AERO 430 – Numerical Simulation

Comparison of Finite Element Methods and Finite Difference Methods for Second-Order Linear Ordinary Differential Equation Boundary-Value Problems

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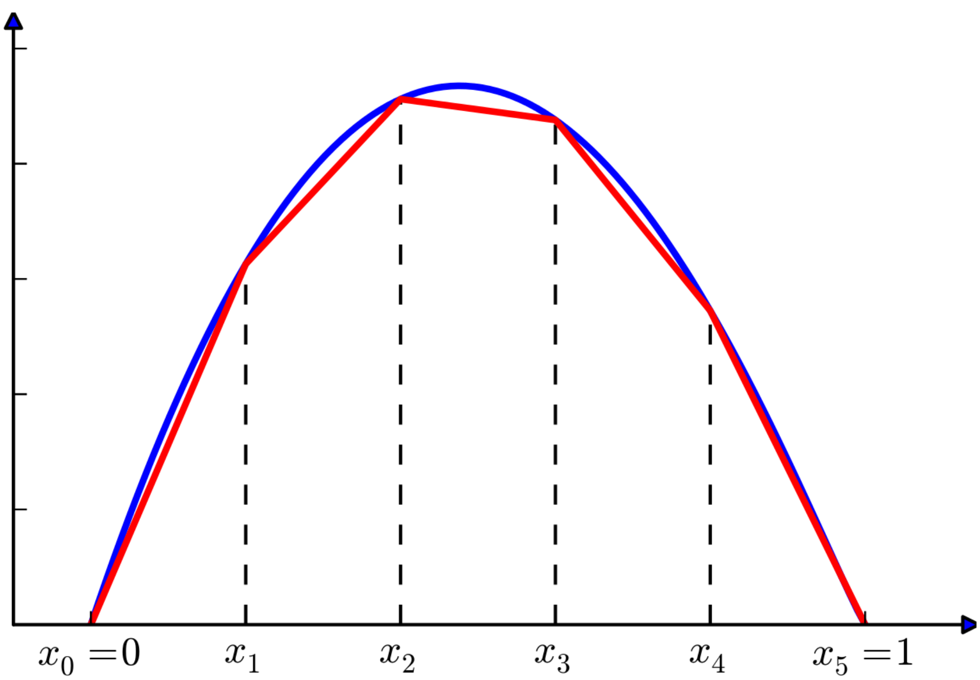


Contents

1 Model Problems

1.1 Diffusion Equation

The *first* model second-order linear ordinary differential equation boundary-value problem consists of:

- the second-order linear ordinary differential equation:

$$-u''(x) + k^2u(x) = k^2x \qquad x \in (0,1) \tag{1.1}$$

- the boundary conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \tag{1.2}$$

The physical model of the second-order linear ordinary differential equation boundary-value problem is that of (1) the temperature of a bar for uniaxial heat conduction, and (2) the deflection of a beam for uniaxial deformation with distributed elastic restraint.

1.2 Harmonic Wave Equation

The *second* model second-order linear ordinary differential equation boundary-value problem consists of:

- the second-order linear ordinary differential equation:

$$u''(x) + k^2u(x) = k^2x \qquad x \in (0,1) \tag{1.3}$$

- the boundary conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \tag{1.4}$$

The physical model of the second-order linear ordinary differential equation boundary-value problem is that of the amplitude of standing waves for uniaxial forced vibration of a bar.

1.3 Convection-Diffusion Equation

The *third* model second-order linear ordinary differential equation boundary-value problem consists of:

- the second-order linear ordinary differential equation:

$$-u''(x) + cu'(x) = 0 \qquad x \in (0,1) \tag{1.5}$$

- the boundary conditions:

$$u(0) = 0 \quad \text{and} \quad u(1) = 1 \tag{1.6}$$

The physical model of the second-order linear ordinary differential equation boundary-value problem is that of the concentration of a flow property that convects and diffuses proportional to the constant c. For example, the convection-diffusion equation could represent the concentration of ink as a function of distance in a quasi-one-dimensional flow.

2 Analytical Solutions

2.1 Analytical Solution of the Diffusion Equation

The following equation is the diffusion equation.

$$-u''(x) + k^2 u(x) = k^2 x \tag{2.1}$$

2.1.1 Homogeneous Solution

Let the homogeneous solution to the diffusion equation be defined as $u_h(x)$. Then, $u_h(x)$ must satisfy the following homogeneous ODE.

$$-u_h''(x) + k^2 u_h(x) = 0 \tag{2.2}$$

The solution of the homogeneous ODE is assumed to be of the form:

$$u_h(x) = e^{\lambda x} \tag{2.3}$$

Taking the second-derivative of $u_h(x)$, substituting the second-derivative into the homogeneous ODE, and reducing the equation yields the **characteristic equation**.

$$u_h'' = \lambda^2 e^{\lambda x} \tag{2.4}$$

$$-\lambda^2 e^{\lambda x} + k^2 e^{\lambda x} = 0 \tag{2.5}$$

$$\boldsymbol{-\lambda^2 + k^2 = 0} \tag{2.6}$$

Solving for λ yields:

$$\lambda = \pm k \tag{2.7}$$

The homogenous solution $u_h(x)$ is then:

$$u_h(x) = \alpha e^{kx} + \beta e^{-kx} \tag{2.8}$$

By making a transformation with the following relations, a more sophistocated solution can be developed:

$$\gamma = \frac{\alpha + \beta}{2} \quad \text{and} \quad \delta = \frac{\alpha - \beta}{2} \tag{2.9}$$

$$u_h(x) = \gamma \frac{e^{kx} + e^{-kx}}{2} + \delta \frac{e^{kx} - e^{-kx}}{2} \tag{2.10}$$

$$\mathbf{u_h(x) = \gamma \cosh(kx) + \delta \sinh(kx)} \tag{2.11}$$

2.1.2 Particular Solution

Let the particular solution to the diffusion equation be defined as $u_p(x)$. Then, $u_p(x)$ must satisfy the ODE:

$$-u_p''(x) + k^2 u_p(x) = k^2 x \tag{2.12}$$

The second-derivative of $u_p(x)$, $u_p''(x)$, is assumed to be zero, and thus yields the particular solution $u_p(x)$:

$$k^2 u_p(x) = k^2 x \tag{2.13}$$

$$\mathbf{u_p(x) = x} \tag{2.14}$$

2.1.3 Application of Boundary Conditions

Given that $u_h(x)$ is a solution to the homogeneous ODE and $u_p(x)$ is a solution to the ODE, then the combination of $u_h(x)$ and $u_p(x)$ is also a solution to the ODE.

$$u(x) = u_h(x) + u_p(x) \tag{2.15}$$

$$u(x) = \gamma \cosh(kx) + \delta \sinh(kx) + x \tag{2.16}$$

The boundary conditions for the model problem are:

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \tag{2.17}$$

Applying the first boundary condition, $u(0) = 0$, we get that $\gamma = 0$:

$$u(0) = 0 = \gamma \cosh(0) + \delta \sinh(0) + 0 \tag{2.18}$$

$$\gamma = 0 \tag{2.19}$$

Applying the second boundary condition, $u(1) = 0$, we get that $\delta = \frac{-1}{\sinh(k)}$:

$$u(1) = 0 = \delta \sinh(k) + 1 \tag{2.20}$$

$$\delta = \frac{-1}{\sinh(k)} \tag{2.21}$$

2.1.4 Analytical Solution

Thus, it is shown that for the negative second-order linear ordinary differential equation with specified boundary conditions (reproduced below) that $u(x)$ is a solution to the differential equation on $x \in (0, 1)$.

$$-u''(x) + k^2 u(x) = k^2 x \qquad x \in (0,1) \tag{2.22}$$

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \tag{2.23}$$

$$\mathbf{u(x) = x - \frac{\sinh(kx)}{\sinh(k)}} \tag{2.24}$$

The values of k tested in this study were $k \in 1, 2, 5, 10, 20, 50$. A plot of the analytical solution for values of k is depicted in Figure 2.1.1:

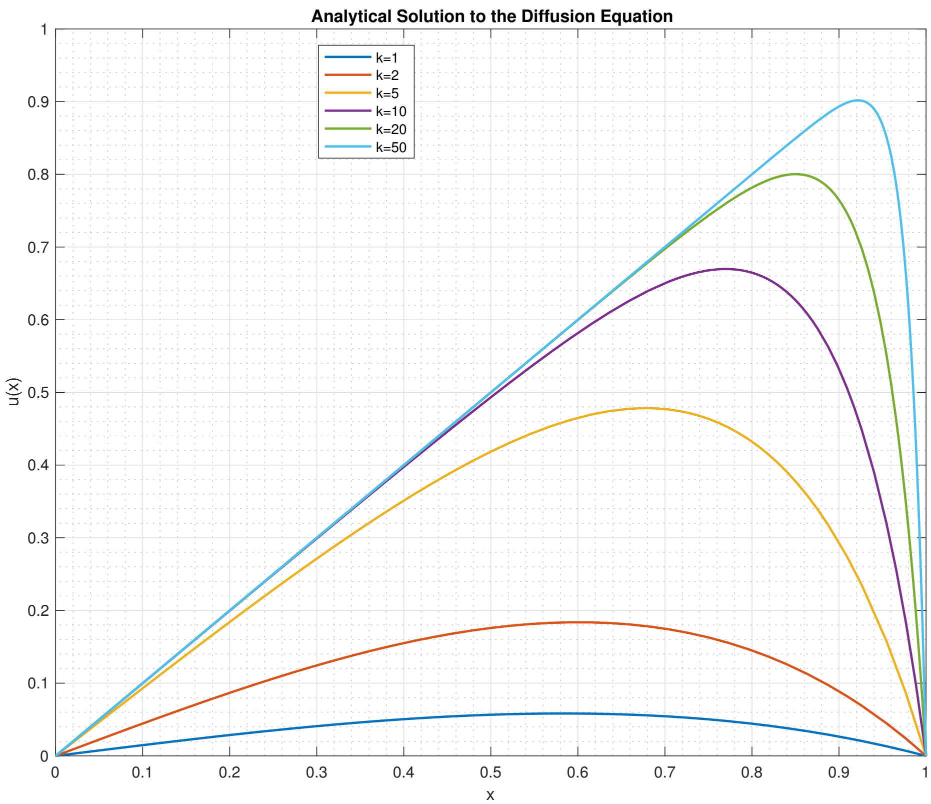


Figure 2.1.1 – Analytical Solution to the Diffusion Equation for Values of k

2.2 Analytical Solution of the Harmonic Wave Equation

The following equation is the harmonic wave equation.

$$u''(x) + k^2 u(x) = k^2 x \tag{2.25}$$

2.2.1 Homogeneous Solution

Let the homogeneous solution to the harmonic wave equation be defined as $u_h(x)$. Then, $u_h(x)$ must satisfy the following homogeneous ODE.

$$u_h''(x) + k^2 u_h(x) = 0 \tag{2.26}$$

The solution of the homogeneous ODE is assumed to be of the form:

$$u_h(x) = e^{\lambda x} \tag{2.27}$$

Taking the second-derivative of $u_h(x)$, substituting the second-derivative into the homogeneous ODE, and reducing the equation yields the **characteristic equation**.

$$u_h''(x) = \lambda^2 e^{\lambda x} \tag{2.28}$$

$$\lambda^2 e^{\lambda x} + k^2 e^{\lambda x} = 0 \tag{2.29}$$

$$\boldsymbol{\lambda^2 + k^2 = 0} \tag{2.30}$$

Solving for λ yields:

$$\lambda = \pm ik \tag{2.31}$$

The homogenous solution $u_h(x)$ is then:

$$u_h(x) = \alpha e^{ikx} + \beta e^{-ikx} \tag{2.32}$$

Making a transformation with the following relations, a more sophistocated solution can be developed:

$$\gamma = \frac{\alpha + \beta}{2} \quad \text{and} \quad \delta = i\frac{\alpha - \beta}{2} \tag{2.33}$$

$$u_h(x) = \gamma \frac{e^{ikx} + e^{-ikx}}{2} + \delta \frac{e^{ikx} - e^{-ikx}}{2i} \tag{2.34}$$

$$\mathbf{u_h(x)} = \gamma \mathbf{cos(kx)} + \delta \mathbf{sin(kx)} \tag{2.35}$$

2.2.2 Particular Solution

Let the particular solution to the harmonic wave equation be defined as $u_p(x)$. Then, $u_p(x)$ must satisfy the ODE:

$$u_p''(x) + k^2 u_p(x) = k^2 x \tag{2.36}$$

The second-derivative of $u_p(x)$, $u_p''(x)$, is assumed to be zero, and thus yields the particular solution $u_p(x)$:

$$k^2 u_p(x) = k^2 x \tag{2.37}$$

$$\mathbf{u_p(x) = x} \tag{2.38}$$

2.2.3 Application of Boundary Conditions

Given that $u_h(x)$ is a solution to the homogeneous ODE and $u_p(x)$ is a solution to the ODE, then the combination of $u_h(x)$ and $u_p(x)$ is also a solution to the ODE.

$$u(x) = u_h(x) + u_p(x) \tag{2.39}$$

$$u(x) = \gamma \cos(kx) + \delta \sin(kx) + x \tag{2.40}$$

The boundary conditions for the model problem are:

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \tag{2.41}$$

Applying the first boundary condition, $u(0) = 0$, we get that $\gamma = 0$:

$$u(0) = 0 = \gamma \cos(0) + \delta \sin(0) + 0 \tag{2.42}$$

$$\gamma = 0 \tag{2.43}$$

Applying the second boundary condition, $u(1) = 0$, we get that $\delta = \frac{-1}{\sin(k)}$:

$$u(1) = 0 = \delta \sin(k) + 1 \tag{2.44}$$

$$\delta = \frac{-1}{\sin(k)} \tag{2.45}$$

2.2.4 Analytical Solution

Thus, it is shown that for the positive second-order linear ordinary differential equation with specified boundary conditions (reproduced below) that $u(x)$ is a solution to the differential equation on $x \in (0, 1)$.

$$u''(x) + k^2 u(x) = k^2 x \qquad x \in (0, 1) \tag{2.46}$$

$$u(0) = 0 \quad \text{and} \quad u(1) = 0 \tag{2.47}$$

$$\mathbf{u(x) = x - \frac{\sin(kx)}{\sin(k)}} \tag{2.48}$$

The values of k tested in this study were $k \in 1, 2, 5, 10, 20, 50$. A plot of the analytical solution for values of k is depicted in Figure 2.2.1:

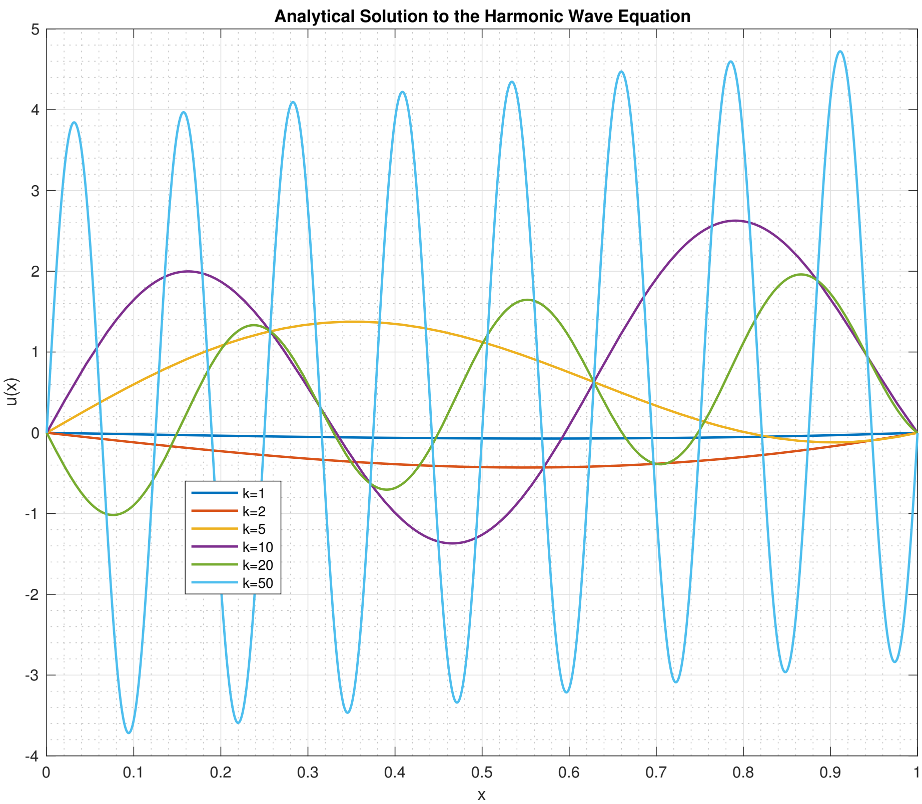


Figure 2.2.1 – Analytical Solution to the Harmonic Wave Equation for Values of k

2.3 Analytical Solution of the Convection-Diffusion Equation

The following equation is the homogeneous second-order linear ordinary differential equation (ODE).

$$-u''(x) + cu'(x) = 0 \tag{2.49}$$

2.3.1 Homogeneous Solution

The solution of the homogeneous ODE, $u_h(x)$ is assumed to be of the form:

$$u_h(x) = e^{\lambda x} \tag{2.50}$$

Taking the derivatives of $u_h(x)$, substituting them into the homogeneous ODE, and reducing the equation yields the **characteristic equation**.

$$u_h'(x) = \lambda e^{\lambda x} \tag{2.51}$$

$$u_h''(x) = \lambda^2 e^{\lambda x} \tag{2.52}$$

$$-\lambda^2 e^{\lambda x} + c\lambda e^{\lambda x} = 0 \tag{2.53}$$

$$\boldsymbol{-\lambda^2 + c\lambda = \lambda(c - \lambda) = 0} \tag{2.54}$$

Solving for λ yields:

$$\lambda = \{0, c\} \tag{2.55}$$

The homogenous solution $u_h(x)$ is then:

$$u_h(x) = ae^{0x} + be^{cx} \tag{2.56}$$

$$\mathbf{u_h(x) = a + be^{cx}} \tag{2.57}$$

2.3.2 Application of Boundary Conditions

Applying the both boundary conditions, $u(0) = 0$ and $u(1) = 1$, we get the system of algebraic equations, which yields a and b:

$$u_h(0) = 0 = a + b \tag{2.58}$$

$$u_h(1) = 1 = a + be^c \tag{2.59}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & e^c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.60}$$

$$a = \frac{-1}{e^c - 1} \quad \text{and} \quad b = \frac{1}{e^c - 1} \tag{2.61}$$

2.3.3 Analytical Solution

Thus, it is shown that for the second-order linear ordinary differential equation with specified boundary conditions, that $u(x)$ is a solution to the differential equation on $x \in (0, 1)$.

$$-u''(x) + cu'(x) = 0 \qquad x \in (0, 1) \tag{2.62}$$

$$u(0) = 0 \quad \text{and} \quad u(1) = 1 \tag{2.63}$$

$$\mathbf{u(x) = \frac{e^{cx} - 1}{e^{c} - 1}} \tag{2.64}$$

The values of c tested in this study were $c \in 1, 2, 5, 10, 20, 50$. A plot of the analytical solution for values of c is depicted in Figure 2.3.1:

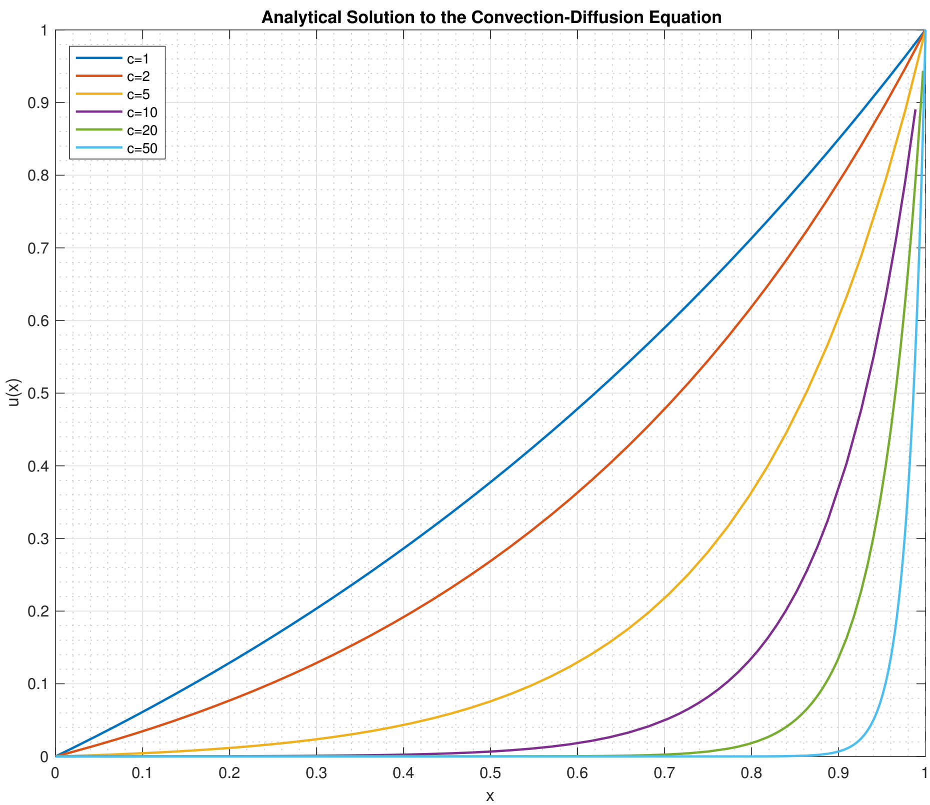


Figure 2.3.1 – Analytical Solution to the Convection-Diffusion Equation for Values of c

2.4 Analytical Solution of the Quantity of Interest for the Diffusion Equation

The quantity of interest for the diffusion equation is the first derivative at the right boundary, or $u'(1)$. The differential equation and the analytical solution to the differential equation, respectively, are reproduced below.

$$-u''(x) + k^2 u(x) = k^2 x \tag{2.65}$$

$$u(x) = x - \frac{\sinh(kx)}{\sinh(k)} \tag{2.66}$$

Thus, taking the first derivative, and solving at $x = 1$, we obtain the exact quantity of interest:

$$u'(x) = 1 - \frac{k\cosh(kx)}{\sinh(k)} \tag{2.67}$$

$$u'(1) = 1 - \frac{k\cosh(k)}{\sinh(k)} \tag{2.68}$$

$$\mathbf{u'(1) = 1 - kcoth(k)} \tag{2.69}$$

As k approaches 0, the quantity of interest approaches 0. As k approaches ∞, $\coth(x)$ approaches 1, and thus for large k, the quantity of interest behaves as $1 - k$. Additionally, since $\coth(x)$ approaches 1 as x approaches ∞; as k approaches ∞, the quantity of interest approaches ∞:

$$\lim_{k\to 0}[1 - k\coth(k)] = 0 \tag{2.70}$$

$$\lim_{k\to\infty}[1 - k\coth(k)] \approx \lim_{k\to\infty}(1-k) = -\infty \tag{2.71}$$

A table of the exact quantity of interest for the tested values of k is included below (for $k = (20, 50)$, error is less than machine epsilon, ϵ):

Table 2.4.1 – Analytical Solution to the Quantity of Interest for the Diffusion Equation for Values of k

k	$\mathbf{u'(1)}$
1	-0.313035
2	-1.074629
5	-4.000454
10	-9.0000000412
20	-19.000000000000000
50	-49.000000000000000

2.5 Analytical Solution of the Quantity of Interest for the Harmonic Wave Equation

The quantity of interest for the harmonic wave equation is the first derivative at the right boundary, or $u'(1)$. The differential equation and the analytical solution to the differential equation, respectively, are reproduced below.

$$u''(x) + k^2 u(x) = k^2 x \tag{2.72}$$

$$u(x) = x - \frac{\sin(kx)}{\sin(k)} \tag{2.73}$$

Thus, taking the first derivative, and solving at $x = 1$, we obtain the exact quantity of interest:

$$u'(x) = 1 - \frac{k\cos(kx)}{\sin(k)} \tag{2.74}$$

$$u'(x) = 1 - \frac{k\cos(k)}{\sin(k)} \tag{2.75}$$

$$\mathbf{u'(1) = 1 - kcot(k)} \tag{2.76}$$

As k approaches 0, the quantity of interest approaches 0. But, since $\cot(x)$ is oscillatory, the quantity of interest does not converge as k approaches ∞:

$$\lim_{k\to 0}[1 - k\cot(k)] = 0 \tag{2.77}$$

$$\lim_{k\to\infty}[1 - k\cot(k)] = -\infty \ \ to \ \ \infty \tag{2.78}$$

A table of the exact quantity of interest for the tested values of k is included below:

Table 2.5.1 – Analytical Solution to the Quantity of Interest for the Harmonic Wave Equation for Values of k

k	$\mathbf{u'(1)}$
1	0.3579
2	1.9153
5	2.4790
10	-14.4235
20	-7.9399
50	184.8907

2.6 Analytical Solution of the Quantity of Interest for the Convection-Diffusion Equation

The quantity of interest for the convection-diffusion equation is the first derivative at the right boundary, or $u'(1)$. The differential equation and the analytical solution to the differential equation, respectively, are reproduced below.

$$-u''(x) + cu'(x) = 0 \tag{2.79}$$

$$u(x) = \frac{e^{cx} - 1}{e^c - 1} \tag{2.80}$$

Thus, taking the first derivative, and solving at $x = 1$, we obtain the exact quantity of interest:

$$u'(x) = \frac{ce^{cx}}{e^c - 1} \tag{2.81}$$

$$\mathbf{u'(1)} = \frac{\mathbf{ce^c}}{\mathbf{e^c - 1}} \tag{2.82}$$

It can be seen that if e^c, or rather c, is sufficiently large ($e^c \gg 1$), then the quantity of interest tends to c given that the ratio of e^c and $(e^c - 1)$ tends to unity:

$$\lim_{c \to \infty} \frac{ce^c}{e^c - 1} = c \tag{2.83}$$

A table of the exact quantity of interest for the tested values of c is included below (for $c = 50$, error is less than machine epsilon, ϵ):

Table 2.6.1 – Analytical Solution to the Quantity of Interest for the Convection-Diffusion Equation for Values of c

c	$\mathbf{u'(1)}$
1	1.58198
2	2.31304
5	5.03392
10	10.00045
20	20.00000004
50	50.00000000

3 Numerical Methods

3.1 Derivations for the Diffusion Equation

3.1.1 2nd-Order Central Difference Scheme Finite Difference Method

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.1}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.2}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.3}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{3.4}$$

From this specific second-derivative formulation using the finite difference method, the approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + k^2 u(x) = k^2 x \tag{3.5}$$

Discretizing using the above formulas and simplifying yields:

$$(-1)\, u_{i-1} + \left(2 + k^2 \Delta x^2\right) u_i + (-1)\, u_{i+1} = \left(k^2 \Delta x^2\right) x_i \tag{3.6}$$

3.1.2 4th-Order Central Difference Scheme Finite Difference Method

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.7}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.8}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \frac{\Delta x^4}{12} u_i^{(4)} + \mathcal{O}(\Delta x^6) \tag{3.9}$$

Rearranging for u_i'', we get:

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12} u_i^{(4)} + \mathcal{O}(\Delta x^4) \tag{3.10}$$

Returning to the differential equation and then taking two additional derivatives, we can arrive at an expression for $u_i^{(4)}$:

$$-u''(x) + k^2 u(x) = k^2 x \qquad x \in (0,1) \tag{3.11}$$

$$-u^{(4)}(x) + k^2 u''(x) = 0 \tag{3.12}$$

$$u^{(4)}(x) = k^2 u''(x) \tag{3.13}$$

Substituting in the fourth-derivative expression:

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12} k^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.14}$$

Now, exchanging the u_i'' term with the earlier derivation and simplifying:

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12} k^2 \left[\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \right] + \mathcal{O}(\Delta x^4) \tag{3.15}$$

$$u_i'' = \frac{1}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}) - \frac{k^2}{12}(u_{i+1} - 2u_i + u_{i-1}) + \mathcal{O}(\Delta x^4) \tag{3.16}$$

$$u_i'' = \left(\frac{1}{\Delta x^2} - \frac{k^2}{12} \right) (u_{i+1} - 2u_i + u_{i-1}) + \mathcal{O}(\Delta x^4) \tag{3.17}$$

From this specific second-derivative formulation using the finite difference method, the approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + k^2 u(x) = k^2 x \tag{3.18}$$

Discretizing using the above formulas and simplifying yields:

$$\left(-1 + \frac{k^2 \Delta x^2}{12} \right) u_{i-1} + \left(2 + \frac{10 k^2 \Delta x^2}{12} \right) u_i + \left(-1 + \frac{k^2 \Delta x^2}{12} \right) u_{i+1} = \left(k^2 \Delta x^2 \right) x_i \tag{3.19}$$

3.1.3 1st-Order (p=1) Galerkin Method Finite Element Method

3.1.4 2nd-Order (p=2) Galerkin Method Finite Element Method

3.2 Derivations for the Harmonic Wave Equation

3.2.1 2nd-Order Central Difference Scheme Finite Difference Method

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.20}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.21}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.22}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{3.23}$$

From this specific second-derivative formulation using the finite difference method, the approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Discretized Differential Equation The differential equation is:

$$u''(x) + k^2 u(x) = k^2 x \tag{3.24}$$

Discretizing using the above formulas and simplifying yields:

$$(1)\, u_{i-1} + \left(-2 + k^2 \Delta x^2\right) u_i + (1)\, u_{i+1} = \left(k^2 \Delta x^2\right) x_i \tag{3.25}$$

3.2.2 4th-Order Central Difference Scheme Finite Difference Method

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.26}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.27}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \frac{\Delta x^4}{12} u_i^{(4)} + \mathcal{O}(\Delta x^6) \tag{3.28}$$

Rearranging for u_i'', we get:

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12} u_i^{(4)} + \mathcal{O}(\Delta x^4) \tag{3.29}$$

Returning to the differential equation and then taking two additional derivatives, we can arrive at an expression for $u_i^{(4)}$:

$$u''(x) + k^2 u(x) = k^2 x \qquad x \in (0, 1) \tag{3.30}$$

$$u^{(4)}(x) + k^2 u''(x) = 0 \tag{3.31}$$

$$u^{(4)}(x) = -k^2 u''(x) \tag{3.32}$$

Substituting in the fourth-derivative expression:

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\Delta x^2}{12} k^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.33}$$

Now, exchanging the u_i'' term with the earlier derivation and simplifying:

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{\Delta x^2}{12} k^2 \left[\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)\right] + \mathcal{O}(\Delta x^4) \tag{3.34}$$

$$u_i'' = \frac{1}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{k^2}{12}(u_{i+1} - 2u_i + u_{i-1}) + \mathcal{O}(\Delta x^4) \tag{3.35}$$

$$u_i'' = \left(\frac{1}{\Delta x^2} + \frac{k^2}{12}\right)(u_{i+1} - 2u_i + u_{i-1}) + \mathcal{O}(\Delta x^4) \tag{3.36}$$

From this specific second-derivative formulation using the finite difference method, the approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

Discretized Differential Equation The differential equation is:

$$u''(x) + k^2 u(x) = k^2 x \tag{3.37}$$

Discretizing using the above formulas and simplifying yields:

$$\left(1 + \frac{k^2 \Delta x^2}{12}\right) u_{i-1} + \left(-2 + \frac{10k^2 \Delta x^2}{12}\right) u_i + \left(1 + \frac{k^2 \Delta x^2}{12}\right) u_{i+1} = \left(k^2 \Delta x^2\right) x_i \tag{3.38}$$

3.2.3 1st-Order (p=1) Galerkin Method Finite Element Method

3.2.4 2nd-Order (p=2) Galerkin Method Finite Element Method

3.3 Derivations for the Convection-Diffusion Equation

3.3.1 2nd-Order Central Difference Scheme Finite Difference Method

First Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.39}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.40}$$

Subtracting the Taylor series for u_{i-1} from u_{i+1} and canceling terms:

$$u_{i+1} - u_{i-1} = 2\Delta x u_i' + \frac{\Delta x^3}{3} u_i^{(3)} + \mathcal{O}(\Delta x^5) \tag{3.41}$$

Solving for u_i':

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2) \tag{3.42}$$

From this specific first-derivative formulation using the finite difference method, the first-derivative approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x = i$:

$$u_{i-1} = u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.43}$$

$$u_{i+1} = u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i^{(3)} + \frac{\Delta x^4}{24} u_i^{(4)} + \mathcal{O}(\Delta x^5) \tag{3.44}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \mathcal{O}(\Delta x^4) \tag{3.45}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{3.46}$$

From this specific second-derivative formulation using the finite difference method, the second-derivative approximation can be observed to be second-order ($\mathcal{O}(\Delta x^2)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + cu'(x) = 0 \tag{3.47}$$

Discretizing using the above formulas and simplifying yields:

$$-\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right) + c\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) = 0 \tag{3.48}$$

$$-\left(u_{i+1} - 2u_i + u_{i-1}\right) + c\Delta x\left(\frac{u_{i+1} - u_{i-1}}{2}\right) = 0 \tag{3.49}$$

$$\left(-1-\frac{c\Delta x}{2}\right)u_{i-1}+(2)\,u_i+\left(-1+\frac{c\Delta x}{2}\right)u_{i+1}=0 \tag{3.50}$$

3.3.2 4th-Order Central Difference Scheme Finite Difference Method

First Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x=i$:

$$u_{i-1}=u_i-\Delta x u_i'+\frac{\Delta x^2}{2}u_i''-\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.51}$$

$$u_{i+1}=u_i+\Delta x u_i'+\frac{\Delta x^2}{2}u_i''+\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.52}$$

Subtracting the Taylor series for u_{i-1} from u_{i+1} and canceling terms:

$$u_{i+1}-u_{i-1}=2\Delta x u_i'+\frac{\Delta x^3}{3}u_i^{(3)}+\mathcal{O}(\Delta x^5) \tag{3.53}$$

Returning to the differential equation and taking one additional derivative:

$$u''(x)=cu'(x) \tag{3.54}$$

$$u^{(3)}(x)=cu''(x) \tag{3.55}$$

Replacing the second-derivative term with the original differential equation:

$$u^{(3)}(x)=c^2u'(x) \tag{3.56}$$

Now replacing the third-derivative term in the Taylor series subtraction and then dividing by $2\Delta x$:

$$u_{i+1}-u_{i-1}=2\Delta x u_i'+\frac{c^2\Delta x^3}{3}u_i'+\mathcal{O}(\Delta x^5) \tag{3.57}$$

$$\frac{u_{i+1}-u_{i-1}}{2\Delta x}=\left(1+\frac{c^2\Delta x^2}{6}\right)u_i'+\mathcal{O}(\Delta x^4) \tag{3.58}$$

Solving for u_i':

$$u_i'=\left(1+\frac{c^2\Delta x^2}{6}\right)^{-1}\frac{u_{i+1}-u_{i-1}}{2\Delta x}+\mathcal{O}(\Delta x^4) \tag{3.59}$$

From this specific first-derivative formulation using the finite difference method, the first-derivative approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

Second Derivative Developing the Taylor series for $u(x)$ in the vicinity of $x=i$:

$$u_{i-1}=u_i-\Delta x u_i'+\frac{\Delta x^2}{2}u_i''-\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.60}$$

$$u_{i+1}=u_i+\Delta x u_i'+\frac{\Delta x^2}{2}u_i''+\frac{\Delta x^3}{6}u_i^{(3)}+\frac{\Delta x^4}{24}u_i^{(4)}+\mathcal{O}(\Delta x^5) \tag{3.61}$$

Adding the Taylor series for u_{i-1} and u_{i+1} and canceling terms:

$$u_{i+1}+u_{i-1}=2u_i+\Delta x^2u_i''+\frac{\Delta x^4}{12}u_i^{(4)}+\mathcal{O}(\Delta x^6) \tag{3.62}$$

Returning to the differential equation and taking two additional derivatives:

$$u''(x) = cu'(x) \tag{3.63}$$

$$u^{(3)}(x) = cu''(x) \tag{3.64}$$

$$u^{(4)}(x) = cu^{(3)}(x) \tag{3.65}$$

Replacing the fourth-derivative and third-derivative terms with the original differential equation, we arrive at:

$$u^{(4)}(x) = c^2 u''(x) \tag{3.66}$$

Now replacing the fourth-derivative term in the Taylor series addition, rearranging, and dividing by Δx^2:

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 u_i'' + \frac{c^2 \Delta x^4}{12} u_i'' + \mathcal{O}(\Delta x^6) \tag{3.67}$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = \left(1 + \frac{c^2 \Delta x^2}{12}\right) u_i'' + \mathcal{O}(\Delta x^4) \tag{3.68}$$

Rearranging terms to solve for u_i'':

$$u_i'' = \left(1 + \frac{c^2 \Delta x^2}{12}\right)^{-1} \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^4) \tag{3.69}$$

From this specific second-derivative formulation using the finite difference method, the second-derivative approximation can be observed to be fourth-order ($\mathcal{O}(\Delta x^4)$).

Discretized Differential Equation The differential equation is:

$$-u''(x) + cu'(x) = 0 \tag{3.70}$$

Discretizing using the above formulas and simplifying yields:

$$-\left(1 + \frac{c^2 \Delta x^2}{12}\right)^{-1} \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + c\left(1 + \frac{c^2 \Delta x^2}{6}\right)^{-1} \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0 \tag{3.71}$$

$$\left(-1 - \frac{c\Delta x}{2} - \frac{c^2 \Delta x^2}{12}\right) u_{i-1} + \left(2 + \frac{c^2 \Delta x^2}{6}\right) u_i + \left(-1 + \frac{c\Delta x}{2} - \frac{c^2 \Delta x^2}{12}\right) u_{i+1} = 0 \tag{3.72}$$

3.3.3 1st-Order (p=1) Galerkin Method Finite Element Method

3.3.4 2nd-Order (p=2) Galerkin Method Finite Element Method

4 Results

4.1 Finite Difference Method – Solution Results

4.1.1 2nd-Order Central Difference Scheme - Diffusion Equation

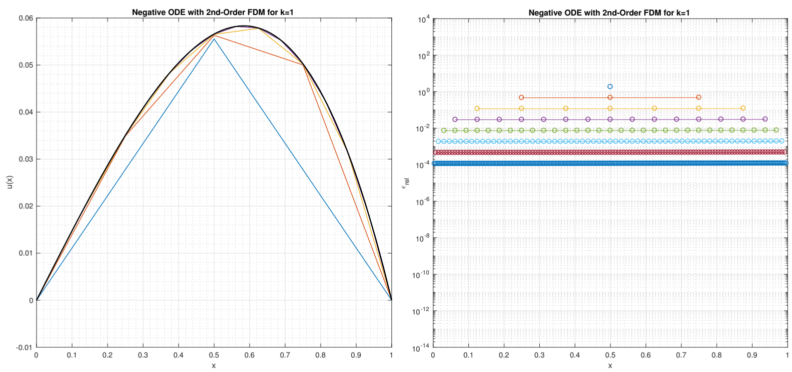


Figure 4.1.1 – 2nd-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 1$

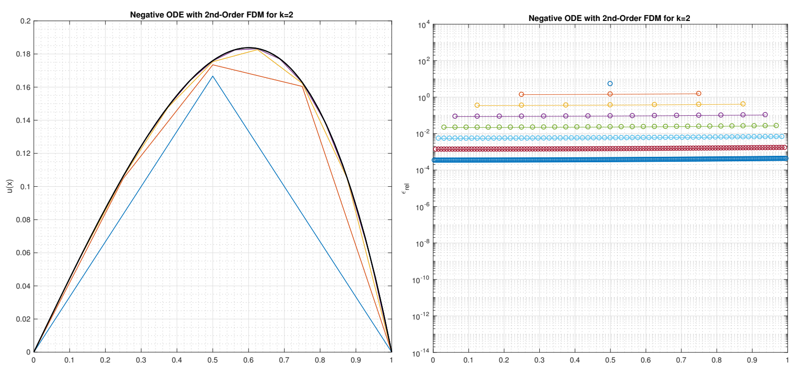


Figure 4.1.2 – 2nd-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 2$

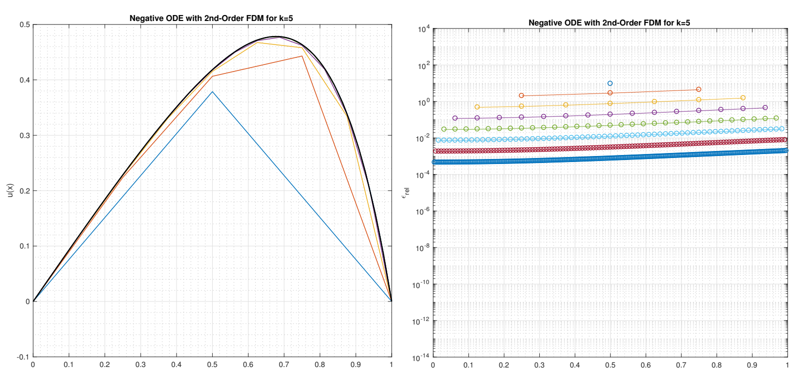


Figure 4.1.3 – 2nd-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 5$

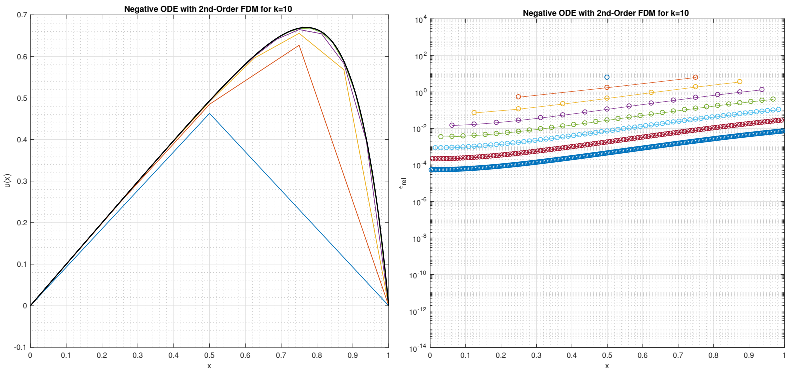


Figure 4.1.4 – 2nd-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 10$

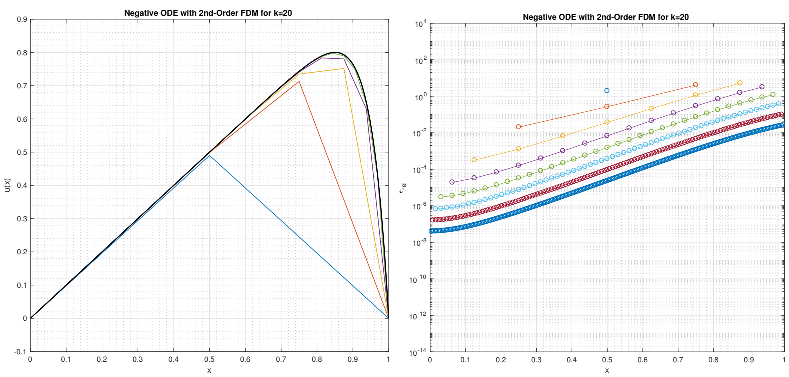


Figure 4.1.5 – 2nd-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 20$

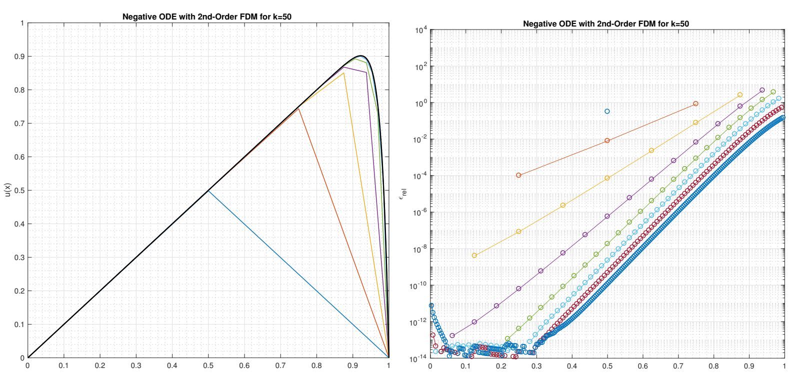
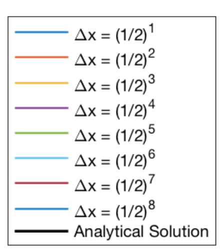


Figure 4.1.6 – 2nd-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 50$



4.1.2 2nd-Order Central Difference Scheme - Harmonic Wave Equation

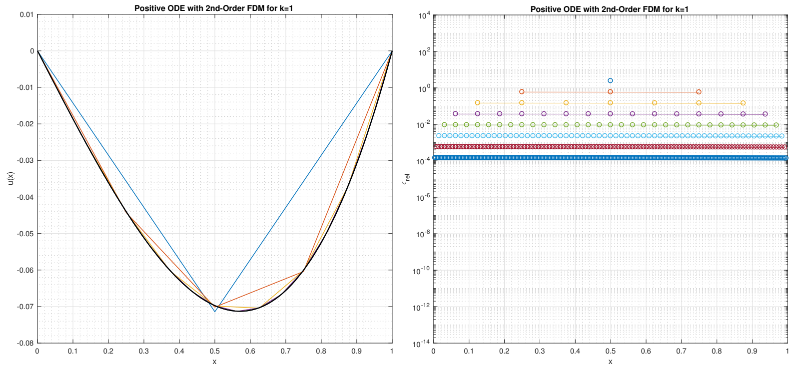


Figure 4.1.7 – 2nd-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 1$

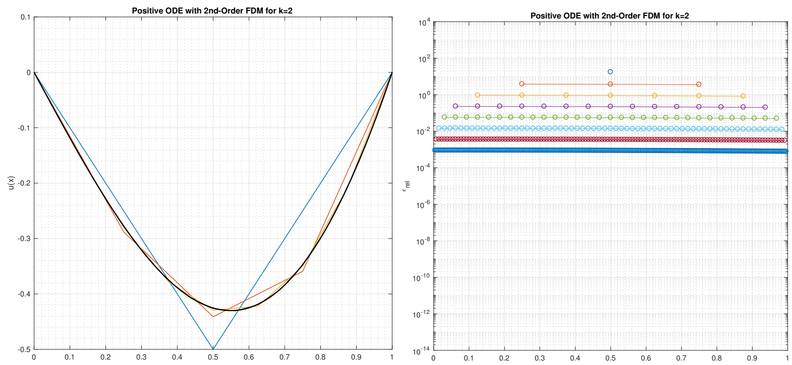


Figure 4.1.8 – 2nd-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 2$

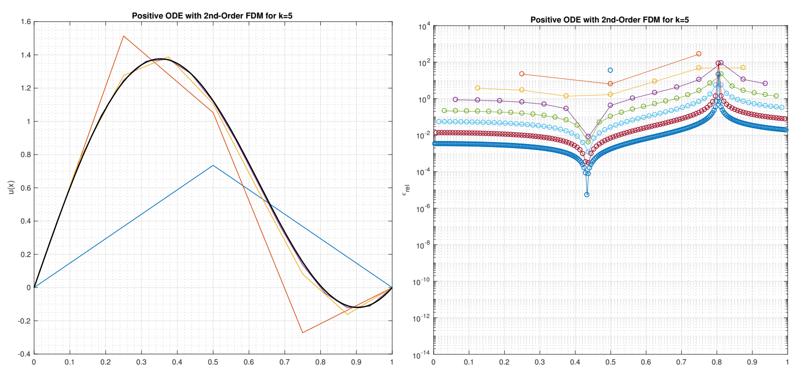


Figure 4.1.9 – 2nd-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 5$

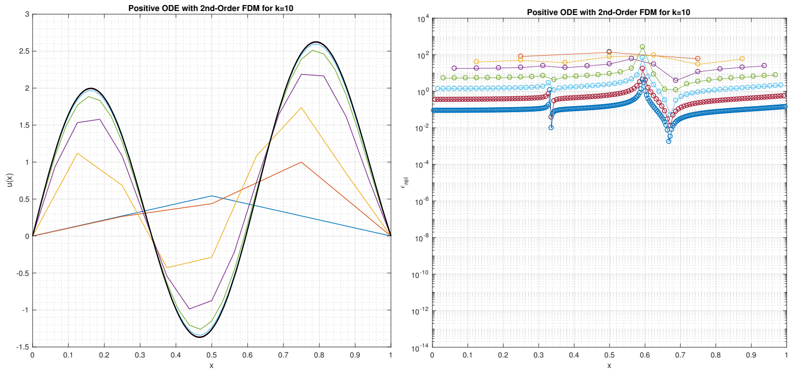


Figure 4.1.10 – 2nd-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 10$

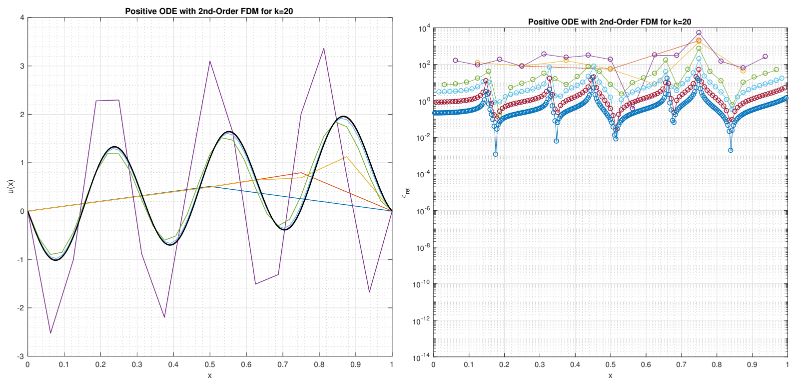


Figure 4.1.11 – 2nd-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 20$

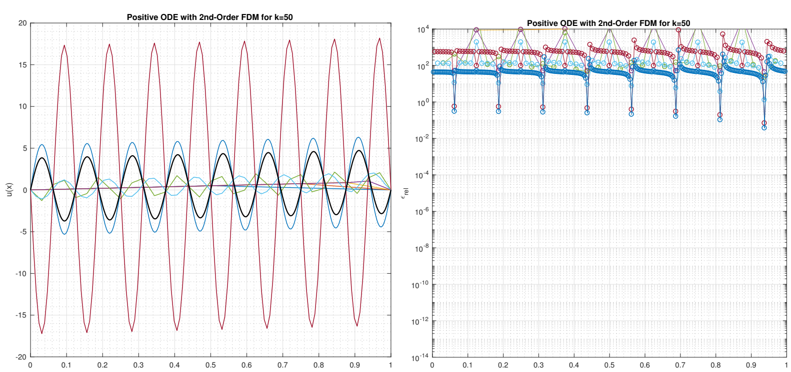
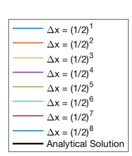


Figure 4.1.12 – 2nd-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 50$



4.1.3 2nd-Order Central Difference Scheme - Convection-Diffusion Equation

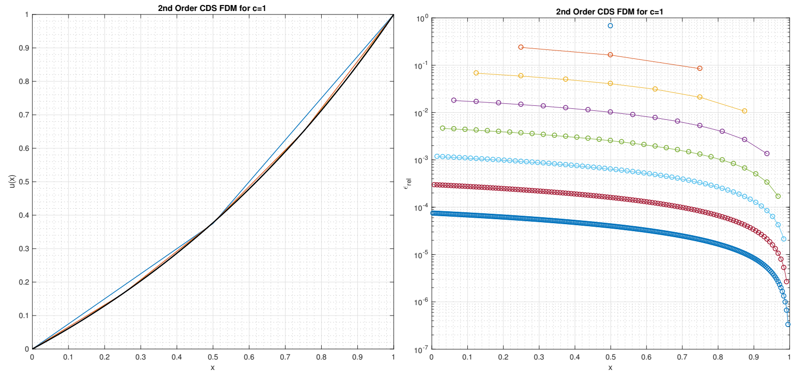


Figure 4.1.13 – 2nd-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 1$

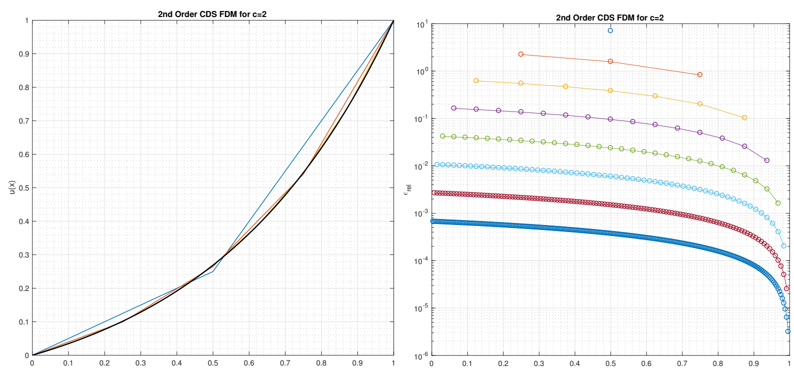


Figure 4.1.14 – 2nd-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 2$

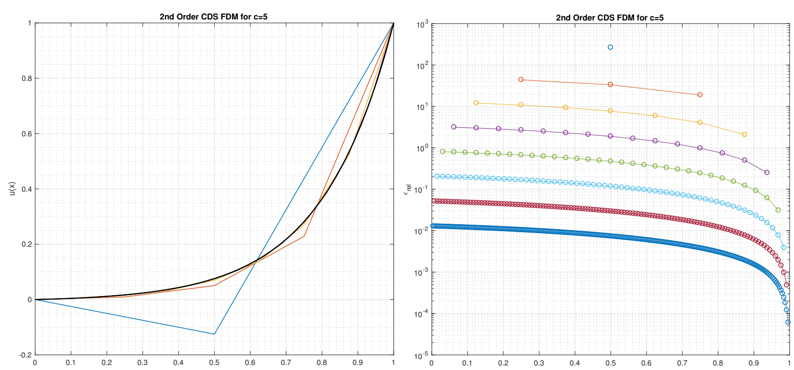


Figure 4.1.15 – 2nd-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 5$

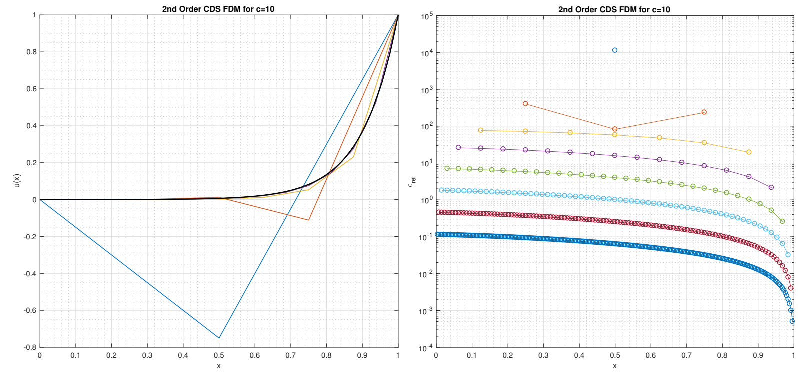


Figure 4.1.16 – 2nd-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 10$

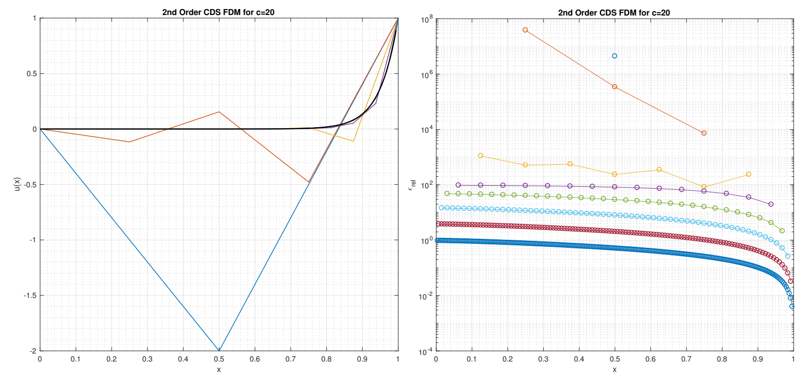


Figure 4.1.17 – 2nd-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 20$

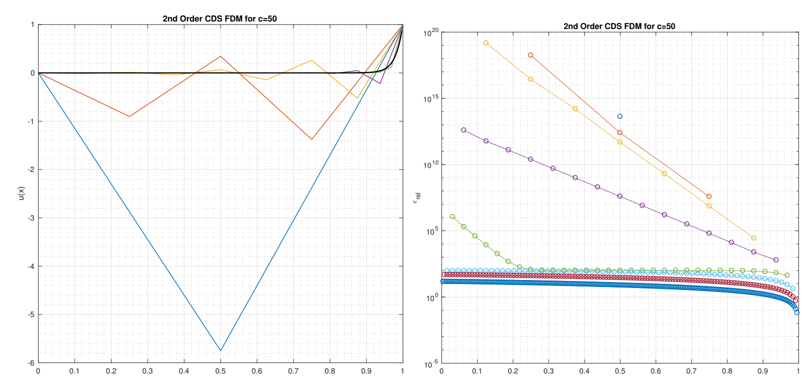
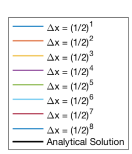


Figure 4.1.18 – 2nd-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 50$



4.1.4 4th-Order Central Difference Scheme - Diffusion Equation

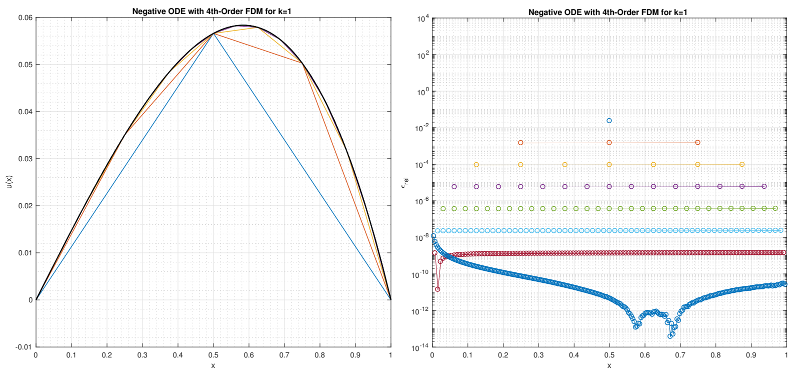


Figure 4.1.19 – 4th-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 1$

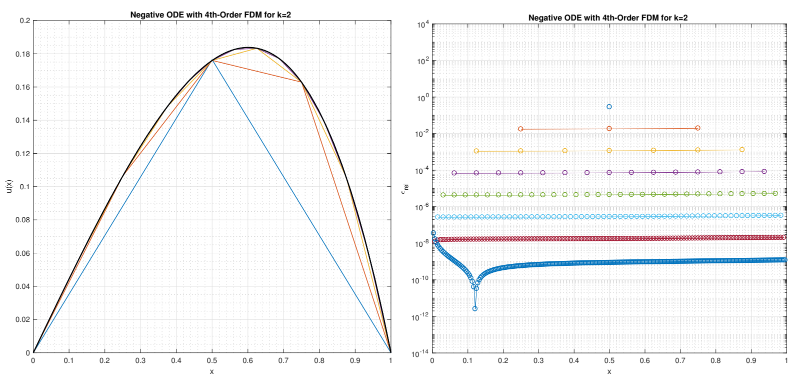


Figure 4.1.20 – 4th-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 2$

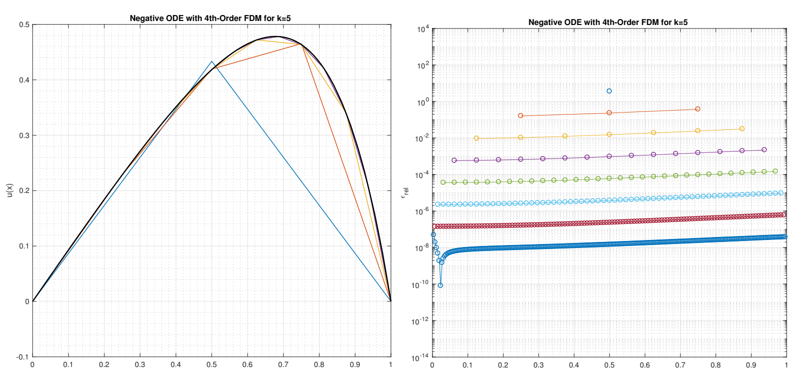


Figure 4.1.21 – 4th-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 5$

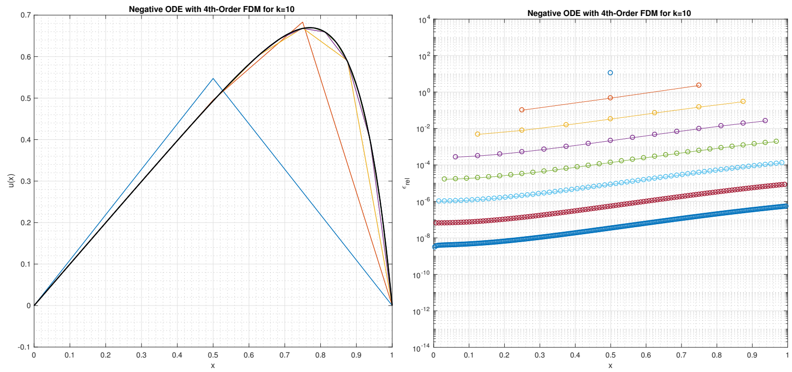


Figure 4.1.22 – 4th-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 10$

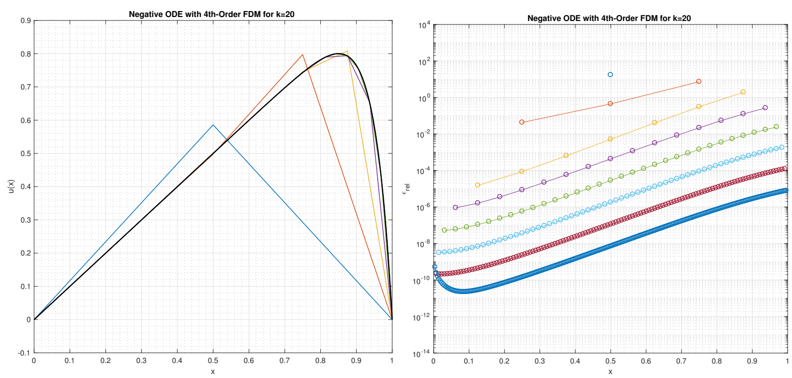


Figure 4.1.23 – 4th-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 20$

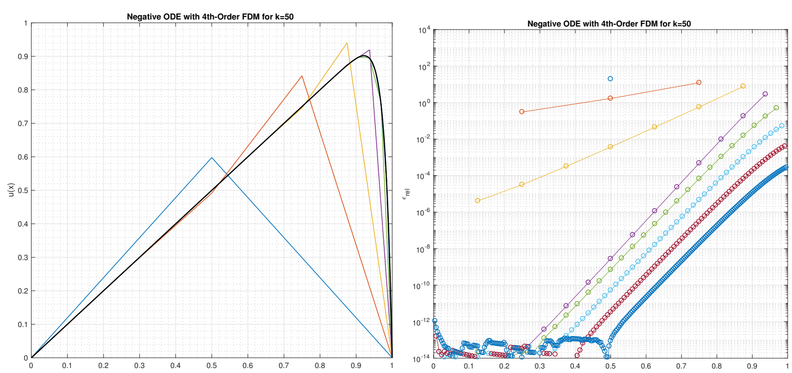
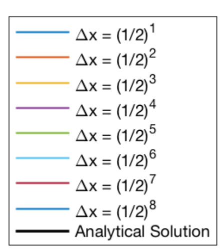


Figure 4.1.24 – 4th-Order CDS FDM and Pointwise Error for Diffusion Equation with $k = 50$



4.1.5 4th-Order Central Difference Scheme - Harmonic Wave Equation

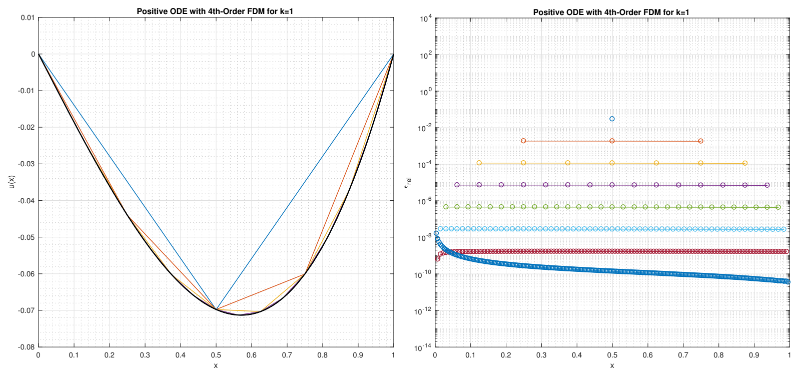


Figure 4.1.25 – 4th-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 1$

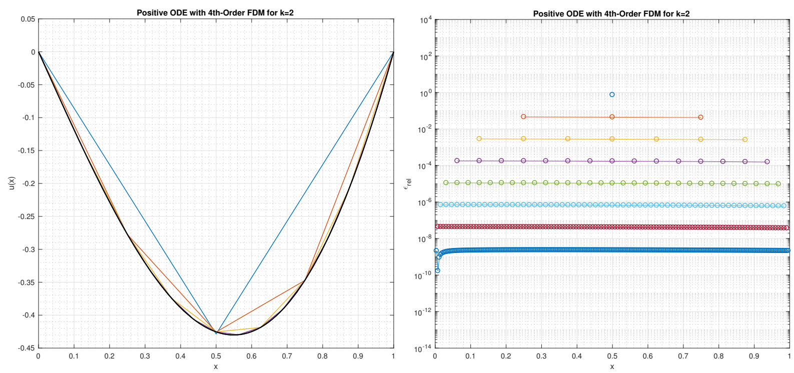


Figure 4.1.26 – 4th-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 2$

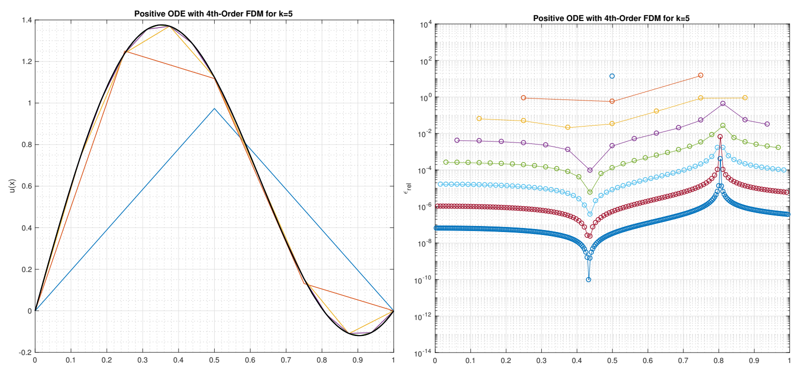


Figure 4.1.27 – 4th-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 5$

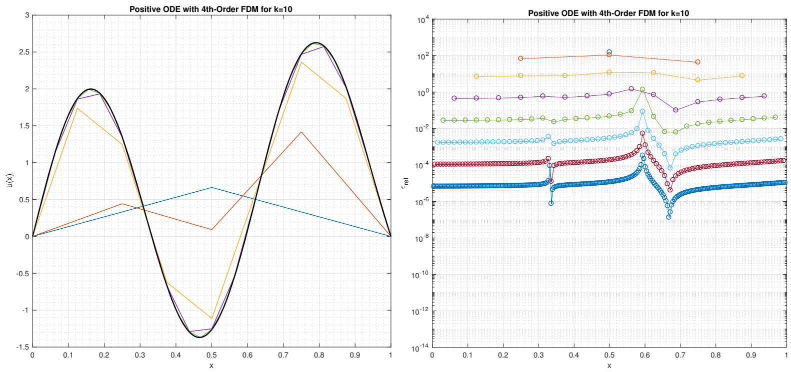


Figure 4.1.28 – 4th-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 10$

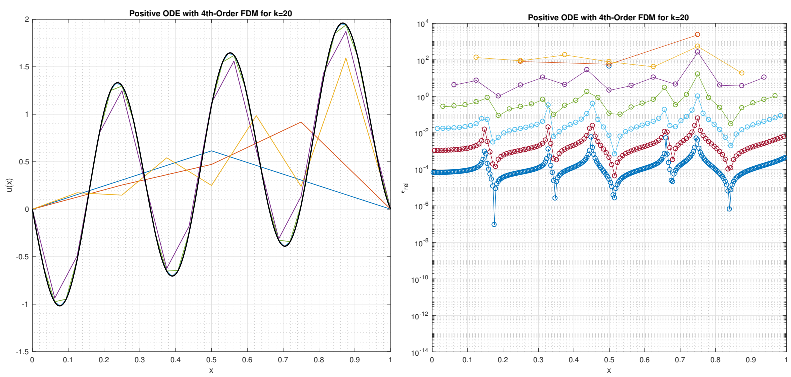


Figure 4.1.29 – 4th-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 20$

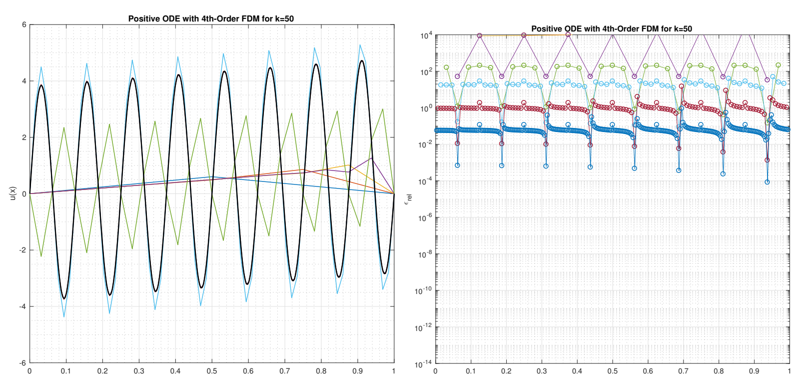
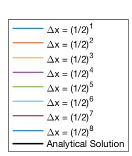


Figure 4.1.30 – 4th-Order CDS FDM and Pointwise Error for Harmonic Wave Equation with $k = 50$



4.1.6 4th-Order Central Difference Scheme - Convection-Diffusion Equation

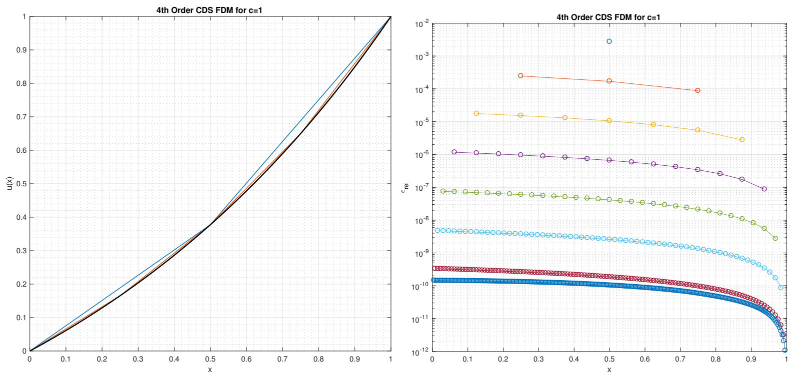


Figure 4.1.31 – 4th-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 1$

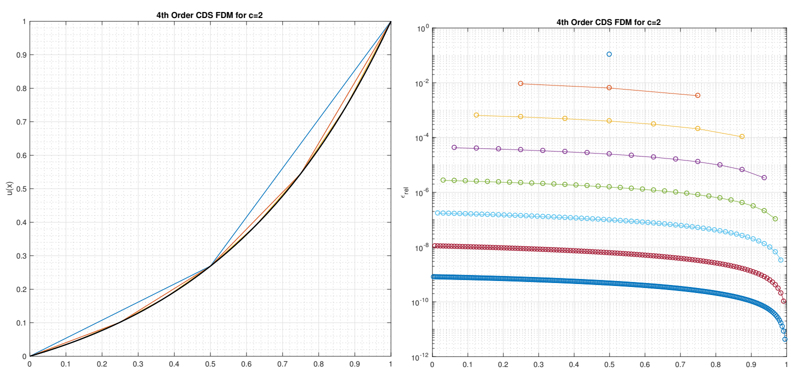


Figure 4.1.32 – 4th-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 2$

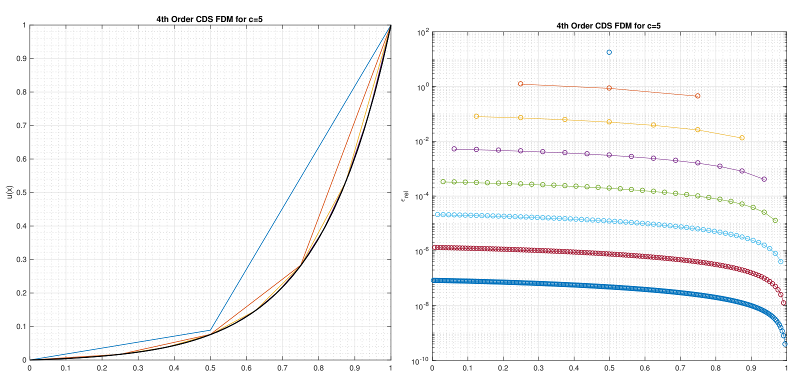


Figure 4.1.33 – 4th-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 5$

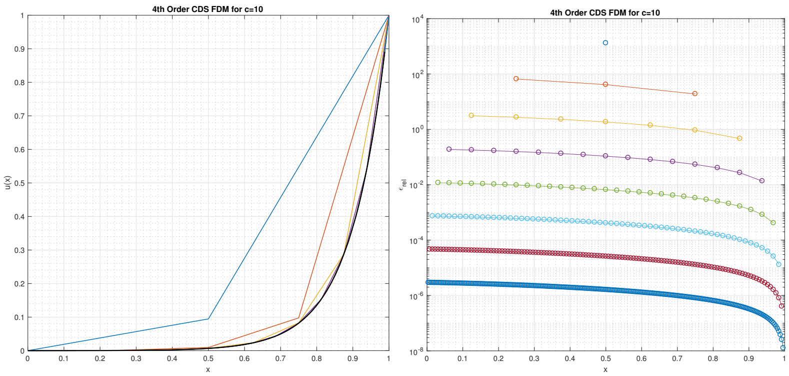


Figure 4.1.34 – 4th-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 10$

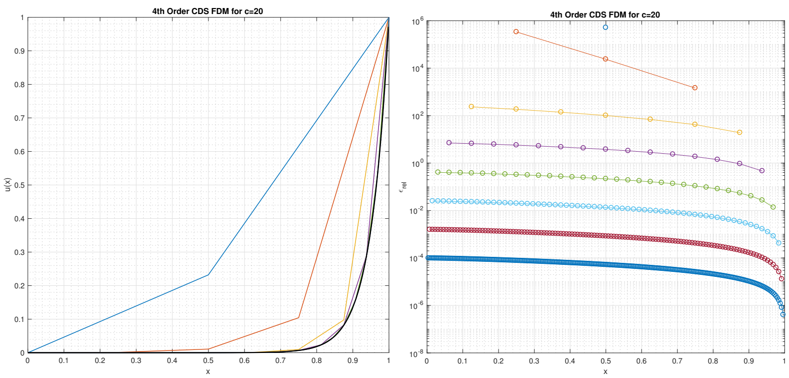


Figure 4.1.35 – 4th-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 20$

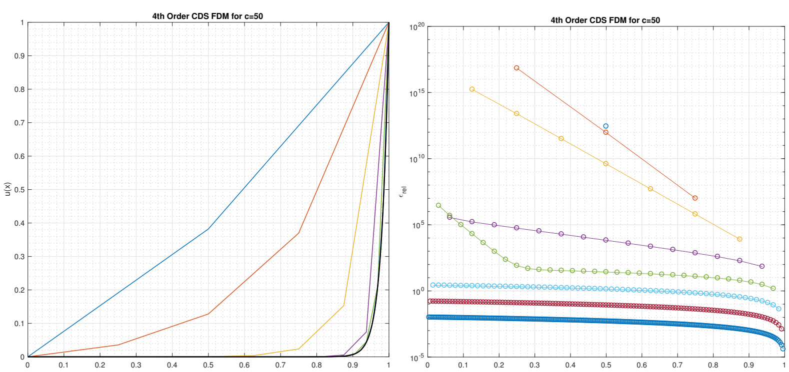
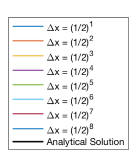


Figure 4.1.36 – 4th-Order CDS FDM and Pointwise Error for Convection-Diffusion Equation with $c = 50$



4.2 Finite Element Method – Solution Results

4.2.1 1st-Order (p=1) Galerkin Method - Diffusion Equation

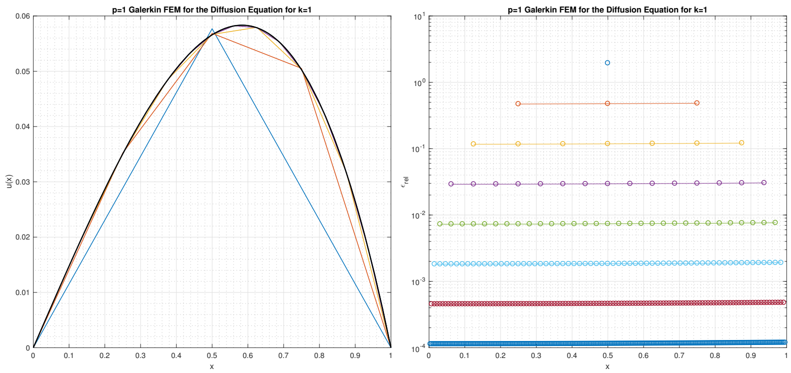


Figure 4.2.1 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 1$

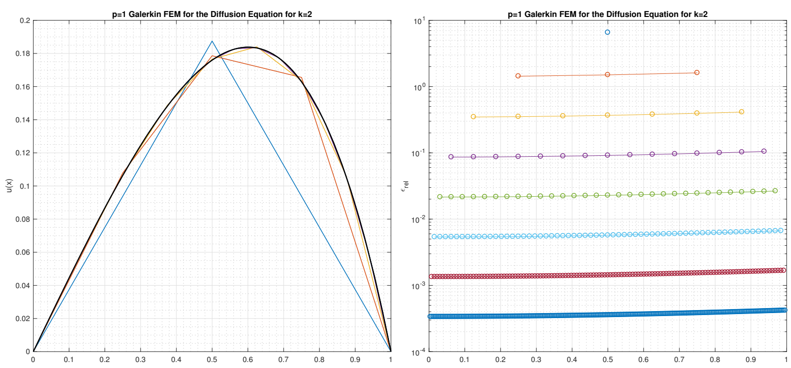


Figure 4.2.2 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 2$

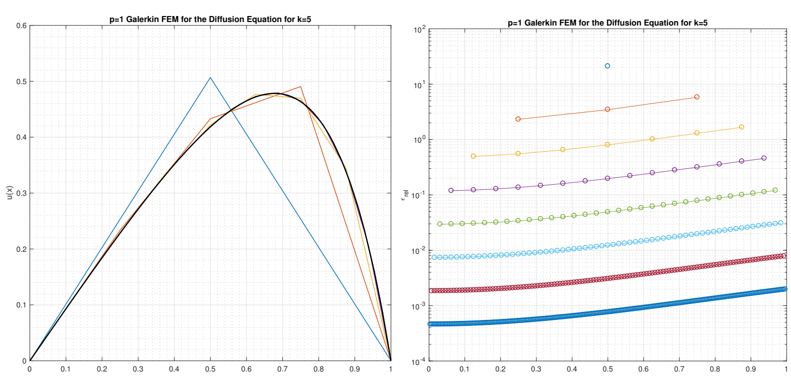


Figure 4.2.3 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 5$

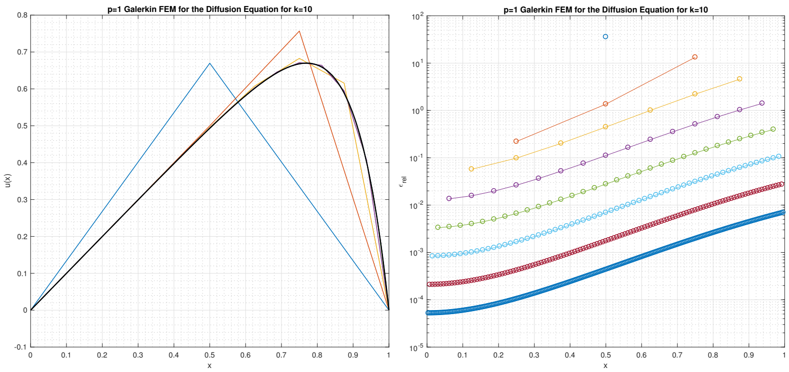


Figure 4.2.4 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 10$

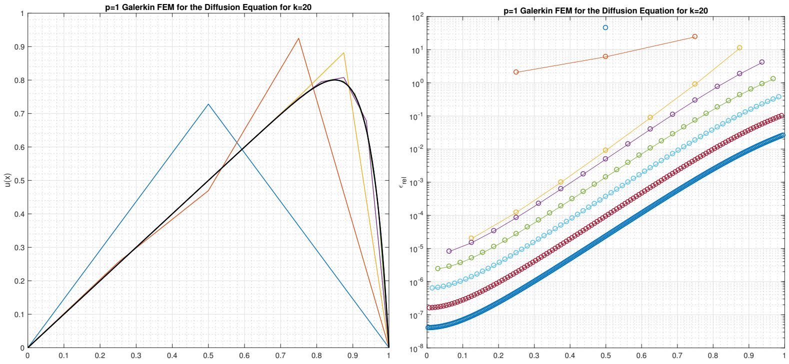


Figure 4.2.5 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 20$

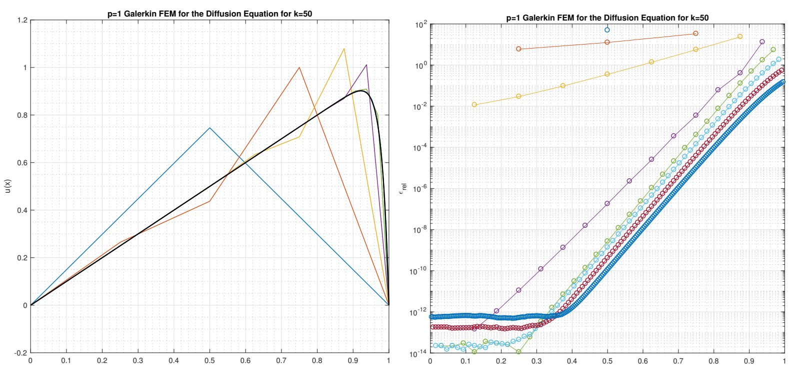
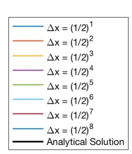


Figure 4.2.6 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 50$



4.2.2 1st-Order (p=1) Galerkin Method - Harmonic Wave Equation

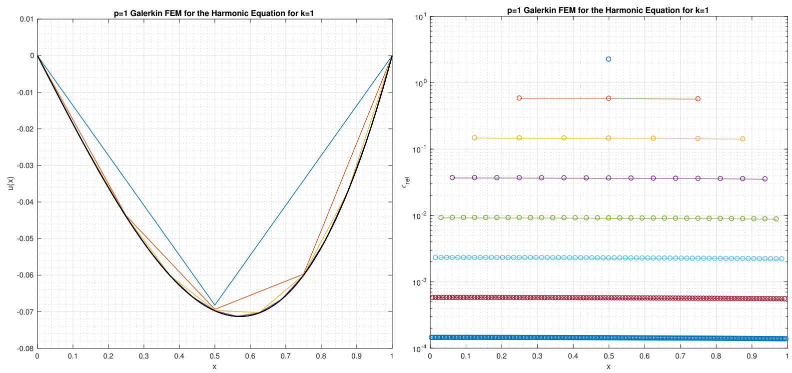


Figure 4.2.7 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 1$

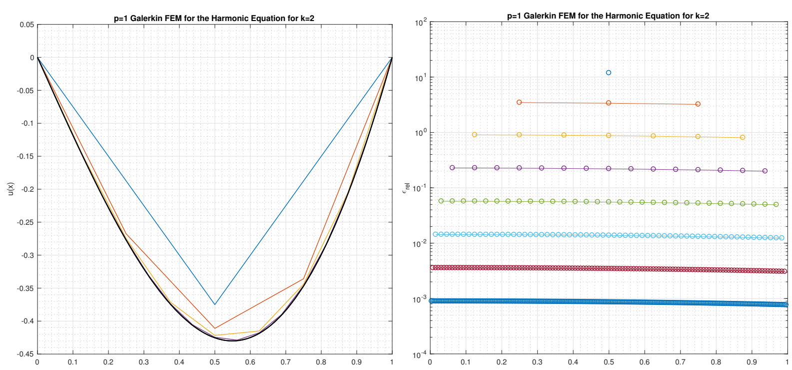


Figure 4.2.8 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 2$

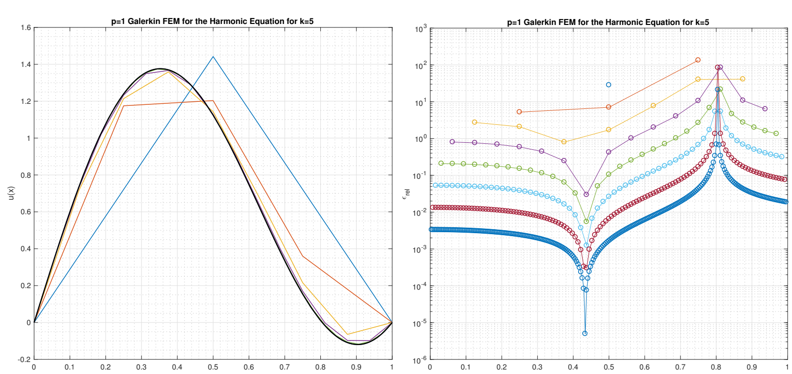


Figure 4.2.9 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 5$

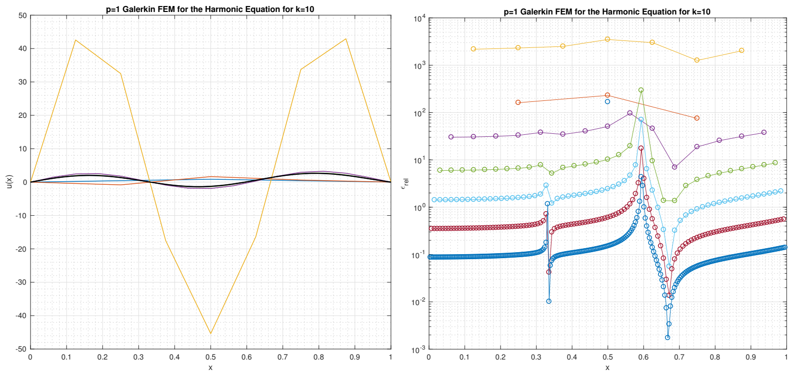


Figure 4.2.10 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 10$

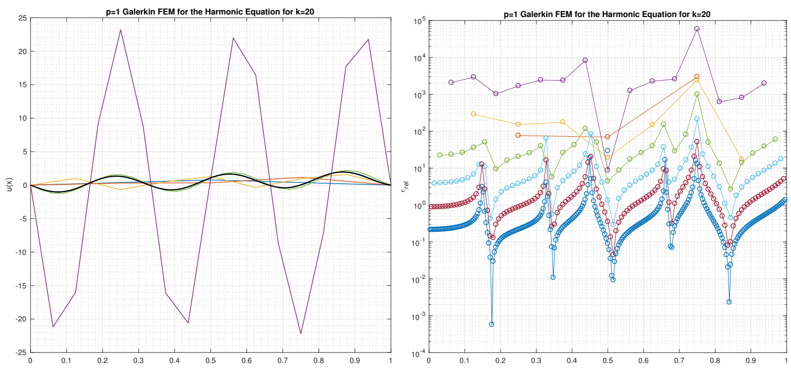


Figure 4.2.11 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 20$

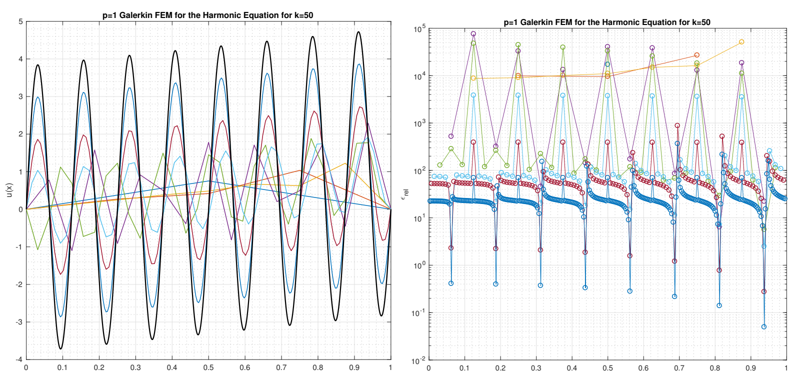
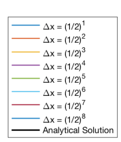


Figure 4.2.12 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 50$



4.2.3 1st-Order (p=1) Galerkin Method - Convection-Diffusion Equation

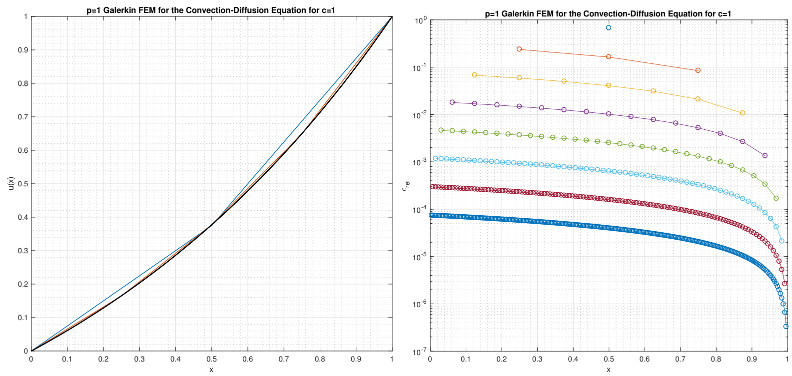


Figure 4.2.13 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 1$

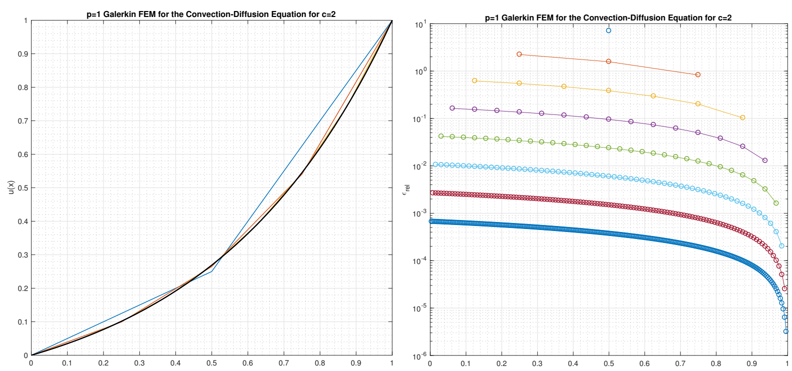


Figure 4.2.14 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 2$

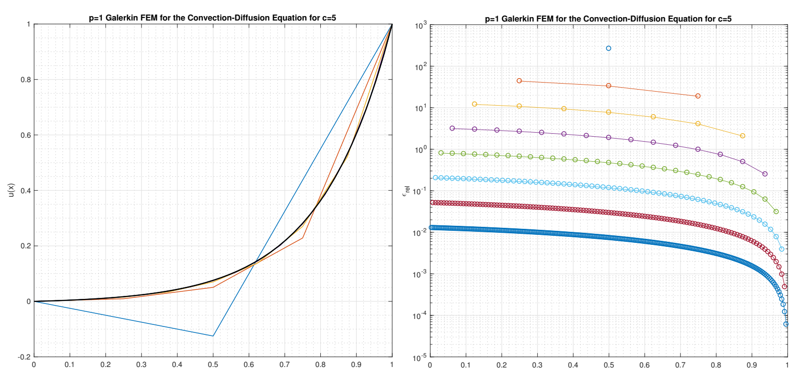


Figure 4.2.15 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 5$

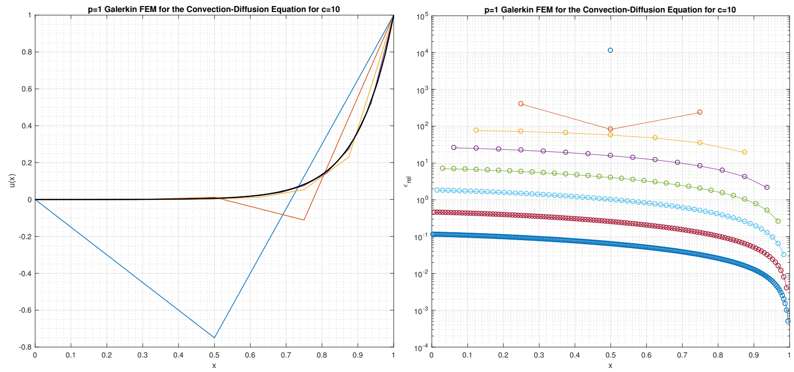


Figure 4.2.16 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 10$

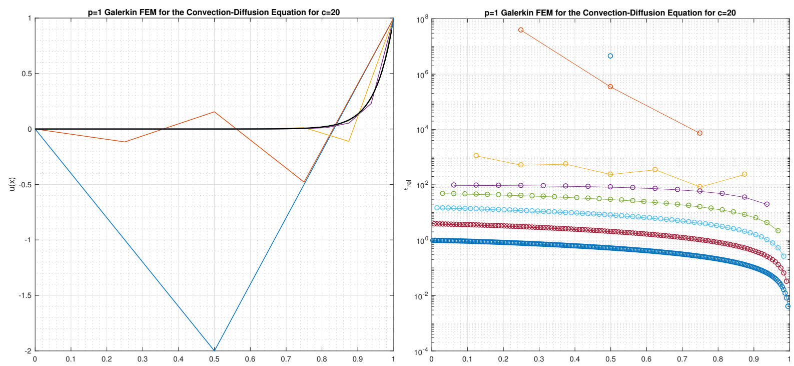


Figure 4.2.17 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 20$

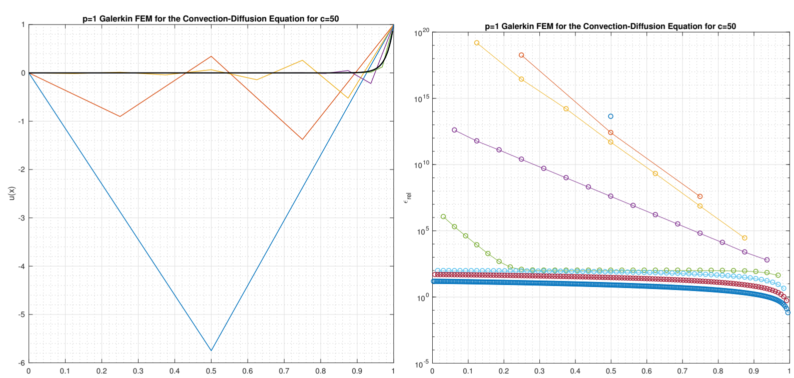
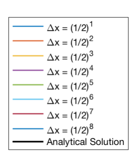


Figure 4.2.18 – 1st-Order (p=1) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 50$



4.2.4 2nd-Order (p=2) Galerkin Method - Diffusion Equation

Figure 4.2.19 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 1$

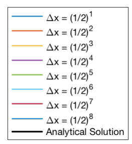
Figure 4.2.20 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 2$

Figure 4.2.21 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 5$

Figure 4.2.22 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 10$

Figure 4.2.23 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 20$

Figure 4.2.24 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Diffusion Equation with $k = 50$



4.2.5 2nd-Order (p=2) Galerkin Method - Harmonic Wave Equation

Figure 4.2.25 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 1$

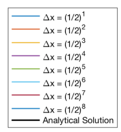
Figure 4.2.26 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 2$

Figure 4.2.27 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 5$

Figure 4.2.28 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 10$

Figure 4.2.29 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 20$

Figure 4.2.30 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Harmonic Wave Equation with $k = 50$



4.2.6 2nd-Order (p=2) Galerkin Method - Convection-Diffusion Equation

Figure 4.2.31 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 1$

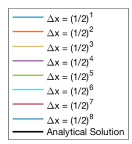
Figure 4.2.32 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 2$

Figure 4.2.33 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 5$

Figure 4.2.34 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 10$

Figure 4.2.35 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 20$

Figure 4.2.36 – 2nd-Order (p=2) Galerkin Method FEM and Pointwise Error for Convection-Diffusion Equation with $c = 50$



4.3 Finite Difference Method – Quantity of Interest Results

4.3.1 2nd-Order Central Difference Scheme - Diffusion Equation

Table 4.3.1 – Quantity of Interest for 2nd-Order CDS FDM for Diffusion Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	-0.3611	-1.3333	-7.0076	-0.3130	-1.0746	-4.0005	15.3580	24.0738	75.1695
0.2500	-0.3251	-1.1417	-4.8972	-0.3130	-1.0746	-4.0005	3.8691	6.2434	22.4154
0.1250	-0.3161	-1.0916	-4.2390	-0.3130	-1.0746	-4.0005	0.9692	1.5765	5.9632
0.0625	-0.3138	-1.0789	-4.0611	-0.3130	-1.0746	-4.0005	0.2424	0.3951	1.5171
0.0312	-0.3132	-1.0757	-4.0157	-0.3130	-1.0746	-4.0005	0.0606	0.0988	0.3810
0.0156	-0.3131	-1.0749	-4.0043	-0.3130	-1.0746	-4.0005	0.0152	0.0247	0.0954
0.0078	-0.3130	-1.0747	-4.0014	-0.3130	-1.0746	-4.0005	0.0038	0.0062	0.0238
0.0039	-0.3130	-1.0746	-4.0007	-0.3130	-1.0746	-4.0005	0.0009	0.0015	0.0060
0.0020	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0002	0.0004	0.0015
0.0010	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0001	0.0001	0.0004
0.0005	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0001
0.0002	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0001	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0001	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-25.9259	-100.9804	-625.9968	-9.0000	-19.0000	-49.0000	188.0658	431.4757	1177.5445
0.2500	-15.0078	-52.8516	-315.4747	-9.0000	-19.0000	-49.0000	66.7535	178.1666	543.8260
0.1250	-10.7925	-31.0156	-163.0551	-9.0000	-19.0000	-49.0000	19.9164	63.2401	232.7654
0.0625	-9.4769	-22.5850	-91.7551	-9.0000	-19.0000	-49.0000	5.2990	18.8682	87.2554
0.0312	-9.1213	-19.9538	-62.4498	-9.0000	-19.0000	-49.0000	1.3482	5.0201	27.4486
0.0156	-9.0305	-19.2427	-52.6793	-9.0000	-19.0000	-49.0000	0.3386	1.2772	7.5088
0.0078	-9.0076	-19.0609	-49.9447	-9.0000	-19.0000	-49.0000	0.0847	0.3207	1.9281
0.0039	-9.0019	-19.0153	-49.2379	-9.0000	-19.0000	-49.0000	0.0212	0.0803	0.4854
0.0020	-9.0005	-19.0038	-49.0596	-9.0000	-19.0000	-49.0000	0.0053	0.0201	0.1216
0.0010	-9.0001	-19.0010	-49.0149	-9.0000	-19.0000	-49.0000	0.0013	0.0050	0.0304
0.0005	-9.0000	-19.0002	-49.0037	-9.0000	-19.0000	-49.0000	0.0003	0.0013	0.0076
0.0002	-9.0000	-19.0001	-49.0009	-9.0000	-19.0000	-49.0000	0.0001	0.0003	0.0019
0.0001	-9.0000	-19.0000	-49.0002	-9.0000	-19.0000	-49.0000	0.0000	0.0001	0.0005
0.0001	-9.0000	-19.0000	-49.0001	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0001
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000

4.3.2 4th-Order Central Difference Scheme - Diffusion Equation

Table 4.3.2 – Quantity of Interest for 4th-Order CDS FDM for Diffusion Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	-0.3137	-1.0882	-4.5701	-0.3130	-1.0746	-4.0005	0.2057	1.2661	14.2392
0.2500	-0.3131	-1.0757	-4.0724	-0.3130	-1.0746	-4.0005	0.0141	0.1006	1.7995
0.1250	-0.3130	-1.0747	-4.0066	-0.3130	-1.0746	-4.0005	0.0009	0.0070	0.1546
0.0625	-0.3130	-1.0746	-4.0009	-0.3130	-1.0746	-4.0005	0.0001	0.0005	0.0111
0.0312	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0007
0.0156	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0078	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0039	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0020	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0010	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0005	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0002	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0001	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0001	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0001	0.0001	0.0000

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-14.3248	-51.9531	-314.4923	-9.0000	-19.0000	-49.0000	59.1646	173.4375	541.8211
0.2500	-10.1396	-29.6497	-161.1896	-9.0000	-19.0000	-49.0000	12.6621	56.0511	228.9584
0.1250	-9.1440	-21.2792	-88.6622	-9.0000	-19.0000	-49.0000	1.6000	11.9957	80.9433
0.0625	-9.0124	-19.2880	-58.9060	-9.0000	-19.0000	-49.0000	0.1375	1.5158	20.2164
0.0312	-9.0009	-19.0247	-50.4779	-9.0000	-19.0000	-49.0000	0.0098	0.1302	3.0161
0.0156	-9.0001	-19.0018	-49.1402	-9.0000	-19.0000	-49.0000	0.0007	0.0093	0.2861
0.0078	-9.0000	-19.0001	-49.0105	-9.0000	-19.0000	-49.0000	0.0000	0.0006	0.0214
0.0039	-9.0000	-19.0000	-49.0007	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0014
0.0020	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0001
0.0010	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0005	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0002	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0001	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0001	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000

4.3.3 2nd-Order Central Difference Scheme - Harmonic Wave Equation

Table 4.3.3 – Quantity of Interest for 2nd-Order CDS FDM for Harmonic Wave Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	0.3929	2.0000	4.7794	0.3579	1.9153	2.4791	9.7650	4.4215	92.7909
0.2500	0.3666	1.9370	4.2126	0.3579	1.9153	2.4791	2.4321	1.1309	69.9281
0.1250	0.3601	1.9208	2.8575	0.3579	1.9153	2.4791	0.6075	0.2840	15.2634
0.0625	0.3585	1.9167	2.5721	0.3579	1.9153	2.4791	0.1518	0.0711	3.7519
0.0312	0.3580	1.9157	2.5022	0.3579	1.9153	2.4791	0.0380	0.0178	0.9346
0.0156	0.3579	1.9154	2.4849	0.3579	1.9153	2.4791	0.0095	0.0044	0.2335
0.0078	0.3579	1.9153	2.4805	0.3579	1.9153	2.4791	0.0024	0.0011	0.0584
0.0039	0.3579	1.9153	2.4794	0.3579	1.9153	2.4791	0.0006	0.0003	0.0146
0.0020	0.3579	1.9153	2.4792	0.3579	1.9153	2.4791	0.0001	0.0001	0.0036
0.0010	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0009
0.0005	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0002
0.0002	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0001
0.0001	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0001	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	23.9130	98.9796	623.9968	-14.4235	-7.9399	184.8907	265.7921	1346.6097	237.4949
0.2500	8.5002	46.8258	309.4741	-14.4235	-7.9399	184.8907	158.9331	689.7523	67.3822
0.1250	-0.5294	16.0000	149.0340	-14.4235	-7.9399	184.8907	96.3296	301.5138	19.3935
0.0625	-9.2875	39.3146	61.0293	-14.4235	-7.9399	184.8907	35.6089	595.1520	66.9917
0.0312	-12.9422	-0.5161	-27.0919	-14.4235	-7.9399	184.8907	10.2700	93.4994	114.6529
0.0156	-14.0381	-5.9449	-22.2961	-14.4235	-7.9399	184.8907	2.6723	25.1268	112.0591
0.0078	-14.3262	-7.4289	-843.6179	-14.4235	-7.9399	184.8907	0.6750	6.4362	556.2792
0.0039	-14.3991	-7.8113	265.9260	-14.4235	-7.9399	184.8907	0.1692	1.6195	43.8287
0.0020	-14.4174	-7.9077	200.2412	-14.4235	-7.9399	184.8907	0.0423	0.4055	8.3025
0.0010	-14.4220	-7.9318	188.5106	-14.4235	-7.9399	184.8907	0.0106	0.1014	1.9579
0.0005	-14.4231	-7.9379	185.7831	-14.4235	-7.9399	184.8907	0.0026	0.0254	0.4826
0.0002	-14.4234	-7.9394	185.1130	-14.4235	-7.9399	184.8907	0.0007	0.0063	0.1202
0.0001	-14.4235	-7.9398	184.9463	-14.4235	-7.9399	184.8907	0.0002	0.0016	0.0300
0.0001	-14.4235	-7.9399	184.9046	-14.4235	-7.9399	184.8907	0.0000	0.0004	0.0075
0.0000	-14.4235	-7.9399	184.8942	-14.4235	-7.9399	184.8907	0.0000	0.0001	0.0019
0.0000	-14.4235	-7.9399	184.8916	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0005
0.0000	-14.4235	-7.9399	184.8909	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0001
0.0000	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000

4.3.4 4th-Order Central Difference Scheme - Harmonic Wave Equation

Table 4.3.4 – Quantity of Interest for 4th-Order CDS FDM for Harmonic Wave Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	0.3576	1.9286	-0.1218	0.3579	1.9153	2.4791	0.0976	0.6921	104.9113
0.2500	0.3579	1.9163	2.6143	0.3579	1.9153	2.4791	0.0051	0.0533	5.4558
0.1250	0.3579	1.9154	2.4886	0.3579	1.9153	2.4791	0.0003	0.0037	0.3832
0.0625	0.3579	1.9153	2.4797	0.3579	1.9153	2.4791	0.0000	0.0002	0.0257
0.0312	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0017
0.0156	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0001
0.0078	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0039	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0020	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0010	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0005	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0002	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0001	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0001	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0001	0.0002	0.0000

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	10.2876	47.9508	310.4923	-14.4235	-7.9399	184.8907	171.3253	703.9220	67.9329
0.2500	17.3278	19.5811	151.1878	-14.4235	-7.9399	184.8907	220.1358	346.6161	18.2286
0.1250	-13.2223	42.9313	66.6054	-14.4235	-7.9399	184.8907	8.3281	640.7031	63.9758
0.0625	-14.3432	-5.8395	11.6553	-14.4235	-7.9399	184.8907	0.5569	26.4536	93.6961
0.0312	-14.4184	-7.8064	-110.4917	-14.4235	-7.9399	184.8907	0.0353	1.6818	159.7606
0.0156	-14.4232	-7.9314	219.6799	-14.4235	-7.9399	184.8907	0.0022	0.1067	18.8161
0.0078	-14.4235	-7.9394	186.7232	-14.4235	-7.9399	184.8907	0.0001	0.0067	0.9911
0.0039	-14.4235	-7.9399	185.0038	-14.4235	-7.9399	184.8907	0.0000	0.0004	0.0612
0.0020	-14.4235	-7.9399	184.8978	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0038
0.0010	-14.4235	-7.9399	184.8912	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0002
0.0005	-14.4235	-7.9399	184.8908	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0002	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0001	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0001	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0000	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0000	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0000	-14.4235	-7.9399	184.8907	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000
0.0000	-14.4235	-7.9399	184.8908	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0000

4.3.5 2nd-Order Central Difference Scheme - Convection-Diffusion Equation

Table 4.3.5 – Quantity of Interest for 2nd-Order CDS FDM for Convection-Diffusion Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	1.6667	3.0000	-9.0000	1.5820	2.3130	5.0339	5.3534	29.6997	278.7872
0.2500	1.6022	2.4510	8.2285	1.5820	2.3130	5.0339	1.2782	5.9638	63.4604
0.1250	1.5870	2.3459	5.5727	1.5820	2.3130	5.0339	0.3160	1.4213	10.7032
0.0625	1.5832	2.3212	5.1585	1.5820	2.3130	5.0339	0.0788	0.3512	2.4744
0.0312	1.5823	2.3151	5.0645	1.5820	2.3130	5.0339	0.0197	0.0876	0.6072
0.0156	1.5821	2.3135	5.0415	1.5820	2.3130	5.0339	0.0049	0.0219	0.1511
0.0078	1.5820	2.3132	5.0358	1.5820	2.3130	5.0339	0.0012	0.0055	0.0377
0.0039	1.5820	2.3131	5.0344	1.5820	2.3130	5.0339	0.0003	0.0014	0.0094
0.0020	1.5820	2.3130	5.0340	1.5820	2.3130	5.0339	0.0001	0.0003	0.0024
0.0010	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0001	0.0006
0.0005	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0001
0.0002	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0002	0.0001	0.0001

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-2.3333	-1.5000	-1.1739	10.0005	20.0000	50.0000	123.3323	107.5000	102.3478
0.2500	-17.7805	-3.9425	-1.8118	10.0005	20.0000	50.0000	277.7968	119.7126	103.6237
0.1250	16.4104	-35.5556	-5.7325	10.0005	20.0000	50.0000	64.0964	277.7778	111.4651
0.0625	11.0826	32.8205	-34.6883	10.0005	20.0000	50.0000	10.8210	64.1026	169.3767
0.0312	10.2507	22.1645	128.3208	10.0005	20.0000	50.0000	2.5021	10.8225	156.6416
0.0156	10.0619	20.5005	59.0032	10.0005	20.0000	50.0000	0.6140	2.5025	18.0063
0.0078	10.0157	20.1228	51.9830	10.0005	20.0000	50.0000	0.1528	0.6141	3.9660
0.0039	10.0043	20.0306	50.4814	10.0005	20.0000	50.0000	0.0382	0.1528	0.9629
0.0020	10.0014	20.0076	50.1195	10.0005	20.0000	50.0000	0.0095	0.0382	0.2390
0.0010	10.0007	20.0019	50.0298	10.0005	20.0000	50.0000	0.0024	0.0095	0.0596
0.0005	10.0005	20.0005	50.0075	10.0005	20.0000	50.0000	0.0006	0.0024	0.0149
0.0002	10.0005	20.0001	50.0019	10.0005	20.0000	50.0000	0.0001	0.0006	0.0037
0.0001	10.0005	20.0000	50.0005	10.0005	20.0000	50.0000	0.0000	0.0001	0.0009
0.0001	10.0005	20.0000	50.0001	10.0005	20.0000	50.0000	0.0000	0.0000	0.0002
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0001
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0002	0.0000
0.0000	10.0005	20.0002	50.0001	10.0005	20.0000	50.0000	0.0002	0.0008	0.0001

4.3.6 4th-Order Central Difference Scheme - Convection-Diffusion Equation

Table 4.3.6 – Quantity of Interest for 4th-Order CDS FDM for Convection-Diffusion Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	1.5829	2.3385	12.9559	1.5820	2.3130	5.0339	0.0594	1.0993	157.3713
0.2500	1.5820	2.3144	5.1772	1.5820	2.3130	5.0339	0.0034	0.0570	2.8461
0.1250	1.5820	2.3131	5.0410	1.5820	2.3130	5.0339	0.0002	0.0032	0.1400
0.0625	1.5820	2.3130	5.0343	1.5820	2.3130	5.0339	0.0000	0.0002	0.0077
0.0312	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0004
0.0156	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0078	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0039	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0020	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0010	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0005	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0002	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0002	0.0001	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0006	0.0004	0.0001
0.0000	1.5819	2.3130	5.0339	1.5820	2.3130	5.0339	0.0025	0.0016	0.0005
0.0000	1.5818	2.3129	5.0338	1.5820	2.3130	5.0339	0.0099	0.0062	0.0018
0.0000	1.5813	2.3125	5.0336	1.5820	2.3130	5.0339	0.0397	0.0248	0.0072

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-0.7125	-0.0530	-0.0022	10.0005	20.0000	50.0000	107.1242	100.2648	100.0044
0.2500	25.6665	-1.4095	-0.0416	10.0005	20.0000	50.0000	156.6534	107.0476	100.0832
0.1250	10.2838	51.3283	-1.1707	10.0005	20.0000	50.0000	2.8333	156.6416	102.3414
0.0625	10.0144	20.5666	-71.7281	10.0005	20.0000	50.0000	0.1393	2.8331	243.4562
0.0312	10.0012	20.0279	53.9799	10.0005	20.0000	50.0000	0.0077	0.1393	7.9598
0.0156	10.0005	20.0015	50.1805	10.0005	20.0000	50.0000	0.0004	0.0077	0.3611
0.0078	10.0005	20.0001	50.0097	10.0005	20.0000	50.0000	0.0000	0.0004	0.0193
0.0039	10.0005	20.0000	50.0006	10.0005	20.0000	50.0000	0.0000	0.0000	0.0011
0.0020	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0001
0.0010	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0005	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0002	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0001	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0001	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0004	20.0000	50.0000	10.0005	20.0000	50.0000	0.0001	0.0001	0.0000
0.0000	10.0004	20.0000	50.0000	10.0005	20.0000	50.0000	0.0005	0.0000	0.0000
0.0000	10.0003	19.9999	49.9999	10.0005	20.0000	50.0000	0.0017	0.0005	0.0001

4.4 Finite Element Method – Quantity of Interest Results

4.4.1 1st-Order (p=1) Galerkin Method - Diffusion Equation

Table 4.4.1 – Quantity of Interest for 1st-Order (p=1) Galerkin Method FEM for Diffusion Equation

Δx	$u'_{k=1}(1)$	$u'_{k=2}(1)$	$u'_{k=5}(1)$	$\bar{u}'_{k=1}(1)$	$\bar{u}'_{k=2}(1)$	$\bar{u}'_{k=5}(1)$	$\epsilon'_{rel,k=1}$	$\epsilon'_{rel,k=2}$	$\epsilon'_{rel,k=5}$
0.5000	-0.3654	-1.3750	-7.2635	-0.3130	-1.0746	-4.0005	16.7231	27.9511	81.5672
0.2500	-0.3271	-1.1622	-5.0872	-0.3130	-1.0746	-4.0005	4.4849	8.1456	27.1652
0.1250	-0.3167	-1.0985	-4.3261	-0.3130	-1.0746	-4.0005	1.1655	2.2199	8.1414
0.0625	-0.3140	-1.0809	-4.0909	-0.3130	-1.0746	-4.0005	0.2974	0.5812	2.2605
0.0312	-0.3133	-1.0762	-4.0244	-0.3130	-1.0746	-4.0005	0.0751	0.1488	0.5983
0.0156	-0.3131	-1.0750	-4.0066	-0.3130	-1.0746	-4.0005	0.0189	0.0377	0.1541
0.0078	-0.3131	-1.0747	-4.0020	-0.3130	-1.0746	-4.0005	0.0047	0.0095	0.0391
0.0039	-0.3130	-1.0747	-4.0008	-0.3130	-1.0746	-4.0005	0.0012	0.0024	0.0099
0.0020	-0.3130	-1.0746	-4.0006	-0.3130	-1.0746	-4.0005	0.0003	0.0006	0.0025
0.0010	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0001	0.0001	0.0006
0.0005	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0002
0.0002	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0001	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0001	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0000	0.0000	0.0000
0.0000	-0.3130	-1.0746	-4.0005	-0.3130	-1.0746	-4.0005	0.0001	0.0001	0.0000

Δx	$u'_{k=10}(1)$	$u'_{k=20}(1)$	$u'_{k=50}(1)$	$\bar{u}'_{k=10}(1)$	$\bar{u}'_{k=20}(1)$	$\bar{u}'_{k=50}(1)$	$\epsilon'_{rel,k=10}$	$\epsilon'_{rel,k=20}$	$\epsilon'_{rel,k=50}$
0.5000	-26.3393	-101.4563	-626.4928	-9.0000	-19.0000	-49.0000	192.6587	433.9806	1178.5568
0.2500	-15.5270	-53.6993	-316.5025	-9.0000	-19.0000	-49.0000	72.5225	182.6279	545.9236
0.1250	-11.1738	-32.0541	-164.8880	-9.0000	-19.0000	-49.0000	24.1531	68.7056	236.5061
0.0625	-9.6515	-23.3476	-94.3114	-9.0000	-19.0000	-49.0000	7.2385	22.8818	92.4723
0.0312	-9.1809	-20.3029	-64.6828	-9.0000	-19.0000	-49.0000	2.0098	6.8575	32.0056
0.0156	-9.0479	-19.3618	-53.8484	-9.0000	-19.0000	-49.0000	0.5319	1.9040	9.8948
0.0078	-9.0123	-19.0957	-50.3751	-9.0000	-19.0000	-49.0000	0.1370	0.5039	2.8063
0.0039	-9.0031	-19.0247	-49.3686	-9.0000	-19.0000	-49.0000	0.0348	0.1298	0.7522
0.0020	-9.0008	-19.0063	-49.0956	-9.0000	-19.0000	-49.0000	0.0088	0.0329	0.1951
0.0010	-9.0002	-19.0016	-49.0244	-9.0000	-19.0000	-49.0000	0.0022	0.0083	0.0497
0.0005	-9.0000	-19.0004	-49.0061	-9.0000	-19.0000	-49.0000	0.0006	0.0021	0.0125
0.0002	-9.0000	-19.0001	-49.0015	-9.0000	-19.0000	-49.0000	0.0001	0.0005	0.0032
0.0001	-9.0000	-19.0000	-49.0004	-9.0000	-19.0000	-49.0000	0.0000	0.0001	0.0008
0.0001	-9.0000	-19.0000	-49.0001	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0002
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000
0.0000	-9.0000	-19.0000	-49.0000	-9.0000	-19.0000	-49.0000	0.0000	0.0000	0.0000

4.4.2 2nd-Order (p=2) Galerkin Method - Diffusion Equation

Table 4.4.2 – Quantity of Interest for 2nd-Order (p=2) Galerkin Method FEM for Diffusion Equation

4.4.3 1st-Order (p=1) Galerkin Method - Harmonic Wave Equation

Table 4.4.3 – Quantity of Interest for 1st-Order (p=1) Galerkin Method FEM for Harmonic Wave Equation

Δx	$u'_{k=1}(1)$	$u'_{k=2}(1)$	$u'_{k=5}(1)$	$\bar{u}'_{k=1}(1)$	$\bar{u}'_{k=2}(1)$	$\bar{u}'_{k=5}(1)$	$\epsilon'_{rel,k=1}$	$\epsilon'_{rel,k=2}$	$\epsilon'_{rel,k=5}$
0.5000	0.3864	1.7500	3.3654	0.3579	1.9153	2.4791	7.9507	8.6312	35.7522
0.2500	0.3639	1.8434	1.6860	0.3579	1.9153	2.4791	1.6636	3.7568	31.9915
0.1250	0.3592	1.8938	2.0780	0.3579	1.9153	2.4791	0.3721	1.1247	16.1781
0.0625	0.3582	1.9095	2.3553	0.3579	1.9153	2.4791	0.0874	0.3017	4.9907
0.0312	0.3580	1.9138	2.4458	0.3579	1.9153	2.4791	0.0212	0.0778	1.3437
0.0156	0.3579	1.9149	2.4705	0.3579	1.9153	2.4791	0.0052	0.0197	0.3460
0.0078	0.3579	1.9152	2.4769	0.3579	1.9153	2.4791	0.0013	0.0050	0.0876
0.0039	0.3579	1.9153	2.4785	0.3579	1.9153	2.4791	0.0003	0.0012	0.0220
0.0020	0.3579	1.9153	2.4789	0.3579	1.9153	2.4791	0.0001	0.0003	0.0055
0.0010	0.3579	1.9153	2.4790	0.3579	1.9153	2.4791	0.0000	0.0001	0.0014
0.0005	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0003
0.0002	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0001
0.0001	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0001	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0000
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0000	0.0000	0.0001
0.0000	0.3579	1.9153	2.4791	0.3579	1.9153	2.4791	0.0001	0.0002	0.0003

Δx	$u'_{k=10}(1)$	$u'_{k=20}(1)$	$u'_{k=50}(1)$	$\bar{u}'_{k=10}(1)$	$\bar{u}'_{k=20}(1)$	$\bar{u}'_{k=50}(1)$	$\epsilon'_{rel,k=10}$	$\epsilon'_{rel,k=20}$	$\epsilon'_{rel,k=50}$
0.5000	23.2955	98.4536	623.4928	-14.4235	-7.9399	184.8907	261.5103	1339.9852	237.2223
0.2500	10.0444	45.3583	308.3550	-14.4235	-7.9399	184.8907	169.6388	671.2705	66.7769
0.1250	-337.1886	12.1114	146.4432	-14.4235	-7.9399	184.8907	2237.7707	252.5390	20.7947
0.0625	-19.5262	-335.7360	41.4567	-14.4235	-7.9399	184.8907	35.3778	4128.4651	77.5777
0.0312	-15.5113	-15.2189	28.2046	-14.4235	-7.9399	184.8907	7.5418	91.6759	84.7453
0.0156	-14.6887	-9.6788	7.9312	-14.4235	-7.9399	184.8907	1.8384	21.9004	95.7103
0.0078	-14.4896	-8.3772	75.5131	-14.4235	-7.9399	184.8907	0.4583	5.5073	59.1580
0.0039	-14.4401	-8.0498	139.2302	-14.4235	-7.9399	184.8907	0.1147	1.3847	24.6960
0.0020	-14.4276	-7.9675	171.1203	-14.4235	-7.9399	184.8907	0.0287	0.3473	7.4479
0.0010	-14.4245	-7.9468	181.2596	-14.4235	-7.9399	184.8907	0.0072	0.0870	1.9639
0.0005	-14.4238	-7.9416	183.9703	-14.4235	-7.9399	184.8907	0.0018	0.0218	0.4978
0.0002	-14.4236	-7.9403	184.6598	-14.4235	-7.9399	184.8907	0.0004	0.0054	0.1249
0.0001	-14.4235	-7.9400	184.8329	-14.4235	-7.9399	184.8907	0.0001	0.0014	0.0313
0.0001	-14.4235	-7.9399	184.8763	-14.4235	-7.9399	184.8907	0.0000	0.0003	0.0078
0.0000	-14.4235	-7.9399	184.8871	-14.4235	-7.9399	184.8907	0.0000	0.0001	0.0020
0.0000	-14.4235	-7.9399	184.8898	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0005
0.0000	-14.4235	-7.9399	184.8905	-14.4235	-7.9399	184.8907	0.0000	0.0000	0.0001
0.0000	-14.4235	-7.9399	184.8908	-14.4235	-7.9399	184.8907	0.0002	0.0001	0.0000

4.4.4 2nd-Order (p=2) Galerkin Method - Harmonic Wave Equation

Table 4.4.4 – Quantity of Interest for 2nd-Order (p=2) Galerkin Method FEM for Harmonic Wave Equation

4.4.5 1st-Order (p=1) Galerkin Method - Convection-Diffusion Equation

Table 4.4.5 – Quantity of Interest for 1st-Order (p=1) Galerkin Method FEM for Convection-Diffusion Equation

Δx	$u'_{c=1}(1)$	$u'_{c=2}(1)$	$u'_{c=5}(1)$	$\bar{u}'_{c=1}(1)$	$\bar{u}'_{c=2}(1)$	$\bar{u}'_{c=5}(1)$	$\epsilon'_{rel,c=1}$	$\epsilon'_{rel,c=2}$	$\epsilon'_{rel,c=5}$
0.5000	1.6667	3.0000	-9.0000	1.5820	2.3130	5.0339	5.3534	29.6997	278.7872
0.2500	1.6022	2.4510	8.2285	1.5820	2.3130	5.0339	1.2782	5.9638	63.4604
0.1250	1.5870	2.3459	5.5727	1.5820	2.3130	5.0339	0.3160	1.4213	10.7032
0.0625	1.5832	2.3212	5.1585	1.5820	2.3130	5.0339	0.0788	0.3512	2.4744
0.0312	1.5823	2.3151	5.0645	1.5820	2.3130	5.0339	0.0197	0.0876	0.6072
0.0156	1.5821	2.3135	5.0415	1.5820	2.3130	5.0339	0.0049	0.0219	0.1511
0.0078	1.5820	2.3132	5.0358	1.5820	2.3130	5.0339	0.0012	0.0055	0.0377
0.0039	1.5820	2.3131	5.0344	1.5820	2.3130	5.0339	0.0003	0.0014	0.0094
0.0020	1.5820	2.3130	5.0340	1.5820	2.3130	5.0339	0.0001	0.0003	0.0024
0.0010	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0001	0.0006
0.0005	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0001
0.0002	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0001	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000
0.0000	1.5820	2.3130	5.0339	1.5820	2.3130	5.0339	0.0000	0.0000	0.0000

Δx	$u'_{c=10}(1)$	$u'_{c=20}(1)$	$u'_{c=50}(1)$	$\bar{u}'_{c=10}(1)$	$\bar{u}'_{c=20}(1)$	$\bar{u}'_{c=50}(1)$	$\epsilon'_{rel,c=10}$	$\epsilon'_{rel,c=20}$	$\epsilon'_{rel,c=50}$
0.5000	-2.3333	-1.5000	-1.1739	10.0005	20.0000	50.0000	123.3323	107.5000	102.3478
0.2500	-17.7805	-3.9425	-1.8118	10.0005	20.0000	50.0000	277.7968	119.7126	103.6237
0.1250	16.4104	-35.5556	-5.7325	10.0005	20.0000	50.0000	64.0964	277.7778	111.4651
0.0625	11.0826	32.8205	-34.6883	10.0005	20.0000	50.0000	10.8210	64.1026	169.3767
0.0312	10.2507	22.1645	128.3208	10.0005	20.0000	50.0000	2.5021	10.8225	156.6416
0.0156	10.0619	20.5005	59.0032	10.0005	20.0000	50.0000	0.6140	2.5025	18.0063
0.0078	10.0157	20.1228	51.9830	10.0005	20.0000	50.0000	0.1528	0.6141	3.9660
0.0039	10.0043	20.0306	50.4814	10.0005	20.0000	50.0000	0.0382	0.1528	0.9629
0.0020	10.0014	20.0076	50.1195	10.0005	20.0000	50.0000	0.0095	0.0382	0.2390
0.0010	10.0007	20.0019	50.0298	10.0005	20.0000	50.0000	0.0024	0.0095	0.0596
0.0005	10.0005	20.0005	50.0075	10.0005	20.0000	50.0000	0.0006	0.0024	0.0149
0.0002	10.0005	20.0001	50.0019	10.0005	20.0000	50.0000	0.0001	0.0006	0.0037
0.0001	10.0005	20.0000	50.0005	10.0005	20.0000	50.0000	0.0000	0.0001	0.0009
0.0001	10.0005	20.0000	50.0001	10.0005	20.0000	50.0000	0.0000	0.0000	0.0002
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0001
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000
0.0000	10.0005	20.0000	50.0000	10.0005	20.0000	50.0000	0.0000	0.0000	0.0000

4.4.6 2nd-Order (p=2) Galerkin Method - Convection-Diffusion Equation

Table 4.4.6 – Quantity of Interest for 2nd-Order (p=2) Galerkin Method FEM for Convection-Diffusion Equation

5 Convergence Analysis

5.1 Rate of Convergence Derivation

Let the error for a particular mesh size Δx be $E(\Delta x)$:

$$E(\Delta x) = C(\Delta x)^{\beta} \tag{5.1}$$

Then for a smaller mesh size $\frac{\Delta x}{2}$ we have:

$$E\left(\frac{\Delta x}{2}\right) = C\left(\frac{\Delta x}{2}\right)^{\beta} \tag{5.2}$$

Dividing the error at each mesh size and taking the logarithm:

$$\frac{E(\Delta x)}{E\left(\frac{\Delta x}{2}\right)} = \frac{C(\Delta x)^{\beta}}{C\left(\frac{\Delta x}{2}\right)^{\beta}} = 2^{\beta} \tag{5.3}$$

$$\log\left[\frac{E(\Delta x)}{E\left(\frac{\Delta x}{2}\right)}\right] = \log(2^{\beta}) \tag{5.4}$$

$$\log\left[\frac{E(\Delta x)}{E\left(\frac{\Delta x}{2}\right)}\right] = \beta \log(2) \tag{5.5}$$

Rearranging for β and simplifying:

$$\beta = \frac{1}{\log(2)}\left[\log\left(E(\Delta x)\right) - \log\left(E\left(\frac{\Delta x}{2}\right)\right)\right] \tag{5.6}$$

Denoting $E^{*}_{\Delta x} = \log\left(E(\Delta x)\right)$:

$$\beta = \frac{\mathbf{E}^{*}_{\boldsymbol{\Delta}\mathbf{x}} - \mathbf{E}^{*}_{\frac{\boldsymbol{\Delta}\mathbf{x}}{\mathbf{2}}}}{\log(\mathbf{2})} \tag{5.7}$$

5.2 Rate of Convergence for the Diffusion Equation – Results

5.2.1 2nd-Order Central Difference Scheme Finite Difference Method

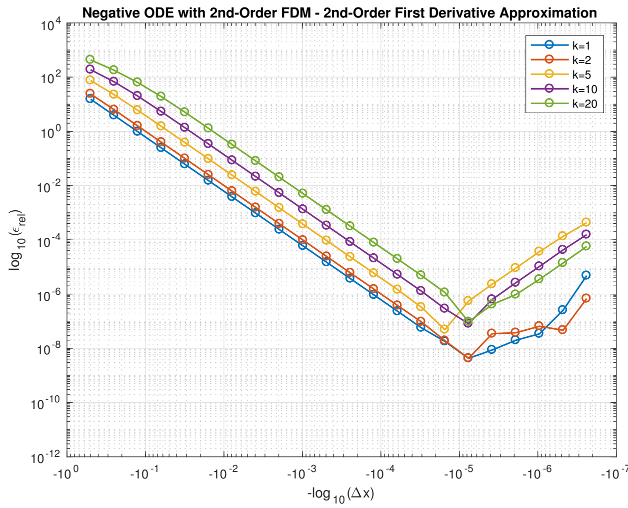


Figure 5.2.1 – 2nd-Order CDS FDM for the Diffusion Equation

Δx	$\beta(k=1)$	$\beta(k=2)$	$\beta(k=5)$	$\beta(k=10)$	$\beta(k=20)$
0.5000	1.9889	1.9470	1.7457	1.4943	1.2761
0.2500	1.9972	1.9856	1.9103	1.7449	1.4943
0.1250	1.9993	1.9963	1.9748	1.9102	1.7449
0.0625	1.9998	1.9991	1.9935	1.9747	1.9102
0.0312	2.0000	1.9998	1.9984	1.9935	1.9747
0.0156	2.0000	1.9999	1.9996	1.9984	1.9935
0.0078	2.0000	2.0000	1.9999	1.9996	1.9984
0.0039	2.0000	2.0000	2.0000	1.9999	1.9996
0.0020	2.0000	2.0000	2.0000	2.0000	1.9999
0.0010	2.0000	2.0000	2.0000	2.0000	2.0000
0.0005	2.0000	2.0000	2.0000	2.0000	2.0000
0.0002	1.9998	2.0000	2.0005	2.0000	2.0000
0.0001	2.0050	2.0021	2.0089	2.0002	2.0004
0.0001	2.0069	1.9918	2.1377	2.0059	2.0058
0.0000	1.6814	2.3036	2.7592	2.1640	2.1067
0.0000	2.0635	2.1656	-3.5045	1.8302	3.5754
0.0000	-1.0192	-3.0149	-2.0596	-2.9679	-2.1197

Table 5.2.1 – 2nd-Order CDS FDM for the Diffusion Equation – Rate of Convergence Values

5.2.2 4th-Order Central Difference Scheme Finite Difference Method

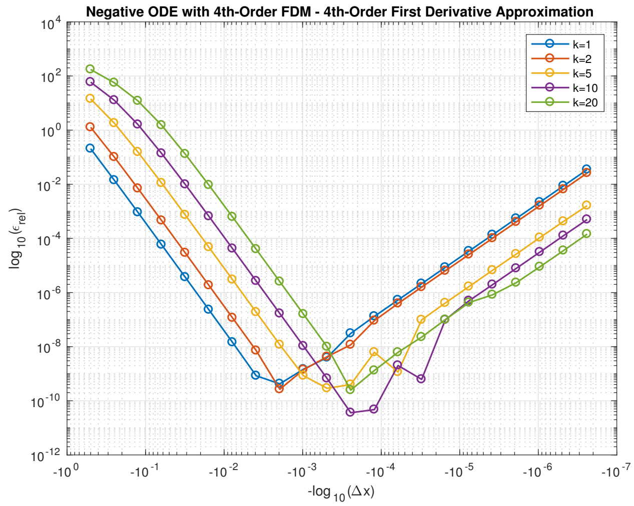


Figure 5.2.2 – 4th-Order CDS FDM for the Diffusion Equation

Δx	$\beta(k=1)$	$\beta(k=2)$	$\beta(k=5)$	$\beta(k=10)$	$\beta(k=20)$
0.5000	3.8716	3.6535	2.9842	2.2242	1.6296
0.2500	3.9420	3.8513	3.5408	2.9844	2.2242
0.1250	3.9726	3.9328	3.8045	3.5408	2.9844
0.0625	3.9867	3.9683	3.9132	3.8045	3.5408
0.0312	3.9935	3.9846	3.9595	3.9132	3.8045
0.0156	3.9998	3.9927	3.9805	3.9595	3.9132
0.0078	4.1029	4.0149	3.9904	3.9805	3.9595
0.0039	0.9883	4.7505	3.9985	3.9904	3.9805
0.0020	-1.7914	-2.3986	3.7985	3.9952	3.9905
0.0010	-1.4284	-1.5994	1.5477	3.9782	4.0083
0.0005	-2.9683	-1.4594	-0.4553	4.2002	5.3157
0.0002	-2.0946	-2.9623	-3.9648	-0.3806	-2.4106
0.0001	-2.0061	-2.1071	2.4296	-5.4173	-2.2287
0.0001	-2.0024	-2.0068	-6.4048	1.7062	-1.8689
0.0000	-2.0007	-1.9995	-2.0766	-7.3010	-2.1411
0.0000	-1.9983	-1.9971	-2.0075	-2.3425	-2.0900
0.0000	-2.0005	-2.0023	-2.0052	-1.9815	-0.9647

Table 5.2.2 – 4th-Order CDS FDM for the Diffusion Equation – Rate of Convergence Values

5.2.3 1st-Order (p=1) Galerkin Method Finite Element Method

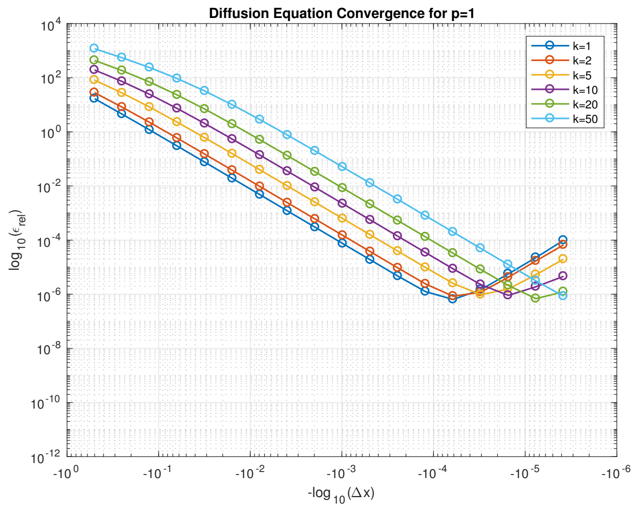


Figure 5.2.3 – 1st-Order (p=1) Galerkin Method FEM for the Diffusion Equation

Δx	$\beta(k=1)$	$\beta(k=2)$	$\beta(k=5)$	$\beta(k=10)$	$\beta(k=20)$	$\beta(k=50)$
0.5000	1.8987	1.7788	1.5862	1.4095	1.2487	1.1103
0.2500	1.9441	1.8755	1.7384	1.5862	1.4104	1.2068
0.1250	1.9706	1.9334	1.8486	1.7385	1.5862	1.3548
0.0625	1.9849	1.9655	1.9177	1.8486	1.7385	1.5307
0.0312	1.9924	1.9824	1.9570	1.9177	1.8486	1.6936
0.0156	1.9962	1.9911	1.9780	1.9570	1.9177	1.8180
0.0078	1.9981	1.9955	1.9889	1.9780	1.9570	1.8994
0.0039	1.9990	1.9978	1.9944	1.9889	1.9780	1.9468
0.0020	1.9995	1.9989	1.9972	1.9944	1.9889	1.9726
0.0010	1.9994	1.9993	1.9986	1.9972	1.9944	1.9861
0.0005	1.9935	1.9974	1.9991	1.9986	1.9972	1.9930
0.0002	1.9009	1.9623	1.9965	1.9991	1.9986	1.9965
0.0001	0.9568	1.5020	1.9511	1.9948	1.9992	1.9982
0.0001	-1.2034	-0.5004	1.3822	1.9252	1.9977	1.9991
0.0000	-1.9372	-1.8261	-0.6943	1.3256	1.9487	1.9982
0.0000	-1.9999	-2.0043	-1.7673	-1.0580	1.6239	1.9854
0.0000	-2.1171	-1.9680	-1.8780	-1.2753	-0.8459	1.8866

Table 5.2.3 – 1st-Order (p=1) Galerkin Method FEM for the Diffusion Equation – Rate of Convergence Values

5.2.4 2nd-Order (p=2) Galerkin Method Finite Element Method

5.3 Rate of Convergence for the Harmonic Wave Equation – Results

5.3.1 2nd-Order Central Difference Scheme Finite Difference Method

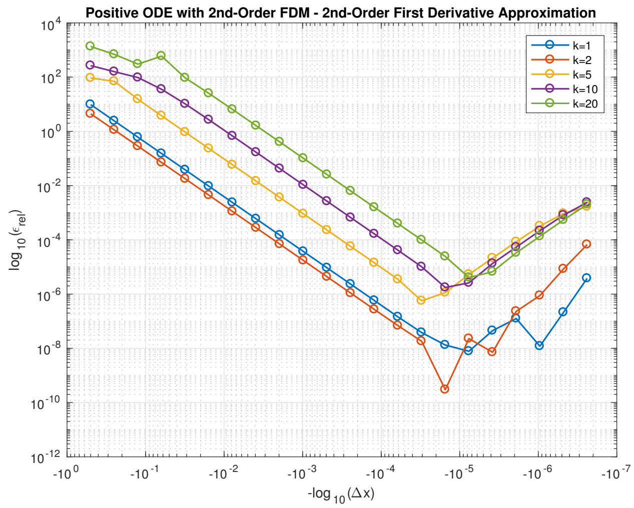


Figure 5.3.1 – 2nd-Order CDS FDM for the Harmonic Wave Equation

Δx	$\beta(k=1)$	$\beta(k=2)$	$\beta(k=5)$	$\beta(k=10)$	$\beta(k=20)$
0.5000	2.0054	1.9671	0.4081	0.7419	0.9652
0.2500	2.0013	1.9934	2.1958	0.7224	1.1939
0.1250	2.0003	1.9984	2.0244	1.4357	-0.9810
0.0625	2.0001	1.9996	2.0052	1.7938	2.6702
0.0312	2.0000	1.9999	2.0012	1.9423	1.8957
0.0156	2.0000	2.0000	2.0003	1.9851	1.9649
0.0078	2.0000	2.0000	2.0001	1.9963	1.9907
0.0039	2.0000	2.0000	2.0000	1.9991	1.9976
0.0020	2.0000	2.0000	2.0000	1.9998	1.9994
0.0010	2.0000	2.0000	2.0000	1.9999	1.9999
0.0005	2.0001	2.0001	2.0001	2.0000	2.0000
0.0002	2.0001	2.0048	2.0019	2.0001	2.0000
0.0001	2.0078	1.9828	2.0304	2.0017	2.0001
0.0001	1.9164	1.9109	2.6199	2.0278	2.0017
0.0000	1.5266	5.9278	-0.9987	2.5302	2.0282
0.0000	0.7639	-6.2533	-2.2282	-0.5680	2.5725
0.0000	-2.5225	1.6689	-2.0098	-2.3722	-0.7079

Table 5.3.1 – 2nd-Order CDS FDM for the Harmonic Wave Equation – Rate of Convergence Values

5.3.2 4th-Order Central Difference Scheme Finite Difference Method

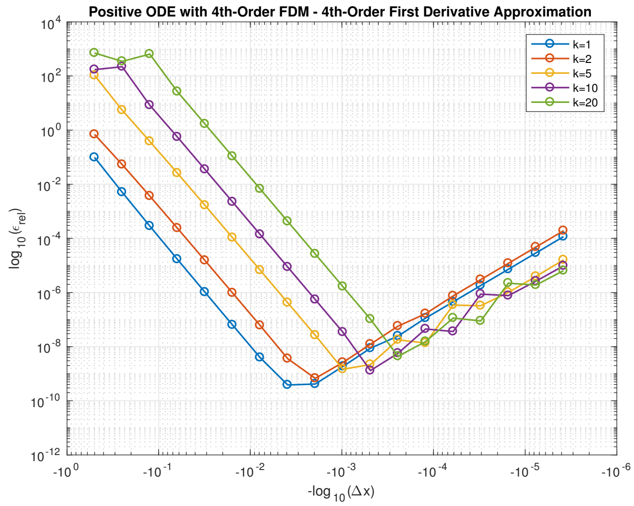


Figure 5.3.2 – 4th-Order CDS FDM for the Harmonic Wave Equation

Δx	$\beta(k=1)$	$\beta(k=2)$	$\beta(k=5)$	$\beta(k=10)$	$\beta(k=20)$
0.5000	4.2697	3.7001	4.2652	-0.3617	1.0221
0.2500	4.1474	3.8644	3.8316	4.7243	-0.8863
0.1250	4.0781	3.9346	3.8964	3.9026	4.5981
0.0625	4.0403	3.9678	3.9468	3.9791	3.9754
0.0312	4.0204	3.9840	3.9732	3.9887	3.9786
0.0156	3.9991	3.9929	3.9865	3.9936	3.9863
0.0078	3.3645	4.0749	3.9931	3.9966	3.9923
0.0039	-0.1136	2.4190	3.9951	3.9980	3.9959
0.0020	-2.1577	-1.9509	4.1698	3.9950	3.9978
0.0010	-2.2335	-2.2401	-0.5720	4.7185	3.9969
0.0005	-1.4971	-2.2205	-3.0356	-2.1267	4.5637
0.0002	-2.2025	-1.5230	0.4008	-2.9793	-1.8116
0.0001	-1.9936	-2.1858	-4.6688	0.3097	-2.8856
0.0001	-2.0067	-2.0021	0.0828	-4.5914	0.3558
0.0000	-1.9920	-1.9930	-1.6301	0.1885	-4.6359
0.0000	-2.0051	-2.0010	-1.9642	-1.7750	0.2354
0.0000	-1.9994	-2.0026	-2.0299	-1.8824	-1.7738

Table 5.3.2 – 2nd-Order CDS FDM for the Harmonic Wave Equation – Rate of Convergence Values

5.3.3 1st-Order (p=1) Galerkin Method Finite Element Method

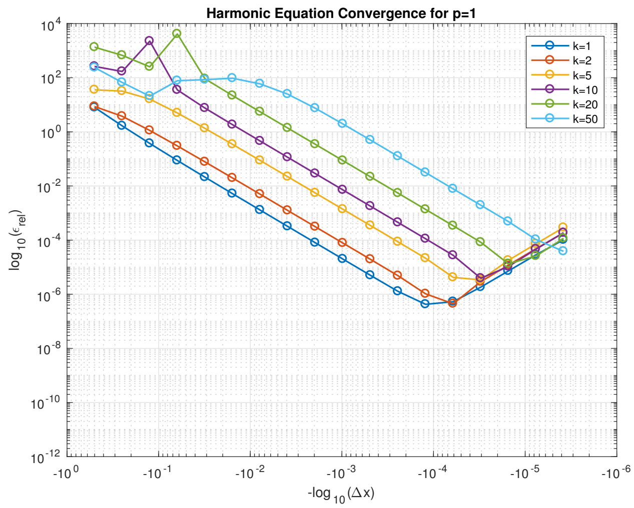


Figure 5.3.3 – 1st-Order (p=1) Galerkin Method FEM for the Harmonic Wave Equation

Δx	$\beta(k=1)$	$\beta(k=2)$	$\beta(k=5)$	$\beta(k=10)$	$\beta(k=20)$	$\beta(k=50)$
0.5000	2.2568	1.2001	0.1603	0.6244	0.9973	1.8288
0.2500	2.1604	1.7400	0.9836	-3.7215	1.4104	1.6831
0.1250	2.0896	1.8981	1.6967	5.9831	-4.0310	-1.8994
0.0625	2.0474	1.9555	1.8931	2.2299	5.4929	-0.1275
0.0312	2.0244	1.9793	1.9572	2.0364	2.0656	-0.1755
0.0156	2.0124	1.9900	1.9812	2.0040	1.9915	0.6941
0.0078	2.0062	1.9951	1.9912	1.9988	1.9918	1.2603
0.0039	2.0031	1.9976	1.9958	1.9986	1.9953	1.7294
0.0020	2.0014	1.9988	1.9979	1.9991	1.9975	1.9231
0.0010	1.9989	2.0002	1.9990	1.9995	1.9987	1.9800
0.0005	1.9701	2.0130	2.0005	1.9999	1.9994	1.9949
0.0002	1.5857	2.2235	2.0164	2.0016	1.9998	1.9987
0.0001	-0.3203	1.2131	2.3435	2.0315	2.0013	1.9997
0.0001	-1.7873	-2.7047	0.3143	2.8149	2.0257	2.0007
0.0000	-1.9821	-2.0352	-2.4024	-1.4364	2.6411	2.0090
0.0000	-1.9793	-1.9996	-2.0249	-2.1288	-0.9494	2.2074
0.0000	-1.8440	-1.9702	-1.9960	-2.0128	-2.1645	1.4451

Table 5.3.3 – 1st-Order (p=1) Galerkin Method FEM for the Harmonic Wave Equation – Rate of Convergence Values

5.3.4 2nd-Order (p=2) Galerkin Method Finite Element Method

5.4 Rate of Convergence for the Convection-Diffusion Equation – Results

5.4.1 2nd-Order Central Difference Scheme Finite Difference Method

Note: The quantity of interest for the 2nd-order CDS FDM is extracted using the 2nd-order first-derivative extraction - yielding the quantity of interest with an error of $\mathcal{O}(\Delta x^2)$.

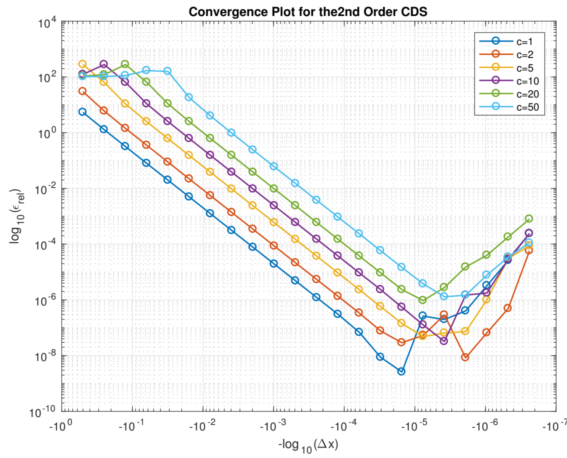


Figure 5.4.1 – 2nd-Order CDS FDM for the Convection-Diffusion Equation

Δx	$\beta(c=1)$	$\beta(c=2)$	$\beta(c=5)$	$\beta(c=10)$	$\beta(c=20)$	$\beta(c=50)$
0.5000	2.0663	2.3161	2.1352	-1.1715	-0.1552	-0.0179
0.2500	2.0161	2.0691	2.5678	2.1157	-1.2144	-0.1052
0.1250	2.0040	2.0168	2.1129	2.5664	2.1155	-0.6036
0.0625	2.0010	2.0042	2.0269	2.1126	2.5663	0.1128
0.0312	2.0002	2.0010	2.0066	2.0268	2.1126	3.1209
0.0156	2.0001	2.0003	2.0017	2.0066	2.0268	2.1828
0.0078	2.0000	2.0001	2.0004	2.0017	2.0066	2.0423
0.0039	2.0000	2.0000	2.0001	2.0004	2.0017	2.0104
0.0020	2.0000	2.0000	2.0000	2.0001	2.0004	2.0026
0.0010	1.9999	2.0000	2.0000	2.0000	2.0001	2.0006
0.0005	2.0004	1.9999	2.0000	2.0000	2.0000	2.0002
0.0002	1.9929	2.0000	2.0001	2.0000	2.0000	2.0000
0.0001	2.1500	1.9898	2.0005	2.0000	2.0000	2.0000
0.0001	2.9660	2.1314	2.0049	1.9994	1.9996	1.9999
0.0000	1.7543	1.4102	2.0078	2.0924	1.9906	1.9967
0.0000	-6.6673	-0.8937	1.5642	2.1065	1.3098	1.9564
0.0000	0.4112	-2.4168	-0.4198	1.9889	-1.5473	1.5401
0.0000	-1.0256	5.1013	-0.1715	-5.5431	-2.4479	-0.1831
0.0000	-3.0136	-2.9764	-3.7817	-0.2395	-1.4021	-2.3934
0.0000	-3.0056	-2.8983	-4.9316	-3.9962	-2.1755	-2.1561
0.0000	-3.2026	-6.8772	-1.5264	-3.1065	-2.1369	-1.6780

Table 5.4.1 – 2nd-Order CDS FDM for the Convection-Diffusion Equation – Rate of Convergence Values

5.4.2 4th-Order Central Difference Scheme Finite Difference Method

Note: The quantity of interest for the 4th-order CDS FDM is extracted using the 4th-order first-derivative extraction - yielding the quantity of interest with an error of $\mathcal{O}(\Delta x^4)$.

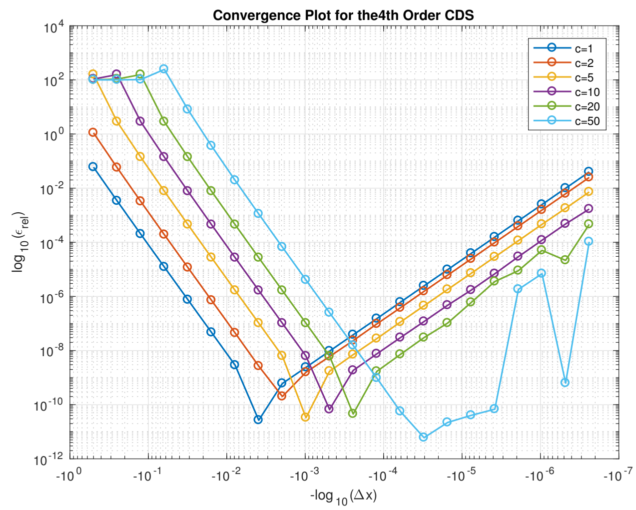


Figure 5.4.2 – 4th-Order CDS FDM for the Convection-Diffusion Equation

Δx	$\beta(c=1)$	$\beta(c=2)$	$\beta(c=5)$	$\beta(c=10)$	$\beta(c=20)$	$\beta(c=50)$
0.5000	4.1405	4.2684	5.7890	-0.5483	-0.0944	-0.0011
0.2500	4.0749	4.1459	4.3454	5.7890	-0.5492	-0.0322
0.1250	4.0389	4.0782	4.1839	4.3464	5.7889	-1.2503
0.0625	4.0198	4.0407	4.1005	4.1847	4.3464	4.9348
0.0312	4.0103	4.0208	4.0529	4.1010	4.1848	4.4625
0.0156	4.0233	4.0112	4.0272	4.0532	4.1010	4.2233
0.0078	6.7476	4.0609	4.0142	4.0274	4.0532	4.1232
0.0039	-4.5113	3.7287	4.0311	4.0143	4.0274	4.0657
0.0020	-1.9775	-2.9750	7.5728	4.0329	4.0143	4.0340
0.0010	-1.9979	-1.9307	-5.7027	6.5530	4.0323	4.0175
0.0005	-1.9992	-1.9571	-2.0388	-4.7944	7.1048	4.0140
0.0002	-1.9781	-2.0422	-1.9624	-2.0272	-5.2245	4.0011
0.0001	-2.0174	-1.9810	-2.0355	-1.9977	-2.0860	4.1025
0.0001	-2.0013	-2.0205	-1.9915	-1.9714	-2.0585	3.2178
0.0000	-1.9917	-1.9825	-2.0030	-2.0012	-1.7842	-1.8639
0.0000	-2.0022	-1.9949	-1.9876	-1.8558	-2.5308	-0.8759
0.0000	-2.0021	-2.0102	-2.0060	-2.0094	-2.5697	-0.7555
0.0000	-1.9991	-1.9939	-1.9989	-2.0981	-1.3343	-14.7059
0.0000	-1.9985	-1.9998	-2.0049	-2.0182	-2.4899	-1.9214
0.0000	-1.9986	-1.9998	-1.9684	-2.0010	1.2231	13.4265
0.0000	-1.9973	-1.9968	-1.9979	-1.8221	-4.4135	-17.3388

Table 5.4.2 – 4th-Order CDS FDM for the Convection-Diffusion Equation – Rate of Convergence Values

5.4.3 1st-Order (p=1) Galerkin Method Finite Element Method

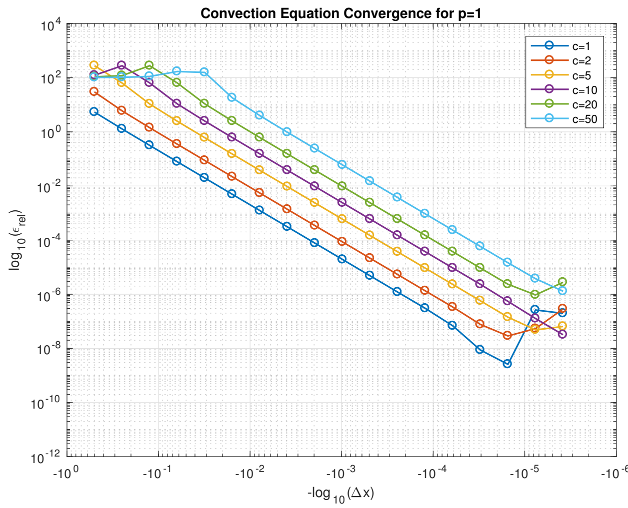


Figure 5.4.3 – 1st-Order (p=1) Galerkin Method FEM for the Convection-Diffusion Equation

Δx	$\beta(c=1)$	$\beta(c=2)$	$\beta(c=5)$	$\beta(c=10)$	$\beta(c=20)$	$\beta(c=50)$
0.5000	2.0663	2.3161	2.1352	-1.1715	-0.1552	-0.0179
0.2500	2.0161	2.0691	2.5678	2.1157	-1.2144	-0.1052
0.1250	2.0040	2.0168	2.1129	2.5664	2.1155	-0.6036
0.0625	2.0010	2.0042	2.0269	2.1126	2.5663	0.1128
0.0312	2.0002	2.0010	2.0066	2.0268	2.1126	3.1209
0.0156	2.0001	2.0003	2.0017	2.0066	2.0268	2.1828
0.0078	2.0000	2.0001	2.0004	2.0017	2.0066	2.0423
0.0039	2.0000	2.0000	2.0001	2.0004	2.0017	2.0104
0.0020	2.0000	2.0000	2.0000	2.0001	2.0004	2.0026
0.0010	1.9999	2.0000	2.0000	2.0000	2.0001	2.0006
0.0005	2.0004	1.9999	2.0000	2.0000	2.0000	2.0002
0.0002	1.9929	2.0000	2.0001	2.0000	2.0000	2.0000
0.0001	2.1500	1.9898	2.0005	2.0000	2.0000	2.0000
0.0001	2.9660	2.1314	2.0049	1.9994	1.9996	1.9999
0.0000	1.7543	1.4102	2.0078	2.0924	1.9906	1.9967
0.0000	-6.6673	-0.8937	1.5642	2.1065	1.3098	1.9564
0.0000	0.4112	-2.4168	-0.4198	1.9889	-1.5473	1.5401

Table 5.4.3 – 1st-Order (p=1) Galerkin Method FEM for the Convection-Diffusion Equation – Rate of Convergence Values

5.4.4 2nd-Order (p=2) Galerkin Method Finite Element Method

6 Discussion

A critical view of the first-order Galerkin method FEM solution results and quantity of interest convergence shows that the rate of convergence β is approximately 2. For the diffusion equation, the first-order Galerkin method FEM has significant error at small mesh size, but decreases quickly until the error tolerance is reached. For the harmonic wave equation, the first-order Galerkin method FEM has significant error at small mesh sizes and in some cases, the error increase due to eigenvalue resonances, but then decreases quickly until the error tolerance is reached. Finally for the convection-diffusion equation, there are large initial errors at small mesh sizes. Once mesh sizes are sufficiently small, error begins to decrease and then plateaus at the error tolerance.

A simultaneous comparison of all of the methods for various values of k and c proves that:

- the second-order central difference scheme finite difference method is quadratically convergent
- the fourth-order central difference scheme finite difference method is quartically convergent
- the first-order Galerkin method finite element method is quadratically convergent

Further analysis would likely yield that the second-order Galerkin method is quartically convergent.

A $u(x)$ v. $u_{exact}(x)$ Tables

Error tables are available for all mesh spacings through the MATLAB code or by request (ross.alexander19@tamu.edu). The tables are not printed here due to printing constraints.

B MATLAB Code

```
clc; close all; clear all;

%odeString = 'diffusion_p_1';
%odeString = 'harmonic_p_1';
odeString = 'convection_p_1';

%odeType = 'Diffusion';
%odeType = 'Harmonic';
odeType = 'Convection';

meshOrder   = 1:18;
meshDx      = 0.5.^meshOrder;

rowID = 0;

plotGen = true;
plotSave = true;
tableSaveUx = false;
tableSaveRoc = false;
tableSaveError = false;

%% FEM for p = 1

for k = [1 2 5 10 20 50]

rowID = rowID + 1;
colID = 0;

if plotGen

fig1 = figure(1);
xlabel('x');     ylabel('u(x)');
grid on;         grid minor;
box on;          hold on;
set(gcf, 'Position', [1 1 624 550])

if strcmpi(odeType, 'diffusion')
titleString = strcat('p=1 Galerkin FEM for the Diffusion Equation for k=', num2str(k));
elseif strcmpi(odeType, 'harmonic')
titleString = strcat('p=1 Galerkin FEM for the Harmonic Equation for k=', num2str(k));
elseif strcmpi(odeType, 'convection')
titleString = strcat('p=1 Galerkin FEM for the Convection-Diffusion Equation for c=', num2str(k));
end

title(titleString)

fig2 = figure(2);
xlabel('x');     ylabel('\epsilon_{rel}');
grid on;         grid minor;
box on;          hold on;
set(gcf, 'Position', [1 1 624 550])

title(titleString)

end

for dx = meshDx
```

```
nx = 1 / dx + 1;
x = linspace(0, 1, nx);
f = zeros(nx, 1);
colID = colID + 1;

for i = 2:nx-1
if strcmpi('diffusion', odeType)
f(i) = k^2*dx^2/6*(x(i-1)+4*x(i)+x(i+1));
elseif strcmpi('harmonic', odeType)
f(i) = k^2*dx^2/6*(x(i-1)+4*x(i)+x(i+1));
elseif strcmpi('convection', odeType)
f(i) = 0;
end
end

if strcmpi('diffusion', odeType)
alpha = -1 + k^2*dx^2/6;
beta  = 2 + 4*k^2*dx^2/6;
gamma = -1 + k^2*dx^2/6;
elseif strcmpi('harmonic', odeType)
alpha = 1 + k^2*dx^2/6;
beta  = -2 + 4*k^2*dx^2/6;
gamma = 1 + k^2*dx^2/6;
elseif strcmpi('convection', odeType)
alpha = -1 - k*dx/2;
beta  = 2;
gamma = -1 + k*dx/2;
end

A = gallery('tridiag', nx, alpha, beta, gamma);

A(1, 1) = 1;    A(1, 2) = 0;
A(nx, nx) = 1;  A(nx, nx-1) = 0;

if strcmpi('diffusion', odeType)
f(1) = 0;       f(nx) = 0;
elseif strcmpi('harmonic', odeType)
f(1) = 0;       f(nx) = 0;
elseif strcmpi('convection', odeType)
f(1) = 0;       f(nx) = 1;
end

u = A\f;

if plotGen && dx >= meshDx(8)

figure(1)
plot(x, u, 'linewidth', 1)

end

if strcmpi(odeType, 'diffusion')
dudx.fem(rowID, colID) = - u(end-1) / dx - dx * k^2 / 2;
dudx.exact(rowID, colID) = 1 - k * cosh(k) / sinh(k);
ux.exact = x - sinh(k*x) ./ sinh(k);
elseif strcmpi(odeType, 'harmonic')
dudx.fem(rowID, colID) = - u(end-1) / dx + dx * k^2 / 2;
ux.exact = x - sin(k*x) ./ sin(k);
dudx.exact(rowID, colID) = 1 - k * cos(k) / sin(k);
elseif strcmpi(odeType, 'convection')
dudx.fem(rowID, colID) = (1 -k*dx/2)^-1 * (1 - u(end-1)) / dx;
```

```
ux.exact = (exp(k*x) - 1) / (exp(k) - 1);
dudx.exact(rowID, colID) = k*exp(k) / (exp(k) - 1);
end

if plotGen && dx >= meshDx(8)

figure(2)
plot(x(2:end-1), abs(ux.exact(2:end-1)'-u(2:end-1)) ./ ...
abs(ux.exact(2:end-1)')*100, 'o-')

end

if tableSaveUx && dx >= meshDx(5)

colLabels = {'$x$', '$u(x)$', '$\bar{u}(x)$', '$\epsilon_{rel}$'};

matrix2latex([x' u ux.exact' (abs(ux.exact'-u)./abs(ux.exact')*100)], ...
strcat('pointwise_error_table_dx_', num2str(nx), '_', odeString, '.tex'), ...
'columnLabels', colLabels, 'alignment', 'c', 'format', '%1.2e')

end

end

if plotGen

figure(1)

if strcmpi(odeType, 'diffusion')
fplot(@(x) x-sinh(k*x)/sinh(k), [0 1], '-k', 'linewidth', 1.5)
elseif strcmpi(odeType, 'harmonic')
fplot(@(x) x-sin(k*x)/sin(k), [0 1], '-k', 'linewidth', 1.5)
elseif strcmpi(odeType, 'convection')
fplot(@(x) (exp(k*x)-1)/(exp(k) - 1), [0 1], '-k', 'linewidth', 1.5)
end

%             legend('\Deltax = (1/2)^1', '\Deltax = (1/2)^2', '\Deltax = (1/2)^3', ...
%                 '\Deltax = (1/2)^4', '\Deltax = (1/2)^5', '\Deltax = (1/2)^6', ...
%                 '\Deltax = (1/2)^7', '\Deltax = (1/2)^8', 'Analytical Solution', ...
%                 'location', 'eastoutside')

drawnow

figure(2)

%             legend('\Deltax = (1/2)^1', '\Deltax = (1/2)^2', '\Deltax = (1/2)^3', ...
%                 '\Deltax = (1/2)^4', '\Deltax = (1/2)^5', '\Deltax = (1/2)^6', ...
%                 '\Deltax = (1/2)^7', '\Deltax = (1/2)^8', 'location', 'eastoutside')
set(gca, 'YScale', 'log')

drawnow

if plotSave

figure(1)
figureString = strcat('solution_', lower(odeString), '_k_', num2str(k));
saveas(gcf, figureString, 'epsc')

figure(2)
figureString = strcat('pointwise_error_', lower(odeString), '_k_', num2str(k));
saveas(gcf, figureString, 'epsc')
```

```
close gcf;  close gcf

end

end

end

%% Convergence Analysis

relError = abs(dudx.exact-dudx.fem) ./ abs(dudx.exact) * 100;

if plotGen

figure
xlabel('-log_{10}(\Deltax)');   ylabel('log_{10}(\epsilon_{rel})');
grid on;                        grid minor;
box on;                         hold on;
ylim([10^-12 10^4])

for kID = 1:6
loglog(-meshDx, relError(kID, :), '-o', 'linewidth', 1.25);
end

if strcmpi(odeType, 'diffusion')
titleString = strcat(odeType, ' Equation Convergence for p=1');
elseif strcmpi(odeType, 'harmonic')
titleString = strcat(odeType, ' Equation Convergence for p=1');
elseif strcmpi(odeType, 'convection')
titleString = strcat(odeType, ' Equation Convergence for p=1');
end

title(titleString)

if ~strcmpi(odeType, 'convection')
legend('k=1', 'k=2', 'k=5', 'k=10', 'k=20', 'k=50')
else
legend('c=1', 'c=2', 'c=5', 'c=10', 'c=20', 'c=50')
end

set(gca, 'XScale', 'log');  set(gca, 'YScale', 'log');
drawnow

if plotSave

figureString = strcat('convergence_', odeString);
saveas(gcf, figureString, 'epsc')
close gcf

end

end

%% Rate of Convergence Analysis

logRelError = log10(relError);

for kID = 1:6

for rocID = 1:length(logRelError) - 1
roc(kID, rocID) = (logRelError(kID, rocID+1) - logRelError(kID, rocID)) / -log10(2);
end
```

```
end

if ~strcmpi(odeType, 'convection')
colLabels = {'$\Delta x$', '$\beta(k=1)$', '$\beta(k=2)$', '$\beta(k=5)$', ...
'$\beta(k=10)$', '$\beta(k=20)$', '$\beta(k=50)$'};
else
colLabels = {'$\Delta x$', '$\beta(c=1)$', '$\beta(c=2)$', '$\beta(c=5)$', ...
'$\beta(c=10)$', '$\beta(c=20)$', '$\beta(c=50)$'};
end

if tableSaveRoc

matrix2latex([meshDx(1:17)' roc'], strcat('roc_', odeString, '.tex'), ...
'columnLabels', colLabels, 'alignment', 'c', 'format', '%5.4f')

end

if ~strcmpi(odeType, 'convection')
colLabels1 = {'$\Delta x$', '$u''_{k=1}(1)$', '$u''_{k=2}(1)$', '$u''_{k=5}(1)$', ...
'$\bar{u}''_{k=1}(1)$', '$\bar{u}''_{k=2}(1)$', '$\bar{u}''_{k=5}(1)$', ...
'$\epsilon''_{rel,k=1}$', '$\epsilon''_{rel,k=2}$', '$\epsilon''_{rel,k=5}$'};
colLabels2 = {'$\Delta x$', '$u''_{k=10}(1)$', '$u''_{k=20}(1)$', '$u''_{k=50}(1)$', ...
'$\bar{u}''_{k=10}(1)$', '$\bar{u}''_{k=20}(1)$', '$\bar{u}''_{k=50}(1)$', ...
'$\epsilon''_{rel,k=10}$', '$\epsilon''_{rel,k=20}$', '$\epsilon''_{rel,k=50}$'};
else
colLabels1 = {'$\Delta x$', '$u''_{c=1}(1)$', '$u''_{c=2}(1)$', '$u''_{c=5}(1)$', ...
'$\bar{u}''_{c=1}(1)$', '$\bar{u}''_{c=2}(1)$', '$\bar{u}''_{c=5}(1)$', ...
'$\epsilon''_{rel,c=1}$', '$\epsilon''_{rel,c=2}$', '$\epsilon''_{rel,c=5}$'};
colLabels2 = {'$\Delta x$', '$u''_{c=10}(1)$', '$u''_{c=20}(1)$', '$u''_{c=50}(1)$', ...
'$\bar{u}''_{c=10}(1)$', '$\bar{u}''_{c=20}(1)$', '$\bar{u}''_{c=50}(1)$', ...
'$\epsilon''_{rel,c=10}$', '$\epsilon''_{rel,c=20}$', '$\epsilon''_{rel,c=50}$'};
end

if tableSaveError

matrix2latex([meshDx(1:end)' dudx.fem(1:3,:)', dudx.exact(1:3, :)', relError(1:3, :)'], ...
strcat('qoi_1_table_', lower(odeString), '.tex'), 'columnLabels', colLabels1, ...
'alignment', 'c', 'format', '%5.4f')
matrix2latex([meshDx(1:end)' dudx.fem(4:6,:)', dudx.exact(4:6, :)', relError(4:6, :)'], ...
strcat('qoi_2_table_', lower(odeString), '.tex'), 'columnLabels', colLabels2, ...
'alignment', 'c', 'format', '%5.4f')

end
```