S is of general type, and such stability conditions are crucial in our approach, as discussed in §1.4.3.

1.4.5. In the rest of this section, we outline a few key general results proved in this thesis that apply to all of the above settings, which are used in constructing the invariants and studying their properties.

1.4.6. The no-pole theorem. The first such result is the *no-pole theorem*, Theorem 5.5.5, which is a key property of the epsilon motives $\epsilon_\alpha(\tau)$ and $\epsilon_\theta^{\mathrm{sd}}(\tau)$. It is roughly the statement that the Euler characteristics (1.4.1.1)–(1.4.1.2) are finite, which is a non-trivial property in light of the discussions in §1.2.6. Proving this property is the main technical difficulty in showing that our invariants are well-defined.

In the linear case, this result was proved by Joyce [83, Theorem 8.7] under a slightly different setting, and the orthosymplectic case of this theorem is a main result of this thesis.

1.4.7. Wall-crossing formulae. A key property of our orthosymplectic Donaldson–Thomas invariants is that they satisfy *wall-crossing formulae*, Theorem 7.3.2, which relate the invariants for different stability conditions.

More precisely, for stability conditions τ_+ and τ_-, under certain assumptions, we prove relations of the form

$$\mathrm{DT}_\alpha(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n}} C(\alpha_1, \ldots, \alpha_n) \cdot \mathrm{DT}_{\alpha_1}(\tau_+) \cdots \mathrm{DT}_{\alpha_n}(\tau_+)\,, \tag{1.4.7.1}$$

$$\mathrm{DT}_\theta^{\mathrm{sd}}(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\};\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho}} C'(\alpha_1, \ldots, \alpha_n) \cdot \mathrm{DT}_{\alpha_1}(\tau_+) \cdots \mathrm{DT}_{\alpha_n}(\tau_+) \cdot \mathrm{DT}_\rho^{\mathrm{sd}}(\tau_+)\,, \tag{1.4.7.2}$$

where $C(\ldots)$ and $C'(\ldots)$ are certain combinatorial coefficients. The linear case (1.4.7.1) was first obtained by Joyce and Song [89, Theorem 5.18] in the case of counting coherent sheaves on Calabi–Yau threefolds, and the orthosymplectic case (1.4.7.2) is a main result of this thesis.

Furthermore, in §7.5, we prove a similar result for wall-crossing in derived categories, where we compare invariants for two Bridgeland stability conditions that are close enough with respect to the metric on the space of stability conditions, and we show that wall-crossing formulae hold in this case.

Wall-crossing formulae are important because they impose a very strong constraint on the structure of the invariants, and can sometimes be used to compute the invariants directly, or to obtain very strong properties of the invariants. See, for example, Feyzbakhsh and Thomas [59–62] for a series of applications of this type in the linear case.

Moreover, we hope that other flavours of enumerative invariants, such as the quasi-smooth invariants mentioned in §1.1.4 (i), (ii), should exhibit the same wall-crossing behaviour, in that they should satisfy wall-crossing formulae with the same combinatorial coefficients. This phenomenon was already observed in the linear case by Gross, Joyce, and Tanaka [65] and Joyce [88]. See also the author [25] and Bojko, Lim, and Moreira [16] for applications of the wall-crossing formulae for quasi-smooth invariants in the linear case. Assuming that this phenomenon generalizes to the orthosymplectic case or more general cases, we can hope to predict the behaviour of these invariants, or even compute them, without necessarily having a general construction of the invariants.

1.4.8. The integral identity. A key technical ingredient in the proof of wall-crossing formulae for our Donaldson–Thomas invariants is the *motivic integral identity* for the Behrend functions that appear in the definitions of the invariants, (7.3.2.1)–(7.3.2.2).

In the linear case, the integral identity is the statement that the motivic version $\nu_{\mathcal{X}}^{\mathrm{mot}}$ of the Behrend function of $\mathcal{X}$ should satisfy, roughly, the relation

$$\nu_{\mathcal{X}}^{\mathrm{mot}}(y) \cdot \nu_{\mathcal{X}}^{\mathrm{mot}}(z) = \mathbb{L}^{-d/2} \cdot \int_{0 \to y \to x \to z \to 0} \nu_{\mathcal{X}}^{\mathrm{mot}}(x)\, dx \tag{1.4.8.1}$$

for given $y, z \in \mathcal{X}$, where the integral is a motivic integral in the sense of §5.2.6, taken over the space of short exact sequences $0 \to y \to x \to z \to 0$, and d is the virtual dimension of this space. This was conjectured by Kontsevich and Soibelman [99, Conjecture 4], and later proved by Lê [104]; a numerical version was proved earlier by Joyce and Song [89, Theorem 5.11] to obtain wall-crossing formulae for Donaldson–Thomas invariants.

In this thesis, we prove a more general version of the integral identity, Theorem 7.4.2, which works for a general class of (-1)-shifted symplectic stacks. It states that we have

$$\nu^{\mathrm{mot}}_{\mathcal{G}rad(\mathcal{X})} = \mathbb{L}^{-d/2} \cdot \mathrm{gr}_! \circ \mathrm{ev}^* (\nu^{\mathrm{mot}}_{\mathcal{X}}) \,, \tag{1.4.8.2}$$

where we consider the *attractor correspondence*

$$\mathcal{G}rad(\mathcal{X}) \xleftarrow{\mathrm{gr}} \mathcal{F}ilt(\mathcal{X}) \xrightarrow{\mathrm{ev}} \mathcal{X} \,, \tag{1.4.8.3}$$

where $\mathcal{G}rad(\mathcal{X})$ and $\mathcal{F}ilt(\mathcal{X})$ are the stacks of graded and filtered points introduced in §1.3.7.

In (1.4.8.2), the pushforward $\mathrm{gr}_!$ can be interpreted as integrating along the fibres of gr. For example, in the linear case, each fibre of gr consists of filtrations with given graded quotients, and restricting to a connected component of $\mathcal{F}ilt(\mathcal{X})$ where all filtrations are two-step filtrations gives the statement (1.4.8.1).

Similarly, in the orthosymplectic case, the integral identity (1.4.8.2) can be written more explicitly, roughly as

$$\nu^{\mathrm{mot}}_{\mathcal{X}}(y) \cdot \nu^{\mathrm{mot}}_{\mathcal{X}^{\mathrm{sd}}}(z) = \mathbb{L}^{-d/2} \cdot \int_{\substack{\text{3-step self-dual filtrations} \\ 0 = x_0 \subset x_1 \subset x_2 \subset x_3 = x \\ \text{with quotients } y,\, z,\, y^\vee}} \nu^{\mathrm{mot}}_{\mathcal{X}^{\mathrm{sd}}}(x) \, dx \tag{1.4.8.4}$$

for given $y \in \mathcal{X}$ and $z \in \mathcal{X}^{\mathrm{sd}}$, where d is the virtual dimension of the space of such self-dual filtrations. See also §2.2.4 for details on self-dual filtrations.

Our result is stronger than previous works on this topic mentioned above, as we are able to remove a technical assumption on weights which is not necessarily satisfied outside the linear case; see Theorem B.2.1 for details. This result will later also be used in a future work [33] to prove wall-crossing formulae for *intrinsic Donaldson–Thomas invariants*, which we discuss in §1.5.1 below.

1.5 Future directions

1.5.1. Intrinsic enumerative geometry. A natural direction to go from the above discussion is to further generalize this theory to more general algebraic stacks.

A satisfactory answer is already available as a framework that we would call *intrinsic enumerative geometry*, developed very recently by the author and collaborators [30–33], based on ideas from the author's earlier preprints [26; 27] on orthosymplectic enumerative geometry and Halpern-Leistner's formalism [67] generalizing important ideas in geometric invariant theory from quotient stacks to general algebraic stacks, involving the notions of *stacks of graded and filtered points*, Θ-*stratifications*, etc.

Substantial progress has already been made in this framework, including defining enumerative invariants in the flavours of §1.1.4 (iii), (iv) for more general stacks, and studying their properties. We hope that this framework will lead to more applications, such as generalizing other types of invariants in §1.1.4, or even applying the framework to other algebraic stacks, such as those arising from Gromov–Witten theory or from K-stability.

1.5.2. Quasi-smooth invariants. One possible direction for future work is to construct enumerative invariants that generalize the virtual fundamental class from Deligne–Mumford stacks to Artin stacks, in the style of §1.1.4 (i), (ii). One obstacle here is that the approach of Joyce [88] using stable pairs does not seem to easily generalize outside the linear case, so more work or a replacement approach is needed.

In the case of principal bundles on curves, Teleman and Woodward [150] recover the conjectural formula of Witten [161] as a large level limit of K-theoretic indices. It would be interesting to explore connections of their approach with the intrinsic framework discussed in §1.5.1. Additionally, in the case of $\mathrm{GL}(n)$, the author [25] obtained a formula involving a regularized divergent series. It would be interesting to explore whether a similar formula can be obtained for more general groups.

1.5.3. Vafa–Witten invariants. Another possible direction is to generalize the theory of Vafa–Witten invariants of Tanaka and Thomas [148; 149], mentioned in §1.1.4 (vi), to the orthosymplectic case. In the linear case, these invariants count coherent sheaves on surfaces equipped with Higgs fields, or *Higgs sheaves*. An orthosymplectic analogue of such Higgs sheaves is what we call *orthosymplectic Higgs complexes*, which we discuss in §4.3 and §8.4, defined using certain Bridgeland stability conditions. Although this thesis constructs a motivic version of Vafa–Witten invariants, we have not yet been able to construct invariants in the same flavour of Tanaka and Thomas using torus localization, and this is a possible direction for future work.

1.5.4. DT4 invariants. A perhaps more difficult problem is to construct DT4 invariants, mentioned in §1.1.4 (vii), in the orthosymplectic case and the general case, although even the linear case is not entirely understood yet. These invariants behave somewhat similarly to the quasi-smooth invariants discussed in §1.5.2, and it seems likely that a replacement of stable pairs mentioned there would also help with this case.

1.5.5. Duality. In many of the flavours of enumerative invariants above, it is often interesting to explore various types of duality relations between invariants for Langlands dual groups. One striking example is the modularity of generating functions of Vafa–Witten invariants, originally proposed by Vafa and Witten [160], which is related to the self-duality of the groups $\mathrm{GL}(n)$. The orthosymplectic setting of this thesis includes an interesting pair of dual groups, $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$, as well as the self-dual groups $\mathrm{SO}(2n)$, where one can hope to explore such duality relations.

1.5.6. Relations with Gromov–Witten theory. Using orthosymplectic Donaldson–Thomas invariants for a Calabi–Yau threefold Y, as in §1.4.3, we can expect to obtain curve-counting invariants by choosing a suitable self-dual structure on $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$, then counting self-dual complexes supported on curves. For a summary of similar constructions in the linear case, see Pandharipande and Thomas [134]. We also hope to be able to compare these invariants with Gromov–Witten invariants in the style of Maulik, Nekrasov, Okounkov, and Pandharipande [114; 115].

Chapter 2

Self-dual categories

This chapter introduces the basic setting of orthosymplectic enumerative geometry, including the notions of *self-dual linear categories* and *self-dual objects* in such categories, which we already briefly discussed in §1.3.2. A basic example to keep in mind is the linear category of vector bundles on a scheme, where self-dual objects are orthogonal or symplectic vector bundles, as described in Example 2.1.5. More examples and details will be provided in Chapter 4 below. For background on the basic theory of additive, abelian, triangulated, and derived categories, we refer to Mac Lane [113] and Gelfand and Manin [63].

The main focus of this thesis later on will be to construct and study orthosymplectic enumerative invariants counting such self-dual objects, based on the geometry of *moduli stacks* of these objects, which we will introduce in Chapter 3.

2.1 Self-dual linear categories

2.1.1. Linear categories. Let K be a field, which we fix throughout this chapter.

By a *K-linear category*, we mean an additive category $\mathscr{A}$ (see, for example, Mac Lane [113, §VIII.2]), together with the structure of a K-vector space on the set of morphisms $\mathscr{A}(x, y)$ for every pair of objects $x, y \in \mathscr{A}$, such that addition of vectors agrees with addition of morphisms using the additive category structure, and the composition map

$$\circ \colon \mathscr{A}(y, z) \times \mathscr{A}(x, y) \longrightarrow \mathscr{A}(x, z)$$

is K-bilinear for any $x, y, z \in \mathscr{A}$.

Examples of K-linear categories include the category of K-vector spaces, the category of modules over a K-algebra, the category of coherent sheaves on a K-scheme, etc.

2.1.2. Self-dual linear categories. For a K-linear category $\mathscr{A}$, define a *self-dual structure* on $\mathscr{A}$ to be the following data:

(i) An equivalence of K-linear categories

$$(-)^\vee \colon \mathscr{A} \xrightarrow{\sim} \mathscr{A}^{\mathrm{op}} , \tag{2.1.2.1}$$

called the *dual functor*.

(ii) A natural isomorphism

$$\eta \colon \mathrm{id}_{\mathscr{A}} \xRightarrow{\sim} (-)^{\vee\vee} , \tag{2.1.2.2}$$

such that for any object $x \in \mathscr{A}$, we have $\eta_{x^\vee} = (\eta_x^\vee)^{-1} \colon x^\vee \xrightarrow{\sim} x^{\vee\vee\vee}$.

A *self-dual K-linear category* is a K-linear category equipped with a self-dual structure.

Given a self-dual K-linear category $\mathscr{A}$, define a *self-dual object* in $\mathscr{A}$ to be a pair (x, ϕ), where $x \in \mathscr{A}$ and $\phi \colon x \xrightarrow{\sim} x^\vee$ is an isomorphism, such that $\phi = \phi^\vee \circ \eta_x$:

$$\begin{tikzcd} x \arrow[d, "\eta_x"', "\wr"] \arrow[dr, "\phi", "\sim"'] & \\ x^{\vee\vee} \arrow[r, "\sim", "\phi^\vee"'] & x^\vee \end{tikzcd} .$$

We denote by $\mathscr{A}^{\mathrm{sd}}$ the groupoid of self-dual objects in $\mathscr{A}$, where morphisms are isomorphisms in $\mathscr{A}$ compatible with the self-dual structures.

2.1.3. Self-dual objects as fixed points. More conceptually, a self-dual K-linear category can be defined as a (2-categorical) fixed point of the $\mathbb{Z}_2$-action on the 2-category of K-linear categories given by taking the opposite category, and a self-dual object in a self-dual K-linear category $\mathscr{A}$ is a (2-categorical) fixed point of the $\mathbb{Z}_2$-action on the underlying groupoid of $\mathscr{A}$ given by its self-dual structure.

Note that as we are considering $\mathbb{Z}_2$-actions on categories and higher categories, the only notion of fixed points that makes sense is the homotopy one, which is defined as the homotopy limit of a functor from the groupoid $\mathrm{B}\mathbb{Z}_2$, and its behaviour is different from the classical notion of fixed points. For example, a fixed point in a category is not only the data of an object in the original category that is fixed by the action, but contains extra data including explicit isomorphisms witnessing the object being fixed, as in the above explicit definitions.

2.1.4. The hyperbolic self-dual object. Let $\mathscr{A}$ be a self-dual K-linear category, and let $x \in \mathscr{A}$ be an object. Then there is a self-dual object $(x \oplus x^\vee, \phi) \in \mathscr{A}^{\mathrm{sd}}$, with the *hyperbolic self-dual structure* given by

$$\phi = \begin{pmatrix} 0 & \mathrm{id}_{x^\vee} \\ \eta_x & 0 \end{pmatrix} \colon x \oplus x^\vee \xrightarrow{\sim} x^\vee \oplus x^{\vee\vee} \, . \tag{2.1.4.1}$$

2.1.5. Example. Vector bundles. Let X be a K-scheme, and let $\mathscr{A} = \mathsf{Vect}(X)$ be the K-linear category of vector bundles on X of finite rank.

For each choice of a sign $\varepsilon \in \{\pm 1\}$, there is a self-dual structure $(-)^\vee \colon \mathscr{A} \xrightarrow{\sim} \mathscr{A}^{\mathrm{op}}$ sending a vector bundle to its dual vector bundle, with the natural isomorphism $\eta \colon (-)^{\vee\vee} \xRightarrow{\sim} \mathrm{id}_{\mathscr{A}}$ given by ε times the usual identification.

A self-dual object in $\mathscr{A}$ is a pair (E, ϕ), where E is a vector bundle on X, and $\phi \colon E \xrightarrow{\sim} E^\vee$ is an isomorphism, satisfying $\phi^\vee = \phi \circ \eta_E$. Equivalently, ϕ is a non-degenerate symmetric (or antisymmetric) bilinear form on E when $\varepsilon = +1$ (or -1). In particular, if K is algebraically closed of characteristic $\neq 2$, then self-dual objects of $\mathscr{A}$ can be identified with principal $\mathrm{O}(n)$-bundles (or $\mathrm{Sp}(n)$-bundles) on X.

2.1.6. Example. Self-dual quivers. Let Q be a *self-dual quiver*, that is, a quiver with an involution $\sigma \colon Q \xrightarrow{\sim} Q^{\mathrm{op}}$, where Q^{op} is the opposite quiver of Q. This notion was due to Derksen and Weyman [51] and Young [163–165]. See §4.1 for details.

Let $\mathscr{A} = \mathsf{Mod}(KQ)$ be the K-linear abelian category of finite-dimensional representations of Q over K. There is a self-dual structure $(-)^\vee \colon \mathscr{A} \xrightarrow{\sim} \mathscr{A}^{\mathrm{op}}$ sending a representation to the representation with the dual vector spaces and dual linear maps. This also involves choosing signs when defining $\eta \colon (-)^{\vee\vee} \xRightarrow{\sim} \mathrm{id}_{\mathscr{A}}$, as in the previous example. Again, see §4.1 for details.

Self-dual objects in $\mathscr{A}$ are called *self-dual representations* of Q, which we think of as analogues of orthogonal or symplectic bundles in the quiver setting.

2.1.7. Non-example. Coherent sheaves. Let X be a connected, smooth, projective K-variety of positive dimension, and let $\mathscr{A} = \mathsf{Coh}(X)$ be the abelian category of coherent sheaves on X.

Then $\mathscr{A}$ does not admit a self-dual structure. This is because $\mathscr{A}$ is *noetherian*, meaning that every ascending chain of subobjects of a given object stabilizes, while it is not *artinian*, in that there exists an infinite descending chain of subobjects $\mathcal{O}_X \supset \mathcal{O}_X(-1) \supset \mathcal{O}_X(-2) \supset \cdots$. Since taking the opposite category exchanges the properties of being noetherian and artinian, the category $\mathcal{A}$ is not equivalent to $\mathcal{A}^{\mathrm{op}}$.

This problem can be fixed, however, by considering the derived category $\mathcal{D} = \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$, which has many interesting self-dual structures. See §4.2 for details.

2.2 Self-dual filtrations

2.2.1. We discuss *self-dual filtrations* in a self-dual exact category $\mathcal{A}$. This notion is the orthosymplectic analogue of filtrations in linear enumerative geometry, and will play a crucial role in orthosymplectic enumerative geometry.

For example, the orthosymplectic version of Harder–Narasimhan filtrations, mentioned in §1.3.4, will be such self-dual filtrations.

2.2.2. Exact categories. For the purpose of describing filtrations later on, we briefly discuss the notion of *exact categories*, originally introduced by Quillen [138]. We present the following definition taken from Keller [92, Appendix A]:

An *exact category* is an additive category $\mathcal{A}$ equipped with a distinguished class of sequences

$$y \overset{i}{\hookrightarrow} x \overset{p}{\twoheadrightarrow} z \tag{2.2.2.1}$$

of morphisms in $\mathcal{A}$, called *short exact sequences*, satisfying the following conditions:

We call morphisms that appear as the first (resp. second) arrow in a short exact sequence an *inclusion* (resp. a *projection*). Then,

(i) Sequences of the form $E \hookrightarrow E \oplus F \twoheadrightarrow F$, called *split exact sequences*, are short exact, where the two arrows are the canonical ones.

(ii) All short exact sequences are kernel–cokernel pairs, that is, in (2.2.2.1), we always have $i = \ker(p)$ and $p = \operatorname{coker}(i)$.

(iii) Inclusions and projections are closed under composition.

(iv) Pushouts along inclusions exist, and inclusions are closed under pushouts. Dually, pullbacks along projections exist, and projections are closed under pullbacks.

For a short exact sequence (2.2.2.1), we also say that y is a *subobject* of x, and that z is a *quotient* of x.

For example, every abelian category has a canonical structure of an exact category, given by the notion of short exact sequences in the abelian category.

A *self-dual K-linear exact category* is a K-linear exact category with a self-dual structure, such that the dual functor $(-)^\vee$ sends short exact sequences $y \hookrightarrow x \twoheadrightarrow z$ to short exact sequences $z^\vee \hookrightarrow x^\vee \twoheadrightarrow y^\vee$.

2.2.3. Categories of filtrations. For a K-linear exact category $\mathscr{A}$ and an integer $n \geqslant 0$, define the K-linear exact category $\mathscr{A}^{(n)}$ of *n-step filtrations* in $\mathscr{A}$ whose objects are diagrams

$$\begin{tikzcd} 0 = x_0 \arrow[r, hook] & x_1 \arrow[r, hook] \arrow[d, two heads] & x_2 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & x_n = x \arrow[d, two heads] \\ & y_1 & y_2 & & y_n , \end{tikzcd} \tag{2.2.3.1}$$

with each sequence $x_{i-1} \hookrightarrow x_i \twoheadrightarrow y_i$ short exact in $\mathscr{A}$, and morphisms are morphisms of diagrams. Short exact sequences in $\mathscr{A}^{(n)}$ are sequences that are term-wise short exact.

2.2.4. Self-dual filtrations. Now, suppose that $\mathscr{A}$ is a self-dual K-linear exact category, and consider the category $\mathscr{A}^{(n)}$ defined above.

For an n-step filtration (2.2.3.1), define its *dual filtration* to be the n-step filtration

$$\begin{tikzcd} 0 = (x/x_n)^\vee \arrow[r, hook] & (x/x_{n-1})^\vee \arrow[r, hook] \arrow[d, two heads] & (x/x_{n-2})^\vee \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & (x/x_0)^\vee = x^\vee \arrow[d, two heads] \\ & y_n^\vee & y_{n-1}^\vee & & y_1^\vee , \end{tikzcd} \tag{2.2.4.1}$$

where x/x_i denotes the cokernel of the inclusion $x_i \hookrightarrow x$, which exists by the axioms of an exact category. We have the short exact sequence $y_i \hookrightarrow x/x_{i-1} \twoheadrightarrow x/x_i$ by the third isomorphism theorem, which holds in any exact category.

This defines a self-dual structure on $\mathscr{A}^{(n)}$. Its self-dual objects are called *n-step self-dual filtrations* in $\mathscr{A}$.

In other words, an n-step self-dual filtration is a filtration of the form (2.2.3.1), where x has a self-dual structure $\phi\colon x \xrightarrow{\sim} x^\vee$, such that ϕ identifies the filtrations (2.2.3.1) and (2.2.4.1).

In particular, ϕ induces isomorphisms $y_i \xrightarrow{\sim} y_{n+1-i}^\vee$ for all i, and if n is odd, then the middle piece $y_{(n+1)/2}$ acquires an induced self-dual structure $y_{(n+1)/2} \xrightarrow{\sim} y_{(n+1)/2}^\vee$.

2.3 Stability conditions on exact categories

2.3.1. We define *stability conditions* on exact categories, generalizing the notion of stability conditions on abelian categories considered by Rudakov [141] and Joyce [83].

As mentioned in §§1.2.3–1.2.4, the purpose of introducing stability conditions is mainly to deal with constructing enumerative invariants when the moduli stack is not quasi-compact, so that a stability condition should produce a stratification of the moduli stack with quasi-compact strata, so it is meaningful to count points in each stratum. For example, if the moduli stack is already quasi-compact, then we can usually use the *trivial stability condition*, giving the trivial stratification.

In the case of self-dual exact categories in the sense of §2.2.4, we also define when a stability condition is compatible with the self-dual structure. In this case, we can also say when a self-dual object is semistable or stable.

These notions of stability will not be essentially used in our main constructions, but will serve as a motivation for the more complicated definition of stability conditions for linear stacks in §3.5 below, and will be easier to work with when studying examples.

We prove two useful results, Theorems 2.3.4 and 2.3.5, which characterize semistability and stability for self-dual objects in self-dual exact categories.

2.3.2. Stability for exact categories. We first define a notion of stability conditions for exact categories, following ideas of Rudakov [141], Joyce [83], and Bridgeland [22].

Let $\mathscr{A}$ be a K-linear exact category (see §2.2.2), which we assume to be essentially small. The *Grothendieck group* of $\mathscr{A}$ is the abelian group $K_0(\mathscr{A})$ generated by isomorphism classes of objects of $\mathscr{A}$, modulo the relations $[x] \sim [y] + [z]$ for short exact sequences $y \hookrightarrow x \twoheadrightarrow z$ in $\mathscr{A}$. We assume that $[x] = 0$ in $K_0(\mathscr{A})$ implies $x \simeq 0$. Let $C(\mathscr{A}) \subset K_0(\mathscr{A})$ be the submonoid consisting of classes of objects in $\mathscr{A}$.

A *weak stability condition* on $\mathscr{A}$ is a map of sets

$$\tau \colon C(\mathscr{A}) \setminus \{0\} \longrightarrow T \,,$$

where T is a totally ordered set, satisfying the following conditions: We say that

- An object $x \in \mathscr{A}$ is *τ-semistable*, if for any short exact sequence $y \hookrightarrow x \twoheadrightarrow z$ in $\mathscr{A}$ with $y, z \neq 0$, we have $\tau(y) \leqslant \tau(x) \leqslant \tau(z)$.
- An object $x \in \mathscr{A}$ is *τ-stable*, if it is non-zero, and for any short exact sequence $y \hookrightarrow x \twoheadrightarrow z$ in $\mathscr{A}$ with $y, z \neq 0$, we have $\tau(y) < \tau(x) < \tau(z)$.

Then, we require the following:

(i) For any short exact sequence $y \hookrightarrow x \twoheadrightarrow z$ of non-zero objects in $\mathscr{A}$, we have either $\tau(y) \leqslant \tau(x) \leqslant \tau(z)$ or $\tau(y) \geqslant \tau(x) \geqslant \tau(z)$.

(ii) For any non-zero τ-semistable objects $x, y \in \mathscr{A}$, if $\tau(x) > \tau(y)$, then $\mathscr{A}(x, y) = 0$.

(iii) Every object $x \in \mathscr{A}$ has a *Harder–Narasimhan filtration*, that is a filtration

$$\begin{tikzcd} 0 = x_0 \arrow[r, hook] & x_1 \arrow[r, hook] \arrow[d, two heads] & x_2 \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & x_k = x \arrow[d, two heads] \\ & y_1 & y_2 & & y_k \,, \end{tikzcd}$$

with each $x_{i-1} \hookrightarrow x_i \twoheadrightarrow y_i$ short exact, and each y_i non-zero and τ-semistable, such that $\tau(y_1) > \tau(y_2) > \cdots > \tau(y_k)$.

Here, the conditions (ii)–(iii) are automatic when $\mathscr{A}$ is an abelian category and is *noetherian* and *τ-artinian* in the sense of Joyce [83, §4]; see there for more details.

In (iii), the Harder–Narasimhan filtration of every object is unique up to a unique isomorphism, which can be deduced from a standard argument.

We say that τ is a *stability condition*, if in addition, we have the following:

(iv) For any $t \in T$, the full subcategory $\mathscr{A}(t) \subset \mathscr{A}$ consisting of τ-semistable objects x with either $\tau(x) = t$ or $x = 0$, with the induced exact structure, is an abelian category, and is closed under taking kernels and cokernels in $\mathscr{A}$.

In this case, for any short exact sequence $y \hookrightarrow x \twoheadrightarrow z$ of non-zero objects in $\mathscr{A}$, we have either $\tau(y) < \tau(x) < \tau(z)$, or $\tau(y) = \tau(x) = \tau(z)$, or $\tau(y) > \tau(x) > \tau(z)$. This condition is equivalent to (iv) when $\mathscr{A}$ itself is an abelian category.

2.3.3. Self-dual stability. Let $\mathscr{A}$ be a small self-dual K-linear exact category, where the duality preserves the exact structure as in §2.2.4, and also the linear structure.

We say that a weak stability condition τ on $\mathscr{A}$ is *self-dual*, if the following condition is satisfied:

- For any non-zero objects $x, y \in \mathscr{A}$, we have $\tau(x) \leqslant \tau(y)$ if and only if $\tau(y^\vee) \leqslant \tau(x^\vee)$.

In this case, shrinking T (replacing it by the image of τ) if necessary and assuming that $\mathscr{A} \not\simeq 0$, we may assume that there is an order-reversing involution on T, denoted by $t \mapsto -t$, with a unique fixed element $0 \in T$. This element exists because for any self-dual object $(x, \phi) \in \mathscr{A}^{\mathrm{sd}}$ with $x \not\simeq 0$, we have $\tau(x) = \tau(x^\vee) = -\tau(x)$, so that we must have $\tau(x) = 0$; such (x, ϕ) exists by §2.1.4.

For a self-dual object $(x, \phi) \in \mathscr{A}^{\mathrm{sd}}$, we introduce the following notions:

- A subobject $i \colon y \hookrightarrow x$ is *isotropic*, if the composition $y \overset{i}{\hookrightarrow} x \xrightarrow{\phi} x^\vee \overset{i^\vee}{\twoheadrightarrow} y^\vee$ is zero.
- We say that (x, ϕ) is τ*-semistable*, if for any non-zero isotropic subobject $y \hookrightarrow x$, we have $\tau(y) \leqslant 0$.
- We say that (x, ϕ) is τ*-stable*, if for any non-zero isotropic subobject $y \hookrightarrow x$, we have $\tau(y) < 0$.

The reason for only considering isotropic subobjects here, instead of all subobjects, is that giving an isotropic subobject $y \hookrightarrow x$ is equivalent to giving a three-term self-dual filtration whose total object is (x, ϕ), in the sense of §2.2.4, which is necessarily of the form

$$
\begin{tikzcd}
0 \arrow[r, hook] & y \arrow[r, hook] \arrow[d, two heads] & y^\perp \arrow[r, hook] \arrow[d, two heads] & x \arrow[d, two heads] \\
& y & z & y^\vee ,
\end{tikzcd}
\tag{2.3.3.1}
$$

where $y^\perp = (x/y)^\vee$, $z = y^\perp / y$, and $y^\vee \simeq x/y^\perp$. The object z has an induced self-dual structure, giving an object $(z, \psi) \in \mathscr{A}^{\mathrm{sd}}$.

From this, one can deduce the following characterizations of semistability and stability for self-dual objects.

2.3.4. Theorem. *Let $\mathscr{A}$ be a self-dual K-linear exact category, and let τ be a self-dual weak stability condition on $\mathscr{A}$. Then an object $(x, \phi) \in \mathscr{A}^{\mathrm{sd}}$ is τ-semistable if and only if its underlying object $x \in \mathscr{A}$ is τ-semistable.*

Proof. Consider the Harder–Narasimhan filtration of x. Its dual filtration in the sense of §2.2.4 is a Harder–Narasimhan filtration of $x^\vee$, and the self-dual structure on x equips this filtration with the structure of a self-dual filtration. Therefore, if this filtration has at least two terms, then the first term must be isotropic, proving that (x, ϕ) being semistable implies x being semistable. The other direction is clear. $\square$

2.3.5. Theorem. *Let $\mathscr{A}$ be a noetherian self-dual K-linear exact category, and let τ be a self-dual stability condition on $\mathscr{A}$. Then an object $(x, \phi) \in \mathscr{A}^{\mathrm{sd}}$ is τ-stable if and only if it is of the form*

$$(x, \phi) \simeq (x_1, \phi_1) \oplus \cdots \oplus (x_n, \phi_n) \, ,$$

where $n \geqslant 0$, $(x_i, \phi_i) \in \mathscr{A}^{\mathrm{sd}}$, and the underlying objects $x_i \in \mathscr{A}$ are τ-stable.

In particular, if -1 has a square root in K, then the objects (x_i, ϕ_i) are pairwise non-isomorphic, and $\mathrm{Aut}(x, \phi) \simeq \mathbb{Z}_2^n$.

Proof. The abelian category $\mathscr{A}(0) \subset \mathscr{A}$ in §2.3.2 (iv) is noetherian and self-dual, so it is also artinian. Let $(x, \phi) \in \mathscr{A}^{\mathrm{sd}}$ be stable, so $x \in \mathscr{A}(0)$. Let

$$0 = x_0 \hookrightarrow x_1 \hookrightarrow \cdots \hookrightarrow x_n = x \tag{2.3.5.1}$$

be a Jordan–Hölder filtration of x in $\mathscr{A}(0)$, where the stepwise quotients are simple. If $n \leqslant 1$, then we are done. Suppose that $n > 1$. Since (x, ϕ) is stable, the inclusion $x_1 \hookrightarrow x$ cannot factor through $x_1^\perp \simeq (x/x_1)^\vee$. The composition $x_1 \hookrightarrow x \twoheadrightarrow x/x_1^\perp \simeq x_1^\vee$ is thus non-zero, and hence an isomorphism, which then splits the inclusion $x_1 \hookrightarrow x$, giving a decomposition $(x, \phi) \simeq (x_1, \phi|_{x_1}) \oplus (x_1^\perp, \phi|_{x_1^\perp})$ into stable self-dual objects. Repeating this process, we obtain the desired decomposition, where the artinian property ensures that the process terminates.

For the last statement, suppose we have an isomorphism $\psi \colon (x_i, \phi_i) \simeq (x_j, \phi_j)$ for some $i \neq j$. Then $\mathrm{id}_{x_i} + \sqrt{-1} \cdot \psi \colon x_i \hookrightarrow x$ is an isotropic subobject of x, a contradiction. $\square$

2.4 Bridgeland stability conditions

2.4.1. We now introduce *Bridgeland stability conditions* following Bridgeland [22], which are a notion of stability defined on triangulated categories, which we will apply to derived categories of coherent sheaves on varieties. Later on, we will also study enumerative invariants counting semistable objects for such stability conditions.

Another reason why we are interested in Bridgeland stability conditions is that we need them to define a category of sheaves on a given smooth projective variety that is self-dual, due to the problem discussed in §2.1.7, so that our theory of orthosymplectic enumerative invariants can be applied.

In the following, we work with *triangulated categories* in the sense of, for example, Gelfand and Manin [63, Chapter IV].

2.4.2. Bridgeland stability conditions. Assume that we are given the following data:

- A K-linear triangulated category $\mathscr{C}$.
- A finitely generated free abelian group Γ, with a surjective homomorphism $K(\mathscr{C}) \twoheadrightarrow \Gamma$ from the Grothendieck group of $\mathscr{C}$.

Then a *Bridgeland stability condition* on $\mathscr{C}$ that factors through Γ is a pair $\tau = (Z, \mathscr{P})$, where

- $Z \colon \Gamma \to \mathbb{C}$ is a group homomorphism.
- $\mathscr{P}$ is a *slicing* on $\mathscr{C}$, meaning a family of K-linear full subcategories $(\mathscr{P}(t) \subset \mathscr{C})_{t \in \mathbb{R}}$, such that the following conditions hold:
  - (i) We have $\mathscr{P}(t+1) = \mathscr{P}(t)[1]$ for all $t \in \mathbb{R}$.
  - (ii) If $t_1 > t_2$, then for any $x_1 \in \mathscr{P}(t_1)$ and $x_2 \in \mathscr{P}(t_2)$, we have $\mathscr{C}(x_1, x_2) = 0$.
  - (iii) Each object $x \in \mathscr{C}$ has a *Harder–Narasimhan filtration*, that is a sequence

$$\begin{tikzcd} 0 = x_0 \arrow[r] & x_1 \arrow[r] \arrow[d] & x_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & x_k = x \arrow[d] \\ & y_1 & y_2 & & y_k\,, \end{tikzcd} \tag{2.4.2.1}$$

    where each $x_{i-1} \to x_i \to y_i$ is an exact triangle, we have $0 \neq y_i \in \mathscr{P}(t_i)$ for some $t_i \in \mathbb{R}$, and we have $t_1 > \cdots > t_k$.

They should satisfy the following condition:

- For any $t \in \mathbb{R}$ and any $0 \neq x \in \mathscr{P}(t)$, we have

$$Z(x) \in \mathbb{R}_{>0} \cdot \mathrm{e}^{\mathrm{i}\pi t}\,, \tag{2.4.2.2}$$

  where $Z(x)$ denotes the value of Z on the image of x in Γ.

- *Support property*. For any $r \in \mathbb{R}_{>0}$, there are only finitely many classes $\alpha \in \Gamma$ admitting a semistable object (see below), such that $|Z(\alpha)| \leqslant r$.

Here, $\mathscr{P}(t)$ is called the subcategory of *semistable objects of phase* t, and is necessarily a K-linear abelian category.

For any interval $I \subset \mathbb{R}$, denote by $\mathscr{P}(I) \subset \mathscr{C}$ the smallest extension-closed subcategory of $\mathscr{C}$ containing all the subcategories $\mathscr{P}(t)$ for $t \in I$, where being *extension-closed* means that for any exact triangle $x \to y \to z$ in $\mathscr{C}$ with $x, z \in \mathscr{P}(I)$, we also have $y \in \mathscr{P}(I)$.

For any object $0 \neq x \in \mathscr{C}$, one can show that its Harder–Narasimhan filtration is unique up to a unique isomorphism. If its Harder–Narasimhan factors are $y_1, \ldots, y_k$ as above, then we define

$$\phi_+(x) = t_1 , \qquad \phi_-(x) = t_k , \qquad m(x) = \sum_{i=1}^{k} |Z(y_i)| . \tag{2.4.2.3}$$

These are called the *maximal phase*, the *minimal phase*, and the *mass* of x.

We denote by $\mathrm{Stab}_\Gamma(\mathscr{C})$ the set of such Bridgeland stability conditions.

2.4.3. The space of stability conditions. The set $\mathrm{Stab}_\Gamma(\mathscr{C})$ has a topology given by a *generalized metric* d, that is, a metric allowing infinite distance, defined as in [22, §8] by

$$d(\tau, \tilde{\tau}) = \sup \Big\{ |\phi^+(x) - \tilde{\phi}^+(x)|, |\phi^-(x) - \tilde{\phi}^-(x)|, |\log m(x) - \log \tilde{m}(x)| \ \Big| \ x \neq 0 \Big\} , \tag{2.4.3.1}$$

where x goes through all non-zero objects of $\mathscr{C}$. The projection

$$\mathrm{Stab}_\Gamma(\mathscr{C}) \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C}) \tag{2.4.3.2}$$

given by $(Z, \mathscr{P}) \mapsto Z$ is a local homeomorphism, and equips $\mathrm{Stab}_\Gamma(\mathscr{C})$ with the structure of a complex manifold.

2.4.4. Self-dual Bridgeland stability conditions. We now discuss self-dual Bridgeland stability conditions in *self-dual triangulated categories.* This type of duality already appeared, for example, in Bayer [8, §3.3] and Bayer, Macrì, and Toda [9, Lemma 4.1.2 ff.], although for different purposes.

Let $\mathscr{C}$ be a *self-dual K-linear triangulated category*, that is a K-linear triangulated category with a self-dual structure in the sense of §2.1.2, such that the dual functor $(-)^\vee$ exchanges shifting by 1 and -1, and sends exact triangles $x \to y \to z \to x[1]$ to exact triangles $x^\vee[-1] \to$

$z^\vee \to y^\vee \to x^\vee$.

Let $K(\mathscr{C}) \to \Gamma$ be a map as in §2.4.2, such that its kernel is preserved by the dual functor $(-)^\vee$. In this case, the group Γ has an induced involution $(-)^\vee \colon \Gamma \xrightarrow{\sim} \Gamma$, whose fixed locus is denoted by Γ^{sd}.

For a Bridgeland stability condition $\tau = (Z, \mathscr{P}) \in \mathrm{Stab}_\Gamma(\mathscr{C})$, define its *dual stability condition* $\tau^\vee = (Z^\vee, \mathscr{P}^\vee)$ by setting

$$Z^\vee(\alpha) = \overline{Z(\alpha^\vee)}\,, \qquad \mathscr{P}^\vee(t) = \mathscr{P}(-t)^\vee \tag{2.4.4.1}$$

for all $\alpha \in \Gamma$ and $t \in \mathbb{R}$, where $\overline{(-)}$ denotes complex conjugation.

If $\tau = \tau^\vee$, then it is called a *self-dual Bridgeland stability condition.*

Taking the dual stability condition defines an anti-holomorphic involution

$$(-)^\vee \colon \mathrm{Stab}_\Gamma(\mathscr{C}) \xrightarrow{\sim} \mathrm{Stab}_\Gamma(\mathscr{C})\,, \tag{2.4.4.2}$$

and its fixed locus $\mathrm{Stab}_\Gamma^{\mathrm{sd}}(\mathscr{C}) \subset \mathrm{Stab}_\Gamma(\mathscr{C})$ is the space of self-dual stability conditions, which is a real analytic manifold.

2.4.5. Remark. Self-dual dg-categories. Finally, we remark that from a higher categorical point of view, the above notion of self-dual structures on triangulated categories might not be the most natural one, and we sometimes also need to impose higher coherence conditions.

For this purpose, we can choose to work with *K-linear dg-categories*, or categories enriched in chain complexes of K-modules, with a notion of equivalence of dg-categories as described in Keller [93, §7.2] or Haugseng [72, Definition 5.6]. All derived categories we are interested in can be seen as K-linear dg-categories. By Lurie [110, §1.3.1], a K-linear dg-category $\mathscr{C}$ has an underlying ∞-category $\mathscr{C}_0$ obtained by taking its *dg-nerve*. Moreover, all (small) K-linear dg-categories form an ∞-category as in Tabuada [146] and Toën [156].

We may now define self-dual K-linear dg-categories, and self-dual objects in such categories, as fixed points of suitable $\mathbb{Z}_2$-actions in the corresponding ∞-categories. Note that in this case, the explicit presentation in §2.1.2 will not be enough, and more coherence data is required: The equivalence $\eta_{x^\vee} \simeq (\eta_x^\vee)^{-1}$ there needs to be self-dual, with the self-duality witnessed by a higher equivalence that is also self-dual, and so on.

Chapter 3

Moduli stacks

In this chapter, we discuss moduli stacks of objects in linear categories, and moduli stacks of self-dual objects in self-dual linear categories, which are algebraic stacks whose points correspond to objects in these categories, and whose stabilizer groups correspond to automorphism groups of these objects.

We will take an intrinsic approach, meaning that we solely work with algebraic stacks that behave like such moduli stacks, without referring to the original categories. Such stacks are called *linear stacks* as in the author et al. [31, §7.1], and we also discuss self-dual structures on linear stacks, from which one can construct the moduli stack of self-dual objects. See Chapter 4 for concrete examples.

3.1 Algebraic spaces and stacks

3.1.1. This section provides background material on algebraic spaces and algebraic stacks, mainly following Olsson [124].

We do not attempt to give a complete introduction to the theory or define every notion involved here. The reader should refer to standard textbooks such as Olsson [124] or *the Stacks project* [144] for more details.

3.1.2. Algebraic spaces. We first give a definition of *algebraic spaces.* See Olsson [124, §5.1], Knutson [97, II.1], or Laumon and Moret-Bailly [103, §1] for more details.

Let Aff be the category of affine schemes, equipped with the étale topology (see [124, Example 2.1.13]). An *algebraic space* is a functor

$$X : \mathsf{Aff}^{\mathrm{op}} \longrightarrow \mathsf{Set} ,$$

satisfying the following properties:

(i) X is a sheaf with respect to the étale topology on Aff.

(ii) There exists a scheme U and a surjective étale morphism $U \to X$ that is representable by schemes.

Here, the property (ii) means more precisely that for any affine scheme T and any morphism $T \to X$, the base change $U_T = U \times_X T \to T$ is a surjective étale morphism of schemes, where we identify a scheme with the functor $\mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ that it represents.

Note that some sources, such as Knutson [97] or Laumon and Moret-Bailly [103], impose the extra condition that X is *quasi-separated*, meaning that the diagonal morphism $\Delta_X : X \to X \times X$ is quasi-compact. More modern sources such as Olsson [124] or *the Stacks project* [144] tend to omit this condition, and say *quasi-separated algebraic spaces* when this condition is satisfied, and we follow this latter approach.

3.1.3. Algebraic stacks. We now give a definition of *algebraic stacks*, also called *Artin stacks*. Standard references on algebraic stacks include Olsson [124], *the Stacks project* [144], and Laumon and Moret-Bailly [103].

Let Grpd be the 2-category of small groupoids. An *algebraic stack* is a functor

$$\mathcal{X} : \mathsf{Aff}^{\mathrm{op}} \longrightarrow \mathsf{Grpd} ,$$

satisfying the following properties:

(i) $\mathcal{X}$ is a sheaf with respect to the étale topology on Aff.

(ii) There exists a scheme U and a surjective smooth morphism $U \to \mathcal{X}$ that is representable by algebraic spaces.

Here, the property (i) involves a 2-categorical notion of sheaves; see, for example, Olsson [124, §4.6]. The property (ii) means that for any affine scheme T and any morphism $T \to \mathcal{X}$, the base change $U_T = U \times_{\mathcal{X}} T \to T$ is a surjective smooth morphism of algebraic spaces.

We denote by $\mathsf{St} \subset \mathsf{Fun}(\mathsf{Aff}^{\mathrm{op}}, \mathsf{Grpd})$ the full sub-2-category of algebraic stacks.

Again, some sources such as [103] impose the extra condition that $\mathcal{X}$ is *quasi-separated* (and/or has *separated diagonal*), meaning that the diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact and quasi-separated (and/or separated). We follow more modern sources, such as [124] or [144], which omit these conditions.

An algebraic stack $\mathcal{X}$ is called a *Deligne–Mumford stack* if it satisfies the following extra property:

(ii′) There exists a scheme U and a surjective étale morphism $U \to \mathcal{X}$ that is representable by algebraic spaces.

Roughly, this condition means that the stabilizer groups of $\mathcal{X}$, as in §3.1.5 below, must be discrete groups.

Many properties of morphisms of schemes generalize to stacks, such as being an *open* (or *closed*, or *locally closed*) *immersion*, being *quasi-compact*, (*quasi-*)*separated*, (*locally*) *of finite type*, *proper*, *affine*, *étale*, *smooth*, etc. We do not list the definitions here, but we refer to Olsson [124] for details. In particular, an *open* (or *closed*, or *locally closed*) *substack* of $\mathcal{X}$ means an algebraic stack with an open (or closed, or locally closed) immersion to $\mathcal{X}$.

Another useful class of morphisms of stacks is those *representable by algebraic spaces*, or *representable morphisms* for short, meaning morphisms $\mathcal{Y} \to \mathcal{X}$ such that for any algebraic space T and any morphism $T \to \mathcal{X}$, the base change $\mathcal{Y}_T = \mathcal{Y} \times_{\mathcal{X}} T$ is an algebraic space.

3.1.4. The underlying topological space. Each algebraic stack $\mathcal{X}$ has an *underlying topological space* $|\mathcal{X}|$, as in Laumon and Moret-Bailly [103, §5], similar to the underlying topological space of a scheme equipped with the Zariski topology.

Namely, a *point* $x \in |\mathcal{X}|$ is an equivalence class of morphisms $\operatorname{Spec} K \to \mathcal{X}$ for fields K, with the equivalence relation generated by the relation that two such morphisms are equivalent if one factors through the other. The set $|\mathcal{X}|$ of points of $\mathcal{X}$ admits a natural topology, called the *Zariski topology*, where the open sets are the sets $|\mathcal{U}| \subset |\mathcal{X}|$ for open immersions $\mathcal{U} \to \mathcal{X}$. The space $|\mathcal{X}|$ is called the *underlying topological space* of $\mathcal{X}$.

As with the case of schemes, open substacks of $\mathcal{X}$ are in bijection with open sets in $|\mathcal{X}|$, whereas different closed substacks of $|\mathcal{X}|$ can correspond to the same closed set in $|\mathcal{X}|$. Nevertheless, each closed set gives a canonical *reduced induced closed substack.*

We will frequently use the *set of connected components* of $\mathcal{X}$, denoted by $\pi_0(\mathcal{X}) = \pi_0(|\mathcal{X}|)$. Each element $\alpha \in \pi_0(\mathcal{X})$ gives an open and closed substack $\mathcal{X}_\alpha \subset \mathcal{X}$.

3.1.5. Stabilizer groups. For an algebraic stack $\mathcal{X}$ and a point $x \colon \operatorname{Spec} K \to \mathcal{X}$, the *stabilizer group* of x is a group algebraic space over K given by the fibre product

$$\operatorname{Aut}(x) = \operatorname{Spec} K \underset{x, \mathcal{X}, x}{\times} \operatorname{Spec} K \, .$$

Its group of K-points is the automorphism group of x as an object in the groupoid $\mathcal{X}(K)$.

All the stabilizer groups of points in $\mathcal{X}$ arrange themselves into a group stack over $\mathcal{X}$, given by

$$\mathcal{I}_{\mathcal{X}} = \mathcal{X} \underset{\mathcal{X} \times \mathcal{X}}{\times} \mathcal{X} \, ,$$

called the *inertia stack* or the *loop stack* of $\mathcal{X}$. The natural morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, given by projection to either factor, has the stabilizer groups $\operatorname{Aut}(x)$ as its fibres.

3.1.6. Quotient stacks. *Quotient stacks* are an important class of motivating examples and one of the most common sources of algebraic stacks.

For an algebraic space X defined over a base algebraic space S, and a smooth group algebraic space G over S acting on X, there is the *quotient stack* X/G as an algebraic stack over S. See Olsson [124, Example 8.1.12] for details.

3.1.7. Mapping stacks. We will make extensive use of *mapping stacks*, or stacks that parametrize morphisms between two given algebraic stacks.

For algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ defined over a base algebraic stack $\mathcal{S}$, define the functor

$$\begin{aligned} \mathcal{M}ap_{\mathcal{S}}(\mathcal{X}, \mathcal{Y}) \colon \mathsf{Aff}_{/\mathcal{S}} &\longrightarrow \mathsf{Grpd} \, , \\ T &\longmapsto \mathsf{St}_{/\mathcal{S}}(\mathcal{X} \times_{\mathcal{S}} T, \mathcal{Y}) \, , \end{aligned}$$

where $\mathsf{Aff}_{/\mathcal{S}}$ is the category of affine schemes with a morphism to $\mathcal{S}$. If this functor is represented by an object of $\mathsf{St}_{/\mathcal{S}}$, this object is called the *mapping stack* from $\mathcal{X}$ to $\mathcal{Y}$, also denoted by $\mathcal{M}ap_{\mathcal{S}}(\mathcal{X}, \mathcal{Y})$. We often omit the base $\mathcal{S}$ from the notation when it is clear from context.

3.2 Graded and filtered points

3.2.1. We introduce the *stacks of graded and filtered points* of an algebraic stack, following Halpern-Leistner [67], as we mentioned in §1.3.7. We will later use this formalism to define *linear stacks* and *self-dual linear stacks*, which will be the main basic setting of this thesis.

3.2.2. Running assumptions. In the following, we work over an algebraically closed field K, and we work with algebraic stacks $\mathcal{X}$ over K with the following properties:

(i) $\mathcal{X}$ is *locally of finite type* over K, meaning that there exists a scheme U locally of finite type over K and a surjective smooth morphism $U \to \mathcal{X}$.

(ii) $\mathcal{X}$ has *affine diagonal*, meaning that the diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is an affine morphism, in that it is so upon a base change to any scheme.

The condition (ii) has the following consequences:

- $\mathcal{X}$ is *quasi-separated*, meaning that the diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact and quasi-separated.
- $\mathcal{X}$ has *affine stabilizers*, meaning that for any point $x \in \mathcal{X}$, the stabilizer group $\mathrm{Aut}(x)$ is an affine algebraic group over the residue field κ_x of x.

These assumptions are almost always satisfied by the moduli stacks that we are interested in.

3.2.3. Graded and filtered points. Let $\mathcal{X}$ be a stack over K as in §3.2.2. Following Halpern-Leistner [67], define the *stack of graded points* and the *stack of filtered points* of $\mathcal{X}$ as the mapping stacks

$$\mathcal{G}rad(\mathcal{X}) = \mathcal{M}ap(*/\mathbb{G}_{\mathrm{m}}, \mathcal{X}) \, , \tag{3.2.3.1}$$

$$\mathcal{F}ilt(\mathcal{X}) = \mathcal{M}ap(\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}, \mathcal{X}) \, , \tag{3.2.3.2}$$

where $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{A}^1$ by scaling. By [67, Proposition 1.1.2], these are again algebraic stacks over K, quasi-separated and locally of finite type.

Consider the morphisms

$$\begin{tikzcd} */\mathbb{G}_{\mathrm{m}} \arrow[r, shift left, "0"] & \mathbb{A}^1/\mathbb{G}_{\mathrm{m}} \arrow[l, shift left, "\mathrm{pr}"] & * \arrow[l, "1"'] \arrow[ll, bend right] \end{tikzcd} ,$$

where pr is induced by the projection $\mathbb{A}^1 \to *$. These induce morphisms of stacks

$$\begin{tikzcd} \mathscr{G}rad(\mathscr{X}) \arrow[r, shift left, leftarrow, "\mathrm{gr}"] \arrow[r, shift right, "\mathrm{sf}"'] & \mathscr{F}ilt(\mathscr{X}) \arrow[r, "\mathrm{ev}"] \arrow[r, bend left, "\mathrm{tot}"] & \mathscr{X} \end{tikzcd} ,$$

where the notations 'gr', 'sf', 'ev', and 'tot' stand for the *associated graded point*, the *split filtration*, the *evaluation*, and the *total point*, respectively.

By [67, Lemma 1.1.5 and Proposition 1.1.13], the morphisms tot and ev are representable by algebraic spaces, under our assumptions in §3.2.2. Moreover, by [67, Lemma 1.3.8], the morphisms gr and sf form an $\mathbb{A}^1$*-deformation retract*, meaning that there is a morphism $h\colon \mathbb{A}^1 \times \mathscr{F}ilt(\mathscr{X}) \to \mathscr{F}ilt(\mathscr{X})$ such that $h(1, -) \simeq \mathrm{id}_{\mathscr{F}ilt(\mathscr{X})}$ and $h(0, -) \simeq \mathrm{sf} \circ \mathrm{gr}$. In particular, the morphisms gr and sf induce a bijection $\pi_0(\mathscr{G}rad(\mathscr{X})) \simeq \pi_0(\mathscr{F}ilt(\mathscr{X}))$.

3.2.4. Example. Quotient stacks. The stacks of graded and filtered points of a quotient stack can be described explicitly, following Halpern-Leistner [67, Theorems 1.4.7 and 1.4.8].

Let $\mathscr{X} = U/G$ be a quotient stack, where U is a quasi-separated algebraic space over K, locally of finite type, acted on by a smooth affine algebraic group G over K with a split maximal torus $T \subset G$.

For a cocharacter $\lambda\colon \mathbb{G}_\mathrm{m} \to G$, define the *Levi subgroup* and the *parabolic subgroup* of G associated to λ by

$$\begin{aligned} L_\lambda &= \{ g \in G \mid g = \lambda(t)\, g\, \lambda(t)^{-1} \text{ for all } t \} , \\ P_\lambda &= \{ g \in G \mid \lim_{t \to 0} \lambda(t)\, g\, \lambda(t)^{-1} \text{ exists} \} , \end{aligned}$$

respectively. Here, that the limit exists means that the corresponding morphism $\mathbb{G}_\mathrm{m} \to G$ can be extended to a morphism $\mathbb{A}^1 \to G$.

For example, if $G = \mathrm{GL}(n)$ and $\lambda(t) = \mathrm{diag}(t^{k_1}, \dots, t^{k_n})$ with $k_1 \geqslant \cdots \geqslant k_n$, then $L_\lambda, P_\lambda \subset G$ are the groups of block diagonal and block upper triangular matrices, respectively, where the i-th and j-th positions belong to the same block if and only if $k_i = k_j$.

Define the *fixed locus* and the *attractor* associated to λ by

$$\begin{aligned} U^\lambda &= \mathrm{Map}^{\mathbb{G}_\mathrm{m}}(*, U) , \\ U^{\lambda,+} &= \mathrm{Map}^{\mathbb{G}_\mathrm{m}}(\mathbb{A}^1, U) , \end{aligned}$$

where $\mathrm{Map}^{\mathbb{G}_\mathrm{m}}(-,-)$ denotes the $\mathbb{G}_\mathrm{m}$-equivariant mapping space, and $\mathbb{G}_\mathrm{m}$ acts on U via λ, and on $\mathbb{A}^1$ by scaling. These are again algebraic spaces over K, as in Drinfeld and Gaitsgory [56, §1]. For example, if U is a G-representation, then $U^\lambda, U^{\lambda,+} \subset U$ are the subspaces with zero and non-negative λ-weights, respectively.

By [67, Theorems 1.4.7 and 1.4.8], we have

$$\mathscr{G}rad(\mathscr{X}) \simeq \coprod_{\lambda\colon \mathbb{G}_\mathrm{m} \to G} U^\lambda / L_\lambda \, , \tag{3.2.4.1}$$

$$\mathscr{F}ilt(\mathscr{X}) \simeq \coprod_{\lambda\colon \mathbb{G}_\mathrm{m} \to G} U^{\lambda,+} / P_\lambda \, , \tag{3.2.4.2}$$

where the coproducts are taken over conjugacy classes of cocharacters λ, and the L_λ-action on U^λ and the P_λ-action on $U^{\lambda,+}$ are induced by the G-action on U.

3.3 Linear stacks

3.3.1. We introduce the notion of *linear stacks*, also called *linear moduli stacks*, as in the author et al. [31, §7.1]. They are algebraic stacks that behave like moduli stacks of objects in linear categories. The reader is recommended to refer to Chapter 4 for concrete examples of such stacks.

This thesis takes an intrinsic approach to moduli stacks, meaning that we base all our constructions on the structure of linear stacks, without referring to the original categories of objects that these stacks are supposed to parametrize.

3.3.2. Linear stacks. Let K be an algebraically closed field. As in the author et al. [31, §7.1], define a *linear stack* over K to be the following data:

- An algebraic stack $\mathscr{X}$ over K.
- A commutative monoid structure $\oplus\colon \mathscr{X} \times \mathscr{X} \to \mathscr{X}$, with unit $0 \in \mathscr{X}(K)$.
- A $*/\mathbb{G}_\mathrm{m}$-action $\odot\colon */\mathbb{G}_\mathrm{m} \times \mathscr{X} \to \mathscr{X}$ respecting the monoid structure.

Note that these structures come with extra coherence data; the compatibility of the monoid structure and the $*/\mathbb{G}_\mathrm{m}$-action means more precisely that we are given a commutative monoid object in the 2-category of $*/\mathbb{G}_\mathrm{m}$-equivariant algebraic stacks over K.

In this case, the set $\pi_0(\mathcal{X})$ of connected components of $\mathcal{X}$ (see §3.1.4) carries the structure of a commutative monoid. We denote its operation by $+$, and its unit by 0.

We require the following additional property:

- There is an equivalence

$$\coprod_{\gamma \colon \mathbb{Z} \to \pi_0(\mathcal{X})} \prod_{n \in \operatorname{supp}(\gamma)} \mathcal{X}_{\gamma(n)} \xrightarrow{\sim} \mathcal{G}rad(\mathcal{X}) \, , \tag{3.3.2.1}$$

  where γ runs through maps of sets $\mathbb{Z} \to \pi_0(\mathcal{X})$ such that $\operatorname{supp}(\gamma) = \mathbb{Z} \setminus \gamma^{-1}(0)$ is finite, and the morphism is defined by the composition

$$*/\mathbb{G}_{\mathrm{m}} \times \prod_{n \in \operatorname{supp}(\gamma)} \mathcal{X}_{\gamma(n)} \xrightarrow{(-)^n} \prod_{n \in \operatorname{supp}(\gamma)} (*/\mathbb{G}_{\mathrm{m}} \times \mathcal{X}_{\gamma(n)}) \xrightarrow{\odot} \prod_{n \in \operatorname{supp}(\gamma)} \mathcal{X}_{\gamma(n)} \xrightarrow{\oplus} \mathcal{X}$$

  on the component corresponding to γ, where the first morphism is given by the n-th power map $(-)^n \colon */\mathbb{G}_{\mathrm{m}} \to */\mathbb{G}_{\mathrm{m}}$ on the factor corresponding to $\mathcal{X}_{\gamma(n)}$.

We can think of (3.3.2.1) roughly as an equivalence $\mathcal{G}rad(\mathcal{X}) \simeq \mathcal{X}^{\mathbb{Z}}$, where we only consider components of $\mathcal{X}^{\mathbb{Z}}$ involving finitely many non-zero classes in $\pi_0(\mathcal{X})$.

3.3.3. Example. Vector spaces. Consider the moduli stack of finite-dimensional K-vector spaces, defined as the coproduct

$$\mathcal{X} = \coprod_{n \in \mathbb{N}} */\mathrm{GL}(n) \, . \tag{3.3.3.1}$$

It is a linear stack over K, with the monoid structure $\oplus$ given by the direct sum of vector spaces, and the $*/\mathbb{G}_{\mathrm{m}}$-action $\odot$ given by scalar multiplication on vector spaces, or equivalently, given by the central cocharacters $\mathbb{G}_{\mathrm{m}} \to \mathrm{GL}(n)$ defined by $t \mapsto \operatorname{diag}(t, \ldots, t)$. Here, the equivalence (3.3.2.1) follows from the explicit description of $\mathcal{G}rad(*/\mathrm{GL}(n))$ in Example 3.2.4.

3.3.4. Stacks of filtrations. For a linear stack $\mathcal{X}$, recall the canonical bijections

$$\pi_0(\mathcal{F}ilt(\mathcal{X})) \simeq \pi_0(\mathcal{G}rad(\mathcal{X})) \simeq \{\gamma \colon \mathbb{Z} \to \pi_0(\mathcal{X}) \mid \operatorname{supp}(\gamma) \text{ finite}\} \, ,$$

where the first bijections is induced by the morphism gr, and the second is given by (3.3.2.1).

For classes $\alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X})$, there is a *stack of filtrations*

$$\mathcal{X}^+_{\alpha_1, \ldots, \alpha_n} \subset \mathcal{F}ilt(\mathcal{X}) \, ,$$

defined as a connected component corresponding to a map γ as above whose non-zero values agree with the non-zero elements in $\alpha_n, \dots, \alpha_1$, preserving order. We think of this as the stack parametrizing n-step filtrations with stepwise quotients of classes $\alpha_1, \dots, \alpha_n$. This stack is independent of the choice of γ up to a canonical equivalence, as in [31, §7.1].

The morphisms defined in §3.2.3 restrict to canonical morphisms

$$\mathrm{gr}\colon \mathcal{X}^+_{\alpha_1,\dots,\alpha_n} \longrightarrow \mathcal{X}_{\alpha_1} \times \dots \times \mathcal{X}_{\alpha_n} , \tag{3.3.4.1}$$

$$\mathrm{ev}\colon \mathcal{X}^+_{\alpha_1,\dots,\alpha_n} \longrightarrow \mathcal{X}_{\alpha_1+\dots+\alpha_n} , \tag{3.3.4.2}$$

sending a filtration to its associated graded object and total object, respectively, where we also restrict the targets to single connected components as the sources are connected. These morphisms also do not depend on the choice of γ, up to the canonical equivalences for $\mathcal{X}^+_{\alpha_1,\dots,\alpha_n}$ mentioned above, and up to canonical 2-isomorphisms.

We say that $\mathcal{X}$ has *quasi-compact filtrations*, if for any $\alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X})$, the morphism $\mathrm{ev}\colon \mathcal{X}^+_{\alpha_1,\dots,\alpha_n} \to \mathcal{X}_{\alpha_1+\dots+\alpha_n}$ is quasi-compact. See also Halpern-Leistner [67, Definition 3.8.1]. This is a very mild condition, and is satisfied by all examples of our interest.

3.4 Self-dual linear stacks

3.4.1. Self-dual linear stacks. We now introduce a notion of *self-dual linear stacks*, which describe moduli stacks of objects in self-dual linear categories.

Let $\mathcal{X}$ be a linear stack over K. A *self-dual structure* on $\mathcal{X}$ is a $\mathbb{Z}_2$-action on $\mathcal{X}$, given by an involution

$$(-)^\vee \colon \mathcal{X} \xrightarrow{\sim} \mathcal{X} ,$$

together with a 2-isomorphism $\eta\colon (-)^{\vee\vee} \Rightarrow \mathrm{id}_{\mathcal{X}}$ with $\eta_{(-)^\vee} = (\eta^\vee_{(-)})^{-1}$ similarly to §2.1.2, such that the involution respects the monoid structure $\oplus$ on $\mathcal{X}$, and inverts the $*/\mathbb{G}_\mathrm{m}$-action $\odot$, meaning that it is equivariant with respect to the involution $(-)^{-1}\colon */\mathbb{G}_\mathrm{m} \to */\mathbb{G}_\mathrm{m}$. More precisely, this means that we require an action of the group object $(*/\mathbb{G}_\mathrm{m}) \rtimes \mathbb{Z}_2$ on $\mathcal{X}$, where $\mathbb{Z}_2$ acts on $*/\mathbb{G}_\mathrm{m}$ by the above involution, and we require that $\oplus$ defines a commutative monoid structure in the 2-category of $(*/\mathbb{G}_\mathrm{m}) \rtimes \mathbb{Z}_2$-equivariant algebraic stacks.

In this case, we call $\mathcal{X}$ a *self-dual linear stack.* Define the *stack of self-dual points* of $\mathcal{X}$ as the fixed locus

$$\mathcal{X}^{\mathrm{sd}} = \mathcal{X}^{\mathbb{Z}_2}$$

for the above $\mathbb{Z}_2$-action, defined as a 2-categorical limit in the 2-category of algebraic stacks. It has affine diagonal by Lemma 3.4.5 below. Note that this is different from the fixed locus of the automorphism $(-)^\vee$ of $\mathcal{X}$, which would give the fixed locus of the corresponding $\mathbb{Z}$-action on $\mathcal{X}$, rather than that of the $\mathbb{Z}_2$-action, and these are different as 2-categorical fixed loci.

There is a monoid action

$$\oplus^{\mathrm{sd}} \colon \mathcal{X} \times \mathcal{X}^{\mathrm{sd}} \longrightarrow \mathcal{X}^{\mathrm{sd}} , \tag{3.4.1.1}$$

given by $(x, y) \mapsto x \oplus y \oplus x^\vee$. This induces a monoid action $\pi_0(\mathcal{X}) \times \pi_0(\mathcal{X}^{\mathrm{sd}}) \to \pi_0(\mathcal{X}^{\mathrm{sd}})$, which we often denote by $(\alpha, \theta) \mapsto \alpha + \theta + \alpha^\vee$, where $\alpha + \alpha^\vee$ can also be seen as a class in $\pi_0(\mathcal{X}^{\mathrm{sd}})$, corresponding to the case when $\theta = 0$.

3.4.2. Example. Vector spaces. Consider the linear stack

$$\mathcal{X} = \coprod_{n \in \mathbb{N}} */\mathrm{GL}(n)$$

in Example 3.3.3. Consider the involution $(-)^\vee \colon */\mathrm{GL}(n) \to */\mathrm{GL}(n)$ sending a vector space to its dual, or equivalently, sending a matrix to its inverse transpose, and choose a sign $\varepsilon \in \{\pm 1\}$ when identifying $(-)^{\vee\vee}$ with $\mathrm{id}_{\mathcal{X}}$, similarly to Example 2.1.5.

Then $\mathcal{X}^{\mathrm{sd}}$ is the classifying stack of non-degenerate symmetric (or anti-symmetric) bilinear forms if $\varepsilon = +1$ (or -1). In particular, if K is of characteristic $\neq 2$, then we have

$$\mathcal{X}^{\mathrm{sd}} = \begin{cases} \coprod_{n \in \mathbb{N}} */\mathrm{O}(n) & \text{if } \varepsilon = +1, \\ \coprod_{n \in \mathbb{N}} */\mathrm{Sp}(2n) & \text{if } \varepsilon = -1. \end{cases} \tag{3.4.2.1}$$

3.4.3. Self-dual graded points. The involution on $\mathcal{X}$ induces an involution on $\mathcal{G}rad(\mathcal{X})$, and we may identify $\mathcal{G}rad(\mathcal{X}^{\mathrm{sd}}) \simeq \mathcal{G}rad(\mathcal{X})^{\mathbb{Z}_2}$. This gives an isomorphism

$$\mathcal{G}rad(\mathcal{X}^{\mathrm{sd}}) \simeq \coprod_{\substack{\gamma \colon \mathbb{Z} \setminus \{0\} \to \pi_0(\mathcal{X}) \\ \text{involutive,} \\ \gamma(0) \in \pi_0(\mathcal{X}^{\mathrm{sd}})}} \Bigg(\mathcal{X}^{\mathrm{sd}}_{\gamma(0)} \times \prod_{n > 0 \colon \gamma(n) \neq 0} \mathcal{X}_{\gamma(n)} \Bigg) , \tag{3.4.3.1}$$

where γ runs through finitely supported maps that are *involutive*, meaning that $\gamma(-n) = \gamma(n)^\vee$ for all $n > 0$, and $\gamma(0)$ is a convenient notation which is independent of the map γ, and $\mathcal{X}^{\mathrm{sd}}_{\gamma(0)} \subset \mathcal{X}^{\mathrm{sd}}$ denotes the component corresponding to $\gamma(0)$.

3.4.4. Self-dual filtrations. For classes $\alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, define the *stack of self-dual filtrations*

$$\mathcal{X}^{\mathrm{sd},+}_{\alpha_1,\ldots,\alpha_n,\theta} \subset \mathcal{F}\mathit{ilt}(\mathcal{X}^{\mathrm{sd}})$$

as a component such that under the isomorphism $\pi_0(\mathcal{F}\mathit{ilt}(\mathcal{X}^{\mathrm{sd}})) \simeq \pi_0(\mathcal{G}\mathit{rad}(\mathcal{X}^{\mathrm{sd}}))$, its corresponding map γ as above has $\gamma(0) = \theta$, and its non-zero values at positive integers agree with the non-zero elements in $\alpha_n, \ldots, \alpha_1$, preserving order. This does not depend on the choice of γ by the constancy theorem of the author et al. [31, Theorem 6.1.2].

The stack $\mathcal{X}^{\mathrm{sd},+}_{\alpha_1,\ldots,\alpha_n,\theta}$ can be thought of as parametrizing *self-dual filtrations* in the sense of §2.2.4, with stepwise quotients of classes $\alpha_1, \ldots, \alpha_n, \theta, \alpha_n^\vee, \ldots, \alpha_1^\vee$.

The morphisms defined in §3.2.3 restrict to morphisms $\mathrm{gr}\colon \mathcal{X}^{\mathrm{sd},+}_{\alpha_1,\ldots,\alpha_n,\theta} \to \mathcal{X}_{\alpha_1} \times \cdots \times \mathcal{X}_{\alpha_n} \times \mathcal{X}^{\mathrm{sd}}_\theta$ and $\mathrm{ev}\colon \mathcal{X}^{\mathrm{sd},+}_{\alpha_1,\ldots,\alpha_n,\theta} \to \mathcal{X}^{\mathrm{sd}}_{\alpha_1+\cdots+\alpha_n+\theta+\alpha_n^\vee+\cdots+\alpha_1^\vee}$. If $\mathcal{X}$ has quasi-compact filtrations as in §3.3.4, then the morphism ev described above is always quasi-compact.

3.4.5. Lemma. *Let $\mathcal{X}$ be a stack as in §3.2.2, acted on by $\mathbb{Z}_2$. Then the forgetful morphism $\mathcal{X}^{\mathbb{Z}_2} \to \mathcal{X}$ is affine.*

Proof. Let $\mathcal{J} = \mathcal{X} \times_{j_0, \mathcal{X}\times\mathcal{X}, j_1} \mathcal{X}$, where j_0 is the diagonal morphism, and $j_1 = (\mathrm{id}, i)$, where i is the involution. Let $\pi\colon \mathcal{J} \to \mathcal{X}$ be the projection to the first factor, which is affine as $\mathcal{X}$ has affine diagonal. Let $\mathbb{Z}_2$ act on $\mathcal{J}$ by the involution on the second factor, so we may identify $\mathcal{J}^{\mathbb{Z}_2} \simeq \mathcal{X}^{\mathbb{Z}_2}$. Then π is equivariant with respect to the trivial $\mathbb{Z}_2$-action on $\mathcal{X}$, so the forgetful morphism $\mathcal{J}^{\mathbb{Z}_2} \to \mathcal{J}$ is a closed immersion, which can be seen by base changing along morphisms from affine schemes to $\mathcal{X}$. The composition $\mathcal{X}^{\mathbb{Z}_2} \simeq \mathcal{J}^{\mathbb{Z}_2} \to \mathcal{J} \to \mathcal{X}$ is thus affine. $\square$

3.5 Stability conditions on linear stacks

3.5.1. We define a notion of *stability conditions* on linear stacks, based on the notion of Θ-*stratifications* of a stack developed by Halpern-Leistner [67]. Such stratifications is a geometric formulation of the existence and uniqueness of Harder–Narasimhan filtrations in various moduli problems, as we discussed in §1.2.4.

The theory of Θ-stratifications has seen important applications, including the construction of good moduli spaces for algebraic stacks in Alper, Halpern-Leistner, and Heinloth [5].

Our notion of stability is a geometric version of the notion of stability conditions on categories introduced in §2.3. In particular, such a stability condition will determine a *semistable locus* in the moduli stack, which will be used to construct enumerative invariants.

3.5.2. Stratifications. For an algebraic stack $\mathcal{X}$, we define a *stratification* of $\mathcal{X}$ as a family of locally closed substacks $(\mathcal{X}_i)_{i \in I}$ of $\mathcal{X}$, satisfying the following properties:

(i) The subspaces $|\mathcal{X}_i| \subset |\mathcal{X}|$ give a partition of the underlying set $|\mathcal{X}|$, meaning that they are disjoint and that their union is $|\mathcal{X}|$.

(ii) There exists a total order $\leqslant$ on I, such that for any $i \in I$, the set $\bigcup_{i' \leqslant i} |\mathcal{X}_{i'}| \subset |\mathcal{X}|$ is open.

(iii) *Local finiteness.* For any quasi-compact open substack $\mathcal{U} \subset \mathcal{X}$, there are only finitely many $i \in I$ such that $|\mathcal{U}| \cap |\mathcal{X}_i| \neq \varnothing$.

Note that these conditions depend entirely on the topological subspaces $|\mathcal{X}_i| \subset |\mathcal{X}|$.

In this case, each $\mathcal{X}_i$ is called a *stratum* of the stratification.

3.5.3. Θ-stratifications. We now define Θ-*stratifications* following Halpern-Leistner [67], but we slightly weaken the original definition by discarding the ordering on the set of strata.

Let $\mathcal{X}$ be a stack as in §3.2.2. A Θ-*stratification* of $\mathcal{X}$ is the following data:

- Open substacks $\mathcal{S} \subset \mathcal{F}ilt(\mathcal{X})$ and $\mathcal{Z} \subset \mathcal{G}rad(\mathcal{X})$, with $\mathcal{S} = \mathrm{gr}^{-1}(\mathcal{Z})$,

such that for each $\lambda \in \pi_0(\mathcal{G}rad(\mathcal{X})) \simeq \pi_0(\mathcal{F}ilt(\mathcal{X}))$, if we write $\mathcal{S}_\lambda \subset \mathcal{S}$ and $\mathcal{Z}_\lambda \subset \mathcal{Z}$ for the parts lying in the components $\mathcal{X}_\lambda^+ \subset \mathcal{F}ilt(\mathcal{X})$ and $\mathcal{X}_\lambda \subset \mathcal{G}rad(\mathcal{X})$, respectively, then:

- The morphism $\mathrm{ev}\colon \mathcal{S}_\lambda \to \mathcal{X}$ is a locally closed immersion, and the family $(\mathcal{S}_\lambda)_\lambda$ defines a stratification of $\mathcal{X}$.

In this case, each $\mathcal{Z}_\lambda$ is called the *centre* of the stratum $\mathcal{S}_\lambda$.

3.5.4. Stability for linear stacks. Let $\mathcal{X}$ be a linear stack, as in §3.3.2. Define a *stability condition* on $\mathcal{X}$ to be a map

$$\tau\colon \pi_0(\mathcal{X}) \setminus \{0\} \longrightarrow T$$

to a totally ordered set T, satisfying the following conditions:

(i) If $\alpha_1, \alpha_2 \in \pi_0(\mathcal{X}) \setminus \{0\}$ and $\tau(\alpha_1) \leqslant \tau(\alpha_2)$, then $\tau(\alpha_1) \leqslant \tau(\alpha_1 + \alpha_2) \leqslant \tau(\alpha_2)$.

(ii) For any class $\alpha \in \pi_0(\mathcal{X})$, the *semistable locus*

$$\mathcal{X}^{\mathrm{ss}}_\alpha(\tau) = \mathcal{X}_\alpha \Bigm\backslash \bigcup_{\substack{\alpha = \alpha_1 + \alpha_2 \\ \tau(\alpha_1) > \tau(\alpha_2)}} \mathrm{ev}_1(\mathcal{X}^+_{\alpha_1, \alpha_2}) \tag{3.5.4.1}$$

is open in $\mathcal{X}_\alpha$, where α_1, α_2 are assumed non-zero. Moreover, for any $t \in T$, the union

$$\mathcal{X}^{\mathrm{ss}}(\tau; t) = \{0\} \cup \coprod_{\substack{\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \tau(\alpha) = t}} \mathcal{X}^{\mathrm{ss}}_\alpha(\tau) \tag{3.5.4.2}$$

is an open linear substack of $\mathcal{X}$.

(iii) The open substacks

$$\begin{aligned} \mathcal{Z}_{\alpha_1, \ldots, \alpha_n}(\tau) &= \mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau) \times \cdots \times \mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau) \subset \mathcal{X}_{\alpha_1} \times \cdots \times \mathcal{X}_{\alpha_n}\,, \\ \mathcal{S}_{\alpha_1, \ldots, \alpha_n}(\tau) &= \mathrm{gr}^{-1}(\mathcal{Z}_{\alpha_1, \ldots, \alpha_n}(\tau)) \qquad \subset \mathcal{X}^+_{\alpha_1, \ldots, \alpha_n} \end{aligned}$$

for all $n \geqslant 0$ and classes $\alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}$ with $\tau(\alpha_1) > \cdots > \tau(\alpha_n)$ define a Θ-stratification of $\mathcal{X}$ in the sense of §3.5.3.

More precisely, the last condition means that for each choice of $\alpha_1, \ldots, \alpha_n$ as above, we choose an element λ as in §3.5.3 such that $\mathcal{X}_\lambda \simeq \mathcal{X}_{\alpha_1} \times \cdots \times \mathcal{X}_{\alpha_n}$ and $\mathcal{X}^+_\lambda \simeq \mathcal{X}^+_{\alpha_1, \ldots, \alpha_n}$, and we set $\mathcal{Z}_\lambda$ and $\mathcal{S}_\lambda$ as above; for all other λ, we set them to be empty.

3.5.5. Examples. Here are some examples of stability conditions on linear stacks.

(i) Let $\mathcal{X}$ be any linear stack. The constant map $\tau\colon \pi_0(\mathcal{X}) \setminus \{0\} \to \{0\}$ is called the *trivial stability condition*, where $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau) = \mathcal{X}_\alpha$ for all α.

(ii) Let $\mathscr{X}$ be the moduli stack of representations of a quiver Q. Then any *slope function* $\mu\colon Q_0 \to \mathbb{Q}$ induces a stability condition on $\mathscr{X}$ given by

$$\tau(d) = \frac{\sum_{i \in Q_0} d_i \cdot \mu(i)}{\sum_{i \in Q_0} d_i}$$

for non-zero dimension vectors $d \in \pi_0(\mathscr{X}) \setminus \{0\}$, where the Θ-stratification exists by Ibáñez Núñez [77, Theorem 2.6.3]. See §4.1 for more details.

(iii) Let $\mathscr{X}$ be the moduli stack of coherent sheaves on a projective scheme Y over an algebraically closed field K of characteristic zero. Then *Gieseker stability* is a stability condition on $\mathscr{X}$, where the choice of τ is described in Joyce [83, Example 4.16], and the Θ-stratification exists by Alper, Halpern-Leistner, and Heinloth [5, Example 7.28].

3.5.6. Stability for self-dual linear stacks. Let $\mathscr{X}$ be a self-dual linear stack over K, and let τ be a stability condition on $\mathscr{X}$. We say that τ is *self-dual*, if the following condition holds:

- For any $\alpha, \beta \in \pi_0(\mathscr{X}) \setminus \{0\}$, we have $\tau(\alpha) \leqslant \tau(\beta)$ if and only if $\tau(\alpha^\vee) \geqslant \tau(\beta^\vee)$.

In this case, for each $\theta \in \pi_0(\mathscr{X}^{\mathrm{sd}})$, writing $\alpha = j(\theta)$ for the corresponding class in $\pi_0(\mathscr{X})$, we have the *semistable locus*

$$\mathscr{X}^{\mathrm{sd,ss}}_\theta(\tau) = \mathscr{X}^{\mathrm{ss}}_\alpha(\tau)^{\mathbb{Z}_2} \cap \mathscr{X}^{\mathrm{sd}}_\theta \subset \mathscr{X}^{\mathrm{sd}}_\theta\,, \tag{3.5.6.1}$$

where $\mathscr{X}^{\mathrm{sd}}_\theta \subset (\mathscr{X}_\alpha)^{\mathbb{Z}_2}$ as an open and closed substack.

Note also that the open linear substack $\mathscr{X}^{\mathrm{ss}}(\tau; 0) \subset \mathscr{X}$ defined in (3.5.4.2) is self-dual.

We have an induced Θ-stratification of $\mathscr{X}^{\mathrm{sd}}$ given by the open substacks

$$\begin{aligned}
\mathscr{Z}^{\mathrm{sd}}_{\alpha_1,\ldots,\alpha_n,\theta}(\tau) &= \mathscr{X}^{\mathrm{ss}}_{\alpha_1}(\tau) \times \cdots \times \mathscr{X}^{\mathrm{ss}}_{\alpha_n}(\tau) \times \mathscr{X}^{\mathrm{sd,ss}}_\theta(\tau) \subset \mathscr{X}_{\alpha_1} \times \cdots \times \mathscr{X}_{\alpha_n} \times \mathscr{X}^{\mathrm{sd}}_\theta\,, \\
\mathscr{S}^{\mathrm{sd}}_{\alpha_1,\ldots,\alpha_n,\theta}(\tau) &= \mathrm{gr}^{-1}(\mathscr{Z}^{\mathrm{sd}}_{\alpha_1,\ldots,\alpha_n,\theta}(\tau)) \qquad\qquad\qquad\qquad \subset \mathscr{X}^{\mathrm{sd},+}_{\alpha_1,\ldots,\alpha_n,\theta}\,,
\end{aligned}$$

where $\alpha_1, \ldots, \alpha_n \in \pi_0(\mathscr{X}) \setminus \{0\}$ and $\theta \in \pi_0(\mathscr{X}^{\mathrm{sd}})$ are classes such that $\tau(\alpha_1) > \cdots > \tau(\alpha_n) > 0$. These strata and centres can also be realized as $\mathbb{Z}_2$-fixed loci in the strata and centres of the Θ-stratification of $\mathscr{X}$ given by τ.

3.5.7. Permissibility. Let $\mathscr{X}$ be a linear stack over K, and let τ be a stability condition on $\mathscr{X}$. We say that τ is *permissible*, if the following condition holds:

- For any $\alpha \in \pi_0(\mathscr{X})$, the semistable locus $\mathscr{X}^{\mathrm{ss}}_\alpha(\tau) \subset \mathscr{X}_\alpha$ is quasi-compact.

This is similar to the notion of permissible weak stability conditions in Joyce [83, Definition 4.7] and Joyce and Song [89, Definition 3.7].

3.5.8. Lemma. *Let $\mathcal{X}$ be a linear stack over K, and let τ be a permissible stability condition on $\mathcal{X}$. Then for any $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$, there are only finitely many decompositions $\alpha = \alpha_1 + \cdots + \alpha_n$ into classes $\alpha_i \in \pi_0(\mathcal{X}) \setminus \{0\}$, such that $\tau(\alpha_i) = \tau(\alpha)$ and $\mathcal{X}^{\mathrm{ss}}_{\alpha_i}(\tau) \neq \varnothing$ for all i.*

Proof. Let $t = \tau(\alpha)$. Then the open substack

$$\mathcal{X}(\tau; t) = \{0\} \cup \coprod_{\substack{\beta \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \tau(\beta) = t}} \mathcal{X}^{\mathrm{ss}}_{\beta}(\tau) \subset \mathcal{X}$$

is again a linear stack. Replacing $\mathcal{X}$ by $\mathcal{X}(\tau; t)$, we may assume that $\mathcal{X}$ has quasi-compact connected components, and that τ is trivial.

By the finiteness theorem of the author et al. [31, Theorem 6.2.3], each connected component of $\mathcal{X}$ has finitely many *special faces*. In this case, this is the statement that for any $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$, there are finitely many decompositions $\alpha = \alpha_1 + \cdots + \alpha_n$, such that all other decompositions can be obtained from combining terms in these decompositions, and hence the total number of decompositions is finite. $\square$

3.6 Derived algebraic geometry

3.6.1. In the remainder of this chapter, we discuss derived enhancements of ideas and concepts discussed above, using derived algebraic geometry. For example, moduli stacks will be upgraded to derived algebraic stacks, and the extra derived structure will be useful later in the thesis.

This section provides background material on derived algebraic geometry. We mainly follow Toën and Vezzosi [158; 159] and Pantev, Toën, Vaquié, and Vezzosi [135], and use their framework as our foundation, but we restate their definitions using the language of ∞-categories instead of model categories. See Khan [94] for a gentle introduction to derived algebraic geometry. Other useful references include Lurie [108; 111] and Calaque, Haugseng, and Scheimbauer [37, Appendix B].

3.6.2. The étale topology. Let dAff be the ∞-category of *derived affine schemes*, defined as the opposite category $\mathsf{sCRing}^{\mathrm{op}}$ of the ∞-category of simplicial commutative rings.

For a morphism $A \to B$ in sCRing, denote by $\operatorname{Spec} B \to \operatorname{Spec} A$ the corresponding morphism in dAff. We say that such a morphism is an *étale surjection*, if the following hold:

(i) For each $n \in \mathbb{N}$, the induced morphism $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_n(B)$ is an isomorphism.

(ii) The morphism $\operatorname{Spec} \pi_0(B) \to \operatorname{Spec} \pi_0(A)$ of usual schemes is an étale surjection.

A finite family $(\operatorname{Spec} B_i \to \operatorname{Spec} A)_{i \in I}$ is an *étale cover* if the induced morphism $\operatorname{Spec} \prod_i B_i \to \operatorname{Spec} A$ is an étale surjection. This notion of covering defines the *étale topology* on dAff.

3.6.3. Derived stacks. A *derived stack* is a functor

$$\mathcal{X} \colon \mathsf{dAff}^{\mathrm{op}} \longrightarrow \infty\text{-}\mathsf{Grpd} ,$$

where ∞-Grpd is the ∞-category of small ∞-groupoids (also known as the ∞*-category of spaces*, such as in Lurie [109, §1.2.16]), such that it is a *hypersheaf* with respect to the étale topology on dAff. This means more precisely the following conditions:

(i) $\mathcal{X}$ preserves finite products, that is, it sends finite coproducts in dAff to products.

(ii) For any *hypercover* $\operatorname{Spec} B^\bullet \to \operatorname{Spec} A$ in dAff, that is a cosimplicial object $B^\bullet \colon \Delta \to \mathsf{sCRing}_{A/}$, where Δ is the category of simplices, such that each induced morphism $(\operatorname{cosk}_A^{n-1} B^\bullet)^n \to B^n$ is an étale surjection, where $\operatorname{cosk}_A^n \colon (\mathsf{sCRing}_{A/})^{\Delta_{\leqslant n}} \to (\mathsf{sCRing}_{A/})^\Delta$ is the right adjoint of the restriction, the induced morphism

$$\mathcal{X}(A) \longrightarrow \lim_{\Delta} \Bigl(\mathcal{X}(B^0) \rightrightarrows \mathcal{X}(B^1) \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}} \cdots \Bigr)$$

in ∞-Grpd is an equivalence.

This is a translation of the definition of Toën and Vezzosi [159, Definition 2.2.2.14]; see [159, Corollary 1.3.2.4] for the model category version of these conditions, and [108, Remark 4.2.3 ff.] for the ∞-categorical notion of hypercovers.

Derived stacks form an ∞-category

$$\mathsf{dSt} \subset \mathsf{Fun}(\mathsf{dAff}^{\mathrm{op}}, \infty\text{-}\mathsf{Grpd}) ,$$

as a full subcategory in the functor category consisting of functors that are derived stacks.

We have a notion of *derived algebraic stacks*, called *locally geometric stacks* in Toën and Vaquié [157], defined as those stacks that admit an open cover by *geometric stacks* in the sense of Toën and Vezzosi [159, Lemma 2.2.3.1 ff.].

For a derived stack $\mathcal{X}$, its *classical truncation* $\mathcal{X}_{\mathrm{cl}}$ is the restriction of $\mathcal{X}$ to the full subcategory $\mathsf{Aff}^{\mathrm{op}} \subset \mathsf{dAff}^{\mathrm{op}}$. If its image lands in $\mathsf{Grpd} \subset \infty\text{-}\mathsf{Grpd}$, and if $\mathcal{X}_{\mathrm{cl}}$ is a classical algebraic stack, we sometimes say that $\mathcal{X}$ is a *derived Artin stack*.

A derived algebraic stack $\mathcal{X}$ that is (*homotopically*) *locally of finite presentation* over a field K admits a *cotangent complex* $\mathbb{L}_{\mathcal{X}}$, which is a perfect complex on $\mathcal{X}$. Its dual is called the *tangent complex* of $\mathcal{X}$, denoted by $\mathbb{T}_{\mathcal{X}}$. However, note that this notion of finite presentation is very different from the classical one; see Khan [94, Warning 1.3.49].

For derived stacks $\mathcal{X}, \mathcal{Y}$ defined over a base derived stack $\mathcal{S}$, we have the *derived mapping stack* ${}^{\mathrm{d}}\mathcal{M}\mathit{ap}_{\mathcal{S}}(\mathcal{X}, \mathcal{Y})$ defined as the object of $\mathsf{dSt}_{/\mathcal{S}}$, if it exists, representing the functor

$$\begin{aligned} {}^{\mathrm{d}}\mathcal{M}\mathit{ap}_{\mathcal{S}}(\mathcal{X}, \mathcal{Y}) \colon \mathsf{dAff}_{/\mathcal{S}} &\longrightarrow \infty\text{-}\mathsf{Grpd} \, , \\ T &\longmapsto \mathsf{dSt}_{/\mathcal{S}}(\mathcal{X} \times_{\mathcal{S}} T, \mathcal{Y}) \, . \end{aligned}$$

We often omit the base $\mathcal{S}$ when it is clear from context.

3.6.4. Moduli of objects in dg-categories. A main source of derived algebraic stacks for us is from moduli stacks of objects in dg-categories, constructed by Toën and Vaquié [157].

Let K be a commutative ring, and let $\mathcal{C}$ be a K-linear dg-category of finite type, in the sense of [157, Definition 2.4]. By [157, Theorem 3.6], there is a moduli stack $\mathcal{M}_{\mathcal{C}}$ of right proper objects in $\mathcal{C}$, which is a derived algebraic stack locally of finite presentation over K, given by the moduli functor

$$\mathcal{M}_{\mathcal{C}}(R) = \mathsf{dgCat}_K(\mathcal{C}^{\mathrm{op}}, \mathsf{Perf}(R)) \tag{3.6.4.1}$$

for simplicial commutative K-algebras R, where $\mathsf{dgCat}_K(-,-)$ denotes the mapping space of K-linear dg-categories, as in [147; 156]. See [157] for the precise definitions.

In particular, we have the moduli stack

$$\mathcal{P}\mathit{erf} = \mathcal{M}_{\mathsf{Perf}(K)} \tag{3.6.4.2}$$

of perfect complexes over K, and for any smooth and proper K-scheme X, we have the moduli stack of perfect complexes on X, defined by

$$\mathscr{P}\!\mathit{erf}(X) = \mathscr{M}_{\mathsf{Perf}(X)} \simeq {}^{\mathrm{d}}\mathscr{M}\!\mathit{ap}(X, \mathscr{P}\!\mathit{erf}) \, , \tag{3.6.4.3}$$

which is also algebraic and locally of finite presentation over K, and equivalent to the derived mapping stack from X to $\mathscr{P}\!\mathit{erf}$. See [157, Definition 3.28 ff.] for details on this.

3.6.5. Total stacks of perfect complexes. Another construction of derived algebraic stacks is as total stacks of perfect complexes on other schemes or stacks.

Let $\mathcal{X}$ be a derived algebraic stack locally of finite presentation over a field K of characteristic zero, and let $E \in \mathsf{Perf}(\mathcal{X})$ be a perfect complex on $\mathcal{X}$. As in Calaque [36, §2], there is a *total stack* of E, which is a derived algebraic stack $\mathscr{E}$ locally of finite presentation over K, given by the relative spectrum

$$\mathscr{E} = \mathscr{S}\!\mathit{pec}_{\mathcal{X}}(\mathrm{Sym}(E^\vee)) \, , \tag{3.6.5.1}$$

as a relative spectrum of a possibly non-connective commutative algebra object over $\mathcal{X}$, meaning that it is defined by the universal property that

$$\mathsf{dSt}_{/\mathcal{X}}(\mathrm{Spec}(A), \mathscr{E}) \simeq \mathsf{Alg}_A(\mathrm{Sym}_A(E^\vee|_{\mathrm{Spec}\,A}), A) \tag{3.6.5.2}$$

for all morphisms $\mathrm{Spec}\,A \to \mathcal{X}$ for $\mathrm{Spec}\,A \in \mathsf{dAff}$, where $\mathsf{Alg}_A(-,-)$ denotes the mapping space of commutative A-algebras. See [36, §2] for the precise formulation.

For example, if $\mathcal{X}$ is a classical smooth scheme and E is a vector bundle on $\mathcal{X}$, then $\mathscr{E}$ is the usual total space of E.

We often consider the *n-shifted cotangent stack* $\mathrm{T}^*[n]\,\mathcal{X}$, defined as the total stack of the shifted cotangent complex $\mathbb{L}_{\mathcal{X}}[n]$, which is perfect.

3.6.6. Derived graded points. The stacks of graded and filtered points defined in §3.2.3 have derived versions, simply by replacing the mapping stack with the derived mapping stack described in §3.6.3.

Namely, for a derived algebraic stack $\mathcal{X}$ locally almost of finite presentation over a field K, whose classical truncation $\mathcal{X}_{\mathrm{cl}}$ satisfies the conditions of §3.2.2, we denote

$$^{\mathrm{d}}\mathcal{G}rad(\mathcal{X}) = {}^{\mathrm{d}}\mathcal{M}ap(*/\mathbb{G}_{\mathrm{m}}, \mathcal{X}) , \tag{3.6.6.1}$$

$$^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}) = {}^{\mathrm{d}}\mathcal{M}ap(\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}, \mathcal{X}) . \tag{3.6.6.2}$$

These exist as derived algebraic stacks locally almost of finite presentation over K, by Halpern-Leistner [68, Theorem 1.2.1].

Note that even if $\mathcal{X}$ is a classical algebraic stack, these derived stacks may have non-trivial derived structure. See [68, Example 1.6.4] for an example of this phenomenon. However, for classical $\mathcal{X}$, we always have ${}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})_{\mathrm{cl}} \simeq \mathcal{G}rad(\mathcal{X})$ and ${}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})_{\mathrm{cl}} \simeq \mathcal{F}ilt(\mathcal{X})$ by their definitions.

3.6.7. Shifted symplectic structures. A main reason for working with derived stacks in this thesis is to work with *shifted symplectic structures*, which are possessed by some moduli stacks of interest, and they contain rich geometric information.

For a derived algebraic stack $\mathcal{X}$ locally finitely presented over a field K of characteristic zero, and an integer $n \in \mathbb{Z}$, an *n-shifted symplectic structure* on $\mathcal{X}$ is an *n-shifted closed 2-form* ω on $\mathcal{X}$ that induces an isomorphism

$$\omega \colon \mathbb{T}_{\mathcal{X}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{X}}[n] . \tag{3.6.7.1}$$

See Pantev, Toën, Vaquié, and Vezzosi [135] or Park and You [136] for precise definitions.

A basic example of n-shifted symplectic stacks is the n-shifted cotangent stack $\mathbb{T}^*[n]\,\mathcal{X}$ of a derived stack $\mathcal{X}$ locally finitely presented over K, defined in §3.6.5, analogous to the canonical symplectic structure on the cotangent bundle of a smooth manifold. See Calaque [36, Theorem 2.4] for the construction of the shifted symplectic structure.

As another example, the moduli stack of objects in a *Calabi–Yau dg-category* of degree $d \in \mathbb{Z}$ admits a $(2-d)$-shifted symplectic structure, by Brav and Dyckerhoff [21, Theorem 5.6].

Finally, as in Pantev, Toën, Vaquié, and Vezzosi [135, §2.1] or Calaque, Haugseng, and Scheimbauer [37], the derived mapping stack from a *d-oriented stack* to an n-shifted symplectic stack, if it is algebraic and locally finitely presented, admits an $(n-d)$-shifted symplectic structure.

3.6.8. Orientations. We now discuss an extra piece of data on shifted symplectic stacks, called *orientations* or *spin structures*, that will be used in various constructions later on.

For an n-shifted symplectic stack $\mathcal{X}$ with n odd, consider its canonical line bundle $K_{\mathcal{X}} = \det \mathbb{L}_{\mathcal{X}}$. The isomorphism (3.6.7.1) induces an isomorphism $K_{\mathcal{X}}^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_{\mathcal{X}}$.

An *orientation* of $\mathcal{X}$ is a pair $(K_{\mathcal{X}}^{1/2}, o_{\mathcal{X}})$, where $K_{\mathcal{X}}^{1/2}$ is a line bundle on $\mathcal{X}$, and

$$o_{\mathcal{X}} \colon (K_{\mathcal{X}}^{1/2})^{\otimes 2} \xrightarrow{\sim} K_{\mathcal{X}} \tag{3.6.8.1}$$

is an isomorphism that squares to the canonical one. We sometimes abbreviate the pair as $o_{\mathcal{X}}$, and we call the pair $(\mathcal{X}, o_{\mathcal{X}})$ an *oriented* n-shifted symplectic stack.

3.6.9. Example. The derived critical locus. Let $\mathcal{U}$ be a smooth algebraic stack over K, and let $f \colon \mathcal{U} \to \mathbb{A}^1$ be a function. The *derived critical locus* of f is the derived algebraic stack

$$^{\mathrm{d}}\mathrm{Crit}(f) = \mathcal{U} \underset{0,\, \mathrm{T}^*\mathcal{U},\, df}{\times} \mathcal{U} \, , \tag{3.6.9.1}$$

whose classical truncation is the classical critical locus $\mathrm{Crit}(f)$. It admits a canonical (-1)-shifted symplectic structure by, for example, Bozec, Calaque, and Scherotzke [19, §4.2.1]. In fact, this construction holds for any derived algebraic stack $\mathcal{U}$ locally of finite presentation over K.

Moreover, $^{\mathrm{d}}\mathrm{Crit}(f)$ admits a canonical orientation given by

$$K^{1/2}_{^{\mathrm{d}}\mathrm{Crit}(f)} = K_{\mathcal{U}}|_{^{\mathrm{d}}\mathrm{Crit}(f)} \, , \tag{3.6.9.2}$$

where we restrict along either projection $^{\mathrm{d}}\mathrm{Crit}(f) \to \mathcal{U}$.

3.7 Derived linear stacks

3.7.1. We now discuss *derived linear stacks*, which are linear stacks introduced in §§3.3–3.4 equipped with compatible derived structure. These stacks will be used to model derived moduli stacks of objects in linear categories.

Note that moduli stacks of objects in dg-categories discussed in §3.6.4 will not be examples of derived linear stacks, since their classical truncations are not classical algebraic stacks, but rather *higher stacks*. Instead, roughly speaking, we will consider open substacks in these stacks that correspond to subcategories of the original dg-category that are 1-categories, such as hearts, and these substacks will be derived linear stacks. See §4.2 below for details.

3.7.2. Derived linear stacks. Let K be an algebraically closed field. Define a *derived linear stack* over K to be the following data:

- A derived algebraic stack $\mathcal{X}$ locally finitely presented over K.
- A commutative monoid structure on $\mathcal{X}$ in the ∞-category dSt_K, with multiplication morphism $\oplus\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and unit $0 \in \mathcal{X}(K)$.
- A $*/\mathbb{G}_\mathrm{m}$-action $\odot\colon */\mathbb{G}_\mathrm{m} \times \mathcal{X} \to \mathcal{X}$ respecting the monoid structure. More precisely, we require the data of a commutative monoid object in the ∞-category of $*/\mathbb{G}_\mathrm{m}$-equivariant derived stacks.

We require the following additional property:

- There is an isomorphism

  $$\coprod_{\gamma\colon \mathbb{Z} \to \pi_0(\mathcal{X})} \prod_{n \in \operatorname{supp}(\gamma)} \mathcal{X}_{\gamma(n)} \xrightarrow{\sim} {}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X}) \, , \tag{3.7.2.1}$$

  defined by the same process as in §3.3.2.

3.7.3. Shifted symplectic linear stacks. Now let K be an algebraically closed field of characteristic zero. As in the author et al. [30, §3.1.7], for an integer $n \in \mathbb{Z}$, define an *n-shifted symplectic linear stack* to be the following data:

- A derived linear stack $\mathcal{X}$ over K.
- An n-shifted symplectic structure ω on $\mathcal{X}$, such that

  $$\oplus^*(\omega) \simeq \omega \boxplus \omega \tag{3.7.3.1}$$

  on $\mathcal{X} \times \mathcal{X}$, where $\omega \boxplus \omega = \mathrm{pr}_1^*(\omega) + \mathrm{pr}_2^*(\omega)$, and $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are the projections.

Note that the requirement (3.7.3.1) is weaker than the perhaps more natural one requiring this equivalence together with higher coherence data. However, this weaker condition is sufficient for our applications in this thesis.

3.7.4. Orientation data. We introduce a compatibility condition for orientations, as in §3.6.8, on shifted symplectic linear stacks, called *orientation data* following Kontsevich and Soibelman [99].

Let $\mathcal{X}$ be an n-shifted symplectic linear stack, with n odd, and let $o_{\mathcal{X}}$ be an orientation of $\mathcal{X}$ as in §3.6.8. By [30, §6.1.6], this induces an orientation $o_{\mathcal{G}rad(\mathcal{X})}$ of $\mathcal{G}rad(\mathcal{X})$. An orientation $o_{\mathcal{X}}$ is called an *orientation data*, if it satisfies the following compatibility condition:

- Under the isomorphism (3.3.2.1), the induced orientation $o_{\mathcal{G}rad(\mathcal{X})}$ of $\mathcal{G}rad(\mathcal{X})$ agrees with the product orientations on the left-hand side.

By Joyce and Upmeier [90, Theorem 3.6], such an orientation data exists canonically on moduli stacks of coherent sheaves on Calabi–Yau threefolds.

3.7.5. Self-dual orientation data. Now, let $\mathcal{X}$ be a *self-dual* (-1)*-shifted symplectic linear stack*, that is, a stack $\mathcal{X}$ as in §3.7.4, equipped with a $\mathbb{Z}_2$-action preserving the symplectic form ω, compatible with the monoid structure $\oplus$ and inverting the $*/\mathbb{G}_{\mathrm{m}}$-action $\odot$.

In this case, the fixed locus $\mathcal{X}^{\mathrm{sd}} = \mathcal{X}^{\mathbb{Z}_2}$ carries an induced (-1)-shifted symplectic structure. However, an orientation of $\mathcal{X}$ does not naturally induce one on $\mathcal{X}^{\mathrm{sd}}$.

We define a *self-dual orientation data* on $\mathcal{X}$ to be a pair $(o_{\mathcal{X}}, o_{\mathcal{X}^{\mathrm{sd}}})$ of orientations of $\mathcal{X}$ and $\mathcal{X}^{\mathrm{sd}}$, respectively, satisfying the following conditions:

(i) $o_{\mathcal{X}}$ is an orientation data.

(ii) Under the isomorphism (3.4.3.1), the induced orientation of $\mathcal{G}rad(\mathcal{X}^{\mathrm{sd}})$ agrees with the product orientations on the right-hand side.

The author does not know if such a self-dual orientation data, or even an orientation, exists in the case of coherent sheaves on Calabi–Yau threefolds, which we will discuss in §4.2 and §8.3 below.

3.8 The attractor correspondence

3.8.1. In this section, we study the *attractor correspondence*

$$\mathcal{G}rad(\mathcal{X}) \xleftarrow{\mathrm{gr}} \mathcal{F}ilt(\mathcal{X}) \xrightarrow{\mathrm{ev}} \mathcal{X} \tag{3.8.1.1}$$

for an algebraic stack $\mathcal{X}$, defined in §3.2.3, together with its derived version, which will be an important tool for constructions later on. We name it this way because of the local description below, involving the fixed and attractor loci discussed in Example 3.2.4.

3.8.2. Local structure. We first discuss how étale local models of an algebraic stack interact with its stacks of graded and filtered points.

Let $\mathcal{X}$ be an algebraic stack over K as in §3.2.2, and let $(\mathcal{X}_i \to \mathcal{X})_{i \in I}$ be a representable étale cover, where each $\mathcal{X}_i \simeq S_i / G_i$, with S_i an algebraic space over K and G_i a reductive group. Then there are commutative diagrams

$$
\begin{tikzcd}
S_i^{\lambda} / L_{i,\lambda} \arrow[d] & S_i^{\lambda,+} / P_{i,\lambda} \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & S_i / G_i \arrow[d] \\
\mathcal{G}rad(\mathcal{X}) & \mathcal{F}ilt(\mathcal{X}) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}"] & \mathcal{X} \, ,
\end{tikzcd}
\tag{3.8.2.1}
$$

where all vertical arrows are representable and étale, $\lambda \colon \mathbb{G}_{\mathrm{m}} \to G_i$ is a cocharacter, and the left-hand square is a pullback square by [31, Theorem 5.2.7]. Moreover, the families

$$
\begin{aligned}
\big(S_i^{\lambda} / L_{i,\lambda} \longrightarrow \mathcal{G}rad(\mathcal{X}) \big)_{i \in I, \ \lambda \colon \mathbb{G}_{\mathrm{m}} \to G_i} \, , \\
\big(S_i^{\lambda,+} / P_{i,\lambda} \longrightarrow \mathcal{F}ilt(\mathcal{X}) \big)_{i \in I, \ \lambda \colon \mathbb{G}_{\mathrm{m}} \to G_i}
\end{aligned}
$$

are representable étale covers of $\mathcal{G}rad(\mathcal{X})$ and $\mathcal{F}ilt(\mathcal{X})$, respectively, which follows from Halpern-Leistner [67, Corollary 1.1.7] and the pullback square in (3.8.2.1).

3.8.3. Deformation theory. For a derived algebraic stack $\mathcal{X}$ locally of finite presentation over K, one can express the tangent complexes of ${}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})$ and ${}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})$ in terms of that of $\mathcal{X}$. Concretely, by Halpern-Leistner and Preygel [69, Proposition 5.1.10], or Halpern-Leistner [67, Lemma 1.2.2], we have

$$
\mathbb{T}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \simeq \mathrm{tot}^* (\mathbb{T}_{\mathcal{X}})_0 \, , \tag{3.8.3.1}
$$

$$
\mathbb{T}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})} \simeq q_* \circ p^* (\mathbb{T}_{\mathcal{X}}) \, , \tag{3.8.3.2}
$$

where $(-)_0$ denotes the weight 0 part with respect to the natural $\mathbb{G}_{\mathrm{m}}$-action, $p \colon \mathbb{A}^1 / \mathbb{G}_{\mathrm{m}} \times {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}) \to \mathcal{X}$ is the evaluation morphism, and $q \colon \mathbb{A}^1 / \mathbb{G}_{\mathrm{m}} \times {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}) \to {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})$ is the projection.

3.8.4. Shifted Lagrangian correspondences. Now suppose that K is algebraically closed of characteristic 0. Let $\mathscr{X}$ and $\mathscr{Y}$ be oriented n-shifted symplectic stacks over K, as in §3.6.8, where n is odd.

As in Pantev, Toën, Vaquié, and Vezzosi [135, Definition 2.8] or Calaque, Haugseng, and Scheimbauer [37, §2.4], a diagram

$$\mathscr{X} \xleftarrow{f} \mathscr{L} \xrightarrow{g} \mathscr{Y} \tag{3.8.4.1}$$

is called an *n-shifted Lagrangian correspondence*, roughly if we have an exact triangle

$$\mathbb{T}_{\mathscr{L}} \longrightarrow f^*(\mathbb{T}_{\mathscr{X}}) \oplus g^*(\mathbb{T}_{\mathscr{Y}}) \longrightarrow \mathbb{L}_{\mathscr{L}}[n] \longrightarrow \mathbb{T}_{\mathscr{L}}[1] \tag{3.8.4.2}$$

of perfect complexes on $\mathscr{L}$, where the first map is $(f_*, -g_*)$, and the second map is $(f^*, g^*)[n]$ composed with the identifications $\mathbb{T}_{\mathscr{X}} \simeq \mathbb{L}_{\mathscr{X}}[n]$ and $\mathbb{T}_{\mathscr{Y}} \simeq \mathbb{L}_{\mathscr{Y}}[n]$ given by the symplectic structures. See [37, §2.4] for details.

An *orientation* of the shifted Lagrangian correspondence (3.8.4.1) is an isomorphism

$$K_{\mathscr{L}} \xrightarrow{\sim} f^*(K_{\mathscr{X}}^{1/2}) \otimes g^*(K_{\mathscr{Y}}^{1/2}) \,, \tag{3.8.4.3}$$

such that it squares to the canonical isomorphism $K_{\mathscr{L}}^{\otimes 2} \simeq f^*(K_{\mathscr{X}}) \otimes g^*(K_{\mathscr{Y}})$ induced by the exact triangle (3.8.4.2).

3.8.5. Theorem. *Let K be an algebraically closed field of characteristic 0, and let $\mathscr{X}$ be an n-shifted symplectic stack over K, with symplectic form ω.*

Then we have an induced n-shifted symplectic structure $\mathrm{tot}^*(\omega)$ on ${}^{\mathrm{d}}\mathscr{G}rad(\mathscr{X})$, and an n-shifted Lagrangian correspondence

$${}^{\mathrm{d}}\mathscr{G}rad(\mathscr{X}) \xleftarrow{\mathrm{gr}} {}^{\mathrm{d}}\mathscr{F}ilt(\mathscr{X}) \xrightarrow{\mathrm{ev}} \mathscr{X} \,. \tag{3.8.5.1}$$

Moreover, if n is odd and $\mathscr{X}$ has an orientation $K_{\mathscr{X}}^{1/2}$, then ${}^{\mathrm{d}}\mathscr{G}rad(\mathscr{X})$ has an induced orientation $K_{{}^{\mathrm{d}}\mathscr{G}rad(\mathscr{X})}^{1/2}$, and the Lagrangian correspondence is oriented.

Proof. The stacks ${}^{\mathrm{d}}\mathscr{G}rad(\mathscr{X})$ and ${}^{\mathrm{d}}\mathscr{F}ilt(\mathscr{X})$ are derived algebraic stacks locally of finitely presentation over K. These follow from Halpern-Leistner and Preygel [69, Theorem 5.1.1 and Remark 5.1.3]; although they work with stacks locally almost of finite presentation, their argument also shows in our case that our stacks are locally of finitely presentation.

To prove that (3.8.5.1) is an n-shifted Lagrangian correspondence, by Calaque [35, Theorem 4.8], it is enough to show that the cospan

$$*/\mathbb{G}_{\mathrm{m}} \xrightarrow{0} \mathbb{A}^1/\mathbb{G}_{\mathrm{m}} \xleftarrow{1} * \tag{3.8.5.2}$$

is a 0-*oriented cospan*, in the sense of [35, §4.2] and Calaque, Haugseng, and Scheimbauer [37, §2.5]. Indeed, $*$ carries a natural 0-orientation, and the 0-orientation on $*/\mathbb{G}_{\mathrm{m}}$ is given by the isomorphism $\mathbb{R}\Gamma(\mathcal{O}_{*/\mathbb{G}_{\mathrm{m}}}) \xrightarrow{\sim} K$. To see that this is indeed a 0-orientation, we check the condition in [135, Definition 2.4]. For $A \in \mathsf{CdgA}_K^{\leqslant 0}$ and a perfect complex $\mathcal{E} \in \mathsf{Perf}(\operatorname{Spec} A \times (*/\mathbb{G}_{\mathrm{m}}))$, one has $p_*(\mathcal{E}^\vee)^\vee \simeq p_*(\mathcal{E})$ on $\operatorname{Spec} A$, where $p\colon \operatorname{Spec} A \times (*/\mathbb{G}_{\mathrm{m}}) \to \operatorname{Spec} A$ is the projection, since both sides are the weight 0 part of the induced $\mathbb{G}_{\mathrm{m}}$-action on $\pi^*(\mathcal{E})$, where $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} A \times (*/\mathbb{G}_{\mathrm{m}})$ is the projection.

To see that (3.8.5.2) is a 0-oriented cospan, we check the condition in [37, Lemma 2.5.5]. For any $A \in \mathsf{CdgA}_K^{\leqslant 0}$ and $\mathcal{E} \in \mathsf{Perf}(\operatorname{Spec} A \times (\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}))$, we need to show that the induced commutative diagram

$$\begin{tikzcd} q_*(\mathcal{E}) \arrow[r] \arrow[d] & p_* \circ 0^*(\mathcal{E}) \arrow[d] \\ 1^*(\mathcal{E}) \arrow[r] & q_*(\mathcal{E}^\vee)^\vee \end{tikzcd} \tag{3.8.5.3}$$

in $\mathsf{Perf}(A)$ is cartesian, where p and q are the projections from $\operatorname{Spec} A \times (*/\mathbb{G}_{\mathrm{m}})$ and $\operatorname{Spec} A \times (\mathbb{A}^1/\mathbb{G}_{\mathrm{m}})$ to $\operatorname{Spec} A$, respectively. Indeed, as in Halpern-Leistner [68, Proposition 1.1.2 ff.], such an object $\mathcal{E}$ can be seen as a filtered object in $\mathsf{Perf}(A)$, that is, a sequence of maps

$$\cdots \longrightarrow E_{\geqslant 1} \longrightarrow E_{\geqslant 0} \longrightarrow E_{\geqslant -1} \longrightarrow \cdots$$

in $\mathsf{Perf}(A)$, where all but finitely many arrows are isomorphisms, such that $E_{\geqslant n} = 0$ for $n \gg 0$. Write $E_n = \operatorname{cofib}(E_{\geqslant n+1} \to E_{\geqslant n})$, and write $E = \operatorname{colim}_{n \to -\infty} E_{\geqslant n}$. Then $0^*(\mathcal{E}) \simeq \bigoplus_n E_n$, with the natural $\mathbb{G}_{\mathrm{m}}$-action having weight n on E_n. One can deduce from [68, Proposition 1.1.2 ff.] that we have natural identifications

$$\begin{aligned} q_*(\mathcal{E}) &\simeq E_{\geqslant 0}\,, \\ p_* \circ 0^*(\mathcal{E}) &\simeq E_0\,, \\ 1^*(\mathcal{E}) &\simeq E\,, \\ q_*(\mathcal{E}^\vee)^\vee \simeq ((E^\vee)_{\geqslant 0})^\vee &\simeq E_{\leqslant 0}\,, \end{aligned}$$

where $E_{\leqslant 0} = \operatorname{cofib}(E_{\geqslant 1} \to E)$, and the arrows in the diagram (3.8.5.3) are the natural ones. This implies that (3.8.5.3) is cartesian.

For the final statement, observe that

$$
\begin{aligned}
\operatorname{tot}^*(K_{\mathcal{X}}) &\simeq \det\big(\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^0\big) \otimes \det\big(\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^+\big) \otimes \det\big(\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^-\big) \\
&\simeq K_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \otimes \det\big(\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^+\big) \otimes \det\big((\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^+)^\vee[-s]\big) \\
&\simeq K_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \otimes \det\big(\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^+\big)^2 ,
\end{aligned}
$$

where $(-)^0, (-)^+, (-)^-$ denote the parts with zero, positive, and negative weights, respectively, with respect to the natural $\mathbb{G}_{\mathrm{m}}$-action. Therefore, we may define

$$
K^{1/2}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} = \operatorname{tot}^*(K^{1/2}_{\mathcal{X}}) \otimes \det\big(\operatorname{tot}^*(\mathbb{L}_{\mathcal{X}})^+\big)^{-1} , \tag{3.8.5.4}
$$

and this gives an orientation on ${}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})$. To see that the n-shifted Lagrangian correspondence is oriented, consider the cartesian diagram

$$
\begin{tikzcd}
\mathbb{T}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{gr}^*(\mathbb{T}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})}) \arrow[d] \\
\operatorname{ev}^*(\mathbb{T}_{\mathcal{X}}) \arrow[r] & \mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}[s]
\end{tikzcd} \tag{3.8.5.5}
$$

in $\operatorname{Perf}({}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}))$, witnessing the n-shifted Lagrangian correspondence structure. Write $\mathcal{E} = r^*(\mathbb{T}_{\mathcal{X}})$, where $r\colon (\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}) \times {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}) \to \mathcal{X}$ is the evaluation morphism. As in the argument above, $\mathcal{E}$ can be seen as a filtered object in $\operatorname{Perf}({}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}))$, and the terms in (3.8.5.5) can be identified with $E_{\geqslant 0}$, E_0, E, and $E_{\leqslant 0}$, respectively. In particular, one has $K_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})} \simeq \operatorname{gr}^*(K^{1/2}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})}) \otimes \operatorname{ev}^*(K^{1/2}_{\mathcal{X}})$, as both sides can be identified with $\det(E_{\geqslant 0})^{-1}$. □

Chapter 4

Examples

4.1 Self-dual quivers

4.1.1. As a basic example of orthosymplectic enumerative geometry, we discuss *self-dual quivers* and their *self-dual representations*. Such representations are analogous to orthogonal and symplectic principal bundles on a variety, similar to how the usual quiver representations are analogous to vector bundles or coherent sheaves on varieties.

Self-dual quivers were first introduced by Derksen and Weyman [51] as a special case of *G-quivers* for $G = \mathrm{O}(n)$ or $\mathrm{Sp}(2n)$, and studied by Young [163–165] in the context of Donaldson–Thomas theory.

Throughout, we fix an algebraically closed field K.

4.1.2. Quivers. Recall that a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where

- Q_0 is a finite set, called the set of *vertices*.
- Q_1 is a finite set, called the set of *edges*.
- $s, t \colon Q_1 \to Q_0$ are maps sending each edge to its source and target, respectively.

A *representation* of Q is the data $E = \big((E_i)_{i \in Q_0}, (e_a)_{a \in Q_1}\big)$, where each E_i is a finite-dimensional K-vector space, and each $e_a \colon E_{s(a)} \to E_{t(a)}$ is a linear map.

We denote by $\mathsf{Mod}(KQ)$ the K-linear abelian category of finite-dimensional representations of Q over K, where KQ stands for the *path algebra* of Q, which is a possibly non-commutative K-algebra whose representations are equivalent to representations of Q.

See, for example, Derksen and Weyman [52], for background on quivers.

4.1.3. Self-dual quivers. We define a *self-dual quiver* to be a quiver Q equipped with the following data:

- A *contravariant involution* $(-)^\vee \colon Q \xrightarrow{\sim} Q^{\mathrm{op}}$, where $Q^{\mathrm{op}} = (Q_0, Q_1, t, s)$ is the opposite quiver of Q, such that $(-)^{\vee\vee} = \mathrm{id}$.
- Choices of signs $u \colon Q_0 \longrightarrow \{\pm 1\}$ and $v \colon Q_1 \longrightarrow \{\pm 1\}$, such that $u(i) = u(i^\vee)$ for all $i \in Q_0$, and $v(a)\, v(a^\vee) = u(s(a))\, u(t(a))$ for all $a \in Q_1$.

This data is called a *self-dual structure* on Q.

In this case, the abelian category $\mathsf{Mod}(KQ)$ admits a self-dual structure in the sense of §2.1.2, defined as follows. For a representation $E = \big((E_i)_{i \in Q_0}, (e_a)_{a \in Q_1}\big)$, define the *dual representation* $E^\vee$ by assigning the vector space $(E_{i^\vee})^\vee$ to the vertex i, and the linear map $v(a) \cdot (e_{a^\vee})^\vee$ to the edge a. Then, identify $E^{\vee\vee}$ with E using the sign $u(i)$ at each vertex i.

As in §2.1.2, we have the groupoid $\mathsf{Mod}(KQ)^{\mathrm{sd}}$ of *self-dual representations* of Q.

4.1.4. Example. Consider the quiver

$$Q = \left(\begin{tikzcd}[sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd} \right),$$

with the involution $(-)^\vee \colon Q \xrightarrow{\sim} Q^{\mathrm{op}}$ given by horizontal flipping. Then the self-dual structure on $\mathsf{Mod}(KQ)$ is given by

$$\begin{tikzcd}[sep=small] & E_2 \arrow[dr, "e_{24}"] & & & & E_2^\vee \arrow[dr, "v_{12} \cdot e_{12}^\vee"] & \\ E_1 \arrow[ur, "e_{12}"] \arrow[dr, "e_{13}"'] & & E_4 & \overset{(-)^\vee}{\leftrightsquigarrow} & E_4^\vee \arrow[ur, "v_{24} \cdot e_{24}^\vee"] \arrow[dr, "v_{13} \cdot e_{34}^\vee"'] & & E_1^\vee \\ & E_3 \arrow[ur, "e_{34}"'] & & & & E_3^\vee \arrow[ur, "v_{13} \cdot e_{13}^\vee"'] & \end{tikzcd},$$

where $v_{12} \in \{\pm 1\}$ is the value of v on the edge e_{12}, etc., and we identify $E_i \simeq E_i^{\vee\vee}$ using the sign $u(i)$.

In particular, a self-dual representation of Q must have $E_4 \simeq E_4^\vee$, and the isomorphisms $E_2 \simeq E_2^\vee$ and $E_3 \simeq E_3^\vee$ define either orthogonal or symplectic structures on the vector spaces E_2 and E_3, depending on the signs u_2 and u_3.

4.1.5. Moduli stacks. Recall that for a quiver Q, the moduli stack $\mathcal{X}_Q$ of representations of Q

over K is given by the disjoint union of quotient stacks

$$\mathcal{X}_Q = \coprod_{\alpha \in \mathbb{N}^{Q_0}} V_\alpha / G_\alpha \ , \tag{4.1.5.1}$$

where $V_\alpha = \prod_{a \in Q_1} \mathrm{Hom}(K^{\alpha_{s(a)}}, K^{\alpha_{t(a)}})$, and $G_\alpha = \prod_{i \in Q_0} \mathrm{GL}(\alpha_i)$. Each factor $\mathrm{Hom}(K^{\alpha_i}, K^{\alpha_j})$ is acted on by $\mathrm{GL}(\alpha_i)$ via right multiplication, and by $\mathrm{GL}(\alpha_j)$ via left multiplication of the inverse matrix.

If Q is equipped with a self-dual structure, the self-dual structure on $\mathsf{Mod}(KQ)$ extends to an involution of $\mathcal{X}_Q$, establishing it as a self-dual linear stack. The fixed locus $\mathcal{X}_Q^{\mathrm{sd}}$ of the involution is the moduli stack of self-dual representations of Q. Explicitly, we have

$$\mathcal{X}_Q^{\mathrm{sd}} \simeq \coprod_{\theta \in (\mathbb{N}^{Q_0})^{\mathrm{sd}}} V_\theta^{\mathrm{sd}} / G_\theta^{\mathrm{sd}} \ , \tag{4.1.5.2}$$

where $(\mathbb{N}^{Q_0})^{\mathrm{sd}} \subset \mathbb{N}^{Q_0}$ is the subset of dimension vectors θ such that $\theta_i = \theta_{i^\vee}$ for all $i \in Q_0$ and θ_i is even if $i = i^\vee$ and $u(i) = -1$. The vector space V_θ^{sd} and the group G_θ^{sd} are given by

$$V_\theta^{\mathrm{sd}} = \prod_{a \in Q_1^\circ / \mathbb{Z}_2} \mathrm{Hom}(K^{\theta_{s(a)}}, K^{\theta_{t(a)}}) \times \prod_{a \in Q_1^+} \mathrm{Sym}^2(K^{\theta_{t(a)}}) \times \prod_{a \in Q_1^-} \wedge^2(K^{\theta_{t(a)}}) \ , \tag{4.1.5.3}$$

$$G_\theta^{\mathrm{sd}} = \prod_{i \in Q_0^\circ / \mathbb{Z}_2} \mathrm{GL}(\theta_i) \times \prod_{i \in Q_0^+} \mathrm{O}(\theta_i) \times \prod_{i \in Q_0^-} \mathrm{Sp}(\theta_i) \ , \tag{4.1.5.4}$$

where Q_0° is the set of vertices i with $i \neq i^\vee$, and $Q_0^\pm$ the sets of vertices i with $i = i^\vee$ and $u(i) = \pm 1$. Similarly, Q_1° is the set of edges a with $a \neq a^\vee$, and $Q_1^\pm$ the sets of edges a with $a = a^\vee$ and $v(a)\, u(t(a)) = \pm 1$.

4.1.6. Stability conditions. We now describe a commonly used class of stability conditions for quiver representations, called *slope stability*, introduced by King [96] and discussed in Rudakov [141, §3].

A *slope function* on a quiver Q is a map $\mu \colon Q_0 \to \mathbb{Q}$. Given such a map, the *slope* of a dimension vector $\alpha \in \mathbb{N}^{Q_0} \setminus \{0\}$ is the number

$$\tau(\alpha) = \frac{\sum_{i \in Q_0} \alpha_i \, \mu(i)}{\sum_{i \in Q_0} \alpha_i} \ .$$

This defines a stability condition on the linear stack $\mathcal{X}_Q$ in the sense of §3.5.4, where the Θ-stratification exists by Ibáñez Núñez [77, Theorem 2.6.3].

If Q is equipped with a self-dual structure, then a slope function μ is said to be *self-dual* if

$\mu(i^\vee) = -\mu(i)$ for all $i \in Q_0$. In this case, the corresponding stability condition on $\mathsf{Mod}(KQ)$ is self-dual, and the corresponding stability condition on $\mathcal{X}_Q$ is also self-dual.

4.1.7. Potentials. The above discussion also generalizes to *quivers with potentials*, considered by Joyce and Song [89] and Kontsevich and Soibelman [99] in the context of Donaldson–Thomas theory, which serve as a local model for the enumerative theory of coherent sheaves on Calabi–Yau threefolds.

We now assume that the base field K is of characteristic zero, as we are going to work with shifted symplectic structures, which are only well-understood in characteristic zero.

For a quiver Q, a *potential* on Q is an element $W \in KQ/[KQ, KQ]$, where $[KQ, KQ] \subset KQ$ is the K-linear subspace spanned by commutators. Such an element can be equivalently described as a formal linear combination of cyclic paths in Q, and there is a trace function $\varphi_W = \mathrm{tr}(W) \colon \mathcal{X}_Q \to \mathbb{A}^1$ defined by taking traces along cyclic paths in a representation. The derived critical locus

$$\mathcal{X}_{Q,W} = {}^{\mathrm{d}}\mathrm{Crit}(\varphi_W) \subset \mathcal{X}_Q \,,$$

as in Example 3.6.9, admits a natural (-1)-shifted symplectic structure, and is a (-1)-shifted symplectic linear stack, equipped with a canonical orientation data.

When Q is equipped with a self-dual structure, the potential W is said to be *self-dual* if it is invariant under the involution of KQ sending a path to its dual path, multiplied by the product of the signs assigned to the edges in the path. In this case, the function φ_W is $\mathbb{Z}_2$-invariant, so $\mathcal{X}_{Q,W}$ is a self-dual linear stack, and the fixed locus $\mathcal{X}_{Q,W}^{\mathrm{sd}}$ admits a natural (-1)-shifted symplectic derived structure and a canonical self-dual orientation data.

When the potential W is zero, $\mathcal{X}_{Q,0} \simeq \mathrm{T}^*[-1]\,\mathcal{X}_Q$ is the (-1)-shifted cotangent stack of the smooth stack $\mathcal{X}_Q$ defined in §3.6.5, and its classical truncation coincides with $\mathcal{X}_Q$.

Any slope function τ on Q defines a stability condition on $\mathcal{X}_{Q,W}$, where the existence of a Θ-stratification follows from Ibáñez Núñez [77, Theorem 2.6.3]. For a self-dual potential W on a self-dual quiver Q, a self-dual slope function τ on Q defines a self-dual stability condition on $\mathcal{X}_{Q,W}$.

4.2 Sheaves on varieties

4.2.1. We now discuss how to apply our framework of orthosymplectic enumerative geometry to coherent sheaves on varieties, with the goal of defining enumerative invariants counting orthosymplectic coherent sheaves, with potential applications as outlined in §1.4.1.

As mentioned in §2.1.7, the category of coherent sheaves on a smooth projective variety is usually not self-dual. Therefore, we take an alternative approach by considering the derived category of coherent sheaves, which is self-dual, and then constructing self-dual subcategories using Bridgeland stability conditions. The self-dual objects, which we call *orthosymplectic complexes*, are then complexes of coherent sheaves which are quasi-isomorphic to their derived duals. We will see that such complexes are parametrized by a nice moduli stack, which can be seen as a variant of the moduli of principal G-bundles, where $G = \mathrm{O}(n)$ or $\mathrm{Sp}(2n)$.

We note that this approach of defining a coherent-sheaf-like version of principal bundles is different from the related construction of Gómez, Fernandez Herrero, and Zamora [64]. Our approach has the advantage that it interacts well with Bridgeland stability conditions, and satisfies wall-crossing formulae under a change of Bridgeland stability, which is an important feature in Donaldson–Thomas theory.

4.2.2. The derived category. Let Y be a connected, smooth, projective $\mathbb{C}$-variety of dimension n, and consider the derived category of coherent sheaves on Y, or equivalently (see, for example, [144, Tag 0FDC]), perfect complexes on Y, denoted by $\mathsf{Perf}(Y)$, which is a $\mathbb{C}$-linear dg-category. We construct self-dual structures on $\mathsf{Perf}(Y)$.

Fix the data (I, L, s, ε), where $I \colon Y \xrightarrow{\sim} Y$ is an involution, $L \to Y$ is a line bundle, $s \in \mathbb{Z}$, and $\varepsilon \colon L \xrightarrow{\sim} I^*(L)$ is an isomorphism such that $I^*(\varepsilon) \circ \varepsilon = \mathrm{id}_L$. Then there is a self-dual structure on $\mathsf{Perf}(Y)$ given by the dual functor

$$\mathbb{D} = \mathbb{R}\mathcal{H}om(I^*(-), L)[s] \colon \mathsf{Perf}(Y) \xrightarrow{\sim} \mathsf{Perf}(Y)^{\mathrm{op}} , \tag{4.2.2.1}$$

and identify $\mathbb{D}(\mathbb{D}(E))$ with E using the isomorphism ε, for all objects $E \in \mathsf{Perf}(Y)$.

4.2.3. The derived moduli stack. Consider the derived moduli stack

$$\mathcal{P}erf(Y) \simeq {}^{\mathrm{d}}\mathcal{M}ap(Y, \mathcal{P}erf) , \tag{4.2.3.1}$$

of perfect complexes on Y, constructed by Toën and Vaquié [157], where $\mathscr{P}\!erf$ on the right-hand side is the classifying stack of perfect complexes. The stack $\mathscr{P}\!erf(Y)$ is a derived algebraic stack locally of finite presentation over $\mathbb{C}$. By Pantev, Toën, Vaquié, and Vezzosi [135, Corollary 2.13], if Y is a *Calabi–Yau n-fold*, meaning that its canonical bundle K_Y is trivial, then $\mathscr{P}\!erf(Y)$ has a $(2-n)$-shifted symplectic structure.

Given the data (I, L, s, ε) in §4.2.2, the self-dual structure on $\mathsf{Perf}(Y)$ induces a $\mathbb{Z}_2$-action on $\mathscr{P}\!erf(Y)$, and the fixed locus $\mathscr{P}\!erf(Y)^{\mathrm{sd}}$ is the moduli stack of self-dual perfect complexes on Y. In particular, when $I = \mathrm{id}_Y$ and $\varepsilon = \pm\mathrm{id}_L$, the stack $\mathscr{P}\!erf(Y)^{\mathrm{sd}}$ parametrizes *$L[s]$-twisted orthogonal* or *symplectic complexes* on Y, respectively. When $L = \mathscr{O}_Y$ and $s = 0$, we simply call them *orthogonal* or *symplectic complexes*.

4.2.4. Next, we wish to construct abelian or quasi-abelian subcategories of $\mathsf{Perf}(Y)$ that are preserved by the dual functor $\mathbb{D}$, so they themselves become self-dual. Their moduli stacks of objects and self-dual objects will form open substacks of $\mathscr{P}\!erf(Y)$ and $\mathscr{P}\!erf(Y)^{\mathrm{sd}}$, respectively, and the latter substack can be seen roughly as a compactification of the stack of orthogonal or symplectic bundles on Y, analogously to how coherent sheaves are a compactification of vector bundles.

4.2.5. Bridgeland stability conditions. Consider the free abelian group

$$K(Y) = \{\mathrm{ch}(E) \mid E \in \mathsf{Perf}(Y)\} \subset \mathrm{H}^{2\bullet}(Y;\mathbb{Q})\,. \tag{4.2.5.1}$$

Define a *Bridgeland stability condition* on Y to be a Bridgeland stability condition on $\mathsf{Perf}(Y)$ in the sense of §2.4, where we use $K(Y)$ as the group Γ there. We have the spaces

$$\mathrm{Stab}(Y)\,, \qquad \mathrm{Stab}^{\mathrm{sd}}(Y)$$

of Bridgeland stability conditions and self-dual Bridgeland stability conditions on Y, which are shorthand notations for $\mathrm{Stab}_{K(Y)}(\mathsf{Perf}(Y))$ and $\mathrm{Stab}^{\mathrm{sd}}_{K(Y)}(\mathsf{Perf}(Y))$, respectively.

4.2.6. Permissibility. We discuss a technical condition on Bridgeland stability conditions which we call *permissibility*, mainly to ensure that moduli stacks have nice behaviours, especially for the purpose of wall-crossing. We follow ideas of Piyaratne and Toda [137].

Define subspaces of *permissible Bridgeland stability conditions*,

$$\mathrm{Stab}^{\circ}(Y) \subset \mathrm{Stab}(Y)\,, \qquad \mathrm{Stab}^{\circ,\mathrm{sd}}(Y) \subset \mathrm{Stab}^{\mathrm{sd}}(Y)$$

as maximal open subsets such that every element $\tau = (Z, \mathscr{P})$ with $Z(K(Y)) \subset \mathbb{Q}+\mathrm{i}\mathbb{Q}$ satisfies the following conditions:

(i) *Generic flatness.* See Abramovich and Polishchuk [1, Problem 3.5.1], Halpern-Leistner [67, Definition 6.2.4], or Piyaratne and Toda [137, Definition 4.4] for the precise formulation. Roughly, this condition says that if we consider the heart of $\mathsf{Perf}(Y)$ induced by $\mathscr{P}$, then for any smooth K-algebra R, any object in an induced heart of $\mathsf{Perf}(Y_R)$ is flat over a dense open subset of Spec R.

(ii) *Boundedness.* For any $t \in \mathbb{R}$ and $\alpha \in K(Y)$ with $Z(\alpha) \in \mathbb{R}_{\geqslant 0} \cdot \mathrm{e}^{\pi \mathrm{i} t}$, there is a quasi-compact open substack $\mathcal{X}(\tau; t)_\alpha \subset \mathscr{P}\!\mathit{erf}(Y)$ whose $\mathbb{C}$-points are the objects of $\mathscr{P}(t)$ of class α.

By Piyaratne and Toda [137, Proposition 4.12], if a stability condition τ satisfies these conditions and has *rational central charge*, meaning that the central charge is valued in $\mathbb{Q} + \mathrm{i}\mathbb{Q} \subset \mathbb{C}$, then a neighbourhood of τ lies in $\mathrm{Stab}^{\circ}(Y)$.

For $\tau \in \mathrm{Stab}^{\circ}(Y)$ and an interval $J \subset \mathbb{R}$ of length $|J| < 1$, there is an open substack

$$\mathcal{X}(\tau; J) \subset \mathscr{P}\!\mathit{erf}(Y) \tag{4.2.6.1}$$

whose $\mathbb{C}$-points are the objects of $\mathscr{P}(J)$, which we construct in §4.2.8 below. It is a derived linear stack in the sense of §3.7.2, and τ defines a permissible stability condition on its classical truncation in the sense of §3.5, where the Θ-stratification is constructed in §4.2.8 below.

In particular, if $\tau \in \mathrm{Stab}^{\circ,\mathrm{sd}}(Y)$ and $J = -J$, then $\mathcal{X}(\tau; J)$ is a self-dual derived linear stack, and the induced stability condition on $\mathcal{X}(\tau; J)$ is self-dual. The stack $\mathcal{X}(\tau; 0)^{\mathrm{sd}}$ is the moduli stack of *τ-semistable self-dual complexes*, which our orthosymplectic DT invariants will count.

4.2.7. Example. We now give examples of permissible self-dual Bridgeland stability conditions with rational central charge, for certain classes of Y, so that the spaces $\mathrm{Stab}^{\circ}(Y)$ and $\mathrm{Stab}^{\circ,\mathrm{sd}}(Y)$ are non-empty in these cases.

In the setting of §4.2.2, let Y be either a curve, a surface, or a threefold satisfying the conjectural Bogomolov–Gieseker inequality of Bayer, Macrì, and Toda [9, Conjecture 3.2.7]. Fix the data (I, L, s, ε) as in §4.2.2. Let $\omega \in \mathrm{H}^{1,1}(Y; \mathbb{Q})$ be an ample class with $I^*(\omega) = \omega$.

Let $\beta = c_1(L)/2 \in \mathrm{H}^2(Y; \mathbb{Q})$. Consider the map $Z_\omega \colon K(Y) \to \mathbb{C}$ given by

$$Z_\omega(\alpha) = \mathrm{i}^{n-s} \cdot \int_Y \exp(-\beta - \mathrm{i}\omega) \cdot \alpha \tag{4.2.7.1}$$

for $\alpha \in K(Y)$, where $n = \dim Y$. We use the notation Z_ω because only ω can be varied if (I, L, s, ε) is fixed. This central charge is compatible with the self-dual structure, because for all $\alpha \in K(Y)$, we have

$$\begin{aligned} Z_\omega(\mathbb{D}(\alpha)) &= \mathrm{i}^{n-s} \cdot \int_Y \exp(-\beta - \mathrm{i}\omega) \cdot I^*(\alpha^\vee) \cdot (-1)^s \exp(2\beta) \\ &= \mathrm{i}^{n+s} \cdot \int_Y \exp(\beta - \mathrm{i}\omega) \cdot \alpha^\vee \\ &= \mathrm{i}^{n+s} \cdot (-1)^n \cdot \int_Y \exp(-\beta + \mathrm{i}\omega) \cdot \alpha = \overline{Z_\omega(\alpha)} \,, \end{aligned}$$

where the second step uses that $I^*(\beta) = \beta$ and $I^*(\omega) = \omega$, and the third step uses that the cohomological degree $2i$ parts of α and $\alpha^\vee$ satisfy $(\alpha^\vee)_i = (-1)^i \alpha_i$ for $i = 0, \ldots, n$.

There is a Bridgeland stability condition $\tau_\omega = (Z_\omega, \mathscr{P}_\omega) \in \mathrm{Stab}^\circ(Y)$ with central charge Z_ω, by the works of Toda [153] and Piyaratne and Toda [137]. See also the earlier works of Bridgeland [23] and Arcara and Bertram [6] in the case of surfaces.

In fact, we can also choose $\mathscr{P}_\omega$ so that τ_ω is self-dual, or equivalently, the slicing $\mathscr{P}_\omega$ coincides with its dual slicing $\mathscr{P}_\omega^\vee$ given by

$$\mathscr{P}_\omega^\vee(t) = \mathbb{D}(\mathscr{P}_\omega(-t)) \,.$$

This can be deduced from Bayer, Macrì, and Toda [9, Remark 4.4.3], where we choose our $\mathscr{P}_\omega$ to be their $\hat{\mathscr{P}}_{\mathscr{B}}^{\omega,\beta}$ with a phase shift of $(1-s)/2$.

4.2.8. Construction of the moduli stack. We now explain the detailed construction of the open substack $\mathscr{X}(\tau; J) \subset \mathscr{P}\!\mathit{erf}(Y)$ and its Θ-stratification mentioned in §4.2.6, where $\tau \in \mathrm{Stab}^\circ(Y)$, and $J \subset \mathbb{R}$ is an interval with $|J| < 1$,

Applying Piyaratne and Toda [137, Proposition 4.12], we may apply a phase shift and assume that $J \subset \left]\varepsilon, 1-\varepsilon\right[$ for some $\varepsilon > 0$. Fix $\alpha \in K(Y)$ of slope within J, and then choose a perturbation $\tau' = (Z', \mathscr{P}')$ of τ satisfying the above properties, with $d(\tau', \tau) < \varepsilon$ and $Z'(K(Y)) \subset \mathbb{Q} + \mathrm{i}\mathbb{Q}$. Then if $\beta \in K(Y)$ is the class of a τ-Harder–Narasimhan factor of an object of $\mathscr{P}(J)$ of class α, then $Z'(\beta)$ must lie in the bounded region

$$\{ r e^{\pi \mathrm{i} t} \mid r \geqslant 0, \ t \in J_\varepsilon \} \cap \{ Z'(\alpha) - r e^{\pi \mathrm{i} t} \mid r \geqslant 0, \ t \in J_\varepsilon \} \subset \mathbb{C} ,$$

where J_ε is the ε-neighbourhood of J, so the set B of such classes β is finite. We then choose ε small enough, possibly changing τ', so that for any $\beta, \beta' \in B$, $\arg Z(\beta) < \arg Z(\beta')$ implies $\arg Z'(\beta) < \arg Z'(\beta')$, where we take phases within J_ε. Now, Halpern-Leistner [67, Theorem 6.5.3] gives the open substack $\mathscr{X}(\tau';]0,1[)$ with a Θ-stratification by τ'-Harder–Narasimhan types. The part of $\mathscr{X}(\tau; J)$ lying in $\mathscr{X}_\alpha$ can be defined as a finite open union of strata.

To construct the Θ-stratification on $\mathscr{X}(\tau; J)$, we follow the proof of [67, Theorem 6.5.3], with the following modifications. Instead of using rational weights for Harder–Narasimhan filtrations, we use *real-weighted filtrations* in the sense of [31, §§7.2–7.3]. As a result, we obtain real-weighted Θ-stratifications, which non-canonically give usual Θ-stratifications by [31, Proposition 7.2.12]. The key ingredients of the proof in [67] are the conditions (R), (S), and (B) there. The rationality condition (R) is no longer needed as we use real weights. The condition (S) needs to be modified to incorporate real weights, but the argument still works to prove it. The condition (B) follows from the quasi-compactness of $\mathscr{X}(\tau; J)$.

This also shows that any $\tau \in \mathrm{Stab}^\circ(Y)$ satisfies the support property and the boundedness property in §4.2.6, where for the support property, fixing $r > 0$ and choosing τ' rational with $d(\tau', \tau) < \varepsilon$ with $\varepsilon < 1/2$, for any class α with $|Z(\alpha)| \leqslant r$ admitting a τ-semistable object E, by considering the τ'-Harder–Narasimhan filtration of E, we see that α is a finite sum of classes β with $|Z'(\beta)| < r e^\varepsilon$ admitting τ'-semistable objects, and these classes lie on the same side of a line in $\mathbb{C}$, so there are only finitely many choices.

4.3 Higgs sheaves on varieties

4.3.1. We apply orthosymplectic enumerative geometry to *Higgs sheaves* on varieties.

A *Higgs sheaf* on a smooth, projective variety Y is a coherent sheaf on Y equipped with a

Higgs field, and can be identified with a compactly supported coherent sheaf on the total space of the canonical line bundle K_Y of Y.

When Y is a curve, there is also the notion of *G-Higgs bundles* on Y for a reductive group G, due to Hitchin [75], which are principal G-bundles with a Higgs field. Here, we would like to study such G-Higgs bundles for $G = \mathrm{O}(n)$ or $\mathrm{Sp}(2n)$.

Moreover, using the approach of orthosymplectic complexes developed in §4.2, in the case when Y is a higher-dimensional variety, it is possible to define a version of G-Higgs sheaves for $G = \mathrm{O}(n)$ or $\mathrm{Sp}(2n)$, which we call *orthosymplectic Higgs complexes* on Y. They can be described as complexes of Higgs sheaves equipped with self-dual structures, and they admit a well-behaved moduli stack which we can use to define enumerative invariants. For example, in the case when Y is a surface, in §8.4, we will define and study a version of *orthosymplectic Vafa–Witten invariants* counting orthosymplectic Higgs complexes on Y.

4.3.2. Higgs complexes. Let Y be a connected, smooth, projective $\mathbb{C}$-variety, and fix the data (I, L, s, ε) as in §4.2.2 defining a self-dual structure $\mathbb{D}$ on $\mathsf{Perf}(Y)$.

For an object $E \in \mathsf{Perf}(Y)$, a *Higgs field* on E is a morphism

$$\psi \colon E \longrightarrow E \otimes K_Y$$

in $\mathsf{Perf}(Y)$. We call such a pair (E, ψ) a *Higgs complex* on Y.

A *self-dual Higgs complex* is then defined as a fixed point of the involution

$$(E, \psi) \longmapsto (\mathbb{D}(E), -\mathbb{D}(\psi) \otimes K_Y)$$

on the ∞-groupoid of Higgs complexes, where $\mathbb{D}(\psi) \colon \mathbb{D}(E) \otimes K_Y^{-1} \to \mathbb{D}(E)$.

More concretely, for a self-dual object $(E, \phi) \in \mathsf{Perf}(Y)^{\mathrm{sd}}$ with $\mathrm{Ext}^i(E, E \otimes K_Y) = 0$ for all $i < 0$, where $\phi \colon E \xrightarrow{\sim} \mathbb{D}(E)$, a self-dual Higgs field on (E, ϕ) is the same data as a Higgs field $\psi \colon E \to E \otimes K_Y$ such that $(\phi \otimes K_Y) \circ \psi = -(\mathbb{D}(\psi) \otimes K_Y) \circ \phi$ as morphisms $E \to \mathbb{D}(E) \otimes K_Y$.

4.3.3. Moduli stacks. Let $\mathscr{Perf}(Y)$ be the derived moduli stack of perfect complexes on Y, and let $n = \dim Y$. Let

$$\mathscr{H}iggs(Y) = \mathrm{T}^*[1-n]\, \mathscr{Perf}(Y) \tag{4.3.3.1}$$

be the $(1-n)$-shifted cotangent stack of $\mathscr{Perf}(Y)$, equipped with the canonical $(1-n)$-shifted symplectic structure as in Calaque [36, Theorem 2.4].

The stack $\mathscr{H}iggs(Y)$ is a derived moduli stack of Higgs complexes on Y, since at a $\mathbb{C}$-point $E \in \mathscr{P}erf(Y)(\mathbb{C})$, we have

$$\mathbb{L}_{\mathscr{P}erf(Y)}[1-n]|_E \simeq \mathbb{R}\mathrm{Hom}_Y(E,E)^\vee[-n] \simeq \mathbb{R}\mathrm{Hom}_Y(E, E \otimes K_Y)\,,$$

parametrizing Higgs fields on E.

The self-dual structure on $\mathsf{Perf}(Y)$ determines a $\mathbb{Z}_2$-action on $\mathscr{P}erf(Y)$, which induces a $\mathbb{Z}_2$-action on $\mathscr{H}iggs(Y)$. We have $\mathscr{H}iggs(Y)^{\mathrm{sd}} \simeq \mathrm{T}^*[1-n]\,\mathscr{P}erf(Y)^{\mathrm{sd}}$, giving $\mathscr{H}iggs(Y)^{\mathrm{sd}}$ a canonical $(1-n)$-shifted symplectic structure.

We regard $\mathscr{H}iggs(Y)^{\mathrm{sd}}$ as a moduli stack of self-dual Higgs complexes on Y. This description agrees with the definition of a self-dual Higgs field, as the $(1-n)$-shifted tangent map of the involution $\mathbb{D}$, as a map $\mathbb{R}\mathrm{Hom}(E,E) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}(\mathbb{D}(E), \mathbb{D}(E))$, is given by $\psi \mapsto -\mathbb{D}(\psi)$.

4.3.4. Stability conditions. We now restrict to the case when the anti-canonical bundle K_Y^{-1} of Y is either ample or trivial. This condition is often also referred to as Y being either *Fano* or *Calabi–Yau*. We abbreviate this condition as $K_Y \leqslant 0$.

In this case, for any $\tau \in \mathrm{Stab}^\circ(Y)$ and any $E \in \mathsf{Perf}(Y)$, every Higgs field $\psi\colon E \to E \otimes K_Y$ respects the τ-Harder–Narasimhan filtration of E, since choosing a non-zero map $\xi\colon K_Y \to \mathscr{O}_Y$, the composition $\xi \circ \psi\colon E \to E \otimes K_Y \to E$ must preserve the Harder–Narasimhan filtration. Therefore, heuristically, a Higgs complex (E, ψ) is τ-semistable if and only if E is τ-semistable. This justifies the following series of definitions:

Let $\tau \in \mathrm{Stab}^\circ(Y)$, and let $J \subset \mathbb{R}$ be an interval of length $|J| < 1$. Let $\mathscr{X}(\tau; J) \subset \mathscr{P}erf(Y)$ be the open substack as in §4.2.6, and define

$$\mathscr{H}(\tau; J) = \mathrm{T}^*[1-n]\,\mathscr{X}(\tau; J) \subset \mathscr{H}iggs(Y) \tag{4.3.4.1}$$

be the corresponding open substack. The stacks $\mathscr{X}(\tau; J)$ and $\mathscr{H}(\tau; J)$ are derived linear stacks. When $J = -J$, they also admit self-dual structures, and $\mathscr{H}(\tau; J)^{\mathrm{sd}} \simeq \mathrm{T}^*[1-n]\,\mathscr{X}(\tau; J)^{\mathrm{sd}}$.

Moreover, τ defines permissible stability conditions on $\mathscr{X}(\tau; J)$ and $\mathscr{H}(\tau; J)$, in the sense of §3.5.6. Here, the Θ-stratification on $\mathscr{H}(\tau; J)$ can be obtained by following the proof of Halpern-Leistner [67, Theorem 6.5.3], similarly to §4.2.8, where the conditions (S) and (B) follow from the respective properties of $\mathscr{X}(\tau; J)$.

Chapter 5

Donaldson–Thomas invariants

This chapter presents a main construction of this thesis, that of *orthosymplectic Donaldson–Thomas invariants*, which are an orthosymplectic analogue of the theory of Donaldson–Thomas invariants in the linear case developed by Donaldson and Thomas [55], Thomas [151], Behrend [10], Joyce [81–85], Joyce and Song [89], and Kontsevich and Soibelman [99]. We explain in §5.1 some of the basic ideas behind this construction in the linear case.

One of the key technical constructions in this thesis is the generalization of *epsilon motives* from the linear case to the orthosymplectic case, which we present in §5.5. These motives satisfy an important property, the *no-pole theorem*, Theorem 5.5.5, which allows us to define Donaldson–Thomas invariants in the orthosymplectic case in §5.6.

5.1 Idea

5.1.1. We start by informally discussing some basic ideas in the usual theory of Donaldson–Thomas invariants, to motivate some of our technical constructions later on.

Consider a smooth projective Calabi–Yau threefold Y over $\mathbb{C}$, and let $\mathcal{X}$ be the moduli stack of coherent sheaves on Y. Then $\mathcal{X}$ is a (-1)-shifted symplectic linear stack in the sense of §3.7.3.

Given a stability condition τ on $\mathcal{X}$, then for each class $\alpha \in \pi_0(\mathcal{X})$, there is an open substack $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau) \subset \mathcal{X}_\alpha$ consisting of τ-semistable sheaves. When $\alpha \neq 0$, we may form the $\mathbb{G}_\mathrm{m}$*-rigidification* $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)/\mathrm{B}\mathbb{G}_\mathrm{m}$, which is a modification of $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)$ where stabilizer groups are replaced by their quotients by $\mathbb{G}_\mathrm{m}$, which corresponds to scalar automorphisms of the sheaves.

If all τ-semistable sheaves of class α are *τ-stable*, which implies in particular that they only have scalar automorphisms, then the rigidification $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)/\mathrm{B}\mathbb{G}_\mathrm{m}$ is usually a proper scheme, and admits a (-1)-shifted symplectic structure. In this case, the *Donaldson–Thomas invariant* $\mathrm{DT}_\alpha(\tau) \in \mathbb{Z}$ is defined as the *virtual fundamental class* of $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)/\mathrm{B}\mathbb{G}_\mathrm{m}$ in the sense of Behrend and Fantechi [12], which is a number as the space has virtual dimension zero.

5.1.2. It was observed by Behrend [10, Theorem 4.18] that in this 'stable = semistable' case, the invariant $\mathrm{DT}_\alpha(\tau)$ can be written as a *weighted Euler characteristic* of the space $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)/\mathrm{B}\mathbb{G}_\mathrm{m}$, weighted by a constructible function $\nu\colon \mathcal{X}_\alpha^{\mathrm{ss}}(\tau)/\mathrm{B}\mathbb{G}_\mathrm{m} \to \mathbb{Z}$, now called the *Behrend function*. This fact may be written as an integral

$$\mathrm{DT}_\alpha(\tau) = \int_{\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)/\mathrm{B}\mathbb{G}_\mathrm{m}} \nu \, d\chi \, ,$$

defined as $\sum_{c \in \mathbb{Z}} c \cdot \chi(\nu^{-1}(c))$, where $\chi(-)$ denotes the usual Euler characteristic.

5.1.3. Epsilon motives. The aforementioned motivic approach to Donaldson–Thomas theory allowed one to also define Donaldson–Thomas invariants for classes where not all τ-semistable sheaves are τ-stable, as was done by Joyce and Song [89] based on earlier works of Joyce [81–85], and by Kontsevich and Soibelman [99]. In this case, the above relation is replaced by an integral of the form

$$\mathrm{DT}_\alpha(\tau) = \int_{\mathcal{X}} (1 - \mathbb{L}) \cdot \epsilon_\alpha(\tau) \cdot \nu_{\mathcal{X}} \, d\chi \, , \tag{5.1.3.1}$$

where $\epsilon_\alpha(\tau)$ is the *epsilon motive*, which is a modification of the motive $[\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)]$ that makes the integral well-defined (see also §1.2.6), and $\nu_{\mathcal{X}}$ is the Behrend function of $\mathcal{X}$. The integral is in the sense of §5.2.9 below. The factor $1 - \mathbb{L}$ accounts for the fact that we are now integrating over the non-rigidified moduli stack, where $\mathbb{L} - 1$ is the motive of $\mathbb{G}_\mathrm{m}$, and the sign difference comes from the fact that $p^*(\nu_{\mathcal{X}/\mathrm{B}\mathbb{G}_\mathrm{m}}) = -\nu_{\mathcal{X}}$, where $p\colon \mathcal{X} \to \mathcal{X}/\mathrm{B}\mathbb{G}_\mathrm{m}$ is the projection.

Note that although we have the Euler characteristic $\chi(1 - \mathbb{L}) = 0$, the above integral can still be non-zero, since $\epsilon_\alpha(\tau)$ has a built-in factor of $(\mathbb{L} - 1)^{-1}$.

5.1.4. Orthosymplectic Donaldson–Thomas invariants. In the orthosymplectic setting, as explained in §4.2, we no longer use the moduli of coherent sheaves, but we use an alternative abelian subcategory $\mathscr{P}(0) \subset \mathrm{D}^{\mathrm{b}}\mathsf{Coh}(Y)$ coming from a Bridgeland stability condition, so that $\mathscr{P}(0)$ is self-dual, and its derived moduli stack of objects $\mathcal{X}$ is a self-dual (-1)-shifted symplectic linear stack.

In fact, we do not need to restrict ourselves to this concrete example, as the construction works for general self-dual (-1)-shifted symplectic linear stacks $\mathcal{X}$.

For a class $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, we would like to define orthosymplectic Donaldson–Thomas invariants

$$\mathrm{DT}^{\mathrm{sd}}_{\theta}(\tau) = \int_{\mathcal{X}^{\mathrm{sd}}} \epsilon^{\mathrm{sd}}_{\theta}(\tau) \cdot \nu_{\mathcal{X}^{\mathrm{sd}}} \, d\chi \, , \tag{5.1.4.1}$$

analogously to (5.1.3.1), where we no longer need the factor $1 - \mathbb{L}$, since points in $\mathcal{X}^{\mathrm{sd}}$ no longer necessarily have a copy of $\mathbb{G}_{\mathrm{m}}$ in their stabilizer groups.

The only remaining difficulty is to define the epsilon motives $\epsilon^{\mathrm{sd}}_{\theta}(\tau)$ in the orthosymplectic case, so that the integral (5.1.4.1) is well-defined and finite. This will be the main focus of this chapter, and the *no-pole theorem*, Theorem 5.5.5, guarantees this desired property.

5.2 Rings of motives

5.2.1. This section provides background material on *rings of motives* over an algebraic stack. Roughly speaking, these are rings generated by classes $[\mathcal{Z}]$ of algebraic stacks $\mathcal{Z}$ defined over $\mathcal{X}$, up to the cut-and-paste relations

$$[\mathcal{Z}] = [\mathcal{Z}'] + [\mathcal{Z} \setminus \mathcal{Z}'] \tag{5.2.1.1}$$

for closed substacks $\mathcal{Z}' \subset \mathcal{Z}$. The class $[\mathcal{Z}]$ is called the *motive* of $\mathcal{Z}$. In the context of stacks, these rings were originally considered by Joyce [84] under the name of *stack functions.*

For technical reasons, we will define multiple versions of rings of motives, which can be roughly arranged into the diagram below:

$$\begin{tikzcd}
\mathbb{M}(\mathcal{X}) \arrow[dr, phantom, "\cup" {rotate=90}] \arrow[dr] & \\
\mathbb{M}^{\mathrm{s}}(\mathcal{X}) \arrow[u, phantom, "\cup" {rotate=90}] \arrow[r] \arrow[dr] & \hat{\mathbb{M}}(\mathcal{X}) \\
& \hat{\mathbb{M}}^{\circ}(\mathcal{X}) \arrow[u, phantom, "\cup" {rotate=90}] \arrow[r, "\chi"] & \mathrm{CF}(\mathcal{X}) \, .
\end{tikzcd} \tag{5.2.1.2}$$

Here, $\mathbb{M}(\mathcal{X})$ is the default version, $\mathbb{M}^{\mathrm{s}}(\mathcal{X})$ is the *schematic* version, $\widehat{\mathbb{M}}(\mathcal{X})$ is the *completed* version, and $\widehat{\mathbb{M}}^{\circ}(\mathcal{X})$ is the subspace of *regular* motives. The middle horizontal map is a localization, the diagonal map is like a retract of the inclusion $\mathbb{M}^{\mathrm{s}}(\mathcal{X}) \hookrightarrow \mathbb{M}(\mathcal{X})$ up to this localization, and χ is the Euler characteristic map, valued in constructible functions on $\mathcal{X}$. As mentioned in §1.2.6, one cannot expect to define the Euler characteristic for all stacks, so it is not defined on the full space $\widehat{\mathbb{M}}(\mathcal{X})$.

Recall from §3.2.2 our running assumptions on algebraic stacks, which we will assume all stacks in this section to satisfy.

5.2.2. The ring of motives. Let K be a field, let $\mathcal{X}$ be an algebraic stack over K satisfying assumptions in §3.2.2, and let A be a commutative ring.

The *ring of motives* over $\mathcal{X}$ with coefficients in A is the A-module

$$\mathbb{M}(\mathcal{X}; A) = \widehat{\bigoplus_{\mathcal{Z} \to \mathcal{X}}} A \cdot [\mathcal{Z}] \Big/ \sim \, , \tag{5.2.2.1}$$

where we run through isomorphism classes of representable morphisms $\mathcal{Z} \to \mathcal{X}$ of finite type, with $\mathcal{Z}$ quasi-compact, and $\hat{\oplus}$ indicates that we take the set of *locally finite sums*, that is, possibly infinite sums $\sum_{\mathcal{Z} \to \mathcal{X}} a_{\mathcal{Z}} \cdot [\mathcal{Z}]$, such that for each quasi-compact open substack $\mathcal{U} \subset \mathcal{X}$, there are only finitely many $\mathcal{Z}$ such that $a_{\mathcal{Z}} \neq 0$ and $\mathcal{Z} \times_{\mathcal{X}} \mathcal{U} \neq \varnothing$. The relation $\sim$ is generated by locally finite sums of elements of the form

$$a \cdot ([\mathcal{Z}] - [\mathcal{Z}'] - [\mathcal{Z} \setminus \mathcal{Z}']) \, ,$$

where $a \in A$, $\mathcal{Z}$ is as above, and $\mathcal{Z}' \subset \mathcal{Z}$ is a closed substack. The class $[\mathcal{Z}] \in \mathbb{M}(\mathcal{X}; A)$ is called the *motive* of $\mathcal{Z}$.

In fact, $\mathbb{M}(\mathcal{X}; A)$ carries a natural topology which is the limit topology of the discrete topologies on $\mathbb{M}(\mathcal{U}; A)$ for quasi-compact open substacks $\mathcal{U} \subset \mathcal{X}$. The locally finite sums are precisely the sums that converge in this topology.

For a representable morphism $\mathcal{Z} \to \mathcal{X}$ of finite type, where $\mathcal{Z}$ is not necessarily quasi-compact, we can still define its motive $[\mathcal{Z}] \in \mathbb{M}(\mathcal{X}; A)$, by stratifying $\mathcal{Z}$ into quasi-compact locally closed substacks, $\mathcal{Z} = \bigcup_{i \in I} \mathcal{Z}_i$, and defining $[\mathcal{Z}] = \sum_{i \in I} [\mathcal{Z}_i]$ as a locally finite sum. It is easy to check that this does not depend on the choice of stratification, by passing to a common refinement of two given stratifications.

The ring structure on $\mathbb{M}(\mathcal{X}; A)$ is given by $[\mathcal{Z}] \cdot [\mathcal{Z}'] = [\mathcal{Z} \times_{\mathcal{X}} \mathcal{Z}']$ on generators, with unit element $[\mathcal{X}]$ defined using the procedure of the previous paragraph.

We also write $\mathbb{M}(\mathcal{X})$ for $\mathbb{M}(\mathcal{X}; \mathbb{Z})$, and $\mathbb{M}(K; A)$ for $\mathbb{M}(\mathrm{Spec}(K); A)$.

The ring $\mathbb{M}(\mathcal{X}; A)$ is naturally an $\mathbb{M}(K; A)$-algebra, with the action given by the product. In particular, it is an $A[\mathbb{L}]$-algebra, where

$$\mathbb{L} = [\mathbb{A}^1] \in \mathbb{M}(K; A) \tag{5.2.2.2}$$

is the class of the affine line over K.

5.2.3. Properties. We list some basic properties of rings of motives.

(i) For a morphism $f: \mathcal{Y} \to \mathcal{X}$, there is a *pullback map*

$$f^*: \mathbb{M}(\mathcal{X}; A) \longrightarrow \mathbb{M}(\mathcal{Y}; A) ,$$

given by $[\mathcal{Z}] \mapsto [\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}]$ on generators, which is a ring homomorphism.

(ii) For a representable quasi-compact morphism $f: \mathcal{Y} \to \mathcal{X}$, there is a *pushforward map*

$$f_!: \mathbb{M}(\mathcal{Y}; A) \longrightarrow \mathbb{M}(\mathcal{X}; A) ,$$

given by $[\mathcal{Z}] \mapsto [\mathcal{Z}]$ on generators. This is not a ring homomorphism in general.

(iii) For stacks $\mathcal{X}$ and $\mathcal{Y}$, there is an *external product*

$$\boxtimes: \mathbb{M}(\mathcal{X}; A) \otimes \mathbb{M}(\mathcal{Y}; A) \longrightarrow \mathbb{M}(\mathcal{X} \times \mathcal{Y}; A) ,$$

given by $[\mathcal{Z}] \otimes [\mathcal{Z}'] \mapsto [\mathcal{Z} \times \mathcal{Z}']$ on generators. The multiplication on $\mathbb{M}(\mathcal{X}; A)$ can be realized as the external product for $\mathcal{X} \times \mathcal{X}$ followed by pulling back along the diagonal $\Delta: \mathcal{X} \to \mathcal{X} \times \mathcal{X}$.

(iv) For a representable quasi-compact morphism $f: \mathcal{Y} \to \mathcal{X}$, we have the *projection formula*

$$f_!(a \cdot f^*(b)) = f_!(a) \cdot b \tag{5.2.3.1}$$

for $a \in \mathbb{M}(\mathcal{Y}; A)$ and $b \in \mathbb{M}(\mathcal{X}; A)$, which can be verified on generators.

(v) For a pullback diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X} \end{tikzcd},$$

where f is representable and quasi-compact, we have the *base change formula*

$$g^* \circ f_! = f'_! \circ (g')^* \colon \mathbb{M}(\mathcal{Y}; A) \longrightarrow \mathbb{M}(\mathcal{X}'; A) . \tag{5.2.3.2}$$

Again, this can be verified directly on generators.

5.2.4. Schematic motives. Let $\mathcal{X}$ and A be as above. Define the A-submodule

$$\mathbb{M}^{\mathrm{s}}(\mathcal{X}; A) \subset \mathbb{M}(\mathcal{X}; A)$$

of *schematic motives* to be the smallest closed submodule containing the motives $[Z]$ for morphisms $Z \to \mathcal{X}$ from a K-scheme Z of finite type. Here, being a closed submodule is equivalent to being closed under taking locally finite sums in the sense of §5.2.2.

For a morphism $Z \to \mathcal{X}$ of finite type from an algebraic space Z locally of finite type over K, we can still define its motive $[Z] \in \mathbb{M}^{\mathrm{s}}(\mathcal{X}; A)$ using a stratification, similarly to §5.2.2.

Note that we have $\mathbb{M}^{\mathrm{s}}(\mathcal{X}; A) = \mathbb{M}(\mathcal{X}; A)$ if and only if $\mathcal{X}$ is an algebraic space. Also, $\mathbb{M}^{\mathrm{s}}(\mathcal{X}; A)$ is closed under multiplication in $\mathbb{M}(\mathcal{X}; A)$, but it does not contain the unit $[\mathcal{X}]$ when $\mathcal{X}$ is not an algebraic space. We also write $\mathbb{M}^{\mathrm{s}}(\mathcal{X})$ for $\mathbb{M}^{\mathrm{s}}(\mathcal{X}; \mathbb{Z})$.

5.2.5. Completed motives. Consider the localization

$$\widehat{\mathbb{M}}(\mathcal{X}; A) = \mathbb{M}^{\mathrm{s}}(\mathcal{X}; A) \mathop{\hat{\otimes}}_{A[\mathbb{L}]} A[\mathbb{L}^{\pm 1}, (\mathbb{L}^k - 1)^{-1} : k > 0] , \tag{5.2.5.1}$$

where $\mathbb{L} = [\mathbb{A}^1]$ is the motive of the affine line, and $\hat{\otimes}$ denotes the completed tensor product with respect to locally finite sums. We call this the *completed ring of motives* over $\mathcal{X}$.

A key property of completed motives is that for a morphism $\mathcal{Z} \to \mathcal{X}$ of finite type, not necessarily representable, we have a class

$$[\mathcal{Z}] \in \widehat{\mathbb{M}}(\mathcal{X}; A) \tag{5.2.5.2}$$

defined as follows. By Kresch [102, Proposition 3.5.9], we may stratify $\mathcal{Z}$ into locally closed substacks of the form $(\mathscr{Z}_i = U_i/G_i)_{i\in I}$, with U_i a quasi-projective K-scheme acted on by a group $G_i = \mathrm{GL}(n_i)$ for some $n_i \in \mathbb{N}$. We then define

$$[\mathscr{Z}] \longmapsto \sum_{i\in I} [G_i]^{-1} \cdot [U_i] \,, \tag{5.2.5.3}$$

where $[G_i]^{-1} \in \mathbb{A}[\mathbb{L}^{-1}, (\mathbb{L}^k - 1)^{-1}]$. To see that this is well-defined, it is enough to check that this does not depend on the choice of stratification, and by choosing common refinements of two given stratifications, we are reduced to the following fact: If $U_1/G_1 \simeq U_2/G_2 \simeq \mathscr{U}$, then $[G_1] \cdot [U_2] = [G_2] \cdot [U_1]$, where $G_i = \mathrm{GL}(n_i)$ for $i = 1, 2$. To see this, set $U = U_1 \times_{\mathscr{U}} U_2$. Then $U \to U_2$ is a G_1-bundle, and since every principal $\mathrm{GL}(n_1)$-bundle is Zariski locally trivial, a further stratification of U_2 shows that $[U] = [G_1] \cdot [U_2]$, and similarly, $[U] = [G_2] \cdot [U_1]$.

Note that here, we have used a special property of the groups $\mathrm{GL}(n)$ that every principal bundle over a scheme is Zariski locally trivial. This property is called being a *special group* in Serre [143].

In particular, there is a natural map

$$\mathbb{M}(\mathscr{X}; A) \longrightarrow \widehat{\mathbb{M}}(\mathscr{X}; A) \,, \tag{5.2.5.4}$$

given on generators by $[\mathscr{Z}] \mapsto [\mathscr{Z}]$ defined as above.

5.2.6. Properties. We collect here some properties of the rings $\widehat{\mathbb{M}}(\mathscr{X}; A)$.

Firstly, all properties in §5.2.3 hold analogously for $\widehat{\mathbb{M}}(\mathscr{X}; A)$. Moreover, we are also allowed to push forward along quasi-compact but possibly non-representable morphisms, given on generators by $[\mathscr{Z}] \mapsto [\mathscr{Z}]$, where the second $[\mathscr{Z}]$ is the class defined in §5.2.5.

In particular, if $\mathscr{X}$ is quasi-compact, there is a *motivic integration map*

$$\int_{\mathscr{X}} (-) \colon \mathbb{M}(\mathscr{X}; A) \longrightarrow \widehat{\mathbb{M}}(K; A) \,,$$

defined as the map (5.2.5.4) followed by pushing forward along the possibly non-representable morphism $\mathscr{X} \to \mathrm{Spec}(K)$.

Another useful property is that for a class $[\mathscr{Z}] \in \widehat{\mathbb{M}}(\mathscr{X}; A)$ as in §5.2.5, a vector bundle $\mathscr{E} \to \mathscr{Z}$ of rank n, and a principal G-bundle $\mathscr{P} \to \mathscr{Z}$ for a *special group* G in the sense of

Serre [143], such as $G = \mathrm{GL}(n)$ for some $n \in \mathbb{N}$, we have the relations

$$[\mathscr{E}] = \mathbb{L}^n \cdot [\mathscr{Z}]\,, \tag{5.2.6.1}$$

$$[\mathscr{P}] = [G] \cdot [\mathscr{Z}]\,. \tag{5.2.6.2}$$

These can be verified using the fact that any vector bundle or principal G-bundle on a scheme is Zariski locally trivial, where the latter property is by the definition of special groups.

5.2.7. Constructible functions. For a stack $\mathcal{X}$ as above, and a commutative ring A, a *constructible function* on $\mathcal{X}$ is a map of sets

$$a \colon |\mathcal{X}| \longrightarrow A\,,$$

where $|\mathcal{X}|$ is the underlying topological space of $\mathcal{X}$ as in §3.1.4, such that for any $c \in A$, the preimage $a^{-1}(c)$ is a locally constructible subset of $|\mathcal{X}|$. The A-algebra of constructible functions on $\mathcal{X}$ is denoted by $\mathrm{CF}(\mathcal{X}; A)$.

5.2.8. Euler characteristics. Define the subspace

$$\hat{\mathbb{M}}^\circ(\mathcal{X}; A) \subset \hat{\mathbb{M}}(\mathcal{X}; A) \tag{5.2.8.1}$$

of *regular motives* as the image of the map

$$\mathbb{M}(\mathcal{X}; A) \mathbin{\hat{\otimes}_{A[\mathbb{L}]}} A[\mathbb{L}^{\pm 1}, (1 + \mathbb{L} + \cdots + \mathbb{L}^k)^{-1} : k > 0] \longrightarrow \hat{\mathbb{M}}(\mathcal{X}; A) \tag{5.2.8.2}$$

naturally extending the map (5.2.5.4).

Roughly speaking, this is the subspace of motives that 'have no poles at $\mathbb{L} = 1$', so that taking the Euler characteristic, which sets $\mathbb{L} = 1$, is a valid operation on this subspace.

When A contains $\mathbb{Q}$, define the *Euler characteristic map*

$$\chi \colon \hat{\mathbb{M}}^\circ(\mathcal{X}; A) \longrightarrow \mathrm{CF}(\mathcal{X}; A)\,,$$

as follows. For a generator $a = f(\mathbb{L}) \cdot [\mathscr{Z}] \in \hat{\mathbb{M}}^\circ(\mathcal{X}; A)$, where $[\mathscr{Z}]$ is a generator of $\mathbb{M}(\mathcal{X}; A)$ as in §5.2.2, and $f(\mathbb{L})$ is a rational function in $\mathbb{L}$ regular at $\mathbb{L} = 1$, define

$$\chi(a)(x) = f(1) \cdot \sum_{i \geqslant 0} (-1)^i \cdot \dim \mathrm{H}^i_{\mathrm{c}}(Z_{\bar{x}}; \mathbb{Q}_\ell)$$

to be the alternating sum of the dimensions of the compactly supported ℓ-adic cohomology groups of the base change of $\mathcal{X}$ to the geometric point $\bar{x}\colon \bar{K}_x \to \mathcal{X}$ of x, where $\bar{K}_x$ is the algebraic closure of the residue field K_x of x, and ℓ is a prime number different from $\operatorname{char}(K)$. This integer is independent of the choice of ℓ, as in Illusie [78, §1.1].

5.2.9. The virtual rank decomposition. Let $\mathcal{X}$ be a stack over K, and let A be a commutative $\mathbb{Q}$-algebra. As in Joyce [84, §5] and the author, Ibáñez Núñez, and Kinjo [32, §5.1], there is a *virtual rank decomposition*

$$\mathbb{M}(\mathcal{X}; A) = \widehat{\bigoplus_{k \geqslant 0}} \mathbb{M}^{(k)}(\mathcal{X}; A) ,$$

where $\hat{\oplus}$ means allowing locally finite sums as in §5.2.2, and each $\mathbb{M}^{(k)}(\mathcal{X}; A) \subset \mathbb{M}(\mathcal{X}; A)$ is the submodule of motives of *pure virtual rank* k.

Roughly speaking, having virtual rank $\leqslant k$ means having a pole of order at most k at $\mathbb{L} = 1$ after motivic integration (see §5.2.6).

Precisely, this decomposition is defined by the projection operators

$$\pi^{(k)}\colon \mathbb{M}(\mathcal{X}; A) \longrightarrow \mathbb{M}(\mathcal{X}; A) , \tag{5.2.9.1}$$

which are $\mathbb{M}(K; A)$-linear maps that are continuous (that is, preserving locally finite sums), characterized by the following properties:

(i) For any $a \in \mathbb{M}(\mathcal{X}; A)$, we have $a = \sum_{k \geqslant 0} \pi^{(k)}(a)$ as a locally finite sum.

(ii) For a generator $[U/G] \in \mathbb{M}(\mathcal{X}; A)$, where U is a quasi-projective K-scheme, acted on by a smooth affine algebraic group G over K with a split maximal torus, such as $G = \mathrm{GL}(n)$, equipped with a representable morphism $U/G \to \mathcal{X}$, we have

$$[U/G] = \sum_{k \geqslant 0;\ T \subset G} \frac{1}{|W_T|} \cdot \pi^{(k)}([U^T/L_T]) , \tag{5.2.9.2}$$

where we sum over conjugacy classes of split tori $T \simeq \mathbb{G}_{\mathrm{m}}^k \subset G$ of dimension k, $W_T = \mathrm{N}_G(T)/\mathrm{Z}_G(T)$ is the relative Weyl group, $U^T \subset U$ is the fixed locus, $L_T = \mathrm{Z}_G(T) \subset G$ is the Levi subgroup given by T, and the sum only has finitely many non-zero terms.

(iii) For a generator $[U/G] \in \mathbb{M}(\mathcal{X}; A)$ as above, if there exists a subtorus $\mathbb{G}_{\mathrm{m}}^k \subset Z(G)$ acting on U trivially, then $\pi^{(k')}([U/G]) = 0$ for all $0 \leqslant k' < k$.

The image of $\pi^{(k)}$ is then defined as $\mathbb{M}^{(k)}(\mathcal{X}; A)$. See [32, §5.1] for details.

Note that this definition uses almost no information about $\mathcal{X}$, and as a consequence, the decomposition §5.2.9 is compatible with pushforwards of motives.

The reader should be warned that this decomposition does not descend to $\hat{\mathbb{M}}(\mathcal{X}; A)$, since for example, when $\mathcal{X} = */\mathbb{G}_{\mathrm{m}}$, the motives $[\mathbb{G}_{\mathrm{m}} \times (*/\mathbb{G}_{\mathrm{m}})]$ (with the natural projection to $*/\mathbb{G}_{\mathrm{m}}$) and $[*]$ (with the unique morphism to $*/\mathbb{G}_{\mathrm{m}}$) get identified in $\hat{\mathbb{M}}(\mathcal{X}; A)$, whereas they have pure virtual ranks 1 and 0, respectively.

When $\mathcal{X}$ is quasi-compact, the motivic integration map satisfies the property that

$$\int_{\mathcal{X}} (-) \colon \mathbb{M}^{(\leqslant k)}(\mathcal{X}; A) \longrightarrow (\mathbb{L} - 1)^{-k} \cdot \hat{\mathbb{M}}^{\circ}(K; A) \subset \hat{\mathbb{M}}(K; A) \,, \tag{5.2.9.3}$$

that is, the image of the space $\mathbb{M}^{(\leqslant k)} = \mathbb{M}^{(0)} \oplus \cdots \oplus \mathbb{M}^{(k)}$ lies in the subspace $(\mathbb{L} - 1)^{-k} \cdot \hat{\mathbb{M}}^{\circ}(K; A)$. In particular, there is an Euler characteristic integration map

$$\int_{\mathcal{X}} (\mathbb{L} - 1)^{k} \cdot (-) \, d\chi = \chi \circ \int_{\mathcal{X}} (\mathbb{L} - 1)^{k} \cdot (-) \colon \; \mathbb{M}^{(\leqslant k)}(\mathcal{X}; A) \longrightarrow A \,. \tag{5.2.9.4}$$

5.3 Descent of motives

5.3.1. We now discuss descent properties of the rings of motives defined above. These rings do *not* satisfy étale descent, since for example, pulling back along the double cover $(-)^2 \colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ identifies the class of the trivial double cover $\mathbb{G}_{\mathrm{m}} \times \mu_2 \to \mathbb{G}_{\mathrm{m}}$ and the non-trivial double cover $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$.

However, we show in Theorem 5.3.3 below that the ring of motives $\hat{\mathbb{M}}(-)$ does satisfy descent under the Nisnevich topology.

5.3.2. The Nisnevich topology. Recall that for an algebraic space X, a *Nisnevich cover* of X is a family of étale morphisms $(f_i \colon X_i \to X)_{i \in I}$, such that for each point $x \in X$, there exists $i \in I$ and a point $x' \in X_i$, such that $f_i(x') = x$, and f_i induces an isomorphism on residue fields at x' and x.

Let $\mathcal{X}$ be an algebraic stack. Define a *Nisnevich cover* of $\mathcal{X}$ to be a representable étale cover $(f_i \colon \mathcal{X}_i \to \mathcal{X})_{i \in I}$ such that its base change to any algebraic space is a Nisnevich cover of algebraic spaces. See also Choudhury, Deshmukh, and Hogadi [39, Definition 1.2 ff.].

For example, for an integer $n > 1$, the morphism $* \to */\mu_n$ is *not* a Nisnevich cover, since its base change $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$, $t \mapsto t^n$ is not a Nisnevich cover.

Algebraic spaces over K that are locally of finite type admit Nisnevich covers by affine K-schemes, which can be deduced from Knutson [97, II, Theorem 6.4].

5.3.3. Theorem. *Let $\mathcal{X}$ be a stack as above, and let $(f_i \colon \mathcal{X}_i \to \mathcal{X})_{i \in I}$ be a Nisnevich cover. Then the map*

$$(f_i^*)_{i \in I} \colon \widehat{\mathbb{M}}(\mathcal{X}) \longrightarrow \mathrm{eq}\Biggl(\prod_{i \in I} \widehat{\mathbb{M}}(\mathcal{X}_i) \rightrightarrows \prod_{i,j \in I} \widehat{\mathbb{M}}(\mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_j) \Biggr)$$

is an isomorphism, where the right-hand side is the equalizer of the two maps induced by pulling back along projections from each $\mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_j$ to $\mathcal{X}_i$ and $\mathcal{X}_j$, respectively.

Proof. We first consider the case when $\mathcal{X}$ is an algebraic space. In this case, one can stratify $\mathcal{X}$ into locally closed subspaces $S_k \subset \mathcal{X}$, such that the map $\coprod_i \mathcal{X}_i \to \mathcal{X}$ admits a section s_k over each S_k. After a base change to each S_k, we can assume that $\coprod_i \mathcal{X}_i \to \mathcal{X}$ admits a global section, in which case the result is clear.

For the general case, by Kresch [102, Proposition 3.5.9], $\mathcal{X}$ can be stratified by quotient stacks of the form U/G, where U is a quasi-projective K-scheme acted on by $G \simeq \mathrm{GL}(n)$ for some n. Therefore, we may assume that $\mathcal{X} = U/G$ is of this form. Let $\pi \colon U \to U/G$ be the projection. Then for all $a \in \widehat{\mathbb{M}}(U/G)$, we have $a = [G]^{-1} \cdot \pi_! \circ \pi^*(a)$, so that π^* is injective. Its image consists of elements $\tilde{a} \in \widehat{\mathbb{M}}(U)$ such that $\pi^* \circ \pi_!(\tilde{a}) = [G] \cdot \tilde{a}$. We call such elements *G-invariant*. In other words, we may identify $\widehat{\mathbb{M}}(U/G)$ with the subring of $\widehat{\mathbb{M}}(U)$ consisting of G-invariant elements. Writing $U_i = U \times_{\mathcal{X}} \mathcal{X}_i$, it suffices to show that $\widehat{\mathbb{M}}(U) \xrightarrow{\sim} \mathrm{eq}\bigl(\prod_{i \in I} \widehat{\mathbb{M}}(U_i) \rightrightarrows \prod_{i,j \in I} \widehat{\mathbb{M}}(U_i \times_U U_j)\bigr)$, since taking G-invariant elements on both sides gives the desired result. We are now reduced to the already known case of algebraic spaces. □

5.4 Motivic Hall algebras and modules

5.4.1. We introduce the *motivic Hall algebra* for a linear stack, originally defined by Joyce [82], which is an associative algebra structure on the ring of motives $\mathbb{M}(\mathcal{X})$.

For self-dual linear stacks, we show that the ring of motives $\mathbb{M}(\mathcal{X}^{\mathrm{sd}})$ is a module for the motivic Hall algebra $\mathbb{M}(\mathcal{X})$, which we call the *motivic Hall module*.

Hall modules have been constructed and studied for other flavours of Hall algebras, such as by Young [163–165] in the context of Ringel's [139; 140] Hall algebras and that of cohomological Hall algebras. A similar construction in the context of Joyce's [87; 88] vertex algebras is obtained by the author [27]. Another closely related work is DeHority and Latyntsev [47], who studied the relation between the cohomological version and the vertex algebra version.

5.4.2. The motivic Hall algebra. Let $\mathcal{X}$ be a linear stack over an algebraically closed field K, with quasi-compact filtrations as in §3.3.4. Define an operation

$$*\colon \mathbb{M}(\mathcal{X}) \otimes \mathbb{M}(\mathcal{X}) \longrightarrow \mathbb{M}(\mathcal{X})$$

by the composition

$$\mathbb{M}(\mathcal{X}) \otimes \mathbb{M}(\mathcal{X}) \xrightarrow{\boxtimes} \mathbb{M}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\mathrm{gr}^*} \mathbb{M}(\mathcal{X}^+) \xrightarrow{\mathrm{ev}_!} \mathbb{M}(\mathcal{X}) ,$$

where $\mathcal{X}^+$ denotes the disjoint union of the stacks of filtrations $\mathcal{X}^+_{\alpha_1,\alpha_2}$ for all $\alpha_1, \alpha_2 \in \pi_0(\mathcal{X})$.

Roughly speaking, for motives $a, b \in \mathbb{M}(\mathcal{X})$, the product $a * b \in \mathbb{M}(\mathcal{X})$ parametrizes all possible extensions of objects parametrized by a and b, respectively. Here, taking gr^* picks out two-step filtrations with given quotients specified by a and b, and taking $\mathrm{ev}_!$ maps these filtrations to their total objects.

We will see in Theorem 5.4.4 that the product $*$ is associative, and that it has a unit element $[\{0\}] \in \mathbb{M}(\mathcal{X})$, which is the motive of the component $\{0\} \subset \mathcal{X}$. This defines an associative algebra structure on $\mathbb{M}(\mathcal{X})$, called the *motivic Hall algebra* of $\mathcal{X}$.

5.4.3. The motivic Hall module. Now, let $\mathcal{X}$ be a self-dual linear stack over K, with quasi-compact filtrations as in §3.3.4. Define an operation

$$\diamond\colon \mathbb{M}(\mathcal{X}) \otimes \mathbb{M}(\mathcal{X}^{\mathrm{sd}}) \longrightarrow \mathbb{M}(\mathcal{X}^{\mathrm{sd}})$$

by the composition

$$\mathbb{M}(\mathcal{X}) \otimes \mathbb{M}(\mathcal{X}^{\mathrm{sd}}) \xrightarrow{\boxtimes} \mathbb{M}(\mathcal{X} \times \mathcal{X}^{\mathrm{sd}}) \xrightarrow{\mathrm{gr}^*} \mathbb{M}(\mathcal{X}^{\mathrm{sd},+}) \xrightarrow{\mathrm{ev}_!} \mathbb{M}(\mathcal{X}^{\mathrm{sd}}) ,$$

where $\mathcal{X}^{\mathrm{sd},+}$ denotes the disjoint union of the stacks of filtrations $\mathcal{X}^+_{\alpha,\theta}$ for all $\alpha \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$.

Again, roughly speaking, for motives $a \in \mathbb{M}(\mathcal{X})$ and $b \in \mathbb{M}(\mathcal{X}^{\mathrm{sd}})$, the product $a \diamond b \in$

$\mathbb{M}(\mathscr{X}^{\mathrm{sd}})$ parametrizes the total objects of all possible three-step self-dual filtrations, as in §2.2.4, whose graded pieces are parametrized by a, b, and $a^\vee$, respectively,

We will prove in Theorem 5.4.4 that the product $\diamond$ establishes $\mathbb{M}(\mathscr{X}^{\mathrm{sd}})$ as a left module for the motivic Hall algebra $\mathbb{M}(\mathscr{X})$. This is called the *motivic Hall module* of $\mathscr{X}$.

5.4.4. Theorem. *Let $\mathscr{X}$ be a linear stack with quasi-compact filtrations as above.*

(i) *Consider the operation $*$ defined in §5.4.2. Then for any $a, b, c \in \mathbb{M}(\mathscr{X})$, we have*

$$[\{0\}] * a = a = a * [\{0\}] \,, \tag{5.4.4.1}$$

$$(a * b) * c = a * (b * c) \,, \tag{5.4.4.2}$$

where $[\{0\}] \in \mathbb{M}(\mathscr{X})$ is the motive of the component $\{0\} \subset \mathscr{X}$.

(ii) *Suppose that $\mathscr{X}$ is equipped with a self-dual structure. Consider the involution $(-)^\vee$ on $\mathbb{M}(\mathscr{X})$ induced by the involution of $\mathscr{X}$, and the operation $\diamond$ defined in §5.4.3. Then for any $a, b \in \mathbb{M}(\mathscr{X})$ and $c \in \mathbb{M}(\mathscr{X}^{\mathrm{sd}})$, we have*

$$a^\vee * b^\vee = (b * a)^\vee \,, \tag{5.4.4.3}$$

$$[\{0\}] \diamond c = c \,, \tag{5.4.4.4}$$

$$a \diamond (b \diamond c) = (a * b) \diamond c \,. \tag{5.4.4.5}$$

Proof. For (5.4.4.1), it is enough to show that for any $\alpha \in \pi_0(\mathscr{X})$, the morphisms $\mathscr{X}^+_{0,\alpha} \to \mathscr{X}_\alpha$ and $\mathscr{X}^+_{\alpha,0} \to \mathscr{X}_\alpha$ are isomorphisms, which follows from the descriptions in §3.3.4.

For (5.4.4.2), we may assume that $a \in \mathbb{M}(\mathscr{X}_{\alpha_1})$, $b \in \mathbb{M}(\mathscr{X}_{\alpha_2})$, and $c \in \mathbb{M}(\mathscr{X}_{\alpha_3})$, for some $\alpha_1, \alpha_2, \alpha_3 \in \pi_0(\mathscr{X})$. Applying the base change formula (5.2.3.2) to the pullback squares in the diagrams

$$\begin{tikzcd}[column sep=tiny] & & \mathscr{X}^+_{\alpha_1,\alpha_2,\alpha_3} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" very near start] & & \\ & \mathscr{X}^+_{\alpha_1,\alpha_2} \times \mathscr{X}_{\alpha_3} \arrow[dl] \arrow[dr] & & \mathscr{X}^+_{\alpha_1+\alpha_2,\alpha_3} \arrow[dl] \arrow[dr] & \\ \mathscr{X}_{\alpha_1} \times \mathscr{X}_{\alpha_2} \times \mathscr{X}_{\alpha_3} & & \mathscr{X}_{\alpha_1+\alpha_2} \times \mathscr{X}_{\alpha_3} & & \mathscr{X}_{\alpha_1+\alpha_2+\alpha_3} \,, \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] & & \mathscr{X}^+_{\alpha_1,\alpha_2,\alpha_3} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" very near start] & & \\ & \mathscr{X}_{\alpha_1} \times \mathscr{X}^+_{\alpha_2,\alpha_3} \arrow[dl] \arrow[dr] & & \mathscr{X}^+_{\alpha_1,\alpha_2+\alpha_3} \arrow[dl] \arrow[dr] & \\ \mathscr{X}_{\alpha_1} \times \mathscr{X}_{\alpha_2} \times \mathscr{X}_{\alpha_3} & & \mathscr{X}_{\alpha_1} \times \mathscr{X}_{\alpha_2+\alpha_3} & & \mathscr{X}_{\alpha_1+\alpha_2+\alpha_3} \,, \end{tikzcd} \tag{5.4.4.6}$$

we see that both sides of (5.4.4.2) are equal to $\mathrm{ev}_! \circ \mathrm{gr}^*(a \boxtimes b \boxtimes c)$, where gr and ev are the outer compositions in both diagrams in (5.4.4.6). These diagrams are special cases of the *associativity theorem* of the author et al. [31, §6.3], as explained in [31, §7.1.7].

The relation (5.4.4.3) follows from the commutativity of the diagram

$$
\begin{tikzcd}
\mathscr{X}_{\alpha_1} \times \mathscr{X}_{\alpha_2} \arrow[d, "(-)^\vee"', "\simeq"] & \mathscr{X}^+_{\alpha_1,\alpha_2} \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}"] \arrow[d, "(-)^\vee"', "\simeq"] & \mathscr{X}_{\alpha_1+\alpha_2} \arrow[d, "(-)^\vee"', "\simeq"] \\
\mathscr{X}_{\alpha_2^\vee} \times \mathscr{X}_{\alpha_1^\vee} & \mathscr{X}^+_{\alpha_2^\vee,\alpha_1^\vee} \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}"] & \mathscr{X}_{\alpha_2^\vee+\alpha_1^\vee} \, ,
\end{tikzcd}
\tag{5.4.4.7}
$$

where $\alpha_1, \alpha_2 \in \pi_0(\mathscr{X})$, and the middle vertical isomorphism is given by the $\mathbb{Z}_2$-action on $\mathscr{F}\!\mathit{ilt}(\mathscr{X})$.

The relation (5.4.4.4) follows from the isomorphism $\mathscr{X}^{\mathrm{sd},+}_{0,\theta} \xrightarrow{\sim} \mathscr{X}^{\mathrm{sd}}_{\theta}$ for $\theta \in \pi_0(\mathscr{X}^{\mathrm{sd}})$.

For (5.4.4.5), we have similar diagrams

$$
\begin{tikzcd}[column sep=tiny]
& & \mathscr{X}^{\mathrm{sd},+}_{\alpha_1,\alpha_2,\theta} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" near start] & & \\
& \mathscr{X}^+_{\alpha_1,\alpha_2} \times \mathscr{X}^{\mathrm{sd}}_{\theta} \arrow[dl] \arrow[dr] & & \mathscr{X}^{\mathrm{sd},+}_{\alpha_1+\alpha_2,\theta} \arrow[dl] \arrow[dr] & \\
\mathscr{X}_{\alpha_1} \times \mathscr{X}_{\alpha_2} \times \mathscr{X}^{\mathrm{sd}}_{\theta} & & \mathscr{X}_{\alpha_1+\alpha_2} \times \mathscr{X}^{\mathrm{sd}}_{\theta} & & \mathscr{X}^{\mathrm{sd},+}_{\alpha_1+\alpha_2+\theta+\alpha_2^\vee+\alpha_1^\vee} \, ,
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & \mathscr{X}^{\mathrm{sd},+}_{\alpha_1,\alpha_2,\theta} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge" near start] & & \\
& \mathscr{X}_{\alpha_1} \times \mathscr{X}^{\mathrm{sd},+}_{\alpha_2,\theta} \arrow[dl] \arrow[dr] & & \mathscr{X}^{\mathrm{sd},+}_{\alpha_1,\alpha_2+\theta+\alpha_2^\vee} \arrow[dl] \arrow[dr] & \\
\mathscr{X}_{\alpha_1} \times \mathscr{X}_{\alpha_2} \times \mathscr{X}^{\mathrm{sd}}_{\theta} & & \mathscr{X}_{\alpha_1} \times \mathscr{X}^{\mathrm{sd}}_{\alpha_2+\theta+\alpha_2^\vee} & & \mathscr{X}^{\mathrm{sd},+}_{\alpha_1+\alpha_2+\theta+\alpha_2^\vee+\alpha_1^\vee} \, ,
\end{tikzcd}
\tag{5.4.4.8}
$$

where the pullback squares follow from the associativity theorem of the author et al. [31, §6.3]. Alternatively, these diagrams can be obtained by taking $\mathbb{Z}_2$-fixed loci in pullback diagrams analogous to (5.4.4.6) for 5-step filtrations. The relation (5.4.4.5) then follows from applying the base change formula (5.2.3.2) to these diagrams. □

5.5 Epsilon motives

5.5.1. We define *epsilon motives* for linear and self-dual linear stacks, following Joyce [85] in the linear case and the construction of the author, Ibáñez Núñez, and Kinjo [32] for general algebraic stacks. These are elements of the rings of motives $\mathbb{M}(\mathscr{X}; \mathbb{Q})$ and $\mathbb{M}(\mathscr{X}^{\mathrm{sd}}; \mathbb{Q})$, depending on a stability condition τ, and are obtained from motives of semistable loci, $[\mathscr{X}^{\mathrm{ss}}_{\alpha}(\tau)]$ and $[\mathscr{X}^{\mathrm{sd,ss}}_{\theta}(\tau)]$, by removing certain parts of the strictly semistable locus. The purpose of doing this step is so that the *no-pole theorem*, Theorem 5.5.5, holds, allowing us to take the Euler characteristics of epsilon motives, which will then be used to define Donaldson–Thomas invariants.

Throughout, we assume that $\mathscr{X}$ is a linear stack over an algebraically closed field K with quasi-compact filtrations as in §3.3.4.

5.5.2. The linear case. Let τ be a permissible stability condition on $\mathcal{X}$. Following Joyce [85], for each class $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$, define the *epsilon motive* $\epsilon_\alpha(\tau) \in \mathbb{M}(\mathcal{X}_\alpha; \mathbb{Q})$ by the formula

$$\epsilon_\alpha(\tau) = \sum_{\substack{n>0;\ \alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n, \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n)}} \frac{(-1)^{n-1}}{n} \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] * \cdots * [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \, , \tag{5.5.2.1}$$

where $*$ denotes multiplication in the motivic Hall algebra $\mathbb{M}(\mathcal{X}; \mathbb{Q})$. By Lemma 3.5.8, only finitely many terms in the sum are non-zero. Note that $\epsilon_\alpha(\tau)$ is supported on $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)$.

Formally inverting the formula (5.5.2.1), we obtain the relation

$$[\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)] = \sum_{\substack{n>0;\ \alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n, \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n)}} \frac{1}{n!} \cdot \epsilon_{\alpha_1}(\tau) * \cdots * \epsilon_{\alpha_n}(\tau) \, . \tag{5.5.2.2}$$

The relation between the coefficients $(-1)^{n-1}/n$ and $1/n!$ are explained in §5.5.4 below.

One can also combine (5.5.2.2) with the relation

$$[\mathcal{X}_\alpha] = \sum_{\substack{n>0;\ \alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n, \\ \tau(\alpha_1) > \cdots > \tau(\alpha_n)}} [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] * \cdots * [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \, , \tag{5.5.2.3}$$

which comes from the Θ-stratification of $\mathcal{X}$, and can be an infinite but locally finite sum, giving the formula

$$[\mathcal{X}_\alpha] = \sum_{\substack{n>0;\ \alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n, \\ \tau(\alpha_1) \geqslant \cdots \geqslant \tau(\alpha_n)}} \frac{1}{|W_{\alpha_1, \ldots, \alpha_n}(\tau)|} \cdot \epsilon_{\alpha_1}(\tau) * \cdots * \epsilon_{\alpha_n}(\tau) \, , \tag{5.5.2.4}$$

where $W_{\alpha_1, \ldots, \alpha_n}(\tau)$ denotes the group of permutations σ of $\{1, \ldots, n\}$ such that $\tau(\alpha_{\sigma(1)}) \geqslant \cdots \geqslant \tau(\alpha_{\sigma(n)})$. This can be taken as an alternative definition of the invariants $\epsilon_\alpha(\tau)$, that is, they are the unique set of motives such that (5.5.2.4) holds for all α.

One can interpret (5.5.2.4) as considering a generalized version of HN filtrations, where the slopes of the quotients are non-increasing rather than strictly decreasing, and the sum is averaged over all possible orderings satisfying the non-increasing condition.

5.5.3. The self-dual case. Suppose that $\mathcal{X}$ is equipped with a self-dual structure, and let τ be a permissible self-dual stability condition on $\mathcal{X}$.

For each class $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, define the *epsilon motive* $\epsilon^{\mathrm{sd}}_\theta(\tau) \in \mathbb{M}(\mathcal{X}^{\mathrm{sd}}_\theta; \mathbb{Q})$ by the formula

$$\epsilon^{\mathrm{sd}}_\theta(\tau) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho, \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n) = 0}} \binom{-1/2}{n} \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \diamond \cdots \diamond [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \diamond [\mathcal{X}^{\mathrm{sd,ss}}_{\rho}(\tau)] \, , \tag{5.5.3.1}$$

where $\diamond$ denotes the multiplication for the motivic Hall module, the notation $\alpha_i + \alpha_i^\vee$ is from §3.4.1, and $\binom{-1/2}{n}$ is the binomial coefficient. The sum only contains finitely many non-zero terms, and $\epsilon^{\mathrm{sd}}_\theta(\tau)$ is supported on the semistable locus $\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau) \subset \mathcal{X}^{\mathrm{sd}}_\theta$.

Formally inverting the formula (5.5.3.1), we obtain the relation

$$[\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau)] = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho, \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n) = 0}} \frac{1}{2^n \, n!} \cdot \epsilon_{\alpha_1}(\tau) \diamond \cdots \diamond \epsilon_{\alpha_n}(\tau) \diamond \epsilon^{\mathrm{sd}}_\rho(\tau) \, , \tag{5.5.3.2}$$

which we explain further in §5.5.4. This can be combined with the relation

$$[\mathcal{X}^{\mathrm{sd}}_\theta] = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho, \\ \tau(\alpha_1) > \cdots > \tau(\alpha_n) > 0}} [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \diamond \cdots \diamond [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \diamond [\mathcal{X}^{\mathrm{sd,ss}}_{\rho}(\tau)] \tag{5.5.3.3}$$

from the Θ-stratification of $\mathcal{X}^{\mathrm{sd}}$, together with (5.5.2.2), to obtain the formula

$$[\mathcal{X}^{\mathrm{sd}}_\theta] = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho, \\ \tau(\alpha_1) \geqslant \cdots \geqslant \tau(\alpha_n) \geqslant 0}} \frac{1}{|W^{\mathrm{sd}}_{\alpha_1, \dots, \alpha_n}(\tau)|} \cdot \epsilon_{\alpha_1}(\tau) \diamond \cdots \diamond \epsilon_{\alpha_n}(\tau) \diamond \epsilon^{\mathrm{sd}}_\rho(\tau) \, , \tag{5.5.3.4}$$

where $W^{\mathrm{sd}}_{\alpha_1, \dots, \alpha_n}(\tau)$ is the group of permutations σ of $\{1, \dots, n, n^\vee, \dots, 1^\vee\}$, such that $\sigma(i)^\vee = \sigma(i^\vee)$ for all i, where we set $(i^\vee)^\vee = i$, satisfying the non-increasing condition $\tau(\alpha_{\sigma(1)}) \geqslant \cdots \geqslant \tau(\alpha_{\sigma(n)}) \geqslant 0$, where we set $\alpha_{i^\vee} = \alpha_i^\vee$. For example, we have $|W^{\mathrm{sd}}_{\alpha_1, \dots, \alpha_n}(\tau)| = 2^n \, n!$ if $\tau(\alpha_1) = \cdots = \tau(\alpha_n) = 0$.

5.5.4. Explanations of the coefficients. The relations between the coefficients in (5.5.2.1), (5.5.2.2), (5.5.3.1), and (5.5.3.2), can be seen more directly by setting

$$\delta(\tau;t) = [\{0\}] + \sum_{\substack{\alpha\in\pi_0(\mathcal{X})\setminus\{0\}:\\ \tau(\alpha)=t}} [\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)] \, , \qquad \delta^{\mathrm{sd}}(\tau) = \sum_{\theta\in\pi_0(\mathcal{X}^{\mathrm{sd}})} [\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau)] \, ,$$

$$\epsilon(\tau;t) = \sum_{\substack{\alpha\in\pi_0(\mathcal{X})\setminus\{0\}:\\ \tau(\alpha)=t}} \epsilon_\alpha(\tau) \, , \qquad \epsilon^{\mathrm{sd}}(\tau) = \sum_{\theta\in\pi_0(\mathcal{X}^{\mathrm{sd}})} \epsilon^{\mathrm{sd}}_\theta(\tau) \, ,$$

as motives on $\mathcal{X}$ or $\mathcal{X}^{\mathrm{sd}}$, where $t \in T$, so that these relations can be rewritten as

$$\epsilon(\tau;t) = \log\delta(\tau;t) \, , \qquad \epsilon^{\mathrm{sd}}(\tau) = \delta(\tau;0)^{-1/2} \diamond \delta^{\mathrm{sd}}(\tau) \, ,$$
$$\delta(\tau;t) = \exp\epsilon(\tau;t) \, , \qquad \delta^{\mathrm{sd}}(\tau) = \exp\Big(\frac{1}{2}\epsilon(\tau;0)\Big) \diamond \epsilon^{\mathrm{sd}}(\tau) \, ,$$

where we take formal power series using the product in the motivic Hall algebra.

The coefficients $(-1)^{n-1}/n$ and $\binom{-1/2}{n}$ in (5.5.2.1) and (5.5.3.1) are determined by the coefficients $1/n!$ and $1/(2^n n!)$ in (5.5.2.2) and (5.5.3.2) in this way. They are the unique choice of coefficients only depending on n, such that the *no-pole theorem*, Theorem 5.5.5, holds for the epsilon motives. The rough reason for this is that they ensure the combinatorial descriptions of the coefficients $1/|W_{\alpha_1,\ldots,\alpha_n}(\tau)|$ and $1/|W^{\mathrm{sd}}_{\alpha_1,\ldots,\alpha_n}(\tau)|$ in (5.5.2.4) and (5.5.3.4), and from the viewpoint of [32], the no-pole theorem corresponds to the property that these coefficients sum up to 1 for all permutations σ as described for each of them, for fixed classes α_i.

5.5.5. The no-pole theorem. A key property of the epsilon motives is the *no-pole theorem*, which states that they have pure virtual ranks in the sense of §5.2.9. This will allow us to define numerical invariants, including Donaldson–Thomas invariants, by taking their Euler characteristics.

Theorem. *Let $\mathcal{X}$ be a linear stack over K, with quasi-compact filtrations.*

(i) *For any permissible stability condition τ on $\mathcal{X}$, and any $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$, the motive $\epsilon_\alpha(\tau)$ has pure virtual rank* 1.

(ii) *If $\mathcal{X}$ is equipped with a self-dual structure, then for any permissible self-dual stability condition τ on $\mathcal{X}$, and any $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, the motive $\epsilon^{\mathrm{sd}}_\theta(\tau)$ has pure virtual rank* 0.

We defer the proof of the theorem to Appendix A. The linear case (i) was originally proved by Joyce [83, Theorem 8.7], under a slightly different setting. The orthosymplectic case (ii) was originally proved in the author's preprint [26, Appendix E], again under a slightly different setting. This theorem is now available in a more general form for intrinsic Donaldson–Thomas invariants in [32, Theorem 5.3.7], but we present a proof in Appendix A that is spiritually closer to the original proofs in [83] and [26], and requires less abstract formalism to set up.

5.6 Donaldson–Thomas invariants

5.6.1. We now turn to the definition of *Donaldson–Thomas invariants* for linear and self-dual linear stacks, where the latter is one of the main constructions of this thesis. The linear case was first due to Joyce and Song [89] and Kontsevich and Soibelman [99].

Throughout this section, we assume that the base field K is algebraically closed and has characteristic zero.

5.6.2. The Behrend function. We now discuss the definition of *Behrend functions* of algebraic stacks, described in §5.1.2.

For an algebraic stack $\mathcal{X}$ over K as in §3.2.2, we would like to define its *Behrend function*

$$\nu_{\mathcal{X}} \in \mathrm{CF}(\mathcal{X}; \mathbb{Z}) \, . \tag{5.6.2.1}$$

There are multiple ways to define it, in different generalities:

(i) The original definition of Behrend [10] works for Deligne–Mumford stacks over $\mathbb{C}$.

(ii) This was later extended by Joyce and Song [89, §4.1] to algebraic stacks locally of finite type over an algebraically closed field K of characteristic zero.

(iii) Alternatively, when $\mathcal{X}$ upgrades to a derived stack with a (-1)-shifted symplectic structure, we can define $\nu_{\mathcal{X}}$ using the *motivic Behrend function*, which we do in §6.2.8 below.

The first two definitions agree when they are defined; the third one agrees with the second one when $K = \mathbb{C}$ and when the former is defined, which can be deduced from Denef and Loeser [50, Theorem 3.10]. Although we expect them to agree for general K, we do not have a proof of this yet.

In the following, we always take (iii) as our definition of $\nu_{\mathcal{X}}$, since we can prove more properties of it, including crucially the *motivic integral identity*, which is important for proving wall-crossing formulae for our Donaldson–Thomas invariants. As mentioned above, when $K = \mathbb{C}$, we can also use (ii) instead.

5.6.3. The linear case. From now on, we fix a (-1)-*shifted symplectic linear stack* $\mathcal{X}$ over K in the sense of §3.7.3. Let τ be a permissible stability condition on $\mathcal{X}$, or more precisely, on the classical truncation of $\mathcal{X}$, as in §3.5.7,

Following Joyce and Song [89, Definition 5.15], but adapting it to our more general setting of linear stacks, for a class $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$, we define the *Donaldson–Thomas invariant* $\mathrm{DT}_\alpha(\tau) \in \mathbb{Q}$ by the formula

$$\mathrm{DT}_\alpha(\tau) = \int_{\mathcal{X}} (1 - \mathbb{L}) \cdot \epsilon_\alpha(\tau) \cdot \nu_{\mathcal{X}} \, d\chi \, , \tag{5.6.3.1}$$

where the notation $\int (-) \, d\chi$ is defined in §5.2.9, and $\nu_{\mathcal{X}}$ is the Behrend function of $\mathcal{X}$.

This integral is well-defined since $\epsilon_\alpha(\tau)$ is supported on the semistable locus $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)$, which is quasi-compact, and by the no-pole theorem, Theorem 5.5.5 (i).

5.6.4. The self-dual case. Assume further that $\mathcal{X}$ is equipped with a *self-dual structure*, that is a $\mathbb{Z}_2$-action preserving the (-1)-shifted symplectic form, and reversing the $*/\mathbb{G}_{\mathrm{m}}$-action, analogously to §3.4.1.

Let τ be a permissible self-dual stability condition on (the classical truncation of) $\mathcal{X}$. For a class $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, define the *self-dual Donaldson–Thomas invariant* $\mathrm{DT}_\theta^{\mathrm{sd}}(\tau) \in \mathbb{Q}$ by the formula

$$\mathrm{DT}_\theta^{\mathrm{sd}}(\tau) = \int_{\mathcal{X}^{\mathrm{sd}}} \epsilon_\theta^{\mathrm{sd}}(\tau) \cdot \nu_{\mathcal{X}^{\mathrm{sd}}} \, d\chi \, . \tag{5.6.4.1}$$

Again, this is well-defined by the fact that $\epsilon_\theta^{\mathrm{sd}}(\tau)$ is supported on $\mathcal{X}_\theta^{\mathrm{sd,ss}}(\tau)$, which is quasi-compact, and by the no-pole theorem, Theorem 5.5.5 (ii).

5.6.5. For smooth stacks. Let $\mathcal{X}$ be a classical linear stack that is smooth over K, and consider its (-1)-shifted cotangent stack $\mathrm{T}^*[-1]\,\mathcal{X}$, which has a canonical (-1)-shifted symplectic structure as in §3.6.7, making it a (-1)-shifted symplectic linear stack. We have $(\mathrm{T}^*[-1]\,\mathcal{X})_{\mathrm{cl}} \simeq \mathcal{X}$. If $\mathcal{X}$ is equipped with a self-dual structure, then the fixed locus $\mathcal{X}^{\mathrm{sd}}$ is also smooth, and $(\mathrm{T}^*[-1]\,\mathcal{X})^{\mathrm{sd}} \simeq \mathrm{T}^*[-1]\,\mathcal{X}^{\mathrm{sd}}$.

In this case, we have $\nu_{\mathcal{X}} = (-1)^{\dim \mathcal{X}}$ and $\nu_{\mathcal{X}^{\mathrm{sd}}} = (-1)^{\dim \mathcal{X}^{\mathrm{sd}}}$, and (5.6.3.1)–(5.6.4.1) become

$$\mathrm{DT}_\alpha(\tau) = (-1)^{\dim \mathcal{X}_\alpha} \cdot \int_{\mathcal{X}_\alpha} (1 - \mathbb{L}) \cdot \epsilon_\alpha(\tau) \, d\chi \, , \tag{5.6.5.1}$$

$$\mathrm{DT}^{\mathrm{sd}}_\theta(\tau) = (-1)^{\dim \mathcal{X}^{\mathrm{sd}}_\theta} \cdot \int_{\mathcal{X}^{\mathrm{sd}}_\theta} \epsilon^{\mathrm{sd}}_\theta(\tau) \, d\chi \, . \tag{5.6.5.2}$$

The invariants $\mathrm{DT}_\alpha(\tau)$ are essentially the same as those defined by Joyce [85, §6.2], denoted by $J^\alpha(\tau)^\Omega$ there, while the invariants $\mathrm{DT}^{\mathrm{sd}}_\theta(\tau)$ are new.

Chapter 6

Motivic Donaldson–Thomas invariants

We introduce motivic enhancements of the orthosymplectic Donaldson–Thomas invariants defined in Chapter 5, living in a ring of *monodromic motives*, parallel to the conjectural construction of Kontsevich and Soibelman [99] in the linear case, later rigorously established through the works of Lê [104], Bussi, Joyce, and Meinhardt [34], Ben-Bassat et al. [13], and others.

In the linear case, these invariants are, in general, not easy to compute, but in cases where they are computable, they often exhibit interesting and rich structures. See Behrend, Bryan, and Szendrői [11], Morrison, Mozgovoy, Nagao, and Szendrői [119], Davison and Meinhardt [44; 45], etc., for examples. It would be interesting to explore if the orthosymplectic case also has similar interesting structures.

6.1 Monodromic motives

6.1.1. Idea. The motivic Donaldson–Thomas invariants are defined using the ring of *monodromic motives*, a variant of the ring of motives defined in §5.2.

To explain the rough idea of monodromic motives, it might be more straightforward to work in the analytic setting. For this purpose, let X be a complex manifold, with a smooth metric $d\colon X \times X \to \mathbb{R}_{\geqslant 0}$, and let $f\colon X \to \mathbb{C}$ be a holomorphic function. Let $x \in X$ be a point such that $f(x) = 0$. Let $0 < \delta \ll \varepsilon \ll 1$ be small positive numbers, and consider the map

$$X^{\times}_{\delta,\varepsilon}(x) = B_\varepsilon(x) \cap f^{-1}(D^{\times}_\delta) \xrightarrow{f} D^{\times}_\delta \, ,$$

where $D_\delta^\times = \{z \in \mathbb{C} \mid 0 < |z| < \delta\}$. This map is a topological fibration, and its fibre $\mathrm{MF}_f(x)$ is called the *Milnor fibre* of f at x. The cohomology of $\mathrm{MF}_f(x)$ is often called the *nearby cycles* of f at x, and carries the action of the *monodromy operator* induced by this fibration.

There is also a motivic analogue of this construction, called the *motivic Milnor fibre*, which we discuss in §6.2 below.

The ring of monodromic motives can then be roughly thought of as having Milnor fibres of functions as above as its elements, equipped with monodromy actions, and these elements satisfy cut-and-paste relations (5.2.1.1) for closed subsets respecting the monodromy action.

6.1.2. Monodromic motives. We define the ring of *monodromic motives* over a stack $\mathcal{X}$, extending the ring of motives $\widehat{\mathbb{M}}(\mathcal{X}; A)$ defined in §5.2.6.

Let $\hat{\mu} = \lim \mu_n$ be the projective limit of the groups of roots of unity. For a scheme Z, a *good action* of $\hat{\mu}$ on Z is one that factors through μ_n for some n, such that each orbit is contained in an affine open subscheme of Z.

For a stack $\mathcal{X}$ over K as in §3.2.2, and a commutative ring A, define

$$K^{\hat{\mu}}(\mathcal{X}; A) = \widehat{\bigoplus_{Z \to \mathcal{X}}} A \cdot [Z] \Big/ \sim\,, \tag{6.1.2.1}$$

$$\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}; A) = K^{\hat{\mu}}(\mathcal{X}; A) \mathop{\hat{\otimes}}_{A[\mathbb{L}]} A[\mathbb{L}^{\pm 1}, (\mathbb{L}^k - 1)^{-1}] \Big/ \approx\,, \tag{6.1.2.2}$$

where $\hat{\oplus}$ and $\hat{\otimes}$ indicate that we allow locally finite sums, as in §5.2.2 and §5.2.6, and we sum over morphisms $Z \to \mathcal{X}$ with a good $\hat{\mu}$-action on Z, called the *monodromy action*, that is compatible with the trivial $\hat{\mu}$-action on $\mathcal{X}$. The relation $\sim$ is generated by $[Z] \sim [Z'] + [Z \setminus Z']$ for $\hat{\mu}$-invariant closed subschemes $Z' \subset Z$, and $[Z \times V] \sim [Z \times \mathbb{A}^n]$ for a $\hat{\mu}$-representation V of dimension n, where the projections to $\mathcal{X}$ factor through Z, and $\hat{\mu}$ acts trivially on $\mathbb{A}^n$. The definition of $\approx$ is slightly more involved, and can be found in Ben-Bassat et al. [13, Definition 5.13], where it is denoted by $I_{\mathcal{X}}^{\mathrm{st},\hat{\mu}}$. There is a map $\widehat{\mathbb{M}}(\mathcal{X}; A) \to \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}; A)$ given by $[Z] \mapsto [Z]$ on generators, with trivial $\hat{\mu}$-action.

There is a commutative multiplication on $\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}; A)$, denoted by '$\odot$' in [13, Definition 5.13], which is *different* from the one given by the fibre product in general. Equipped with this multiplication, $\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}; A)$ is a commutative A-algebra, called the ring of *monodromic motives* over $\mathcal{X}$.

6.1.3. Properties. There are pullback and pushforward maps for monodromic motives, similar to those defined in §5.2.3, and they satisfy the base change and projection formulae (5.2.3.1)–(5.2.3.2).

Define a subring

$$\widehat{\mathbb{M}}^{\hat{\mu},\circ}(\mathscr{X}; A) \subset \widehat{\mathbb{M}}^{\hat{\mu}}(\mathscr{X}; A) \tag{6.1.3.1}$$

as the set of motives whose underlying motives live in the subring $\widehat{\mathbb{M}}^{\circ}(\mathscr{X}; A)$ defined in §5.2.8. When A contains $\mathbb{Q}$, there is an Euler characteristic map

$$\chi \colon \widehat{\mathbb{M}}^{\hat{\mu},\circ}(\mathscr{X}; A) \to \mathrm{CF}(\mathscr{X}; A) \,, \tag{6.1.3.2}$$

defined via the underlying non-monodromic motive.

6.1.4. Motives of double covers. There is an element

$$\mathbb{L}^{1/2} = 1 - [\mu_2] \in \widehat{\mathbb{M}}^{\hat{\mu}}(K) \,, \tag{6.1.4.1}$$

where μ_2 is equipped with the unique non-trivial $\hat{\mu}$-action. This element satisfies $(\mathbb{L}^{1/2})^2 = \mathbb{L}$. We also write $\mathbb{L}^{-1/2} = \mathbb{L}^{-1} \cdot \mathbb{L}^{1/2}$. In particular, we have $\chi(\mathbb{L}^{1/2}) = -1$.

As in Bussi, Joyce, and Meinhardt [34, §2.5] and Ben-Bassat et al. [13, Definition 5.13], for a principal μ_2-bundle $\mathscr{P} \to \mathscr{X}$, we have an element

$$\Upsilon(\mathscr{P}) = ([\mathscr{X}] - [\mathscr{P}]) \in \widehat{\mathbb{M}}^{\hat{\mu}}(\mathscr{X}) \,, \tag{6.1.4.2}$$

where $\hat{\mu}$ acts trivially on $\mathscr{X}$ and via the μ_2-action on $\mathscr{P}$, and monodromic motives of stacks are defined by a similar process to that in §5.2.5. This construction satisfies $\Upsilon(\mathscr{P} \otimes \mathscr{Q}) = \Upsilon(\mathscr{P}) \cdot \Upsilon(\mathscr{Q})$ for principal μ_2-bundles $\mathscr{P}, \mathscr{Q} \to \mathscr{X}$, where $\mathscr{P} \otimes \mathscr{Q}$ denotes the tensor product principal μ_2-bundle.

6.2 The motivic Behrend function

6.2.1. We introduce the *motivic Behrend function*, which is a motivic enhancement of the Behrend function introduced in §5.6.2. It can be seen as a motivic version and a globalization of the construction of Milnor fibres mentioned in §6.1.1.

For functions on smooth varieties, the Milnor fibre was constructed by Denef and Loeser

[48–50], and is a monodromic motive on the zero locus of the function. See also Looijenga [106]. This was later generalized to the case of stacks by Ben-Bassat et al. [13]. Here, we slightly generalize this construction by weakening the assumptions on the stack.

6.2.2. Local structure. For our constructions, we will need our stacks to satisfy local conditions, which we introduce now.

Recall from §5.3.2 the definition of a *Nisnevich cover* of an algebraic stack.

For an algebraic stack $\mathcal{X}$ satisfying the assumptions in §3.2.2, we say that $\mathcal{X}$ is *étale* (or *Nisnevich*) *locally a quotient stack*, if it admits a representable étale (or Nisnevich) cover by quotient stacks of the form $U/\mathrm{GL}(n)$, with U an algebraic space.

We say that $\mathcal{X}$ is *étale* (or *Nisnevich*) *locally fundamental*, if it admits a representable étale (or Nisnevich) cover by quotient stacks of the form $U/\mathrm{GL}(n)$, with U an affine scheme.

These conditions are preserved by taking $\mathbb{Z}_2$-fixed points by Lemma 3.4.5.

For example, by Alper, Hall, and Rydh [4, Theorem 6.1], $\mathcal{X}$ is Nisnevich locally fundamental if it admits a *good moduli space* in the sense of Alper [2]. Also, by Alper, Hall, and Rydh [3, Theorem 1.1], $\mathcal{X}$ is étale locally fundamental if closed points of $\mathcal{X}$ have linearly reductive stabilizers, and if every point of $\mathcal{X}$ specializes to a closed point.

6.2.3. The motivic Milnor fibre. Let K be an algebraically closed field of characteristic zero. By a *smooth K-variety*, we mean an integral, separated K-scheme of finite type that is smooth over Spec K.

Let X be a smooth K-variety, and let $f \colon X \to \mathbb{A}^1$ be a morphism. Write $X_0 = f^{-1}(0)$. Following Denef and Loeser [48–50], we define the *motivic Milnor fibre* of f, which is an element

$$\mathrm{MF}_f \in \mathbb{M}^{\hat{\mu}}(X_0) \ ,$$

as follows.

If f is constant, define $\mathrm{MF}_f = 0$. Otherwise, choose a resolution $\pi \colon \widetilde{X} \to X$ of f, in the sense that $\widetilde{X}$ is a smooth K-variety, π is a proper morphism that restricts to an isomorphism on $\pi^{-1}(X \setminus X_0)$, and $\pi^{-1}(X_0)$ is a simple normal crossings divisor in $\widetilde{X}$. See, for example, Kollár [98] for the existence of such resolutions and their properties.

Let $(E_i)_{i \in J}$ be the irreducible components of $\pi^{-1}(X_0)$, and write N_i for the multiplicity of E_i in the divisor of $f \circ \pi$ on $\widetilde{X}$. For a non-empty subset $I \subset J$, write $E_I^\circ = \bigcap_{i \in I} E_i \setminus \bigcup_{i \notin I} E_i$. Let $m_I = \gcd_{i \in I} N_i$, and define an m_I-fold cover $\widetilde{E}_I^\circ \to E_I^\circ$ as follows. For each open set $U \subset \widetilde{X}$ such that $f \circ \pi = u v^{m_I}$ on U for $u \colon U \to \mathbb{A}^1 \setminus \{0\}$ and $v \colon U \to \mathbb{A}^1$, define the restriction of $\widetilde{E}_I^\circ$ on $E_I^\circ \cap U$ as

$$\widetilde{E}_I^\circ|_{E_I^\circ \cap U} = \left\{ (z, y) \in \mathbb{A}^1 \times (E_I^\circ \cap U) \;\middle|\; z^{m_I} = u^{-1} \right\}. \tag{6.2.3.1}$$

Since E_I° can be covered by such open sets U, (6.2.3.1) can be glued together to obtain a cover $\widetilde{E}_I^\circ \to E_I^\circ$, with a natural μ_{m_I}-action given by scaling the z-coordinate, which induces a $\hat{\mu}$-action on $\widetilde{E}_I^\circ$. The motivic Milnor fibre MF_f is then given by

$$\mathrm{MF}_f = \sum_{\varnothing \neq I \subset J} (1 - \mathbb{L})^{|I|-1} \, [\widetilde{E}_I^\circ] \,. \tag{6.2.3.2}$$

It can be shown [49, Definition 3.8] that this is independent of the choice of the resolution π.

6.2.4. d-critical stacks. We now introduce a convenient tool, the notion of *d-critical structures* due to Joyce [86] and Ben-Bassat et al. [13], which describe some information about (-1)-shifted symplectic structures but entirely on the classical stack.

More precisely, given a (-1)-shifted symplectic stack $\mathcal{X}$ over K, Ben-Bassat et al. [13, §3.3] define an induced *d-critical structure* on its classical truncation $\mathcal{X}_{\mathrm{cl}}$, so that $\mathcal{X}_{\mathrm{cl}}$ is a *d-critical stack*. See there and Joyce [86] for the precise definitions. For our purposes, it suffices to know the following properties:

(i) For a smooth K-variety U and a function $f \colon U \to \mathbb{A}^1$, the critical locus $\mathrm{Crit}(f) \subset U$ carries a canonical d-critical structure.

(ii) d-critical structures can be pulled back along smooth morphisms of algebraic stacks over K.

(iii) If a K-scheme X carries a d-critical structure, then it can be covered by open subschemes called *critical charts*, each of which with the induced d-critical structure has the form $\mathrm{Crit}(f)$ as in (i), such that $\mathrm{Crit}(f) \subset f^{-1}(0)$. We denote such a critical chart by $i \colon \mathrm{Crit}(f) \hookrightarrow X$.

(iv) Given a d-critical stack $\mathcal{X}$, we have its *canonical bundle* $K_{\mathcal{X}}$, which models the canonical bundle of the original (-1)-shifted symplectic stack as in §3.6.8. We can define *orient-*

ations of d-critical stacks using this canonical bundle, as in §3.6.8. Moreover, by Joyce [86, Lemma 2.58], for a smooth morphism $g\colon \mathcal{Y} \to \mathcal{X}$ of d-critical stacks, compatible with the d-critical structures, an orientation $(K_{\mathcal{X}}^{1/2}, o_{\mathcal{X}})$ of $\mathcal{X}$ induces an orientation $(K_{\mathcal{Y}}^{1/2}, o_{\mathcal{Y}})$ of $\mathcal{Y}$ given by $K_{\mathcal{Y}}^{1/2} = g^*(K_{\mathcal{X}}^{1/2}) \otimes \det \mathbb{L}_{\mathcal{Y}/\mathcal{X}}|_{\mathcal{Y}^{\mathrm{red}}}$.

6.2.5. Definition for schemes. Let X be an oriented d-critical K-scheme. Its *motivic Behrend function* $\nu_X^{\mathrm{mot}} \in \mathbb{M}^{\hat{\mu}}(X)$ is defined by the following property:

- For any critical chart $i\colon \mathrm{Crit}(f) \hookrightarrow X$, where $f\colon U \to \mathbb{A}^1$ and U is a smooth K-variety, we have

$$i^*(\nu_X^{\mathrm{mot}}) = -\mathbb{L}^{-\dim U/2} \cdot (\mathrm{MF}_f - [U_0]) \cdot \Upsilon\big(i^*(K_X^{1/2}) \otimes K_U^{-1}|_{\mathrm{Crit}(f)^{\mathrm{red}}}\big)\,, \tag{6.2.5.1}$$

  in $\mathbb{M}^{\hat{\mu}}(\mathrm{Crit}(f))$, where $U_0 = f^{-1}(0)$, and $\mathrm{MF}_f - [U_0]$ is supported on $\mathrm{Crit}(f)$. The map Υ is as in §6.1.4, and the part inside $\Upsilon(\ldots)$ is a line bundle on $\mathrm{Crit}(f)^{\mathrm{red}}$ whose square is trivial, so it can be seen as a μ_2-bundle.

This is well-defined due to Bussi, Joyce, and Meinhardt [34, Theorem 5.10].

For X as above, and a smooth morphism $g\colon Y \to X$ of relative dimension d, where Y is equipped with the induced oriented d-critical structure, we have the relation

$$g^*(\nu_X^{\mathrm{mot}}) = \mathbb{L}^{d/2} \cdot \nu_Y^{\mathrm{mot}}\,, \tag{6.2.5.2}$$

which follows from Ben-Bassat et al. [13, Theorem 5.14].

6.2.6. Definition for stacks. Let $\mathcal{X}$ be an oriented d-critical stack over K, and assume that $\mathcal{X}$ is Nisnevich locally a quotient stack in the sense of §6.2.2.

We define the *motivic Behrend function* of $\mathcal{X}$ below, slightly generalizing the construction of [13, Theorem 5.14], who only considered stacks that are Zariski locally quotient stacks.

Theorem. *Let $\mathcal{X}$ be as above. Then there exists a unique element*

$$\nu_{\mathcal{X}}^{\mathrm{mot}} \in \mathbb{M}^{\hat{\mu}}(\mathcal{X})\,,$$

called the motivic Behrend function of $\mathcal{X}$, such that for any K-scheme Y and any smooth morphism $f\colon Y \to \mathcal{X}$ of relative dimension d, we have

$$f^*(\nu_{\mathcal{X}}^{\mathrm{mot}}) = \mathbb{L}^{d/2} \cdot \nu_Y^{\mathrm{mot}} \tag{6.2.6.1}$$

in $\mathbb{M}^{\hat{\mu}}(Y)$, where ν_Y^{mot} is defined in §6.2.5, and Y is equipped with the induced oriented d-critical structure.

Proof. We first show that the theorem holds when $\mathcal{X} = X$ is an algebraic space. Indeed, this follows formally from Theorem 5.3.3 and the relation (6.2.5.2) for schemes, since X has a Nisnevich cover by affine schemes.

Also, note that if the element $\nu_{\mathcal{X}}^{\mathrm{mot}}$ exists, then the relation (6.2.6.1) must also hold for smooth morphisms from algebraic spaces Y to $\mathcal{X}$, by passing to a Nisnevich cover of Y by affine schemes.

Now, the proof of [13, Theorem 5.14] can be repeated word-by-word to show that the theorem is true when $\mathcal{X} \simeq S/G$ is a quotient stack, where S is an algebraic space over K and $G = \mathrm{GL}(n)$ for some n.

For the general case, let $(j_i \colon \mathcal{X}_i \hookrightarrow \mathcal{X})_{i \in I}$ be a Nisnevich cover by quotient stacks. The condition on $\nu_{\mathcal{X}}^{\mathrm{mot}}$ forces that $j_i^*(\nu_{\mathcal{X}}^{\mathrm{mot}}) = \nu_{\mathcal{X}_i}^{\mathrm{mot}}$ for all i. We show that the elements $\nu_{\mathcal{X}_i}^{\mathrm{mot}}$ agree on overlaps. Indeed, let $1, 2 \in I$ be two indices, and let $\mathcal{X}_{1,2} = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$. Then $\mathcal{X}_{1,2}$ is also a quotient stack, so the theorem holds for $\mathcal{X}_{1,2}$. Let $j_i' \colon \mathcal{X}_{1,2} \to \mathcal{X}_i$ be the projections, where $i = 1, 2$. Then we have $(j_i')^*(\nu_{\mathcal{X}_i}^{\mathrm{mot}}) = \nu_{\mathcal{X}_{1,2}}^{\mathrm{mot}}$ for $i = 1, 2$, since the left-hand side satisfies the characterizing property of $\nu_{\mathcal{X}_{1,2}}^{\mathrm{mot}}$. By Theorem 5.3.3, it then follows that the elements $\nu_{\mathcal{X}_i}^{\mathrm{mot}}$ for $i \in I$ glue to a unique element $\nu_{\mathcal{X}}^{\mathrm{mot}}$, and a standard argument verifies that it satisfies the relation (6.2.6.1). □

6.2.7. Compatibility with smooth pullbacks. We now show that the smooth pullback relation (6.2.6.1) holds for all smooth morphisms of d-critical stacks.

Theorem. *Let $\mathcal{X}, \mathcal{Y}$ be oriented d-critical stacks over K that are Nisnevich locally quotient stacks, and let $f \colon \mathcal{Y} \to \mathcal{X}$ be a smooth morphism of relative dimension d which is compatible with the oriented d-critical structures. Then we have the relation*

$$f^*(\nu_{\mathcal{X}}^{\mathrm{mot}}) = \mathbb{L}^{d/2} \cdot \nu_{\mathcal{Y}}^{\mathrm{mot}} \,. \tag{6.2.7.1}$$

Proof. It is straightforward to verify that the element $\mathbb{L}^{-d/2} \cdot f^*(\nu_{\mathcal{X}}^{\mathrm{mot}})$ satisfies the characterizing property of $\nu_{\mathcal{Y}}^{\mathrm{mot}}$. □

6.2.8. The numerical Behrend function. Let $\mathcal{X}$ be an algebraic stack over K that is Nisnevich locally a quotient stack, equipped with an oriented d-critical structure. The *Behrend function* of $\mathcal{X}$ is the constructible function

$$\nu_{\mathcal{X}} = \chi(\nu_{\mathcal{X}}^{\mathrm{mot}}) \in \mathrm{CF}(\mathcal{X}) ,$$

where χ denotes taking the pointwise Euler characteristic, as in §5.2.7.

In fact, we can define $\nu_{\mathcal{X}}$ for any stack $\mathcal{X}$ satisfying the assumptions in §3.2.2 in this way, without the local condition or the orientability assumption. Indeed, we may define $\nu_{\mathcal{X}}$ by the property that

$$f^*(\nu_{\mathcal{X}}) = (-1)^d \cdot \nu_Y$$

for any smooth morphism $f \colon Y \to \mathcal{X}$ of relative dimension d such that Y is a K-scheme and the pullback d-critical structure is orientable, where $\nu_Y = \chi(\nu_Y^{\mathrm{mot}})$. The function ν_Y does not depend on the choice of orientation, because changing the orientation only affects the term $\Upsilon(\ldots)$ in (6.2.5.1), which always has Euler characteristic 1. Given two such smooth morphisms, we may pass to the fibre product and use (6.2.5.2) to conclude that the values agree on overlaps.

When $K = \mathbb{C}$, the Behrend function $\nu_{\mathcal{X}}$ agrees with the original definitions by Behrend [10] and Joyce and Song [89, §4.1]. This follows from the compatibility of both versions with smooth pullbacks, namely Theorem 6.2.7 and [89, Theorem 4.3], and the case of critical loci on smooth varieties, which relies on an analytic argument, and follows from Denef and Loeser [50, Theorem 3.10] and Joyce and Song [89, Theorem 4.7].

6.3 Motivic Donaldson–Thomas invariants

6.3.1. Motivic Donaldson–Thomas invariants. Let $\mathcal{X}$ be a (-1)-shifted symplectic linear stack over K, equipped with an orientation data as in §3.7.4. Assume that its classical truncation $\mathcal{X}_{\mathrm{cl}}$ is Nisnevich locally a quotient stack, as in §6.2.2.

For a permissible stability condition τ on $\mathcal{X}$, and a class $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$, following the construction of Kontsevich and Soibelman [99], define the *motivic Donaldson–Thomas invari-*

ant $\mathrm{DT}^{\mathrm{mot}}_{\alpha}(\tau) \in \widehat{\mathbb{M}}^{\mathrm{mot}}(K;\mathbb{Q})$ by the formula

$$\mathrm{DT}^{\mathrm{mot}}_{\alpha}(\tau) = \int_{\mathcal{X}_\alpha} (\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}) \cdot \epsilon_\alpha(\tau) \cdot \nu^{\mathrm{mot}}_{\mathcal{X}} \,, \tag{6.3.1.1}$$

where $\nu^{\mathrm{mot}}_{\mathcal{X}}$ is the motivic Behrend function of $\mathcal{X}$ defined in §3.6.8.

Now, suppose further that $\mathcal{X}$ is equipped with a self-dual structure as in §3.7.5, together with a self-dual orientation data.

For a self-dual permissible stability condition τ and a class $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, define the *self-dual motivic Donaldson–Thomas invariant* $\mathrm{DT}^{\mathrm{mot,sd}}_{\theta}(\tau) \in \widehat{\mathbb{M}}^{\mathrm{mot}}(K;\mathbb{Q})$ by

$$\mathrm{DT}^{\mathrm{mot,sd}}_{\theta}(\tau) = \int_{\mathcal{X}^{\mathrm{sd}}_\theta} \epsilon^{\mathrm{sd}}_\theta(\tau) \cdot \nu^{\mathrm{mot}}_{\mathcal{X}^{\mathrm{sd}}} \,. \tag{6.3.1.2}$$

This is the main construction of this chapter.

6.3.2. For smooth stacks. Let $\mathcal{X}$ be a linear stack which is smooth and Nisnevich locally a quotient stack, and consider its (-1)-shifted cotangent stack $\mathrm{T}^*[-1]\,\mathcal{X}$, as in §5.6.5. It has a canonical (-1)-shifted symplectic linear structure and orientation data, and in the self-dual case, also a canonical self-dual orientation data.

The motivic Behrend function of $\mathcal{X}$ is $\nu^{\mathrm{mot}}_{\mathcal{X}} = \mathbb{L}^{-\dim \mathcal{X}/2}$ by Theorem 6.2.7, where $\dim \mathcal{X}$ refers to the dimension of the classical smooth stack $\mathcal{X}$. The formulae (6.3.1.1)–(6.3.1.2) can be simplified to

$$\mathrm{DT}^{\mathrm{mot}}_{\alpha}(\tau) = \frac{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}}{\mathbb{L}^{\dim \mathcal{X}_\alpha/2}} \cdot \int_{\mathcal{X}_\alpha} \epsilon_\alpha(\tau) \,, \tag{6.3.2.1}$$

$$\mathrm{DT}^{\mathrm{mot,sd}}_{\theta}(\tau) = \mathbb{L}^{-\dim \mathcal{X}^{\mathrm{sd}}_\theta/2} \cdot \int_{\mathcal{X}^{\mathrm{sd}}_\theta} \epsilon^{\mathrm{sd}}_\theta(\tau) \,. \tag{6.3.2.2}$$

Chapter 7

Wall-crossing formulae

This chapter discusses *wall-crossing formulae* for our orthosymplectic Donaldson–Thomas invariants defined in Chapters 5 and 6, which are formulae that characterize the change of these invariants when we change the stability condition. These formulae are an important feature of the invariants, and impose a strong constraint on the structure of the invariants, as we mentioned in §1.4.7.

We first prove wall-crossing formulae for epsilon motives in Theorem 7.1.3, which we then use in §7.3 to obtain wall-crossing formulae for Donaldson–Thomas invariants. Finally, in §7.5, we prove wall-crossing formulae for Donaldson–Thomas invariants when changing Bridgeland stability conditions in the derived category.

7.1 Wall-crossing for epsilon motives

7.1.1. Throughout, let $\mathcal{X}$ be a self-dual linear stack with quasi-compact filtrations as in §3.3.4. Results in the linear case will not need the self-dual structure on $\mathcal{X}$, and we will indicate this when it is the case.

7.1.2. Dominance of stability conditions. For stability conditions τ_0, τ on $\mathcal{X}$, following Joyce [83, Definition 4.10], we say that τ_0 *dominates* τ, if $\tau(\alpha_1) \leqslant \tau(\alpha_2)$ implies $\tau_0(\alpha_1) \leqslant \tau_0(\alpha_2)$ for all $\alpha_1, \alpha_2 \in \pi_0(\mathcal{X}) \setminus \{0\}$.

In this case, the Θ-stratification of $\mathcal{X}$ given by τ refines the one given by τ_0, and in particular, we have $\mathcal{X}_\alpha^{\mathrm{ss}}(\tau) \subset \mathcal{X}_\alpha^{\mathrm{ss}}(\tau_0)$ for all $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$.

For example, every stability condition is dominated by the trivial stability condition.

7.1.3. Theorem. *Let τ_+, τ_-, τ_0 be permissible self-dual stability conditions on $\mathcal{X}$, with τ_0 dominating both τ_+ and τ_-. Then for any $\alpha \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, we have the relations*

$$[\mathcal{X}^{\mathrm{ss}}_\alpha(\tau_-)] = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n}} S(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau_+)] * \cdots * [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau_+)] \, , \tag{7.1.3.1}$$

$$[\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau_-)] = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho}} S^{\mathrm{sd}}(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau_+)] \diamond \cdots \diamond [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau_+)] \diamond [\mathcal{X}^{\mathrm{sd,ss}}_\rho(\tau_+)] \, , \tag{7.1.3.2}$$

$$\epsilon_\alpha(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n}} U(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot \epsilon_{\alpha_1}(\tau_+) * \cdots * \epsilon_{\alpha_n}(\tau_+) \, , \tag{7.1.3.3}$$

$$\epsilon^{\mathrm{sd}}_\theta(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho}} U^{\mathrm{sd}}(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot \epsilon_{\alpha_1}(\tau_+) \diamond \cdots \diamond \epsilon_{\alpha_n}(\tau_+) \diamond \epsilon^{\mathrm{sd}}_\rho(\tau_+) \, , \tag{7.1.3.4}$$

in $\mathbb{M}(\mathcal{X}_\alpha; \mathbb{Q})$ and $\mathbb{M}(\mathcal{X}^{\mathrm{sd}}_\theta; \mathbb{Q})$, where the sums are finite, and

$$S(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) = \prod_{i=1}^{n-1} \left\{ \begin{array}{ll} 1, & \tau_+(\alpha_i) > \tau_+(\alpha_{i+1}) \text{ and} \\ & \qquad \tau_-(\alpha_1 + \cdots + \alpha_i) \leqslant \tau_-(\alpha_{i+1} + \cdots + \alpha_n) \\ -1, & \tau_+(\alpha_i) \leqslant \tau_+(\alpha_{i+1}) \text{ and} \\ & \qquad \tau_-(\alpha_1 + \cdots + \alpha_i) > \tau_-(\alpha_{i+1} + \cdots + \alpha_n) \\ 0, & \text{otherwise} \end{array} \right\} , \tag{7.1.3.5}$$

$$S^{\mathrm{sd}}(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) = \prod_{i=1}^{n} \left\{ \begin{array}{ll} 1, & \tau_+(\alpha_i) > \tau_+(\alpha_{i+1}) \text{ and } \tau_-(\alpha_1 + \cdots + \alpha_i) \leqslant 0 \\ -1, & \tau_+(\alpha_i) \leqslant \tau_+(\alpha_{i+1}) \text{ and } \tau_-(\alpha_1 + \cdots + \alpha_i) > 0 \\ 0, & \text{otherwise} \end{array} \right\} , \tag{7.1.3.6}$$

$$U(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) = \sum_{\substack{0 = a_0 < \cdots < a_m = n,\ 0 = b_0 < \cdots < b_\ell = m: \\ \text{Writing } \beta_i = \alpha_{a_{i-1}+1} + \cdots + \alpha_{a_i} \text{ for } i = 1, \dots, m, \\ \text{and } \gamma_i = \beta_{b_{i-1}+1} + \cdots + \beta_{b_i} \text{ for } i = 1, \dots, \ell, \\ \text{we have } \tau_+(\alpha_j) = \tau_+(\beta_i) \text{ for all } a_{i-1} < j \leqslant a_i, \\ \text{and } \tau_-(\gamma_i) = \tau_-(\alpha_1 + \cdots + \alpha_n) \text{ for all } i = 1, \dots, \ell}} \frac{(-1)^{\ell-1}}{\ell} \cdot \biggl(\prod_{i=1}^{\ell} S(\beta_{b_{i-1}+1}, \dots, \beta_{b_i}; \tau_+, \tau_-) \biggr) \cdot \biggl(\prod_{i=1}^{m} \frac{1}{(a_i - a_{i-1})!} \biggr) , \tag{7.1.3.7}$$

$$U^{\mathrm{sd}}(\alpha_1,\dots,\alpha_n;\tau_+,\tau_-) = \sum_{\substack{0=a_0<\cdots<a_m\leqslant n,\ 0=b_0<\cdots<b_\ell\leqslant m:\\ \text{Writing } \beta_i=\alpha_{a_{i-1}+1}+\cdots+\alpha_{a_i} \text{ for } i=1,\dots,m,\\ \text{and } \gamma_i=\beta_{b_{i-1}+1}+\cdots+\beta_{b_i} \text{ for } i=1,\dots,\ell,\\ \text{we have } \tau_+(\alpha_j)=\tau_+(\beta_i) \text{ for all } a_{i-1}<j\leqslant a_i,\\ \tau_+(\alpha_j)=0 \text{ for all } j>a_m,\\ \text{and } \tau_-(\gamma_i)=0 \text{ for all } i=1,\dots,\ell}} \binom{-1/2}{\ell}\cdot\Bigl(\prod_{i=1}^{\ell} S(\beta_{b_{i-1}+1},\dots,\beta_{b_i};\tau_+,\tau_-)\Bigr)\cdot S^{\mathrm{sd}}(\beta_{b_\ell+1},\dots,\beta_m;\tau_+,\tau_-)\cdot \Bigl(\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}\Bigr)\cdot\frac{1}{2^{n-a_m}\,(n-a_m)!}\,, \tag{7.1.3.8}$$

where we set $\tau_+(\alpha_{n+1})=0$ *in* (7.1.3.6).

For (7.1.3.1) *and* (7.1.3.3)*, we do not need* $\mathcal{X}$ *or* τ_+,τ_-,τ_0 *to be self-dual.*

The formulae (7.1.3.1) and (7.1.3.3) were originally due to Joyce [85, Theorem 5.2], under a slightly different setting. The self-dual versions (7.1.3.2) and (7.1.3.4) are new.

The coefficients (7.1.3.5)–(7.1.3.8) are combinatorial, and are defined whenever $\tau_\pm$ are maps from the set $C=\{\alpha_i+\cdots+\alpha_j \mid 1\leqslant i\leqslant j\leqslant n\}$ of symbolic sums to totally ordered sets $T_\pm$, such that $\tau_+(\gamma_1)\leqslant\tau_+(\gamma_2)$ implies $\tau_+(\gamma_1)\leqslant\tau_+(\gamma_1+\gamma_2)\leqslant\tau_+(\gamma_2)$ whenever $\gamma_1,\gamma_2,\gamma_1+\gamma_2\in C$, and similarly for τ_-. For (7.1.3.6) and (7.1.3.8), we also require distinguished elements $0\in T_\pm$.

Proof. The Θ-stratifications of $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau_0)$ and $\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau_0)$ defined by τ_+ and τ_- give the relations

$$[\mathcal{X}^{\mathrm{ss}}_\alpha(\tau_0)] = \sum_{\substack{n>0;\ \alpha_1,\dots,\alpha_n\in\pi_0(\mathcal{X})\setminus\{0\}:\\ \alpha=\alpha_1+\cdots+\alpha_n,\\ \tau_0(\alpha_1)=\cdots=\tau_0(\alpha_n),\\ \tau_\pm(\alpha_1)>\cdots>\tau_\pm(\alpha_n)}} [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau_\pm)]*\cdots*[\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau_\pm)]\,, \tag{7.1.3.9}$$

$$[\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau_0)] = \sum_{\substack{n\geqslant 0;\ \alpha_1,\dots,\alpha_n\in\pi_0(\mathcal{X})\setminus\{0\},\ \rho\in\pi_0(\mathcal{X}^{\mathrm{sd}}):\\ \theta=\alpha_1+\alpha_1^\vee+\cdots+\alpha_n+\alpha_n^\vee+\rho,\\ \tau_0(\alpha_1)=\cdots=\tau_0(\alpha_n)=0,\\ \tau_\pm(\alpha_1)>\cdots>\tau_\pm(\alpha_n)>0}} [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau_\pm)]\diamond\cdots\diamond[\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau_\pm)]\diamond[\mathcal{X}^{\mathrm{sd,ss}}_{\rho}(\tau_\pm)]\,, \tag{7.1.3.10}$$

where the '$\pm$' signs mean that we have a relation for τ_+, and another for τ_-. These are finite sums by Lemma 3.5.8, and agree with (7.1.3.1)–(7.1.3.2) with $\tau_\pm,\tau_0$ in place of τ_+,τ_-.

These relations then imply the relations

$$[\mathcal{X}^{\mathrm{ss}}_\alpha(\tau_\pm)] = \sum_{\substack{n>0;\ \alpha_1,\dots,\alpha_n\in\pi_0(\mathcal{X})\setminus\{0\}:\\ \alpha=\alpha_1+\cdots+\alpha_n,\\ \tau_0(\alpha_1)=\cdots=\tau_0(\alpha_n),\\ \tau_\pm(\alpha_1+\cdots+\alpha_i)>\tau_\pm(\alpha_{i+1}+\cdots+\alpha_n) \text{ for } i=1,\dots,n-1}} (-1)^{n-1}\cdot[\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau_0)]*\cdots*[\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau_0)]\,, \tag{7.1.3.11}$$

$$[\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau_\pm)] = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho, \\ \tau_0(\alpha_1) = \cdots = \tau_0(\alpha_n) = 0, \\ \tau_\pm(\alpha_1 + \cdots + \alpha_i) > 0 \text{ for } i = 1, \dots, n}} (-1)^n \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau_0)] \diamond \cdots \diamond [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau_0)] \diamond [\mathcal{X}^{\mathrm{sd,ss}}_\rho(\tau_0)] , \qquad (7.1.3.12)$$

which agree with (7.1.3.1)–(7.1.3.2) with $\tau_0, \tau_\pm$ in place of τ_+, τ_-. Indeed, these can be verified by expanding the right-hand sides of (7.1.3.11)–(7.1.3.12) using (7.1.3.9)–(7.1.3.10), then applying Lemma 7.1.4 below to see that the results are equal to the left-hand sides.

Now, expanding the right-hand sides of (7.1.3.11)–(7.1.3.12) for τ_+ using (7.1.3.9)–(7.1.3.10) for τ_-, then applying Lemma 7.1.4 below, gives the general case of (7.1.3.1)–(7.1.3.2).

To verify the relations (7.1.3.3)–(7.1.3.4), we first substitute the relations (7.1.3.1)–(7.1.3.2), in (5.5.2.1), (5.5.3.1) for τ_-, then substitute in (5.5.2.2), (5.5.3.2) for τ_+. Keeping track of the coefficients gives the desired relations. □

7.1.4. Lemma. *For symbols $\alpha_1, \dots, \alpha_n$ and maps τ_1, τ_2, τ_3 from $\{\alpha_i + \cdots + \alpha_j \mid 1 \leqslant i \leqslant j \leqslant n\}$ to totally ordered sets with distinguished elements 0, we have the identities*

$$S(\alpha_1, \dots, \alpha_n; \tau_1, \tau_3) = \sum_{(\beta_1,\dots,\beta_m) \in Q} S(\beta_1, \dots, \beta_m; \tau_2, \tau_3) \cdot \prod_{i=1}^m S(\alpha_{a_{i-1}+1}, \dots, \alpha_{a_i}; \tau_1, \tau_2) , \qquad (7.1.4.1)$$

$$\begin{aligned} S^{\mathrm{sd}}(\alpha_1, \dots, \alpha_n; \tau_1, \tau_3) = \sum_{(\beta_1,\dots,\beta_m) \in Q^{\mathrm{sd}}} & S^{\mathrm{sd}}(\beta_1, \dots, \beta_m; \tau_2, \tau_3) \cdot \\ & \biggl(\prod_{i=1}^m S(\alpha_{a_{i-1}+1}, \dots, \alpha_{a_i}; \tau_1, \tau_2) \biggr) \cdot S^{\mathrm{sd}}(\alpha_{a_m+1}, \dots, \alpha_n; \tau_1, \tau_2) , \end{aligned} \qquad (7.1.4.2)$$

where

$$Q = \left\{ (\beta_1, \dots, \beta_m) \;\middle|\; \begin{array}{l} m \geqslant 1,\ 0 = a_0 < \cdots < a_m = n, \\ \beta_i = \alpha_{a_{i-1}+1} + \cdots + \alpha_{a_i} \text{ for all } i \end{array} \right\},$$

$$Q^{\mathrm{sd}} = \left\{ (\beta_1, \dots, \beta_m) \;\middle|\; \begin{array}{l} m \geqslant 0,\ 0 = a_0 < \cdots < a_m \leqslant n, \\ \beta_i = \alpha_{a_{i-1}+1} + \cdots + \alpha_{a_i} \text{ for all } i \end{array} \right\}.$$

Proof. The identity (7.1.4.1) was proved in Joyce [85, Theorem 4.5]. The identity (7.1.4.2) follows from (7.1.4.1) and the fact that $S^{\mathrm{sd}}(\alpha_1, \dots, \alpha_n; \tau_i, \tau_j) = S(\alpha_1, \dots, \alpha_n, \infty; \tau_i, \tau_j)$, where we set $\tau_i(\alpha_j + \cdots + \alpha_n + \infty) = 0$ for all i and all $1 \leqslant j \leqslant n + 1$. □

7.1.5. Weakening the assumptions. In Theorem 7.1.3, we can slightly weaken the assumptions by allowing τ_0 to be non-permissible, so that $\mathcal{X}^{\mathrm{ss}}_\alpha(\tau_0)$ can be non-quasi-compact, and we add the extra assumption that the sums (7.1.3.11)–(7.1.3.12) are locally finite for all classes α, θ. In this case, the relations (7.1.3.9)–(7.1.3.10) are always valid as locally finite sums, and the proof shows that the relations (7.1.3.1)–(7.1.3.4) still hold as locally finite sums.

7.2 An anti-symmetric version

7.2.1. In this section, we rewrite the relations (7.1.3.3)–(7.1.3.4) in terms of anti-symmetric product operations, instead of the operations $*$ and $\diamond$. This will be useful in writing down wall-crossing formulae for Donaldson–Thomas invariants in §7.3 below.

As in §7.1.1, let $\mathcal{X}$ be a self-dual linear stack with quasi-compact filtrations.

7.2.2. Lie algebras and twisted modules. The motivic Hall algebra $\mathbb{M}(\mathcal{X})$ can be seen as a Lie algebra using the commutator

$$[a, b] = a * b - b * a \,. \tag{7.2.2.1}$$

This was considered in Joyce [82, §5.2]. It is equipped with a contravariant involution $(-)^\vee$, meaning that $[a^\vee, b^\vee] = [b, a]^\vee$ for $a, b \in \mathbb{M}(\mathcal{X})$, which follows from Theorem 5.4.4.

We define a similar anti-symmetric operation $\heartsuit \colon \mathbb{M}(\mathcal{X}) \otimes \mathbb{M}(\mathcal{X}^{\mathrm{sd}}) \to \mathbb{M}(\mathcal{X}^{\mathrm{sd}})$ by

$$a \heartsuit m = a \diamond m - a^\vee \diamond m \,, \tag{7.2.2.2}$$

where $\diamond$ is the multiplication in the motivic Hall module. This does not define a Lie algebra module, but a *twisted module*, in that it satisfies the relations

$$a \heartsuit m = -a^\vee \heartsuit m \,, \tag{7.2.2.3}$$

$$a \heartsuit (b \heartsuit m) - b \heartsuit (a \heartsuit m) = [a, b] \heartsuit m - [a^\vee, b] \heartsuit m \,. \tag{7.2.2.4}$$

We see (7.2.2.4) as a Jacobi identity twisted by the contravariant involution of the Lie algebra, giving the extra term $[a^\vee, b] \heartsuit m$.

Note that over $\mathbb{Q}$, a twisted module in this sense is equivalent to a usual module for the Lie subalgebra consisting of elements a with $a^\vee = -a$, with the action $a \cdot m = (1/2)(a \heartsuit m)$.

7.2.3. Theorem. *The relations (7.1.3.3)–(7.1.3.4) can be written only using the Lie bracket $[-, -]$ and the operation $\heartsuit$, without using the products $*$ or $\diamond$.*

More precisely, using the notations of Theorem 7.1.3, we have the relations

$$\epsilon_\alpha(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n}} \tilde{U}(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot \big[\dots \big[\big[\epsilon_{\alpha_1}(\tau_+), \epsilon_{\alpha_2}(\tau_+) \big], \dots \big], \epsilon_{\alpha_n}(\tau_+) \big] , \tag{7.2.3.1}$$

$$\epsilon^{\mathrm{sd}}_\theta(\tau_-) = \sum_{\substack{n \geqslant 0;\ m_1, \dots, m_n > 0; \\ \alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots ; \alpha_{n,1}, \dots, \alpha_{n,m_n} \in \pi_0(\mathcal{X}) \setminus \{0\};\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = (\alpha_{1,1} + \alpha_{1,1}^\vee + \cdots + \alpha_{1,m_1} + \alpha_{1,m_1}^\vee) + \cdots + (\alpha_{n,1} + \alpha_{n,1}^\vee + \cdots + \alpha_{n,m_n} + \alpha_{n,m_n}^\vee) + \rho}} \tilde{U}^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots ; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \tau_+, \tau_-) \cdot \tag{7.2.3.2}$$

$$\big[\big[\epsilon_{\alpha_{1,1}}(\tau_+), \dots \big], \epsilon_{\alpha_{1,m_1}}(\tau_+) \big] \mathbin{\heartsuit} \cdots \mathbin{\heartsuit} \big[\big[\epsilon_{\alpha_{n,1}}(\tau_+), \dots \big], \epsilon_{\alpha_{n,m_n}}(\tau_+) \big] \mathbin{\heartsuit} \epsilon^{\mathrm{sd}}_\rho(\tau_+) ,$$

where $\tilde{U}(\dots)$ and $\tilde{U}^{\mathrm{sd}}(\dots)$ are certain combinatorial coefficients, whose choices are not unique.

Here, the formulae (7.2.3.1)–(7.2.3.2) are just (7.1.3.3)–(7.1.3.4) with the terms grouped differently, and this theorem is essentially a combinatorial property of the coefficients $U(\dots)$ and $U^{\mathrm{sd}}(\dots)$ stating that such regrouping is always possible. The non-uniqueness of the coefficients is due to relations in the Lie brackets and the twisted module operation, such as the Jacobi identity and (7.2.2.3)–(7.2.2.4).

The linear case (7.2.3.1) was shown in Joyce [85, Theorem 5.4]. The self-dual case (7.2.3.2) is a main result of this thesis, and its proof will be given in Appendix C.

7.3 Wall-crossing for Donaldson–Thomas invariants

7.3.1. In this section, we prove wall-crossing formulae for our self-dual Donaldson–Thomas invariants defined in §5.6 and §6.3, using the wall-crossing formulae for epsilon motives established in Theorems 7.1.3 and 7.2.3. A key ingredient is the *motivic integral identity* for Behrend functions, which we discuss in §7.4 below, generalizing the integral identities in the linear case of Kontsevich and Soibelman [99, Conjecture 4], proved by Lê [104], and Joyce and Song [89, Theorem 5.11].

Throughout, let K be an algebraically closed field of characteristic 0, and let $\mathcal{X}$ be a self-dual (-1)-shifted symplectic linear stack over K, as in §3.7.2. We further assume that the classical truncation $\mathcal{X}_{\mathrm{cl}}$ is Nisnevich locally fundamental, as in §6.2.2.

7.3.2. Theorem. *Let τ_+, τ_-, τ_0 be permissible self-dual stability conditions on $\mathcal{X}$, with τ_0 domin-*

ating both τ_+ and τ_-. Then for any $\alpha \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, we have the wall-crossing formulae

$$\mathrm{DT}_\alpha(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n}} \tilde{U}(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot \ell(\alpha_1, \dots, \alpha_n) \cdot \mathrm{DT}_{\alpha_1}(\tau_+) \cdots \mathrm{DT}_{\alpha_n}(\tau_+) , \tag{7.3.2.1}$$

$$\begin{aligned} \mathrm{DT}^{\mathrm{sd}}_\theta(\tau_-) = \sum_{\substack{n \geqslant 0;\ m_1, \dots, m_n > 0; \\ \alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n} \in \pi_0(\mathcal{X}) \setminus \{0\};\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = (\alpha_{1,1} + \alpha_{1,1}^\vee + \cdots + \alpha_{1,m_1} + \alpha_{1,m_1}^\vee) + \cdots + (\alpha_{n,1} + \alpha_{n,1}^\vee + \cdots + \alpha_{n,m_n} + \alpha_{n,m_n}^\vee) + \rho}} & \tilde{U}^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \tau_+, \tau_-) \cdot \\ & \ell^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \rho) \cdot \\ & \big(\mathrm{DT}_{\alpha_{1,1}}(\tau_+) \cdots \mathrm{DT}_{\alpha_{1,m_1}}(\tau_+)\big) \cdots \big(\mathrm{DT}_{\alpha_{n,1}}(\tau_+) \cdots \mathrm{DT}_{\alpha_{n,m_n}}(\tau_+)\big) \cdot \mathrm{DT}^{\mathrm{sd}}_\rho(\tau_+) , \end{aligned} \tag{7.3.2.2}$$

where the sums contain finitely many non-zero terms, the coefficients $\tilde{U}(\dots), \tilde{U}^{\mathrm{sd}}(\dots) \in \mathbb{Q}$ are defined in Theorem 7.2.3, and the coefficients $\ell(\dots), \ell^{\mathrm{sd}}(\dots) \in \mathbb{Z}$ are defined in §7.3.6 below.

If, moreover, $\mathcal{X}$ is equipped with an orientation data $o_{\mathcal{X}}$ or a self-dual orientation data $(o_{\mathcal{X}}, o_{\mathcal{X}^{\mathrm{sd}}})$, then we have the wall-crossing formulae

$$\mathrm{DT}^{\mathrm{mot}}_\alpha(\tau_-) = \sum_{\substack{n \geqslant 0;\ \alpha_1, \dots, \alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n}} \tilde{U}(\alpha_1, \dots, \alpha_n; \tau_+, \tau_-) \cdot L(\alpha_1, \dots, \alpha_n) \cdot \mathrm{DT}^{\mathrm{mot}}_{\alpha_1}(\tau_+) \cdots \mathrm{DT}^{\mathrm{mot}}_{\alpha_n}(\tau_+) , \tag{7.3.2.3}$$

$$\begin{aligned} \mathrm{DT}^{\mathrm{mot,sd}}_\theta(\tau_-) = \sum_{\substack{n \geqslant 0;\ m_1, \dots, m_n > 0; \\ \alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n} \in \pi_0(\mathcal{X}) \setminus \{0\};\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = (\alpha_{1,1} + \alpha_{1,1}^\vee + \cdots + \alpha_{1,m_1} + \alpha_{1,m_1}^\vee) + \cdots + (\alpha_{n,1} + \alpha_{n,1}^\vee + \cdots + \alpha_{n,m_n} + \alpha_{n,m_n}^\vee) + \rho}} & \tilde{U}^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \tau_+, \tau_-) \cdot \\ & L^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \rho) \cdot \\ & \big(\mathrm{DT}^{\mathrm{mot}}_{\alpha_{1,1}}(\tau_+) \cdots \mathrm{DT}^{\mathrm{mot}}_{\alpha_{1,m_1}}(\tau_+)\big) \cdots \big(\mathrm{DT}^{\mathrm{mot}}_{\alpha_{n,1}}(\tau_+) \cdots \mathrm{DT}^{\mathrm{mot}}_{\alpha_{n,m_n}}(\tau_+)\big) \cdot \mathrm{DT}^{\mathrm{mot,sd}}_\rho(\tau_+) , \end{aligned} \tag{7.3.2.4}$$

respectively, where the coefficients $L(\dots), L^{\mathrm{sd}}(\dots) \in \mathbb{Z}[\mathbb{L}^{\pm 1/2}]$ are defined in §7.3.6 below.

The theorem will be proved in §7.4.7, using various integral identities that we introduce in the next section.

7.3.3. Symmetric stacks. The wall-crossing formulae in Theorem 7.3.2 provide a condition for the Donaldson–Thomas invariants to be independent of the choice of the stability condition.

We say that a (-1)-shifted symplectic stack $\mathcal{X}$ is *numerically symmetric*, if every connected component of $\mathscr{F}ilt(\mathscr{X})$ has virtual dimension 0. See the author et al. [30, §4.3] for examples of stacks satisfying this condition.

For example, if $\mathscr{X}$ is a self-dual (-1)-shifted symplectic linear stack, then $\mathscr{X}$ is numerically symmetric if and only if $\operatorname{vdim} \mathscr{X}^{+}_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \pi_0(\mathscr{X})$, and $\mathscr{X}^{\mathrm{sd}}$ is numerically symmetric if and only if $\operatorname{vdim} \mathscr{X}^{\mathrm{sd},+}_{\alpha,\theta} = 0$ for all $\alpha \in \pi_0(\mathscr{X})$ and $\theta \in \pi_0(\mathscr{X}^{\mathrm{sd}})$.

When $\mathscr{X}$ and $\mathscr{X}^{\mathrm{sd}}$ are numerically symmetric, the coefficients $L(\ldots), \ell(\ldots)$ are zero unless $n \leqslant 1$, and the coefficients $L^{\mathrm{sd}}(\ldots), \ell^{\mathrm{sd}}(\ldots)$ are zero unless $n = 0$, which follow from their definitions. This immediately implies the following:

7.3.4. Corollary. *In the situation of Theorem 7.3.2, assume that $\mathscr{X}$ and $\mathscr{X}^{\mathrm{sd}}$ are numerically symmetric. Then the relations (7.3.2.1)–(7.3.2.4) simplify to*

$$\mathrm{DT}_\alpha(\tau_-) = \mathrm{DT}_\alpha(\tau_+)\,, \qquad \mathrm{DT}^{\mathrm{sd}}_\theta(\tau_-) = \mathrm{DT}^{\mathrm{sd}}_\theta(\tau_+)\,, \tag{7.3.4.1}$$

$$\mathrm{DT}^{\mathrm{mot}}_\alpha(\tau_-) = \mathrm{DT}^{\mathrm{mot}}_\alpha(\tau_+)\,, \qquad \mathrm{DT}^{\mathrm{mot,sd}}_\theta(\tau_-) = \mathrm{DT}^{\mathrm{mot,sd}}_\theta(\tau_+)\,. \tag{7.3.4.2}$$

In particular, if $\mathscr{X}$ has quasi-compact connected components, then all the above invariants are independent of the choice of the stability condition.

Here, the final claim follows from taking τ_0 and τ_+ to be the trivial stability condition, which is permissible when $\mathscr{X}$ has quasi-compact connected components.

The remaining part of this section is devoted to the proof of Theorem 7.3.2.

7.3.5. Lattice algebras and modules. Define

$$\Lambda_{\mathscr{X}} = \bigoplus_{\alpha \in \pi_0(\mathscr{X})} \hat{\mathbb{M}}^{\hat{\mu}}(K;\mathbb{Q}) \cdot \lambda_\alpha\,, \qquad \Lambda^{\mathrm{sd}}_{\mathscr{X}} = \bigoplus_{\theta \in \pi_0(\mathscr{X}^{\mathrm{sd}})} \hat{\mathbb{M}}^{\hat{\mu}}(K;\mathbb{Q}) \cdot \lambda^{\mathrm{sd}}_\theta\,,$$

where $\hat{\mathbb{M}}^{\hat{\mu}}(K;\mathbb{Q})$ is the ring of monodromic motives defined in §6.1.2. We define a product $*$ on $\Lambda_{\mathscr{X}}$, and a $\Lambda_{\mathscr{X}}$-module structure $\diamond$ on $\Lambda^{\mathrm{sd}}_{\mathscr{X}}$, by setting

$$\lambda_\alpha * \lambda_\beta = \frac{\mathbb{L}^{\operatorname{vdim} \mathscr{X}^{+}_{\alpha,\beta}/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \cdot \lambda_{\alpha+\beta}\,, \qquad \lambda_\alpha \diamond \lambda^{\mathrm{sd}}_\theta = \frac{\mathbb{L}^{\operatorname{vdim} \mathscr{X}^{\mathrm{sd},+}_{\alpha,\theta}/2}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} \cdot \lambda^{\mathrm{sd}}_{\alpha+\theta+\alpha^\vee} \tag{7.3.5.1}$$

for $\alpha, \beta \in \pi_0(\mathscr{X})$ and $\theta \in \pi_0(\mathscr{X}^{\mathrm{sd}})$. The associativity of these operations follow from the relations

$$\operatorname{vdim} \mathcal{X}^+_{\alpha,\beta} + \operatorname{vdim} \mathcal{X}^+_{\alpha+\beta,\gamma} = \operatorname{vdim} \mathcal{X}^+_{\alpha,\beta,\gamma} = \operatorname{vdim} \mathcal{X}^+_{\alpha,\beta+\gamma} + \operatorname{vdim} \mathcal{X}^+_{\beta,\gamma} \ ,$$
$$\operatorname{vdim} \mathcal{X}^+_{\alpha,\beta} + \operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha+\beta,\theta} = \operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha,\beta,\theta} = \operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha,\beta+\theta+\beta^\vee} + \operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\beta,\theta} \ ,$$

which follow from the derived versions of the associativity diagrams (5.4.4.6) and (5.4.4.8). The algebra $\Lambda_{\mathcal{X}}$ is often called the *quantum torus* in the literature, such as in Kontsevich and Soibelman [99, §6.2].

The map $\lambda_\alpha \mapsto \lambda_{\alpha^\vee}$ defines a contravariant involution $(-)^\vee$ of $\Lambda_{\mathcal{X}}$. We also write $a \heartsuit m = a \diamond m - a^\vee \diamond m$ for $a \in \Lambda_{\mathcal{X}}$ and $m \in \Lambda^{\mathrm{sd}}_{\mathcal{X}}$, as in §7.2.2, which gives $\Lambda^{\mathrm{sd}}_{\mathcal{X}}$ the structure of a twisted module over the involutive Lie algebra $\Lambda_{\mathcal{X}}$, with the commutator Lie bracket.

We also define the numerical versions

$$\bar{\Lambda}_{\mathcal{X}} = \bigoplus_{\alpha \in \pi_0(\mathcal{X})} \mathbb{Q} \cdot \bar{\lambda}_\alpha \ , \qquad \bar{\Lambda}^{\mathrm{sd}}_{\mathcal{X}} = \bigoplus_{\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})} \mathbb{Q} \cdot \bar{\lambda}^{\mathrm{sd}}_\theta \ ,$$

which are no longer equipped with algebra structures, but have a Lie bracket and a twisted module operation $\heartsuit$, respectively, given by

$$[\bar{\lambda}_\alpha, \bar{\lambda}_\beta] = (-1)^{1+\operatorname{vdim} \mathcal{X}^+_{\alpha,\beta}} \cdot \operatorname{vdim} \mathcal{X}^+_{\alpha,\beta} \cdot \bar{\lambda}_{\alpha+\beta} \ , \tag{7.3.5.2}$$
$$\bar{\lambda}_\alpha \heartsuit \bar{\lambda}^{\mathrm{sd}}_\theta = (-1)^{1+\operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha,\theta}} \cdot \operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha,\theta} \cdot \bar{\lambda}^{\mathrm{sd}}_{\alpha+\theta+\alpha^\vee} \ . \tag{7.3.5.3}$$

By Lemma B.3.4 below, we have $\operatorname{vdim} \mathcal{X}^+_{\beta,\alpha} = -\operatorname{vdim} \mathcal{X}^+_{\alpha,\beta}$ and $\operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha^\vee,\theta} = -\operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha,\theta}$, establishing (7.3.5.2) and (7.3.5.3) as limits of (7.3.5.1) as $\mathbb{L}^{1/2} \to -1$.

7.3.6. Coefficients. We can now define the coefficients $L(\ldots), L^{\mathrm{sd}}(\ldots)$, etc., which appear in (7.3.2.1).

For $\alpha_1, \ldots, \alpha_n \in \pi_0(\mathcal{X})$, we record the coefficients of the Lie brackets in $\Lambda_{\mathcal{X}}$ and $\bar{\Lambda}_{\mathcal{X}}$ as

$$[[\ldots[\lambda_{\alpha_1}, \lambda_{\alpha_2}], \ldots], \lambda_{\alpha_n}] = L(\alpha_1, \ldots, \alpha_n) \cdot \lambda_{\alpha_1+\cdots+\alpha_n} \ , \tag{7.3.6.1}$$
$$[[\ldots[\bar{\lambda}_{\alpha_1}, \bar{\lambda}_{\alpha_2}], \ldots], \bar{\lambda}_{\alpha_n}] = \ell(\alpha_1, \ldots, \alpha_n) \cdot \bar{\lambda}_{\alpha_1+\cdots+\alpha_n} \ , \tag{7.3.6.2}$$

where $L(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}[\mathbb{L}^{\pm 1/2}]$ and $\ell(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}$.

Similarly, for $\alpha_{1,1}, \ldots, \alpha_{1,m_1}; \ldots ; \alpha_{n,1}, \ldots, \alpha_{n,m_n} \in \pi_0(\mathcal{X})$ and $\rho \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, we also record the coefficients in

$$
\begin{aligned}
&[[\dots[\lambda_{\alpha_{1,1}}, \lambda_{\alpha_{1,2}}], \dots], \lambda_{\alpha_{1,m_1}}] \heartsuit \cdots \heartsuit [[\dots[\lambda_{\alpha_{n,1}}, \lambda_{\alpha_{n,2}}], \dots], \lambda_{\alpha_{n,m_n}}] \heartsuit \lambda^{\mathrm{sd}}_{\rho} \\
&\qquad = L^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \rho) \cdot \lambda^{\mathrm{sd}}_{\alpha_{1,1}+\alpha_{1,1}^\vee+\cdots+\alpha_{n,m_n}+\alpha_{n,m_n}^\vee+\rho}, \qquad (7.3.6.3)
\end{aligned}
$$

$$
\begin{aligned}
&[[\dots[\bar\lambda_{\alpha_{1,1}}, \bar\lambda_{\alpha_{1,2}}], \dots], \bar\lambda_{\alpha_{1,m_1}}] \heartsuit \cdots \heartsuit [[\dots[\bar\lambda_{\alpha_{n,1}}, \bar\lambda_{\alpha_{n,2}}], \dots], \bar\lambda_{\alpha_{n,m_n}}] \heartsuit \bar\lambda^{\mathrm{sd}}_{\rho} \\
&\qquad = \ell^{\mathrm{sd}}(\alpha_{1,1}, \dots, \alpha_{1,m_1}; \dots; \alpha_{n,1}, \dots, \alpha_{n,m_n}; \rho) \cdot \bar\lambda^{\mathrm{sd}}_{\alpha_{1,1}+\alpha_{1,1}^\vee+\cdots+\alpha_{n,m_n}+\alpha_{n,m_n}^\vee+\rho}, \qquad (7.3.6.4)
\end{aligned}
$$

where $L^{\mathrm{sd}}(\dots) \in \mathbb{Z}[\mathbb{L}^{\pm 1/2}]$ and $\ell^{\mathrm{sd}}(\dots) \in \mathbb{Z}$.

These coefficients only depend on the numbers $\operatorname{vdim} \mathcal{X}^+_{\alpha,\beta}$ and $\operatorname{vdim} \mathcal{X}^{\mathrm{sd},+}_{\alpha,\theta}$ for $\alpha, \beta \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$. They have straightforward explicit expressions, which we omit.

We have the relations $\ell(\dots) = L(\dots)|_{\mathbb{L}^{1/2}=-1}$ and $\ell^{\mathrm{sd}}(\dots) = L^{\mathrm{sd}}(\dots)|_{\mathbb{L}^{1/2}=-1}$. Also, $L(\dots)$ and $L^{\mathrm{sd}}(\dots)$ are symmetric Laurent polynomials in $\mathbb{L}^{1/2}$, in that they are invariant under the transformation $\mathbb{L}^{1/2} \mapsto \mathbb{L}^{-1/2}$.

7.4 Integral identities

7.4.1. The motivic integral identity. A crucial ingredient in proving wall-crossing formulae for Donaldson–Thomas invariants, Theorem 7.3.2, is the *motivic integral identity* for the motivic Behrend function, first conjectured by Kontsevich and Soibelman [99, Conjecture 4] in the linear case, and proved by Lê [104] in that case.

For our applications in the orthosymplectic case, however, we will need the following stronger and global version of the integral identity.

7.4.2. Theorem. *Let $\mathcal{X}$ be an oriented (-1)-shifted symplectic stack over K, such that its classical truncation is an algebraic stack that is Nisnevich locally fundamental in the sense of §6.2.2.*

Consider the attractor correspondence

$$
\mathcal{G}rad(\mathcal{X}) \xleftarrow{\mathrm{gr}} \mathcal{F}ilt(\mathcal{X}) \xrightarrow{\mathrm{ev}} \mathcal{X} \qquad (7.4.2.1)
$$

as in §3.2.3. Then we have the identity

$$
\mathrm{gr}_! \circ \mathrm{ev}^*(\nu^{\mathrm{mot}}_{\mathcal{X}}) = \mathbb{L}^{\operatorname{vdim} \mathcal{F}ilt(\mathcal{X})/2} \cdot \nu^{\mathrm{mot}}_{\mathcal{G}rad(\mathcal{X})} \qquad (7.4.2.2)
$$

in $\widehat{\mathbb{M}}^{\hat\mu}(\mathcal{G}rad(\mathcal{X}))$, where $\operatorname{vdim} \mathcal{F}ilt(\mathcal{X})$ is the virtual dimension of the derived stack ${}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})$, *seen as a function* $\pi_0(\mathscr{G}rad(\mathcal{X})) \simeq \pi_0(\mathscr{F}ilt(\mathcal{X})) \to \mathbb{Z}$.

Here, the statement of the theorem treats $\mathcal{X}$ as a classical stack via the classical truncation, except when taking the stack ${}^{\mathrm{d}}\mathscr{F}ilt(\mathcal{X})$. The proof of this theorem is deferred to Appendix B.

7.4.3. We will use Theorem 7.4.2 to prove Theorem 7.3.2 in the following way. Let $\mathcal{X}$ be as in Theorem 7.3.2, and suppose that we are given a self-dual orientation data $(o_{\mathcal{X}}, o_{\mathcal{X}^{\mathrm{sd}}})$ on $\mathcal{X}$. Then Theorem 7.4.2 implies that

$$\nu_{\mathcal{X}}^{\mathrm{mot}} \boxtimes \nu_{\mathcal{X}}^{\mathrm{mot}} = \mathbb{L}^{-\mathrm{vdim}\,\mathcal{X}_{\alpha,\beta}^{+}/2} \cdot \mathrm{gr}_! \circ \mathrm{ev}^*(\nu_{\mathcal{X}}^{\mathrm{mot}}) \quad \text{in } \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_\alpha \times \mathcal{X}_\beta)\,, \tag{7.4.3.1}$$

$$\nu_{\mathcal{X}}^{\mathrm{mot}} \boxtimes \nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}} = \mathbb{L}^{-\mathrm{vdim}\,\mathcal{X}_{\alpha,\theta}^{\mathrm{sd},+}/2} \cdot \mathrm{gr}_! \circ \mathrm{ev}^*(\nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}}) \quad \text{in } \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_\alpha \times \mathcal{X}_\theta^{\mathrm{sd}})\,, \tag{7.4.3.2}$$

where $\alpha, \beta \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, and the compositions are through $\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_{\alpha,\beta}^{+})$ and $\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_{\alpha,\theta}^{\mathrm{sd},+})$, respectively. These identities imply the relations

$$\biggl(\int_{\mathcal{X}_\alpha} a \cdot \nu_{\mathcal{X}}^{\mathrm{mot}}\biggr) \cdot \biggl(\int_{\mathcal{X}_\beta} b \cdot \nu_{\mathcal{X}}^{\mathrm{mot}}\biggr) = \mathbb{L}^{-\mathrm{vdim}\,\mathcal{X}_{\alpha,\beta}^{+}/2} \cdot \int_{\mathcal{X}_{\alpha+\beta}} (a * b) \cdot \nu_{\mathcal{X}}^{\mathrm{mot}}\,, \tag{7.4.3.3}$$

$$\biggl(\int_{\mathcal{X}_\alpha} a \cdot \nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}}\biggr) \cdot \biggl(\int_{\mathcal{X}_\theta^{\mathrm{sd}}} m \cdot \nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}}\biggr) = \mathbb{L}^{-\mathrm{vdim}\,\mathcal{X}_{\alpha,\theta}^{\mathrm{sd},+}/2} \cdot \int_{\mathcal{X}_{\alpha+\theta+\alpha^\vee}^{\mathrm{sd}}} (a \diamond m) \cdot \nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}}\,, \tag{7.4.3.4}$$

where $a \in \mathbb{M}_{\mathrm{qc}}(\mathcal{X}_\alpha; \mathbb{Q})$, $b \in \mathbb{M}_{\mathrm{qc}}(\mathcal{X}_\beta; \mathbb{Q})$, and $m \in \mathbb{M}_{\mathrm{qc}}(\mathcal{X}_\theta^{\mathrm{sd}}; \mathbb{Q})$, and the subscripts 'qc' indicate quasi-compactly supported motives. These follow from identifying both sides of each relation with the integrals

$$\mathbb{L}^{-\mathrm{vdim}\,\mathcal{X}_{\alpha,\beta}^{+}/2} \cdot \int_{\mathcal{X}_{\alpha,\beta}^{+}} \mathrm{gr}^*(a \boxtimes b) \cdot \mathrm{ev}^*(\nu_{\mathcal{X}}^{\mathrm{mot}})\,,$$

$$\mathbb{L}^{-\mathrm{vdim}\,\mathcal{X}_{\alpha,\theta}^{\mathrm{sd},+}/2} \cdot \int_{\mathcal{X}_{\alpha,\theta}^{\mathrm{sd},+}} \mathrm{gr}^*(a \boxtimes m) \cdot \mathrm{ev}^*(\nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}})\,,$$

respectively, using the projection formula (5.2.3.1). The relation (7.4.3.3) was first described by Kontsevich and Soibelman [99, Theorem 8].

7.4.4. The numeric integral identity. To prove the numeric wall-crossing formulae (7.3.2.1)–(7.3.2.2), we will need a corollary of Theorem 7.4.2, which is an integral identity for the numeric Behrend functions.

We first introduce a notation used in the statement of the corollary. For a graded point

$\gamma \in \mathscr{G}rad(\mathcal{X})(K)$, write

$$\mathbb{P}(\mathrm{gr}^{-1}(\gamma)) = \Big(*/\mathbb{G}_{\mathrm{m}} \underset{\mathscr{G}rad(\mathcal{X})}{\times} \mathscr{F}ilt(\mathcal{X}) \Big) \setminus \{\mathrm{sf}(\gamma)\} \, ,$$

where the map $*/\mathbb{G}_{\mathrm{m}} \to \mathscr{G}rad(\mathcal{X})$ is given by the tautological $\mathbb{G}_{\mathrm{m}}$-action on γ. The K-point $\mathrm{sf}(\gamma)$ is closed in the fibre product, and $\{\mathrm{sf}(\gamma)\}$ denotes the corresponding closed substack. The space $\mathbb{P}(\mathrm{gr}^{-1}(\gamma))$ can be seen as the projectivized space of filtrations of a given associated graded point.

7.4.5. Theorem. *Let $\mathcal{X}$ be an oriented (-1)-shifted symplectic stack over K, such that its classical truncation is an algebraic stack that is étale locally fundamental in the sense of §6.2.2.*

Let $\gamma \in \mathscr{G}rad(\mathcal{X})(K)$ be a graded point, and let $\bar{\gamma}$ be its opposite graded point, given by precomposing with the morphism $(-)^{-1} \colon */\mathbb{G}_{\mathrm{m}} \to */\mathbb{G}_{\mathrm{m}}$.

Then we have the identities

$$\nu_{\mathcal{X}}(\mathrm{tot}(\gamma)) = (-1)^{\mathrm{rank}^{[0,1]}(\mathbb{L}_{\mathscr{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\gamma)}) - \mathrm{rank}^{[0,1]}(\mathbb{L}_{\mathscr{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\bar{\gamma})})} \cdot \nu_{\mathscr{G}rad(\mathcal{X})}(\gamma) \, , \tag{7.4.5.1}$$

$$\begin{aligned} \int_{\varphi \in \mathbb{P}(\mathrm{gr}^{-1}(\gamma))} & \nu_{\mathcal{X}}(\mathrm{ev}(\varphi)) \, d\chi - \int_{\varphi \in \mathbb{P}(\mathrm{gr}^{-1}(\bar{\gamma}))} \nu_{\mathcal{X}}(\mathrm{ev}(\varphi)) \, d\chi \\ & = \Big(\dim \mathrm{H}^0(\mathbb{L}_{\mathscr{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\gamma)}) - \dim \mathrm{H}^0(\mathbb{L}_{\mathscr{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\bar{\gamma})}) \Big) \cdot \nu_{\mathcal{X}}(\mathrm{tot}(\gamma)) \, , \end{aligned} \tag{7.4.5.2}$$

where $\mathrm{rank}^{[0,1]} = \dim \mathrm{H}^0 - \dim \mathrm{H}^1$.

This theorem is a generalization of Joyce and Song [89, Theorem 5.11], who considered the case when $\mathcal{X}$ is the moduli stack of objects in a 3-Calabi–Yau abelian category. The proof is deferred to Appendix B.

7.4.6. In the setting of Theorem 7.3.2, we can also obtain from Theorem 7.4.5 numerical integral relations analogous to (7.4.3.3)–(7.4.3.4),

$$\begin{aligned} \int_{\mathcal{X}_{\alpha+\beta}} (1 - \mathbb{L}) \cdot [a, b] \cdot \nu_{\mathcal{X}} \, d\chi & = (-1)^{1 + \mathrm{vdim} \, \mathcal{X}^+_{\alpha,\beta}} \cdot \mathrm{vdim} \, \mathcal{X}^+_{\alpha,\beta} \cdot \\ & \bigg(\int_{\mathcal{X}_\alpha} (1 - \mathbb{L}) \cdot a \cdot \nu_{\mathcal{X}} \, d\chi \bigg) \cdot \bigg(\int_{\mathcal{X}_\beta} (1 - \mathbb{L}) \cdot b \cdot \nu_{\mathcal{X}} \, d\chi \bigg) \, , \end{aligned} \tag{7.4.6.1}$$

$$\begin{aligned} \int_{\mathcal{X}^{\mathrm{sd}}_{\alpha+\theta+\alpha^\vee}} (a \heartsuit m) \cdot \nu_{\mathcal{X}^{\mathrm{sd}}} \, d\chi & = (-1)^{1 + \mathrm{vdim} \, \mathcal{X}^{\mathrm{sd},+}_{\alpha,\theta}} \cdot \mathrm{vdim} \, \mathcal{X}^{\mathrm{sd},+}_{\alpha,\theta} \cdot \\ & \bigg(\int_{\mathcal{X}_\alpha} (1 - \mathbb{L}) \cdot a \cdot \nu_{\mathcal{X}} \, d\chi \bigg) \cdot \bigg(\int_{\mathcal{X}^{\mathrm{sd}}_\theta} m \cdot \nu_{\mathcal{X}^{\mathrm{sd}}} \, d\chi \bigg) \, , \end{aligned} \tag{7.4.6.2}$$

provided that the motives a, b, m are chosen so that the integrals on the right-hand sides are finite, that is, they lie in $\widehat{\mathbb{M}}^{\hat{\mu},\circ}(K; A)$ as in §6.1.3 before taking the Euler characteristics. These identities do not require orientations on $\mathcal{X}$ or $\mathcal{X}^{\mathrm{sd}}$. The identity (7.4.6.1) was proved by Joyce and Song [89, Theorem 5.14] in the setting of Calabi–Yau threefolds.

To prove them, we use a similar argument as in §7.4.1. Namely, for (7.4.6.1), we identify the left-hand side with

$$\chi\bigg((1-\mathbb{L})^2 \cdot \Big(-\int_{\mathbb{P}(\mathcal{X}^+_{\alpha,\beta})} \mathrm{gr}^*(a \boxtimes b) \cdot \mathrm{ev}^*(\nu_{\mathcal{X}}) + \int_{\mathbb{P}(\mathcal{X}^+_{\beta,\alpha})} \bar{\mathrm{gr}}^*(a \boxtimes b) \cdot \bar{\mathrm{ev}}^*(\nu_{\mathcal{X}})\Big)\bigg)$$
$$+ \int_{\mathcal{X}_\alpha \times \mathcal{X}_\beta} (1-\mathbb{L}) \cdot (a \boxtimes b) \cdot \big(\mathbb{L}^{-h^1(\mathbb{L}_{\mathrm{gr}})} - \mathbb{L}^{-h^1(\mathbb{L}_{\bar{\mathrm{gr}}})}\big) \cdot \oplus^*(\nu_{\mathcal{X}})\, d\chi\,, \tag{7.4.6.3}$$

where $\mathbb{P}(\mathcal{X}^+_{\alpha,\beta}) = (\mathcal{X}^+_{\alpha,\beta} \setminus \mathrm{sf}(\mathcal{X}_\alpha \times \mathcal{X}_\beta))/\mathbb{G}_{\mathrm{m}}$, with the $\mathbb{G}_{\mathrm{m}}$-action given by choosing an identification of $\mathcal{X}^+_{\alpha,\beta}$ with a component of $\mathcal{F}\mathit{ilt}(\mathcal{X})$, and $\mathbb{P}(\mathcal{X}^+_{\beta,\alpha})$ is defined similarly, using the opposite component. We denote by $\bar{\mathrm{gr}}, \bar{\mathrm{ev}}$ the maps gr, ev for $\mathcal{X}^+_{\beta,\alpha}$, and by $\mathbb{L}_{\mathrm{gr}}$ the relative cotangent complex of $\mathcal{X}^+_{\alpha,\beta}$ over $\mathcal{X}_\alpha \times \mathcal{X}_\beta$. We regard $h^1(\mathbb{L}_{\mathrm{gr}}) = \dim \mathrm{H}^1(\mathbb{L}_{\mathrm{gr}})$ as a constructible function on $\mathcal{X}^+_{\alpha,\beta}$, which can be pulled back to $\mathcal{X}_\alpha \times \mathcal{X}_\beta$. The factors $\mathbb{L}^{-h^1(\mathbb{L}_{\mathrm{gr}})}$ and $\mathbb{L}^{-h^1(\mathbb{L}_{\bar{\mathrm{gr}}})}$ are due to the difference of stabilizer groups in $\mathcal{X}^+_{\alpha,\beta}$ and $\mathcal{X}_\alpha \times \mathcal{X}_\beta$. Applying (7.4.5.2) turns (7.4.6.3) into

$$\int_{\mathcal{X}_\alpha \times \mathcal{X}_\beta} (1-\mathbb{L})^2 \cdot (a \boxtimes b) \cdot \big(h^1(\mathbb{L}_{\mathrm{gr}}) - h^0(\mathbb{L}_{\mathrm{gr}}) + h^0(\mathbb{L}_{\bar{\mathrm{gr}}}) - h^1(\mathbb{L}_{\bar{\mathrm{gr}}})\big) \cdot \oplus^*(\nu_{\mathcal{X}})\, d\chi\,, \tag{7.4.6.4}$$

where we also replaced $\mathbb{L}^{-h^1(\mathbb{L}_{\mathrm{gr}})} - \mathbb{L}^{-h^1(\mathbb{L}_{\bar{\mathrm{gr}}})}$ by $(1-\mathbb{L}) \cdot (h^1(\mathbb{L}_{\mathrm{gr}}) - h^1(\mathbb{L}_{\bar{\mathrm{gr}}}))$, as they are equal modulo $(1-\mathbb{L})^2$, so this will not affect the integral. By Lemma B.3.4, the alternating sum in (7.4.6.4) is equal to $-\mathrm{vdim}\, \mathcal{X}^+_{\alpha,\beta}$. Finally, by (7.4.5.1), we have $\oplus^*(\nu_{\mathcal{X}}) = (-1)^{\mathrm{vdim}\, \mathcal{X}^+_{\alpha,\beta}} \cdot (\nu_{\mathcal{X}} \boxtimes \nu_{\mathcal{X}})$, which identifies (7.4.6.4) with the right-hand side of (7.4.6.1).

The identity (7.4.6.2) can be proved analogously.

7.4.7. Proof of Theorem 7.3.2. Consider the integration maps

$$(\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}) \cdot \int_{\mathcal{X}} (-) \cdot \nu^{\mathrm{mot}}_{\mathcal{X}} \colon \quad \mathbb{M}_{\mathrm{qc}}(\mathcal{X}; \mathbb{Q}) \longrightarrow \Lambda_{\mathcal{X}}\,,$$
$$\int_{\mathcal{X}^{\mathrm{sd}}} (-) \cdot \nu^{\mathrm{mot}}_{\mathcal{X}^{\mathrm{sd}}} \colon \; \mathbb{M}_{\mathrm{qc}}(\mathcal{X}^{\mathrm{sd}}; \mathbb{Q}) \longrightarrow \Lambda^{\mathrm{sd}}_{\mathcal{X}}\,,$$

where the generators λ_α and $\lambda^{\mathrm{sd}}_\theta$ record which components the motives are supported on. The relations (7.4.3.3)–(7.4.3.4) imply that these maps are algebra and module homomorphisms.

Similarly, the relations (7.4.6.1)–(7.4.6.2) imply that the integration maps

$$\int_{\mathcal{X}} (1 - \mathbb{L}) \cdot (-) \cdot \nu_{\mathcal{X}} \, d\chi : \quad \mathbb{M}^*_{\mathrm{qc}}(\mathcal{X}; \mathbb{Q}) \longrightarrow \bar{\Lambda}_{\mathcal{X}} \, ,$$
$$\int_{\mathcal{X}^{\mathrm{sd}}} (-) \cdot \nu_{\mathcal{X}^{\mathrm{sd}}} : \ \mathbb{M}^*_{\mathrm{qc}}(\mathcal{X}^{\mathrm{sd}}; \mathbb{Q}) \longrightarrow \bar{\Lambda}^{\mathrm{sd}}_{\mathcal{X}} \, ,$$

are Lie algebra and twisted module homomorphisms, where the superscripts $*$ indicate subspaces of motives for which the integrals are finite. It follows from (7.4.6.1)–(7.4.6.2) that these subspaces are a Lie subalgebra and a sub-twisted module for this subalgebra, respectively.

The theorem is now a direct consequence of Theorem 7.2.3, by applying the above integration homomorphisms to the relations (7.2.3.1)–(7.2.3.2). □

7.5 Wall-crossing in derived categories

7.5.1. We now discuss wall-crossing formulae for linear and orthosymplectic Donaldson–Thomas invariants when changing Bridgeland stability conditions in the derived category.

These wall-crossing formulae are similar to those in Theorem 7.3.2, but since changing the Bridgeland stability condition also changes the heart of the derived category, the theorem does not directly apply. However, we show that if the stability condition is not changed by too much, as measured by the metric on the space of stability conditions, then the wall-crossing formulae still hold.

7.5.2. The setting. Throughout, we fix an algebraically closed field K of characteristic 0.

Let $\mathscr{C}$ be a K-linear dg-category of finite type, and let $\bar{\mathcal{X}}$ be the derived moduli stack of objects in $\mathscr{C}$, as in Toën and Vaquié [157, Theorem 3.6]. We assume that $\mathscr{C}$ is equipped with a self-dual structure, which induces an involution of $\bar{\mathcal{X}}$.

We fix a surjection $K(\mathscr{C}) \to \Gamma$ to a finitely generated free abelian group Γ, as in §2.4.2, and consider the space $\mathrm{Stab}_\Gamma(\mathscr{C})$ of Bridgeland stability conditions as in §2.4.3. It is equipped with an involution as in §2.4.4.

We assume given an open subset $\mathrm{Stab}^\circ_\Gamma(\mathscr{C}) \subset \mathrm{Stab}_\Gamma(\mathscr{C})$, invariant under the involution, satisfying the following conditions:

- For any stability condition $\tau = (Z, \mathscr{P}) \in \mathrm{Stab}^\circ_\Gamma(\mathscr{C})$, and any interval $J \subset \mathbb{R}$ such that

$J \cap (J+1) = \varnothing$, there exists an open substack

$$\mathcal{X}(\tau; J) \subset \bar{\mathcal{X}} \tag{7.5.2.1}$$

consisting of objects in $\mathcal{P}(J)$, which is a derived linear stack in the sense of §3.7.2. Moreover, τ defines a permissible stability condition on $\mathcal{X}(\tau; J)$ in the sense of §3.5.

- *Support property*. For any $r \in \mathbb{R}_{>0}$, there are only finitely many classes $\alpha \in \Gamma$ such that there exist τ-semistable objects in $\mathcal{C}$ of class α, and $|Z(\alpha)| \leqslant r$.

Denote by $\mathrm{Stab}_\Gamma^{\circ,\mathrm{sd}}(\mathcal{C}) \subset \mathrm{Stab}_\Gamma^{\circ}(\mathcal{C})$ the fixed locus of the involution, which is a real analytic manifold.

In particular, if $\tau \in \mathrm{Stab}_\Gamma^{\circ,\mathrm{sd}}(\mathcal{C})$, and $J \subset \mathbb{R}$ is an interval with $J = -J$ and $J \cap (J+1) = \varnothing$, then $\mathcal{X}(\tau; J)$ is a self-dual derived linear stack.

Finally, we assume that $\mathcal{C}$ is equipped with a Calabi–Yau structure of degree 3, preserved by the self-dual structure. By Brav and Dyckerhoff [21, Theorem 5.6], this also defines a (-1)-shifted symplectic structure on $\bar{\mathcal{X}}$, preserved by its involution, establishing the derived linear stacks $\mathcal{X}(\tau; J)$ as (-1)-shifted symplectic linear stacks.

In §8.3 and §8.4 below, we will verify these conditions for some choices of $\mathcal{C}$ and $\mathrm{Stab}_\Gamma^{\circ}(\mathcal{C})$.

7.5.3. Theorem. *Let $\mathcal{C}$ be as above, and let $\tau = (Z, \mathcal{P})$, $\tilde{\tau} = (\tilde{Z}, \tilde{\mathcal{P}}) \in \mathrm{Stab}_\Gamma^{\circ}(\mathcal{C})$ be Bridgeland stability conditions.*

(i) *If $\tau, \tilde{\tau}$ can be connected by a path of length $< 1/4$ in $\mathrm{Stab}_\Gamma^{\circ}(\mathcal{C})$, then for any class $\alpha \in \Gamma$ with $Z(\alpha) \neq 0$, the wall-crossing formula (7.3.2.1) holds.*

(ii) *If $\tau, \tilde{\tau} \in \mathrm{Stab}_\Gamma^{\circ,\mathrm{sd}}(\mathcal{C})$, and they can be connected by a path of length $< 1/4$ in $\mathrm{Stab}_\Gamma^{\circ,\mathrm{sd}}(\mathcal{C})$, then for any class $\alpha \in \Gamma^{\mathrm{sd}}$ with $Z(\alpha) \in \mathbb{R}_{>0}$, the wall-crossing formula (7.3.2.2) holds.*

Here, the length of a path is defined as the supremum of sums of distances over all subdivisions. In (7.3.2.1)–(7.3.2.2), we use $\tau, \tilde{\tau}$ in place of τ_+, τ_-. The sets $\pi_0(\mathcal{X})$, $\pi_0(\mathcal{X}^{\mathrm{sd}})$ in the formulae are defined using $\mathcal{X} = \mathcal{X}(\tau;]t - 1/4, t + 1/4[)$, where t is a phase of $Z(\alpha)$ in (i) or $t = 0$ in (ii). The coefficients $\tilde{U}(\ldots)$, $\tilde{U}^{\mathrm{sd}}(\ldots)$ are defined using the total order on phases in $]t - 1/2, t + 1/2[$.

Moreover, if we are given an orientation data on $\mathcal{X}(\tau;]t - 1/2, t + 1/2[)$, or a self-dual orientation data on $\mathcal{X}(\tau;]-1/2, 1/2[)$, respectively, then (i)–(ii) also hold for the motivic versions (7.3.2.3)–(7.3.2.4), where α has phase t.

Proof. To avoid repetition, we prove (i)–(ii) using a common argument.

We first prove the following claim: For fixed τ and a fixed class α or θ, there exists $\delta > 0$ such that the wall-crossing formulae hold whenever $d(\tau, \tilde{\tau}) < \delta$, with the sets $\pi_0(\mathcal{X}), \pi_0(\mathcal{X}^{\mathrm{sd}})$ defined using $\mathcal{X} = \mathcal{X}(\tau; t)$, and we may take τ_+, τ_- in the formulae to be either $\tau, \tilde{\tau}$ or $\tilde{\tau}, \tau$.

Write $\mathcal{A} = \mathcal{P}(t)$. Let $K \subset \Gamma$ be the set of classes of τ-semistable objects in $\mathcal{C}$, and $C \subset K$ the set of classes realized by objects in $\mathcal{A}$.

We choose $0 < \delta < 1/8$ such that $K \cap Z^{-1}(V_{4\delta}(\mathrm{e}^{2\delta} \cdot Z(\alpha))) \subset C$, where

$$V_u(z) = \{r \mathrm{e}^{\pi \mathrm{i} \phi} \mid 0 \leqslant r \leqslant |z| \, , \; |\phi| \leqslant u\} \subset \mathbb{C} \, .$$

If $\beta \in \Gamma$ is the class of a τ-Harder–Narasimhan factor of a $\tilde{\tau}$-semistable object of class α, then $Z(\beta)$ must lie in $V_{2\delta}(Z(\alpha))$. By the choice of δ, all such classes β have phase t, and are hence equal to α. This implies that all $\tilde{\tau}$-semistable objects of class α are τ-semistable and are in $\mathcal{A}$.

Similarly, we may assume that all $\tilde{\tau}$-semistable objects with phase in $[t - \delta, t + \delta]$ and norm $\leqslant \mathrm{e}^{\delta} \cdot |Z(\alpha)|$ are in $\mathcal{A}$. Indeed, such objects have τ-phase in $[t - 2\delta, t + 2\delta]$ and τ-norm $\leqslant \mathrm{e}^{2\delta} \cdot |Z(\alpha)|$, and this property holds by the choice of δ.

It follows that for any object in $\mathcal{A}$ of class α, its $\tilde{\tau}$-Harder–Narasimhan factors also belong to $\mathcal{A}$. In other words, $\tilde{\tau}$ almost defines a stability condition on $\mathcal{X}(\tau; t)$ in the sense of §3.5.4, except that the Θ-stratification is only defined on $\mathcal{X}(\tau; t)_\beta$ for classes $\beta \in C$ with $|Z(\beta)| \leqslant |Z(\alpha)|$. However, this is enough to prove wall-crossing for α, as the other classes are irrelevant in the argument. The claim thus follows from Theorem 7.3.2, where τ corresponds to trivial stability on $\mathcal{X}(\tau; t)$.

We now turn to the original statement of the theorem. Choose a path $(\tau_s = (Z_s, \mathcal{P}_s))_{s \in [0,1]}$ of length $\ell < 1/4$, with $\tau_0 = \tau$ and $\tau_1 = \tilde{\tau}$. By the compactness of $[0, 1]$, our claim implies that we can choose $0 = s_0 < \cdots < s_n = 1$ such that there are wall-crossing formulae between each τ_{s_i} and $\tau_{s_{i+1}}$. We may thus apply (7.3.2.1), etc., to express $\mathrm{DT}_\alpha(\tau_{s_0})$, etc., in terms of invariants for τ_{s_1}, and so on, finally in terms of invariants for $\tau_{s_n} = \tilde{\tau}$. In each step, the involved invariants $\mathrm{DT}_\beta(\tau_{s_i})$ must satisfy that $Z(\beta)$ lies in the bounded region

$$\{r \mathrm{e}^{\pi \mathrm{i} \phi} \mid r \geqslant 0 \, , \; |\phi - t| \leqslant \ell\} \cap \{Z(\alpha) - r \mathrm{e}^{\pi \mathrm{i} \phi} \mid r \geqslant 0 \, , \; |\phi - t| \leqslant \ell\} \subset \mathbb{C} \, ,$$

so that the sums (7.3.2.1), etc., can not only be written using some $\pi_0(\mathcal{X}(\tau_{s_i}; t_i))$ and its self-dual version, as in the argument above, but also using the larger set $\pi_0(\mathscr{X}(\tau;]t-1/2, t+1/2[))$ and its self-dual version, where the coefficients $\tilde{U}(\dots)$, $\tilde{U}^{\mathrm{sd}}(\dots)$ are zero for the newly introduced terms. The support property of τ ensures that only finitely many non-zero terms appear in each step.

It remains to prove that the coefficients $\tilde{U}(\dots)$, $\tilde{U}^{\mathrm{sd}}(\dots)$ respect composition of wall-crossing formulae, so that the wall-crossing formulae obtained from the above process are equivalent to (7.3.2.1), etc., from τ directly to $\tilde{\tau}$. This follows from the fact that the coefficients $S(\dots)$, $S^{\mathrm{sd}}(\dots)$ respect composition, which was proved in Lemma 7.1.4. $\square$

Chapter 8
Applications

In this chapter, we study orthosymplectic Donaldson–Thomas theory in several concrete examples, based on the settings introduced in Chapter 4.

In §8.1, we study Donaldson–Thomas invariants counting self-dual representations of self-dual quivers with potential; in §8.2 and §8.3, we study orthosymplectic Donaldson–Thomas invariants for curves and threefolds. Finally, §8.4, we define a version of Vafa–Witten invariants counting orthosymplectic Higgs complexes on surfaces.

8.1 Self-dual quivers

8.1.1. In this section, we continue the discussion of self-dual quivers and their self-dual representations from §4.1, and study their orthosymplectic Donaldson–Thomas invariants. We also provide an explicit algorithm that computes these invariants for any self-dual quiver, provided that the potential is zero.

Throughout, we work over an algebraically closed field K of characteristic zero.

8.1.2. Donaldson–Thomas invariants. For a quiver Q, a potential W, a slope function τ on Q, and a dimension vector $\alpha \in \mathbb{N}^{Q_0} \setminus \{0\}$, we have the Donaldson–Thomas invariants

$$\mathrm{DT}_\alpha(\tau) \in \mathbb{Q} \, , \qquad \mathrm{DT}_\alpha^{\mathrm{mot}}(\tau) \in \widehat{\mathbb{M}}^{\hat{\mu}}(K; \mathbb{Q}) \, ,$$

defined as in §5.6.3 and §6.3.1, for the (-1)-shifted symplectic linear stack $\mathcal{X}_{Q,W}$ defined in §4.1.7. These invariants were studied by Joyce and Song [89], Kontsevich and Soibelman [99], and others.

When Q is equipped with a self-dual structure and W, τ are self-dual, we have the self-dual Donaldson–Thomas invariants

$$\mathrm{DT}^{\mathrm{sd}}_{\theta}(\tau) \in \mathbb{Q}\,, \qquad \mathrm{DT}^{\mathrm{mot,sd}}_{\theta}(\tau) \in \hat{\mathbb{M}}^{\hat{\mu}}(K;\mathbb{Q})\,,$$

defined as in §5.6.4 and §6.3.1 for the self-dual (-1)-shifted symplectic linear stack $\mathcal{X}_{Q,W}$. These are new constructions in this thesis.

When the potential W is zero, we have $\mathcal{X}_{Q,0} \simeq \mathrm{T}^*[-1]\,\mathcal{X}_Q$ as in §4.1.7, and the discussions in §5.6.5 and §6.3.2 apply, which provide more straightforward formulae for the Donaldson–Thomas invariants.

8.1.3. Wall-crossing formulae. For a self-dual quiver Q with a self-dual potential W, Theorem 7.3.2 applies to the self-dual (-1)-shifted symplectic linear stack $\mathcal{X}_{Q,W}$, proving wall-crossing formulae for the DT invariants defined in §8.1.2. We are allowed to take τ_+, τ_- in the theorem to be any two self-dual slope functions, since we can take τ_0 in the theorem to be the trivial stability condition, which is permissible.

8.1.4. An algorithm for computing Donaldson–Thomas invariants. For a self-dual quiver Q, in the case when the potential W is zero, we describe an algorithm for computing all the invariants $\mathrm{DT}_\alpha(\tau)$, $\mathrm{DT}^{\mathrm{mot}}_\alpha(\tau)$, $\mathrm{DT}_\theta(\tau)$, and $\mathrm{DT}^{\mathrm{mot}}_\theta(\tau)$, for any self-dual slope function τ.

First, we compute the motives of $\mathcal{X}_\alpha = V_\alpha / G_\alpha$ and $\mathcal{X}^{\mathrm{sd}}_\theta = V^{\mathrm{sd}}_\theta / G^{\mathrm{sd}}_\theta$, as in (4.1.5.1)–(4.1.5.2), in $\mathbb{M}(K)$. We use the relation (5.2.6.1) for the vector bundles $\mathcal{X}_\alpha \to */G_\alpha$ and $\mathcal{X}^{\mathrm{sd}}_\theta \to */G^{\mathrm{sd}}_\theta$, and the motives

$$[*/\mathrm{GL}(n)] = \prod_{i=0}^{n-1} \frac{1}{\mathbb{L}^n - \mathbb{L}^i}\,, \tag{8.1.4.1}$$

$$[*/\mathrm{O}(2n)] = \mathbb{L}^n \cdot \prod_{i=0}^{n-1} \frac{1}{\mathbb{L}^{2n} - \mathbb{L}^{2i}}\,, \tag{8.1.4.2}$$

$$[*/\mathrm{O}(2n+1)] = [*/\mathrm{Sp}(2n)] = \mathbb{L}^{-n} \cdot \prod_{i=0}^{n-1} \frac{1}{\mathbb{L}^{2n} - \mathbb{L}^{2i}}\,, \tag{8.1.4.3}$$

where the linear and symplectic cases follow from Joyce [84, Theorem 4.10], as these are *special groups* in the sense there, while the orthogonal cases are due to Dhillon and Young [53, Theorem 3.7]. We then have

$$\int_{\mathcal{X}_\alpha} \nu_{\mathcal{X}}^{\mathrm{mot}} = \mathbb{L}^{-(\dim V_\alpha - \dim G_\alpha)/2} \cdot [\mathcal{X}_\alpha] \quad = \mathbb{L}^{(\dim V_\alpha + \dim G_\alpha)/2} \cdot [*/G_\alpha] \,, \tag{8.1.4.4}$$

$$\int_{\mathcal{X}_\theta^{\mathrm{sd}}} \nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}} = \mathbb{L}^{-(\dim V_\theta^{\mathrm{sd}} - \dim G_\theta^{\mathrm{sd}})/2} \cdot [\mathcal{X}_\theta^{\mathrm{sd}}] = \mathbb{L}^{(\dim V_\theta^{\mathrm{sd}} + \dim G_\theta^{\mathrm{sd}})/2} \cdot [*/G_\theta^{\mathrm{sd}}] \,, \tag{8.1.4.5}$$

where $[*/G_\alpha]$ and $[*/G_\theta^{\mathrm{sd}}]$ are products of the rational functions in (8.1.4.1)–(8.1.4.3).

Next, we compute the invariants $\mathrm{DT}_\alpha^{\mathrm{mot}}(0)$ and $\mathrm{DT}_\theta^{\mathrm{sd,mot}}(0)$ for the trivial slope function 0. These can be obtained from (6.3.2.1)–(6.3.2.2) by substituting in (5.5.2.1) and (5.5.3.1), then using the relations (7.4.3.3)–(7.4.3.4) to reduce to the known integrals (8.1.4.4)–(8.1.4.5). This process also shows that $\mathrm{DT}_\alpha^{\mathrm{mot}}(0)$ and $\mathrm{DT}_\theta^{\mathrm{sd,mot}}(0)$ are rational functions in $\mathbb{L}^{1/2}$, and evaluating them at $\mathbb{L}^{1/2} = -1$ gives the numerical invariants $\mathrm{DT}_\alpha(0)$ and $\mathrm{DT}_\theta^{\mathrm{sd}}(0)$.

Finally, for a general self-dual slope function τ, we may apply the wall-crossing formulae (7.3.2.1)–(7.3.2.4) to compute the invariants $\mathrm{DT}_\alpha(\tau)$, $\mathrm{DT}_\theta^{\mathrm{sd}}(\tau)$, $\mathrm{DT}_\alpha^{\mathrm{mot}}(\tau)$, and $\mathrm{DT}_\theta^{\mathrm{sd,mot}}(\tau)$ from the case when $\tau = 0$, which is already known.

As an alternative to the final step, we may first compute the integrals $\int_{\mathcal{X}_\alpha^{\mathrm{ss}}(\tau)} \nu_{\mathcal{X}}^{\mathrm{mot}}$ and $\int_{\mathcal{X}_\theta^{\mathrm{sd,ss}}(\tau)} \nu_{\mathcal{X}^{\mathrm{sd}}}^{\mathrm{mot}}$ using the relations (7.1.3.11)–(7.1.3.12), together with (7.4.3.3)–(7.4.3.4) to reduce to the known integrals (8.1.4.4)–(8.1.4.5), then repeat the process above to obtain the invariants $\mathrm{DT}_\alpha^{\mathrm{mot}}(\tau)$ and $\mathrm{DT}_\theta^{\mathrm{sd,mot}}(\tau)$, which are rational functions in $\mathbb{L}^{1/2}$. We then evaluate them at $\mathbb{L}^{1/2} = -1$ to obtain the numerical invariants $\mathrm{DT}_\alpha(\tau)$ and $\mathrm{DT}_\theta^{\mathrm{sd}}(\tau)$.

The author has implemented the above algorithm using a computer program, and some numerical results are presented below.

8.1.5. Example. The point quiver. Consider the point quiver $Q = (\bullet)$, with a single vertex and no edges, with the trivial slope function $\tau = 0$. There are two self-dual structures on Q, with the signs $+1$ and -1 assigned to the vertex, respectively.

We have the moduli stack $\mathcal{X}_Q = \coprod_{n \geqslant 0} */\mathrm{GL}(n)$, and its fixed loci $\mathcal{X}_Q^{\mathrm{sd}} = \coprod_{n \geqslant 0} */\mathrm{O}(n)$ or $\coprod_{n \geqslant 0} */\mathrm{Sp}(2n)$, depending on the sign of the vertex. As in Joyce and Song [89, Example 7.19], the usual Donaldson–Thomas invariants of Q are given by

$$\mathrm{DT}_{\mathsf{A}_{n-1}} = \frac{1}{n^2}$$

for all $n \geqslant 1$, where the subscript A_{n-1} refers to the Dynkin type of $\mathrm{GL}(n)$.

Based on explicit computation following the algorithm in §8.1.4, we conjecture that

$$\mathrm{DT}^{\mathrm{sd}}_{\mathsf{B}_n} = \mathrm{DT}^{\mathrm{sd}}_{\mathsf{C}_n} = (-1)^n \binom{-1/4}{n}, \qquad \mathrm{DT}^{\mathrm{sd}}_{\mathsf{D}_n} = (-1)^n \binom{1/4}{n},$$

where the subscripts B_n, C_n, and D_n refer to the Dynkin types of $\mathrm{O}(2n+1)$, $\mathrm{Sp}(2n)$, and $\mathrm{O}(2n)$, respectively. Equivalently, we have the generating series

$$\sum_{n\geqslant 0} q^n \cdot \mathrm{DT}^{\mathrm{sd}}_{\mathsf{B}_n} = \sum_{n\geqslant 0} q^n \cdot \mathrm{DT}^{\mathrm{sd}}_{\mathsf{C}_n} = (1-q)^{-1/4}, \qquad \sum_{n\geqslant 0} q^n \cdot \mathrm{DT}^{\mathrm{sd}}_{\mathsf{D}_n} = (1-q)^{1/4}.$$

We expect to prove this conjecture in a future paper [33], and we expect that the coincidence of the type B and C invariants here should be related to the fact that these groups are Langlands dual to each other.

8.1.6. Example. The $\tilde{\mathsf{A}}_1$ quiver. Let $Q = (\bullet \rightrightarrows \bullet)$ be the quiver with two vertices and two arrows pointing in the same direction, called the $\tilde{\mathsf{A}}_1$ *quiver*. Consider the contravariant involution of Q that exchanges the two vertices but fixes the edges. We use the simplified notation $\tilde{\mathsf{A}}_1^{u,v}$, where u, v are the signs in the self-dual structure. For example, $\tilde{\mathsf{A}}_1^{+,++}$ means that we take the sign $+1$ on all vertices and edges. Note that both vertices must have the same sign. We use the slope function $\tau = (1,-1)$.

Based on numerical evidence from applying the algorithm in §8.1.4, we conjecture that we have the generating series

$$\sum_{n=0}^{\infty} q^{n/2} \cdot \mathrm{DT}^{\mathrm{sd,mot}}_{(n,n)}(\tau) = \begin{cases} \dfrac{(1-q)^{1/2}}{(1-q^{1/2}\,\mathbb{L}^{-1/2})(1-q^{1/2}\,\mathbb{L}^{1/2})} & \text{for } \tilde{\mathsf{A}}_1^{+,++} \text{ and } \tilde{\mathsf{A}}_1^{-,--}, \quad (8.1.6.1) \\ \left(\dfrac{1+q^{1/2}}{1-q^{1/2}}\right)^{1/2} & \text{for } \tilde{\mathsf{A}}_1^{+,+-} \text{ and } \tilde{\mathsf{A}}_1^{-,+-}, \quad (8.1.6.2) \\ (1-q)^{1/2} & \text{for } \tilde{\mathsf{A}}_1^{+,--} \text{ and } \tilde{\mathsf{A}}_1^{-,++}. \quad (8.1.6.3) \end{cases}$$

This example is related to coherent sheaves on $\mathbb{P}^1$, as we will discuss in Example 8.2.5.

8.2 Donaldson–Thomas invariants for curves

8.2.1. We define Donaldson–Thomas invariants counting orthogonal and symplectic bundles on a curve. These are orthosymplectic versions of Joyce's motivic invariants counting vector bundles on a curve, as in [85, §6.3].

8.2.2. Let C be a connected, smooth, projective curve over $\mathbb{C}$, and fix the data (I, L, s, ε) as in §4.2.2. This defines a self-dual structure on $\mathsf{Perf}(C)$.

Let $\tau = (Z, \mathscr{P})$ be the Bridgeland stability condition on C defined as in Example 4.2.7, where we choose the unique element $\omega \in \mathrm{H}^2(C; \mathbb{Q})$ with $\int_C \omega = 1$. Explicitly, we have

$$Z(E) = \mathrm{i}^{-s} \cdot \biggl(\biggl(1 - \mathrm{i} \, \frac{\deg L}{2} \biggr) \, r + d \biggr) \tag{8.2.2.1}$$

for $E \in \mathsf{Perf}(C)$ with rank r and degree d, so that $\mathrm{ch}(E) = r + d\omega$. Note that the choices of L and s do not affect which objects are semistable, although they affect which objects are self-dual. The subcategory $\mathsf{Vect}(C) \subset \mathsf{Perf}(C)$ of vector bundles on C satisfies $\mathsf{Vect}(C) = \mathscr{P}(](-1-s)/2, (1-s)/2[)$.

8.2.3. The even case. When s is even, the abelian category $\mathscr{P}(0)$ consists of objects $E[s/2]$ for semistable vector bundles E on C in the usual sense, whose rank r and degree d satisfy $d = r \deg L/2$. The self-dual objects are such E with isomorphisms $\phi \colon E \xrightarrow{\sim} \mathscr{H}om(I^*(E), L)$ with $I^*(\phi)^\vee \circ \phi = (-1)^{s/2} \cdot \varepsilon$.

In particular, when $L = \mathscr{O}_C$, semistable self-dual complexes can be identified, up to a shift, with orthogonal or symplectic bundles on C, depending on whether $(-1)^{s/2} \cdot \varepsilon = 1$ or -1, whose underlying vector bundles are semistable in the usual sense.

For each rank $r > 0$, we have the self-dual Donaldson–Thomas invariants

$$\mathrm{DT}^{\mathrm{sd}}_r \in \mathbb{Q} \, , \qquad \mathrm{DT}^{\mathrm{sd,mot}}_r \in \widehat{\mathbb{M}}^{\hat{\mu}}(K; \mathbb{Q}) \, ,$$

counting semistable self-dual vector bundles of rank r as above, defined as in §5.6.5 and §6.3.2 using the self-dual linear stack $\mathscr{X}(\tau; 0)$ defined in §4.2.6 with the trivial stability condition.

8.2.4. The odd case. When s is odd, $\mathscr{P}(0)$ consists of objects $E[(s-1)/2]$ for torsion sheaves E on C, and the semistable self-dual objects are such E with isomorphisms $\phi \colon E \xrightarrow{\sim} \mathbb{R}\mathscr{H}om(I^*(E), L[1])$ with $I^*(\phi)^\vee \circ \phi = (-1)^{(s-1)/2} \cdot \varepsilon$. For each degree $d > 0$, we have the self-dual Donaldson–Thomas invariants

$$\mathrm{DT}^{\mathrm{sd}}_{0,d} \in \mathbb{Q} \, , \qquad \mathrm{DT}^{\mathrm{sd,mot}}_{0,d} \in \widehat{\mathbb{M}}^{\hat{\mu}}(K; \mathbb{Q}) \, ,$$

counting these self-dual torsion sheaves, defined similarly as above.

In fact, these invariants do not depend on the choice of L, since choosing a suitable I-

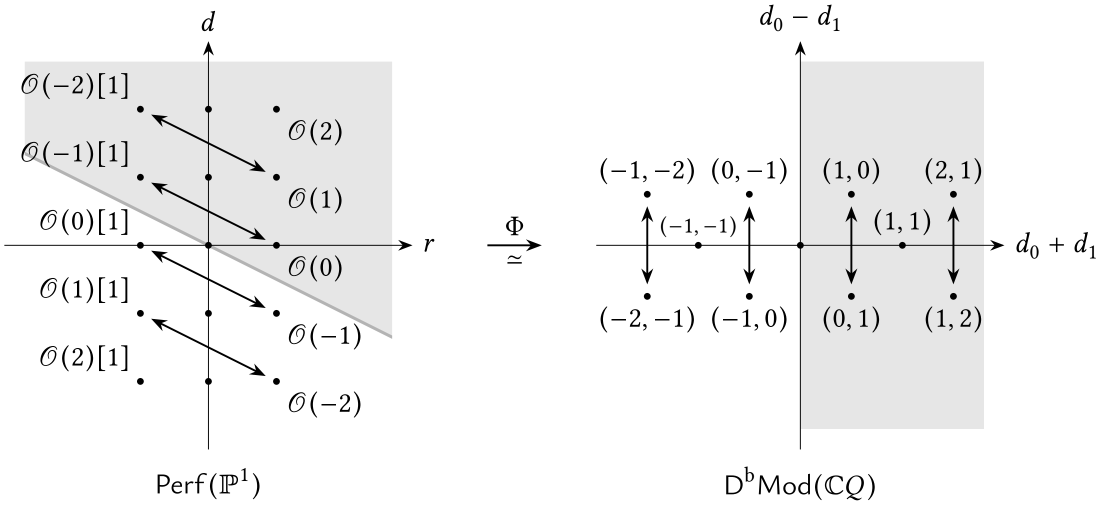

Figure 8.1. An equivalence of categories

invariant open cover of C trivializing L, torsion sheaves supported on the open sets give an open cover of the moduli stacks, where pieces and intersections do not depend on L. It then follows from [32, Theorem 5.2.10 (i)] that the invariants do not depend on L.

8.2.5. Example. Invariants for $\mathbb{P}^1$. Consider the case when $C = \mathbb{P}^1$ and $I = \mathrm{id}_{\mathbb{P}^1}$. We describe the invariants in two situations.

When $s = 0$ and $L = \mathscr{O}_{\mathbb{P}^1}$, since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, all semistable vector bundles of slope 0 are trivial bundles. The self-dual abelian category $\mathscr{P}(0)$ is thus equivalent to the category of finite-dimensional $\mathbb{C}$-vector spaces, with one of the two self-dual structures described in Example 8.1.5, depending on the sign ε. The DT invariants agree with the ones given there.

When $s = 1$, invariants for $\mathbb{P}^1$ are related to Donaldson–Thomas invariants for self-dual quivers. Indeed, as a special case of Bondal [17, Theorem 6.2], we have an equivalence

$$\Phi \colon \mathsf{Perf}(\mathbb{P}^1) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}\mathsf{Mod}(\mathbb{C}Q)\,, \tag{8.2.5.1}$$

where Q is the $\tilde{\mathrm{A}}_1$ quiver in Example 8.1.6, and $\Phi(E) = \big(\mathbb{R}\Gamma(\mathbb{P}^1, E(-1)) \rightrightarrows \mathbb{R}\Gamma(\mathbb{P}^1, E)\big)$, with the two maps given by multiplying with the coordinate functions $x_0, x_1 \in \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$.

In fact, under the isomorphism Φ, the self-dual structure on $\mathsf{Perf}(\mathbb{P}^1)$ given by ($I = \mathrm{id}_{\mathbb{P}^1}$, $L = \mathscr{O}_{\mathbb{P}^1}(-1), s = 1, \varepsilon$) corresponds to the self-dual structure on $\mathsf{D}^{\mathrm{b}}\mathsf{Mod}(\mathbb{C}Q)$ given by the signs $(\varepsilon, {+}{+})$ in the notation of Example 8.1.6, as shown in Fig. 8.1. Here, r and d denote the rank and degree of a complex on $\mathbb{P}^1$, and (d_0, d_1) is the dimension vector of a representation of Q. The two-way arrows indicate the dual operation, and the self-dual objects lie on the vertical axis on the left-hand side, or the horizontal axis on the right-hand side. The shaded regions indicate the usual heart of $\mathsf{D}^{\mathsf{b}}\mathsf{Mod}(\mathbb{C}Q)$ and the corresponding heart of $\mathsf{Perf}(\mathbb{P}^1)$. The right-hand side can also be viewed either as the central charge of τ, or that of the stability condition on Q given by the slope function $(1, -1)$.

In particular, the Donaldson–Thomas invariants in this case coincide with those in Example 8.1.6, and should be given by the conjectural formulae (8.1.6.1) and (8.1.6.3).

8.3 Donaldson–Thomas invariants for threefolds

8.3.1. We define Donaldson–Thomas invariants counting orthogonal or symplectic complexes on a Calabi–Yau threefold. These invariants are one of the main applications of our theory, and are an extension of the usual Donaldson–Thomas invariants studied by Thomas [151], Joyce and Song [89], Kontsevich and Soibelman [99], and many others. We expect our invariants to be related to counting D-branes on Calabi–Yau 3-orientifolds, as discussed in Witten [162, §5.2], Diaconescu, Garcia-Raboso, Karp, and Sinha [54], and Hori and Walcher [76].

We also prove wall-crossing formulae for these invariants in Theorem 8.3.3, which relate the invariants for different Bridgeland stability conditions.

8.3.2. Invariants. Let Y be a Calabi–Yau threefold over $\mathbb{C}$, and fix the data (I, L, s, ε) as in §4.2.2. Let $\tau = (Z, \mathscr{P}) \in \mathsf{Stab}^{\circ,\mathrm{sd}}(Y)$ be a self-dual Bridgeland stability condition on Y as in §4.2.6, which always exists in the situation of Example 4.2.7.

Recall from §4.2.6 the derived moduli stacks

$$\mathscr{X}(\tau; J) \subset \mathscr{P}\!\mathit{erf}(Y)$$

of τ-semistable complexes on Y of phase within an interval $J \subset \mathbb{R}$ with $|J| < 1$, which is a (-1)-shifted symplectic linear stack, and is self-dual if $J = -J$.

Moreover, the stack $\mathscr{X}(\tau; J)$ has an orientation data in the sense of §3.7.4, by Joyce and Upmeier [90, Theorem 3.6]. However, we do not know, in the case when $J = -J$, if the moduli stack $\mathscr{X}(\tau; J)^{\mathrm{sd}}$ of orthosymplectic complexes has an orientation in general.

Given a class $\alpha \in K(Y)$ with $Z_\omega(\alpha) \in \mathbb{R}_{>0} \cdot \mathrm{e}^{\pi i t}$ for some phase $t \in \mathbb{R}$, define the numerical and motivic Donaldson–Thomas invariants

$$\mathrm{DT}_\alpha(\tau) \in \mathbb{Q}\,, \qquad \mathrm{DT}^{\mathrm{mot}}_\alpha(\tau) \in \widehat{\mathbb{M}}^{\hat{\mu}}(\mathbb{C};\mathbb{Q})\,,$$

as in §5.6.3 and §6.3.1 for the stack $\mathcal{X}(\tau;t)$ with the trivial stability condition, where we take the sum of Donaldson–Thomas invariants of connected components of the open and closed substack $\mathcal{X}(\tau;t)_\alpha \subset \mathcal{X}(\tau;t)$, and we use the orientation of Joyce and Upmeier [90] for the motivic version. These invariants are not new, and can be constructed from the formalisms of Joyce and Song [89] and Kontsevich and Soibelman [99].

When $t = 0$, for each $\theta \in K^{\mathrm{sd}}(Y)$ with $Z_\omega(\theta) \in \mathbb{R}_{>0}$, we have the numerical self-dual Donaldson–Thomas invariant

$$\mathrm{DT}^{\mathrm{sd}}_\theta(\tau) \in \mathbb{Q}\,,$$

defined as in §5.6.4 for the self-dual linear stack $\mathcal{X}(\tau;0)$ with the trivial stability condition, where we sum over connected components of $\mathcal{X}(\tau;0)^{\mathrm{sd}}_\theta$. These are new invariants for Calabi–Yau threefolds, and are one of the main constructions of this thesis.

If one can construct a self-dual orientation data on $\mathcal{X}(\tau;0)$ in the sense of §3.7.5, then the motivic self-dual Donaldson–Thomas invariant $\mathrm{DT}^{\mathrm{mot,sd}}_\theta(\tau)$ will also be defined, as in §6.3.1.

8.3.3. Theorem. *Let Y be a Calabi–Yau threefold over $\mathbb{C}$. Choose the data (I, L, s, ε) as in §4.2.2. Let $\tau = (Z, \mathcal{P})$, $\tilde{\tau} = (\tilde{Z}, \tilde{\mathcal{P}}) \in \mathrm{Stab}^\circ(Y)$ be Bridgeland stability conditions.*

(i) *If $\tau, \tilde{\tau}$ can be connected by a path of length $< 1/4$ in $\mathrm{Stab}^\circ(Y)$, then for any class $\alpha \in K(Y)$ with $Z(\alpha) \neq 0$, the wall-crossing formula (7.3.2.1) holds.*

(ii) *If $\tau, \tilde{\tau} \in \mathrm{Stab}^{\circ,\mathrm{sd}}(Y)$, and they can be connected by a path of length $< 1/4$ in $\mathrm{Stab}^{\circ,\mathrm{sd}}(Y)$, then for any class $\theta \in K^{\mathrm{sd}}(Y)$ with $Z(\theta) \in \mathbb{R}_{>0}$, the wall-crossing formula (7.3.2.2) holds.*

Here, the precise formulations of the wall-crossing formulae are as in Theorem 7.5.3.

Moreover, if we are given an orientation data on $\mathcal{X}(\tau;]t - 1/2, t + 1/2[)$, or a self-dual orientation data on $\mathcal{X}(\tau;]-1/2, 1/2[)$, respectively, then (i)–(ii) also hold for the motivic versions (7.3.2.3)–(7.3.2.4), where α has phase t.

Proof. This is a special case of Theorem 7.5.3. □

8.3.4. Generic stability conditions. Following Joyce and Song [89, Conjecture 6.12], we say that a stability condition τ as above is *generic*, if for any $\alpha, \beta \in K(Y)$ with $Z(\alpha) = \lambda Z(\beta) \neq 0$ for some $\lambda \in \mathbb{R}_{>0}$, we have the numerical condition $\operatorname{vdim} \bar{\mathcal{X}}^{+}_{\alpha,\beta} = 0$.

Similarly, when τ is self-dual, we say that it is *generic* as a self-dual stability condition, if it is generic as above, and for any $\alpha \in K(Y)$ of phase 0 and $\theta \in K^{\mathrm{sd}}(Y)$, we have $\operatorname{vdim} \bar{\mathcal{X}}^{\mathrm{sd},+}_{\alpha,\theta} = 0$.

By the first part in the proof of Theorem 8.3.3, combined with Corollary 7.3.4, we see that if $\tau \in \mathrm{Stab}^{\circ,\mathrm{sd}}(Y)$ is generic, then for each class α or θ as in Theorem 8.3.3, there exists $\delta > 0$, such that the invariant $\mathrm{DT}_{\alpha}(\tau)$ or $\mathrm{DT}^{\mathrm{sd}}_{\theta}(\tau)$ does not change if we move τ inside its δ-neighbourhood. Moreover, this also holds for the motivic versions $\mathrm{DT}^{\mathrm{mot}}_{\alpha}(\tau)$ or $\mathrm{DT}^{\mathrm{mot,sd}}_{\theta}(\tau)$, where the self-dual version requires a self-dual orientation data.

8.3.5. Expectations on deformation invariance. We expect that the numeric version of the orthosymplectic Donaldson–Thomas invariants, $\mathrm{DT}^{\mathrm{sd}}_{\theta}(\tau)$, should satisfy *deformation invariance*, analogously to Joyce and Song [89, Corollary 5.28] in the linear case, that is, they should stay constant under deformations of the complex structure of the threefold Y. However, we have not yet been able to prove this, as it does not seem straightforward to adapt the strategy of [89] using Joyce–Song pairs to our case, and further work is needed.

We do not expect the motivic version, $\mathrm{DT}^{\mathrm{sd,mot}}_{\theta}(\tau)$, to satisfy deformation invariance.

8.4 Vafa–Witten type invariants for surfaces

8.4.1. We construct a motivic version of orthosymplectic analogues of *Vafa–Witten invariants* for algebraic surfaces, studied by Tanaka and Thomas [148; 149], Maulik and Thomas [116], and Thomas [152]. We define our invariants for surfaces S with $K_S \leqslant 0$.

Our invariants count *orthosymplectic Higgs complexes* on a surface introduced in §4.3, generalizing the notion of *G-Higgs bundles* for $G = \mathrm{O}(n)$ or $\mathrm{Sp}(2n)$.

Via the spectral construction, these invariants can be seen as a version of orthosymplectic Donaldson–Thomas invariants in §8.3, for the non-compact Calabi–Yau threefold K_S, the total space of the canonical bundle of the surface S, with an involution that reverses the fibre direction.

8.4.2. Invariants. Let S be an algebraic surface over $\mathscr{C}$ with $K_S \leqslant 0$, equipped with the data (I, L, s, ε) as in §4.3.2, and let $\tau \in \mathrm{Stab}^{\circ,\mathrm{sd}}(S)$ be a permissible self-dual Bridgeland stability condition. Recall from §4.3.4 the derived moduli stacks

$$\mathscr{H}(\tau; t) \subset \mathscr{H}iggs(S)$$

of τ-semistable Higgs complexes of phase t, for $t \in \mathbb{R}$, which is a (-1)-shifted symplectic linear stack, and is self-dual if $t = 0$.

For a class $\alpha \in K(S)$ with $Z(\alpha) \neq 0$ or $\theta \in K^{\mathrm{sd}}(S)$ of phase 0, define the *Vafa–Witten type invariants*

$$\mathrm{vw}_\alpha(\tau) \in \mathbb{Q}\,, \qquad \mathrm{vw}_\theta^{\mathrm{sd}}(\tau) \in \mathbb{Q}$$

counting semistable Higgs complexes of class α or semistable self-dual Higgs complexes of class θ, as the Donaldson–Thomas invariants in §5.6 for the stack $\mathscr{H}(\tau; t)$ equipped with the trivial stability condition, where $t \in \mathbb{R}$ is a phase of $Z(\alpha)$ or $t = 0$ for θ.

Moreover, since $\mathscr{H}(\tau; t)$ and $\mathscr{H}(\tau; 0)^{\mathrm{sd}}$ are (-1)-shifted cotangent stacks, they come with canonical orientations, which define an orientation data on $\mathscr{H}(\tau; t)$ and a self-dual orientation data on $\mathscr{H}(\tau; 0)$. We use them to define *motivic Vafa–Witten type invariants*

$$\mathrm{vw}_\alpha^{\mathrm{mot}}(\tau)\,, \qquad \mathrm{vw}_\theta^{\mathrm{mot,sd}}(\tau) \in \widehat{\mathbb{M}}^{\hat{\mu}}(\mathbb{C}; \mathbb{Q})\,.$$

8.4.3. Wall-crossing. We have the following theorem stating the wall-crossing formulae for our Vafa–Witten invariants, which is analogous to Theorem 8.3.3.

Theorem. *Let S be a surface with $K_S \leqslant 0$, and choose the data (I, L, s, ε) as in* §4.3.2*. Let $\tau, \tilde{\tau} \in \mathrm{Stab}^{\circ}(S)$ be Bridgeland stability conditions.*

(i) *If $\tau, \tilde{\tau}$ can be connected by a path of length $< 1/4$ in $\mathrm{Stab}^{\circ}(S)$, then for any class $\alpha \in K(S)$ with $Z(\alpha) \neq 0$, the wall-crossing formulae* (7.3.2.1) *and* (7.3.2.3) *hold for the invariants $\mathrm{vw}_\alpha(-)$, $\mathrm{vw}_\alpha^{\mathrm{mot}}(-)$ when changing between τ and $\tilde{\tau}$.*

(ii) *If $\tau, \tilde{\tau} \in \mathrm{Stab}^{\circ,\mathrm{sd}}(S)$, and they can be connected by a path of length $< 1/4$ in $\mathrm{Stab}^{\circ,\mathrm{sd}}(S)$, then for any class $\theta \in K^{\mathrm{sd}}(S)$ with $Z(\theta) \in \mathbb{R}_{>0}$, the wall-crossing formulae* (7.3.2.2) *and* (7.3.2.4) *hold for the invariants $\mathrm{vw}_\theta^{\mathrm{sd}}(-)$, $\mathrm{vw}_\theta^{\mathrm{mot,sd}}(-)$ when changing between τ and $\tilde{\tau}$.*

Here, the precise formulations of the wall-crossing formulae are as in Theorem 7.5.3.

Proof. This is a special case of Theorem 7.5.3. □

8.4.4. The case of K3 surfaces. We now specialize to the case when S is a K3 surface or an abelian surface. In this case, for any $E, F \in \mathsf{Perf}(S)$, we have the numerical relations

$$\operatorname{rk} \operatorname{Ext}^{\bullet}_S(E, F) = \operatorname{rk} \operatorname{Ext}^{\bullet}_S(F, E) \,, \tag{8.4.4.1}$$

$$\operatorname{rk} \operatorname{Ext}^{\bullet}_S(E, \mathbb{D}(E))^{\mathbb{Z}_2} = \operatorname{rk} \operatorname{Ext}^{\bullet}_S(\mathbb{D}(E), E)^{\mathbb{Z}_2} \,, \tag{8.4.4.2}$$

where 'rk' denotes the alternating sum of dimensions, and $(-)^{\mathbb{Z}_2}$ denotes the fixed part of the involution $\phi \mapsto \mathbb{D}(\phi)$.

These relations imply that $\bar{\mathcal{X}}$ and $\bar{\mathcal{X}}^{\mathrm{sd}}$ are numerically symmetric in the sense of §7.3.3, and therefore, by Corollary 7.3.4 and Theorem 8.4.3, the invariants $\mathrm{vw}_\alpha(-)$, $\mathrm{vw}^{\mathrm{sd}}_\theta(-)$, $\mathrm{vw}^{\mathrm{mot}}_\alpha(-)$, and $\mathrm{vw}^{\mathrm{mot,sd}}_\theta(-)$ are locally constant functions on $\mathrm{Stab}^\circ(S)$ or $\mathrm{Stab}^{\circ,\mathrm{sd}}(S)$.

Appendix A

Proof of the no-pole theorem

This appendix contains the proof of the no-pole theorem, Theorem 5.5.5.

A.1 Virtual rank projections

A.1.1. To prove the no-pole theorem, we first give a useful alternative description of the virtual rank projection operators

$$\pi^{(k)} \colon \mathbb{M}(\mathscr{X}) \longrightarrow \mathbb{M}^{(k)}(\mathscr{X}) \, , \tag{A.1.1.1}$$

$$\pi^{(k)} \colon \mathbb{M}(\mathscr{X}^{\mathrm{sd}}) \longrightarrow \mathbb{M}^{(k)}(\mathscr{X}^{\mathrm{sd}}) \, , \tag{A.1.1.2}$$

introduced in §5.2.9, in the special case when $\mathscr{X}$ is a self-dual linear stack over a field K. This alternative description will be better suited for interacting with operations in the motivic Hall algebra and module introduced in §5.4.

The following description is a specialization of the formulation in the author, Ibáñez Núñez, and Kinjo [32, §5.1] to the case of self-dual linear stacks.

A.1.2. The description. Define operations

$$\circledast = \oplus_! \ \colon \quad \mathbb{M}(\mathscr{X}) \otimes \mathbb{M}(\mathscr{X}) \longrightarrow \mathbb{M}(\mathscr{X}) \, , \tag{A.1.2.1}$$

$$\circledcirc = \oplus_!^{\mathrm{sd}} \colon \ \mathbb{M}(\mathscr{X}) \otimes \mathbb{M}(\mathscr{X}^{\mathrm{sd}}) \longrightarrow \mathbb{M}(\mathscr{X}^{\mathrm{sd}}) \, , \tag{A.1.2.2}$$

where $\oplus \colon \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is the direct sum morphism, and $\oplus^{\mathrm{sd}}$ is the morphism defined in (3.4.1.1). These operations $\mathbb{M}(\mathscr{X})$ into a commutative algebra, and $\mathbb{M}(\mathscr{X}^{\mathrm{sd}})$ into a module over this algebra, which are similar to but different from the motivic Hall algebra and module.

For classes $\alpha \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, define elements $\sigma_\alpha(\tau) \in \mathbb{M}(\mathcal{X}^{\mathrm{ss}}_\alpha(\tau))$ and $\sigma^{\mathrm{sd}}_\theta(\tau) \in \mathbb{M}(\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau))$ by

$$\sigma_\alpha(\tau) = \sum_{\substack{n>0;\ \alpha_1,\dots,\alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n\,, \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n)}} \frac{(-1)^{n-1}}{n} \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \circledast \cdots \circledast [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)]\,, \tag{A.1.2.3}$$

$$\sigma^{\mathrm{sd}}_\theta(\tau) = \sum_{\substack{n \geqslant 0;\ \alpha_1,\dots,\alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n) = 0}} \binom{-1/2}{n} \cdot [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \mathbin{\circledcirc} \cdots \mathbin{\circledcirc} [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \mathbin{\circledcirc} [\mathcal{X}^{\mathrm{sd,ss}}_{\rho}(\tau)]\,. \tag{A.1.2.4}$$

Compare with (5.5.2.1) and (5.5.3.1). The element $\sigma_\alpha(\tau)$ was denoted by $\bar{\delta}^\alpha_{\mathrm{si}}(\tau)$ in Joyce [83, Definition 8.1].

Then, for any $\alpha \in \pi_0(\mathcal{X})$ and $\theta \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, and any $n \in \mathbb{N}$, we have

$$\pi^{(n)}([\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)]) = \frac{1}{n!} \cdot \sum_{\substack{\alpha_1,\dots,\alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_n\,, \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n)}} \sigma_{\alpha_1}(\tau) \circledast \cdots \circledast \sigma_{\alpha_n}(\tau)\,, \tag{A.1.2.5}$$

$$\pi^{(n)}([\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau)]) = \frac{1}{2^n n!} \cdot \sum_{\substack{\alpha_1,\dots,\alpha_n \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_n + \alpha_n^\vee + \rho \\ \tau(\alpha_1) = \cdots = \tau(\alpha_n) = 0}} \sigma_{\alpha_1}(\tau) \mathbin{\circledcirc} \cdots \mathbin{\circledcirc} \sigma_{\alpha_n}(\tau) \mathbin{\circledcirc} \sigma^{\mathrm{sd}}_\rho(\tau)\,. \tag{A.1.2.6}$$

In particular, we have

$$\pi^{(1)}([\mathcal{X}^{\mathrm{ss}}_\alpha(\tau)]) = \sigma_\alpha(\tau)\,, \tag{A.1.2.7}$$

$$\pi^{(0)}([\mathcal{X}^{\mathrm{sd,ss}}_\theta(\tau)]) = \sigma^{\mathrm{sd}}_\theta(\tau)\,. \tag{A.1.2.8}$$

A.2 The no-pole theorem

A.2.1. For a linear stack $\mathcal{X}$, a permissible stability condition τ on $\mathcal{X}$, and a slope $t \in T$, where T is the target of the map τ, recall the linear substack $\mathcal{X}^{\mathrm{ss}}(\tau; t) \subset \mathcal{X}$ defined in (3.5.4.2).

A key idea of the proof of the no-pole theorem is to consider the linear stack

$$\mathcal{X}^{\mathrm{ss},(n)}(\tau; t) = \coprod_{\alpha_1,\dots,\alpha_n \in \tau^{-1}(t) \cup \{0\}} \mathcal{X}^{\mathrm{ss},+}_{\alpha_1,\dots,\alpha_n}(\tau) \tag{A.2.1.1}$$

of n-step filtrations in $\mathcal{X}^{\mathrm{ss}}(\tau; t)$, where $n \in \mathbb{N}$ is a fixed integer, $\alpha_1, \dots, \alpha_n$ are classes in $\pi_0(\mathcal{X})$ that are either zero or of slope t, and we define

$$\mathcal{X}^{\mathrm{ss},+}_{\alpha_1,\dots,\alpha_n}(\tau) = \mathcal{X}^{\mathrm{ss}}(\tau; t)^+_{\alpha_1,\dots,\alpha_n} \subset \mathcal{F}\mathit{ilt}(\mathcal{X}^{\mathrm{ss}}(\tau; t)) \tag{A.2.1.2}$$

as the preimage of $\mathcal{X}^+_{\alpha_1,\dots,\alpha_n}$ under the induced map $\mathcal{F}\mathit{ilt}(\mathcal{X}^{\mathrm{ss}}(\tau; t)) \to \mathcal{F}\mathit{ilt}(\mathcal{X})$. Note that as in §3.3.4, we do not choose a canonical connected component of $\mathcal{F}\mathit{ilt}(\mathcal{X})$, but any choice will give the same construction up to a canonical isomorphism.

For $\alpha_1, \dots, \alpha_n$ as above, we also define an element

$$\sigma^{(n)}_{\alpha_1,\dots,\alpha_n}(\tau) \in \mathbb{M}(\mathcal{X}^{\mathrm{ss},+}_{\alpha_1,\dots,\alpha_n}(\tau)) \tag{A.2.1.3}$$

by the formula (A.1.2.3) for the linear stack $\mathcal{X}^{\mathrm{ss},(n)}(\tau; t)$ and its connected components lying in $\mathcal{X}^{\mathrm{ss},+}_{\alpha_1,\dots,\alpha_n}(\tau)$.

We also use the notation $\sigma^{(I)}_{(\alpha_i)_{i\in I}}(\tau)$ for a totally ordered set I with $|I| = n$ and a tuple $(\alpha_i \in \pi_0(\mathcal{X}))_{i \in I}$ for the element (A.2.1.3).

Similarly, if $\mathcal{X}$ is equipped with a self-dual structure, and τ is self-dual, then the linear stack $\mathcal{X}^{\mathrm{sd,ss},(n)}(\tau; 0)$ is equipped with an induced self-dual structure, where we identify

$$\mathcal{X}^{\mathrm{ss},+}_{\alpha_1,\dots,\alpha_n}(\tau) \simeq \mathcal{X}^{\mathrm{ss},+}_{\alpha_n^\vee,\dots,\alpha_1^\vee}(\tau) \, . \tag{A.2.1.4}$$

For classes $\alpha_1, \dots, \alpha_n \in \tau^{-1}(0) \cup \{0\}$ and $\rho \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, we also define an element

$$\sigma^{(2n+1),\mathrm{sd}}_{\alpha_1,\dots,\alpha_n,\rho}(\tau) \in \mathbb{M}(\mathcal{X}^{\mathrm{sd,ss},+}_{\alpha_1,\dots,\alpha_n,\rho}(\tau)) \tag{A.2.1.5}$$

as in (A.1.2.4) for the self-dual linear stack $\mathcal{X}^{\mathrm{ss},(2n+1)}(\tau; 0)$ and its connected components lying in $\mathcal{X}^{\mathrm{sd,ss},+}_{\alpha_1,\dots,\alpha_n,\rho}(\tau)$, the preimage of $\mathcal{X}^{\mathrm{sd},+}_{\alpha_1,\dots,\alpha_n,\rho}(\tau)$ as in §3.4.4 under the induced map $\mathcal{F}\mathit{ilt}(\mathcal{X}^{\mathrm{sd,ss}}(\tau; 0)) \to \mathcal{F}\mathit{ilt}(\mathcal{X}^{\mathrm{sd}})$, where we write $\mathcal{X}^{\mathrm{sd,ss}}(\tau; 0) = \mathcal{X}^{\mathrm{ss}}(\tau; 0)^{\mathrm{sd}}$.

We also denote (A.2.1.5) as $\sigma^{(I),\mathrm{sd}}_{(\alpha_i)_{i\in I}}(\tau)$, where I is a totally ordered set with $|I| = 2n + 1$, equipped with a (unique) order-reversing involution $(-)^\vee$, and the tuple $(\alpha_i)_{i\in I}$ satisfies $\alpha_0 \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, where $0 \in I$ is the unique fixed element of the involution, and $\alpha_i \in \pi_0(\mathcal{X})$ for $i \neq 0$, with $\alpha_{i^\vee} = \alpha_i^\vee$ for all such i, so that the element (A.2.1.5) corresponds to the tuple $(\alpha_1, \dots, \alpha_n, \rho, \alpha_n^\vee, \dots, \alpha_1^\vee)$ written using the order of I.

A.2.2. Proof of Theorem 5.5.5. We first spell out the proof of (ii), which is the main new result, and then explain how to modify the argument to prove (i), which is easier, and is a variant of Joyce's no-pole theorem in the linear case.

For $\alpha_1, \ldots, \alpha_k \in \pi_0(\mathcal{X}) \setminus \{0\}$ and $\rho \in \pi_0(\mathcal{X}^{\mathrm{sd}})$, we have

$$\mathrm{ev}_!([\mathcal{X}^{\mathrm{sd,ss,+}}_{\alpha_1,\ldots,\alpha_k,\rho}(\tau)]) = [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \diamond \cdots \diamond [\mathcal{X}^{\mathrm{ss}}_{\alpha_k}(\tau)] \diamond [\mathcal{X}^{\mathrm{sd,ss}}_{\rho}(\tau)] \,, \tag{A.2.2.1}$$

as motives on $\mathcal{X}^{\mathrm{sd,ss}}_{\alpha_1+\alpha_1^\vee+\cdots+\alpha_k+\alpha_k^\vee+\rho}(\tau)$, where $\diamond$ is the multiplication in the motivic Hall module. Recall from (5.5.3.1) the definition of $\epsilon^{\mathrm{sd}}_\theta(\tau)$, which can now be written as

$$\epsilon^{\mathrm{sd}}_\theta(\tau) = \sum_{\substack{k \geqslant 0;\ \alpha_1, \ldots, \alpha_k \in \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \theta = \alpha_1 + \alpha_1^\vee + \cdots + \alpha_k + \alpha_k^\vee + \rho \\ \tau(\alpha_1) = \cdots = \tau(\alpha_k) = 0}} \binom{-1/2}{k} \cdot \mathrm{ev}_!([\mathcal{X}^{\mathrm{sd,ss,+}}_{\alpha_1,\ldots,\alpha_k,\rho}(\tau)]) \,. \tag{A.2.2.2}$$

To prove the theorem, it is enough to prove that

$$\pi^{(n)}(\epsilon^{\mathrm{sd}}_\theta(\tau)) = 0 \tag{A.2.2.3}$$

for all integers $n > 0$.

By [84, Proposition 5.14], the virtual rank projection $\pi^{(n)}$ commutes with the pushforward $\mathrm{ev}_!$. We therefore study the virtual rank projections of $[\mathcal{X}^{\mathrm{sd,ss,+}}_{\alpha_1,\ldots,\alpha_k,\rho}(\tau)]$.

Write $I = \{1, \ldots, k, 0, k^\vee, \ldots, 1^\vee\}$ and $J = \{1, \ldots, n, 0, n^\vee, \ldots, 1^\vee\}$, with total orders given by the written order, and the obvious involutions. Applying (A.1.2.6) to the self-dual linear stack $\mathcal{X}^{\mathrm{ss},(I)}(\tau, 0)$, we obtain

$$\pi^{(n)}([\mathcal{X}^{\mathrm{sd,ss,+}}_{\alpha_1,\ldots,\alpha_k,\rho}(\tau)]) = \frac{1}{2^n n!} \cdot \sum_{\substack{\alpha_{i,j} \in \pi_0(\mathcal{X}) \text{ for } (i,j) \in (I \times J) \setminus (0,0);\ \rho_{0,0} \in \pi_0(\mathcal{X}^{\mathrm{sd}}): \\ \tau(\alpha_{i,j}) = 0 \text{ for all } (i,j) \text{ with } \alpha_{i,j} \neq 0, \\ \alpha_{i^\vee, j^\vee} = \alpha_{i,j}^\vee \text{ for all } (i,j), \\ \alpha_i = \sum_{j \in J} \alpha_{i,j} \text{ for all } i \in I, \\ \rho = \alpha_{0,1} + \alpha_{0,1}^\vee + \cdots + \alpha_{0,n} + \alpha_{0,n}^\vee + \rho_{0,0}\,, \\ \sum_{i \in I} \alpha_{i,j} \neq 0 \text{ for all } j \in J \setminus 0}} \sigma^{(I)}_{(\alpha_{i,1})_{i \in I}}(\tau) \circledcirc \cdots \circledcirc \sigma^{(I)}_{(\alpha_{i,n})_{i \in I}}(\tau) \circledcirc \sigma^{(I),\mathrm{sd}}_{\alpha_{1,0},\ldots,\alpha_{k,0},\rho_{0,0}}(\tau) \,. \tag{A.2.2.4}$$

We abbreviate each term in the sum (A.2.2.4) as $\sigma^{(I),\mathrm{sd}}_{\underline{\alpha}}(\tau)$, where $\underline{\alpha} = (\alpha_{i,j})_{i \in I,\, j \in J}$ is a matrix with $\alpha_{0,0} = \rho_{0,0}$, and write

$$\sigma^{\mathrm{sd}}_{\underline{\alpha}}(\tau) = (\pi^{(I),\mathrm{sd}})_!\, \sigma^{(I),\mathrm{sd}}_{\underline{\alpha}}(\tau) \,.$$

Let $A_{n,k}$ be the set of matrices $\underline{\alpha}$ that appear in (A.2.2.4) for some choice of $\alpha_1, \ldots, \alpha_k, \rho$. Then (A.2.2.2) and (A.2.2.4) imply that

$$\pi^{(n)}(\epsilon_\theta^{\mathrm{sd}}(\tau)) = \frac{1}{2^n n!} \cdot \sum_{k \geqslant 0} \sum_{\underline{\alpha} \in A_{n,k}} \binom{-1/2}{k} \cdot \sigma_{\underline{\alpha}}^{\mathrm{sd}}(\tau) \, . \tag{A.2.2.5}$$

Note that the element $\sigma_{\underline{\alpha}}^{\mathrm{sd}}(\tau)$ only depends on the equivalent class of $\underline{\alpha}$, where two matrices $\underline{\alpha} \in A_{n,k}$ and $\underline{\alpha}' \in A_{n,k'}$ are equivalent if for all $j \in J$, the subsequence of $(\alpha_{i,j})_{i \in I}$ with $\alpha_{i,j} \neq 0$ is the same as the subsequence of $(\alpha'_{i,j})_{i \in I'}$ with $\alpha'_{i,j} \neq 0$, where $I' = \{1, \ldots, k', 0, k'^\vee, \ldots, 1^\vee\}$. It is then enough to prove that for a fixed $\underline{\alpha}$, we have

$$\sum_{k \geqslant 0} \binom{-1/2}{k} \cdot \sum_{\substack{\underline{\alpha}' \in A_{n,k}: \\ \underline{\alpha}' \sim \underline{\alpha}}} 1 = 0 \, . \tag{A.2.2.6}$$

To prove this, we first observe that the number $\underline{\alpha}' \in A_{n,k}$ with $\underline{\alpha}' \sim \underline{\alpha}$ is equal to the number of subsets of $I \times J$, invariant under the involution $(i, j) \mapsto (i^\vee, j^\vee)$, such that the number of elements in each row $\{i\} \times J$ is non-zero unless $i = 0$, and the number of elements in each column $I \times \{j\}$ is equal to a_j, where a_j is the number of non-zero entries $\alpha_{i,j}$ in $\underline{\alpha}$ in that column. Note that $a_j = a_{j^\vee}$ for all $j \in J$.

Consider the generating series of this counting problem, with a formal variable x_j assigned to each a_j for $j \in \{1, \ldots, n, 0\}$. For convenience, we write $x_{j^\vee} = x_j$ for $j \in J$. Summing over $k \geqslant 0$, we obtain the generating series

$$\begin{aligned} F(x_1, \ldots, x_n, x_0) &= \sum_{k \geqslant 0} \binom{-1/2}{k} \cdot \Bigg(\sum_{\varnothing \neq J' \subset J} \prod_{j \in J'} x_j \Bigg)^k \cdot \sum_{J' \subset \{1, \ldots, n, 0\}} \prod_{j \in J'} x_j \\ &= \Bigg(\prod_{j \in J} (1 + x_j) \Bigg)^{-1/2} \cdot \prod_{j=0}^{n} (1 + x_j) \\ &= (1 + x_0)^{1/2} \, . \end{aligned} \tag{A.2.2.7}$$

Therefore, the left-hand side of (A.2.2.6) is zero unless $a_j = 0$ for all $j \neq 0$. But this cannot happen, as we assumed that $n > 0$. This proves (A.2.2.6), and hence (A.2.2.3).

Finally, for the linear case (i), we use an analogous argument. Fix a class $\alpha \in \pi_0(\mathscr{X}) \setminus \{0\}$, and set $t = \tau(\alpha)$. The definition (5.5.2.1) can be rewritten as

$$\epsilon_\alpha(\tau) = \sum_{\substack{k \geqslant 0;\ \alpha_1, \dots, \alpha_k \in \pi_0(\mathcal{X}) \setminus \{0\}: \\ \alpha = \alpha_1 + \cdots + \alpha_k \\ \tau(\alpha_1) = \cdots = \tau(\alpha_k)}} \frac{(-1)^{k-1}}{k} \cdot \mathrm{ev}_!([\mathcal{X}^{\mathrm{ss},+}_{\alpha_1,\dots,\alpha_k}(\tau)]) \, . \tag{A.2.2.8}$$

It follows from this and (A.1.2.5) for the linear stack $\mathcal{X}^{\mathrm{ss},(k)}(\tau;t)$ that $\pi^{(0)}(\epsilon_\alpha(\tau)) = 0$, and it suffices to show that $\pi^{(n)}(\epsilon_\alpha(\tau)) = 0$ for all $n > 1$. The key combinatorial identity (A.2.2.6) that we need to prove now becomes

$$\sum_{k \geqslant 0} \frac{(-1)^{k-1}}{k} \cdot b_k = 0 \, , \tag{A.2.2.9}$$

where we write $I = \{1, \dots, k\}$ and $J = \{1, \dots, n\}$, and we fix a sequence $(a_j \in \mathbb{N}_{>0})_{j \in J}$, and b_k is the number of subsets of $I \times J$ whose intersection with each row $\{i\} \times J$ is non-empty, and whose intersection with each column $I \times \{j\}$ has size precisely a_j. Again, consider the generating series of this counting problem, with a formal variable x_j assigned to each a_j, we obtain the series

$$\begin{aligned} G(x_1, \dots, x_n) &= \sum_{k \geqslant 0} \frac{(-1)^{k-1}}{k} \cdot \biggl(\sum_{\varnothing \neq J' \subset J} \prod_{j \in J'} x_j \biggr)^k \\ &= \log \biggl(\prod_{j \in J} (1 + x_j) \biggr) \\ &= \sum_{j \in J} \log(1 + x_j) \, . \end{aligned} \tag{A.2.2.10}$$

Therefore, the left-hand side of (A.2.2.9) is zero unless $a_j = 0$ for all but one $j \in J$, as the above expression is a linear combination of monomials of the form x_j^ℓ. Again, this is impossible, as we assumed that $n > 1$. □

Appendix B

Proof of the integral identity

This appendix is devoted to the proof of the motivic integral identities, Theorems 7.4.2 and 7.4.5, which we used to prove wall-crossing formulae for our Donaldson–Thomas invariants. We prove three versions of the identity: First, in §B.2, we prove a local version of the main theorem, using the theory of motivic nearby and vanishing cycles for stacks developed in §B.1. Then, in §B.3, we glue the local versions together to prove the global version, Theorem 7.4.2. Finally, in §B.4, we take Euler characteristics in the previous identity to obtain identities involving the numerical Behrend functions, Theorem 7.4.5.

Throughout this appendix, we work over an algebraically closed field K of characteristic zero. We mean by a *K-variety* a reduced, separated K-scheme of finite type.

B.1 Motivic vanishing cycles

B.1.1. In this section, we set up an important technical tool that will be used in the proof of the integral identity, the *motivic vanishing cycle map* for stacks, generalizing the construction of Bittner [15] from varieties to stacks.

B.1.2. For varieties. Let X be a K-variety, and let $f\colon X \to \mathbb{A}^1$ be a morphism. Write $X_0 = f^{-1}(0)$. Define the *nearby cycle map* of f, denoted by

$$\Psi_f\colon \mathbb{M}(X) \longrightarrow \mathbb{M}^{\hat{\mu}}(X_0)\ ,$$

to be the unique $\mathbb{M}(K)$-linear map such that for any smooth K-variety Z and any proper morphism $\rho\colon Z \to X$, we have

$$\Psi_f([Z]) = (\rho_0)_! \, (\mathrm{MF}_{f\circ\rho}) \in \mathbb{M}^{\hat{\mu}}(X_0) \, ,$$

where $\rho_0\colon Z_0 \to X_0$ is the restriction of ρ to $Z_0 = (f \circ \rho)^{-1}(0)$, and $\mathrm{MF}_{f\circ\rho} \in \mathbb{M}^{\hat{\mu}}(Z_0)$ is the motivic Milnor fibre of $f \circ \rho$. It follows from Bittner [15, Claim 8.2] that the map Ψ_f is well-defined.

Define the *vanishing cycle map* of f to be the map

$$\Phi_f = \Psi_f - \iota^{\hat{\mu}} \circ i^* \colon \mathbb{M}(X) \longrightarrow \mathbb{M}^{\hat{\mu}}(X_0) \, ,$$

where $i\colon X_0 \hookrightarrow X$ is the inclusion, and $\iota^{\hat{\mu}}\colon \mathbb{M}(X_0) \hookrightarrow \mathbb{M}^{\hat{\mu}}(X_0)$ is the inclusion.

B.1.3. For algebraic spaces. We now generalize the motivic nearby and vanishing cycle maps from varieties to algebraic spaces.

As in Bittner [15, Theorem 8.4], the nearby and vanishing cycle maps are compatible with pulling back along smooth morphisms. In particular, these maps define morphisms $\Psi, \Phi\colon \mathbb{M}(-) \to \mathbb{M}^{\hat{\mu}}((-)_0)$ of sheaves on the category of reduced, separated K-schemes of finite type with a morphism to $\mathbb{A}^1$, with the Nisnevich topology. Since algebraic spaces admit Nisnevich covers by affine K-varieties, as mentioned in §5.3.2, these morphisms of sheaves induce maps on their evaluations on algebraic spaces over K.

More precisely, for a quasi-separated algebraic space X locally of finite type over K, and a morphism $f\colon X \to \mathbb{A}^1$, we have the nearby and vanishing cycle maps

$$\Psi_f, \Phi_f \colon \mathbb{M}(X) \longrightarrow \mathbb{M}^{\hat{\mu}}(X_0) \, .$$

We state some of their properties below.

B.1.4. Theorem. *Let X, Y be quasi-separated algebraic spaces locally of finite type over K.*

(i) *Let $g\colon Y \to X$ be a proper morphism, and $f\colon X \to \mathbb{A}^1$ a morphism. Then we have a commutative diagram*

$$\begin{tikzcd} \mathbb{M}(Y) \arrow[r, "g_!"] \arrow[d, "\Psi_{f\circ g}"'] & \mathbb{M}(X) \arrow[d, "\Psi_f"] \\ \mathbb{M}^{\hat{\mu}}(Y_0) \arrow[r, "g_!"] & \mathbb{M}^{\hat{\mu}}(X_0) \, . \end{tikzcd} \tag{B.1.4.1}$$

(ii) *Let $g\colon Y \to X$ be a smooth morphism, and $f\colon X \to \mathbb{A}^1$ a morphism. Then we have a commutative diagram*

$$\begin{tikzcd} \mathbb{M}(X) \arrow[r, "g^*"] \arrow[d, "\Psi_f"'] & \mathbb{M}(Y) \arrow[d, "\Psi_{f\circ g}"] \\ \mathbb{M}^{\hat{\mu}}(X_0) \arrow[r, "g^*"] & \mathbb{M}^{\hat{\mu}}(Y_0)\,. \end{tikzcd} \tag{B.1.4.2}$$

Proof. The case when X and Y are K-varieties was proved in Bittner [15, Theorem 8.4]. The verification of (ii) for algebraic spaces is completely formal, by passing to Nisnevich covers by K-varieties.

We now prove (i) for algebraic spaces. Again, passing to a Nisnevich cover, we may assume that X is a K-variety. We claim that $\mathbb{M}(Y)$ is spanned over $\mathbb{M}(K)$ by classes $[Z]$ of proper morphisms $Z \to Y$, where Z is a smooth K-variety. Indeed, let $u\colon U \to Y$ be an arbitrary morphism, where U is an integral K-variety. By Nagata compactification, as in Conrad, Lieblich, and Olsson [41, Theorem 1.2.1], u can be factored as a dense open immersion $U \hookrightarrow V$ followed by a proper morphism $V \to Y$, where V is an integral algebraic space over K. By Chow's lemma for algebraic spaces, as in Knutson [97, IV, Theorem 3.1], there exists a K-variety W and a projective birational morphism $W \to V$. Applying a resolution of singularities, we may assume that W is smooth. Now $W \to Y$ is proper, and the difference $[W] - [U]$ is a sum of lower dimensional classes. An induction on the dimension of U verifies the claim.

Now, let $h\colon Z \to Y$ be a proper morphism, where Z is a smooth K-variety. Passing to a Nisnevich cover of Y by K-varieties, one can show that $\Psi_{f\circ g}([Z]) = h_!(\mathrm{MF}_{f\circ g\circ h})$. On the other hand, we have $\Psi_f([Z]) = (g \circ h)_!(\mathrm{MF}_{f\circ g\circ h})$ by definition. This completes the proof since such classes $[Z]$ span $\mathbb{M}(Y)$ over $\mathbb{M}(K)$. □

B.1.5. Theorem. *Let $\mathcal{X}$ be a stack over K that is Nisnevich locally a quotient stack in the sense of §6.2.2, and let $f\colon \mathcal{X} \to \mathbb{A}^1$ be a morphism. Write $\mathcal{X}_0 = f^{-1}(0)$. Then there is a unique $\widehat{\mathbb{M}}(K)$-linear map*

$$\Psi_f\colon \widehat{\mathbb{M}}(\mathcal{X}) \longrightarrow \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0)\,,$$

called the nearby cycle map *of f, such that for any K-scheme Y and any smooth morphism*

$g\colon Y \to \mathcal{X}$, we have a commutative diagram

$$\begin{tikzcd} \widehat{\mathbb{M}}(\mathcal{X}) \arrow[r, "g^*"] \arrow[d, "\Psi_f"'] & \widehat{\mathbb{M}}(Y) \arrow[d, "\Psi_{f\circ g}"] \\ \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0) \arrow[r, "g^*"] & \widehat{\mathbb{M}}^{\hat{\mu}}(Y_0) \, , \end{tikzcd} \tag{B.1.5.1}$$

where the right-hand map is defined in §B.1.2.

We then define the vanishing cycle map of f to be the map

$$\Phi_f = \Psi_f - \iota^{\hat{\mu}} \circ i^* \colon \widehat{\mathbb{M}}(\mathcal{X}) \longrightarrow \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0) \, ,$$

where $i\colon \mathcal{X}_0 \hookrightarrow \mathcal{X}$ and $\iota^{\hat{\mu}}\colon \widehat{\mathbb{M}}(\mathcal{X}_0) \hookrightarrow \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0)$ are the inclusions.

Proof. Let $(j_i\colon \mathcal{X}_i \to \mathcal{X})_{i\in I}$ be a Nisnevich cover, where each $\mathcal{X}_i \simeq U_i/G_i$, with U_i an algebraic space over K, acted on by a group $G_i \simeq \mathrm{GL}(n_i)$ for some n_i. Let $\pi_i\colon U_i \to \mathcal{X}_i$ be the projection.

First, note that the condition on Ψ_f implies that the same condition holds when Y is an algebraic space, with the right-hand map in (B.1.5.1) defined in §B.1.3. This can be seen by passing to a Nisnevich cover of Y by K-varieties, and applying Theorem 5.3.3 to this cover.

To define the map Ψ_f, by Theorem 5.3.3, it is enough to define it on each $\mathcal{X}_i$, and then verify that they agree on overlaps. Let $a \in \widehat{\mathbb{M}}(\mathcal{X})$ be an element. We define the element $\Psi_f(a) \in \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0)$ by giving its pullbacks $\Psi_f(a)_i = j_i^* \circ \Psi_f(a) \in \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_{i,0})$ for each i, where $\mathcal{X}_{i,0} = \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_0$. The condition on Ψ_f forces

$$\begin{aligned} \Psi_f(a)_i = j_i^* \circ \Psi_f(a) &= [G_i]^{-1} \cdot (\pi_i)_! \circ \pi_i^* \circ j_i^* \circ \Psi_f(a) \\ &= [G_i]^{-1} \cdot (\pi_i)_! \circ \Psi_{f\circ j_i \circ \pi_i} \circ \pi_i^* \circ j_i^*(a) \, , \end{aligned}$$

where $[G_i] \in \widehat{\mathbb{M}}(K)$ is the class of G_i and is invertible in $\widehat{\mathbb{M}}(K)$, and we applied (5.2.6.2) to π_i, using the fact that G_i is special. This shows that if the map Ψ_f exists, then it is unique.

To check that the elements $\Psi_f(a)_i$ agree on overlaps, let $1, 2 \in I$ be two indices, and form the pullback squares

$$\begin{tikzcd} U'' \arrow[r, "\pi_2''"] \arrow[d, "\pi_1''"'] \arrow[dr, phantom, "\ulcorner", very near start] & U_1' \arrow[r, "j_2''"] \arrow[d, "\pi_1'"'] \arrow[dr, phantom, "\ulcorner", very near start] & U_1 \arrow[d, "\pi_1"] \\ U_2' \arrow[r, "\pi_2'"] \arrow[d, "j_1''"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X}_{1,2} \arrow[r, "j_2'"] \arrow[d, "j_1'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X}_1 \arrow[d, "j_1"] \\ U_2 \arrow[r, "\pi_2"] & \mathscr{X}_2 \arrow[r, "j_2"] & \mathscr{X} , \end{tikzcd} \tag{B.1.5.2}$$

where U_1', U_2', U'' are algebraic spaces. We need to show that $(j_2')^*(\Psi_f(a)_1) = (j_1')^*(\Psi_f(a)_2)$. We have

$$\begin{aligned} & (j_2')^*(\Psi_f(a)_1) \\ &= [G_1]^{-1} \cdot (j_2')^* \circ (\pi_1)_! \circ \Psi_{f \circ j_1 \circ \pi_1} \circ (j_1 \circ \pi_1)^*(a) \\ &= [G_1]^{-1} \cdot (\pi_1')_! \circ (j_2'')^* \circ \Psi_{f \circ j_1 \circ \pi_1} \circ (j_1 \circ \pi_1)^*(a) \\ &= [G_1]^{-1} \cdot (\pi_1')_! \circ \Psi_{f \circ j_1 \circ \pi_1 \circ j_2''} \circ (j_1 \circ \pi_1 \circ j_2'')^*(a) \\ &= [G_1]^{-1} \cdot [G_2]^{-1} \cdot (\pi_1')_! \circ (\pi_2'')_! \circ (\pi_2'')^* \circ \Psi_{f \circ j_1 \circ \pi_1 \circ j_2''} \circ (j_1 \circ \pi_1 \circ j_2'')^*(a) \\ &= [G_1]^{-1} \cdot [G_2]^{-1} \cdot (\pi_1' \circ \pi_2'')_! \circ \Psi_{f \circ j_1 \circ \pi_1 \circ j_2'' \circ \pi_2''} \circ (j_1 \circ \pi_1 \circ j_2'' \circ \pi_2'')^*(a) , \end{aligned}$$

where we applied (5.2.3.2) in the second step, Theorem B.1.4 (ii) in the third and fifth steps, and (5.2.6.2) in the fourth step. This expression is now symmetric in the indices 1 and 2, so the element $\Psi_f(a)$ is well-defined.

It now remains to show that the map Ψ_f satisfies the required condition. Let Y be a K-variety and $\pi \colon Y \to \mathscr{X}$ a smooth morphism. For each $i \in I$, write $Y_i = Y \times_{\mathscr{X}} \mathscr{X}_i$. Then $(k_i \colon Y_i \to Y)_{i \in I}$ is a Nisnevich cover by algebraic spaces, and it suffices to show that

$$k_i^* \circ g^* \circ \Psi_f = k_i^* \circ \Psi_{f \circ g} \circ g^* \tag{B.1.5.3}$$

for each i. Consider the diagram

$$\begin{tikzcd} V_i \arrow[r, "\rho_i"] \arrow[d, "g_i'"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y_i \arrow[r, "k_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "g"] \\ U_i \arrow[r, "\pi_i"] & \mathscr{X}_i \arrow[r, "j_i"] & \mathscr{X} , \end{tikzcd} \tag{B.1.5.4}$$

where all squares are pullback squares. In particular, ρ_i is a principal G_i-bundle. For any

$a \in \widehat{\mathbb{M}}(\mathcal{X})$, we have

$$\begin{aligned}
& k_i^* \circ g^* \circ \Psi_f(a) \\
&= g_i^*(\Psi_f(a)_i) \\
&= [G_i]^{-1} \cdot g_i^* \circ (\pi_i)_! \circ \Psi_{f \circ j_i \circ \pi_i} \circ (j_i \circ \pi_i)^*(a) \\
&= [G_i]^{-1} \cdot (\rho_i)_! \circ (g_i')^* \circ \Psi_{f \circ j_i \circ \pi_i} \circ (j_i \circ \pi_i)^*(a) \\
&= [G_i]^{-1} \cdot (\rho_i)_! \circ \Psi_{f \circ j_i \circ \pi_i \circ g_i'} \circ (j_i \circ \pi_i \circ g_i')^*(a) \\
&= [G_i]^{-1} \cdot (\rho_i)_! \circ \Psi_{f \circ g \circ k_i \circ \rho_i} \circ (g \circ k_i \circ \rho_i)^*(a) \\
&= [G_i]^{-1} \cdot (\rho_i)_! \circ \rho_i^* \circ \Psi_{f \circ g \circ k_i} \circ (g \circ k_i)^*(a) \\
&= \Psi_{f \circ g \circ k_i} \circ (g \circ k_i)^*(a) \\
&= k_i^* \circ \Psi_{f \circ g} \circ g^*(a) \, ,
\end{aligned}$$

where we applied the monodromic version of (5.2.3.2) in the third step, Theorem B.1.4 (ii) in the fourth, sixth, and eighth steps, and the monodromic version of (5.2.6.2) in the seventh step. This proves the desired identity (B.1.5.3). □

B.1.6. The motivic Milnor fibre. Let $\mathcal{X}$ be a stack over K that is Nisnevich locally a quotient stack as in §6.2.2, and let $f \colon \mathcal{X} \to \mathbb{A}^1$ be a morphism. Write $\mathcal{X}_0 = f^{-1}(0)$. The *motivic Milnor fibre* of f is the element

$$\mathrm{MF}_f = \Psi_f([\mathcal{X}]) \in \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0) \, .$$

We relate this to the description of the motivic Milnor fibre for schemes in §6.2.3. Suppose that we are given a *resolution* of f, which is a representable proper morphism $\pi \colon \widetilde{\mathcal{X}} \to \mathcal{X}$, such that it restricts to an isomorphism on $\pi^{-1}(\mathcal{X} \setminus \mathcal{X}_0)$, and $\pi^{-1}(\mathcal{X}_0)$ is a simple normal crossings divisor in $\widetilde{\mathcal{X}}$, in the sense that it is so after pulling back along smooth morphisms from schemes. Let $(\mathcal{E}_i)_{i \in J}$ be the family of irreducible components of $\pi^{-1}(\mathcal{X}_0)$, and define $\mathcal{E}_I^\circ$ and $\widetilde{\mathcal{E}}_I^\circ$ for non-empty $I \subset J$ similarly to §6.2.3, where $\widetilde{\mathcal{E}}_I^\circ$ carries a natural $\hat{\mu}$-action. We then claim that

$$\mathrm{MF}_f = \sum_{\varnothing \neq I \subset J} (1 - \mathbb{L})^{|I|-1} \, [\widetilde{\mathcal{E}}_I^\circ] \, . \tag{B.1.6.1}$$

Indeed, this can be shown by a similar argument as in the proof of Theorem B.1.5, by first passing to a Nisnevich cover by quotient stacks, then using the relation (5.2.6.2) to further reduce to the case of algebraic spaces, and finally passing to a Nisnevich cover again to reduce to the case of affine varieties.

B.1.7. Theorem. *Let $\mathcal{X}$, $\mathcal{Y}$ be stacks over K that are Nisnevich locally quotient stacks as in §6.2.2.*

(i) *Let $g\colon \mathcal{Y} \to \mathcal{X}$ be a proper morphism, and $f\colon \mathcal{X} \to \mathbb{A}^1$ a morphism. Then we have a commutative diagram*

$$\begin{tikzcd} \widehat{\mathbb{M}}(\mathcal{Y}) \arrow[r, "g_!"] \arrow[d, "\Psi_{f \circ g}"'] & \widehat{\mathbb{M}}(\mathcal{X}) \arrow[d, "\Psi_f"] \\ \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{Y}_0) \arrow[r, "g_!"] & \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0) \, . \end{tikzcd} \tag{B.1.7.1}$$

(ii) *Let $g\colon \mathcal{Y} \to \mathcal{X}$ be a smooth morphism, and $f\colon \mathcal{X} \to \mathbb{A}^1$ a morphism. Then we have a commutative diagram*

$$\begin{tikzcd} \widehat{\mathbb{M}}(\mathcal{X}) \arrow[r, "g^*"] \arrow[d, "\Psi_f"'] & \widehat{\mathbb{M}}(\mathcal{Y}) \arrow[d, "\Psi_{f \circ g}"] \\ \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_0) \arrow[r, "g^*"] & \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{Y}_0) \, . \end{tikzcd} \tag{B.1.7.2}$$

In particular, we have $\mathrm{MF}_{f \circ g} = g^*(\mathrm{MF}_f)$.

Proof. For (i), we first restrict to the case when g is representable. By Theorem B.1.5, the map Ψ_f is determined by pullbacks along smooth morphisms from K-varieties to $\mathcal{X}$, so we may assume that $\mathcal{X}$ is a K-variety, and $\mathcal{Y}$ is an algebraic space that is proper over $\mathcal{X}$. This case is covered by Theorem B.1.4 (i).

For the general case, similarly, we may assume that $\mathcal{X} = X$ is a K-variety. It suffices to show that $g_! \circ \Psi_{f \circ g}([Z]) = \Psi_f \circ g_!([Z])$ for smooth K-varieties Z mapping to $\mathcal{Y}$, as these classes span $\widehat{\mathbb{M}}(\mathcal{Y})$ over $\widehat{\mathbb{M}}(K)$. Since $\mathcal{Y}$ is proper over X and has affine stabilizers, it has finite inertia, and admits a coarse space $\pi_{\mathcal{Y}}\colon \mathcal{Y} \to \overline{Y}$ by the Keel–Mori theorem [40; 91]. The morphism $\pi_{\mathcal{Y}}$ is a proper universal homeomorphism.

By Rydh's compactification theorem for representable morphisms of Deligne–Mumford stacks [142, Theorem B], we may choose a relative compactification $\mathcal{Z}$ of Z over $\mathcal{Y}$, such that there is a dense open immersion $i\colon Z \hookrightarrow \mathcal{Z}$ and a proper representable morphism $h\colon \mathcal{Z} \to \mathcal{Y}$. In particular, $\mathcal{Z}$ also has finite inertia, and admits a coarse space $\pi_{\mathcal{Z}}\colon \mathcal{Z} \to \overline{Z}$, which can be seen as a relative compactification of Z over $\overline{Y}$. We have a commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "i"] & \mathscr{Z} \arrow[r, "\pi_{\mathscr{Z}}"] \arrow[d, "h"'] & \overline{Z} \arrow[d, "\bar{h}"] \\ & \mathscr{Y} \arrow[r, "\pi_{\mathscr{Y}}"] \arrow[dr, "g"'] & \overline{Y} \arrow[d, "\bar{g}"] \\ & & X , \end{tikzcd} \tag{B.1.7.3}$$

where $\bar{g}$ and $\bar{h}$ are the induced morphisms, and all morphisms except i are proper. It is then enough to show that

$$(\pi_{\mathscr{Z}})_! \circ \Psi_{f\circ g\circ h}([Z]) = \Psi_{f\circ\bar{g}\circ\bar{h}} \circ (\pi_{\mathscr{Z}})_!([Z]) \,, \tag{B.1.7.4}$$

since the compatibility with pushing forward along h and $\bar{g} \circ \bar{h}$ is covered by the previous case.

We now apply Bergh and Rydh's *divisorialification theorem* [14, Theorem A] to a desingularization of the pair $(\mathscr{Z}, \mathscr{Z} \setminus Z)$ (see, for example, [58]), which gives a representable proper morphism $\overline{\mathscr{Z}} \to \mathscr{Z}$ that is an isomorphism on the preimage of Z, such that $\overline{\mathscr{Z}} \setminus Z = \mathscr{D}$ is a simple normal crossings divisor on $\overline{\mathscr{Z}}$, with smooth irreducible components $\mathscr{D}_i \subset \overline{\mathscr{Z}}$, and for each $x \in \overline{\mathscr{Z}}$, writing $I_x = \{i \in I \mid x \in \mathscr{D}_i\}$, étale locally around x, one has $\overline{\mathscr{Z}} \sim \prod_{i\in I_x} [\mathbb{A}^1/\mu_{n_i}] \times \mathbb{A}^{d-|I_x|}$, where $d = \dim \overline{\mathscr{Z}}$, each μ_{n_i} acts on $\mathbb{A}^1$ by scaling, and $\mathscr{D}_i$ corresponds to the locus where the i-th factor is zero; the number n_i is the order of the generic stabilizer of $\mathscr{D}_i$.

From now on, we assume that $\mathscr{Z} = \overline{\mathscr{Z}}$, since again, pushing forward along the representable morphism $\overline{\mathscr{Z}} \to \mathscr{Z}$ and the corresponding morphism of coarse spaces is already dealt with.

Now, choose a resolution $\pi \colon \widetilde{\mathscr{Z}} \to \mathscr{Z}$ for the morphism $\mathscr{Z} \to \mathbb{A}^1$, which is a composition of blow-ups along smooth centres. Then $\widetilde{\mathscr{Z}}$ still has the same local description as before. The local description implies that the coarse space of $\widetilde{\mathscr{Z}}$, denoted $\widetilde{Z}$, is a smooth algebraic space, and can be seen as a resolution for the morphism $\overline{Z} \to \mathbb{A}^1$.

For each $i \in I$, let $\widetilde{\mathscr{D}}_i \subset \widetilde{\mathscr{Z}}$ be the strict transform of $\mathscr{D}_i$, which is a smooth divisor, and let $(\mathscr{E}_j \subset \widetilde{\mathscr{Z}})_{j\in J}$ be the family of irreducible components of $\widetilde{\mathscr{Z}}_0$. Then by construction, all the divisors $\widetilde{\mathscr{D}}_i, \mathscr{E}_j \subset \widetilde{\mathscr{Z}}$ have simple normal crossings, and $\widetilde{\mathscr{Z}} \setminus \bigcup_{i\in I} \widetilde{\mathscr{D}}_i$ is an algebraic space. Let $\widetilde{D}_i, E_j \subset \widetilde{Z}$ be the corresponding divisors in the coarse spaces. For $I' \subset I$, write $\mathscr{D}_{I'} = \bigcap_{i \in I'} \mathscr{D}_i$ and $\widetilde{\mathscr{D}}_{I'} = \bigcap_{i \in I'} \widetilde{\mathscr{D}}_i$, with the convention that $\mathscr{D}_{\varnothing} = \mathscr{Z}$ and $\widetilde{\mathscr{D}}_{\varnothing} = \widetilde{\mathscr{Z}}$. Then, each $\widetilde{\mathscr{D}}_{I'}$ can be seen as a resolution for the morphism $\mathscr{D}_{I'} \to \mathbb{A}^1$. By §B.1.6, we have

$$
\begin{aligned}
(\pi_{\mathscr{Z}})_! \circ \Psi_{f \circ g \circ h}([Z]) &= \sum_{I' \subset I} (-1)^{|I'|} \cdot (\pi_{\mathscr{Z}})_! \circ \Psi_{f \circ g \circ h}([\mathscr{D}_{I'}]) \\
&= \sum_{I' \subset I} (-1)^{|I'|} \cdot \sum_{\varnothing \neq J' \subset J} (1 - \mathbb{L})^{|J'|-1} \, [\widetilde{\mathscr{E}}^{\circ}_{J'} \cap \widetilde{\mathscr{D}}_{I'}] \\
&= \sum_{\varnothing \neq J' \subset J} (1 - \mathbb{L})^{|J'|-1} \Big[\widetilde{\mathscr{E}}^{\circ}_{J'} \Big\backslash \bigcup_{i \in I} \widetilde{\mathscr{D}}_i \Big] \\
&= \sum_{\varnothing \neq J' \subset J} (1 - \mathbb{L})^{|J'|-1} \Big[\widetilde{E}^{\circ}_{J'} \Big\backslash \bigcup_{i \in I} \widetilde{D}_i \Big] \\
&= \Psi_{f \circ \bar{g} \circ \bar{h}} \circ (\pi_{\mathscr{Z}})_!([Z]) \, ,
\end{aligned}
$$

where the second last step used the fact that each $\widetilde{\mathscr{E}}^{\circ}_{J'} \setminus \bigcup_{i \in I} \widetilde{\mathscr{D}}_i$ is an algebraic space.

For (ii), similarly, the case when g is representable follows from Theorem B.1.4 (ii). For the general case, we may assume that $\mathscr{X}$ is a K-variety. Since $\Psi_{f \circ g}$ is determined by pullbacks along smooth morphisms from schemes to $\mathscr{Y}$, we can also assume that $\mathscr{Y}$ is a K-variety, and the result follows from Theorem B.1.4 (ii).

The final statement follows from applying (ii) to the element $[\mathscr{X}] \in \widehat{\mathbb{M}}(\mathscr{X})$. □

B.1.8. Remark. The length of the proof of Theorem B.1.7 is primarily due to the case of pushing forward along proper morphisms that are not necessarily representable. We will indeed need this general case in the proof of one of our main results, Theorem B.2.1, where g will be taken to be a *weighted blow-up* in the sense of §B.2.3.

B.2 The local model

B.2.1. Theorem. *Suppose that we are given the following data:*

- *A finite-dimensional $\mathbb{G}_{\mathrm{m}}$-representation V over K. Let*

$$
V = \bigoplus_{k \in \mathbb{Z}} V_k
$$

be the decomposition into weight spaces. Write $V_+ = \bigoplus_{k > 0} V_k$.

- *A K-variety U acted on by $\mathbb{G}_\mathrm{m}$, and a $\mathbb{G}_\mathrm{m}$-equivariant étale morphism $\iota \colon U \to V$. Write $U^0 = U^{\mathbb{G}_\mathrm{m}}$ for the fixed locus, and $U^+ = \mathrm{Map}^{\mathbb{G}_\mathrm{m}}(\mathbb{A}^1, U)$ for the attractor. For a point $u_0 \in U^0(K)$, write*

$$U^+(u_0) = \big\{ u \in U \mid \lim_{t \to 0} t \cdot u = u_0 \big\}$$

for the fibre of the limit map $U^+ \to U^0$ at u_0, which is canonically isomorphic to V_+.
- *A $\mathbb{G}_\mathrm{m}$-invariant function $f \colon U \to \mathbb{A}^1$, with $f(u_0) = 0$.*

Then we have the identities

$$\int_{u \in U^+(u_0)} \Psi_f([U])(u) = \mathbb{L}^{\dim V_+} \cdot \Psi_f([U^0])(u_0)\,, \tag{B.2.1.1}$$

$$\int_{u \in U^+(u_0)} \Phi_f([U])(u) = \mathbb{L}^{\dim V_+} \cdot \Phi_f([U^0])(u_0)\,. \tag{B.2.1.2}$$

Moreover, these hold as identities in $\mathbb{M}^{\hat{\mu}}(U^0)$, where we vary $u_0 \in U^0$.

This theorem can be seen as a generalization of the integral identity conjectured by Kontsevich and Soibelman [99, Conjecture 4], and proved by Lê [104], who restricted to the case when the $\mathbb{G}_\mathrm{m}$-action on V only has weights -1, 0, and 1. Compare also Joyce and Song [89, Theorem 5.11], where a similar identity involving Euler characteristics is proved, with the same restriction on the weights.

The rest of this section is devoted to the proof of Theorem B.2.1. In the following, we first provide preliminaries on weighted projective spaces and weighted blow-ups, and prove some preparatory results. Then, in Lemma B.2.8, we establish a weaker version of the theorem, using the theory of motivic nearby cycles for stacks developed in §B.1. Finally, in §B.2.9, we show that the weaker version implies the stronger version.

B.2.2. Weighted projective spaces. Let V be a finite-dimensional $\mathbb{G}_\mathrm{m}$-representation over K, with only positive weights. The *weighted projective space* of V is the quotient stack

$$^{\mathrm{w}}\mathbb{P}(V) = (V \setminus \{0\}) / \mathbb{G}_\mathrm{m}\,.$$

This is a proper Deligne–Mumford stack over K, since we have the identification

$$ {}^{\mathrm{w}}\mathbb{P}(V) \simeq \mathbb{P}(V) \Big/ \prod_{k=1}^{\dim V} \mu_{n_k} \, , $$

where $\mathbb{P}(V)$ is the usual projective space, and using a basis of eigenvectors of V, each n_k is the weight of the k-th coordinate, and μ_{n_k} acts by scaling the k-th coordinate.

We also consider the coarse space ${}^{\mathrm{cw}}\mathbb{P}(V)$ of ${}^{\mathrm{w}}\mathbb{P}(V)$, which is also given by

$$ {}^{\mathrm{cw}}\mathbb{P}(V) = \operatorname{Proj} K[V] \, , $$

where $K[V]$ is the free polynomial algebra on V, with $\mathbb{Z}$-grading given by the weights of V. It is an integral, normal, projective K-variety.

B.2.3. Weighted blow-ups. Let V be a finite-dimensional $\mathbb{G}_{\mathrm{m}}$-representation over K, with only positive weights. Let U be a smooth K-scheme, $U_0 \subset U$ a reduced closed subscheme, and let $p \colon U \to V$ be a smooth morphism such that $U_0 = p^{-1}(0)$.

Define the *weighted blow-up* of U along U_0, with weights given by those of V, as the quotient stack

$$ {}^{\mathrm{w}}\mathrm{Bl}_{U_0}(U) = \Big\{ (t, v, u) \in \mathbb{A}^1 \times (V \setminus \{0\}) \times U \;\Big|\; p(u) = t \cdot v \Big\} \Big/ \mathbb{G}_{\mathrm{m}} \, , \tag{B.2.3.1} $$

where $t \cdot (-)$ denotes the $\mathbb{G}_{\mathrm{m}}$-action naturally extended to $t \in \mathbb{A}^1$, and $\mathbb{G}_{\mathrm{m}}$ acts with weight -1 on $\mathbb{A}^1$, with the given weights on V, and trivially on U. Note that we have an isomorphism ${}^{\mathrm{w}}\mathrm{Bl}_{U_0}(U) \simeq U \times_V {}^{\mathrm{w}}\mathrm{Bl}_{\{0\}}(V)$.

The natural projection ${}^{\mathrm{w}}\mathrm{Bl}_{U_0}(U) \to U$ is proper. It restricts to an isomorphism over $U \setminus U_0$, and has fibres ${}^{\mathrm{w}}\mathbb{P}(V)$ over points in U_0. In particular, we have the relation

$$ [{}^{\mathrm{w}}\mathrm{Bl}_{U_0}(U)] = \frac{\mathbb{L}^{\dim V} - 1}{\mathbb{L} - 1} \cdot [U_0] + [U \setminus U_0] \tag{B.2.3.2} $$

of motives in $\widehat{\mathbb{M}}(U)$.

B.2.4. Lemma. *Let U be a separated algebraic space of finite type over K, acted on by a torus $T \simeq \mathbb{G}_{\mathrm{m}}^n$ for some n, such that points in U have finite stabilizers. Let $\mathcal{X} = U/T$ be the quotient stack.*

Then $\mathcal{X}$ admits a coarse space $\pi \colon \mathcal{X} \to X$ which is a proper universal homeomorphism, and *we have an isomorphism*

$$\pi_! = (\pi^*)^{-1} \colon \widehat{\mathbb{M}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathbb{M}}(X) \,. \tag{B.2.4.1}$$

A similar statement holds for $\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X})$.

Proof. Since U is separated, the inertia $\mathcal{I}_{\mathcal{X}}$ is a closed substack of $H \times \mathcal{X}$ for some finite group $H \subset T$, and is thus finite over $\mathcal{X}$. It then follows from the Keel–Mori theorem [40; 91] that $\mathcal{X}$ admits a coarse space $\pi \colon \mathcal{X} \to X$, and that π is a proper universal homeomorphism.

To prove (B.2.4.1), stratifying U by locally closed subspaces with constant stabilizers, we may assume that all points in U have the same stabilizers $H \subset T$, so that $X \simeq U/(T/H)$. To show that $\pi_! \circ \pi^* = \mathrm{id}$, it is enough to show that for any $\mathbb{K}$-variety Z and any morphism $g \colon Z \to X$, we have $[\mathcal{Z}] = [Z]$ in $\widehat{\mathbb{M}}(X)$, where $\mathcal{Z} = [Z \times_X \mathcal{X}]$. Writing $V = Z \times_X U$, we have $\mathcal{Z} \simeq V/T$ and $Z \simeq V/(T/H)$, so that $[\mathcal{Z}] = (\mathbb{L} - 1)^{-\dim T} \cdot [V] = [Z]$, where we used the fact that T/H is a torus of the same dimension as T. A similar argument shows that $\pi^* \circ \pi_! = \mathrm{id}$. □

B.2.5. Lemma. *In the situation of Theorem B.2.1, the locus in* U *where the morphism* ι *preserves* $\mathbb{G}_{\mathrm{m}}$*-stabilizers is open.*

Proof. For each $n > 1$, let $\zeta_n \in \mathbb{G}_{\mathrm{m}}(K)$ be a primitive n-th root of unity. It is enough to show that the locus of $u \in U$ such that $\zeta_n \cdot u \neq u$ and $\iota(\zeta_n \cdot u) = \iota(u)$ is closed. The latter condition is equivalent to $\iota(u) \in V_{(n)}$, where $V_{(n)} = \bigoplus_{k \in \mathbb{Z}} V_{kn} \subset V$. Write $U_{(n)} = \iota^{-1}(V_{(n)})$, which is étale over $V_{(n)}$, with a μ_n-action on its fibres, induced from the $\mathbb{G}_{\mathrm{m}}$-action on U. The locus where this action is trivial is open in $U_{(n)}$, proving the claim. □

B.2.6. Lemma. *In the situation of Theorem B.2.1, suppose that* U *is affine, and* ι *preserves* $\mathbb{G}_{\mathrm{m}}$*-stabilizers and sends closed* $\mathbb{G}_{\mathrm{m}}$*-orbits to closed* $\mathbb{G}_{\mathrm{m}}$*-orbits. Then the affine GIT quotient* $U /\!/ \mathbb{G}_{\mathrm{m}}$ *is normal.*

Proof. By Alper [2, Theorem 5.1], since ι is étale and preserves $\mathbb{G}_{\mathrm{m}}$-stabilizers, the induced morphism $\bar{\iota} \colon U /\!/ \mathbb{G}_{\mathrm{m}} \to V /\!/ \mathbb{G}_{\mathrm{m}}$ is étale at $[u] \in (U /\!/ \mathbb{G}_{\mathrm{m}})(K)$ for points $u \in U(K)$ such that the $\mathbb{G}_{\mathrm{m}}$-orbits of u and $\iota(u)$ are closed. By the assumption on closed orbits, it is enough to require that the $\mathbb{G}_{\mathrm{m}}$-orbit of u is closed. Since every S-equivalence class in U contains a closed orbit, the morphism $\bar{\iota}$ is étale, and it is enough to check that $V /\!/ \mathbb{G}_{\mathrm{m}}$ is normal. This follows from a standard fact in toric geometry, as in Cox, Little, and Schenck [43, Theorem 1.3.5], since $V /\!/ \mathbb{G}_{\mathrm{m}} \simeq \operatorname{Spec} K[S]$ for a saturated submonoid $S \subset \mathbb{Z}^{\dim V}$. $\square$

B.2.7. Lemma. *Let $f \colon X \to Y$ be a morphism of integral K-varieties. If f is bijective on K-points and Y is normal, then f is an isomorphism.*

Proof. By generic flatness and generic reducedness, f is flat over a dense open subset $U \subset Y$ with fibres $\operatorname{Spec} K$, and hence étale, hence an isomorphism $f^{-1}(U) \xrightarrow{\sim} U$. It follows that f is birational. Now, a version of Zariski's main theorem [66, IV-3, Corollary 8.12.10] implies that f is an open immersion, hence an isomorphism. $\square$

B.2.8. Lemma. *In the situation of Theorem B.2.1, write $V_- = \bigoplus_{k<0} V_k$, and for a point $u_0 \in U^0(K)$, consider the repeller*

$$U^-(u_0) = \Big\{ u \in U \;\Big|\; \lim_{t \to \infty} t \cdot u = u_0 \Big\} ,$$

defined in the same way as $U^+(u_0)$ for the opposite $\mathbb{G}_{\mathrm{m}}$-action on U.

Then we have the identity

$$\int_{u \in U^+(u_0)} \Psi_f([U])(u) - \int_{u \in U^-(u_0)} \Psi_f([U])(u) = (\mathbb{L}^{\dim V_+} - \mathbb{L}^{\dim V_-}) \cdot \Psi_f([U^0])(u_0) \, . \quad \text{(B.2.8.1)}$$

Moreover, this holds as an identity of monodromic motives on U^0, where we vary $u_0 \in U^0$.

Proof. Since U is smooth, by Sumihiro [145, Corollary 2], U admits a $\mathbb{G}_{\mathrm{m}}$-invariant affine open cover. We may thus assume that U is affine. Moreover, we apply this result whenever we shrink U, so we may assume that U is affine and connected throughout the proof.

Write U^+, U^- for the attractor and repeller of the $\mathbb{G}_{\mathrm{m}}$-action on U. By Halpern-Leistner [67, Propositions 1.3.1 and 1.3.2], the morphism $U^+ \to \iota^{-1}(V_+ \times V_0)$ is étale and a closed immersion, and hence an open immersion. We may thus remove the closed subsets $\iota^{-1}(V_+ \times V_0) \setminus U^+$ and $\iota^{-1}(V_- \times V_0) \setminus U^-$ from U, and assume that $U^\pm = \iota^{-1}(V_\pm \times V_0)$. The morphism ι now sends closed $\mathbb{G}_{\mathrm{m}}$-orbits to closed $\mathbb{G}_{\mathrm{m}}$-orbits.

By Lemma B.2.5, we may also assume that ι preserves $\mathbb{G}_{\mathrm{m}}$-stabilizers, by replacing U with a $\mathbb{G}_{\mathrm{m}}$-invariant open neighbourhood of U^0.

Let $U_\ominus = U \setminus U^-$, and let $U_\ominus^+ = U^+ \setminus U^0 \subset U_\ominus$. Consider the weighted blow-up

$$\pi_\ominus \colon \tilde{U}_\ominus = {}^{\mathrm{w}}\mathrm{Bl}_{U_\ominus^+}(U_\ominus) \longrightarrow U_\ominus \, ,$$

with weight k along the V_{-k}-direction for $k > 0$, and write $\tilde{f}_\ominus = f \circ \pi_\ominus$. Explicitly, as in §B.2.3, we may write

$$\begin{aligned} W_\ominus &= \Big\{ (t, v_-, u) \in \mathbb{A}^1 \times (V_- \setminus \{0\}) \times U_\ominus \;\Big|\; \iota(u)_- = t^{-1} \cdot v_- \Big\} \, , \\ \tilde{U}_\ominus &= W_\ominus / \mathbb{G}_\mathrm{m} \, , \end{aligned}$$

where $\iota(u)_-$ is the projection of $\iota(u)$ to V_-, and $\mathbb{G}_\mathrm{m}$ acts on $W_\ominus$ by $s \cdot (t, v_-, u) = (s^{-1} t, s^{-1} \cdot v_-, u)$. Note that $W_\ominus$ is smooth over $\mathbb{A}^1 \times (V_- \setminus \{0\})$, and hence over K. For any $u \in U_\ominus^+$, by Theorem B.1.7 (i), we have

$$\begin{aligned} & \int_{[v_-] \in {}^{\mathrm{w}}\mathbb{P}(V_-)} \Psi_{\tilde{f}_\ominus}([\tilde{U}_\ominus])([v_-], u) \\ &= \Psi_f([\tilde{U}_\ominus])(u) \\ &= \Psi_f\big([{}^{\mathrm{w}}\mathbb{P}(V_-) \times U_\ominus^+] + [U_\ominus \setminus U_\ominus^+]\big)(u) \\ &= \big([{}^{\mathrm{w}}\mathbb{P}(V_-)] - 1\big) \cdot \Psi_f([U_\ominus^+])(u) + \Psi_f([U_\ominus])(u) \\ &= \bigg(\frac{\mathbb{L}^{\dim V_-} - 1}{\mathbb{L} - 1} - 1 \bigg) \cdot \Psi_f([U_\ominus^+])(u) + \Psi_f([U])(u) \, , \end{aligned} \tag{B.2.8.2}$$

and this holds as an identity of monodromic motives on $U_\ominus^+$.

Define $p^+ \colon U_\ominus^+ \to U^0$ by $p^+(u) = \lim_{t \to 0} t \cdot u$. Then $f(u) = f(p^+(u))$ for all $u \in U_\ominus^+$, and by Theorem B.1.4 (ii), we have

$$\Psi_f([U_\ominus^+])(u) = \Psi_f([U^0])(p^+(u)) \tag{B.2.8.3}$$

for all $u \in U_\ominus^+$. Again, this holds as an identity of monodromic motives on $U_\ominus^+$, where the right-hand side means $(p^+)^* \circ \Psi_f([U^0])$.

Now, consider the quotient stack

$$\check{U}_\ominus = W_\ominus / \mathbb{G}_\mathrm{m}^2 \, , \tag{B.2.8.4}$$

where $\mathbb{G}_\mathrm{m}^2$ acts on $W_\ominus$ by $(s_1, s_2) \cdot (t, v_-, u) = (s_1^{-1} t, s_1^{-1} s_2 \cdot v_-, s_2 \cdot u)$. There is, by definition, a principal $\mathbb{G}_\mathrm{m}$-bundle $\tilde{\pi}_\ominus \colon \tilde{U}_\ominus \to \check{U}_\ominus$. There is a morphism $\check{f}_\ominus \colon \check{U}_\ominus \to \mathbb{A}^1$ induced by $\tilde{f}_\ominus$.

Let $U /\!\!/ \mathbb{G}_{\mathrm{m}}$ be the affine GIT quotient, and consider the reduced closed subscheme

$$\tilde{U} \subset {}^{\mathrm{cw}}\mathbb{P}(V_+) \times {}^{\mathrm{cw}}\mathbb{P}(V_-) \times (U /\!\!/ \mathbb{G}_{\mathrm{m}})$$

consisting of points $([\iota(u)_+], [\iota(u)_-], [u])$ and $([v_+], [v_-], [u_0])$ for $u \in U$, $v_\pm \in V_\pm \setminus \{0\}$, and $u_0 \in U^0$. There is a morphism $\tilde{f} \colon \tilde{U} \to \mathbb{A}^1$ induced by f.

Consider the projection $\check{\pi}_\ominus \colon \check{U}_\ominus \to \tilde{U}$ given by $(t, v_-, u) \mapsto ([\iota(u)_+], [v_-], [u])$. One can check that fibres of the composition $W_\ominus \to \tilde{U}$ are single $\mathbb{G}_{\mathrm{m}}^2$-orbits. We thus have an induced morphism $W_\ominus /\!\!/ \mathbb{G}_{\mathrm{m}}^2 \xrightarrow{\sim} \tilde{U}$, which is an isomorphism by Lemma B.2.7. Here, we used the fact that $\tilde{U}$ is normal by Lemma B.2.6, and the fact that $W_\ominus$ is integral since it is smooth and connected. In other words, the morphism $\check{\pi}_\ominus$ is a coarse space map. In particular, it is proper by Lemma B.2.4.

Since the projection $\tilde{\pi}_\ominus \colon \tilde{U}_\ominus \to \check{U}_\ominus$ is smooth and $\check{\pi}_\ominus$ is proper, by Theorem B.1.7 and Lemma B.2.4, for any $u \in U_\ominus^+$ and $[v_-] \in {}^{\mathrm{w}}\mathbb{P}(V_-)$, we have

$$\begin{aligned} \Psi_{\tilde{f}_\ominus}([\tilde{U}_\ominus])([0, v_-, u]) &= \Psi_{\check{f}_\ominus}([\check{U}_\ominus])([0, v_-, u]) \\ &= \Psi_{\tilde{f}}([\tilde{U}])([\iota(u)_+], [v_-], [p^+(u)]) \, , \end{aligned} \tag{B.2.8.5}$$

where $[u] = [p^+(u)]$ in $U /\!\!/ \mathbb{G}_{\mathrm{m}}$. Moreover, this holds as an identity of monodromic motives on ${}^{\mathrm{w}}\mathbb{P}(V_-) \times U_\ominus^+$.

Combining (B.2.8.2), (B.2.8.3), and (B.2.8.5), we obtain the identity

$$\begin{aligned} \Psi_f([U])(u) = & \int_{[v_-] \in {}^{\mathrm{w}}\mathbb{P}(V_-)} \Psi_{\tilde{f}}([\tilde{U}])([\iota(u)_+], [v_-], [p^+(u)]) \\ & + \left(1 - \frac{\mathbb{L}^{\dim V_-} - 1}{\mathbb{L} - 1}\right) \cdot \Psi_f([U^0])(p^+(u)) \, , \end{aligned} \tag{B.2.8.6}$$

where $u \in U_\oplus^+$ and $[v_-] \in {}^{\mathrm{w}}\mathbb{P}(V_-)$. Integrating over $u \in U^+(u_0) \setminus \{u_0\}$, we obtain

$$\begin{aligned} \int_{u \in U^+(u_0) \setminus \{u_0\}} \Psi_f([U])(u) = (\mathbb{L} - 1) \cdot & \int_{([v_+], [v_-]) \in {}^{\mathrm{w}}\mathbb{P}(V_+) \times {}^{\mathrm{w}}\mathbb{P}(V_-)} \Psi_{\tilde{f}}([\tilde{U}])([v_+], [v_-], [u_0]) \\ & + (\mathbb{L}^{\dim V_+} - 1) \cdot \left(1 - \frac{\mathbb{L}^{\dim V_-} - 1}{\mathbb{L} - 1}\right) \cdot \Psi_f([U^0])(u_0) \, . \end{aligned} \tag{B.2.8.7}$$

Subtracting the analogous identity for integrating over $U_-(u_0) \setminus \{u_0\}$, we arrive at the desired identity (B.2.8.1). □

B.2.9. Proof of Theorem B.2.1. Consider the $\mathbb{G}_{\mathrm{m}}$-representation $V' = V \times \mathbb{A}^1$, with the $\mathbb{G}_{\mathrm{m}}$-action on V as given, and on $\mathbb{A}^1$ by scaling. Let $U' = U \times \mathbb{A}^1$, with the $\mathbb{G}_{\mathrm{m}}$-action on U as given, and on $\mathbb{A}^1$ by scaling, and let $f' = f \circ \mathrm{pr}_1 \colon U' \to \mathbb{A}^1$, where $\mathrm{pr}_1 \colon U' \to U$ is the projection. Let $u_0' = (u_0, 0) \in U'^0 = U^0 \times \{0\}$. By Theorem B.1.4 (ii), we have $\Psi_{f'}([U']) = \mathrm{pr}_1^* \circ \Psi_f([U])$, and similarly, $\Psi_{f'}([U'^0]) = \mathrm{pr}_1^* \circ \Psi_f([U^0])$.

Applying Lemma B.2.8 to this new set of data, and simplifying the expression by the observations above, we obtain

$$\mathbb{L} \cdot \int_{u \in U^+(u_0)} \Psi_f([U])(u) - \int_{u \in U^-(u_0)} \Psi_f([U])(u) = (\mathbb{L}^{\dim V_+ + 1} - \mathbb{L}^{\dim V_-}) \cdot \Psi_f([U^0])(u_0) \,.$$

Subtracting the original identity (B.2.8.1) from this, and dividing by $\mathbb{L} - 1$, we obtain the desired identity (B.2.1.1).

Finally, (B.2.1.2) follows from (B.2.1.1) by the definition of Φ_f. $\square$

B.3 The motivic identity

B.3.1. In this section, we prove the main version of the integral identity, Theorem 7.4.2 using the local model, Theorem B.2.1.

In the following, let $\mathcal{X}$ be a (-1)-shifted symplectic stack over K as in Theorem 7.4.2. For convenience, when dealing with rings of motives, we always take classical truncations, and omit the subscript $(-)_{\mathrm{cl}}$ if no ambiguity is present.

We will prove the theorem in two steps. First, in Lemma B.3.2, we show that the theorem holds for a stack if it holds for a Nisnevich cover of the stack, reducing it to the case of fundamental stacks. Then, we deduce the case of fundamental stacks from the local version, Theorem B.2.1.

B.3.2. Lemma. *Let $(\mathcal{X}_{i,\mathrm{cl}} \to \mathcal{X}_{\mathrm{cl}})_{i \in I}$ be a Nisnevich cover, and set $\mathcal{X}_i = \mathcal{X}_{i,\mathrm{cl}} \times_{\mathcal{X}_{\mathrm{cl}}} \mathcal{X}_{\mathrm{cl}}$, so that each $\mathcal{X}_i$ is equipped with an induced (-1)-shifted symplectic structure and orientation. Then, if Theorem 7.4.2 holds for each $\mathcal{X}_i$, then it holds for $\mathcal{X}$.*

Proof. For each i, consider the diagram

$$\begin{tikzcd} \mathscr{G}rad(\mathcal{X}_{i,\mathrm{cl}}) \arrow[d] & \mathscr{F}ilt(\mathcal{X}_{i,\mathrm{cl}}) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}"] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & \mathcal{X}_{i,\mathrm{cl}} \arrow[d] \\ \mathscr{G}rad(\mathcal{X}_{\mathrm{cl}}) & \mathscr{F}ilt(\mathcal{X}_{\mathrm{cl}}) \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}"] & \mathcal{X}_{\mathrm{cl}} \, , \end{tikzcd} \tag{B.3.2.1}$$

where the left-hand square is a pullback square as in §3.8.2. Therefore, there is a commutative diagram

$$\begin{tikzcd} \widehat{\mathbb{M}}^{\hat{\mu}}(\mathscr{G}rad(\mathcal{X}_{i,\mathrm{cl}})) & \widehat{\mathbb{M}}^{\hat{\mu}}(\mathscr{F}ilt(\mathcal{X}_{i,\mathrm{cl}})) \arrow[l, "\mathrm{gr}_!"'] & \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_{i,\mathrm{cl}}) \arrow[l, "\mathrm{ev}^*"'] \\ \widehat{\mathbb{M}}^{\hat{\mu}}(\mathscr{G}rad(\mathcal{X}_{\mathrm{cl}})) \arrow[u] & \widehat{\mathbb{M}}^{\hat{\mu}}(\mathscr{F}ilt(\mathcal{X}_{\mathrm{cl}})) \arrow[l, "\mathrm{gr}_!"'] \arrow[u] & \widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{X}_{\mathrm{cl}}) \, , \arrow[l, "\mathrm{ev}^*"'] \arrow[u] \end{tikzcd} \tag{B.3.2.2}$$

where the vertical maps are the pullback maps.

By Halpern-Leistner [67, Corollary 1.1.7], we have $\mathscr{G}rad(\mathcal{X}_{i,\mathrm{cl}}) \xrightarrow{\sim} \mathscr{G}rad(\mathcal{X}_{\mathrm{cl}}) \times_{\mathcal{X}} \mathcal{X}_{i,\mathrm{cl}}$ for all i. Therefore, the family $(\mathscr{G}rad(\mathcal{X}_i) \to \mathscr{G}rad(\mathcal{X}))_{i \in I}$ is a Nisnevich cover on classical truncations. By Theorem 5.3.3, it is enough to check the identity (7.4.2.2) after pulling back to each $\mathscr{G}rad(\mathcal{X}_{i,\mathrm{cl}})$. But this follows from the identity (7.4.2.2) for each $\mathcal{X}_i$, the commutativity of (B.3.2.2), the relation (6.2.7.1) establishing the compatibility of the motivic Behrend function with smooth pullbacks, and the fact that the rank of the tangent complex of ${}^{\mathrm{d}}\mathscr{F}ilt(\mathcal{X}_i)$ agrees with that of ${}^{\mathrm{d}}\mathscr{F}ilt(\mathcal{X})$ on the corresponding components, which follows from (3.8.3.2). □

B.3.3. Lemma. *Suppose we have a pullback diagram of d-critical stacks*

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"] & \mathcal{X} \, , \end{tikzcd} \tag{B.3.3.1}$$

where all morphisms are smooth and compatible with the d-critical structures.

Let $K_{\mathcal{X}}^{1/2} \to \mathcal{X}$ and $K_{\mathcal{Y}}^{1/2} \to \mathcal{Y}$ be orientations, not necessarily compatible with f. Let $K_{\mathcal{X}'}^{1/2} \to \mathcal{X}'$ and $K_{\mathcal{Y}'}^{1/2} \to \mathcal{Y}'$ be the orientations induced by $K_{\mathcal{X}}^{1/2}$ and $K_{\mathcal{Y}}^{1/2}$, respectively, as mentioned in §6.2.4. Then we have

$$g'^* \circ \Upsilon\big(K_{\mathcal{Y}}^{1/2} \otimes f^*(K_{\mathcal{X}}^{-1/2}) \otimes \det(\mathbb{L}_{\mathcal{Y}/\mathcal{X}})^{-1}\big) = \Upsilon\big(K_{\mathcal{Y}'}^{1/2} \otimes f'^*(K_{\mathcal{X}'}^{-1/2}) \otimes \det(\mathbb{L}_{\mathcal{Y}'/\mathcal{X}'})^{-1}\big) \tag{B.3.3.2}$$

in $\widehat{\mathbb{M}}^{\hat{\mu}}(\mathcal{Y}')$, where Υ is the map from §6.1.4, and the parts in $\Upsilon(\dots)$ are line bundles with trivial square, and can be seen as μ_2-bundles.

Proof. These line bundles have trivial square by Joyce [86, Lemma 2.58]. We have

$$\begin{aligned}
& g'^*\big(K_{\mathcal{Y}}^{1/2} \otimes f^*(K_{\mathcal{X}}^{-1/2}) \otimes \det(\mathbb{L}_{\mathcal{Y}/\mathcal{X}})^{-1}\big) \\
\simeq{} & g'^*(K_{\mathcal{Y}}^{1/2}) \otimes f'^* \circ g^*(K_{\mathcal{X}}^{-1/2}) \otimes \det(g'^*(\mathbb{L}_{\mathcal{Y}/\mathcal{X}}))^{-1} \\
\simeq{} & K_{\mathcal{Y}'}^{1/2} \otimes \det(\mathbb{L}_{\mathcal{Y}'/\mathcal{Y}})^{-1} \otimes f'^*(K_{\mathcal{X}'}^{-1/2}) \otimes f'^* \circ \det(\mathbb{L}_{\mathcal{X}'/\mathcal{X}}) \otimes \det(\mathbb{L}_{\mathcal{Y}'/\mathcal{X}'})^{-1} \\
\simeq{} & K_{\mathcal{Y}'}^{1/2} \otimes f'^*(K_{\mathcal{X}'}^{-1/2}) \otimes \det(\mathbb{L}_{\mathcal{Y}'/\mathcal{X}'})^{-1} \, ,
\end{aligned}$$

and applying Υ gives the desired identity. □

B.3.4. Lemma. *Let $\mathcal{X}$ be an n-shifted symplectic stack over K. Then we have an isomorphism*

$$\mathrm{sf}^*(\mathbb{T}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}) \simeq \mathrm{op}^* \circ \mathrm{sf}^*(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}[n])$$

of perfect complexes on ${}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})$, where op is the involution of ${}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})$ induced by the morphism $(-)^{-1} \colon */\mathbb{G}_{\mathrm{m}} \to */\mathbb{G}_{\mathrm{m}}$.

Proof. By Halpern-Leistner [67, Lemma 1.2.3], we have $\mathrm{sf}^*(\mathbb{T}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}) \simeq \mathrm{tot}^*(\mathbb{T}_{\mathcal{X}})_{\geqslant 0}$, where $(-)_{\geqslant 0}$ denotes taking the part with non-negative weights with respect to the natural $\mathbb{G}_{\mathrm{m}}$-action. Consequently, we have $\mathrm{op}^* \circ \mathrm{sf}^*(\mathbb{T}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}) \simeq \mathrm{tot}^*(\mathbb{T}_{\mathcal{X}})_{\leqslant 0}$. Its dual shifted by n becomes $\mathrm{tot}^*(\mathbb{L}_{\mathcal{X}}[n])_{\geqslant 0} \simeq \mathrm{tot}^*(\mathbb{T}_{\mathcal{X}})_{\geqslant 0}$. □

B.3.5. Proof of Theorem 7.4.2. By Lemma B.3.2, we may assume that $\mathcal{X}$ is fundamental. Let $\mathcal{X} \simeq S/G$, where S is an affine K-variety, and $G = \mathrm{GL}(n)$ for some n. The classical truncation of the correspondence (7.4.2.1) can be written as

$$\coprod_{\lambda\colon \mathbb{G}_{\mathrm{m}} \to G} S^{\lambda}/L_{\lambda} \xleftarrow{\mathrm{gr}} \coprod_{\lambda\colon \mathbb{G}_{\mathrm{m}} \to G} S^{\lambda,+}/P_{\lambda} \xrightarrow{\mathrm{ev}} S/G \, ,$$

with notations as in Example 3.2.4. The assumption on G implies that all the groups L_{λ} and P_{λ} are special groups.

We fix a cocharacter $\lambda\colon \mathbb{G}_{\mathrm{m}} \to G$, and prove the identity on the component $S^{\lambda,+}/P_{\lambda}$. We may assume that $S^{\lambda,+} \neq \varnothing$.

By Joyce [86, Remark 2.47], shrinking S if necessary, we may assume that there exists a smooth affine K-scheme U acted on by G, and a G-invariant function $f\colon U \to \mathbb{A}^1$, such that $\mathcal{X}$ is isomorphic as a d-critical stack to the critical locus $\mathrm{Crit}(f)/G$, and $S \simeq \mathrm{Crit}(f)$. We now have a commutative diagram

$$
\begin{tikzcd}
U^\lambda \arrow[d, "\pi^0"'] & U^{\lambda,+} \arrow[l, "p"'] \arrow[r, "i"] \arrow[d, "\pi^+"] & U \arrow[d, "\pi"] \\
U^\lambda/L_\lambda & U^{\lambda,+}/P_\lambda \arrow[l, "\mathrm{gr}"'] \arrow[r, "\mathrm{ev}"] & U/G\,.
\end{tikzcd}
\tag{B.3.5.1}
$$

Let $0 \in S^\lambda$ be a K-point, and let $V = \mathbb{T}_U|_0$ be the tangent space. Consider the $\mathbb{G}_\mathrm{m}$-actions on U and V via the cocharacter λ. By Luna [107, Lemma in §III.1], shrinking U if necessary, we may choose a $\mathbb{G}_\mathrm{m}$-equivariant étale morphism $\iota \colon U \to V$ such that $\iota(0) = 0$. Applying Theorem B.2.1 gives the identity

$$
p_! \circ i^* \circ \Phi_f([U]) = \mathbb{L}^{\dim V_+^\lambda} \cdot \Phi_f([U^\lambda])\,, \tag{B.3.5.2}
$$

where $V_+^\lambda \subset V$ is the subspace where $\mathbb{G}_\mathrm{m}$ acts with positive weights. Note that $\Phi_f(U)$ is supported on S by its definition. Let $K_S^{1/2}$ be the orientation of the d-critical scheme S induced from that of $\mathcal{X}$. One computes that

$$
\begin{aligned}
&\mathrm{gr}_! \circ \mathrm{ev}^*(\nu_\mathcal{X}^\mathrm{mot}) \\
&\quad = [P_\lambda]^{-1} \cdot \mathrm{gr}_! \circ \pi_!^+ \circ (\pi^+)^* \circ \mathrm{ev}^*(\nu_\mathcal{X}^\mathrm{mot}) \\
&\quad = [P_\lambda]^{-1} \cdot \pi_!^0 \circ p_! \circ i^* \circ \pi^*(\nu_\mathcal{X}^\mathrm{mot}) \\
&\quad = \mathbb{L}^{\dim G/2} \cdot [P_\lambda]^{-1} \cdot \pi_!^0 \circ p_! \circ i^*(\nu_S^\mathrm{mot}) \\
&\quad = -\mathbb{L}^{\dim G/2 - \dim V/2} \cdot [P_\lambda]^{-1} \cdot \pi_!^0 \circ p_! \circ i^*\big(\Phi_f([U]) \cdot \Upsilon(K_S^{1/2} \otimes K_U^{-1}|_S)\big) \\
&\quad = -\mathbb{L}^{\dim G/2 - \dim V/2} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad\qquad \pi_!^0 \circ p_!\Big(i^* \circ \Phi_f([U]) \cdot i^* \circ \pi^* \circ \Upsilon\big(K_\mathcal{X}^{1/2} \otimes K_{U/G}^{-1}|_\mathcal{X}\big)\Big) \\
&\quad = -\mathbb{L}^{\dim G/2 - \dim V/2} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad\qquad \pi_!^0 \circ p_!\Big(i^* \circ \Phi_f([U]) \cdot (\pi^+)^* \circ \mathrm{ev}^* \circ \Upsilon\big(K_\mathcal{X}^{1/2} \otimes K_{U/G}^{-1}|_\mathcal{X}\big)\Big) \\
&\quad = -\mathbb{L}^{\dim G/2 - \dim V/2} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad\qquad \pi_!^0 \circ p_!\Big(i^* \circ \Phi_f([U]) \cdot (\pi^+)^* \circ \mathrm{gr}^* \circ \Upsilon\big(K_{\mathrm{d}\mathcal{G}rad(\mathcal{X})}^{1/2} \otimes K_{U^\lambda/L_\lambda}^{-1}|_{S^\lambda/L_\lambda}\big)\Big) \\
&\quad = -\mathbb{L}^{\dim G/2 - \dim V/2} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad\qquad \pi_!^0 \circ p_!\Big(i^* \circ \Phi_f([U]) \cdot p^* \circ (\pi^0)^* \circ \Upsilon\big(K_{\mathrm{d}\mathcal{G}rad(\mathcal{X})}^{1/2} \otimes K_{U^\lambda/L_\lambda}^{-1}|_{S^\lambda/L_\lambda}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\mathbb{L}^{\dim G/2-\dim V/2} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad \pi^0_!\Big(p_! \circ i^* \circ \Phi_f([U]) \cdot (\pi^0)^* \circ \Upsilon\big(K^{1/2}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \otimes K^{-1}_{U^\lambda/L_\lambda}|_{S^\lambda/L_\lambda}\big)\Big) \\
&= -\mathbb{L}^{\dim G/2-\dim V/2+\dim V^\lambda_+} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad \pi^0_!\Big(\Phi_f([U^\lambda]) \cdot (\pi^0)^* \circ \Upsilon\big(K^{1/2}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \otimes K^{-1}_{U^\lambda/L_\lambda}|_{S^\lambda/L_\lambda}\big)\Big) \\
&= -\mathbb{L}^{\dim G/2-\dim V/2+\dim V^\lambda_+} \cdot [P_\lambda]^{-1} \cdot \\
&\qquad \pi^0_!\Big(\Phi_f([U^\lambda]) \cdot \Upsilon\big(K^{1/2}_{S^\lambda} \otimes K^{-1}_{U^\lambda}|_{S^\lambda}\big)\Big) \\
&= \mathbb{L}^{\dim G/2-\dim V/2+\dim V^\lambda_+-\dim V^\lambda_0/2} \cdot [P_\lambda]^{-1} \cdot \pi^0_!(\nu^{\mathrm{mot}}_{S^\lambda}) \\
&= \mathbb{L}^{(\dim G-\dim L_\lambda)/2+(\dim V^\lambda_+-\dim V^\lambda_-)/2} \cdot [P_\lambda]^{-1} \cdot \pi^0_! \circ (\pi^0)^*(\nu^{\mathrm{mot}}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})}) \\
&= \mathbb{L}^{(\dim G-\dim L_\lambda)/2+(\dim V^\lambda_+-\dim V^\lambda_-)/2} \cdot [P_\lambda]^{-1} \cdot [L_\lambda] \cdot \nu^{\mathrm{mot}}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \\
&= \mathbb{L}^{(\dim V^\lambda_+-\dim V^\lambda_-)/2} \cdot \nu^{\mathrm{mot}}_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})} \, .
\end{aligned}
$$

Here, the first step uses (5.2.6.2); the third uses (6.2.6.1); the fourth uses (6.2.5.1); the fifth uses Lemma B.3.3, where the morphism f there is taken to be an isomorphism; the seventh uses the fact that the shifted Lagrangian correspondence (7.4.2.1) is oriented by Theorem 3.8.5, and the fact that the orientation for ${}^{\mathrm{d}}\mathcal{G}rad({}^{\mathrm{d}}\mathrm{Crit}(f \colon U/G \to \mathbb{A}^1))$ induced by the canonical one $K_{U/G}$ is given by K_{U^λ/L_λ}; the ninth uses (5.2.3.1); the tenth is the key step, and uses (B.3.5.2); the eleventh is analogous to the fifth; the twelfth uses (6.2.5.1) again; the thirteenth uses (6.2.6.1) again; the fourteenth uses (5.2.6.2) again; and the final step uses the relation $[P_\lambda] = [L_\lambda] \cdot \mathbb{L}^{(\dim G-\dim L_\lambda)/2}$.

Finally, we verify that $\operatorname{vdim} {}^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X}) = \dim V^\lambda_+ - \dim V^\lambda_-$, where ${}^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X}) \subset {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})$ is the open and closed substack corresponding to the cocharacter λ. Indeed, let $\mathcal{X}' = {}^{\mathrm{d}}\mathrm{Crit}(f \colon U/G \to \mathbb{A}^1)$ be the derived critical locus, with the natural (-1)-shifted symplectic structure, so $\mathcal{X}'_{\mathrm{cl}} \simeq \mathcal{X}_{\mathrm{cl}}$. For $x \in S^\lambda(K)$, by Lemma B.3.4, one has

$$
\begin{aligned}
\operatorname{rank}(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X})}|_x) &= \operatorname{rank}^{[0,1]}(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X})}|_x) - \operatorname{rank}^{[0,1]}(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt^{-\lambda}(\mathcal{X})}|_x) \\
&= \operatorname{rank}^{[0,1]}(\mathbb{L}_{\mathcal{F}ilt^\lambda(\mathcal{X}_{\mathrm{cl}})}|_x) - \operatorname{rank}^{[0,1]}(\mathbb{L}_{\mathcal{F}ilt^{-\lambda}(\mathcal{X}_{\mathrm{cl}})}|_x) \\
&= \operatorname{rank}(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X}')}|_x) \, , \qquad \text{(B.3.5.3)}
\end{aligned}
$$

where $\mathrm{rank}^{[0,1]} = \dim \mathrm{H}^0 - \dim \mathrm{H}^1$. We have a presentation

$$\mathbb{L}_{\mathcal{X}'}|_x \simeq \Big(\mathfrak{g} \longrightarrow \mathbb{T}_U|_x \longrightarrow \mathbb{L}_U|_x \longrightarrow \mathfrak{g}^\vee \Big) \tag{B.3.5.4}$$

with degrees in $[-2, 1]$, where $\mathfrak{g}$ is the Lie algebra of G. By Halpern-Leistner [67, Lemma 1.2.3], we have $\mathrm{sf}^*(\mathbb{L}_{^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X})}) \simeq \mathrm{tot}^*(\mathbb{L}_{\mathcal{X}})_{\leqslant 0}$, where $(-)_{\leqslant 0}$ denotes the part of non-positive weights with respect to the natural $\mathbb{G}_\mathrm{m}$-action. This now gives

$$\mathbb{L}_{^{\mathrm{d}}\mathcal{F}ilt^\lambda(\mathcal{X}')}|_x \simeq \Big(\mathfrak{p}_\lambda \longrightarrow \mathbb{T}_{U^{\lambda,-}}|_x \longrightarrow \mathbb{L}_{U^{\lambda,+}}|_x \longrightarrow \mathfrak{p}_{-\lambda}^\vee \Big) , \tag{B.3.5.5}$$

where $\mathfrak{p}_\lambda$ is the Lie algebra of P_λ, and $-\lambda$ is the opposite cocharacter of λ. Note that $\dim P_\lambda = \dim P_{-\lambda}$ and that $\dim U^{\lambda,\pm} = \dim V_\pm^\lambda + \dim V_0^\lambda$. It follows that $\mathrm{vdim}\, {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})$, which is equal to the rank of (B.3.5.5) by (B.3.5.3), is $\dim V_+^\lambda - \dim V_-^\lambda$. □

B.4 The numeric identity

B.4.1. In this section, we deduce the numeric version of the integral identity, Theorem 7.4.5 from the motivic identity, Theorem 7.4.2.

In the following, let $\mathcal{X}$ be a (-1)-shifted symplectic stack over K as in Theorem 7.4.5.

B.4.2. Proof of Theorem 7.4.5. By a similar argument as in the proof of Lemma B.3.2, passing to a representable étale cover of $\mathcal{X}$ by fundamental stacks, which induces representable étale covers of $\mathcal{G}rad(\mathcal{X})$ and $\mathcal{F}ilt(\mathcal{X})$ as in §3.8.2, it is enough to prove the theorem when $\mathcal{X} \simeq S/G$ is fundamental, where S is an affine K-scheme acted on by a reductive group G. Here, we are using étale descent for constructible functions, instead of Nisnevich descent for rings of motives.

As in §B.3.5, shrinking S if necessary, we may assume that there exists a smooth affine K-scheme U acted on by G, and a G-invariant function $f \colon U \to \mathbb{A}^1$, such that $\mathcal{X}$ is isomorphic as a d-critical stack to the critical locus $\mathrm{Crit}(f)/G$. Now, $\mathcal{X}$ comes with a natural orientation, and the motivic Behrend function $\nu_{\mathcal{X}}^{\mathrm{mot}}$ is defined.

Applying Theorem 7.4.2, then evaluating the Euler characteristics at γ, we obtain the identity

$$\int_{\varphi\in\mathrm{gr}^{-1}(\gamma)} \nu_{\mathcal{X}}(\mathrm{ev}(\varphi))\, d\chi = (-1)^{\mathrm{vdim}_\gamma\, {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})} \cdot \nu_{\mathcal{G}rad(\mathcal{X})}(\gamma)\,. \tag{B.4.2.1}$$

Let $\varphi_0 = \mathrm{sf}(\gamma)$. Then the left-hand side of (B.4.2.1) is equal to $\nu_{\mathcal{X}}(\mathrm{ev}(\varphi_0)) = \nu_{\mathcal{X}}(\mathrm{tot}(\gamma))$, since the integrand is $\mathbb{G}_{\mathrm{m}}$-invariant and φ_0 is in the closure of all $\mathbb{G}_{\mathrm{m}}$-orbits. Also, by Lemma B.3.4, we have

$$\mathrm{vdim}_\gamma\, {}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X}) = \mathrm{rank}^{[0,1]}\, \mathbb{L}_{\mathcal{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\gamma)} - \mathrm{rank}^{[0,1]}\, \mathbb{L}_{\mathcal{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\bar{\gamma})}\,. \tag{B.4.2.2}$$

This verifies (7.4.5.1).

For (7.4.5.2), apply Theorem 7.4.2 again, then take the difference of the evaluations at γ and $\bar{\gamma}$. This gives the identity

$$\begin{aligned}
&(\mathbb{L}-1)\cdot\Bigg[\int_{\varphi\in\mathbb{P}(\mathrm{gr}^{-1}(\gamma))} \nu_{\mathcal{X}}^{\mathrm{mot}}(\mathrm{ev}(\varphi)) - \int_{\varphi\in\mathbb{P}(\mathrm{gr}^{-1}(\bar{\gamma}))} \nu_{\mathcal{X}}^{\mathrm{mot}}(\mathrm{ev}(\varphi))\Bigg] \\
&\quad + \mathbb{L}^{\dim \mathrm{H}^1(\mathbb{L}_{\mathcal{G}rad(\mathcal{X})}|_\gamma)}\cdot\Big(\mathbb{L}^{-\dim \mathrm{H}^1(\mathbb{L}_{\mathcal{F}ilt(\mathcal{X}_{\mathrm{cl}})}|_{\mathrm{sf}(\gamma)})} - \mathbb{L}^{-\dim \mathrm{H}^1(\mathbb{L}_{\mathcal{F}ilt(\mathcal{X}_{\mathrm{cl}})}|_{\mathrm{sf}(\bar{\gamma})})}\Big)\cdot \nu_{\mathcal{X}}^{\mathrm{mot}}(\mathrm{tot}(\gamma)) \\
&\qquad = \Big(\mathbb{L}^{\mathrm{rank}(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\gamma)})/2} - \mathbb{L}^{-\mathrm{rank}(\mathbb{L}_{{}^{\mathrm{d}}\mathcal{F}ilt(\mathcal{X})}|_{\mathrm{sf}(\gamma)})/2}\Big)\cdot \nu_{{}^{\mathrm{d}}\mathcal{G}rad(\mathcal{X})}^{\mathrm{mot}}(\gamma)
\end{aligned} \tag{B.4.2.3}$$

of monodromic motives over K. Here, we used the fact that the stabilizer group G_γ of γ in $\mathrm{gr}^{-1}(\gamma)$ is special and has motive $\mathbb{L}^{\dim G_\gamma}$, since G_γ is a subgroup of the fibre of the projection $P_\lambda \to L_\lambda$, and can be obtained by repeated extensions of $\mathbb{G}_{\mathrm{a}}$. All of this can be seen by, for example, equivariantly embedding S into an affine space with a linear G-action.

Starting from (B.4.2.3), we divide both sides by $\mathbb{L}-1$, and then take the Euler characteristic, which sets $\mathbb{L}^{1/2}$ to -1. We then apply the identity (7.4.5.1) to convert $\nu_{\mathcal{G}rad(\mathcal{X})}(\gamma)$ to $\nu_{\mathcal{X}}(\mathrm{tot}(\gamma))$. This gives the desired identity (7.4.5.2). □

Appendix C

Proof of anti-symmetric wall-crossing

This appendix is dedicated to the proof of Theorem 7.2.3, through a complicated combinatorial argument. The proof of the theorem is given in §C.3.7.

C.1 The setting

C.1.1. Throughout this section, let I be a finite set, and let

$$C_I = \{e_i\,, e_i^\vee \mid i \in I\} \tag{C.1.1.1}$$

be a set of symbols. We define a map $(-)^\vee\colon C_I \to C_I$ by sending e_i to $e_i^\vee$ and $e_i^\vee$ to e_i for all $i \in I$.

Let A_I be the free associative algebra over $\mathbb{Q}$ generated by elements of C_I, where we denote the multiplication by $*$. There is an involution $(-)^\vee\colon A_I \to A_I^{\mathrm{op}}$, given by

$$(x_1 * \cdots * x_k)^\vee = x_k^\vee * \cdots * x_1^\vee \tag{C.1.1.2}$$

for $x_1, \ldots, x_k \in C_I$.

Let L_I be the free Lie algebra over $\mathbb{Q}$ generated by elements of C_I. Let L_I^{op} be the opposite Lie algebra of L_I, i.e. the Lie algebra with the same underlying vector space and with Lie bracket $[x, y]_{L_I^{\mathrm{op}}} = [y, x]_{L_I}$. There is an involution $(-)^\vee\colon L_I \to L_I^{\mathrm{op}}$, defined inductively by

$$x \mapsto x^\vee \qquad \text{for } x \in C_I\,, \tag{C.1.1.3}$$

$$[x, y] \mapsto [y^\vee, x^\vee] \quad \text{for } x, y \in L_I\,. \tag{C.1.1.4}$$

There is a natural inclusion of vector spaces $L_I \hookrightarrow A_I$, which identifies A_I with the universal enveloping algebra of L_I.

C.1.2. Define linear subspaces

$$L_I^+ = \{x \in L_I \mid x = -x^\vee\}, \tag{C.1.2.1}$$

$$L_I^- = \{x \in L_I \mid x = x^\vee\}. \tag{C.1.2.2}$$

Then $L_I = L_I^+ \oplus L_I^-$, and this makes L_I into a $\mathbb{Z}_2$-graded Lie algebra. In other words, we have $[L_I^+, L_I^\pm] \subset L_I^\pm$ and $[L_I^-, L_I^\pm] \subset L_I^\mp$. In particular, $L_I^+ \subset L_I$ is a Lie subalgebra, and there is a natural embedding

$$U(L_I^+) \hookrightarrow A_I \tag{C.1.2.3}$$

of associative algebras, where the left-hand side is the universal enveloping algebra of L_I^+. This is important, as our main goal is to show that the wall-crossing formula in Theorem 7.2.3, which was originally expressed in terms of the A_I-module structure given by the operation $\diamond$, using the coefficients $U^{\mathrm{sd}}(\dots)$, can actually be expressed solely in terms of the L_I^+-module structure given by the operation $\heartsuit$, using the coefficients $\tilde{U}^{\mathrm{sd}}(\dots)$.

Define linear maps $(-)^+ \colon L_I \to L_I^+$, $(-)^- \colon L_I \to L_I^-$ by

$$x^+ = x - x^\vee, \tag{C.1.2.4}$$

$$x^- = x + x^\vee. \tag{C.1.2.5}$$

We have the relations

$$(x^+)^- = (x^-)^+ = 0, \tag{C.1.2.6}$$

$$[x, y]^+ = \frac{1}{2}\big([x^+, y^+] + [x^-, y^-]\big), \tag{C.1.2.7}$$

$$[x, y]^- = \frac{1}{2}\big([x^+, y^-] + [x^-, y^+]\big) \tag{C.1.2.8}$$

for $x, y \in L_I$.

C.1.3. Write $n = |I|$. Define a set

$$P_I = \coprod_{\substack{\sigma\colon \{1,\ldots,n\}\to I \\ \text{bijective}}} \bigl\{ x = (x_1, \ldots, x_n) \bigm| x_i \in \{ e_{\sigma(i)} , e^\vee_{\sigma(i)} \} \text{ for all } i \bigr\} , \tag{C.1.3.1}$$

as a subset of C_I^n.

Let $K_I = \mathbb{Z}^{C_I}$ be the free abelian group generated by elements of C_I, and let $K_I^+ = \mathbb{N}^{C_I} \setminus \{0\} \subset K_I$.

C.1.4. Define a *self-dual weak stability condition* on I to be a map $\tau\colon K_I^+ \to T$, where T is a totally ordered set, equipped with a distinguished element $0 \in T$, and an order-reversing involution $t \mapsto -t$ fixing the element 0, such that

(i) For any $\alpha, \beta, \gamma \in K_I^+$, such that $\beta = \alpha + \gamma$, either

$$\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma) , \quad \text{or} \quad \tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma) .$$

(ii) For any $\alpha \in K_I^+$,

$$\tau(\alpha^\vee) = -\tau(\alpha) .$$

C.2 Wall-crossing to trivial stability

C.2.1. The goal of this section is to prove Theorem C.2.2 below.

Let τ be a self-dual weak stability condition on I, as in §C.1.4. Define elements

$$T(I;\tau) = \sum_{\sigma \in \mathfrak{S}_n} U(e_{\sigma(1)}, \ldots, e_{\sigma(n)}; \tau, 0) \cdot e_{\sigma(1)} * \cdots * e_{\sigma(n)} , \tag{C.2.1.1}$$

$$\bar{T}(I;\tau) = \sum_{x \in P_I} U(x_1, \ldots, x_n; \tau, 0) \cdot x_1 * \cdots * x_n , \tag{C.2.1.2}$$

$$T^{\mathrm{sd}}(I;\tau) = \sum_{x \in P_I} U^{\mathrm{sd}}(x_1, \ldots, x_n; \tau, 0) \cdot x_1 * \cdots * x_n \tag{C.2.1.3}$$

in the algebra A_I, where the coefficients $U(\ldots)$ and $U^{\mathrm{sd}}(\ldots)$ are defined as in (7.1.3.7) and (7.1.3.8).

C.2.2. Theorem. *We have*

$$T(I;\tau) \in L_I \,, \tag{C.2.2.1}$$

$$\bar{T}(I;\tau) \in L_I^- \,, \tag{C.2.2.2}$$

$$T^{\mathrm{sd}}(I;\tau) \in U(L_I^+) \,. \tag{C.2.2.3}$$

The proof will be given at the end of this section.

C.2.3. From now on, we take $I = \{1, \dots, n\}$. For a subset $J = \{i_1, \dots, i_k\} \subset I$, where $k \geqslant 1$ and $i_1 > \cdots > i_k$, define elements $F(J), \bar{F}(J) \in L_I$ and $G(J) \in L_I^+$ by

$$F(J) = \frac{(-1)^{k-1}}{(k-1)!}\, B_{k-1} \cdot \sum_{\substack{\sigma \in \mathfrak{S}_k: \\ \sigma(1)=1}} [[\dots [e_{i_1}, e_{i_{\sigma(2)}}], \dots], e_{i_{\sigma(k)}}] \,, \tag{C.2.3.1}$$

$$\bar{F}(J) = \frac{(-1)^{k-1}}{(k-1)!}\, B_{k-1} \cdot \sum_{\substack{\sigma \in \mathfrak{S}_k: \\ \sigma(1)=1}} [[\dots [e_{i_1}, e^-_{i_{\sigma(2)}}], \dots], e^-_{i_{\sigma(k)}}] \,, \tag{C.2.3.2}$$

$$G(J) = \frac{(-1)^{k}}{k!} \Big(B_k - B_k\Big(\frac{1}{2}\Big)\Big) \cdot \sum_{\substack{\sigma \in \mathfrak{S}_k: \\ \sigma(1)=1}} [[\dots [e^{\mp}_{i_1}, e^-_{i_{\sigma(2)}}], \dots], e^-_{i_{\sigma(k)}}] \,, \tag{C.2.3.3}$$

where B_k denotes the k-th Bernoulli number, and $B_k(-)$ denotes the k-th Bernoulli polynomial. The sign '$\mp$' is '$+$' if and only if k is odd.

Note that $F(J) = \bar{F}(J) = 0$ whenever $k > 2$ is even, and $G(J) = 0$ whenever $k > 1$ is odd.

For $x_1, \dots, x_k \in A_I$, we denote

$$s_k(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma(1)} * \cdots * x_{\sigma(k)} \,, \tag{C.2.3.4}$$

$$\bar{s}_k(x_1, \dots, x_k) = \frac{1}{2^k\, k!}\, s_k(x_1^-, \dots, x_k^-) \,. \tag{C.2.3.5}$$

C.2.4. Lemma. *We have combinatorial identities*

$$e_n * s_{n-1}(e_1, \dots, e_{n-1}) = \sum_{\substack{J \subset I: \\ n \in J}} s_{n-|J|+1}\big(F(J),\ e_i : i \in I \setminus J\big) \,, \tag{C.2.4.1}$$

$$e_n * \bar{s}_{n-1}(e_1, \dots, e_{n-1}) = \tag{C.2.4.2}$$

$$\sum_{\substack{J \subset I: \\ n \in J}} \bar{s}_{n-|J|+1}\big(\bar{F}(J),\ e_i : i \in I \setminus J\big) + \sum_{\substack{J \subset I: \\ n \in J}} \bar{s}_{n-|J|}\big(e_i : i \in I \setminus J\big) * G(J) .$$

Proof. For (C.2.4.1), for $1 \leqslant i \leqslant k \leqslant n$, write

$$E_{i,k} = \frac{1}{(n-1)!} \sum_{\substack{\sigma \in \mathfrak{S}_n: \\ \sigma(i) = n}} e_{\sigma(1)} * \cdots * e_{\sigma(k)} . \tag{C.2.4.3}$$

For $k = 1, \dots, n$, using the invariance under the $\mathfrak{S}_{n-1}$-action permuting the elements $e_1, \dots, e_{n-1}$, we find that

$$\frac{1}{\binom{n-1}{k-1}} \sum_{\substack{J \subset I: \\ n \in J, |J| = k}} F(J) = \sum_{i=1}^{k} (-1)^{i-1} \binom{k-1}{i-1} \cdot \frac{(-1)^{k-1}}{(k-1)!} B_{k-1} \cdot (k-1)!\, E_{i,k} . \tag{C.2.4.4}$$

Simplifying this, we obtain

$$\sum_{\substack{J \subset I: \\ n \in J, |J| = k}} F(J) = \sum_{i=1}^{k} \frac{(-1)^{k-i}\,(n-1)!}{(n-k)!\,(k-i)!\,(i-1)!} B_{k-1} \cdot E_{i,k} . \tag{C.2.4.5}$$

Therefore,

$$\begin{aligned} &\sum_{\substack{J \subset I: \\ n \in J,\ |J| = k}} s_{n-k+1}\big(F(J),\ e_i : i \in I \setminus J\big) \\ &= \frac{1}{n-k+1} \cdot \sum_{j=0}^{n-k} \sum_{i=1}^{k} \frac{(-1)^{k-i}\,(n-1)!}{(n-k)!\,(k-i)!\,(i-1)!} B_{k-1} \cdot E_{i+j,n} \\ &= \sum_{i=1}^{n} \sum_{j=i-n+k}^{i} \frac{(-1)^{k-j}\,(n-1)!}{(n-k+1)!\,(k-j)!\,(j-1)!} B_{k-1} \cdot E_{i,n} . \end{aligned} \tag{C.2.4.6}$$

The left-hand side of (C.2.4.1) is just $E_{1,n}$, so it suffices to prove that for any $i = 1, \dots, n$,

$$\sum_{k=1}^{n} \sum_{j=i-n+k}^{i} \frac{(-1)^{k-j}\,(n-1)!}{(n-k+1)!\,(k-j)!\,(j-1)!} B_{k-1} = \begin{cases} 1, & i = 1 , \\ 0, & i > 1 . \end{cases} \tag{C.2.4.7}$$

Setting $p = j - 1$ and $q = k - j$, the above reduces to

$$\sum_{p=0}^{i-1} \sum_{q=0}^{n-i} \frac{(-1)^{q}\,(n-1)!}{(n-p-q)!\,p!\,q!} B_{p+q} = \begin{cases} 1, & i = 1 , \\ 0, & i > 1 . \end{cases} \tag{C.2.4.8}$$

This follows from taking $x = 0$ in Lemma C.4.2 below.

For (C.2.4.2), for $1 \leqslant i \leqslant k \leqslant n$, write

$$E^{\pm}_{i,k} = \frac{1}{2^{k-1}\,(n-1)!} \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma(i)=n}} e^{-}_{\sigma(1)} * \cdots * e^{\pm}_{\sigma(i)} * \cdots * e^{-}_{\sigma(k)}\,. \tag{C.2.4.9}$$

Similarly to the previous case, we find that

$$\sum_{\substack{J\subset I:\\ n\in J,\,|J|=k}} \bar{F}(J)^{-} = \sum_{i=1}^{k} \frac{(-1)^{k-i+1}\,2^{k-2}\,(n-1)!}{(n-k)!\,(k-i)!\,(i-1)!}\,B_{k-1}\cdot E^{(-1)^k}_{i,k}\,, \tag{C.2.4.10}$$

$$\sum_{\substack{J\subset I:\\ n\in J,\,|J|=k}} G(J) = \sum_{i=1}^{k} \frac{(-1)^{k-i}\,2^{k-1}\,(n-1)!}{k\,(n-k)!\,(k-i)!\,(i-1)!}\,B'_{k}\cdot E^{(-1)^{k-1}}_{i,k}\,, \tag{C.2.4.11}$$

where $B'_k = B_k - B_k(1/2)$. Proceeding as before, we have

$$\sum_{\substack{J\subset I:\\ n\in J,\,|J|=k}} \bar{s}_{n-k+1}\big(\bar{F}(J),\ e_i : i\in I\setminus J\big) \tag{C.2.4.12}$$

$$= \sum_{i=1}^{n} \sum_{j=i-n+k}^{i} \frac{(-1)^{k-j}\,2^{k-2}\,(n-1)!}{(n-k+1)!\,(k-j)!\,(j-1)!}\,B_{k-1}\cdot E^{(-1)^k}_{i,n}\,, \tag{C.2.4.13}$$

$$\sum_{\substack{J\subset I:\\ n\in J,\,|J|=k}} \bar{s}_{n-k}\big(e_i : i\in I\setminus J\big) * G(J) \tag{C.2.4.14}$$

$$= \sum_{i=n-k+1}^{n} \frac{(-1)^{n-i+1}\,2^{k-1}\,(n-1)!}{k\,(n-k)!\,(n-i)!\,(i+k-n-1)!}\,B'_{k}\cdot E^{(-1)^{k-1}}_{i,n}\,. \tag{C.2.4.15}$$

The left-hand side of (C.2.4.2) is just $(1/2)\,(E^{+}_{1,n} + E^{-}_{1,n})$. Collecting the coefficients of each $E^{\pm}_{i,n}$, we see that to prove (C.2.4.2), it is enough to prove that for any $i = 1, \ldots, n$ and $\varepsilon = \pm 1$,

$$\sum_{k=1}^{n} \varepsilon^{k-1}\cdot \sum_{j=i-n+k}^{i} \frac{(-1)^{k-j}\,2^{k-2}\,(n-1)!}{(n-k+1)!\,(k-j)!\,(j-1)!}\,B_{k-1}$$
$$+ \sum_{k=n-i+1}^{n} \varepsilon^{k}\cdot \frac{(-1)^{n-i+1}\,2^{k-1}\,(n-1)!}{k\,(n-k)!\,(n-i)!\,(i+k-n-1)!}\,B'_{k} = \begin{cases} 1, & i=1 \text{ and } \varepsilon = 1\,,\\ 0, & i>1 \text{ or } \varepsilon=-1\,. \end{cases} \tag{C.2.4.16}$$

The case $\varepsilon = 1$ follows from taking $x = 0$ in Lemma C.4.3 below. The case $\varepsilon = -1$ follows from taking $x = 0$ in Lemma C.4.4 below. □

C.2.5. For $J \subset I$, we denote

$$I\,/\,J = (I\setminus J)\sqcup\{J\}\,.$$

For any element $x \in A_J$, there is a homomorphism

$$\begin{aligned} A_{I/J} &\longrightarrow A_I \,, \\ y &\longmapsto y\,|_{e_J \mapsto x} \,, \end{aligned}$$

defined by mapping e_i to e_i, and $e_i^\vee$ to $e_i^\vee$, for $i \in I \setminus J$, and mapping e_J to the image of x in A_I, and $e_J^\vee$ to the image of $x^\vee$ in A_I.

If, moreover, $x \in L_J$, then this map sends the subspace $L_{I/J}$ to L_I, preserving the $\mathbb{Z}_2$-grading. In particular, this map sends the subalgebra $U(L_{I/J}^+)$ into $U(L_I^+)$.

If τ is a self-dual weak stability condition on I such that the restriction of τ to $K^+(J)$ is a constant map, then τ induces a self-dual weak stability condition on $I \,/\, J$.

To avoid nested subscripts, for I, J, τ, x as above, we denote

$$T(I, J; \tau; x) = T(I \,/\, J; \tau)\,|_{e_J \mapsto x} \,, \tag{C.2.5.1}$$

$$T^{\mathrm{sd}}(I, J; \tau; x) = T^{\mathrm{sd}}(I \,/\, J; \tau)\,|_{e_J \mapsto x} \,. \tag{C.2.5.2}$$

We also define auxiliary coefficients

$$U'(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) = \sum_{\substack{0 = a_0 < \cdots < a_m = n. \\ \text{Define } \beta_1, \dots, \beta_m \text{ by } \beta_i = \alpha_{a_{i-1}+1} + \cdots + \alpha_{a_i}. \\ \text{We require } \tau(\beta_i) = \tau(\alpha_j) \text{ for all } a_{i-1} < j \leqslant a_i}} S(\beta_1, \dots, \beta_m; \tau, \tilde{\tau}) \cdot \left(\prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!} \right), \tag{C.2.5.3}$$

$$U'^{\mathrm{sd}}(\alpha_1, \dots, \alpha_n; \tau, \tilde{\tau}) = \sum_{\substack{0 = a_0 < \cdots < a_m \leqslant n. \\ \text{Define } \beta_1, \dots, \beta_m \text{ by } \beta_i = \alpha_{a_{i-1}+1} + \cdots + \alpha_{a_i}. \\ \text{We require } \tau(\beta_i) = \tau(\alpha_j) \text{ for all } a_{i-1} < j \leqslant a_i, \\ \text{and } \tau(\alpha_j) = 0 \text{ for all } j > a_m}} S^{\mathrm{sd}}(\beta_1, \dots, \beta_m; \tau, \tilde{\tau}) \cdot \left(\prod_{i=1}^m \frac{1}{(a_i - a_{i-1})!} \right) \cdot \frac{1}{2^{n-a_m}\,(n - a_m)!} \,. \tag{C.2.5.4}$$

In other words, in (C.2.5.3), we take the sum of all terms in (7.1.3.7) with $l = 1$; in (C.2.5.4), we take the sum of all terms in (7.1.3.8) with $l = 0$.

Define

$$T'(I; \tau) = \sum_{\sigma \in \mathfrak{S}_n} U'(e_{\sigma(1)}, \dots, e_{\sigma(n)}; \tau, 0) \cdot e_{\sigma(1)} * \cdots * e_{\sigma(n)} \,, \tag{C.2.5.5}$$

$$\bar{T}'(I; \tau) = \sum_{x \in P_I} U'(x_1, \dots, x_n; \tau, 0) \cdot x_1 * \cdots * x_n \,, \tag{C.2.5.6}$$

$$T'^{\mathrm{sd}}(I;\tau) = \sum_{x \in P_I} U'^{\mathrm{sd}}(x_1, \dots, x_n; \tau, 0) \cdot x_1 * \cdots * x_n \, , \tag{C.2.5.7}$$

as elements of A_I. By definition, we have the relations

$$T(I;\tau) = \sum_{\substack{I = I_1 \sqcup \cdots \sqcup I_l : \\ I_i \neq \varnothing \text{ for any } i}} \frac{(-1)^{l-1}}{l} \cdot T'(I_1;\tau) * \cdots * T'(I_l;\tau) \, , \tag{C.2.5.8}$$

$$\bar{T}(I;\tau) = \sum_{\substack{I = I_1 \sqcup \cdots \sqcup I_l : \\ I_i \neq \varnothing \text{ for any } i}} \frac{(-1)^{l-1}}{l} \cdot \bar{T}'(I_1;\tau) * \cdots * \bar{T}'(I_l;\tau) \, , \tag{C.2.5.9}$$

$$T^{\mathrm{sd}}(I;\tau) = \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_l : \\ I_i \neq \varnothing \text{ for } i = 1, \dots, l}} \binom{-1/2}{l} \cdot \bar{T}'(I_1;\tau) * \cdots * \bar{T}'(I_l;\tau) * T'^{\mathrm{sd}}(I_0;\tau) \, . \tag{C.2.5.10}$$

For $J \subset I$ as above, a self-dual weak stability condition τ on I that is constant on J, and an element $x \in A_I$, we denote

$$T'(I, J; \tau; x) = T'(I \,/\, J; \tau)\,|_{e_J \mapsto x} \, , \tag{C.2.5.11}$$

$$\bar{T}'(I, J; \tau; x) = \bar{T}'(I \,/\, J; \tau)\,|_{e_J \mapsto x} \, , \tag{C.2.5.12}$$

$$T'^{\mathrm{sd}}(I, J; \tau; x) = T'^{\mathrm{sd}}(I \,/\, J; \tau)\,|_{e_J \mapsto x} \, , \tag{C.2.5.13}$$

as is similar to the above.

C.2.6. Lemma. *Let* $0 \leqslant l < m \leqslant n$ *with* $m \geqslant l + 2$. *Let* τ_1, τ_2 *be two self-dual weak stability conditions on* I, *satisfying*

$$\tau_1(e_1) \leqslant \cdots \leqslant \tau_1(e_l) < \tau_1(e_{l+1}) = \cdots = \tau_1(e_{m-1}) < \tau_1(e_m) < \tau_1(e_{m+1}) \leqslant \cdots \leqslant \tau_1(e_n) \, ,$$

$$\tau_2(e_1) \leqslant \cdots \leqslant \tau_2(e_l) < \tau_2(e_{l+1}) = \cdots = \tau_2(e_{m-1}) = \tau_2(e_m) < \tau_2(e_{m+1}) \leqslant \cdots \leqslant \tau_2(e_n) \, ,$$

where each '$\leqslant$' sign is '$=$' for τ_1 if and only if the corresponding '$\leqslant$' sign is '$=$' for τ_2. Then

$$T'(I;\tau_1) = \sum_{\substack{J \subset \{l+1, \dots, m\}: \\ m \in J}} T'(I, J; \tau_2; F(J)) \, , \tag{C.2.6.1}$$

$$T(I;\tau_1) = \sum_{\substack{J \subset \{l+1, \dots, m\}: \\ m \in J}} T(I, J; \tau_2; F(J)) \, , \tag{C.2.6.2}$$

where $F(J)$ *is given by* (C.2.3.1).

Proof. First, let us prove (C.2.6.1). By the definitions, both sides of (C.2.6.1) lie in the subspace of A_I spanned by the elements $e_{\sigma(1)} * \cdots * e_{\sigma(n)}$, where $\sigma \in \mathfrak{S}_n$, such that $\tau_2(e_{\sigma(1)}) \geqslant \cdots \geqslant \tau_2(e_{\sigma(n)})$. Therefore, it suffices to prove that for each of these monomials, its coefficients on both sides are equal.

Let $I_0 = \{l+1, \ldots, m\}$. Then (C.2.6.1) can be rewritten as

$$T'(I, I_0; \tau_2; T'(I_0; \tau_1)) = \sum_{\substack{J \subset I_0: \\ m \in J}} T'\big(I, I_0; \tau_2; T'(I_0, J; \tau_2; F(J))\big), \tag{C.2.6.3}$$

by an elementary combinatorial argument. Therefore, it is enough to show that

$$T'(I_0; \tau_1) = \sum_{\substack{J \subset I_0: \\ m \in J}} T'(I_0, J; \tau_2; F(J)), \tag{C.2.6.4}$$

which is precisely (C.2.6.1) with $I = I_0$. Thus, we may ease the notation by setting $l = 0$, $m = n$, and $I_0 = I$. Expanding both sides of (C.2.6.4), we see that it is equivalent to

$$\begin{aligned} &\sum_{\substack{\sigma \in \mathfrak{S}_n: \\ \sigma(1) = n}} \frac{1}{(n-1)!} \cdot e_{\sigma(1)} * \cdots * e_{\sigma(n)} = \\ &\qquad \sum_{\substack{J \subset I_0: \\ n \in J}} \sum_{\substack{\sigma:\, \{1,\ldots,n-|J|+1\} \to I/J \\ \text{bijective}}} \frac{1}{(n-|J|+1)!} \cdot e_{\sigma(1)} * \cdots * e_{\sigma(n-|J|+1)} \Big|_{e_J \mapsto F(J)}, \end{aligned} \tag{C.2.6.5}$$

which is precisely (C.2.4.1). Therefore, we have proved (C.2.6.1).

For (C.2.6.2), using (C.2.5.8), we see that the right-hand side of (C.2.6.2) equals

$$\begin{aligned} &\sum_{\substack{J \subset \{l+1, \ldots, m\}: \\ m \in J}} \sum_{\substack{I = I_1 \sqcup \cdots \sqcup I_k: \\ I_i \neq \varnothing \text{ for any } i, \\ J \subset I_j \text{ for some } j}} \frac{(-1)^{k-1}}{k} \cdot T'(I_1; \tau_2) * \cdots * T'(I_j, J; \tau_2; F(J)) * \\ &\hspace{24em} \cdots * T'(I_k; \tau_2) \\ = &\sum_{\substack{I = I_1 \sqcup \cdots \sqcup I_k: \\ I_i \neq \varnothing \text{ for any } i. \\ \text{Let } j \text{ satisfy } m \in I_j}} \sum_{\substack{J \subset \{l+1, \ldots, m\}: \\ m \in J}} \frac{(-1)^{k-1}}{k} \cdot T'(I_1; \tau_2) * \cdots * T'(I_j, J; \tau_2; F(J)) * \\ &\hspace{24em} \cdots * T'(I_k; \tau_2) \\ = &\sum_{\substack{I = I_1 \sqcup \cdots \sqcup I_k: \\ I_i \neq \varnothing \text{ for any } i. \\ \text{Let } j \text{ satisfy } m \in I_j}} T'(I_1; \tau_1) * \cdots * T'(I_k; \tau_1) = T(I; \tau_1), \end{aligned} \tag{C.2.6.6}$$

where the second equal sign uses that $T'(I_i; \tau_1) = T'(I_i; \tau_2)$ if $i \neq j$, by the definitions. $\square$

C.2.7. Lemma. *Let* $0 \leqslant l < m \leqslant n$ *and* τ_1, τ_2 *satisfy the assumptions of* Lemma C.2.6. *Then*

$$\bar{T}'(I;\tau_1) = \sum_{\substack{J\subset\{l+1,\,\dots,\,m\}:\\ m\in J}} \bar{T}'(I,J;\tau_2;F(J))\,, \tag{C.2.7.1}$$

$$\bar{T}(I;\tau_1) = \sum_{\substack{J\subset\{l+1,\,\dots,\,m\}:\\ m\in J}} \bar{T}(I,J;\tau_2;F(J))\,, \tag{C.2.7.2}$$

where $F(J)$ *is given by* (C.2.3.1).

Proof. We observe that for any $x = (x_1, \dots, x_n) \in P_I$, we have

$$U'(x_1,\dots,x_n;\tau,0) = U'(x_n^\vee,\dots,x_1^\vee;\tau,0)\,, \tag{C.2.7.3}$$

$$U(x_1,\dots,x_n;\tau,0) = U(x_n^\vee,\dots,x_1^\vee;\tau,0)\,, \tag{C.2.7.4}$$

which follow from the definition of these coefficients. Therefore, if we write

$$T'(x;\tau) = \sum_{\sigma\in\mathfrak{S}_n} U'(x_{\sigma(1)},\dots,x_{\sigma(n)};\tau,0)\cdot x_{\sigma(1)} * \cdots * x_{\sigma(n)}\,, \tag{C.2.7.5}$$

$$T(x;\tau) = \sum_{\sigma\in\mathfrak{S}_n} U(x_{\sigma(1)},\dots,x_{\sigma(n)};\tau,0)\cdot x_{\sigma(1)} * \cdots * x_{\sigma(n)}\,, \tag{C.2.7.6}$$

and write $x^\vee = (x_n^\vee, \dots, x_1^\vee)$, then

$$T'(x^\vee;\tau) = T'(x,\tau)^\vee\,, \tag{C.2.7.7}$$

$$T(x^\vee;\tau) = T(x,\tau)^\vee\,. \tag{C.2.7.8}$$

To prove (C.2.7.1), note that both sides are self-dual by the above observation, so it is enough to prove that the coefficients of monomials $x_1 * \cdots * x_n$ that involve e_m (rather than $e_m^\vee$) are equal on both sides. We divide such monomials into 2^{n-1} classes, according to whether they involve e_i or $e_i^\vee$ for $i \in \{1,\dots,n\} \setminus \{m\}$. For each of these classes, let $I' \subset I$ be the set of $i \in I$ such that $e_i^\vee$ is involved in that class. Let $\xi\colon K_I \to K_I$ be the automorphism exchanging e_i and $e_i^\vee$ for all $i \in I'$. Applying Lemma C.2.6 to the weak stability condition $\alpha \mapsto \tau(\xi(\alpha))$, we see that the coefficients of these monomials on both sides of (C.2.7.1) are equal to the coefficients of the corresponding monomials in (C.2.6.1) using the modified weak stability condition. This proves (C.2.7.1).

Finally, (C.2.7.2) follows from an analogous argument using (C.2.6.2). □

C.2.8. Lemma. *Let* $0 \leqslant l < m \leqslant n$ *with* $m \geqslant l+2$. *Let* τ_1, τ_2 *be two self-dual weak stability conditions on* I, *satisfying*

$$0 \leqslant \tau_1(e_1) \leqslant \cdots \leqslant \tau_1(e_l) < \tau_1(e_{l+1}) = \cdots = \tau_1(e_{m-1}) < \tau_1(e_m) < \tau_1(e_{m+1}) \leqslant \cdots \leqslant \tau_1(e_n) \, ,$$

$$0 \leqslant \tau_2(e_1) \leqslant \cdots \leqslant \tau_2(e_l) < \tau_2(e_{l+1}) = \cdots = \tau_2(e_{m-1}) = \tau_2(e_m) < \tau_2(e_{m+1}) \leqslant \cdots \leqslant \tau_2(e_n) \, ,$$

where each '$\leqslant$*' sign is '*$=$*' for* τ_1 *if and only if the corresponding '*$\leqslant$*' sign is '*$=$*' for* τ_2. *Then*

$$T'^{\mathrm{sd}}(I;\tau_1) = \sum_{\substack{J \subset \{l+1, \dots, m\}: \\ m \in J}} T'^{\mathrm{sd}}(I, J; \tau_2; F(J)) \, , \tag{C.2.8.1}$$

$$T^{\mathrm{sd}}(I;\tau_1) = \sum_{\substack{J \subset \{l+1, \dots, m\}: \\ m \in J}} T^{\mathrm{sd}}(I, J; \tau_2; F(J)) \, , \tag{C.2.8.2}$$

where $F(J)$ *is given by* (C.2.3.1).

Proof. The proof is similar to that of Lemma C.2.6.

First, let us prove (C.2.8.1). By the definitions, both sides of (C.2.6.1) lie in the subspace of A_I spanned by the elements $e_{\sigma(1)} * \cdots * e_{\sigma(n)}$, where $\sigma \in \mathfrak{S}_n$, such that $\tau_2(e_{\sigma(1)}) \geqslant \cdots \geqslant \tau_2(e_{\sigma(n)})$. Note that the $e_i^\vee$ cannot appear. Therefore, it suffices to prove that for each of these monomials, its coefficients on both sides are equal.

Let $I_0 = \{l+1, \dots, m\}$. We rewrite (C.2.8.1) as

$$T'^{\mathrm{sd}}(I, I_0; \tau_2; T'(I_0; \tau_1)) = \sum_{\substack{J \subset I_0: \\ m \in J}} T'^{\mathrm{sd}}\big(I, I_0; \tau_2; T'(I_0, J; \tau_2; F(J))\big) \, , \tag{C.2.8.3}$$

which follows from (C.2.6.4). This proves (C.2.8.1).

For (C.2.8.2), using (C.2.5.10), we see that the right-hand side of (C.2.8.2) equals

$$\begin{aligned}
&\sum_{\substack{J \subset \{l+1, \dots, m\}: \\ m \in J}} \ \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k: \\ I_i \neq \varnothing \text{ for } i = 1, \dots, k, \\ J \subset I_j \text{ for some } j > 0}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_2) * \cdots * \bar{T}'(I_j, J; \tau_2; F(J)) * \\
&\hspace{20em} \cdots * \bar{T}'(I_k; \tau_2) * T'^{\mathrm{sd}}(I_0; \tau_2) \\
&+ \sum_{\substack{J \subset \{l+1, \dots, m\}: \\ m \in J}} \ \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k: \\ I_i \neq \varnothing \text{ for } i = 1, \dots, k, \\ J \subset I_0}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_2) * \cdots * \bar{T}'(I_k; \tau_2) * \\
&\hspace{20em} T'^{\mathrm{sd}}(I_j, J; \tau_2; F(J))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k : \\ I_i \neq \varnothing \text{ for } i = 1, \ldots, k, \\ m \in I_j \text{ for some } j > 0}} \sum_{\substack{J \subset \{l+1, \ldots, m\} \cap I_j : \\ m \in J}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_2) * \cdots * \bar{T}'(I_j, J; \tau_2; F(J)) * \\
&\qquad\qquad \cdots * \bar{T}'(I_k; \tau_2) * T'^{\mathrm{sd}}(I_0; \tau_2) \\
&\quad + \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k : \\ I_i \neq \varnothing \text{ for } i = 1, \ldots, k, \\ m \in I_0}} \sum_{\substack{J \subset \{l+1, \ldots, m\} \cap I_0 : \\ m \in J}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_2) * \cdots * \bar{T}'(I_k; \tau_2) * \\
&\qquad\qquad T'^{\mathrm{sd}}(I_j, J; \tau_2; F(J)) \\
&= \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k : \\ I_i \neq \varnothing \text{ for } i = 1, \ldots, k}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_1) * \cdots * \bar{T}'(I_k; \tau_1) * T'^{\mathrm{sd}}(I_0; \tau_1) \\
&= T^{\mathrm{sd}}(I; \tau_1) \, , \qquad \text{(C.2.8.4)}
\end{aligned}
$$

where the second equal sign is by (C.2.7.1) and (C.2.8.1). This proves (C.2.8.2). □

C.2.9. Lemma. *Let $1 \leqslant m \leqslant n$. Let τ_1, τ_2 be two self-dual weak stability conditions on I, satisfying*

$$
\begin{aligned}
0 = \tau_1(e_1) = \cdots = \tau_1(e_{m-1}) < \tau_1(e_m) < \tau_1(e_{m+1}) < \cdots < \tau_1(e_n) \, , \\
0 = \tau_2(e_1) = \cdots = \tau_2(e_{m-1}) = \tau_2(e_m) < \tau_2(e_{m+1}) < \cdots < \tau_2(e_n) \, .
\end{aligned}
$$

Then

$$
T'^{\mathrm{sd}}(I; \tau_1) = \sum_{\substack{J \subset \{1, \ldots, m\}: \\ m \in J}} T'^{\mathrm{sd}}(I, J; \tau_2; \bar{F}(J)) + \sum_{\substack{J \subset \{1, \ldots, m\}: \\ m \in J}} T'^{\mathrm{sd}}(I \setminus J; \tau_2) * G(J) \, , \qquad \text{(C.2.9.1)}
$$

$$
T^{\mathrm{sd}}(I; \tau_1) = \sum_{\substack{J \subset \{1, \ldots, m\}: \\ m \in J}} T^{\mathrm{sd}}(I, J; \tau_2; \bar{F}(J)) + \sum_{\substack{J \subset \{1, \ldots, m\}: \\ m \in J}} T^{\mathrm{sd}}(I \setminus J; \tau_2) * G(J) \, , \qquad \text{(C.2.9.2)}
$$

where $\bar{F}(J)$ and $G(J)$ are given by (C.2.3.2) *and* (C.2.3.3).

Proof. The proof is similar to that of Lemma C.2.6.

First, let us prove (C.2.9.1). By the definitions, both sides of (C.2.6.1) lie in the subspace of A_I spanned by the elements $e_{\sigma(1)} * \cdots * e_{\sigma(n)}$, where $\sigma \in \mathfrak{S}_n$, such that $\tau_2(e_{\sigma(1)}) \geqslant \cdots \geqslant \tau_2(e_{\sigma(n)})$. Therefore, it suffices to prove that for each of these monomials, its coefficients on both sides are equal.

Let $I_0 = \{1, \ldots, m\}$. We rewrite (C.2.9.1) as

$$T'^{\mathrm{sd}}(I, I_0; \tau_2; T'^{\mathrm{sd}}(I_0; \tau_1)) = \sum_{\substack{J \subset I_0: \\ m \in J}} \Big(T'^{\mathrm{sd}}(I, I_0; \tau_2; T'^{\mathrm{sd}}(I_0, J; \tau_2; \bar{F}(J))) \\ + T'^{\mathrm{sd}}(I, I_0; \tau_2; T'^{\mathrm{sd}}(I_0 \setminus J; \tau_2)) * G(J) \Big) . \quad \text{(C.2.9.3)}$$

Therefore, as before, it suffices to prove (C.2.9.1) in the case when $m = n$. But this is precisely (C.2.4.2). This proves (C.2.9.1).

For (C.2.9.2), using (C.2.5.10), as is similar to the proof of Lemma C.2.8, we see that the right-hand side of (C.2.9.2) equals

$$\begin{aligned}
&\sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k : \\ I_i \neq \varnothing \text{ for } i = 1, \dots, k, \\ m \in I_j \text{ for some } j > 0}} \ \sum_{\substack{J \subset \{1, \dots, m\} \cap I_j : \\ m \in J}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_2) * \cdots * \bar{T}'(I_j, J; \tau_2; F(J)) * \\
&\qquad\qquad \cdots * \bar{T}'(I_k; \tau_2) * T'^{\mathrm{sd}}(I_0; \tau_2) \\
&\quad + \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k : \\ I_i \neq \varnothing \text{ for } i = 1, \dots, k, \\ m \in I_0}} \ \sum_{\substack{J \subset \{1, \dots, m\} \cap I_0 : \\ m \in J}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_2) * \cdots * \bar{T}'(I_k; \tau_2) * \\
&\qquad\qquad \Big(T'^{\mathrm{sd}}(I_0, J; \tau_2; \bar{F}(J)) + T'^{\mathrm{sd}}(I_0 \setminus J; \tau_2) * G(J) \Big) \\
&= \sum_{\substack{I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_k : \\ I_i \neq \varnothing \text{ for } i = 1, \dots, k}} \binom{-1/2}{k} \cdot \bar{T}'(I_1; \tau_1) * \cdots * \bar{T}'(I_k; \tau_1) * T'^{\mathrm{sd}}(I_0; \tau_1) \\
&= T^{\mathrm{sd}}(I; \tau_1) \, , \qquad \text{(C.2.9.4)}
\end{aligned}$$

where the first equal sign is by (C.2.7.1) and (C.2.9.1). This proves (C.2.9.2). □

Now, we are ready to prove Theorem C.2.2.

C.2.10. Proof of Theorem C.2.2. We only write down a proof of the more difficult part (C.2.2.3), as the proof of (C.2.2.1) is analogous and easier, and (C.2.2.2) follows from (C.2.2.1) together with the fact that $U(x_1, \dots, x_m; \tau, 0) = U(x_m^\vee, \dots, x_1^\vee; \tau, 0)$ for all $x \in P_I$.

Let S be the set of self-dual weak stability conditions on I. For $\tau \in S$, let T_τ be its codomain, which is a totally ordered set. For $t, t' \in T_\tau$, write

$$\operatorname{sgn}(t, t') = \begin{cases} 1, & t > t' \, , \\ 0, & t = t' \, , \\ -1, & t < t' \, , \end{cases}$$

and write $\operatorname{sgn}(t) = \operatorname{sgn}(t, 0)$.

Define an equivalence relation $\sim$ on S to be generated by the following relations:

(i) $\tau_1 \sim \tau_2$ if for any $i, j \in \{1, \dots, n\}$, $\operatorname{sgn}\tau_1(e_i) = \operatorname{sgn}\tau_2(e_i)$, $\operatorname{sgn}\bigl(\tau_1(e_i), \tau_1(e_j)\bigr) = \operatorname{sgn}\bigl(\tau_2(e_i), \tau_2(e_j)\bigr)$, and $\operatorname{sgn}\bigl(\tau_1(e_i), \tau_1(e_j^\vee)\bigr) = \operatorname{sgn}\bigl(\tau_2(e_i), \tau_2(e_j^\vee)\bigr)$.

(ii) $\tau_1 \sim \tau_2$ if there exists $\sigma \in \mathfrak{S}_n$ with $\tau_1(e_i) = \pm\tau_2(e_{\sigma(i)})$ for all i, where the '$\pm$' signs are arbitrary.

(iii) $\tau_1 \sim \tau_2$ if they satisfy the assumption of Lemma C.2.8.

(iv) $\tau_1 \sim \tau_2$ if they satisfy the assumption of Lemma C.2.9.

We claim that $\sim$ is trivial, that is, all elements of S are equivalent under $\sim$.

Indeed, every element $\tau \in S$ is equivalent to the trivial stability condition $0 \in S$. To see this, using (ii), we may assume that

$$0 \leqslant \tau(e_1) \leqslant \cdots \leqslant \tau(e_n)\ .$$

If all the inequality signs are equalities, then $\tau = 0$ and we are done. If not, suppose that

$$0 = \tau(e_1) = \cdots = \tau(e_l) < \tau(e_{l+1}) = \cdots = \tau(e_m) < \tau(e_{m+1}) \leqslant \cdots \leqslant \tau(e_n)\ ,$$

where $0 \leqslant l < m \leqslant n$. Using (iii), we may increase the values of $\tau(e_m)$ by a small amount, as long as it stays below $\tau(e_{m+1})$. We then do the same thing to $e_{m-1}, \dots, e_{l+2}$, so that we can now assume that

$$0 = \tau(e_1) = \cdots = \tau(e_l) < \tau(e_{l+1}) < \tau(e_{l+2}) \leqslant \cdots \leqslant \tau(e_n)\ ,$$

where $0 \leqslant l < n$. We can then use (iv) to modify $\tau(e_{l+1})$, so that now we have $\tau(e_{l+1}) = 0$. Repeating this process, we see that we eventually reach a point where $\tau = 0$.

Therefore, what is left to prove is that if $\tau_1 \sim \tau_2$, and τ_1 satisfies (C.2.2.3), then so does τ_2. To see this, we only need to check the cases (i)–(iv) individually. By induction on n, we can assume that this is already true for all smaller values of n, as the case when $n = 0$ is trivial.

For (i), we see that $U^{\mathrm{sd}}(x_1, \dots, x_n; \tau_1, 0) = U^{\mathrm{sd}}(x_1, \dots, x_n; \tau_2, 0)$ for all $(x_1, \dots, x_n) \in P_I$, so that $T^{\mathrm{sd}}(I; \tau_1) = T^{\mathrm{sd}}(I; \tau_2)$. For (ii), permutations does not affect $T^{\mathrm{sd}}(I; \tau)$ either, due to the permutation symmetry of P_I. Switching the sign of $\tau(e_i)$ amounts to exchanging the roles of e_i and $e_i^\vee$, i.e., its effect on $T^{\mathrm{sd}}(I; \tau)$ is swapping e_i with $e_i^\vee$ in the expression. However, since the subspace $L_I^+ \subset L_I$ is fixed under this operation, (C.2.2.3) is preserved. For (iii), we use (C.2.8.2), whose right-hand side contains $T^{\mathrm{sd}}(I; \tau_2)$ as the term with $J = \{m\}$. All other terms involve index sets of size $< n$, and hence, after replacing $e_J \mapsto F(J)$, lie in $U(L_I^+)$, by our induction hypothesis. For (iv), similarly, we see from (C.2.9.2) that the difference between $T^{\mathrm{sd}}(I;\tau_1)$ and $T^{\mathrm{sd}}(I;\tau_2)$ lies in $U(L_I^+)$. □

C.3 General wall-crossing

C.3.1. Let notation be as in the previous subsection. For $x = (x_1, \dots, x_n) \in P_I$, define

$$Q(x) = \left\{ y = (y_1, \dots, y_m) \;\middle|\; \begin{array}{l} m \geqslant 1,\ 0 = a_0 < \cdots < a_m = n, \\ y_i = x_{a_{i-1}+1} + \cdots + x_{a_i} \text{ for all } i \end{array} \right\}, \tag{C.3.1.1}$$

$$Q'(x) = \left\{ y = (y_1, \dots, y_m) \;\middle|\; \begin{array}{l} m \geqslant 0,\ 0 = a_0 < \cdots < a_m \leqslant n, \\ y_i = x_{a_{i-1}+1} + \cdots + x_{a_i} \text{ for all } i \end{array} \right\}, \tag{C.3.1.2}$$

where each element y_i is regarded as an element of $K_I \simeq \bigoplus_i (\mathbb{Z}e_i \oplus \mathbb{Z}e_i^\vee)$.

C.3.2. Lemma. *For self-dual weak stability conditions $\tau, \hat{\tau}, \tilde{\tau}$ on I, we have combinatorial identities*

$$S(x_1, \dots, x_n; \tau, \tau) = \begin{cases} 1, & n = 1\,, \\ 0, & n > 1\,, \end{cases} \tag{C.3.2.1}$$

$$S(x_1, \dots, x_n; \tau, \tilde{\tau}) = \sum_{y \in Q(x)} S(y_1, \dots, y_m; \hat{\tau}, \tilde{\tau}) \cdot \prod_{i=1}^m S(x_{a_{i-1}+1}, \dots, x_{a_i}; \tau, \hat{\tau})\,, \tag{C.3.2.2}$$

$$S^{\mathrm{sd}}(x_1, \dots, x_n; \tau, \tau) = \begin{cases} 1, & n = 0\,, \\ 0, & n > 0\,, \end{cases} \tag{C.3.2.3}$$

$$\begin{aligned} S^{\mathrm{sd}}(x_1, \dots, x_n; \tau, \tilde{\tau}) = {} & \sum_{y \in Q'(x)} S^{\mathrm{sd}}(y_1, \dots, y_m; \hat{\tau}, \tilde{\tau}) \cdot \\ & \left(\prod_{i=1}^m S(x_{a_{i-1}+1}, \dots, x_{a_i}; \tau, \hat{\tau}) \right) \cdot S^{\mathrm{sd}}(x_{a_m+1}, \dots, x_n; \tau, \hat{\tau})\,, \end{aligned} \tag{C.3.2.4}$$

$$U(x_1, \dots, x_n; \tau, \tau) = \begin{cases} 1, & n = 1\,, \\ 0, & n > 1\,, \end{cases} \tag{C.3.2.5}$$

$$U(x_1, \dots, x_n; \tau, \tilde{\tau}) = \sum_{y \in Q(x)} U(y_1, \dots, y_m; \hat{\tau}, \tilde{\tau}) \cdot \prod_{i=1}^m U(x_{a_{i-1}+1}, \dots, x_{a_i}; \tau, \hat{\tau})\,, \tag{C.3.2.6}$$

$$U^{\mathrm{sd}}(x_1, \dots, x_n; \tau, \tau) = \begin{cases} 1, & n = 0\,, \\ 0, & n > 0\,, \end{cases} \tag{C.3.2.7}$$

$$U^{\mathrm{sd}}(x_1, \dots, x_n; \tau, \tilde{\tau}) = \sum_{y \in Q'(x)} U^{\mathrm{sd}}(y_1, \dots, y_m; \hat{\tau}, \tilde{\tau}) \cdot \\ \left(\prod_{i=1}^{m} U(x_{a_{i-1}+1}, \dots, x_{a_i}; \tau, \hat{\tau}) \right) \cdot U^{\mathrm{sd}}(x_{a_m+1}, \dots, x_n; \tau, \hat{\tau}), \qquad \text{(C.3.2.8)}$$

where a_i is as in (C.3.1.1) and (C.3.1.2).

Proof. The identities (C.3.2.1), (C.3.2.2), (C.3.2.5), and (C.3.2.6) were proved in [85, Theorems 4.5 and 4.8], using purely combinatorial methods. The identities (C.3.2.3) and (C.3.2.7) follow from the definitions easily.

One could also prove the other two identities, (C.3.2.4) and (C.3.2.8), using combinatorics. However, we take a more intuitive approach and deduce them from Theorem 7.1.3.

Consider a self-dual quiver Q defined as follows. The set of vertices of Q is $Q_0 = I \sqcup I^\vee = \{1, 1^\vee, \dots, n, n^\vee\}$. There is a unique arrow $i \to j$ for any $i, j \in Q_0$, making a total of $4n^2$ arrows. Define the involution $(-)^\vee \colon Q \xrightarrow{\sim} Q^{\mathrm{op}}$ by exchanging the vertices i and $i^\vee$ for all $i \in \{1, \dots, n\}$. The action on the arrows is determined accordingly. Let u, v assign the sign $+1$ to all vertices and arrows.

Let $\mathcal{X}$ be the moduli of representations of Q as in §4.1.

For convenience, for vertices $i^\vee \in Q_0$ with $i \in I$, we write $e_{i^\vee} = e_i^\vee \in C_I$ and $e_{i^\vee}^\vee = e_i \in C_I$.

Let C_I' be the set of all $\alpha \in \pi_0(\mathcal{X})$ that is a non-zero sum of distinct elements of C_I. For such α, let $C_\alpha \subset C_I$ be the set of terms appearing in α. Define an object $(E^\alpha, e^\alpha) \in \mathcal{A}$ by

$$E_i^\alpha = \begin{cases} K, & i \in C_\alpha\,, \\ 0, & \text{otherwise}\,, \end{cases} \qquad e_{i \to j}^\alpha = \begin{cases} 1, & i, j \in C_\alpha\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

One can see that (E^α, e^α) is a simple object, and hence is semistable under any weak stability condition.

Let Σ be the set of all $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ with $m \geqslant 0$ and $\alpha_s \in C_I'$ for all s, such that $\alpha_1 + \cdots + \alpha_m \in C_I'$. For each $\underline{\alpha} \in \Sigma$, define an object $(E^{\underline{\alpha}}, e^{\underline{\alpha}}) \in \mathcal{A}$ by

$$(E^{\underline{\alpha}})_i = \begin{cases} K, & i \in C_{\alpha_s} \text{ for some } s\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

$$(e^{\underline{\alpha}})_{i \to j} = \begin{cases} 1, & i \in C_{\alpha_s} \text{ and } j \in C_{\alpha_t} \text{ for some } s \geqslant t\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Then $(E^{\underline{\alpha}}, e^{\underline{\alpha}})$ has a unique Jordan–Hölder filtration with quotients $(E^{\alpha_1}, e^{\alpha_1}), \ldots, (E^{\alpha_m}, e^{\alpha_m})$.

Define a partial order $\preceq$ on Σ such that $\underline{\alpha} \preceq \underline{\alpha}'$ if and only if there exists $0 = s_0 < \cdots < s_{m'} = m$ such that $\alpha'_t = \alpha_{s_{t-1}+1} + \cdots + \alpha_{s_t}$ for all t, where m and m' are the lengths of $\underline{\alpha}$ and $\underline{\alpha}'$.

For a fixed weak stability condition τ, and for $\underline{\alpha} \in \Sigma$, write $\delta_{\underline{\alpha}}(\tau) = [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] * \cdots * [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \in \mathbb{M}(\mathcal{X}; \mathbb{Q})$. Then $\delta_{\underline{\alpha}'}(\tau)$ being non-zero at $(E^{\underline{\alpha}}, e^{\underline{\alpha}})$ implies $\underline{\alpha} \preceq \underline{\alpha}'$, since any filtration is refined by the Jordan–Hölder filtration. In particular, $\delta_{\underline{\alpha}}(\tau)$ is not in the linear span of $\delta_{\underline{\alpha}'}(\tau)$ with $\underline{\alpha} \npreceq \underline{\alpha}'$. Since $\preceq$ can be extended to a total order on the finite set Σ, we conclude that the motives $\delta_{\underline{\alpha}}(\tau) \in \mathbb{M}(\mathcal{X}; \mathbb{Q})$ for all $\underline{\alpha} \in \Sigma$ are linearly independent.

As a result, the motives $\epsilon_{\alpha_1}(\tau) * \cdots * \epsilon_{\alpha_m}(\tau) \in \mathbb{M}(\mathcal{X}; \mathbb{Q})$ are also linearly independent, essentially because upper triangular matrices with 1's on the diagonal are invertible.

At this point, as a side note, if we apply Theorem 7.1.3 to express $[\mathcal{X}^{\mathrm{ss}}_{\alpha}(\tilde{\tau})]$ in terms of motives of τ-semistable loci, where $\alpha = e_1 + \cdots + e_n$, and compare the result with first converting $\tilde{\tau}$-semistable loci to $\hat{\tau}$-semistable loci, and then converting to τ-semistable loci, we have reproved (C.3.2.2). Applying this to $\epsilon_\alpha(\tau)$ reproves (C.3.2.6).

Now, let Σ^{sd} be the set of $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m, \rho)$ such that each α_s is in C'_I, $\rho \in C'_I \cup \{0\}$, $\rho = \rho^\vee$, and $\alpha_1 + \alpha_1^\vee + \cdots + \alpha_m + \alpha_m^\vee + \rho \in C'_I$. For a fixed self-dual weak stability condition τ, and for $\underline{\alpha} \in \Sigma^{\mathrm{sd}}$, write $\delta^{\mathrm{sd}}_{\underline{\alpha}}(\tau) = [\mathcal{X}^{\mathrm{ss}}_{\alpha_1}(\tau)] \diamond \cdots \diamond [\mathcal{X}^{\mathrm{ss}}_{\alpha_n}(\tau)] \diamond [\mathcal{X}^{\mathrm{sd,ss}}_{\rho}(\tau)] \in \mathbb{M}(\mathcal{X}^{\mathrm{sd}}; \mathbb{Q})$. Similarly, we claim that the stack functions $\delta^{\mathrm{sd}}_{\underline{\alpha}}(\tau)$ for all $\underline{\alpha} \in \Sigma^{\mathrm{sd}}$ are linearly independent.

Indeed, for $\underline{\alpha} \in \Sigma^{\mathrm{sd}}$, define $\underline{\bar{\alpha}} = (\alpha_1, \ldots, \alpha_m, (\rho,)\ \alpha_m^\vee, \ldots, \alpha_1^\vee) \in \Sigma$, where ρ appears only when it is non-zero. The object $(E^{\underline{\bar{\alpha}}}, e^{\underline{\bar{\alpha}}})$ has a natural self-dual structure. Using its unique Jordan–Hölder filtration, we can show that $\delta^{\mathrm{sd}}_{\underline{\alpha}}(\tau)$ is not in the linear span of $\delta^{\mathrm{sd}}_{\underline{\alpha}'}(\tau)$ with $\underline{\bar{\alpha}} \npreceq \underline{\bar{\alpha}}'$. Since the map $\underline{\alpha} \mapsto \underline{\bar{\alpha}}$ is injective, it follows that the $\delta^{\mathrm{sd}}_{\underline{\alpha}}(\tau)$ are linearly independent.

Similarly, it follows that the stack functions $\epsilon_{\alpha_1}(\tau) \diamond \cdots \diamond \epsilon_{\alpha_m}(\tau) \diamond \epsilon^{\mathrm{sd}}_{\rho}$ for $\underline{\alpha} \in \Sigma^{\mathrm{sd}}$ are linearly independent.

Applying Theorem 7.1.3 to express $[\mathcal{X}^{\mathrm{sd,ss}}_{\theta}(\tilde{\tau})]$ in terms of motives of τ-semistable loci, where $\theta = \bar{e}_1 + \cdots + \bar{e}_n$, and comparing the result with first converting $\tilde{\tau}$-semistable loci to $\hat{\tau}$-semistable loci, and then converting to τ-semistable loci, we have proved (C.3.2.4). Applying this to $\epsilon^{\mathrm{sd}}_{\theta}(\tau)$ proves (C.3.2.8). $\square$

C.3.3. Now, define elements

$$V(I;\tau) = \sum_{\sigma\in\mathfrak{S}_n} U(e_{\sigma(1)},\dots,e_{\sigma(n)};0,\tau)\cdot e_{\sigma(1)} * \cdots * e_{\sigma(n)}\ , \tag{C.3.3.1}$$

$$V^{\mathrm{sd}}(I;\tau) = \sum_{x\in P_I} U^{\mathrm{sd}}(x_1,\dots,x_n;0,\tau)\cdot x_1 * \cdots * x_n \tag{C.3.3.2}$$

in the algebra A_I.

C.3.4. Theorem. *We have*

$$V(I;\tau)\in L_I\ , \tag{C.3.4.1}$$

$$V^{\mathrm{sd}}(I;\tau)\in U(L_I^+)\ . \tag{C.3.4.2}$$

Proof. The proof is essentially by formally inverting the results in Theorem C.2.2.

We use induction on n, and assume that the theorem is true for all smaller values of n. The cases when $n = 1$ in (C.3.4.1) and when $n = 0$ in (C.3.4.2) are trivial, since there is nothing to prove.

To prove (C.3.4.1), we may assume that $n > 1$. By (C.3.2.5) and (C.3.2.6), for any $x \in P_I$, we have

$$\sum_{y\in Q(x)} U(y_1,\dots,y_m;0,\tau)\cdot\prod_{i=1}^{m} U(x_{a_{i-1}+1},\dots,x_{a_i};\tau,0) = 0\ . \tag{C.3.4.3}$$

Summing over all possibilities of $x = (e_{\sigma(1)},\dots,e_{\sigma(n)})$ for $\sigma\in\mathfrak{S}_n$, we obtain that

$$\sum_{\substack{m\geqslant 1,\ I = J_1\sqcup\cdots\sqcup J_m:\\ J_i\neq\varnothing \text{ for all } i.\\ \text{Write } y_i=\sum_{j\in J_i} e_j}} U(y_1,\dots,y_m;0,\tau)\cdot T(J_1;\tau) * \cdots * T(J_m;\tau) = 0\ , \tag{C.3.4.4}$$

where the $T(J_i;\tau)$ are as in (C.2.1.1). By Theorem C.2.2, $T(J_i;\tau)\in L_{J_i}$. Therefore, by the induction hypothesis, that is, by (C.3.4.1) applied to m elements, if $m < n$, then for a fixed choice of $J_1,\dots,J_m$, the sum of all the $m!$ terms in (C.3.4.4) involving a permutation of $J_1,\dots,J_m$ is in L_I. Since (C.3.4.4) equals zero, the sum of the terms that were not involved above must lie in L_I as well. These are precisely the terms with $m = n$. This gives that

$$\sum_{\sigma\in\mathfrak{S}_n} U(e_{\sigma(1)},\dots,e_{\sigma(n)};0,\tau)\cdot e_{\sigma(1)} * \cdots * e_{\sigma(n)}\in L_I\ , \tag{C.3.4.5}$$

which is a restatement of (C.3.4.1).

To prove (C.3.4.2), we assume that $n > 0$, and proceed as before. By (C.3.2.7)–(C.3.2.8), for any $x \in P_I$, we have

$$\sum_{y \in Q'(x)} U^{\mathrm{sd}}(y_1, \dots, y_m; 0, \tau) \cdot \left(\prod_{i=1}^{m} U(x_{a_{i-1}+1}, \dots, x_{a_i}; \tau, 0) \right) \cdot U^{\mathrm{sd}}(x_{a_m+1}, \dots, x_n; \tau, 0) = 0 \,. \tag{C.3.4.6}$$

Summing over all possibilities of $x \in P_I$, we obtain that

$$\sum_{\substack{m \geqslant 0,\ I = J_1 \sqcup \dots \sqcup J_m \sqcup J',\ x^i \in P_{J_i}: \\ J_i \neq \varnothing \text{ for all } i. \\ \text{Write } y_i = \sum_{j \in J_i} x_j}} U^{\mathrm{sd}}(y_1, \dots, y_m; 0, \tau) \cdot T(x^1; \tau) * \dots * T(x^m; \tau) * T^{\mathrm{sd}}(J'; \tau) = 0 \,, \tag{C.3.4.7}$$

where $T^{\mathrm{sd}}(J'; \tau)$ is as in (C.2.1.3), and $T(x^i; \tau)$ is as in (C.2.7.6). By Theorem C.2.2, $T(x^i; \tau) \in L_{J_i}$, and $T^{\mathrm{sd}}(J_i; \tau) \in U(L^+_{J'})$. For $m < n$, fix a choice of $J_1, \dots, J_m, J'$, and a choice of the x^i. Let Σ be the sum of all the $2^m m!$ terms in (C.3.4.7) involving a permutation of $J_1, \dots, J_m$, and for each i, either x^i or $(x^i)^\vee$, where if $x^i = (x_1, \dots, x_k)$, then $(x^i)^\vee = (x_k^\vee, \dots, x_1^\vee)$. Note that as in the proof of (C.2.2.2), we have $T((x^i)^\vee; \tau) = T(x^i; \tau)^\vee$, where the latter $(-)^\vee$ is the involution on A_I. Now, we can apply the induction hypothesis, or (C.3.4.2) applied to m elements, to see that Σ is a linear combination of products of elements either in $L^+_{J'}$, or of the form $T(x^i; \tau) - T(x^i; \tau)^\vee$, which lie in $L^+_{J_i}$. Therefore, $\Sigma \in U(L^+_I)$.

It then follows that the sum of the terms in (C.3.4.7) with $m = n$ lies in $U(L^+_I)$ as well, which is, again, a restatement of (C.3.4.2). □

C.3.5. Next, for two self-dual weak stability conditions $\tau, \tilde{\tau}$ on I, define elements

$$W(I; \tau, \tilde{\tau}) = \sum_{\sigma \in \mathfrak{S}_n} U(e_{\sigma(1)}, \dots, e_{\sigma(n)}; \tau, \tilde{\tau}) \cdot e_{\sigma(1)} * \dots * e_{\sigma(n)} \,, \tag{C.3.5.1}$$

$$W^{\mathrm{sd}}(I; \tau, \tilde{\tau}) = \sum_{x \in P_I} U^{\mathrm{sd}}(x_1, \dots, x_n; \tau, \tilde{\tau}) \cdot x_1 * \dots * x_n \tag{C.3.5.2}$$

in the algebra A_I.

C.3.6. Theorem. *We have*

$$W(I; \tau, \tilde{\tau}) \in L_I \,, \tag{C.3.6.1}$$

$$W^{\mathrm{sd}}(I; \tau, \tilde{\tau}) \in U(L^+_I) \,. \tag{C.3.6.2}$$

Proof. Applying (C.3.2.6) and (C.3.2.8) with $\hat{\tau} = 0$, we may rewrite

$$W(I;\tau,\tilde{\tau}) = \sum_{\substack{m \geqslant 1,\ I = J_1 \sqcup \cdots \sqcup J_m :\\ J_i \neq \varnothing \text{ for all } i.\\ \text{Write } y_i = \sum_{j \in J_i} e_j}} U(y_1, \ldots, y_m; 0, \tilde{\tau}) \cdot T(J_1;\tau) * \cdots * T(J_m;\tau)\,, \tag{C.3.6.3}$$

$$W^{\mathrm{sd}}(I;\tau,\tilde{\tau}) = \sum_{\substack{m \geqslant 0,\ I = J_1 \sqcup \cdots \sqcup J_m \sqcup J',\ x^i \in P_{J_i} :\\ J_i \neq \varnothing \text{ for all } i.\\ \text{Write } y_i = \sum_{j \in J_i} x_j}} U^{\mathrm{sd}}(y_1, \ldots, y_m; 0, \tilde{\tau}) \cdot T(x^1;\tau) * \cdots * T(x^m;\tau) * T^{\mathrm{sd}}(J';\tau)\,. \tag{C.3.6.4}$$

Reasoning as in the proof of Theorem C.3.4, we can deduce (C.3.6.1) and (C.3.6.2) from Theorems C.2.2 and C.3.4. Indeed, we no longer need to use induction, and instead of proving that some of the terms lie in L_I or $U(L_I^+)$, the argument now shows that all the terms are in L_I or $U(L_I^+)$. □

Finally, we deduce Theorem 7.2.3 from Theorem C.3.6.

C.3.7. Proof of Theorem 7.2.3. We only prove the self-dual case, as the linear case is already proved by Joyce [85, Theorem 5.4], and can alternatively be shown using a similar argument.

For a permissible self-dual stability condition τ on $\mathcal{X}$, Let $L_\tau \subset \mathbb{M}(\mathcal{X};\mathbb{Q})$ be the smallest $\mathbb{Q}$-linear subspace containing $\epsilon_\alpha(\tau)$ for all $\alpha \in \pi_0(\mathcal{X}) \setminus \{0\}$ and closed under the involution $(-)^\vee$ and Lie brackets as in §7.2.2, and let $M_\tau \subset \mathbb{M}(\mathcal{X}^{\mathrm{sd}};\mathbb{Q})$ be the smallest subspace containing $\epsilon^{\mathrm{sd}}_\rho(\tau)$ for all $\rho \in \pi_0(\mathcal{X}^{\mathrm{sd}})$ and closed under the operation $a \heartsuit (-)$ for $a \in L_\tau$.

We may rewrite (7.1.3.4) as

$$\epsilon^{\mathrm{sd}}_\theta(\tau_-) = \sum_{\substack{n \geqslant 0\\ \text{Write } I = \{1, \ldots, n\}}} \frac{1}{2^n n!} \cdot \sum_{\substack{\kappa\colon I \to \pi_0(\mathcal{X}) \setminus \{0\},\ \rho \in \pi_0(\mathcal{X}^{\mathrm{sd}}):\\ \theta = \sum_{i \in I} (\kappa(i) + \kappa(i)^\vee) + \rho}}$$

$$\Biggl[\sum_{\substack{x \in P_I\\ \text{Write } \alpha_i = \kappa(j) \text{ if } x_i = e_j,\\ \text{or write } \alpha_i = \kappa(j)^\vee \text{ if } x_i = e_j^\vee}} U^{\mathrm{sd}}(\alpha_1, \ldots, \alpha_n; \tau_+, \tau_-) \cdot \epsilon_{\alpha_1}(\tau_+) \diamond \cdots \diamond \epsilon_{\alpha_n}(\tau_+) \diamond \epsilon^{\mathrm{sd}}_\rho(\tau_+) \Biggr], \tag{C.3.7.1}$$

since every term in (7.1.3.4) appears $2^n n!$ times in (C.3.7.1).

Now, every sum in the square brackets in (C.3.7.1) lies in M_{τ_+}. This is because we can define an involutive algebra homomorphism $\varphi\colon A_I \to \mathbb{M}(\mathcal{X};\mathbb{Q})$ by sending e_i to $\epsilon_{\kappa(i)}(\tau_+)$ and $e_i^\vee$ to $\epsilon_{\kappa(i)^\vee}(\tau_+)$. We have $\varphi(L_I) \subset L_{\tau_+}$, so $\varphi(L_I^+) \subset L_{\tau_+}^+$. The sum in the square brackets is $\varphi(W^{\mathrm{sd}}(I;\tau_+,\tau_-)) \diamond \epsilon_\rho^{\mathrm{sd}}(\tau_+)$, so by Theorem C.3.6, it lies in $U(L_{\tau_+}^+) \diamond \epsilon_\rho^{\mathrm{sd}}(\tau_+) \subset M_{\tau_+}$.

Moreover, the above also shows that every sum in the square brackets can be written as a sum of terms of the form in (7.2.3.2), with the $\tilde{U}(\dots)$ coefficients. Each of these new terms appears $2^n n!$ times in (C.3.7.1). This proves the theorem.

C.4 Some combinatorial identities

Finally, we prove some combinatorial identities that were used in the arguments above. It is interesting that by working with wall-crossing structures, we are able to write down several combinatorial identities involving the Bernoulli numbers, namely, Lemmas C.4.2 to C.4.4, and it is unclear whether there are deeper reasons why these identities are true.

C.4.1. Lemma. *For any integers i, k, n such that $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n-1$, we have*

$$\sum_{\substack{q\colon 0 \leqslant q \leqslant n-i,\\ 0 \leqslant k-q \leqslant i-1}} (-1)^q \binom{k}{q} = (-1)^{i+k-1} \binom{k-1}{i-1} + (-1)^{n-i} \binom{k-1}{n-i}. \tag{C.4.1.1}$$

Proof. We have

$$\begin{aligned} \text{l.h.s.} &= \sum_{q=k-i+1}^{n-i} (-1)^q \left[\binom{k-1}{q-1} + \binom{k-1}{q} \right] \\ &= (-1)^{k-i+1} \binom{k-1}{k-i} + (-1)^{n-i} \binom{k-1}{n-i} \\ &= \text{r.h.s.} \end{aligned}$$

$\square$

C.4.2. Lemma. *For any integers $1 \leqslant i \leqslant n$, we have*

$$\sum_{p=0}^{i-1} \sum_{q=0}^{n-i} \frac{(-1)^q \, (n-1)!}{(n-p-q)! \, p! \, q!} \, B_{p+q}(x) = \binom{n-1}{i-1} x^{i-1} \, (1-x)^{n-i}, \tag{C.4.2.1}$$

where $B_k(x)$ denotes the k-th Bernoulli polynomial.

Proof. Let $l(x)$ and $r(x)$ denote the left and right sides of (C.4.2.1), respectively. By (C.4.1.1), we have

$$
\begin{aligned}
l(x) &= \frac{1}{n}\, B_0(x) + \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-k)!\,k!} \left[(-1)^{i+k-1}\binom{k-1}{i-1} + (-1)^{n-i}\binom{k-1}{n-i}\right] B_k(x) \\
&= \frac{1}{n} + \binom{n-1}{i-1}\cdot\left[\sum_{k=i}^{n-1} \frac{(-1)^{i+k-1}}{k}\binom{n-i}{n-k} B_k(x) + \sum_{k=n-i+1}^{n-1} \frac{(-1)^{n-i}}{k}\binom{i-1}{n-k} B_k(x)\right].
\end{aligned}
$$

Since $B_k(x+1) - B_k(x) = k\,x^{k-1}$, we have

$$
l(x+1) - l(x) = \binom{n-1}{i-1}\cdot\left[\sum_{k=i}^{n-1} (-1)^i \binom{n-i}{n-k} (-x)^{k-1} + \sum_{k=n-i+1}^{n-1} (-1)^{n-i}\binom{i-1}{n-k} x^{k-1}\right].
$$

On the other hand,

$$
\begin{aligned}
r(x+1) - r(x) &= \binom{n-1}{i-1}\cdot\left[(x+1)^{i-1}(-x)^{n-i} - x^{i-1}(1-x)^{n-i}\right] \\
&= \binom{n-1}{i-1}\cdot\left[\sum_{k=i}^{n-1} (-1)^i \binom{n-i}{n-k} (-x)^{k-1} + \sum_{k=n-i+1}^{n-1} (-1)^{n-i}\binom{i-1}{n-k} x^{k-1}\right].
\end{aligned}
$$

Therefore, $l(x+1) - l(x) = r(x+1) - r(x)$, which means that

$$
l(x) - r(x) = c
$$

for some constant c. To show that $c = 0$, we use the fact that

$$
\int_0^1 B_k(x)\,dx = \begin{cases} 1, & k = 0\,, \\ 0, & k > 0\,, \end{cases}
$$

so

$$
\int_0^1 l(x)\,dx = \frac{1}{n}\,.
$$

On the other hand,

$$
\begin{aligned}
\int_0^1 r(x)\,dx &= \binom{n-1}{i-1}\cdot \mathrm{B}(i, n-i+1) \\
&= \binom{n-1}{i-1}\cdot\frac{(i-1)!\,(n-i)!}{n!} \\
&= \frac{1}{n}\,,
\end{aligned}
$$

where B denotes the beta function. This shows that $c = 0$. $\square$

C.4.3. Lemma. *For any integers $1 \leqslant i \leqslant n$,*

$$\sum_{p=0}^{i-1} \sum_{q=0}^{n-i} \frac{(-1)^q \, 2^{p+q-1} \, (n-1)!}{(n-p-q)! \, p! \, q!} \, B_{p+q}\Big(\frac{x}{2}\Big)$$
$$+ \sum_{k=n-i+1}^{n} \frac{(-1)^{n-i+1} \, 2^{k-1} \, (n-1)!}{k \, (n-k)! \, (n-i)! \, (i+k-n-1)!} \Big[B_k\Big(\frac{x}{2}\Big) - B_k\Big(\frac{x+1}{2}\Big) \Big]$$
$$= \binom{n-1}{i-1} x^{i-1} \, (1-x)^{n-i} \,. \quad \text{(C.4.3.1)}$$

Proof. Let $l_1(x)$, $l_2(x)$ and $r(x)$ denote the first and second terms on the left-hand side of (C.4.3.1), and the right-hand side, respectively.

By (C.4.1.1), we have

$$l_1(x) = \frac{1}{2n} B_0\Big(\frac{x}{2}\Big) + \sum_{k=1}^{n-1} \frac{2^{k-1} \, (n-1)!}{(n-k)! \, k!} \Big[(-1)^{i+k-1} \binom{k-1}{i-1} + (-1)^{n-i} \binom{k-1}{n-i} \Big] B_k\Big(\frac{x}{2}\Big)$$
$$= \frac{1}{2n} + \binom{n-1}{i-1} \cdot \Bigg[\sum_{k=i}^{n-1} \frac{(-1)^{i+k-1} \, 2^{k-1}}{k} \binom{n-i}{n-k} B_k\Big(\frac{x}{2}\Big) +$$
$$\sum_{k=n-i+1}^{n-1} \frac{(-1)^{n-i} \, 2^{k-1}}{k} \binom{i-1}{n-k} B_k\Big(\frac{x}{2}\Big) \Bigg] .$$

Proceeding as in the proof of Lemma C.4.2, we see that

$$l_1(x+2) - l_1(x) =$$
$$\binom{n-1}{i-1} \cdot \Bigg[\sum_{k=i}^{n-1} (-1)^i \binom{n-i}{n-k} (-x)^{k-1} + \sum_{k=n-i+1}^{n-1} (-1)^{n-i} \binom{i-1}{n-k} x^{k-1} \Bigg] .$$

Similarly,

$$l_2(x+2) - l_2(x) = \sum_{k=n-i+1}^{n} \frac{(-1)^{n-i+1} \, (n-1)!}{(n-k)! \, (n-i)! \, (i+k-n-1)!} \big[x^{k-1} - (x+1)^{k-1} \big]$$
$$= (-1)^{n-i+1} \binom{n-1}{i-1} \cdot \sum_{k=n-i+1}^{n} \binom{i-1}{n-k} \big[x^{k-1} - (x+1)^{k-1} \big] .$$

Setting $l(x) = l_1(x) + l_2(x)$, we see that

$$l(x+2) - l(x)$$
$$= \binom{n-1}{i-1} \cdot \Bigg[\sum_{k=i}^{n-1} (-1)^i \binom{n-i}{n-k} (-x)^{k-1} + \sum_{k=n-i+1}^{n-1} (-1)^{n-i} \binom{i-1}{n-k} (x+1)^{k-1}$$
$$+ (-1)^{n-i+1} \big[x^{n-1} - (x+1)^{n-1} \big] \Bigg]$$

$$= \binom{n-1}{i-1} \cdot \left[\sum_{k=i}^{n} (-1)^i \binom{n-i}{n-k} (-x)^{k-1} + \sum_{k=n-i+1}^{n} (-1)^{n-i} \binom{i-1}{n-k} (x+1)^{k-1} \right].$$

On the other hand,

$$\begin{aligned} r(x+2) - r(x) &= \binom{n-1}{i-1} \cdot \left[\left((x+1)+1\right)^{i-1} \left(-(x+1)\right)^{n-i} - x^{i-1} (1-x)^{n-i} \right] \\ &= l(x+2) - l(x) \, . \end{aligned}$$

This means that

$$l(x) - r(x) = c$$

for some constant c. To show that $c = 0$, we use the facts that

$$\begin{aligned} \int_0^2 B_k\left(\frac{x}{2}\right) dx &= \begin{cases} 2, & k = 0 \, , \\ 0, & k > 0 \, , \end{cases} \\ \int_0^2 B_k\left(\frac{x+1}{2}\right) dx &= \frac{1}{2^{k-1}} \, , \end{aligned}$$

so

$$\int_0^2 l(x) \, dx = \frac{1}{n} + \sum_{k=n-i+1}^{n} \frac{(-1)^{n-i} (n-1)!}{k \, (n-k)! \, (n-i)! \, (i+k-n-1)!} \, .$$

On the other hand, we have seen in the proof of Lemma C.4.2 that

$$\int_0^1 r(x) \, dx = \frac{1}{n} \, .$$

We then calculate

$$\begin{aligned} \int_1^2 r(x) \, dx &= \binom{n-1}{i-1} \cdot \int_0^1 (-x)^{n-i} (1+x)^{i-1} \, dx \\ &= \binom{n-1}{i-1} \cdot \int_0^1 \left[\sum_{k=n-i+1}^{n} (-1)^{n-i} \binom{i-1}{n-k} x^{k-1} \right] dx \\ &= \sum_{k=n-i+1}^{n} \frac{(-1)^{n-i} (n-1)!}{k \, (n-k)! \, (n-i)! \, (i+k-n-1)!} \, . \end{aligned}$$

This shows that $c = 0$ and we are done. □

C.4.4. Lemma. *For any integers $1 \leqslant i \leqslant n$,*

$$\sum_{p=0}^{i-1} \sum_{q=0}^{n-i} \frac{(-1)^p \, 2^{p+q-1} \, (n-1)!}{(n-p-q)! \, p! \, q!} \, B_{p+q}\Big(\frac{x}{2}\Big)$$
$$+ \sum_{k=n-i+1}^{n} \frac{(-1)^{i+k-n-1} \, 2^{k-1} \, (n-1)!}{k \, (n-k)! \, (n-i)! \, (i+k-n-1)!} \Big[B_k\Big(\frac{x}{2}\Big) - B_k\Big(\frac{x+1}{2}\Big) \Big] = 0 \, . \quad \text{(C.4.4.1)}$$

Proof. Let $l(x)$ denote the left-hand side. From a similar calculation as in the previous lemma, we have

$$\begin{aligned} & l(x+2) - l(x) \\ & \quad = \binom{n-1}{i-1} \cdot \Bigg[\sum_{k=i}^{n} (-1)^{i-1} \binom{n-i}{n-k} x^{k-1} + \sum_{k=n-i+1}^{n} (-1)^{n-i+1} \binom{i-1}{n-k} (-x-1)^{k-1} \Bigg]. \\ & \quad = \binom{n-1}{i-1} \cdot \Big[(-x)^{i-1} \, (1+x)^{n-i} - \big((-x-1) + 1 \big)^{i-1} \, (x+1)^{n-i} \Big] \\ & \quad = 0 \, . \end{aligned}$$

Therefore, $l(x)$ is constant. Again, calculating as before,

$$\begin{aligned} \int_0^2 l(x) \, dx &= \frac{1}{n} + \sum_{k=n-i+1}^{n} \frac{(-1)^{i+k-n} \, (n-1)!}{k \, (n-k)! \, (n-i)! \, (i+k-n-1)!} \\ &= \frac{1}{n} - \binom{n-1}{n-i} \cdot \int_0^1 x^{n-i} \, (1-x)^{i-1} \, dx \\ &= \frac{1}{n} - \frac{1}{n} = 0 \, , \end{aligned}$$

where the second integral was evaluated as in the proof of Lemma C.4.2. This shows that $l(x) \equiv 0$. □

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