BOSTON COLLEGE

MATHEMATICS DEPARTMENT

MT 470 MATHEMATICAL MODELING

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GRADED PROBLEM SET (SPECIAL): MEASLES EPIDEMIC MODEL

Work Individually. No Consultations with Others

1. Below is a 2-dimensional discrete map for the spread of measles around the world (Fulford page 164).

$$I_{k+1} = aS_k I_k$$

$$S_{k+1} = S_k - aS_k I_k + B$$

Here, I_{k+1} and S_{k+1} are the numbers of Infectives and Susceptibles, respectively, in week $(k+1)$. A single infective is assumed to infect a susceptible at a constant fraction a of the total number of susceptibles. The term B denotes the number of births each week k and is assumed to be constant.

(A) With $k \geq 0$ and $k \in \mathrm{Z}$, use recursive enumeration to fill in the blanks of the closed form solution below to S_k as a function of S_0, I_0, B, I_k; that is, $S_k = f(S_0, I_0, B, I_k)$ with S_0, I_0, and B given and with $I_1, I_2, I_3, \ldots$ derived in advance.

$$S_k = \text{blank1} \prod_{\text{blank3}}^{\text{blank2}} (1 - \text{blank4}) + \text{blank5}\left(1 + \sum_{\text{blank7}}^{\text{blank6}} \prod_{\text{blank9}}^{\text{blank8}} (1 - \text{blank10})\right)$$

(B) Find the fixed points associated with this 2-dimensional discrete map. (C) Apply the Jacobian matrix to check

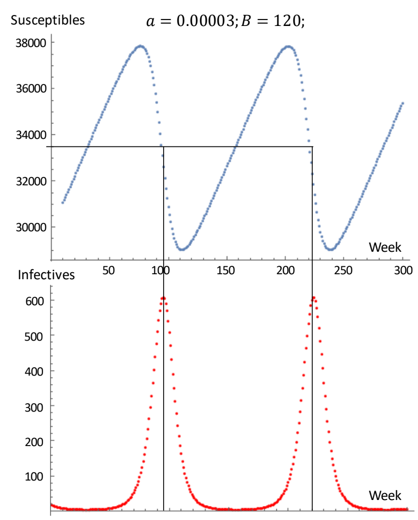
$$J\begin{pmatrix} f(S_k, I_k) \\ g(S_k, I_k) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial S_k} & \dfrac{\partial f}{\partial I_k} \\ \dfrac{\partial g}{\partial S_k} & \dfrac{\partial g}{\partial I_k} \end{pmatrix}$$

the stability of model by testing the eigenvalues of J at the fixed points and with $a = 0.00003, B = 120.$

(D) With $S_0 = 30{,}000$ and $I_0 = 20$, build a spreadsheet model of your results from Part (A) that replicates the graphs (on the right) of S_k and I_k over the period 0 to 300 weeks. From your data, what do you conclude about the number of persons in the population susceptible to measles, when the ratio I_{k+1}/I_k falls just below 1 throughout this time period?

In the infective curve, what is the number of persons susceptible at the peaks shown? At these peaks, what does the ratio I_{k+1}/I_k approximately equal?

What model parameter is responsible for keeping the susceptible population above zero, even when the number of infectives nearly bottoms out – as shown by the minimum regions along the infectives curve?



2. Below is a 3-dimensional discrete map for the spread of measles around the world.

$$I_{k+1} = {\color{red}a}S_k I_k$$
$$S_{k+1} = S_k - {\color{red}a}S_k I_k + B_k$$
$$B_{k+1} = {\color{red}c}B_k$$

In this model, I_{k+1}, S_{k+1}, and B_{k+1} are the numbers of Infectives, Susceptibles, and Births, respectively, in week $(k + 1)$. A single infective is assumed to infect a susceptible at a constant fraction ${\color{red}a}$ of the total number of susceptibles. The number of births increases each week by a constant fraction ${\color{red}c}$.

(A) With $k \geq 0$ and $k \in$ Z, use recursive enumeration to fill in the blanks of the closed form solution below to S_k as a function of S_0, I_0, B_k, I_k; that is, $S_k = f(S_0, I_0, B_k, I_k)$ with S_0, I_0, and B_0 given and with $I_1, I_2, I_3, \ldots$ and $B_1, B_2, B_3, \ldots$ derived in advance.

$$S_k = \text{blank1} \prod_{\text{blank3}}^{\text{blank2}} (1 - \text{blank4}) + \left(\sum_{\text{blank6}}^{\text{blank5}} \text{blank7} \prod_{\text{blank9}}^{\text{blank8}} (1 - \text{blank10}) \right)$$

(B) Build a spreadsheet model of your results from Part (A) that replicates the graphs below over the period 0 to 100 weeks. From your data, compute the first finite differences along the infective curve. Interpret what the sign (+/-) of these differences indicate What is the number of susceptible persons at each peak of the infective curve?

(C) Explain why the Susceptible and Infective curves overlap around the last 7 weeks or so, as shown below on the far right of the graphs?