University of California, Berkeley
Department of EECS

EE120: SIGNALS AND SYSTEMS (Spring 2021)

Discussion 1 Solutions

Issued: January 22, 2021

Problem A. Review of complex numbers. Consider a complex number with magnitude $r > 0$ and phase θ, which can be written in polar form as $z = re^{i\theta}$.

1. Express each of the following complex numbers in polar form:
   (a) z^* the complex conjugate of z.
   (b) z^2
   (c) iz
   (d) zz^*
   (e) $\frac{z}{z^*}$
   (f) $\frac{1}{z}$
2. Let $r = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$. For each of your results in part 1, plot the corresponding vectors in the complex plane.

Solution:

1. (a) If we write $z = a + bi$, then the conjugate is defined as $z^* = a - bi$. Magnitude (absolute value) is $|z*| = |z| = \sqrt{a^a + b^2} = r$. Phase (angle) is $\angle(z^*) = \arctan\frac{-b}{a} = -\arctan\frac{b}{a} = -\angle(z) = -\theta$.
   (b) $z^2 = (re^{i\theta})^2 = r^2e^{i2\theta}$.
   (c) $iz = i \cdot re^{i\theta} = e^{i\pi/2} \cdot re^{i\theta} = re^{i(\theta+\pi/2)}$.
   (d) $zz^* = re^{i\theta} \cdot re^{-i\theta} = r^2$.
   (e) $\frac{z}{z^*} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{i2\theta}$
   (f) $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$
2. For $z = \frac{1}{2}e^{i\frac{\pi}{3}}$, we have:
   (a) $z^* = \frac{1}{2}e^{-i\frac{\pi}{3}}$.

(b) $z^2 = \frac{1}{4}e^{i\frac{2\pi}{3}}$.

(c) $iz = \frac{1}{2}e^{i\frac{5\pi}{6}}$.

(d) $zz^* = \frac{1}{4}$.

(e) $\frac{z}{z^*} = e^{i\frac{2\pi}{3}}$.

(f) $\frac{1}{z} = 2e^{-i\frac{\pi}{3}}$.

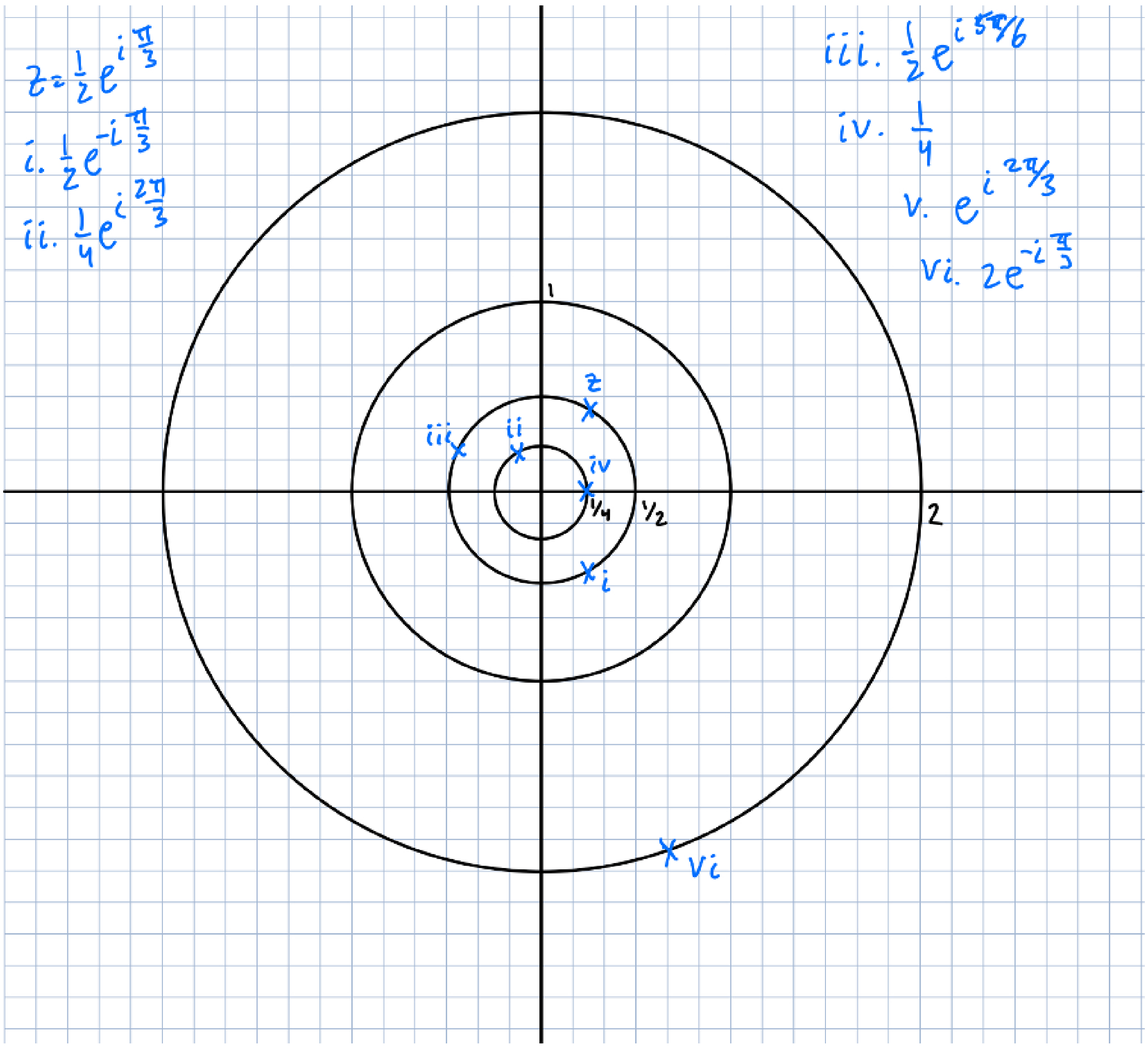

Figure 1: Problem A.2

Problem B. Review on delta function properties. The discrete time and continuous time delta functions are defined as follows:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$$(t - T) = 0, \text{ for } t \neq T$$

$$\int_{-\infty}^{\infty} \delta(t - T) dt = 1$$

1. $x[n]$ is a discrete time signal. Find an equivalent expression to $x[n]\delta[n - T]$ where $N \in \mathcal{Z}$.

2. The derivative of unit step function $\frac{du(t)}{dt}$ is delta function.

3. Express the Kronecker delta in terms of scaled and shifted unit steps.

4. $x(t)$ is a continuous time signal. Find and equivilent expression for $x(t)\delta(t - T)$ where $T \in \mathcal{R}$. What is $\int_{T-1}^{T+1} x(t)\delta(t - T)dt$?

Solution:

1. $\delta[n - N]$ is 0 except when $n = N$, thus,

$$x[n]\delta[n - N] = \begin{cases} x(N) & n = N \\ 0 & \text{otherwise} \end{cases} = x[N]\delta[n - N]$$

2. $\frac{du(t)}{dt} = \lim_{\varepsilon \to 0} \frac{u(t) - u(t - \epsilon)}{\epsilon} = \begin{cases} \lim_{\varepsilon \to 0} \frac{1}{\epsilon} & \epsilon > t > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \infty & \epsilon > t > 0 \\ 0 & \text{otherwise} \end{cases}.$

   $\int_{-infty}^{\infty} \frac{du(t)}{dt} = \int_{-infty}^{\infty} \lim_{\varepsilon \to 0} \frac{u(t)-u(t-\epsilon}{\epsilon)} = \lim_{\varepsilon \to 0} int_{0^-}^{\epsilon} \frac{1}{\epsilon} = 1.$ $\square$

3. $\delta[n] = \sum_{k=0}^{\infty} \delta[n - k] - \sum_{k=1}^{\infty} \delta[n - k] = u[n] - u[n - 1].$

4. $x(t)\delta(t - T) = x(T)\delta(t - T),$

$$\int_{T-1}^{T+1} x(t)\delta(t - T)dt = \int_{T-1}^{T+1} x(T)\delta(t - T)dt = x(T)$$

Problem C. Even and odd part of signal. For a continuous time signal $x(t)$, derive the even part and odd part of the signal.

Solution:

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}(x(t) + x(-t))$$
$$\mathcal{O}d\{x(t)\} = \frac{1}{2}(x(t) - x(-t))$$

Problem D. Properties of signals. Determine whether or not each of the following continuous-time or discrete-time signal is periodic. If the signal is periodic, determine it's fundamental period.

1. $x(t) = [cos(2t - \frac{\pi}{3})]^2$
2. $\mathcal{E}v\{\cos(4\pi t)u(t)\}$
3. $\mathcal{E}v\{\sin(4\pi t)u(t)\}$
4. $x[n] = \cos(\frac{n}{8} - \pi)$
5. $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$.

Solution:

1. Periodic. $T = \frac{\pi}{2}$. $(\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
2. Periodic. $T = \frac{1}{2}$.
3. Aperiodic.
4. Aperiodic.
5. Periodic. $T = 16$.

Problem E. Consider a periodic signal

$$x(t) \begin{cases} 1, & 0 \leq t \leq 1 \\ -2, & 1 < t < 2 \end{cases}$$

with period $T = 2$. The derivative of this signal is related to the "impulse train"

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

with period $T = 2$. It can be shown that

$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2).$$

Determine the values of A_1, t_1, A_2, and t_2.

Solution:

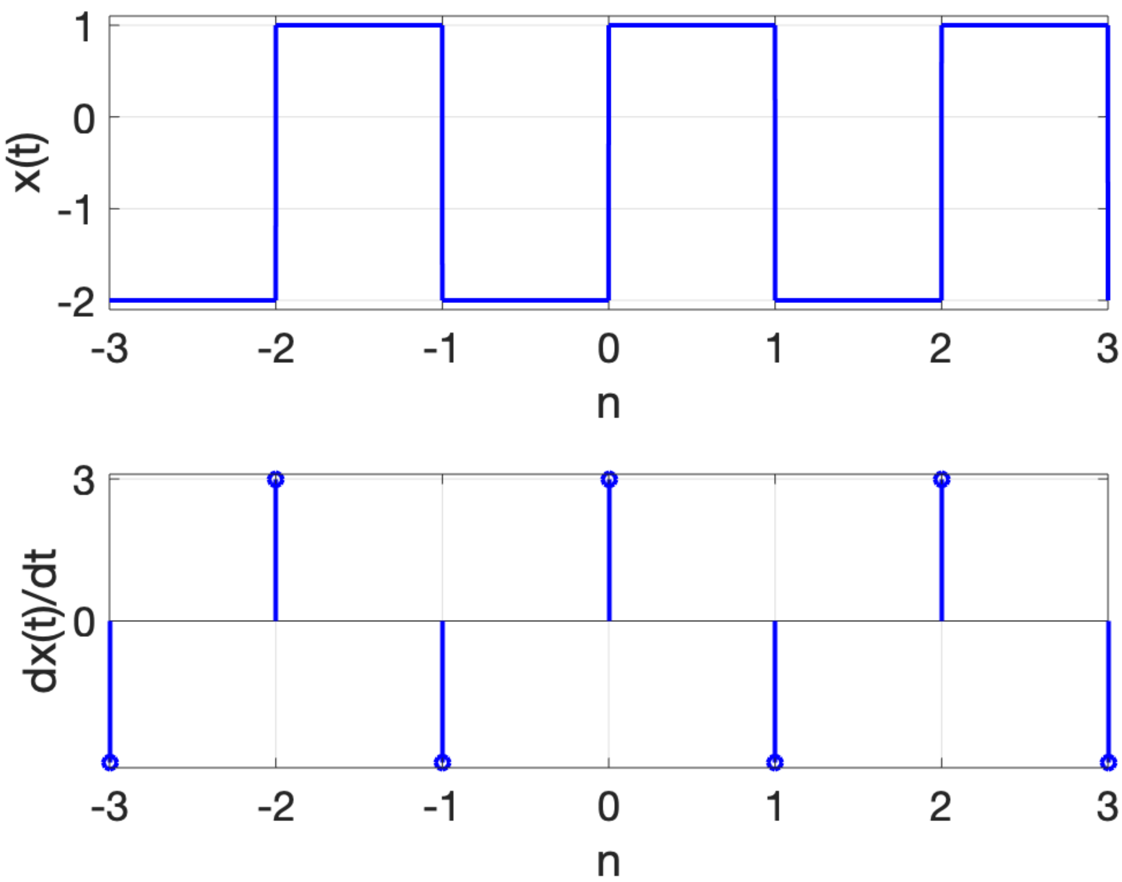

Figure 2: Problem E

Comparing two plots, $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$ (not unique).

Problem F. Function drawing practice

1. $\int_{-\infty}^{\infty} \text{rect}(t - \tau)\text{rect}(\tau)d\tau$.

2. $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$ (note the locations of nodes)

3. Let $x(t) = e^{2t}u(-t)$, $h(t) = u(t-3)$. Plot $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$.

Solution:

1. $\int_{-\infty}^{\infty}\text{rect}(x-\tau)\text{rect}(\tau)d\tau = \begin{cases} 1-|x| & |x|<1 \\ 0 & \text{otherwise} \end{cases} = tri(x)$.
2. $\text{sinc}(x) = \frac{\sin x}{x}$ (note the locations of nodes)
3. Let $x(t) = e^{2t}u(-t)$, $h(t) = u(t-3)$. Plot $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$.