Exam #2

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Question 1:

$S_{n+1} = qS_n + cR_n$
$I_{n+1} = (1-q)S_n + bI_n$
$R_{n+1} = (1-c)R_n + (1-b)I_n$

Part A)

$I^* = (1-q)S^* + bI^*$

$$I^* - bI^* = (1-q)S^* \Rightarrow (1-b)I^* = (1-q)S^* \Rightarrow \frac{S^*}{I^*} = \frac{1-b}{1-q} = \frac{b-1}{q-1}\checkmark$$

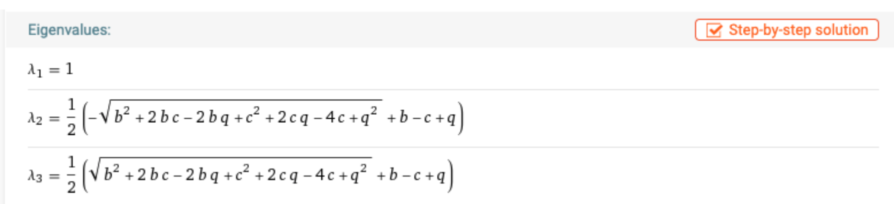
$$f(S_k, I_k, R_k) = qS_k + cR_k \implies \quad \frac{\partial f}{\partial S_k} = q \quad \frac{\partial f}{\partial I_k} = 0 \quad \frac{\partial f}{\partial R_k} = c$$

$$g(S_k, I_k, R_k) = (1-q)S_n + bI_n \implies \quad \frac{\partial g}{\partial S_k} = 1-q \quad \frac{\partial g}{\partial I_k} = b \quad \frac{\partial g}{\partial R_k} = 0$$

$$h(S_k, I_k, R_k) = (1-c)R_n + (1-b)I_n \Rightarrow \frac{\partial h}{\partial S_k} = 0 \quad \frac{\partial h}{\partial I_k} = 1-b \quad \frac{\partial h}{\partial R_k} = 1-c$$

$$J\begin{pmatrix} f(S_k, I_k, R_k) \\ g(S_k, I_k, R_k) \\ h(S_k, I_k, R_k) \end{pmatrix} = \begin{pmatrix} q & 0 & c \\ 1-q & b & 0 \\ 0 & 1-b & 1-c \end{pmatrix}$$

Using the WolframAlpha Eigenvalue Calculator:



The first eigenvalue has value 1.
By the Linearized Stability Theorem for Maps:

If λ_2 and/or λ_3 has modulus greater than 1, then the model (S^*, I^*, R^*) is unstable. If λ_2 and/or λ_3 has modulus less than or equal to 1, then the model (S^*, I^*, R^*), the system neither diverges from nor converges to the fixed points (no conclusion).

Part B)

$S_0 = 9000, I_0 = 900, R_0 = 300$
$q = \frac{1}{4}, b = \frac{4}{5}, c = \frac{1}{100}$

Part B1)

$V_k = A^{*k}V = \begin{pmatrix} S_k \\ I_k \\ R_k \end{pmatrix}$
$A^{*k} = \begin{pmatrix} q & 0 & c \\ 1-q & b & 0 \\ 0 & 1-b & 1-c \end{pmatrix} = \begin{pmatrix} 0.25 & 0 & 0.01 \\ 0.75 & 0.8 & 0 \\ 0 & 0.2 & 0.99 \end{pmatrix}$

Using WolframAlpha:
Eigenvalues:
$\lambda_1 = 1$
$\lambda_2 = 1/100(52 + \sqrt{709}) = 0.78627053911$
$\lambda_3 = 1/100(52 - \sqrt{709}) = 0.25372946088$

Eigenvectors:
$v_1 = \left(\frac{1}{75}, \frac{1}{20}, 1\right) = (0.01333, 0.05, 1)$
$v_2 = \left(\frac{1}{20}(27 - \sqrt{709}), \frac{1}{20}(-47 + \sqrt{709}), 1\right) = (0.01864730443, -1.01864730443, 1)$
$v_3 = \left(\frac{1}{20}(27 + \sqrt{709}), \frac{1}{20}(-47 - \sqrt{709}), 1\right) = (2.68135269557, -3.68135269557, 1)$

$S_0 = 9000 = c_1(0.01333) + c_2(0.01864730443) + c_3(2.68135269557)$
$I_0 = 900 = c_1(0.05) + c_2(-1.01864730443) + c_3(-3.68135269557)$
$R_0 = 300 = c_1 + c_2 + c_3$

Using WolframAlpha's System Calculator
$c_1 = 9592.51, c_2 = -12689.6, c_3 = 3397.08$

Therefore:
$S_k = 9592.51(0.01333)+(-12689.6)(0.01864730443)(0.78627053911)^k+3397.08(2.68135269557)(0.25372946088)^k = 127.868 - 236.627(0.786)^k + 9108.770(0.254)^k$
$I_k = 9592.51(0.05)+-12689.6(-1.01864730443)(0.78627053911)^k+3397.08(-3.68135269557)(0.25372946088)^k = 479.626 - 12926.227(0.786)^k + -12505.850(0.254)^k$
$R_k = 9592.51 - 12689.6(0.786)^k + 3397.08(0.254)^k$

Answer: $S_k = 127.868 - 236.627(0.786)^k + 9108.770(0.254)^k$
$I_k = 479.626 - 12926.227(0.786)^k + -12505.850(0.254)^k$
$R_k = 9592.51 - 12689.6(0.786)^k + 3397.08(0.254)^k$

Part B2)

Looking at the answer above, as $k \to \infty$, all the terms to the k power that are less than 1 collapse to 0.
For example: $S_\infty = 127.868 - 236.627(0.786)^\infty + 9108.770(0.254)^\infty = 127.868 + 0 + 0$
Therefore: $(S_\infty, I_\infty, R_\infty) = (127.868, 479.626, 9592.51)$

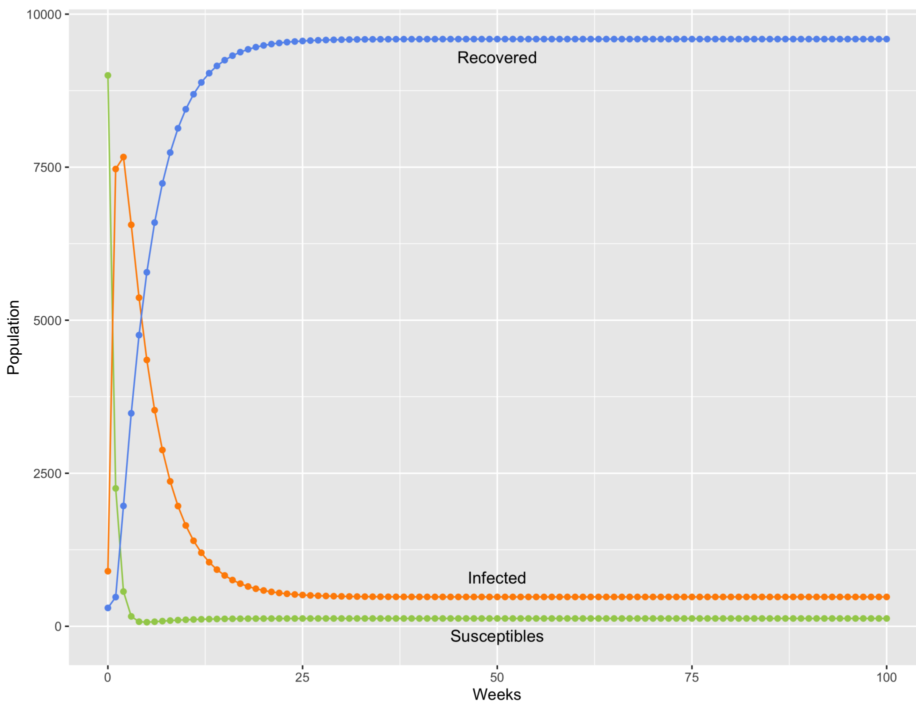
```r
data <- readr::read_csv("SIR_data.csv")
tail(data, 5)
```

```
## # A tibble: 5 x 5
##       n     S     I     R `S/I`
##   <dbl> <dbl> <dbl> <dbl> <dbl>
## 1    96  128.  480. 9592. 0.267
## 2    97  128.  480. 9592. 0.267
## 3    98  128.  480. 9592. 0.267
## 4    99  128.  480. 9592. 0.267
## 5   100  128.  480. 9592. 0.267
```

Looking at the data, the values at $n = 100$,
$(S_{100}, I_{100}, R_{100}) = (127.89969, 479.6238, 9592.476)$
This is very similar to our values at $(S_\infty, I_\infty, R_\infty)$.

Part B3)

Here's the graph: Dataset can be found here




```r
q=1/4
b=4/5
(b-1)/(q-1)
```

```
## [1] 0.2666667
```

Looking at the dataset, in the column 'S/I', if rounding to the 7th digit, the ratio converges and remains at 0.2666667. The is the desired convergent point as calculated above.

This means that the ratio between suceptibles and infectives from this point on remains 0.2666667. Judging from the graph and the data, this follows as the suceptible and infective popultions are also fixed after this point n.

Question 2:

Part A)

Use substitution such that $u_n = \dfrac{1}{x_n}$

Therefore, $\dfrac{1}{u_{n+1}} = \dfrac{1}{u_n}\dfrac{r}{\dfrac{1}{u_n}+A} \Rightarrow u_{n+1} = u_n\dfrac{1+Au_n}{ru_n} = \dfrac{1+Au_n}{r} = \dfrac{A}{r}u_n + \dfrac{1}{r}$

This is a first order linear difference equation where $A^* = \dfrac{A}{r}$ and $B^* = \dfrac{1}{r}$

Therefore the solution to $u_{n+1} = \left(\dfrac{A}{r}\right)^{n+1} u_0 + \dfrac{\frac{1}{r}}{1-\frac{A}{r}}(1-\left(\dfrac{A}{r}\right)^{n+1}) = \left(\dfrac{A}{r}\right)^{n+1} u_0 + \dfrac{1}{r-A}\left(1-\left(\dfrac{A}{r}\right)^{n+1}\right)$

Substitute back in x_n:

$$\frac{1}{x_{n+1}} = \left(\frac{A}{r}\right)^{n+1}\frac{1}{x_0} + \frac{1}{r-A}\left(1-\left(\frac{A}{r}\right)^{n+1}\right) = \left(\frac{A}{r}\right)^{n+1}\left(\frac{1}{x_0}\right) - \left(\frac{A}{r}\right)^{n+1}\left(\frac{1}{r-A}\right) + \frac{1}{r-A}$$

Denominators have $r^{n+1}, x_0, (r-a)$ in common. Simplifies to:

$$\frac{1}{x_{n+1}} = \frac{A^{n+1}(r-A) - A^{n+1}x_0 + r^{n+1}x_0}{r^{n+1}x_0(r-A)} \implies$$
$$x_{n+1} = \frac{r^{n+1}x_0(r-A)}{A^{n+1}(r-A) - A^{n+1}x_0 + r^{n+1}x_0} = \frac{r^{n+1}x_0(r-A)}{A^{n+1}(r-A) + \left(r^{n+1}-A^{n+1}\right)x_0} = \frac{r^{n+1}x_0}{A^{n+1} + \left(r^n - A^n\right)x_0}$$

Part B)

Let $x^* = x_n = x_n + 1$

$x^* = \dfrac{rx^*}{x^*+A} \Rightarrow {x^*}^2 + Ax* - rx^* = 0 \Rightarrow x^*(x^*+A-r) = 0$

Fixed points are at $x^* = 0$ and $x^* = r - A$

If $f(x) = \dfrac{rx^*}{x^*+A}$, then $f'(x) = \dfrac{(x^*+A)r - 1(rx^*)}{(x^*+A)^2} = \dfrac{Ar}{(x^*+A)^2}$

$f'(0) = r/A$ and $f'(r-A) = \dfrac{Ar}{(r-A+A)^2} = \dfrac{A}{r}$

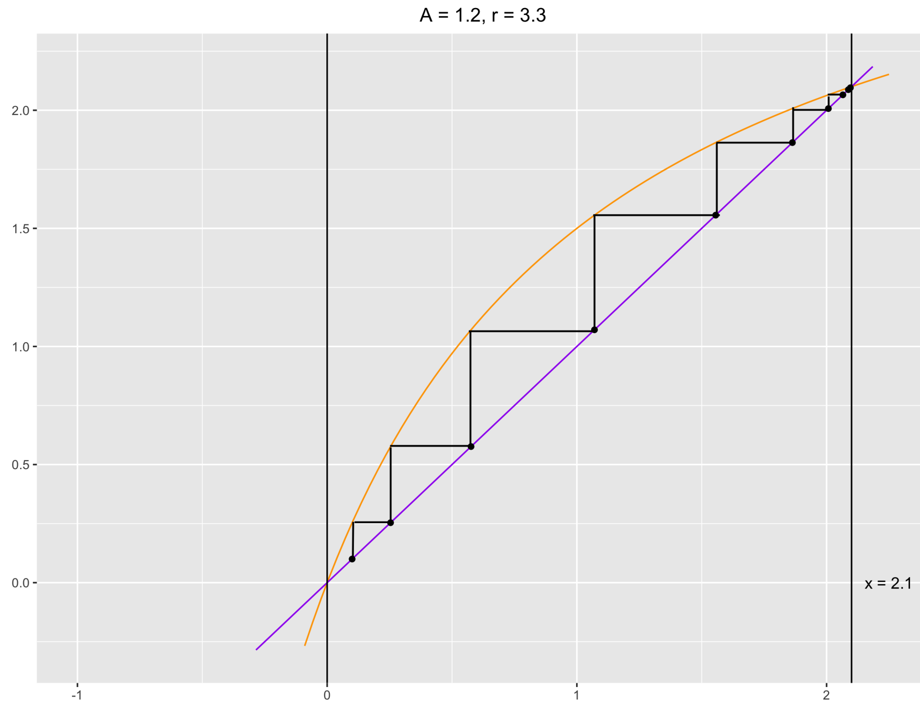
If $|A| > |r|$, point $x^* = (r-A)$ is a stable fixed point and point $x^* = 0$ is unstable.

If $|A| < |r|$, point $x^* = 0$ is a stable fixed point and point $x^* = (r-a)$ is unstable.

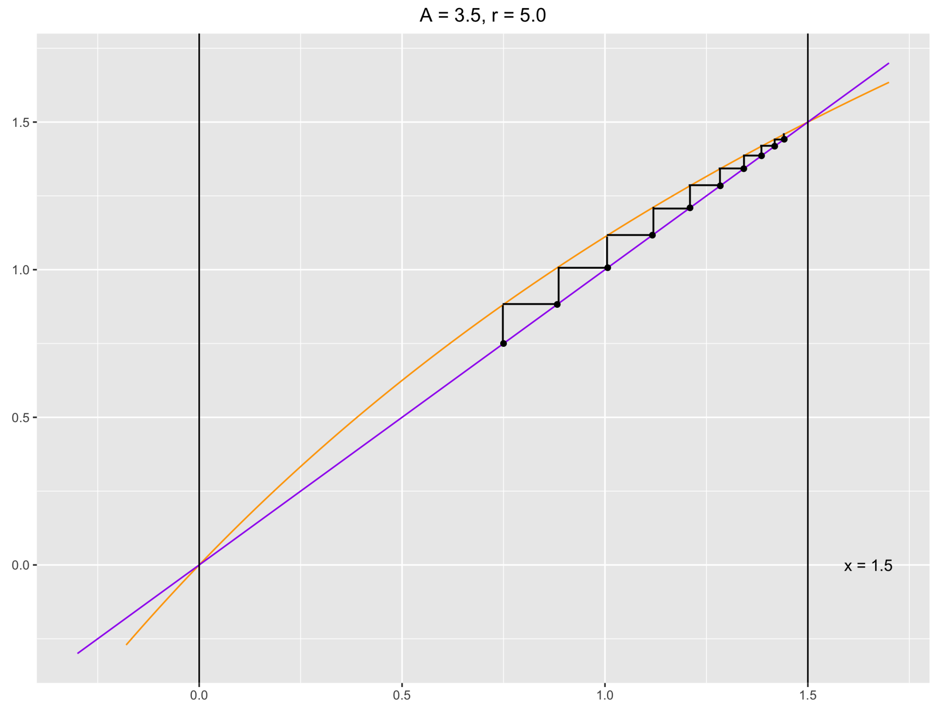
Part C)

Cobweb points were found using R. The rest of the cobweb was drawn manually.

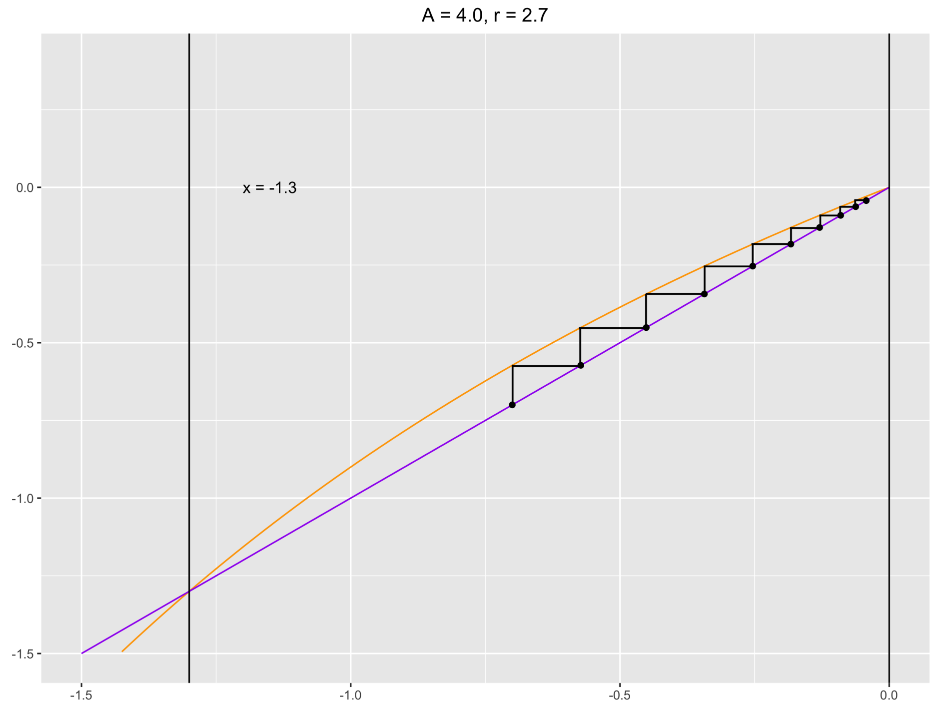
Here are three cobweb diagrams with different As and rs to verify the results.




```
#     x         y
#1    1 0.1000000
#2    2 0.2538462
#3    3 0.5761905
#4    4 1.0705094
#5    5 1.5558980
#6    6 1.8630818
#7    7 2.0071844
#8    8 2.0652721
#9    9 2.0872374
#10  10 2.0953410
```




```
#    x         y
#1   1 0.7500000
#2   2 0.8823529
#3   3 1.0067114
#4   4 1.1169025
#5   5 1.2095799
#6   6 1.2841696
#7   7 1.3421029
#8   8 1.3858678
#9   9 1.4182412
#10 10 1.4418175
```




```
#     x           y
#1    1 -0.70000000
#2    2 -0.57272727
#3    3 -0.45119363
#4    4 -0.34327678
#5    5 -0.25346389
#6    6 -0.18266273
#7    7 -0.12919722
#8    8 -0.09011890
#9    9 -0.06223234
#10 10 -0.04267070
```

Notice in the first two cases, when $r - A > 0$, the points go to $x = r - A$.
However, in the last case where $A - r < 0$, the points go to $x = 0$

Question 3:

$S_t = I_t$
$\frac{S_t}{Y_t} = \alpha$
$\frac{I_{t+1}}{\Delta Y_t} = \frac{I_{t+1}}{Y_{t+1} - Y_t} = \beta$

Part A)

Assume $\beta > \alpha > 0$
$S_t = \alpha Y_t$
Solving for Y_t:
$Y_{t+1} - Y_t = \frac{I_{t+1}}{\beta} \implies Y_{t+1} = \frac{I_{t+1}}{\beta} + Y_t$
$Y_{t+1} = \frac{\alpha Y_{t+1}}{\beta} + Y_t \implies Y_{t+1} - \frac{\alpha Y_{t+1}}{\beta} = Y_t \implies Y_{t+1}(1 - \frac{\alpha}{\beta}) = Y_t \implies Y_{t+1}\frac{\beta - \alpha}{\beta} = Y_t \implies Y_{t+1} = Y_t \frac{\beta}{\beta - \alpha}$
Therefore, $Y_t = \left(\frac{\beta}{\beta - \alpha}\right)^t Y_0$

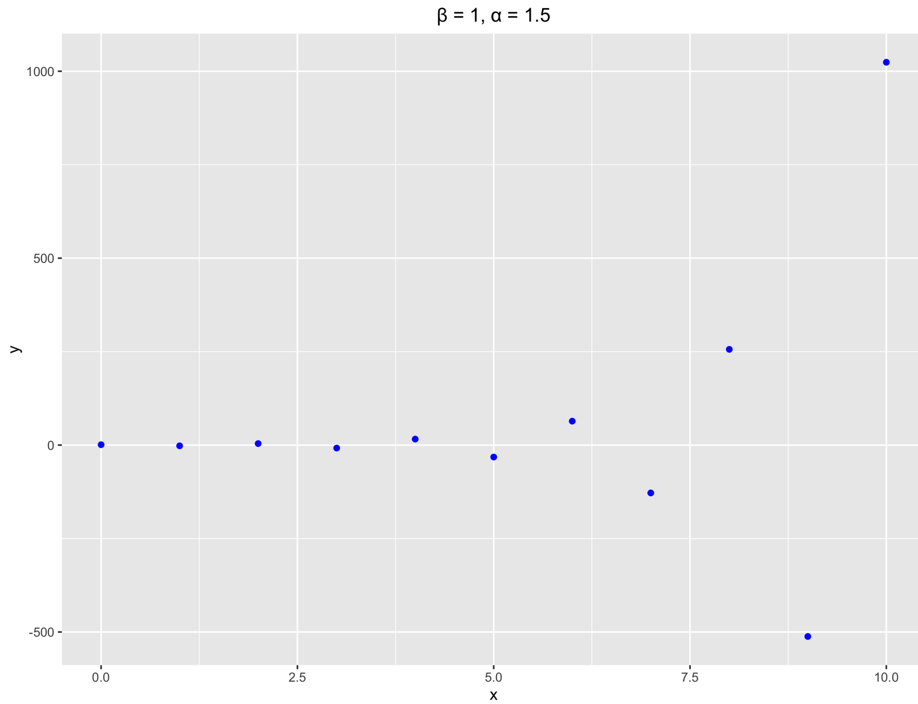
Solving for I_t:
$Y_{t+1} = \frac{\alpha Y_{t+1}}{\beta} + Y_t$:
$\frac{I_{t+1}}{\alpha} = \frac{I_{t+1}}{\beta} + \frac{I_t}{\alpha} \implies \frac{I_{t+1}}{\alpha} - \frac{I_{t+1}}{\beta} = \frac{I_t}{\alpha} \implies I_{t+1}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) = \frac{I_t}{\alpha} \implies I_{t+1}\left(\frac{\beta - \alpha}{\alpha\beta}\right) = \frac{I_t}{\alpha}$
$\implies I_{t+1} = \left(\frac{\beta}{\beta - \alpha}\right) I_t$
Therefore, $I_t = \left(\frac{\beta}{\beta - \alpha}\right)^t I_0$
This checks out with our previous result for Y_t.

Part B)

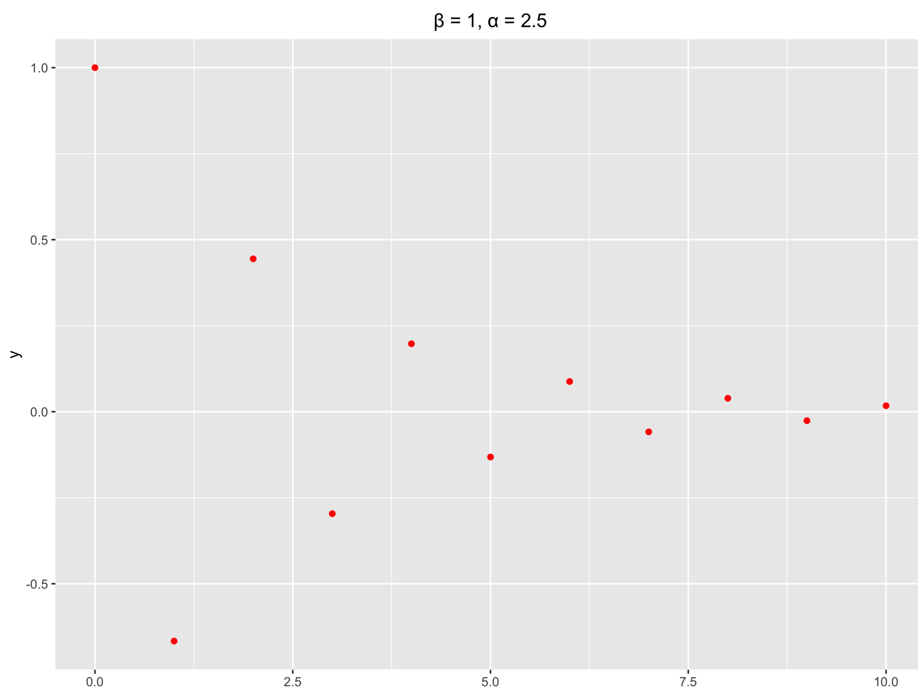
The national income grows exponentially by a ratio of $\frac{\beta}{\beta - \alpha}$

If $\alpha > \beta > 0$, then $\frac{\beta}{\beta - \alpha}$ is a negative number.

Here is an example of $-1 < \frac{\beta}{\beta - \alpha} < 0$. It oscillates and diverges.



Here is an example of $\dfrac{\beta}{\beta-\alpha} < -1$. It oscillates and converges to 0.



Note: If $\dfrac{\beta}{\beta-\alpha} = -1$, it just oscillates between Y_0 and $-Y_0$.