frequently. Numerical procedures for solving initial-value ODEs are presented in Chapter 7.

II.6 BOUNDARY-VALUE ORDINARY DIFFERENTIAL EQUATIONS

A classical example of a boundary-value ODE is the general second-order ODE:

$$\boxed{y'' + P(x, y)y' + Q(x, y)y = F(x) \qquad y(x_1) = y_1 \text{ and } y(x_2) = y_2} \tag{II.27}$$

Equation (II.27) applies to many problems in engineering and science. In the following discussion, the general features of Eq. (II.27) are illustrated for the problem of steady one-dimensional heat diffusion (i.e., conduction) in a rod.

Consider the constant cross-sectional-area rod illustrated in Figure II.7. Heat diffusion transfers energy along the rod and energy is transferred from the rod to the surroundings by convection. An energy balance on the differential control volume yields

$$\dot{q}(x) = \dot{q}(x + dx) + \dot{q}_c(x) \tag{II.28}$$

which can be written as

$$\dot{q}(x) = \dot{q}(x) + \frac{d}{dx}[\dot{q}(x)]\, dx + \dot{q}_c(x) \tag{II.29}$$

which yields

$$\frac{d}{dx}[\dot{q}(x)]\, dx + \dot{q}_c(x) = 0 \tag{II.30}$$

Heat diffusion is governed by *Fourier's law of conduction*, which states that

$$\dot{q}(x) = -kA\frac{dT}{dx} \tag{II.31}$$

where $\dot{q}(x)$ is the energy transfer rate (J/s), k is the thermal conductivity of the solid (J/s-m-K), A is the cross-sectional area of the rod (m^2), and dT/dx is the temperature gradient (K/m). Heat transfer by convection is governed by *Newton's law of cooling*:

$$\dot{q}_c(x) = hA(T - T_a) \tag{II.32}$$

where h is an empirical heat transfer coefficient (J/s-m^2-K), A is the surface area of the rod ($A = P\,dx$, m^2), P is the perimeter of the rod (m), and T_a is the ambient temperature (K) (i.e., the temperature of the surroundings). Substituting Eqs. (II.31) and (II.32) into Eq. (II.30) gives

$$\frac{d}{dx}\left(-kA\frac{dT}{dx}\right) dx + h(P\,dx)(T - T_a) = 0 \tag{II.33}$$

For constant k, A, and P, Eq. (II.33) yields

$$\frac{d^2T}{dx^2} = \frac{hP}{kA}(T - T_a) = 0 \tag{II.34}$$

which can be written as

$$\boxed{T'' - \alpha^2 T = -\alpha^2 T_a} \tag{II.35}$$

where $\alpha^2 = hP/kA$.

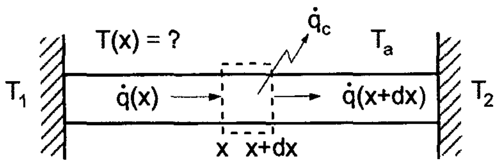


Figure II.7 Steady heat conduction in a rod.

Equation (II.35) is in the general form of Eq. (II.27). Equation (II.35) is a linear second-order boundary-value ODE. The solution of Eq. (II.35) is the function $T(x)$, which describes the temperature distribution in the rod corresponding to the boundary conditions

$$T(x_1) = T_1 \qquad \text{and} \qquad T(x_2) = T_2 \tag{II.36}$$

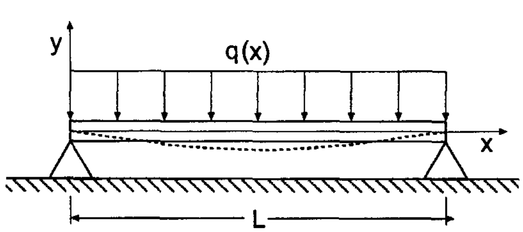
Equation (II.35) is an example of a second-order linear boundary-value problem.

An example of a higher-order boundary-value ODE is given by the fourth-order ODE governing the deflection of a laterally loaded symmetrical beam. The physical system is illustrated in Figure II.8. Bending takes place in the plane of symmetry, which causes deflections in the beam. The neutral axis of the beam is the axis along which the fibers do not undergo strain during bending. When no load is applied (i.e., neglecting the weight of the beam itself), the neutral axis is coincident with the x axis. When a distributed load $q(x)$ is applied, the beam deflects, and the neutral axis is displaced, as illustrated by the dashed line in Figure II.8. The shape of the neutral axis is called the deflection curve.

As shown in many strength of materials books (e.g., Timoshenko, 1955), the differential equation of the deflection curve is

$$EI(x)\frac{d^2y}{dx^2} = -M(x) \tag{II.37}$$

where E is the modulus of elasticity of the beam material, $I(x)$ is the moment of inertia of the beam cross section, which can vary along the length of the beam, and $M(x)$ is the bending moment due to transverse forces acting on the beam, which can vary along the



$$EI(x)\frac{d^4y}{dx^4} = q(x)$$

$y(0) = 0,\ y''(0) = 0;\ y(L) = 0,\ \text{and}\ y''(L) = 0$

Figure II.8 Deflection of a beam.