AERO 430 – Numerical Simulation

Second-Order Linear Ordinary Differential Equation Boundary-Value Problem for 1-D Convection-Diffusion Equation

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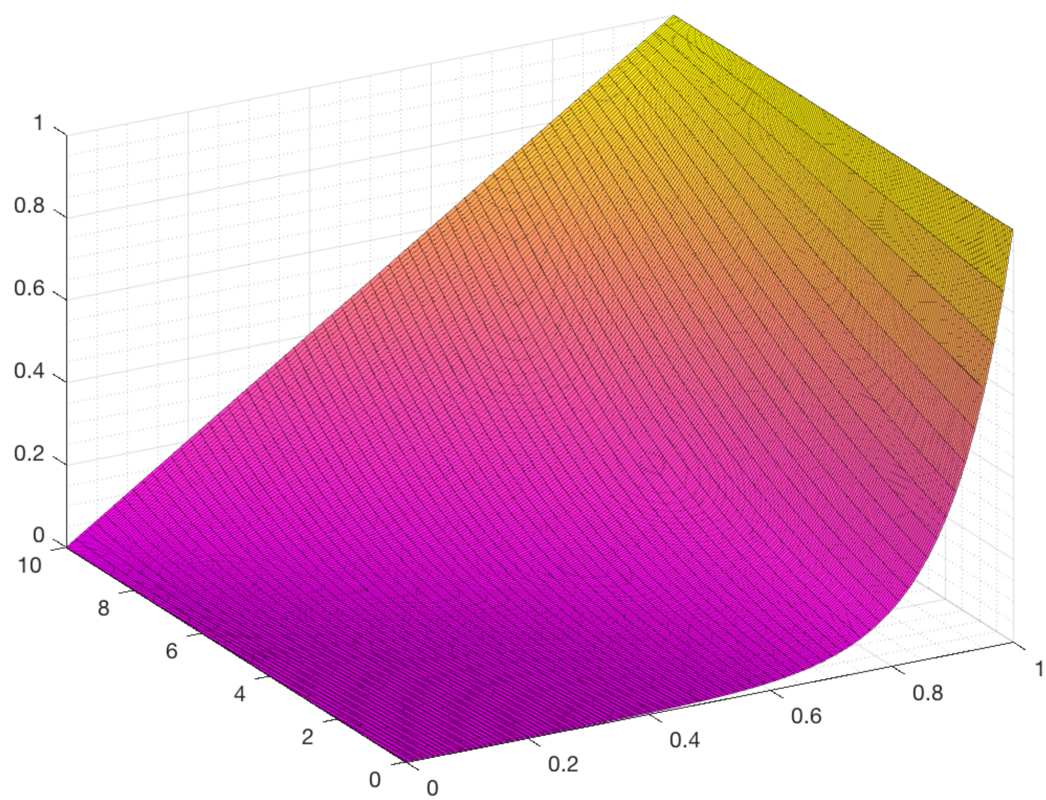

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1 Model Problem

The model second-order linear ordinary differential equation boundary-value problem consists of:

- the second-order linear ordinary differential equation:

$$-u''(x) + cu'(x) = 0 \qquad x \in (0,1) \tag{1.1}$$

- the boundary conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = 1 \tag{1.2}$$

The physical model of the second-order linear ordinary differential equation boundary-value problem is that of the concentration of a flow property that convects and diffuses proportional to the constant c. For example, the convection-diffusion equation could represent the concentration of ink as a function of distance in a quasi-one-dimensional flow.

2 Analytical Solution

2.1 Analytical Solution to the Differential Equation

The following equation is the homogeneous second-order linear ordinary differential equation (ODE).

$$-u''(x) + cu'(x) = 0 \tag{2.1}$$

The solution of the homogeneous ODE, $u_h(x)$ is assumed to be of the form:

$$u_h(x) = e^{\lambda x} \tag{2.2}$$

Taking the derivatives of $u_h(x)$, substituting them into the homogeneous ODE, and reducing the equation yields the **characteristic equation**.

$$u_h'(x) = \lambda e^{\lambda x} \tag{2.3}$$

$$u_h''(x) = \lambda^2 e^{\lambda x} \tag{2.4}$$

$$-\lambda^2 e^{\lambda x} + c\lambda e^{\lambda x} = 0 \tag{2.5}$$

$$\boldsymbol{-\lambda^2 + c\lambda = \lambda(c - \lambda) = 0} \tag{2.6}$$

Solving for λ yields:

$$\lambda = \{0, c\} \tag{2.7}$$

The homogenous solution $u_h(x)$ is then:

$$u_h(x) = ae^{0x} + be^{cx} \tag{2.8}$$

$$\mathbf{u_h(x) = a + be^{cx}} \tag{2.9}$$

Applying the both boundary conditions, $u(0) = 0$ and $u(1) = 1$, we get the system of algebraic equations, which yields a and b:

$$u_h(0) = 0 = a + b \tag{2.10}$$

$$u_h(1) = 1 = a + be^c \tag{2.11}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & e^c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.12}$$

$$a = \frac{-1}{e^c - 1} \quad \text{and} \quad b = \frac{1}{e^c - 1} \tag{2.13}$$

Thus, it is shown that for the second-order linear ordinary differential equation with specified boundary conditions, that $u(x)$ is a solution to the differential equation on $x \in (0, 1)$.

$$-u''(x) + cu'(x) = 0 \qquad x \in (0, 1) \tag{2.14}$$

$$u(0) = 0 \quad \text{and} \quad u(1) = 1 \tag{2.15}$$

$$\mathbf{u(x) = \frac{e^{cx} - 1}{e^c - 1}} \tag{2.16}$$

The values of c tested in this study were $c \in 1, 2, 5, 10, 20, 50$. A plot of the analytical solution for values of c is depicted below:

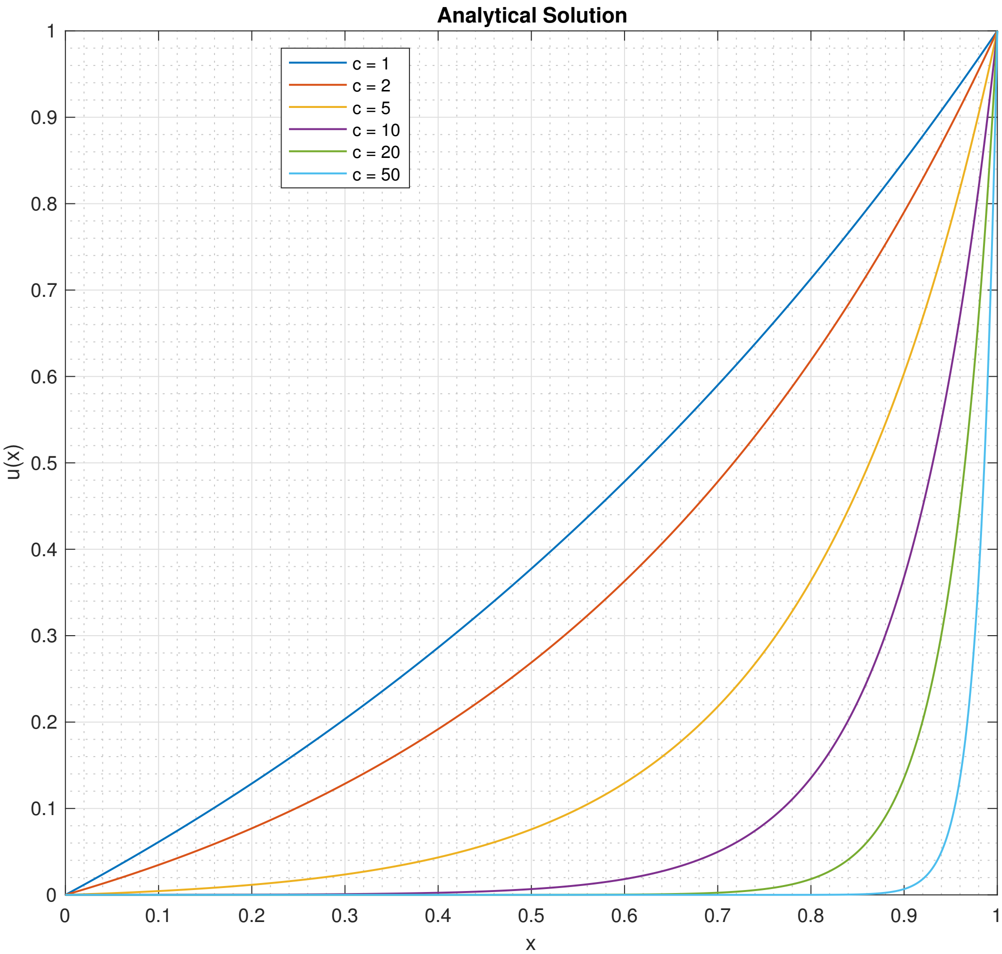


Figure 2.1.1 – Analytical Solution to the Differential Equation for Values of c

2.2 Analytical Solution to the Quantity of Interest

The quantity of interest for this particular problem is the first derivative at the right boundary, or $u'(1)$. The differential equation and the analytical solution to the differential equation, respectively, are reproduced below.

$$-u''(x) + cu'(x) = 0 \tag{2.17}$$

$$u(x) = \frac{e^{cx} - 1}{e^c - 1} \tag{2.18}$$

Thus, taking the first derivative, and solving at $x = 1$, we obtain the exact quantity of interest:

$$u'(x) = \frac{ce^{cx}}{e^c - 1} \tag{2.19}$$

$$\mathbf{u'(1)} = \frac{\mathbf{ce^c}}{\mathbf{e^c - 1}} \tag{2.20}$$

It can be seen that if e^c, or rather c, is sufficiently large ($e^c \gg 1$), then the quantity of interest tends to c given that the ratio of e^c and $(e^c - 1)$ tends to unity:

$$\lim_{c \to \infty} \frac{ce^c}{e^c - 1} = \infty \approxeq c \tag{2.21}$$

A table of the exact quantity of interest for the tested values of c is included below (for $c = 50$, error is less than machine epsilon, ϵ):

Table 2.2.1 – Analytical Solution to the Quantity of Interest for Values of c

c	$\mathbf{u'(1)}$
1	1.58198
2	2.31304
5	5.03392
10	10.00045
20	20.00000004
50	50.00000000

3 Numerical Methods

3.1 Derivations

3.1.1 2nd-Order Central Difference Scheme Finite Difference Method

First Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.1}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.2}$$

Subtracting the Taylor series for u_{i-1} from u_{i+1} and canceling terms:

$$u_{i+1} - u_{i-1} = 2\Delta x u_i' + \frac{\Delta x^3}{3} u_i^{(3)} + \mathcal{O}(\Delta x^5) \tag{3.3}$$

Solving for u_i':

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2) \tag{3.4}$$

From this specific first-derivative formulation using the finite difference method, the first-derivative approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.5}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.6}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.7}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{3.8}$$

From this specific second-derivative formulation using the finite difference method, the second-derivative approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + cu'(x) = 0 \tag{3.9}$$

Discretizing using the above formulas and simplifying yields:

$$-\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) + c\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) = 0 \tag{3.10}$$

$$-\left(u_{i+1}-2u_i+u_{i-1}\right)+c\Delta x\left(\frac{u_{i+1}-u_{i-1}}{2}\right)=0 \tag{3.11}$$

$$\left(-1-\frac{c\Delta x}{2}\right)u_{i-1}+(2)\,u_i+\left(-1+\frac{c\Delta x}{2}\right)u_{i+1}=0 \tag{3.12}$$

3.1.2 4th-Order Central Difference Scheme Finite Difference Method

First Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x=i$:

$$u_{i-1}=u_i-\Delta x u_i'+\frac{\Delta x^2}{2}u_i''-\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.13}$$

$$u_{i+1}=u_i+\Delta x u_i'+\frac{\Delta x^2}{2}u_i''+\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.14}$$

Subtracting the Taylor series for u_{i-1} from u_{i+1} and canceling terms:

$$u_{i+1}-u_{i-1}=2\Delta x u_i'+\frac{\Delta x^3}{3}u_i^{(3)}+\mathcal{O}(\Delta x^5) \tag{3.15}$$

Returning to the differential equation and taking one additional derivative:

$$u''(x)=cu'(x) \tag{3.16}$$

$$u^{(3)}(x)=cu''(x) \tag{3.17}$$

Replacing the second-derivative term with the original differential equation:

$$u^{(3)}(x)=c^2u'(x) \tag{3.18}$$

Now replacing the third-derivative term in the Taylor series subtraction and then dividing by $2\Delta x$:

$$u_{i+1}-u_{i-1}=2\Delta x u_i'+\frac{c^2\Delta x^3}{3}u_i'+\mathcal{O}(\Delta x^5) \tag{3.19}$$

$$\frac{u_{i+1}-u_{i-1}}{2\Delta x}=\left(1+\frac{c^2\Delta x^2}{6}\right)u_i'+\mathcal{O}(\Delta x^4) \tag{3.20}$$

Solving for u_i':

$$u_i'=\left(1+\frac{c^2\Delta x^2}{6}\right)^{-1}\frac{u_{i+1}-u_{i-1}}{2\Delta x}+\mathcal{O}(\Delta x^4) \tag{3.21}$$

From this specific first-derivative formulation using the finite difference method, the first-derivative approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x=i$:

$$u_{i-1}=u_i-\Delta x u_i'+\frac{\Delta x^2}{2}u_i''-\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.22}$$

$$u_{i+1}=u_i+\Delta x u_i'+\frac{\Delta x^2}{2}u_i''+\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.23}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \frac{\Delta x^4}{12} u_i^{(4)} + \mathcal{O}(\Delta x^6) \tag{3.24}$$

Returning to the differential equation and taking two additional derivatives:

$$u''(x) = cu'(x) \tag{3.25}$$

$$u^{(3)}(x) = cu''(x) \tag{3.26}$$

$$u^{(4)}(x) = cu^{(3)}(x) \tag{3.27}$$

Replacing the fourth-derivative and third-derivative terms with the original differential equation, we arrive at:

$$u^{(4)}(x) = c^2 u''(x) \tag{3.28}$$

Now replacing the fourth-derivative term in the Taylor series addition, rearranging, and dividing by Δx^2:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \frac{c^2 \Delta x^4}{12} u_i'' + \mathcal{O}(\Delta x^6) \tag{3.29}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = \left(1 + \frac{c^2 \Delta x^2}{12}\right) u_i'' + \mathcal{O}(\Delta x^4) \tag{3.30}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \left(1 + \frac{c^2 \Delta x^2}{12}\right)^{-1} \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^4) \tag{3.31}$$

From this specific second-derivative formulation using the finite difference method, the second-derivative approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + cu'(x) = 0 \tag{3.32}$$

Discretizing using the above formulas and simplifying yields:

$$-\left(1 + \frac{c^2 \Delta x^2}{12}\right)^{-1} \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + c\left(1 + \frac{c^2 \Delta x^2}{6}\right)^{-1} \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0 \tag{3.33}$$

$$\left(-1 - \frac{c\Delta x}{2} - \frac{c^2 \Delta x^2}{12}\right) u_{i-1} + \left(2 + \frac{c^2 \Delta x^2}{6}\right) u_i + \left(-1 + \frac{c\Delta x}{2} - \frac{c^2 \Delta x^2}{12}\right) u_{i+1} = 0 \tag{3.34}$$

3.1.3 1st-Order Upwind Scheme Finite Difference Method

First Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i - 1$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.35}$$

Rearranging the Taylor series for u_i':

$$u_i' = \frac{u_i - u_{i-1}}{\Delta x} + \mathcal{O}(\Delta x) \tag{3.36}$$

From this specific first-derivative formulation using the finite difference method, the first-derivative approximation can be observed to be first-order ($\mathcal{O}(\Delta x)$).

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.37}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.38}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.39}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{3.40}$$

From this specific second-derivative formulation using the finite difference method, the second-derivative approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + cu'(x) = 0 \tag{3.41}$$

Discretizing using the above formulas and simplifying yields:

$$-\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) + c\left(\frac{u_i - u_{i-1}}{\Delta x}\right) = 0 \tag{3.42}$$

$$-\left(u_{i+1} - 2u_i + u_{i-1}\right) + c\Delta x \left(u_i - u_{i-1}\right) = 0 \tag{3.43}$$

$$\left(-1 - c\Delta x\right) u_{i-1} + \left(2 + c\Delta x\right) u_i + \left(-1\right) u_{i+1} = 0 \tag{3.44}$$

3.2 Finite Difference Method Results

3.2.1 2nd-Order Central Difference Scheme Finite Difference Method

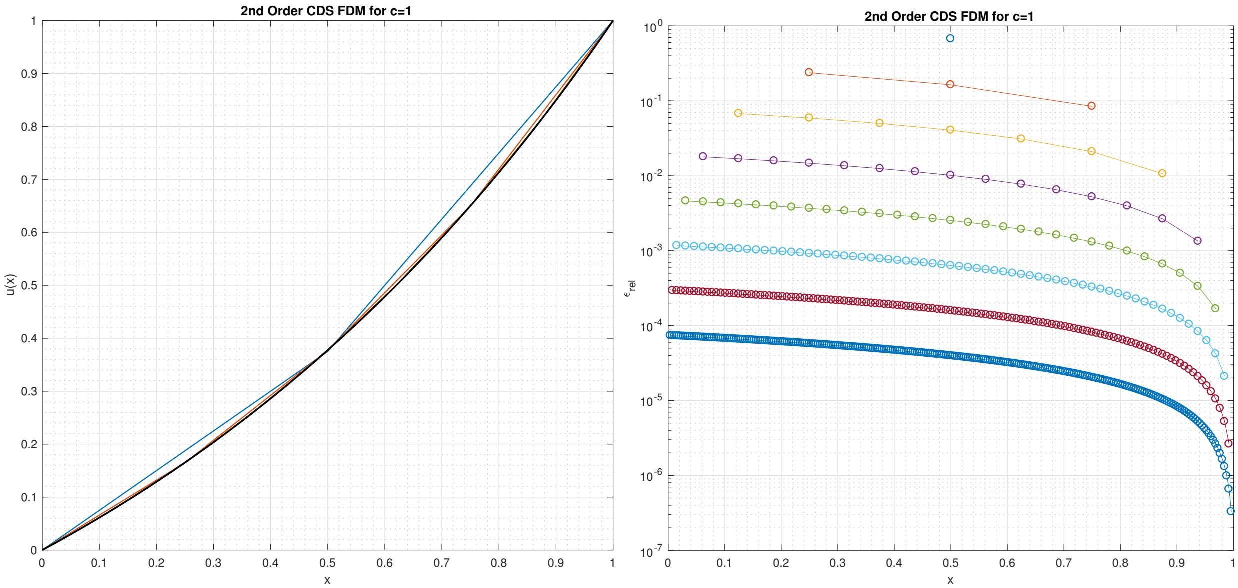


Figure 3.2.1 – 2nd-Order CDS FDM and Pointwise Error for $c = 1$

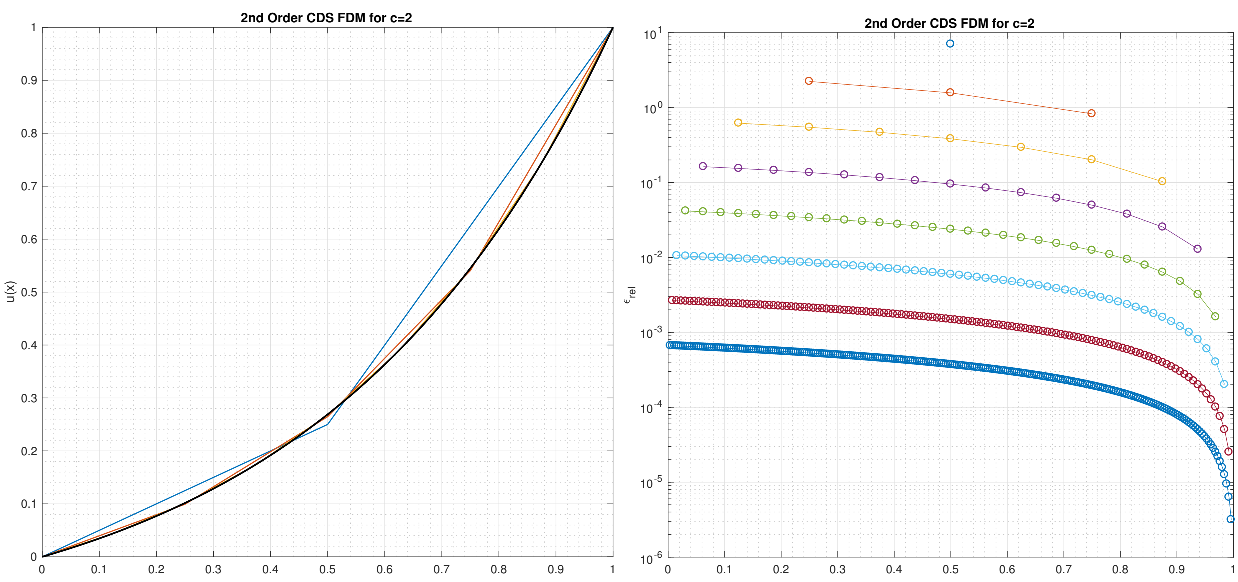
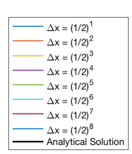


Figure 3.2.2 – 2nd-Order CDS FDM and Pointwise Error for $c = 2$



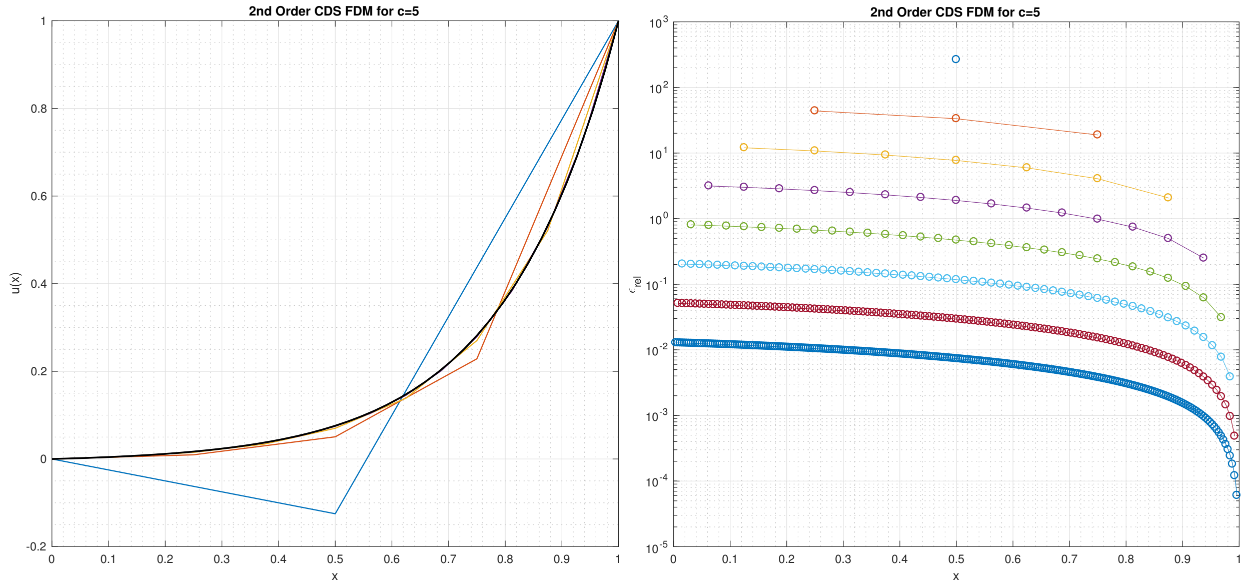


Figure 3.2.3 – 2nd-Order CDS FDM and Pointwise Error for $c = 5$

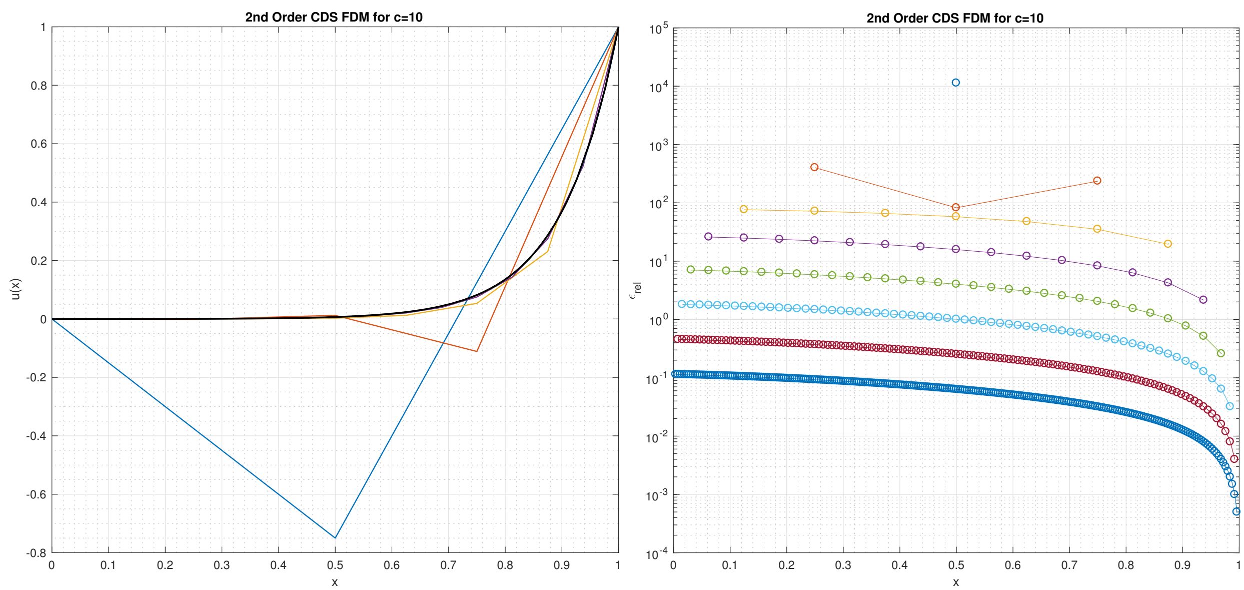
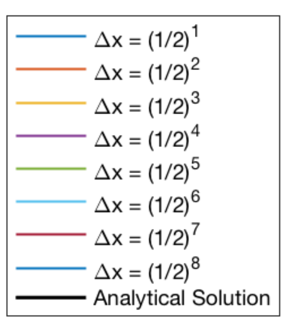


Figure 3.2.4 – 2nd-Order CDS FDM and Pointwise Error for $c = 10$



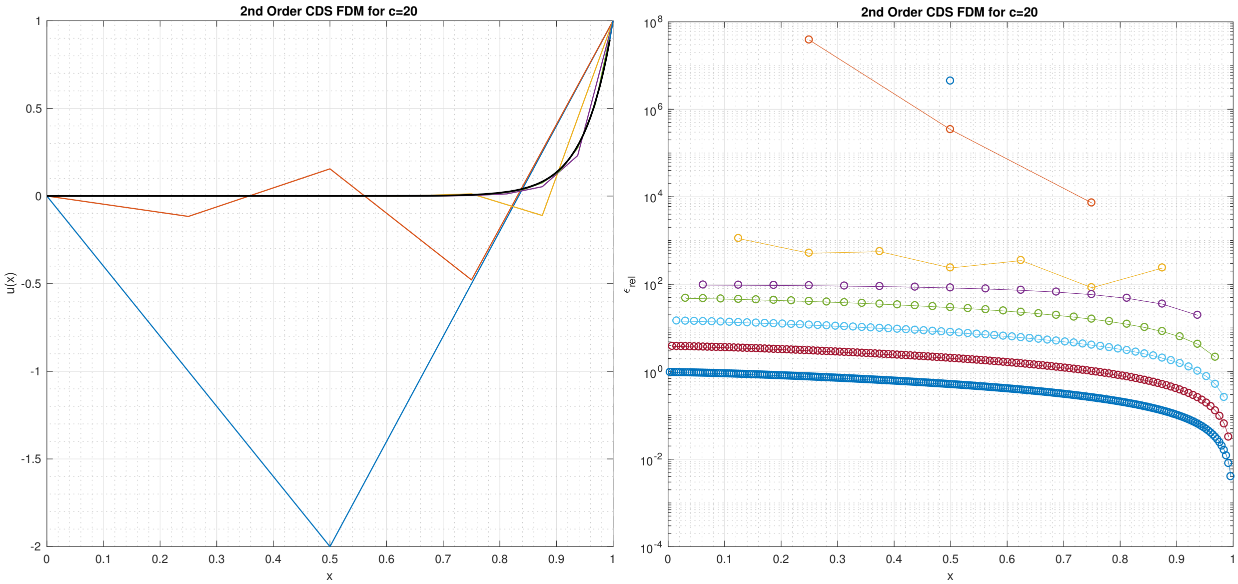


Figure 3.2.5 – 2nd-Order CDS FDM and Pointwise Error for $c = 20$

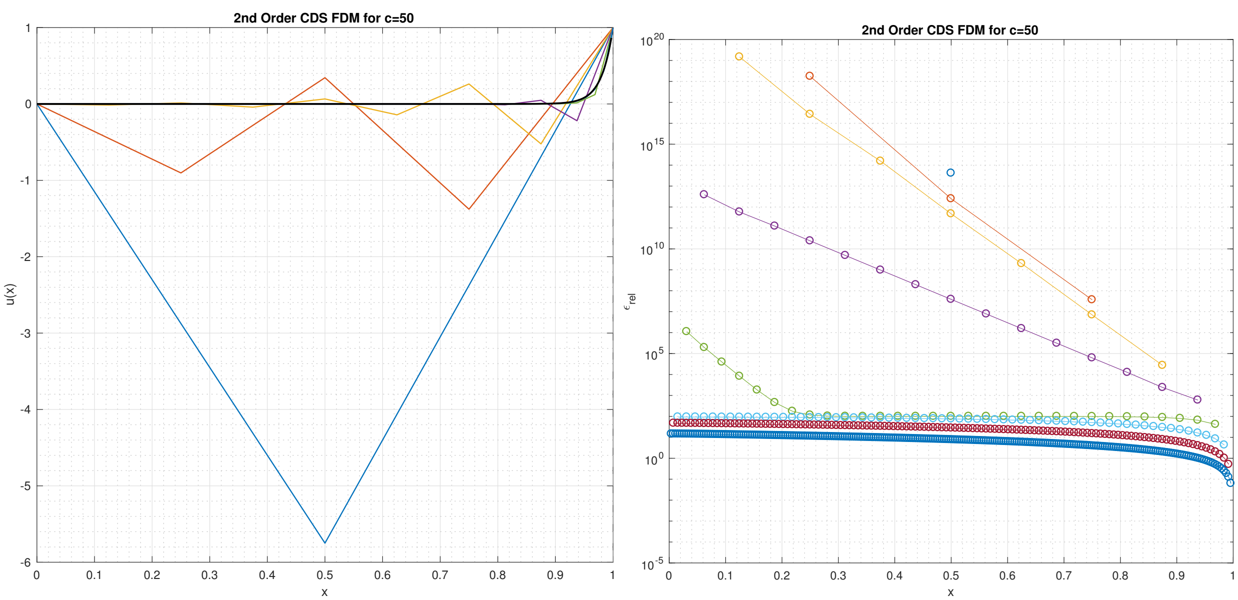
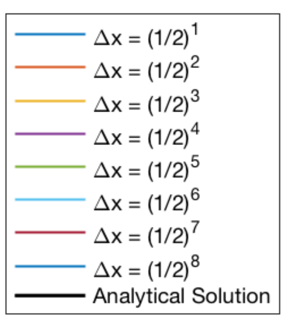


Figure 3.2.6 – 2nd-Order CDS FDM and Pointwise Error for $c = 50$



3.2.2 4th-Order Central Difference Scheme Finite Difference Method

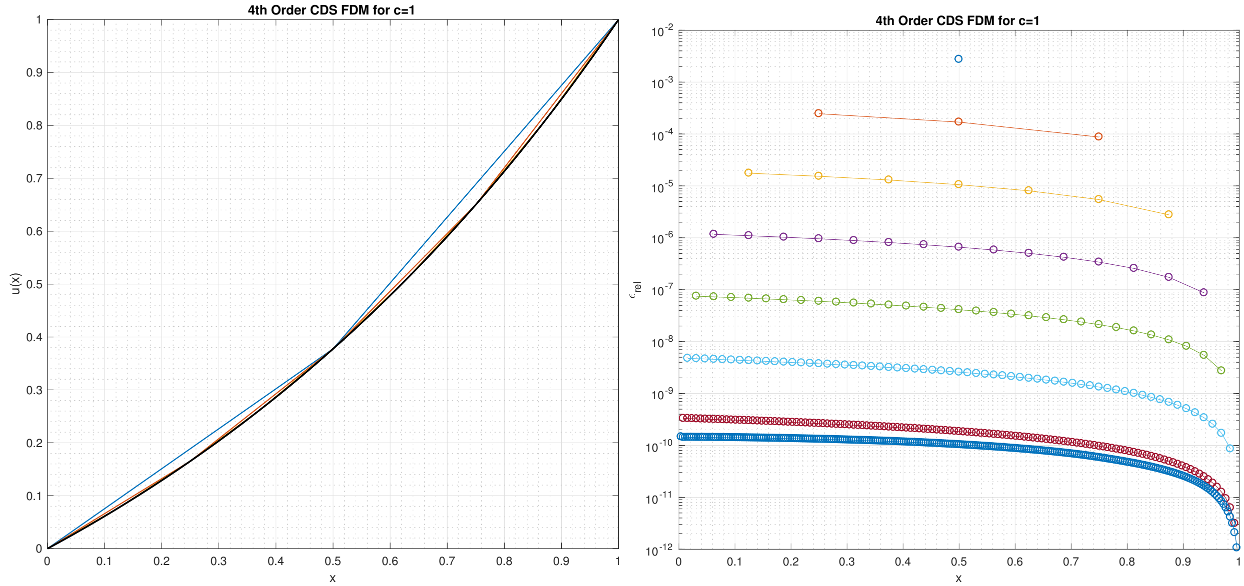


Figure 3.2.7 – 4th-Order CDS FDM and Pointwise Error for $c = 1$

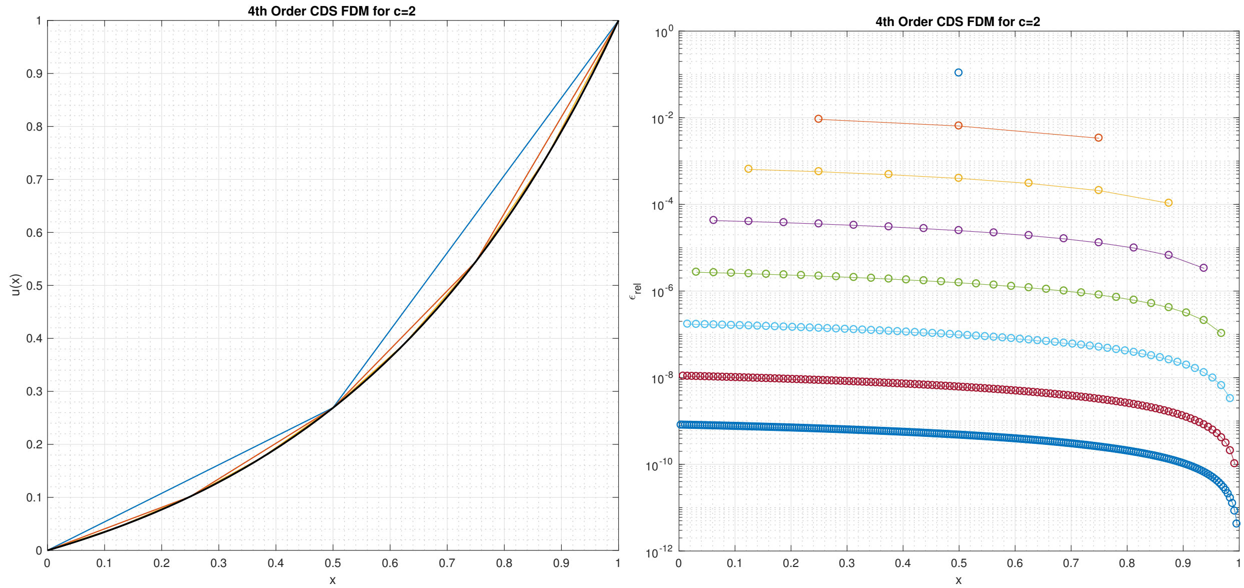
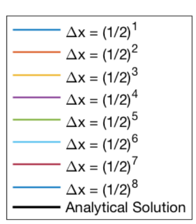


Figure 3.2.8 – 4th-Order CDS FDM and Pointwise Error for $c = 2$



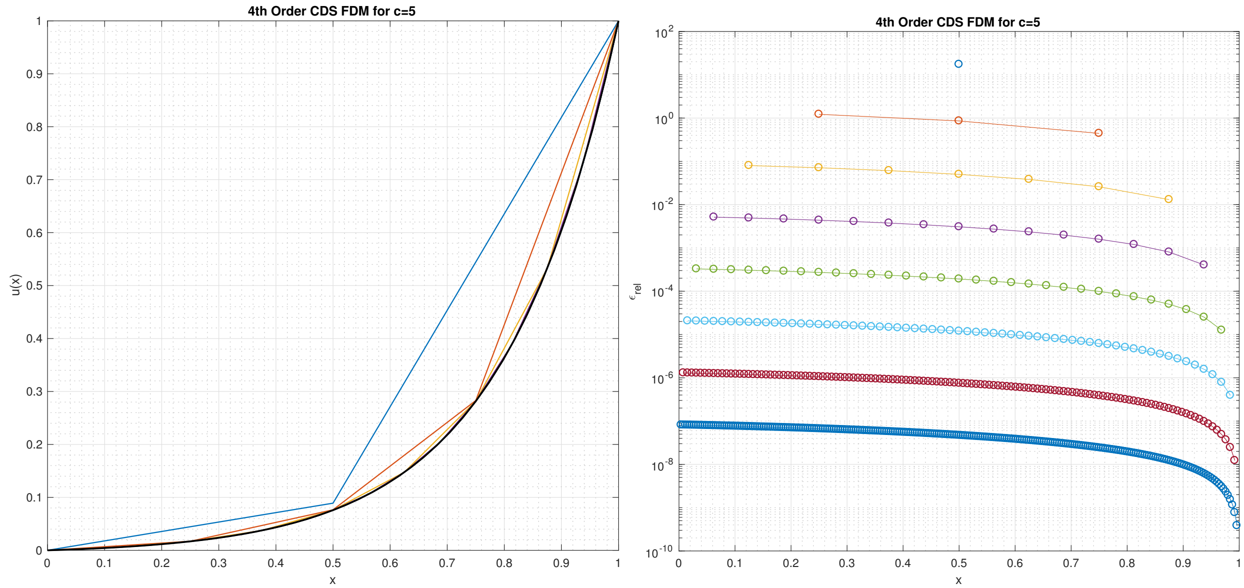


Figure 3.2.9 – 4th-Order CDS FDM and Pointwise Error for $c = 5$

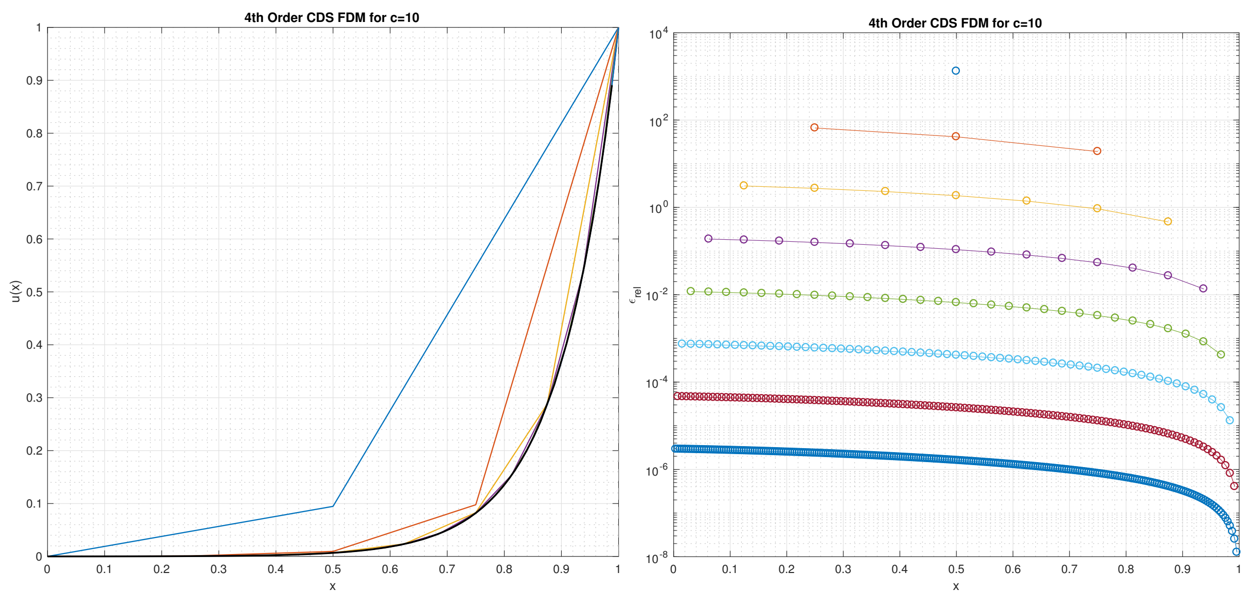
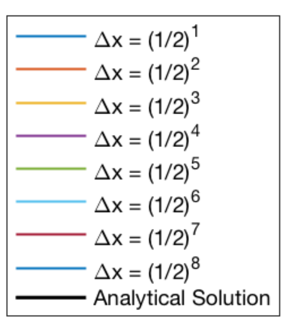


Figure 3.2.10 – 4th-Order CDS FDM and Pointwise Error for $c = 10$



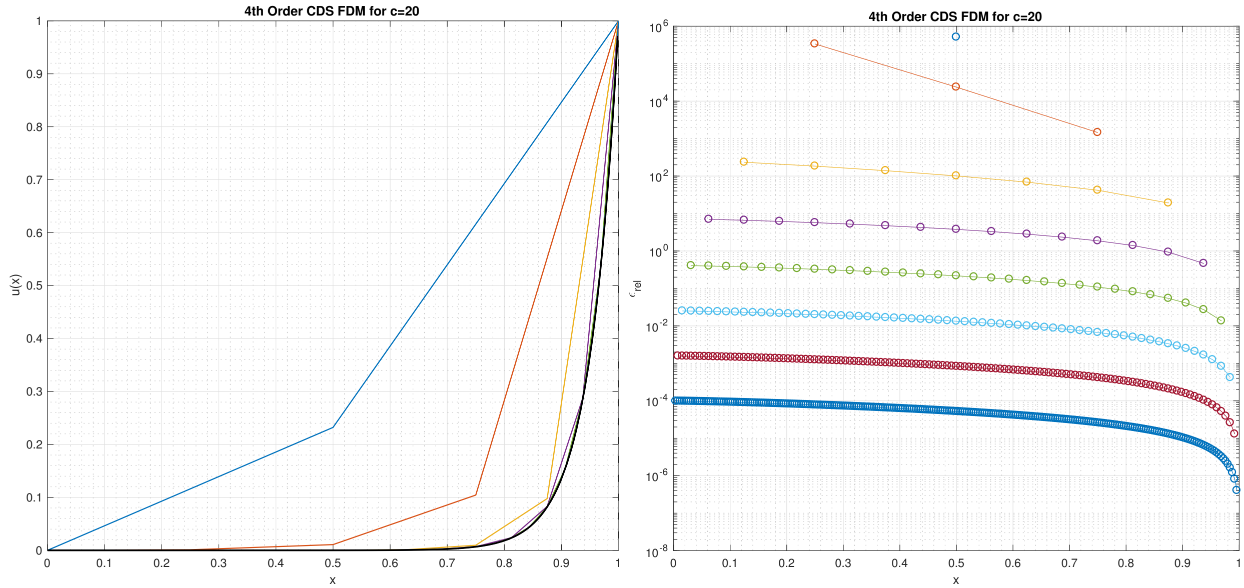


Figure 3.2.11 – 4th-Order CDS FDM and Pointwise Error for $c = 20$

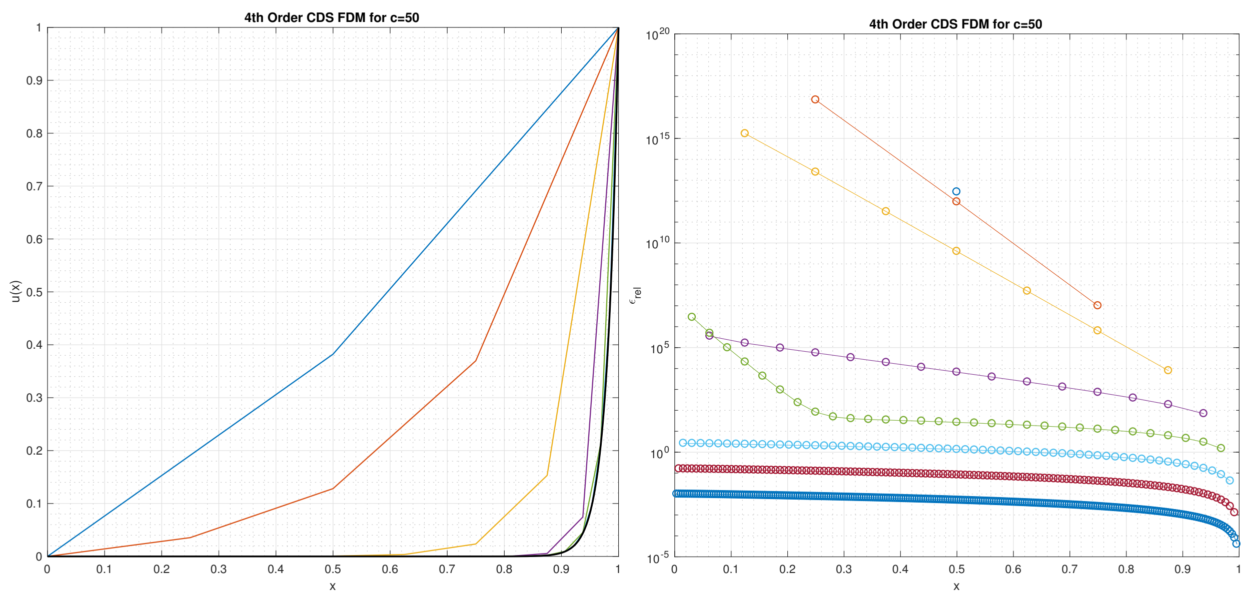
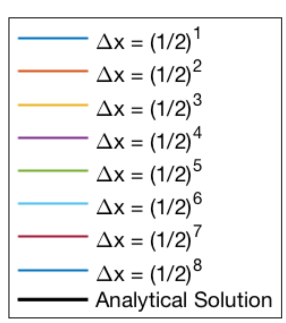


Figure 3.2.12 – 4th-Order CDS FDM and Pointwise Error for $c = 50$



3.2.3 1st-Order Upwind Scheme Finite Difference Method

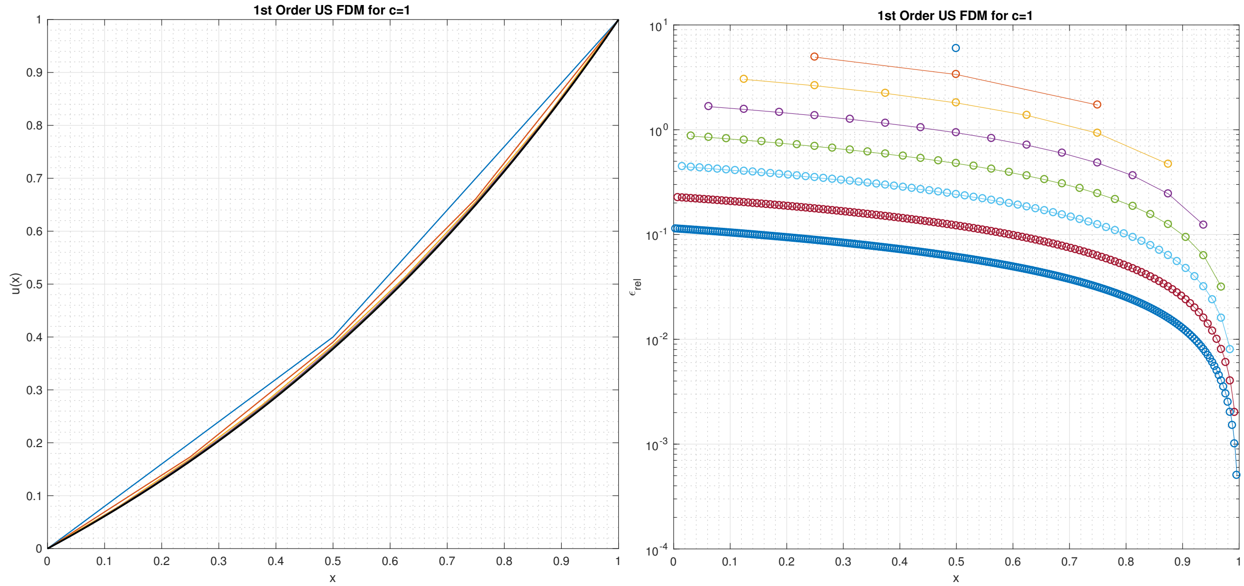


Figure 3.2.13 – 1st-Order US FDM and Pointwise Error for $c = 1$

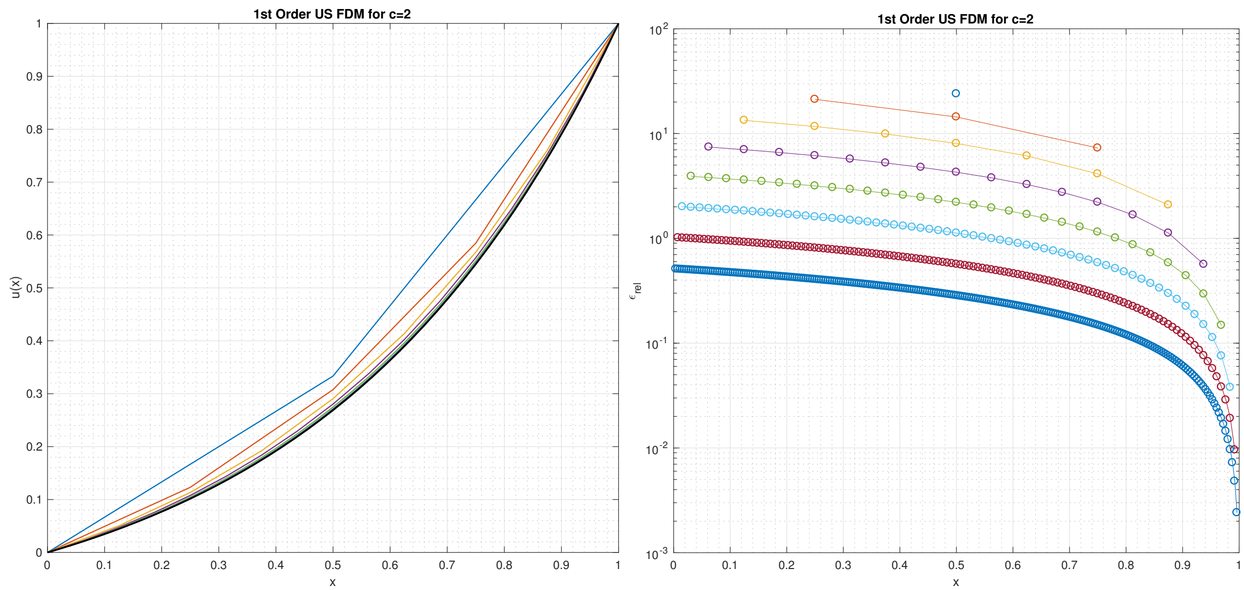
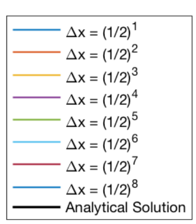


Figure 3.2.14 – 1st-Order US FDM and Pointwise Error for $c = 2$



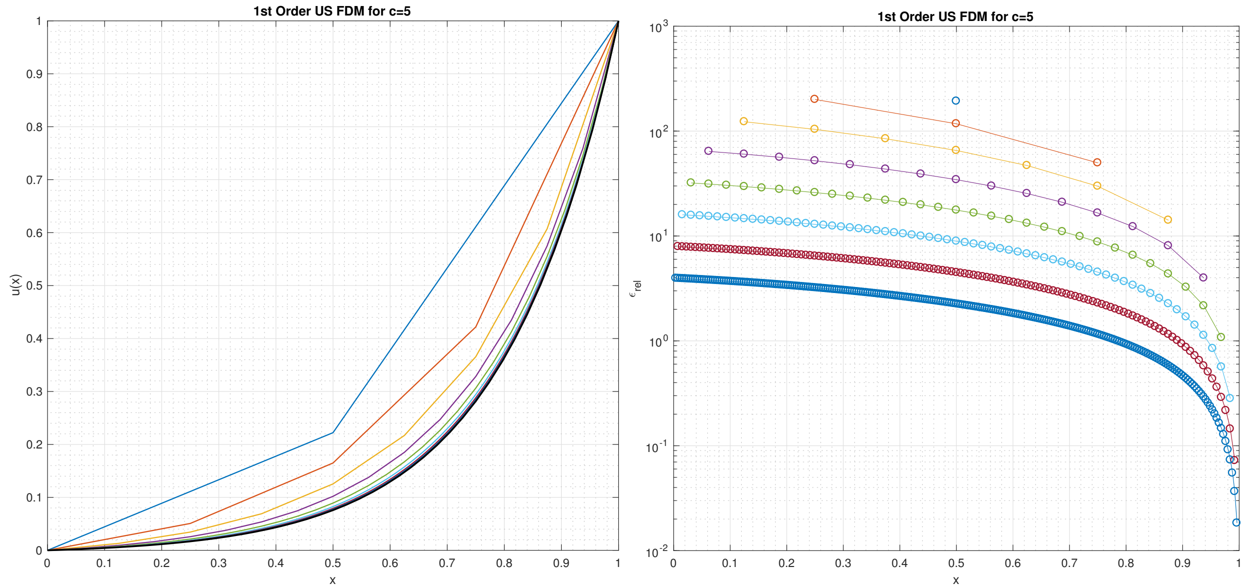


Figure 3.2.15 – 1st-Order US FDM and Pointwise Error for $c = 5$

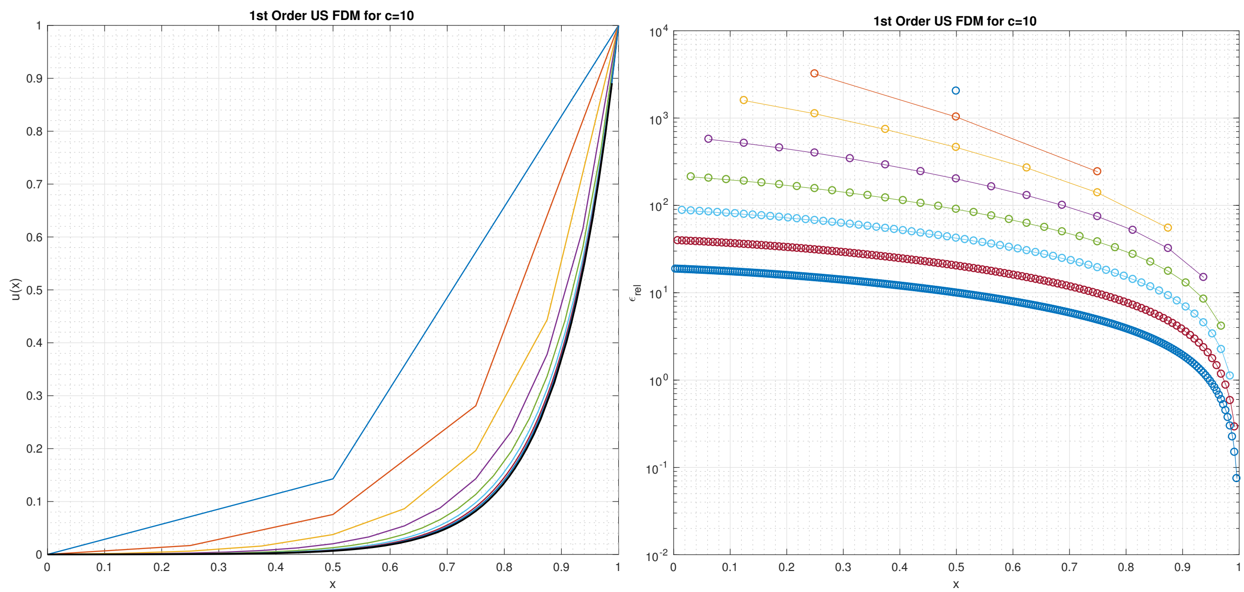
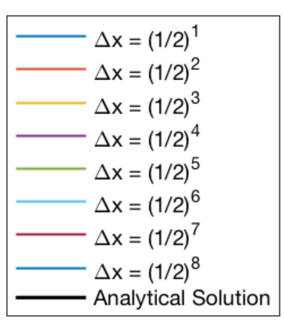


Figure 3.2.16 – 1st-Order US FDM and Pointwise Error for $c = 10$



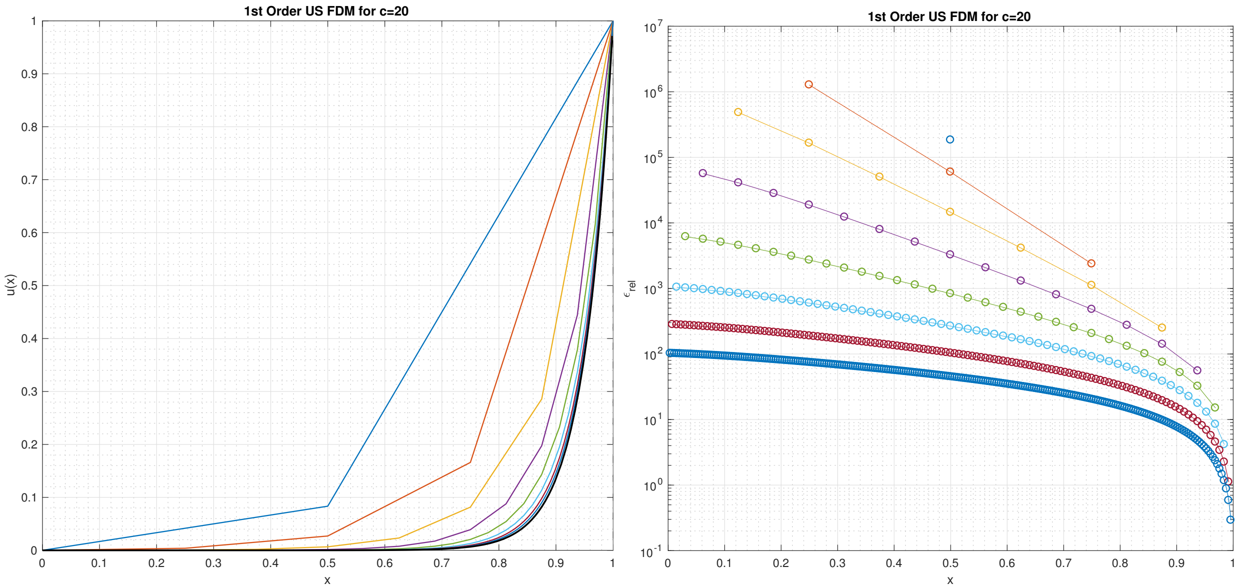


Figure 3.2.17 – 1st-Order US FDM and Pointwise Error for $c = 20$

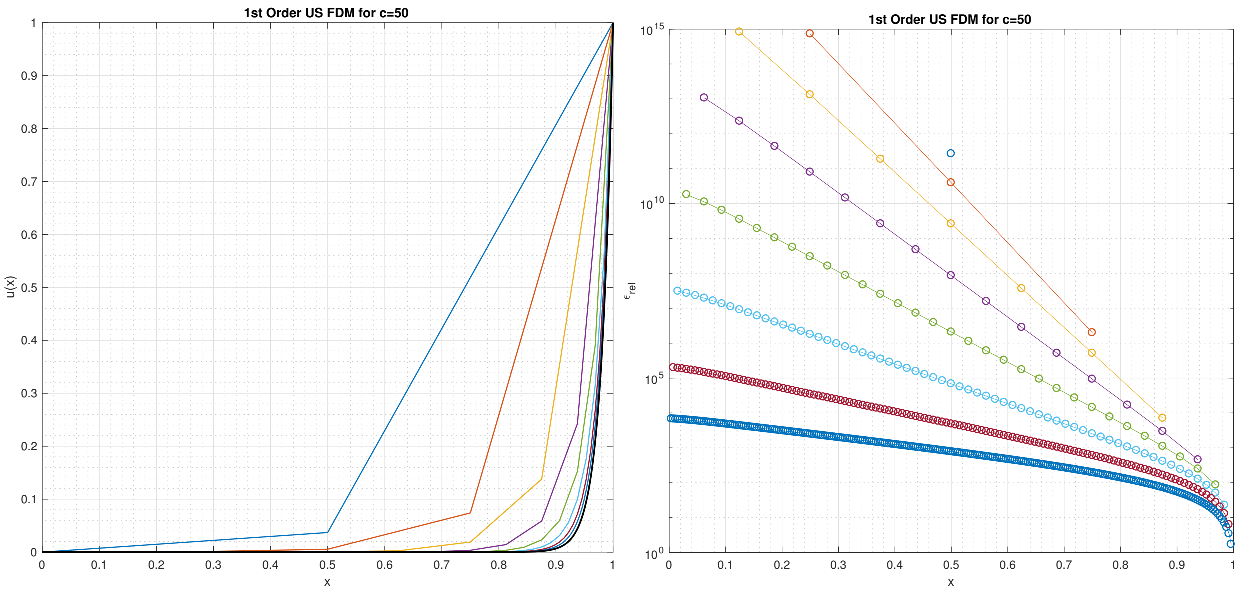
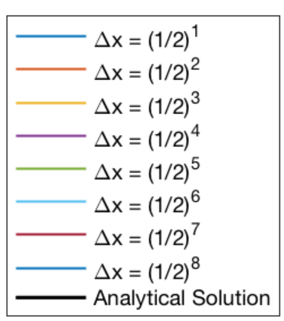


Figure 3.2.18 – 1st-Order US FDM and Pointwise Error for $c = 50$



3.3 Quantity of Interest Results

3.3.1 2nd-Order Central Difference Scheme Finite Difference Method

Table 3.3.1 – Quantity of Interest Values for 2nd-Order CDS FDM - I

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	1.6667	3.0000	-9.0000	1.5820	2.3130	5.0339	5.3534	29.6997	278.7872
0.2500	1.6022	2.4510	8.2285	1.5820	2.3130	5.0339	1.2782	5.9638	63.4604
0.1250	1.5870	2.3459	5.5727	1.5820	2.3130	5.0339	0.3160	1.4213	10.7032
0.0625	1.5832	2.3212	5.1585	1.5820	2.3130	5.0339	0.0788	0.3512	2.4744
0.0312	1.5823	2.3151	5.0645	1.5820	2.3130	5.0339	0.0197	0.0876	0.6072
0.0156	1.5821	2.3135	5.0415	1.5820	2.3130	5.0339	0.0049	0.0219	0.1511
0.0078	1.5820	2.3132	5.0358	1.5820	2.3130	5.0339	0.0012	0.0055	0.0377
0.0039	1.5820	2.3131	5.0344	1.5820	2.3130	5.0339	0.0003	0.0014	0.0094
0.0020	1.5820	2.3130	5.0340	1.5820	2.3130	5.0339	0.0001	0.0003	0.0024
0.0010	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0001	0.0006
0.0005	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0001
0.0002	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0002	0.0001	0.0001

Table 3.3.2 – Quantity of Interest Values for 2nd-Order CDS FDM - II

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-2.3333	-1.5000	-1.1739	10.0005	20.0000	50.0000	123.3323	107.5000	102.3478
0.2500	-17.7805	-3.9425	-1.8118	10.0005	20.0000	50.0000	277.7968	119.7126	103.6237
0.1250	16.4104	-35.5556	-5.7325	10.0005	20.0000	50.0000	64.0964	277.7778	111.4651
0.0625	11.0826	32.8205	-34.6883	10.0005	20.0000	50.0000	10.8210	64.1026	169.3767
0.0312	10.2507	22.1645	128.3208	10.0005	20.0000	50.0000	2.5021	10.8225	156.6416
0.0156	10.0619	20.5005	59.0032	10.0005	20.0000	50.0000	0.6140	2.5025	18.0063
0.0078	10.0157	20.1228	51.9830	10.0005	20.0000	50.0000	0.1528	0.6141	3.9660
0.0039	10.0043	20.0306	50.4814	10.0005	20.0000	50.0000	0.0382	0.1528	0.9629
0.0020	10.0014	20.0076	50.1195	10.0005	20.0000	50.0000	0.0095	0.0382	0.2390
0.0010	10.0007	20.0019	50.0298	10.0005	20.0000	50.0000	0.0024	0.0095	0.0596
0.0005	10.0005	20.0005	50.0075	10.0005	20.0000	50.0000	0.0006	0.0024	0.0149
0.0002	10.0005	20.0001	50.0019	10.0005	20.0000	50.0000	0.0001	0.0006	0.0037
0.0001	10.0005	20.0000	50.0005	10.0005	20.0000	50.0000	0.0000	0.0001	0.0009
0.0001	10.0005	20.0000	50.0001	10.0005	20.0000	50.0000	0.0000	0.0000	0.0002
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0001
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0002	0.0000
0.0000	10.0005	20.0002	50.0001	10.0005	20.0000	50.0000	0.0002	0.0008	0.0001

3.3.2 4th-Order Central Difference Scheme Finite Difference Method

Table 3.3.3 – Quantity of Interest Values for 4th-Order CDS FDM - I

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	1.5829	2.3385	12.9559	1.5820	2.3130	5.0339	0.0594	1.0993	157.3713
0.2500	1.5820	2.3144	5.1772	1.5820	2.3130	5.0339	0.0034	0.0570	2.8461
0.1250	1.5820	2.3131	5.0410	1.5820	2.3130	5.0339	0.0002	0.0032	0.1400
0.0625	1.5820	2.3130	5.0343	1.5820	2.3130	5.0339	0.0000	0.0002	0.0077
0.0312	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0004
0.0156	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0078	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0039	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0020	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0010	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0005	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0002	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0002	0.0001	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0006	0.0004	0.0001
0.0000	1.5819	2.3130	5.0339	1.5820	2.3130	5.0339	0.0025	0.0016	0.0005
0.0000	1.5818	2.3129	5.0338	1.5820	2.3130	5.0339	0.0099	0.0062	0.0018
0.0000	1.5813	2.3125	5.0336	1.5820	2.3130	5.0339	0.0397	0.0248	0.0072

Table 3.3.4 – Quantity of Interest Values for 4th-Order CDS FDM - II

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-0.7125	-0.0530	-0.0022	10.0005	20.0000	50.0000	107.1242	100.2648	100.0044
0.2500	25.6665	-1.4095	-0.0416	10.0005	20.0000	50.0000	156.6534	107.0476	100.0832
0.1250	10.2838	51.3283	-1.1707	10.0005	20.0000	50.0000	2.8333	156.6416	102.3414
0.0625	10.0144	20.5666	-71.7281	10.0005	20.0000	50.0000	0.1393	2.8331	243.4562
0.0312	10.0012	20.0279	53.9799	10.0005	20.0000	50.0000	0.0077	0.1393	7.9598
0.0156	10.0005	20.0015	50.1805	10.0005	20.0000	50.0000	0.0004	0.0077	0.3611
0.0078	10.0005	20.0001	50.0097	10.0005	20.0000	50.0000	0.0000	0.0004	0.0193
0.0039	10.0005	20.0000	50.0006	10.0005	20.0000	50.0000	0.0000	0.0000	0.0011
0.0020	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0001
0.0010	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0005	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0002	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0001	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0001	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0004	20.0000	50.0000	10.0005	20.0000	50.0000	0.0001	0.0001	0.0000
0.0000	10.0004	20.0000	50.0000	10.0005	20.0000	50.0000	0.0005	0.0000	0.0000
0.0000	10.0003	19.9999	49.9999	10.0005	20.0000	50.0000	0.0017	0.0005	0.0001

3.3.3 1st-Order Upwind Scheme Finite Difference Method

Table 3.3.5 – Quantity of Interest Values for 1st-Order US FDM - I

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	1.2000	1.3333	1.5556	1.5820	2.3130	5.0339	24.1455	42.3557	69.0985
0.2500	1.3550	1.6615	2.3125	1.5820	2.3130	5.0339	14.3468	28.1663	54.0626
0.1250	1.4566	1.9226	3.1415	1.5820	2.3130	5.0339	7.9263	16.8819	37.5926
0.0625	1.5158	2.0962	3.8593	1.5820	2.3130	5.0339	4.1838	9.3749	23.3343
0.0312	1.5479	2.1983	4.3662	1.5820	2.3130	5.0339	2.1519	4.9623	13.2634
0.0156	1.5647	2.2539	4.6756	1.5820	2.3130	5.0339	1.0917	2.5562	7.1178
0.0078	1.5733	2.2830	4.8480	1.5820	2.3130	5.0339	0.5498	1.2977	3.6942
0.0039	1.5776	2.2979	4.9391	1.5820	2.3130	5.0339	0.2759	0.6539	1.8828
0.0020	1.5798	2.3054	4.9861	1.5820	2.3130	5.0339	0.1382	0.3282	0.9506
0.0010	1.5809	2.3092	5.0099	1.5820	2.3130	5.0339	0.0692	0.1644	0.4776
0.0005	1.5814	2.3111	5.0219	1.5820	2.3130	5.0339	0.0346	0.0823	0.2394
0.0002	1.5817	2.3121	5.0279	1.5820	2.3130	5.0339	0.0173	0.0412	0.1199
0.0001	1.5818	2.3126	5.0309	1.5820	2.3130	5.0339	0.0087	0.0206	0.0600
0.0001	1.5819	2.3128	5.0324	1.5820	2.3130	5.0339	0.0043	0.0103	0.0300
0.0000	1.5819	2.3129	5.0332	1.5820	2.3130	5.0339	0.0022	0.0051	0.0150
0.0000	1.5820	2.3130	5.0335	1.5820	2.3130	5.0339	0.0011	0.0026	0.0075
0.0000	1.5820	2.3130	5.0337	1.5820	2.3130	5.0339	0.0005	0.0013	0.0037
0.0000	1.5820	2.3130	5.0338	1.5820	2.3130	5.0339	0.0003	0.0007	0.0019
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0002	0.0003	0.0009
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0001	0.0002	0.0005
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0001	0.0001	0.0002
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0001

Table 3.3.6 – Quantity of Interest Values for 1st-Order US FDM - II

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	1.7143	1.8333	1.9259	10.0005	20.0000	50.0000	82.8579	90.8333	96.1481
0.2500	2.8763	3.3359	3.7038	10.0005	20.0000	50.0000	71.2382	83.3205	92.5924
0.1250	4.4512	5.7145	6.8966	10.0005	20.0000	50.0000	55.4898	71.4273	86.2069
0.0625	6.1565	8.8889	12.1212	10.0005	20.0000	50.0000	38.4383	55.5555	75.7576
0.0312	7.6203	12.3077	19.5122	10.0005	20.0000	50.0000	23.8003	38.4615	60.9756
0.0156	8.6494	15.2381	28.0702	10.0005	20.0000	50.0000	13.5095	23.8095	43.8596
0.0078	9.2760	17.2973	35.9551	10.0005	20.0000	50.0000	7.2445	13.5135	28.0899
0.0039	9.6246	18.5507	41.8301	10.0005	20.0000	50.0000	3.7585	7.2464	16.3399
0.0020	9.8089	19.2481	45.5516	10.0005	20.0000	50.0000	1.9153	3.7594	8.8968
0.0010	9.9038	19.6169	47.6723	10.0005	20.0000	50.0000	0.9669	1.9157	4.6555
0.0005	9.9519	19.8066	48.8084	10.0005	20.0000	50.0000	0.4858	0.9671	2.3832
0.0002	9.9761	19.9028	49.3970	10.0005	20.0000	50.0000	0.2435	0.4859	1.2060
0.0001	9.9883	19.9513	49.6967	10.0005	20.0000	50.0000	0.1219	0.2435	0.6066
0.0001	9.9944	19.9756	49.8479	10.0005	20.0000	50.0000	0.0610	0.1219	0.3042
0.0000	9.9974	19.9878	49.9238	10.0005	20.0000	50.0000	0.0305	0.0610	0.1524
0.0000	9.9989	19.9939	49.9619	10.0005	20.0000	50.0000	0.0153	0.0305	0.0762
0.0000	9.9997	19.9969	49.9809	10.0005	20.0000	50.0000	0.0076	0.0153	0.0381
0.0000	10.0001	19.9985	49.9905	10.0005	20.0000	50.0000	0.0038	0.0076	0.0191
0.0000	10.0003	19.9992	49.9952	10.0005	20.0000	50.0000	0.0019	0.0038	0.0095
0.0000	10.0004	19.9996	49.9976	10.0005	20.0000	50.0000	0.0010	0.0018	0.0048
0.0000	10.0004	19.9998	49.9988	10.0005	20.0000	50.0000	0.0005	0.0008	0.0023
0.0000	10.0004	20.0001	49.9995	10.0005	20.0000	50.0000	0.0004	0.0004	0.0010

3.4 Discussion

Meshes were calculated for $\Delta x = 0.5^{22}$, but only meshes up to $\Delta x = 0.5^{8}$ are shown in the tables and figures due to sheer size and readability.

For the 2nd-order CDS FDM and the 4th-order CDS FDM, the pointwise relative error in $u(x)$ is generally not a great measure for the accuracy of the method, as the solution is close to zero, thus amplifying the relative differences.

The 2nd-order CDS FDM is unstable for coarse mesh sizes, but more dependent on the Peclet condition, which relates the mesh spacing to the convective-diffusive constant c. These instabilities arise for $c = 2$ and $\Delta x = 1/2$, but includes even smaller mesh spacings at higher values of c. These spurious oscillations are nonphysical, which means that the 2nd-order method should only be used if the Peclet condition is satisfied, which guarantees stability.

The 4th-order CDS FDM is not unstable for coarse mesh sizes, but error grows siginificantly for higher values of c and coarser mesh spacings. These inaccuracies can be seen when considering the variation in the solution using mesh spacing $\Delta x = 1/2$ for all values of c. These errors are more acceptable over the errors from the 2nd-order CDS FDM, which means that the 4th-order method can be used even if the Peclet condition is not satisfied. In general the solution developed from the 4th-order method is much more accurate than the second-order method on refined meshes.

The 1st-order US FDM is not unstable for any mesh size, since the upwind scheme is a physical model that prevents spurious oscillations from being produced. For even the coarsest mesh, the scheme is stable. In general though, the method is not very accurate, which is a significant reason to investigate using higher-order upwind schemes (i.e. a second-order quadratic upwind scheme) or higher-order central difference schemes despite their constraints on stability.

Additionally, we can see that for the quantity of interest, the second-order CDS FDM produces increasingly accurate values of the first derivative at 1, though the beginnings of error propagation are seen at the smallest meshes tested. The fourth-order CDS FDM produces increasingly accurate values of the first derivative at 1, but error propagation is seen in the smallest meshes tested in greater amounts than the second-order method. For the first-order US FDM, accurate values of the first derivative are produced, but require a much more refined mesh to produce similar values for the relative error.

Ultimately, the selection between 2nd-order CDS FDM, 4th-order CDS FDM, and 1st-order US FDM is a trade-off between stability, computational speed, and mesh size, respectively.

4 Convergence Analysis

4.1 Rate of Convergence Derivation

Let the error for a particular mesh size Δx be $E(\Delta x)$:

$$E(\Delta x) = C(\Delta x)^{\beta} \tag{4.1}$$

Then for a smaller mesh size $\frac{\Delta x}{2}$ we have:

$$E\left(\frac{\Delta x}{2}\right) = C\left(\frac{\Delta x}{2}\right)^{\beta} \tag{4.2}$$

Dividing the error at each mesh size and taking the logarithm:

$$\frac{E(\Delta x)}{E\left(\frac{\Delta x}{2}\right)} = \frac{C(\Delta x)^{\beta}}{C\left(\frac{\Delta x}{2}\right)^{\beta}} = 2^{\beta} \tag{4.3}$$

$$\log\left[\frac{E(\Delta x)}{E\left(\frac{\Delta x}{2}\right)}\right] = \log(2^{\beta}) \tag{4.4}$$

$$\log\left[\frac{E(\Delta x)}{E\left(\frac{\Delta x}{2}\right)}\right] = \beta\log(2) \tag{4.5}$$

Rearranging for β and simplifying:

$$\beta = \frac{1}{\log(2)}\left[\log(E(\Delta x)) - \log\left(E\left(\frac{\Delta x}{2}\right)\right)\right] \tag{4.6}$$

Denoting $E^{*}_{\Delta x} = \log(E(\Delta x))$:

$$\beta = \frac{\mathbf{E}^{*}_{\boldsymbol{\Delta}\mathbf{x}} - \mathbf{E}^{*}_{\frac{\boldsymbol{\Delta}\mathbf{x}}{\mathbf{2}}}}{\log(\mathbf{2})} \tag{4.7}$$

4.2 2nd-Order Central Difference Scheme First-Derivative Extraction

4.2.1 Derivation

Developing the Taylor series for $u(x)$ in the vicinity of $x = 1$:

$$u_{N-1} = u_N - \Delta x u'_N + \frac{\Delta x^2}{2} u''_N + \mathcal{O}(\Delta x^3) \tag{4.8}$$

Returning to the differential equation and rearranging for the second derivative:

$$-u''(x) + cu'(x) = 0 \tag{4.9}$$

$$u''(x) = cu'(x) \tag{4.10}$$

Substituing the expression for the second-derivative in the Taylor series expansion and rearranging:

$$u_{N-1} = u_N - \Delta x u'_N + \frac{c\Delta x^2}{2} u'_N + \mathcal{O}(\Delta x^3) \tag{4.11}$$

$$u_{N-1} = u_N + \left(-\Delta x + \frac{c\Delta x^2}{2}\right) u'_N + \mathcal{O}(\Delta x^3) \tag{4.12}$$

$$u'_N = \left(1 - \frac{c\Delta x}{2}\right)^{-1} \left(\frac{u_N - u_{N-1}}{\Delta x}\right) + \mathcal{O}(\Delta x^2) \tag{4.13}$$

Applying the boundary condition $u(1) = u_N = 1$:

$$u'_N = \left(1 - \frac{c\Delta x}{2}\right)^{-1} \left(\frac{1 - u_{N-1}}{\Delta x}\right) + \mathcal{O}(\Delta x^2) \tag{4.14}$$

From this specific first-derivative formulation at the boundary $x = 1$ using the finite difference method, the approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

4.2.2 Results

Note: The quantity of interest for the 2nd-order CDS FDM is extracted using the 2nd-order first-derivative extraction - yielding the quantity of interest with an error of $\mathcal{O}(\Delta x^2)$.

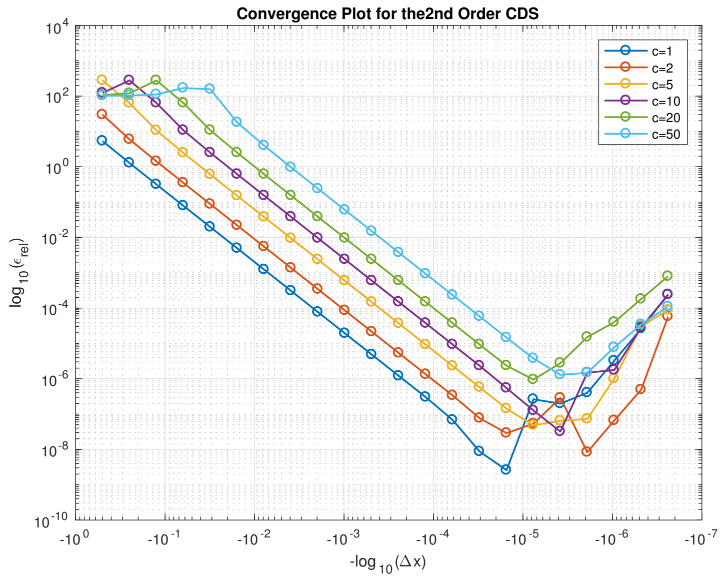


Figure 4.2.1 – Convergence Plot for 2nd-Order CDS FDM

Δx	$\beta(c=1)$	$\beta(c=2)$	$\beta(c=5)$	$\beta(c=10)$	$\beta(c=20)$	$\beta(c=50)$
0.5000	2.0663	2.3161	2.1352	-1.1715	-0.1552	-0.0179
0.2500	2.0161	2.0691	2.5678	2.1157	-1.2144	-0.1052
0.1250	2.0040	2.0168	2.1129	2.5664	2.1155	-0.6036
0.0625	2.0010	2.0042	2.0269	2.1126	2.5663	0.1128
0.0312	2.0002	2.0010	2.0066	2.0268	2.1126	3.1209
0.0156	2.0001	2.0003	2.0017	2.0066	2.0268	2.1828
0.0078	2.0000	2.0001	2.0004	2.0017	2.0066	2.0423
0.0039	2.0000	2.0000	2.0001	2.0004	2.0017	2.0104
0.0020	2.0000	2.0000	2.0000	2.0001	2.0004	2.0026
0.0010	1.9999	2.0000	2.0000	2.0000	2.0001	2.0006
0.0005	2.0004	1.9999	2.0000	2.0000	2.0000	2.0002
0.0002	1.9929	2.0000	2.0001	2.0000	2.0000	2.0000
0.0001	2.1500	1.9898	2.0005	2.0000	2.0000	2.0000
0.0001	2.9660	2.1314	2.0049	1.9994	1.9996	1.9999
0.0000	1.7543	1.4102	2.0078	2.0924	1.9906	1.9967
0.0000	-6.6673	-0.8937	1.5642	2.1065	1.3098	1.9564
0.0000	0.4112	-2.4168	-0.4198	1.9889	-1.5473	1.5401
0.0000	-1.0256	5.1013	-0.1715	-5.5431	-2.4479	-0.1831
0.0000	-3.0136	-2.9764	-3.7817	-0.2395	-1.4021	-2.3934
0.0000	-3.0056	-2.8983	-4.9316	-3.9962	-2.1755	-2.1561
0.0000	-3.2026	-6.8772	-1.5264	-3.1065	-2.1369	-1.6780

Table 4.2.1 – Rate of Convergence Values for 2nd-Order CDS FDM

4.3 4th-Order Central Difference Scheme First-Derivative Extraction

4.3.1 Derivation

Developing the Taylor series for $u(x)$ in the vicinity of $x = 1$:

$$u_{N-1} = u_N - \Delta x u'_N + \frac{\Delta x^2}{2} u''_N - \frac{\Delta x^3}{6} u_N^{(3)} + \frac{\Delta x^4}{24} u_N^{(4)} + \mathcal{O}(\Delta x^5) \tag{4.15}$$

Returning to the differential equation, rearranging for the second derivative, and taking two additional derivatives:

$$-u''(x) + cu'(x) = 0 \tag{4.16}$$

$$u''(x) = cu'(x) \tag{4.17}$$

$$u^{(3)}(x) = c^2 u'(x) \tag{4.18}$$

$$u^{(4)}(x) = c^3 u'(x) \tag{4.19}$$

Replacing the second-, third-, and fourth-derivative terms in the Taylor series expansion:

$$u_{N-1} = u_N - \Delta x u'_N + \frac{c\Delta x^2}{2} u'_N - \frac{c^2 \Delta x^3}{6} u'_N + \frac{c^3 \Delta x^4}{24} u'_N + \mathcal{O}(\Delta x^5) \tag{4.20}$$

$$\frac{u_N - u_{N-1}}{\Delta x} = \left(1 - \frac{c\Delta x}{2} + \frac{c^2 \Delta x^2}{6} - \frac{c^3 \Delta x^3}{24}\right) u'_N + \mathcal{O}(\Delta x^4) \tag{4.21}$$

$$u'_N = \left(1 - \frac{c\Delta x}{2} + \frac{c^2 \Delta x^2}{6} - \frac{c^3 \Delta x^3}{24}\right)^{-1} \frac{u_N - u_{N-1}}{\Delta x} + \mathcal{O}(\Delta x^4) \tag{4.22}$$

Applying the boundary condition $u(1) = u_N = 1$:

$$u'_N = \left(1 - \frac{c\Delta x}{2} + \frac{c^2 \Delta x^2}{6} - \frac{c^3 \Delta x^3}{24}\right)^{-1} \frac{1 - u_{N-1}}{\Delta x} + \mathcal{O}(\Delta x^4) \tag{4.23}$$

From this specific first-derivative formulation at the boundary $x = 1$ using the finite difference method, the approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

4.3.2 Results

Note: The quantity of interest for the 4th-order CDS FDM is extracted using the 4th-order first-derivative extraction - yielding the quantity of interest with an error of $\mathcal{O}(\Delta x^4)$.

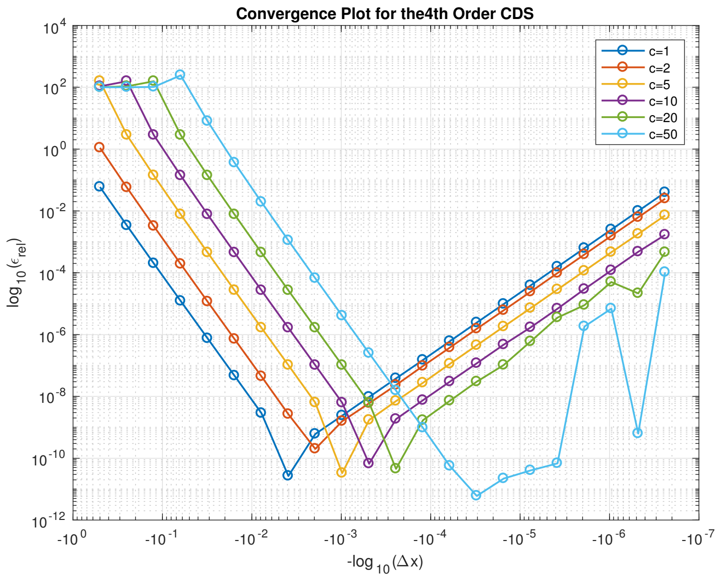


Figure 4.3.1 – Convergence Plot for 4th-Order CDS FDM

Δx	$\beta(c=1)$	$\beta(c=2)$	$\beta(c=5)$	$\beta(c=10)$	$\beta(c=20)$	$\beta(c=50)$
0.5000	4.1405	4.2684	5.7890	-0.5483	-0.0944	-0.0011
0.2500	4.0749	4.1459	4.3454	5.7890	-0.5492	-0.0322
0.1250	4.0389	4.0782	4.1839	4.3464	5.7889	-1.2503
0.0625	4.0198	4.0407	4.1005	4.1847	4.3464	4.9348
0.0312	4.0103	4.0208	4.0529	4.1010	4.1848	4.4625
0.0156	4.0233	4.0112	4.0272	4.0532	4.1010	4.2233
0.0078	6.7476	4.0609	4.0142	4.0274	4.0532	4.1232
0.0039	-4.5113	3.7287	4.0311	4.0143	4.0274	4.0657
0.0020	-1.9775	-2.9750	7.5728	4.0329	4.0143	4.0340
0.0010	-1.9979	-1.9307	-5.7027	6.5530	4.0323	4.0175
0.0005	-1.9992	-1.9571	-2.0388	-4.7944	7.1048	4.0140
0.0002	-1.9781	-2.0422	-1.9624	-2.0272	-5.2245	4.0011
0.0001	-2.0174	-1.9810	-2.0355	-1.9977	-2.0860	4.1025
0.0001	-2.0013	-2.0205	-1.9915	-1.9714	-2.0585	3.2178
0.0000	-1.9917	-1.9825	-2.0030	-2.0012	-1.7842	-1.8639
0.0000	-2.0022	-1.9949	-1.9876	-1.8558	-2.5308	-0.8759
0.0000	-2.0021	-2.0102	-2.0060	-2.0094	-2.5697	-0.7555
0.0000	-1.9991	-1.9939	-1.9989	-2.0981	-1.3343	-14.7059
0.0000	-1.9985	-1.9998	-2.0049	-2.0182	-2.4899	-1.9214
0.0000	-1.9986	-1.9998	-1.9684	-2.0010	1.2231	13.4265
0.0000	-1.9973	-1.9968	-1.9979	-1.8221	-4.4135	-17.3388

Table 4.3.1 – Rate of Convergence Values for 4th-Order CDS FDM

4.4 1st-Order Upwind Scheme First-Derivative Extraction

4.4.1 Derivation

Developing the Taylor series for $u(x)$ in the vicinity of $x = 1$:

$$u_{N-1} = u_N - \Delta x u'_N + \mathcal{O}(\Delta x^2) \tag{4.24}$$

Rearranging terms to solve for u'_N:

$$u'_N = \frac{u_N - u_{N-1}}{\Delta x} + \mathcal{O}(\Delta x) \tag{4.25}$$

Applying the boundary condition $u(1) = u_N = 1$:

$$u'_N = \frac{1 - u_{N-1}}{\Delta x} + \mathcal{O}(\Delta x) \tag{4.26}$$

From this specific first-derivative formulation at the boundary $x = 1$ using the finite difference method, the approximation can be observed to be first-order ($\mathcal{O}(\Delta x)$).

4.4.2 Results

Note: The quantity of interest for the 1st-order US FDM is extracted using the 1st-order first-derivative extraction - yielding the quantity of interest with an error of $\mathcal{O}(\Delta x)$.

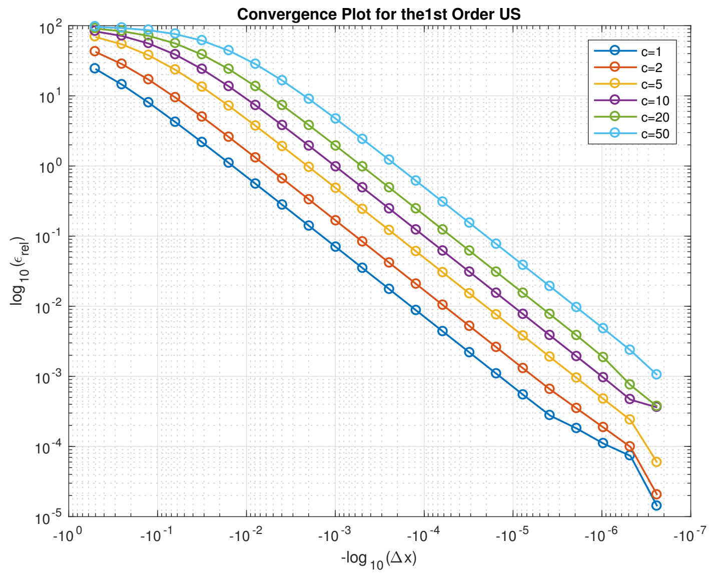
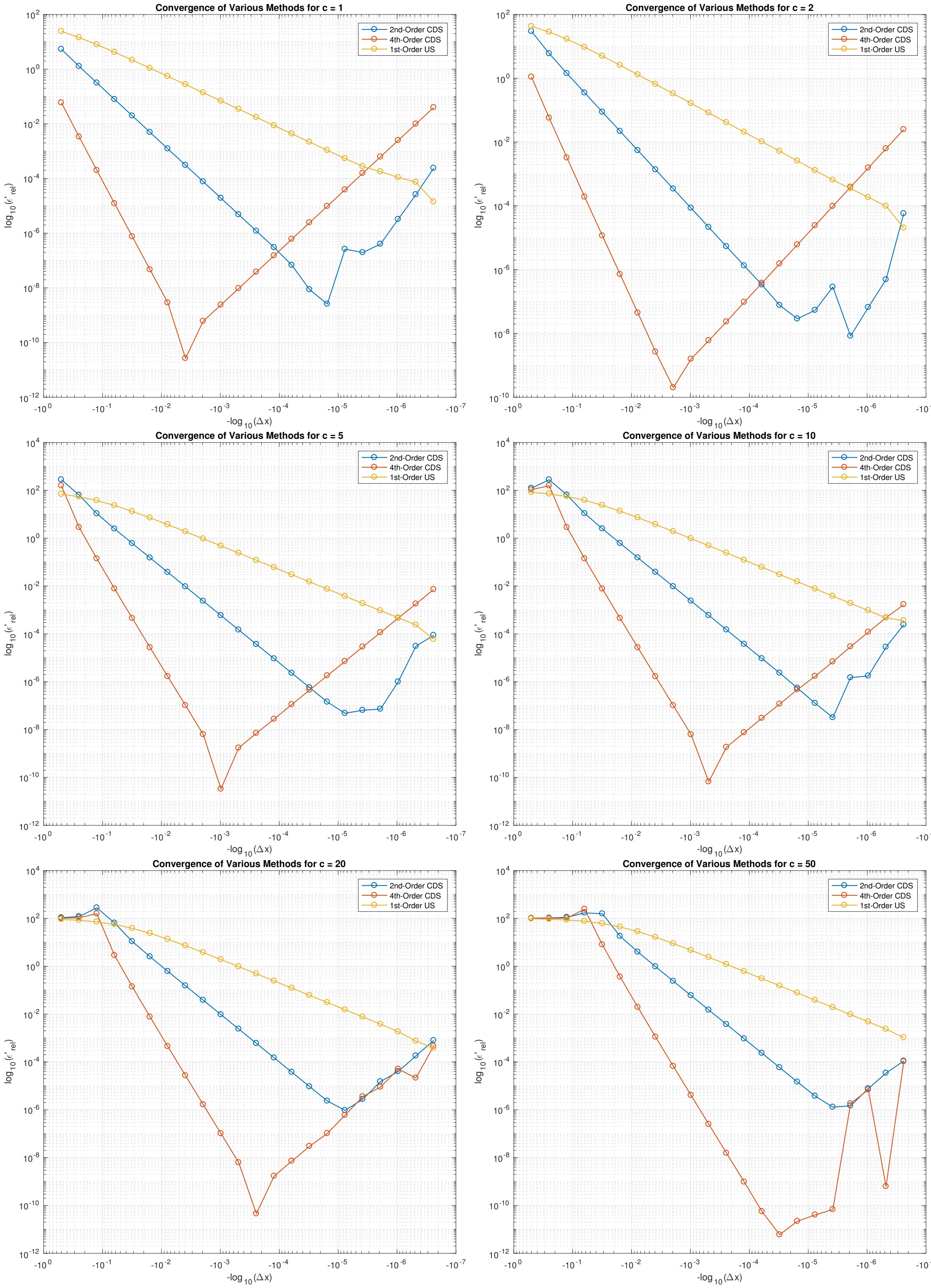


Figure 4.4.1 – Convergence Plot 1st-Order US FDM

Δx	$\beta(c=1)$	$\beta(c=2)$	$\beta(c=5)$	$\beta(c=10)$	$\beta(c=20)$	$\beta(c=50)$
0.5000	0.7510	0.5886	0.3540	0.2180	0.1246	0.0544
0.2500	0.8560	0.7385	0.5242	0.3604	0.2222	0.1031
0.1250	0.9218	0.8486	0.6880	0.5297	0.3625	0.1864
0.0625	0.9592	0.9178	0.8150	0.6916	0.5305	0.3132
0.0312	0.9791	0.9570	0.8980	0.8170	0.6919	0.4753
0.0156	0.9894	0.9780	0.9462	0.8990	0.8171	0.6428
0.0078	0.9947	0.9889	0.9723	0.9467	0.8991	0.7817
0.0039	0.9973	0.9944	0.9860	0.9726	0.9468	0.8770
0.0020	0.9987	0.9972	0.9929	0.9861	0.9726	0.9344
0.0010	0.9993	0.9986	0.9965	0.9930	0.9861	0.9660
0.0005	0.9997	0.9993	0.9982	0.9965	0.9930	0.9827
0.0002	0.9998	0.9996	0.9991	0.9982	0.9965	0.9913
0.0001	0.9999	0.9998	0.9996	0.9991	0.9982	0.9956
0.0001	1.0000	0.9999	0.9998	0.9996	0.9991	0.9978
0.0000	1.0001	1.0000	0.9999	0.9998	0.9996	0.9989
0.0000	0.9985	0.9998	0.9999	0.9999	0.9998	0.9995
0.0000	0.9748	0.9855	0.9980	0.9999	1.0004	0.9998
0.0000	0.6167	0.9060	0.9954	1.0000	1.0044	1.0001
0.0000	0.7135	0.9028	0.9956	1.0010	1.0418	1.0019
0.0000	0.5778	0.9135	0.9886	1.0342	1.2966	1.0190
0.0000	2.3818	2.2734	2.0125	0.3676	1.0159	1.1684

Table 4.4.1 – Rate of Convergence Values for 1st-Order US FDM

4.5 Comparison of 2nd-Order CDS, 4th-Order CDS, and 1st-Order US Finite Difference Methods



4.6 Discussion

As was explored earlier, the composite order of the solution to the boundary-value problem is dependent on the lowest-order method used across the entire solution – in this case, the lowest-order method of the finite difference method and the extraction method. Thus, the extraction method was developed to be equal to the highest-order method used, so that the rates of convergence could be easily seen.

For the 2nd-order CDS FDM, the rate of convergence is shown to be approximately 2 (**quadratic convergence**), which is predicted by order of the error calculations. At small mesh sizes, the error begins to increase as a function of increasing mesh size – typical of such small mesh sizes. We can see that a number of simulations, for example, those at $c = 50$ at the coarsest mesh sizes actually have more than 100% relative error. This is due to the violation of the Peclet condition and earlier described unstable oscillations that develop in the scheme. These oscillations and their errors disappear once the mesh size is sufficiently small that the Peclet condition is satified.

For the 4th-order CDS FDM, the rate of convergence is shown to be approximately 4 and sometimes a little more than 4 (**quartic to superquartic convergence**), which is predicted by order of the error calculations. At moderately small mesh sizes, the error begins to increase as a function of increasing mesh size – typical of such small mesh sizes. However, compared to the 2nd-order CDS FDM, the onset of the error increase is much earlier - this is because of the increased accuracy provided earlier on for coarser meshes. We also observe some odd trends for the case $c = 50$.

For the 1st-order US FDM, the rate of convergence is shown to be approximately 1 and sometimes a little less than 1 (**sublinear to linear convergence**), which is predicted by order of the error calculations. Even at very small mesh sizes, the error never increases as a function of increasing mesh size – though at the smallest mesh size tested, there appears to be the development of the trend reversal, as seen for other methods.

A simultaneous comparison of all of the methods for various values of c proves that:

- the second-order central difference scheme finite difference method is quadratically convergent
- the fourth-order central difference scheme finite difference method is quartically convergent
- the first-order upwind scheme finite difference method is linearly convergent

A $u(x)$ v. $u_{exact}(x)$ Tables

Error tables are included here up to a mesh spacing of $\Delta x = \left(\frac{1}{2}\right)^5$, but are available for all mesh spacings through the MATLAB code.

A.1 2nd-Order CDS FDM

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	0.00e+00	0.00e+00	NaN
5.00e-01	-5.75e+00	1.39e-11	4.14e+13
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-6.13e-17	0.00e+00	Inf
2.50e-01	-9.03e-01	5.18e-17	1.74e+18
5.00e-01	3.44e-01	1.39e-11	2.48e+12
7.50e-01	-1.38e+00	3.73e-06	3.70e+07
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	2.61e-17	0.00e+00	Inf
1.25e-01	-1.47e-02	9.97e-20	1.47e+19
2.50e-01	1.38e-02	5.18e-17	2.67e+16
3.75e-01	-4.14e-02	2.68e-14	1.55e+14
5.00e-01	6.58e-02	1.39e-11	4.74e+11
6.25e-01	-1.42e-01	7.19e-09	1.98e+09
7.50e-01	2.62e-01	3.73e-06	7.02e+06
8.75e-01	-5.23e-01	1.93e-03	2.72e+04
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	7.93e-18	0.00e+00	Inf
6.25e-02	-1.61e-10	4.20e-21	3.85e+12
1.25e-01	5.74e-10	9.97e-20	5.76e+11
1.88e-01	-2.78e-09	2.27e-18	1.22e+11
2.50e-01	1.25e-08	5.18e-17	2.41e+10
3.12e-01	-5.71e-08	1.18e-15	4.84e+09
3.75e-01	2.60e-07	2.68e-14	9.69e+08
4.38e-01	-1.18e-06	6.10e-13	1.94e+08
5.00e-01	5.39e-06	1.39e-11	3.88e+07
5.62e-01	-2.46e-05	3.16e-10	7.77e+06
6.25e-01	1.12e-04	7.19e-09	1.56e+06
6.88e-01	-5.10e-04	1.64e-07	3.11e+05
7.50e-01	2.32e-03	3.73e-06	6.22e+04
8.12e-01	-1.06e-02	8.48e-05	1.26e+04
8.75e-01	4.82e-02	1.93e-03	2.40e+03
9.38e-01	-2.20e-01	4.39e-02	6.00e+02
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-8.12e-18	0.00e+00	Inf
3.12e-02	-8.12e-18	7.27e-22	1.12e+06
6.25e-02	-8.12e-18	4.20e-21	1.94e+05
9.38e-02	-8.12e-18	2.07e-20	3.92e+04
1.25e-01	-8.12e-18	9.97e-20	8.24e+03
1.56e-01	-8.12e-18	4.76e-19	1.80e+03
1.88e-01	-8.12e-18	2.27e-18	4.57e+02
2.19e-01	-8.12e-18	1.08e-17	1.75e+02
2.50e-01	-8.12e-18	5.18e-17	1.16e+02
2.81e-01	-8.12e-18	2.47e-16	1.03e+02
3.12e-01	-8.11e-18	1.18e-15	1.01e+02
3.44e-01	-8.04e-18	5.62e-15	1.00e+02
3.75e-01	-7.51e-18	2.68e-14	1.00e+02
4.06e-01	-3.16e-18	1.28e-13	1.00e+02
4.38e-01	3.22e-17	6.10e-13	1.00e+02
4.69e-01	3.21e-16	2.91e-12	1.00e+02
5.00e-01	2.67e-15	1.39e-11	1.00e+02
5.31e-01	2.18e-14	6.63e-11	1.00e+02
5.62e-01	1.77e-13	3.16e-10	9.99e+01
5.94e-01	1.45e-12	1.51e-09	9.99e+01
6.25e-01	1.18e-11	7.19e-09	9.98e+01
6.56e-01	9.58e-11	3.43e-08	9.97e+01
6.88e-01	7.80e-10	1.64e-07	9.95e+01
7.19e-01	6.35e-09	7.81e-07	9.92e+01
7.50e-01	5.17e-08	3.73e-06	9.86e+01
7.81e-01	4.21e-07	1.78e-05	9.76e+01
8.12e-01	3.43e-06	8.48e-05	9.60e+01
8.44e-01	2.79e-05	4.05e-04	9.31e+01
8.75e-01	2.27e-04	1.93e-03	8.82e+01
9.06e-01	1.85e-03	9.21e-03	7.99e+01
9.38e-01	1.51e-02	4.39e-02	6.57e+01
9.69e-01	1.23e-01	2.10e-01	4.14e+01
1.00e+00	1.00e+00	1.00e+00	0.00e+00

A.2 4th-Order CDS FDM

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	1.09e-18	0.00e+00	Inf
5.00e-01	3.82e-01	1.39e-11	2.75e+12
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-1.47e-17	0.00e+00	Inf
1.25e-01	1.67e-06	9.97e-20	1.68e+15
2.50e-01	1.26e-05	5.18e-17	2.43e+13
3.75e-01	8.39e-05	2.68e-14	3.13e+11
5.00e-01	5.50e-04	1.39e-11	3.96e+09
6.25e-01	3.59e-03	7.19e-09	4.99e+07
7.50e-01	2.35e-02	3.73e-06	6.29e+05
8.75e-01	1.53e-01	1.93e-03	7.83e+03
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	1.26e-17	0.00e+00	Inf
2.50e-01	3.55e-02	5.18e-17	6.86e+16
5.00e-01	1.28e-01	1.39e-11	9.23e+11
7.50e-01	3.70e-01	3.73e-06	9.92e+06
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	3.68e-18	0.00e+00	Inf
6.25e-02	1.47e-17	4.20e-21	3.51e+05
1.25e-01	1.63e-16	9.97e-20	1.63e+05
1.88e-01	2.15e-15	2.27e-18	9.47e+04
2.50e-01	2.89e-14	5.18e-17	5.58e+04
3.12e-01	3.88e-13	1.18e-15	3.29e+04
3.75e-01	5.22e-12	2.68e-14	1.94e+04
4.38e-01	7.01e-11	6.10e-13	1.14e+04
5.00e-01	9.42e-10	1.39e-11	6.68e+03
5.62e-01	1.27e-08	3.16e-10	3.90e+03
6.25e-01	1.70e-07	7.19e-09	2.26e+03
6.88e-01	2.28e-06	1.64e-07	1.30e+03
7.50e-01	3.07e-05	3.73e-06	7.24e+02
8.12e-01	4.12e-04	8.48e-05	3.86e+02
8.75e-01	5.54e-03	1.93e-03	1.87e+02
9.38e-01	7.44e-02	4.39e-02	6.94e+01
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	2.03e-17	0.00e+00	Inf
3.12e-02	2.03e-17	7.27e-22	2.79e+06
6.25e-02	2.03e-17	4.20e-21	4.83e+05
9.38e-02	2.03e-17	2.07e-20	9.77e+04
1.25e-01	2.04e-17	9.97e-20	2.04e+04
1.56e-01	2.10e-17	4.76e-19	4.30e+03
1.88e-01	2.36e-17	2.27e-18	9.38e+02
2.19e-01	3.60e-17	1.08e-17	2.31e+02
2.50e-01	9.40e-17	5.18e-17	8.17e+01
2.81e-01	3.67e-16	2.47e-16	4.86e+01
3.12e-01	1.65e-15	1.18e-15	4.01e+01
3.44e-01	7.68e-15	5.62e-15	3.67e+01
3.75e-01	3.60e-14	2.68e-14	3.44e+01
4.06e-01	1.69e-13	1.28e-13	3.24e+01
4.38e-01	7.96e-13	6.10e-13	3.04e+01
4.69e-01	3.74e-12	2.91e-12	2.85e+01
5.00e-01	1.76e-11	1.39e-11	2.66e+01
5.31e-01	8.27e-11	6.63e-11	2.48e+01
5.62e-01	3.89e-10	3.16e-10	2.29e+01
5.94e-01	1.83e-09	1.51e-09	2.11e+01
6.25e-01	8.59e-09	7.19e-09	1.94e+01
6.56e-01	4.04e-08	3.43e-08	1.76e+01
6.88e-01	1.90e-07	1.64e-07	1.59e+01
7.19e-01	8.92e-07	7.81e-07	1.42e+01
7.50e-01	4.19e-06	3.73e-06	1.25e+01
7.81e-01	1.97e-05	1.78e-05	1.09e+01
8.12e-01	9.27e-05	8.48e-05	9.26e+00
8.44e-01	4.36e-04	4.05e-04	7.66e+00
8.75e-01	2.05e-03	1.93e-03	6.08e+00
9.06e-01	9.63e-03	9.21e-03	4.53e+00
9.38e-01	4.53e-02	4.39e-02	3.00e+00
9.69e-01	2.13e-01	2.10e-01	1.49e+00
1.00e+00	1.00e+00	1.00e+00	0.00e+00

A.3 1st-Order US FDM

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-2.14e-18	0.00e+00	Inf
5.00e-01	3.70e-02	1.39e-11	2.67e+11
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-5.15e-18	0.00e+00	Inf
1.25e-01	8.19e-07	9.97e-20	8.21e+14
2.50e-01	6.76e-06	5.18e-17	1.31e+13
3.75e-01	4.98e-05	2.68e-14	1.86e+11
5.00e-01	3.62e-04	1.39e-11	2.61e+09
6.25e-01	2.62e-03	7.19e-09	3.65e+07
7.50e-01	1.90e-02	3.73e-06	5.10e+05
8.75e-01	1.38e-01	1.93e-03	7.04e+03
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-6.91e-18	0.00e+00	Inf
2.50e-01	3.76e-04	5.18e-17	7.27e+14
5.00e-01	5.46e-03	1.39e-11	3.93e+10
7.50e-01	7.40e-02	3.73e-06	1.99e+06
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	-9.76e-18	0.00e+00	Inf
6.25e-02	4.45e-10	4.20e-21	1.06e+13
1.25e-01	2.28e-09	9.97e-20	2.29e+12
1.88e-01	9.85e-09	2.27e-18	4.33e+11
2.50e-01	4.11e-08	5.18e-17	7.93e+10
3.12e-01	1.70e-07	1.18e-15	1.44e+10
3.75e-01	7.01e-07	2.68e-14	2.61e+09
4.38e-01	2.89e-06	6.10e-13	4.74e+08
5.00e-01	1.19e-05	1.39e-11	8.59e+07
5.62e-01	4.92e-05	3.16e-10	1.56e+07
6.25e-01	2.03e-04	7.19e-09	2.82e+06
6.88e-01	8.37e-04	1.64e-07	5.11e+05
7.50e-01	3.45e-03	3.73e-06	9.26e+04
8.12e-01	1.42e-02	8.48e-05	1.67e+04
8.75e-01	5.88e-02	1.93e-03	2.94e+03
9.38e-01	2.42e-01	4.39e-02	4.52e+02
1.00e+00	1.00e+00	1.00e+00	0.00e+00

x	$u(x)$	$\bar{u}(x)$	ϵ_{rel}
0.00e+00	5.76e-17	0.00e+00	Inf
3.12e-02	1.31e-13	7.27e-22	1.80e+10
6.25e-02	4.66e-13	4.20e-21	1.11e+10
9.38e-02	1.32e-12	2.07e-20	6.38e+09
1.25e-01	3.53e-12	9.97e-20	3.54e+09
1.56e-01	9.16e-12	4.76e-19	1.92e+09
1.88e-01	2.36e-11	2.27e-18	1.04e+09
2.19e-01	6.06e-11	1.08e-17	5.59e+08
2.50e-01	1.56e-10	5.18e-17	3.01e+08
2.81e-01	3.99e-10	2.47e-16	1.61e+08
3.12e-01	1.02e-09	1.18e-15	8.67e+07
3.44e-01	2.62e-09	5.62e-15	4.66e+07
3.75e-01	6.71e-09	2.68e-14	2.50e+07
4.06e-01	1.72e-08	1.28e-13	1.34e+07
4.38e-01	4.41e-08	6.10e-13	7.22e+06
4.69e-01	1.13e-07	2.91e-12	3.88e+06
5.00e-01	2.89e-07	1.39e-11	2.08e+06
5.31e-01	7.41e-07	6.63e-11	1.12e+06
5.62e-01	1.90e-06	3.16e-10	6.01e+05
5.94e-01	4.87e-06	1.51e-09	3.23e+05
6.25e-01	1.25e-05	7.19e-09	1.73e+05
6.56e-01	3.20e-05	3.43e-08	9.30e+04
6.88e-01	8.19e-05	1.64e-07	4.99e+04
7.19e-01	2.10e-04	7.81e-07	2.68e+04
7.50e-01	5.38e-04	3.73e-06	1.43e+04
7.81e-01	1.38e-03	1.78e-05	7.65e+03
8.12e-01	3.53e-03	8.48e-05	4.06e+03
8.44e-01	9.05e-03	4.05e-04	2.14e+03
8.75e-01	2.32e-02	1.93e-03	1.10e+03
9.06e-01	5.94e-02	9.21e-03	5.45e+02
9.38e-01	1.52e-01	4.39e-02	2.47e+02
9.69e-01	3.90e-01	2.10e-01	8.62e+01
1.00e+00	1.00e+00	1.00e+00	0.00e+00

B MATLAB Code

```
clear all; close all; clc

%% Initial Conditions

plotGen     = false;
plotSave    = false;
tableSave   = false;
tableSave2  = false;
tableSave3  = false;

odeType = '1st Order US';
%odeType = '2nd Order CDS';
%odeType = '4th Order CDS';

odeString = '1st_Order_US';

mesh.order = 1:22;
mesh.dx = 0.5.^mesh.order;

rowID = 0;

%% Boundary Value Problem Solution

for c = [50]

    rowID = rowID + 1;
    colID = 0;

    if plotGen

        fig1 = figure(1);
        xlabel('x');    ylabel('u(x)');
        grid on;        grid minor;
        box on;         hold on;
        set(gcf, 'Position', [1 1 624 550])

        if strcmpi(odeType, '2nd Order CDS')
            titleString = strcat('2nd Order CDS FDM for c=', num2str(c));
        elseif strcmpi(odeType, '4th Order CDS')
            titleString = strcat('4th Order CDS FDM for c=', num2str(c));
        elseif strcmpi(odeType, '1st Order US')
            titleString = strcat('1st Order US FDM for c=', num2str(c));
        end

        title(titleString)

        fig2 = figure(2);
        xlabel('x');    ylabel('\epsilon_{rel}');
        grid on;        grid minor;
        box on;         hold on;
        set(gcf, 'Position', [1 1 624 550])

        title(titleString)

    end

    for dx = mesh.dx
```

```matlab
nx = 1 / dx + 1;
x = linspace(0, 1, nx);
b = zeros(nx, 1);
colID = colID + 1;

if strcmpi(odeType, '2nd Order CDS')
    alpha = -1 - c*dx/2;
    beta = 2;
    gamma = -1 + c*dx/2;
elseif strcmpi(odeType, '4th Order CDS')
    alpha = -1 - c*dx/2 - c^2*dx^2/12;
    beta = 2 + c^2*dx^2/6;
    gamma = -1 + c*dx/2 - c^2*dx^2/12;
elseif strcmpi(odeType, '1st Order US')
    alpha = -1 - c*dx;
    beta = 2 + c*dx;
    gamma = -1;
end

A = gallery('tridiag', nx, alpha, beta, gamma);

A(1, 1) = 1;    A(1, 2) = 0;        b(1) = 0;
A(nx, nx) = 1;  A(nx, nx-1) = 0;    b(nx) = 1;
b(2:nx-1) = 0;

u = A\b;

if plotGen && dx >= mesh.dx(8)

    figure(1)
    plot(x, u, 'linewidth', 1)

end

if strcmpi(odeType, '2nd Order CDS')
    dudx.fdm(rowID, colID) = (1 -c*dx/2)^-1 * (1 - u(end-1)) / dx;
elseif strcmpi(odeType, '4th Order CDS')
    dudx.fdm(rowID, colID) = (1 -c*dx/2 + c^2*dx^2/6 - c^3*dx^3/24)^-1 * ...
     (1 - u(end-1)) / dx;
elseif strcmpi(odeType, '1st Order US')
    dudx.fdm(rowID, colID) = (1 - u(end-1)) / dx;
end

ux.exact = (exp(c*x) - 1) / (exp(c) - 1);
dudx.exact(rowID, colID) = c*exp(c) / (exp(c) - 1);

if plotGen && dx >= mesh.dx(8)

    figure(2)
    plot(x(2:end-1), abs(ux.exact(2:end-1)'-u(2:end-1)) ./ ...
    abs(ux.exact(2:end-1)')*100, 'o-')

end

if tableSave2 && dx >= mesh.dx(5)

    colLabels = {'$x$', '$u(x)$', '$\bar{u}(x)$', '$\epsilon_{rel}$'};

    matrix2latex([x' u ux.exact' (abs(ux.exact'-u)./abs(ux.exact')*100)], ...
    strcat('pointwise_error_table_dx_', num2str(nx), '_', lower(odeString), '.tex'), ...
    'columnLabels', colLabels, 'alignment', 'c', 'format', '%1.2e')
```

```
        end

    end

    if plotGen

        figure(1)

        fplot(@(x) (exp(c*x)-1)/(exp(c) - 1), [0 1], '-k', 'linewidth', 1.5)

        legend('\Deltax = (1/2)^1', '\Deltax = (1/2)^2', '\Deltax = (1/2)^3', ...
        '\Deltax = (1/2)^4', '\Deltax = (1/2)^5', '\Deltax = (1/2)^6', ...
        '\Deltax = (1/2)^7', '\Deltax = (1/2)^8', 'Analytical Solution', ...
        'location', 'eastoutside')

        drawnow

        figure(2)

        legend('\Deltax = (1/2)^1', '\Deltax = (1/2)^2', '\Deltax = (1/2)^3', ...
        '\Deltax = (1/2)^4', '\Deltax = (1/2)^5', '\Deltax = (1/2)^6', ...
        '\Deltax = (1/2)^7', '\Deltax = (1/2)^8', 'location', 'eastoutside')
        set(gca, 'YScale', 'log')

        drawnow

        if plotSave

            figure(1)
            figureString = strcat('solution_', lower(odeString), '_c_', num2str(c));
            saveas(gcf, figureString, 'epsc')

            figure(2)
            figureString = strcat('pointwise_error_', lower(odeString), '_c_', num2str(c));
            saveas(gcf, figureString, 'epsc')

            close gcf;  close gcf

        end

    end

end


%% Convergence Analysis

relError = abs(dudx.exact-dudx.fdm) ./ abs(dudx.exact) * 100;

if plotGen

    figure
    xlabel('-log_{10}(\Deltax)');   ylabel('log_{10}(\epsilon_{rel})');
    grid on;                        grid minor;
    box on;                         hold on;

    for kID = 1:6
        loglog(-mesh.dx, relError(kID, :), '-o', 'linewidth', 1.25);
    end

    title(strcat('Convergence Plot for the ', odeType))
```

```
    legend('c=1', 'c=2', 'c=5', 'c=10', 'c=20', 'c=50')
    set(gca, 'XScale', 'log');  set(gca, 'YScale', 'log');
    drawnow

    if plotSave

        figureString = strcat('convergence_', lower(odeString));
        saveas(gcf, figureString, 'epsc')
        close gcf

    end

end

%% Rate of Convergence Analysis

logRelError = log10(relError);

for kID = 1:6

    for rocID = 1:length(logRelError) - 1
        roc(kID, rocID) = (logRelError(kID, rocID+1) - logRelError(kID, rocID)) / -log10(2);
    end

end

colLabels = {'$\Delta x$', '$\beta(c=1)$', '$\beta(c=2)$', '$\beta(c=5)$', ...
'$\beta(c=10)$', '$\beta(c=20)$', '$\beta(c=50)$'};

if tableSave

    matrix2latex([mesh.dx(1:end-1)' roc'], strcat('roc_table_', lower(odeString), '.tex'), ...
    'columnLabels', colLabels, 'alignment', 'c', 'format', '%5.4f')

end

colLabels1 = {'$\Delta x$', '$u''_{c=1}(1)$', '$u''_{c=2}(1)$', '$u''_{c=5}(1)$', ...
'$\bar{u}''_{c=1}(1)$', '$\bar{u}''_{c=2}(1)$', '$\bar{u}''_{c=5}(1)$', ...
'$\epsilon''_{rel,c=1}$', '$\epsilon''_{rel,c=2}$', '$\epsilon''_{rel,c=5}$'};
colLabels2 = {'$\Delta x$', '$u''_{c=10}(1)$', '$u''_{c=20}(1)$', '$u''_{c=50}(1)$', ...
'$\bar{u}''_{c=10}(1)$', '$\bar{u}''_{c=20}(1)$', '$\bar{u}''_{c=50}(1)$', ...
'$\epsilon''_{rel,c=10}$', '$\epsilon''_{rel,c=20}$', '$\epsilon''_{rel,c=50}$'};

if tableSave3

    matrix2latex([mesh.dx(1:end)' dudx.fdm(1:3,:)', dudx.exact(1:3, :)', relError(1:3, :)'], ...
    strcat('qoi_1_table_', lower(odeString), '.tex'), 'columnLabels', colLabels1, ...
    'alignment', 'c', 'format', '%5.4f')
    matrix2latex([mesh.dx(1:end)' dudx.fdm(4:6,:)', dudx.exact(4:6, :)', relError(4:6, :)'], ...
    strcat('qoi_2_table_', lower(odeString), '.tex'), 'columnLabels', colLabels2, ...
    'alignment', 'c', 'format', '%5.4f')

end
```