University of California, Berkeley
Department of EECS
EE120: SIGNALS AND SYSTEMS (Spring 2021)
Homework 0 Solutions

Issued: January 22, 2021 **Due:** 11:59 PM, January 29, 2021

Problem A. Euler formula. The Euler's formula is

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

1. Derive the following identities using Euler's formula
   (a) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$
   (b) $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
2. Derive **de Moivre's theorem**: for any real number θ, integer n,
$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$
3. Show that any linear combination of a set of sine waves of frequency ω is always a sine wave of the same frequency, *even if* each sine wave has a distinct phase. In particular, show that
$$\sum_{k=1}^{N} A_k \cos(\omega t + \phi_k) = A\cos(\omega t + \phi).$$

Solution:

1. (a) $\cos\theta = \cos\theta + i\frac{1}{2}\sin\theta - i\frac{1}{2}\sin\theta = \frac{\cos\theta + i\sin\theta}{2} + \frac{\cos -\theta + i\sin -\theta}{2} = \frac{e^{i\theta}+e^{-i\theta}}{2}$
   (b) $\sin\theta = \sin\theta + \frac{1}{2i}\cos\theta - i\frac{1}{2i}\cos\theta = \frac{\cos\theta + i\sin\theta}{2i} + \frac{-\cos\theta + i\sin\theta}{2i} = \frac{e^{i\theta}-e^{-i\theta}}{2i}$
2. $(\cos(\theta) + i\sin(\theta))^n = (e^{i\theta})^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta)$
3. To show the result, we need to prove there exist a certain A and a certain ϕ, such that the equation holds.

   Consider n complex numbers written in polar form as follows:
$$A_1 e^{i\phi_1}, \ldots, A_k e^{i\phi_k}, \ldots, A_n e^{i\phi_n}.$$

Adding these complex numbers together, the result will still be a complex number, thus

$$\sum_{k=1}^{n} A_k e^{i\phi_k} = Ae^{i\phi}.$$

To find the expressions of A and ϕ, we expand equation (1) with Euler's formula:

$$\begin{aligned} Ae^{i\phi} &= \sum_{k=1}^{n} A_k \cos\phi_k + i\sum_{k=1}^{n} A_k \sin\phi_k \\ &= \sum_{k=1}^{n}(A_k \cos\phi_k + iA_k \sin\phi_k) \\ &= \sum_{k=1}^{n} A_k \cos\phi_k + \sum_{k=1}^{n} A_k i \sin\phi_k \end{aligned}$$

Take the magnitude (or absolute value) of both side,

$$A = \sqrt{(\sum_{k=1}^{n} A_k \cos\phi_k)^2 + (\sum_{k=1}^{n} A_k i \sin\phi_k)^2}.$$

Take the phase (or angle) of both side,

$$\phi = \arctan \frac{\sum_{k=1}^{n} A_k i \sin\phi_k}{\sum_{k=1}^{n} A_k \cos\phi_k}.$$

$\square$

Problem B. Periodicity of signals. Determine whether or not each of the following continuous-time or discrete-time signals is periodic. If the signal is periodic, determine it's fundamental period.

1. $x(t) = 3\cos(4t + \frac{\pi}{3})$
2. $x(t) = e^{i(\pi t - 1)}$
3. $x(t) = \sum_{n=-\infty}^{\infty} e^{-(2t-n)} u(2t - n)$
4. $x[n] = \sin(\frac{n}{8} - \pi)$
5. $x[n] = \cos(\frac{\pi}{8} n^2)$
6. $x[n] = \cos(\frac{6\pi}{7} n + 1)$

Solution:

1. Periodic. $T = \frac{\pi}{2}$.
2. Periodic. $T = 2$.
3. Aperiodic.
4. Aperiodic.
5. Aperiodic.
6. Aperiodic.

Problem C. Signal transformation. For a continuous-time signal $x(t)$ in Figure 1, carefully sketch the following:

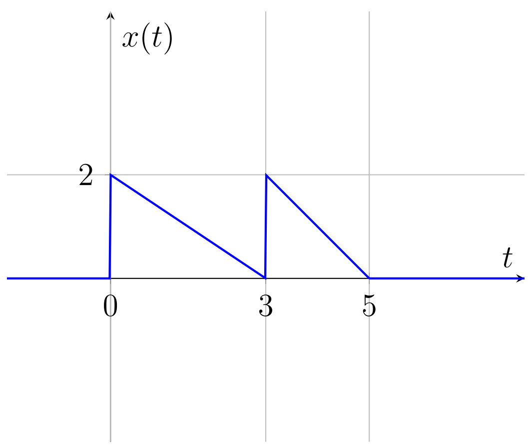

Figure 1

1. $x(-t)$
2. $x(2t)$
3. $x(t+2)$
4. $x(\frac{x}{2} - 1)$
5. $x(1-3t)$

Solution:

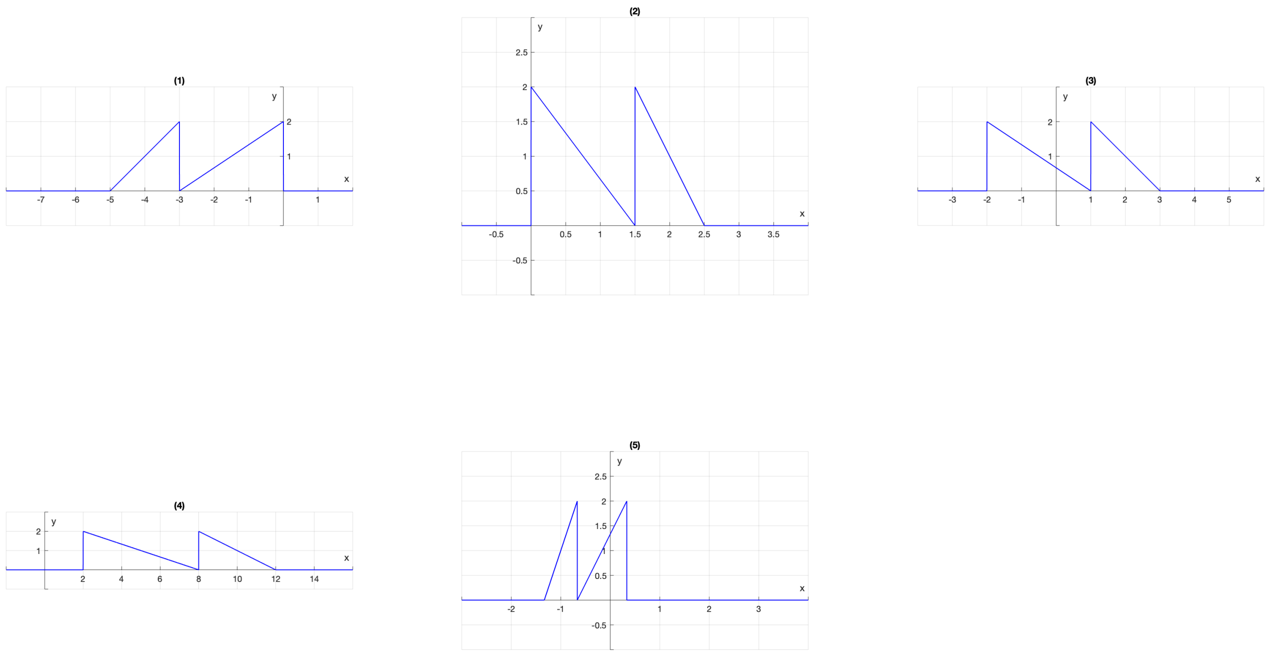

Figure 2: Solution to Problem C

Problem D. Integral review. Evaluate the following integrals.

1. $\int_{-1}^{\infty} e^{-2t} dt$
2. $y(t) = \int_{-\infty}^{\infty} u(\tau) e^{-a(t-\tau)} u(t-\tau) d\tau$
3. $X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t} dt$

Solution:

1. $\int_{-1}^{\infty} e^{-2t} dt = -\frac{1}{2} e^{-2t}|_{-1}^{\infty} = \frac{1}{2} e^2$
2. When $t < 0, \forall \tau \in (-\infty, \infty), u(\tau)u(t-\tau) = 0$, thus, $y(t) = 0$.

   When $t > 0, u(\tau)u(t-\tau) = 1$, when $\tau \in (0, t)$.

$$y(t) = \int_{-\infty}^{\infty} u(\tau) e^{-a(t-\tau)} u(t-\tau) d\tau = \int_{0}^{\infty} e^{-a(t-\tau)} d\tau = e^{-at} \int_{0}^{\infty} e^{a\tau} d\tau$$

$$= e^{-at} (\frac{1}{a} e^{at} - \frac{1}{a}) = \frac{1}{a}(1 - e^{-at}).$$

Therefore,

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{a}(1 - e^{-at}) & t > 0 \end{cases}$$

3.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-i\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-i\omega t}dt \\ &= \int_{0}^{\infty} (e^{(-a-i\omega)t} + e^{(-a+i\omega)t})dt \\ &= -\frac{1}{a+i\omega}e^{(-a-i\omega)t}|_0^\infty - \frac{1}{a-i\omega}e^{(-a+i\omega)t}|_0^\infty \\ &= \frac{1}{a+i\omega} + \frac{1}{a-i\omega} = \frac{2a}{a^2+\omega^2} \end{aligned}$$

Problem E. Functions and signals. Sketch the following functions or signals as described. Label your sketches carefully.

1. Discrete delta function (a.k.a. Kronecker delta) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$.
2. Comb function (a.k.a. Shah function) $Ш_T(t) = \frac{1}{T}Ш(\frac{t}{T}) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$.
3. Rectangular function $\text{rect}(\frac{t+2.5}{3})$
4. Sinusoidal function $x(t) = \sin(\pi t + \frac{\pi}{4})$
5. $x(t) = e^{2t}u(-t)$
6. A discrete signal: $x[n] = e^{-n}u[n]$

Solution:

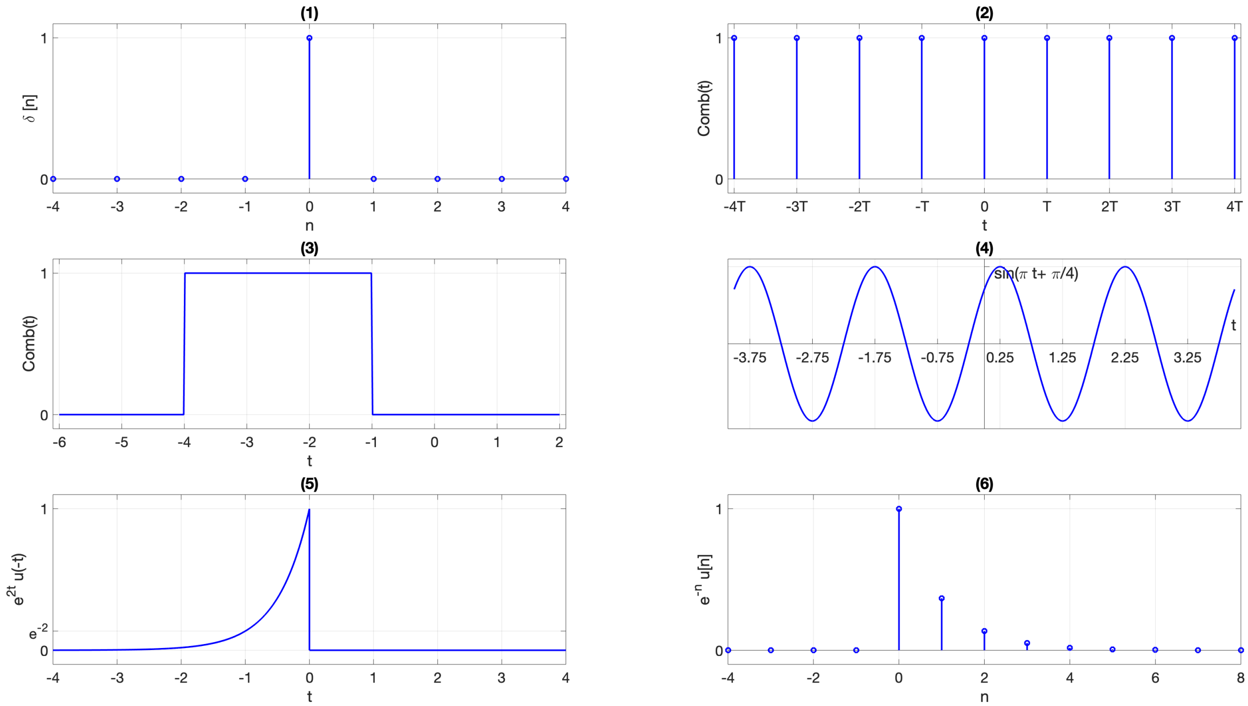

Figure 3: Answer to Problem E