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GPS 5: Measles Model
Mathematical Modeling

Question 1:
$S_{k+1} = S_k - \alpha I_k S_k + B = S_k(1 - \alpha I_k) + B$
$I_{k+1} = \alpha I_k S_k$

a) Complete the closed form:
$S_1 = S_0(1 - \alpha I_0) + B$

$S_2 = S_1(1 - \alpha I_1) + B = [S_0(1 - \alpha I_0) + B](1 - \alpha I_1) + B$
$\Rightarrow S_2 = S_0(1 - \alpha I_0)(1 - \alpha I_1) + (1 - \alpha I_1)B + B$

$S_3 = S_2(1 - \alpha I_2) + B = [S_0(1 - \alpha I_0)(1 - \alpha I_1) + (1 - \alpha I_1)B + B](1 - \alpha I_2) + B$
$\Rightarrow S_3 = S_0(1 - \alpha I_0)(1 - \alpha I_1)(1 - \alpha I_2) + (1 - \alpha I_2)(1 - \alpha I_1)B + (1 - \alpha I_2)B + B$

$$\Rightarrow S_3 = S_0 \sum_{i=0}^{2}(1 - \alpha I_i) + B(1 + (1 - \alpha I_2)(1 - \alpha I_1) + (1 - \alpha I_2))$$
$$= S_0 \sum_{i=0}^{2}(1 - \alpha I_i) + B\left(1 + \sum_{j=1}^{2}\prod_{i=j}^{2}(1 - \alpha I_i)\right)$$

$$\text{Generalized Solution: } S_k = S_0 \sum_{i=0}^{k-1}(1 - \alpha I_i) + B\left(1 + \sum_{j=1}^{k-1}\prod_{i=j}^{k-1}(1 - \alpha I_i)\right)$$

blank 1: S_0 blank 2: $k-1$ blank 3: $i = 0$ blank 4: αI_i blank 5 = B
blank 6: $k-1$ blank 7: $j = 1$ blank 8: $k-1$ blank 9: $i = j$ blank 10: αI_i

b) Find the fixed points:
Let $S^* = S_k = S_{k+1}$
Let $I^* = I_k = I_{k+1}$

$S^* = S^*(1 - \alpha I^*) + B$
$I^* = \alpha I^* S^*$
From the second equation, we can derive that $S^* = \frac{1}{\alpha}$
If we multiply the first equation by α, we get $1 = (1 - \alpha I^*) + B\alpha$
$\alpha I^* = B\alpha$
$I^* = B$

At the fixed points:

$$S^* = \frac{1}{\alpha} \qquad\qquad I^* = B$$

c) Apply the Jacobian to check stability by testing eigenvalues at fixed points.

$$\alpha = 0.00003 \qquad B = 120$$

$$f(S_k, I_k) = S_k - \alpha I_k S_k + B \quad \Rightarrow \qquad \frac{\partial f}{\partial S_k} = 1 - \alpha I_k \qquad \frac{\partial f}{\partial I_k} = -\alpha S_k$$

$$g(S_k, I_k) = \alpha I_k S_k \qquad\qquad \Rightarrow \qquad\qquad \frac{\partial g}{\partial S_k} = \alpha I_k \qquad \frac{\partial g}{\partial I_k} = \alpha S_k$$

$$J\begin{pmatrix} f(S_k, I_k) \\ g(S_k, I_k) \end{pmatrix} = \begin{pmatrix} 1 - \alpha I_k & -\alpha S_k \\ \alpha I_k & \alpha S_k \end{pmatrix}$$

At the fixed point $\left(S^* = \dfrac{1}{\alpha}, I^* = B\right)$: $\qquad J\begin{pmatrix} f(S_k, I_k) \\ g(S_k, I_k) \end{pmatrix} = \begin{pmatrix} 1 - \alpha B & -1 \\ \alpha B & 1 \end{pmatrix}$

After plugging in α and B: $\qquad J\begin{pmatrix} f(S_k, I_k) \\ g(S_k, I_k) \end{pmatrix} = \begin{pmatrix} 0.9964 & -1 \\ 0.0036 & 1 \end{pmatrix}$

Using a matrix calculator, we can find eigenvalues satisfy the equation: $\lambda^2 - 1.9964\lambda + 1 = 0$

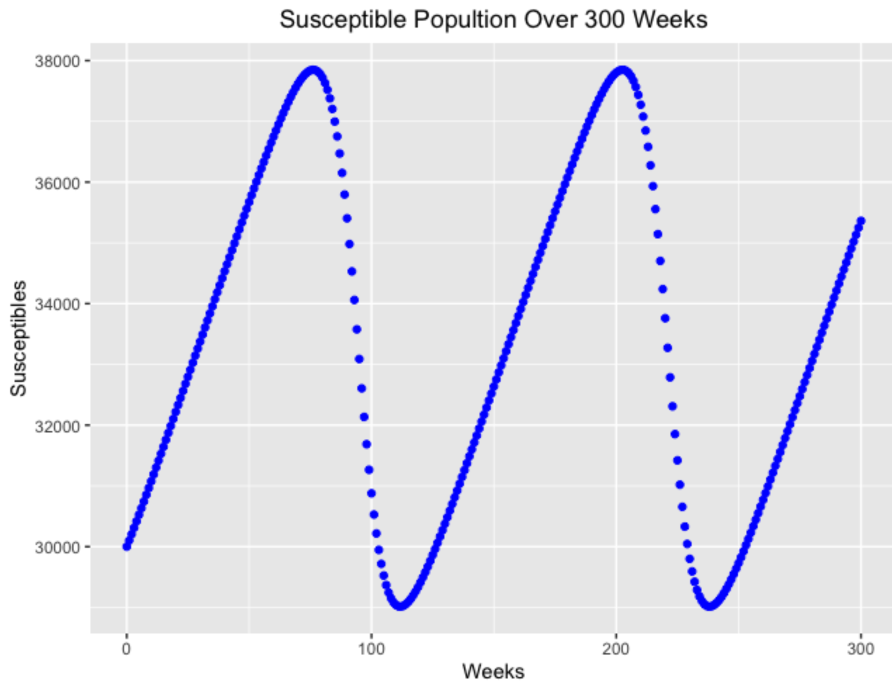
$\lambda = 0.9982 + 0.05997\ldots i, \ \lambda = 0.9982 - 0.05997\ldots i$

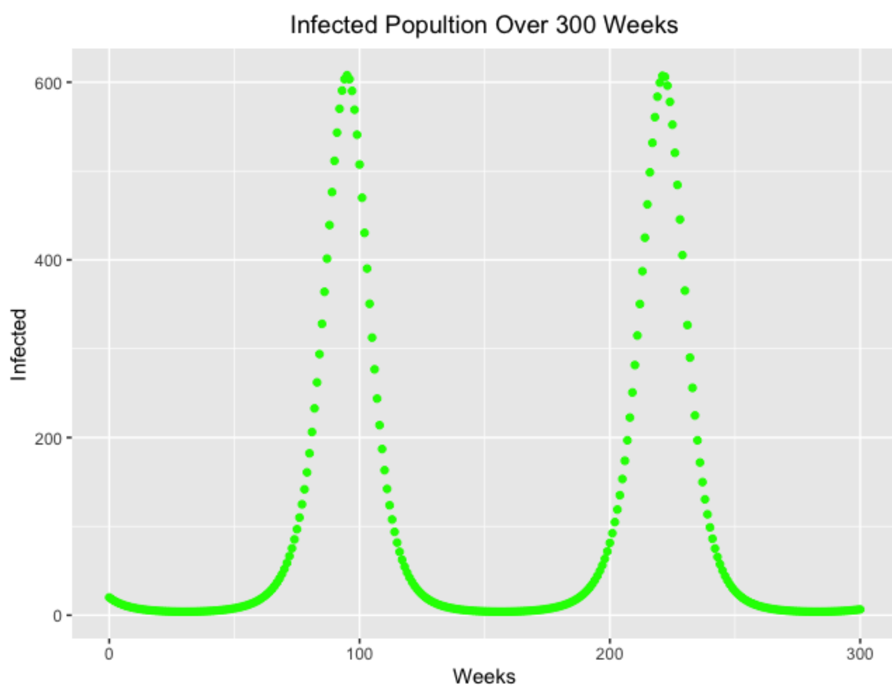
Modulus = $\sqrt{(0.9982 - 0.05997i)(0.9982 + 0.05997i)} = 0.79797$

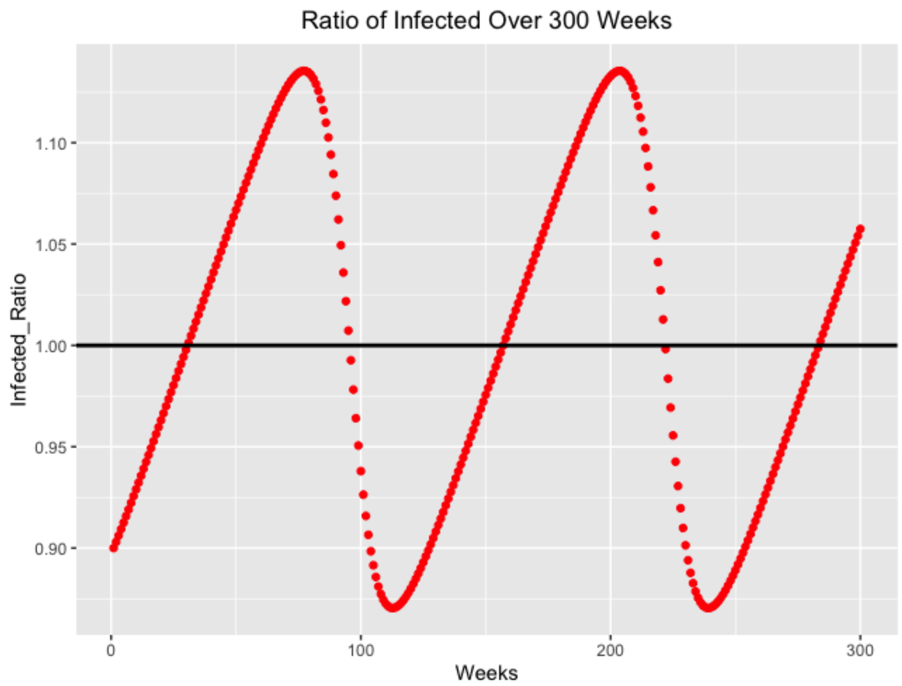
$0.79797 < 1$ so the model is stable at the fixed points.

d) Graph results:

$$S_0 = 30,000 \quad I_0 = 20 \quad \alpha = 0.00003 \quad B = 120$$







$\frac{I_{k+1}}{I_k}$ falls just below 1 at weeks 96 and 222. At these points, the number of infected people starts to decrease and the population of susceptible people starts to accelerate.

At the peaks, the number of infected is around 608 people. The number of susceptible people at this point is just under 32800 and the ratio $\frac{I_{k+1}}{I_k} \approx 1$

A constant birth rate of 120 people stops the number of susceptible people from ever reaching zero.

The dataset used to construct the graphs can be accessed through this link.

Question 2:
$S_{k+1} = S_k - \alpha I_k S_k + B_k = S_k(1-\alpha I_k) + B_k$
$I_{k+1} = \alpha I_k S_k$
$B_{k+1} = cB_k$

a) Complete the closed form:

$S_1 = S_0(1-\alpha I_0) + B_0$

$S_2 = S_1(1-\alpha I_1) + B_1 = [S_0(1-\alpha I_0) + B_0](1-\alpha I_1) + B_1$
$\Rightarrow S_2 = S_0(1-\alpha I_0)(1-\alpha I_1) + (1-\alpha I_1)B_0 + B_1$

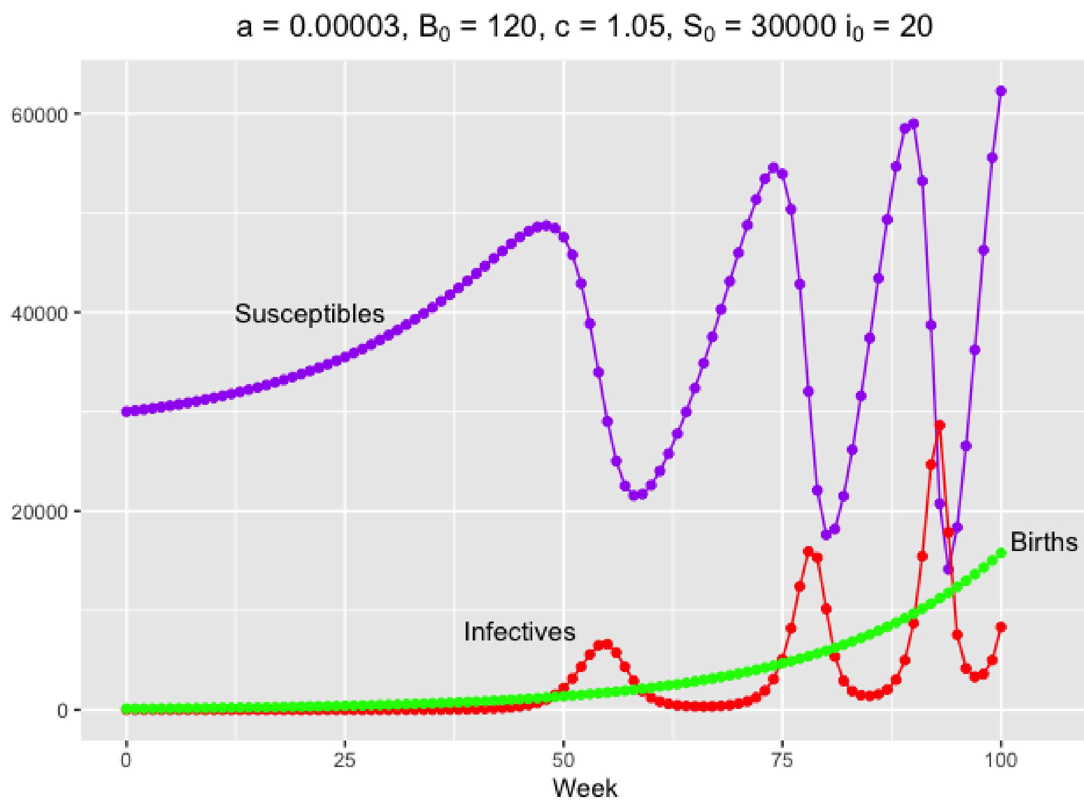
$S_3 = S_2(1-\alpha I_2) + B_2 = [S_0(1-\alpha I_0)(1-\alpha I_1) + (1-\alpha I_1)B_0 + B_1](1-\alpha I_2) + B_2$
$\Rightarrow S_3 = S_0(1-\alpha I_0)(1-\alpha I_1)(1-\alpha I_2) + (1-\alpha I_2)(1-\alpha I_1)B_0 + (1-\alpha I_2)B_1 + B_2$

$$\Rightarrow S_3 = S_0 \sum_{i=0}^{2}(1-\alpha I_i) + B_2 + (1-\alpha I_2)B_1 + (1-\alpha I_2)(1-\alpha I_1)B_0$$
$$= S_0 \sum_{i=0}^{2}(1-\alpha I_i) + \left(\sum_{j=0}^{2} B_j \prod_{i=j+1}^{2}(1-\alpha I_i)\right)$$

$$\text{Generalized Solution: } S_k = S_0 \sum_{i=0}^{k-1}(1-\alpha I_i) + \left(\sum_{j=0}^{k-1} B_j \prod_{i=j+1}^{k-1}(1-\alpha I_i)\right)$$

blank 1: S_0 blank 2: $k-1$ blank 3: $i=0$ blank 4: αI_i blank 5 = k-1
blank 6: $j=0$ blank 7: B_j blank 8: $k-1$ blank 9: $i=j+1$ blank 10: αI_i

b) Graph produced:



At each peak:

Week	48	74	90	101
Susceptible	48570.28	53428.76	58469.86	62254.86

The first finite difference is roughly equivalent to the first derivative. Therefore, when the sign becomes negative at these points, the number of infectives begins to decrease.

c) Explain the graph's behavior in the last 7 weeks:
Because of S_k and I_k are all dependent on B_k and B_k is an exponentially growing function, toards the end of the graph, I_k and S_k oscillate more and more towards the end, due to more and more births. Eventually, when the birth rate is high enough, the population of susceptible and infective will begin to intersect, as we see in the last 7 weeks.

The dataset used to construct the graphs can be accessed through this link.

All datasets and graphs were constructed in R through RStudio, These files, as well as the LaTeX files, can be found here.