Exam #2

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Question 1:

$S_{n+1} = qS_n + cR_n$
$I_{n+1} = (1-q)S_n + bI_n$
$R_{n+1} = (1-c)R_n + (1-b)I_n$

Part A)

$I^* = (1-q)S^* + bI^*$

$I^* - bI^* = (1-q)S^* \Rightarrow (1-b)I^* = (1-q)S^* \Rightarrow \dfrac{S^*}{I^*} = \dfrac{1-b}{1-q} = \dfrac{b-1}{q-1}\checkmark$

$f(S_k, I_k, R_k) = qS_k + cR_k \implies \quad \dfrac{\partial f}{\partial S_k} = q \quad \dfrac{\partial f}{\partial I_k} = 0 \quad \dfrac{\partial f}{\partial R_k} = c$

$g(S_k, I_k, R_k) = (1-q)S_n + bI_n \implies \quad \dfrac{\partial g}{\partial S_k} = 1-q \quad \dfrac{\partial g}{\partial I_k} = b \quad \dfrac{\partial g}{\partial R_k} = 0$

$h(S_k, I_k, R_k) = (1-c)R_n + (1-b)I_n \Rightarrow \dfrac{\partial h}{\partial S_k} = 0 \quad \dfrac{\partial h}{\partial I_k} = 1-b \quad \dfrac{\partial h}{\partial R_k} = 1-c$

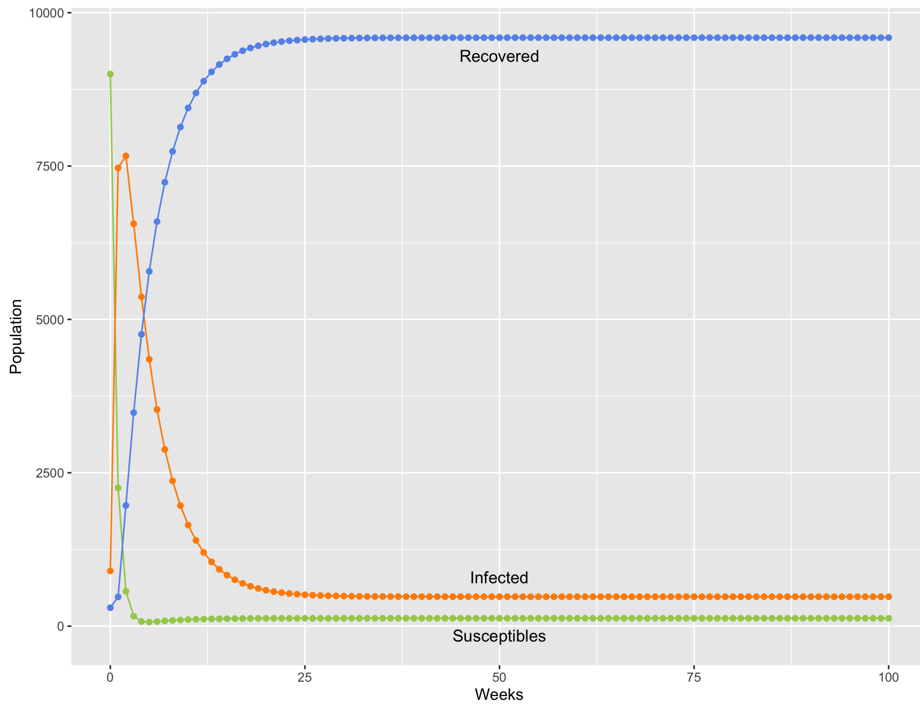
$$J\begin{pmatrix} f(S_k, I_k, R_k) \\ g(S_k, I_k, R_k) \\ h(S_k, I_k, R_k) \end{pmatrix} = \begin{pmatrix} q & 0 & c \\ 1-q & b & 0 \\ 0 & 1-b & 1-c \end{pmatrix}$$

Part B)

$S_0 = 9000, I_0 = 900, R_0 = 300$
$q = \dfrac{1}{4}, b = \dfrac{4}{5}, c = \dfrac{1}{100}$

Part B3)

Here's the graph:




```r
data <- readr::read_csv("/Users/joshuachen/exam_2_modeling/SIR_data.csv")
round(data[80:90, "S/I"], digits = 7)
```

```
## # A tibble: 11 x 1
##    `S/I`
##    <dbl>
##  1 0.267
##  2 0.267
##  3 0.267
##  4 0.267
##  5 0.267
##  6 0.267
##  7 0.267
##  8 0.267
##  9 0.267
## 10 0.267
## 11 0.267
```

Question 2:

Part A)

Use substitution such that $u_n = \dfrac{1}{x_n}$

Therefore, $\dfrac{1}{u_{n+1}} = \dfrac{1}{u_n}\dfrac{r}{\dfrac{1}{u_n} + A} \Rightarrow u_{n+1} = u_n\dfrac{1 + Au_n}{ru_n} = \dfrac{1 + Au_n}{r} = \dfrac{A}{r}u_n + \dfrac{1}{r}$

This is a first order linear difference equation where $A^* = \dfrac{A}{r}$ and $B^* = \dfrac{1}{r}$

Therefore the solution to $u_{n+1} = \left(\frac{A}{r}\right)^{n+1} u_0 + \frac{\frac{1}{r}}{1-\frac{A}{r}}(1 - \left(\frac{A}{r}\right)^{n+1}) = \left(\frac{A}{r}\right)^{n+1} u_0 + \frac{1}{r-A}\left(1 - \left(\frac{A}{r}\right)^{n+1}\right)$

Substitute back in x_n:

$$\frac{1}{x_{n+1}} = \left(\frac{A}{r}\right)^{n+1}\frac{1}{x_0} + \frac{1}{r-A}\left(1 - \left(\frac{A}{r}\right)^{n+1}\right) = \left(\frac{A}{r}\right)^{n+1}\left(\frac{1}{x_0}\right) - \left(\frac{A}{r}\right)^{n+1}\left(\frac{1}{r-A}\right) + \frac{1}{r-A}$$

Denominators have $r^{n+1}, x_0, (r-a)$ in common. Simplifies to:

$$\frac{1}{x_{n+1}} = \frac{A^{n+1}(r-A) - A^{n+1}x_0 + r^{n+1}x_0}{r^{n+1}x_0(r-A)} \Longrightarrow$$

$$x_{n+1} = \frac{r^{n+1}x_0(r-A)}{A^{n+1}(r-A) - A^{n+1}x_0 + r^{n+1}x_0} = \frac{r^{n+1}x_0(r-A)}{A^{n+1}(r-A) + \left(r^{n+1} - A^{n+1}\right)x_0} = \frac{r^{n+1}x_0}{A^{n+1} + \left(r^n - A^n\right)x_0}$$

Part B)

Let $x^* = x_n = x_n + 1$

$x^* = \frac{rx^*}{x^* + A} \Rightarrow {x^*}^2 + Ax * -rx^* = 0 \Rightarrow x^*(x^* + A - r) = 0$

Fixed points are at $x^* = 0$ and $x^* = r - A$

If $f(x) = \frac{rx^*}{x^* + A}$, then $f'(x) = \frac{(x^* + A)r - 1(rx^*)}{(x^* + A)^2} = \frac{Ar}{(x^* + A)^2}$

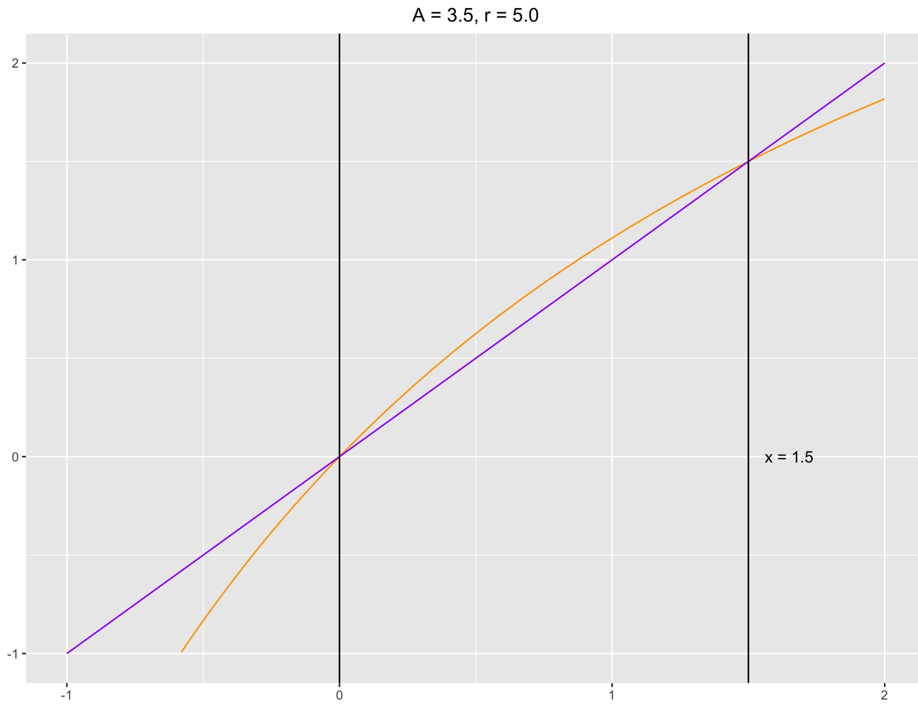
$f'(0) = r/A$ and $f'(r - A) = \frac{Ar}{(r - A + A)^2} = \frac{A}{r}$

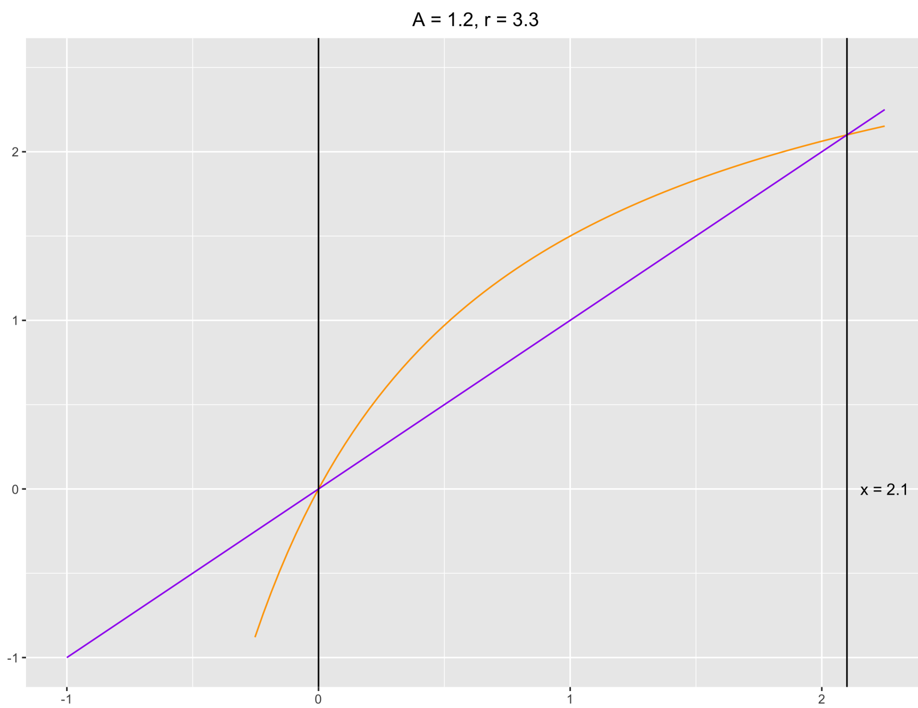
If $|A| > |r|$, point $x^* = (r - A)$ is a stable fixed point and point $x^* = 0$ is unstable.

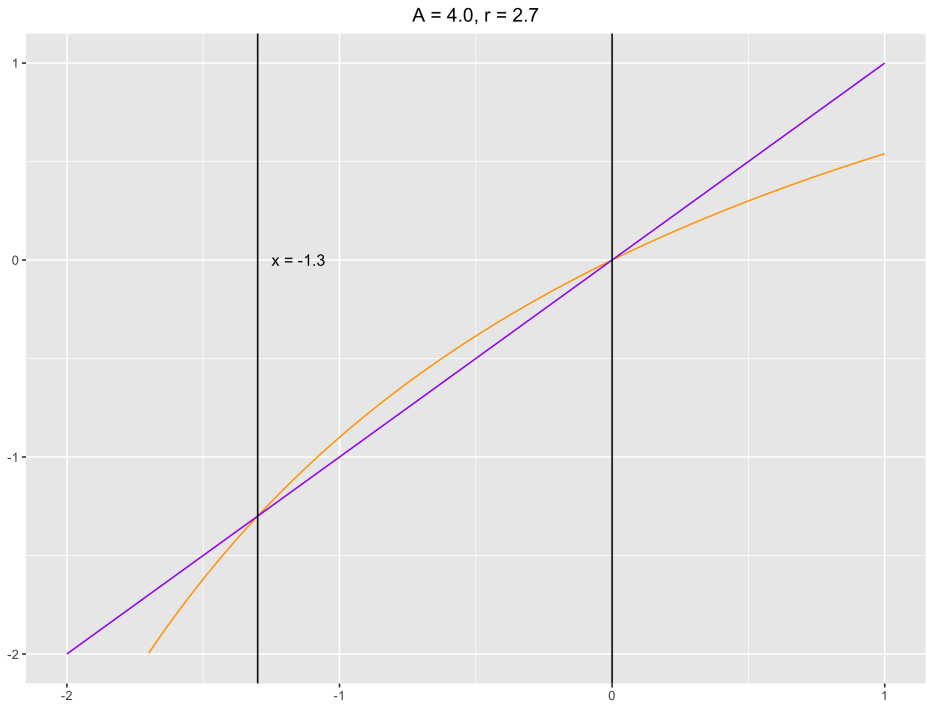
If $|A| < |r|$, point $x^* = 0$ is a stable fixed point and point $x^* = (r - a)$ is unstable.

Part C)

Here are three cobweb diagrams with different As and rs to verify the results.


A = 3.5, r = 5.0
x = 1.5
2
1
0
-1
-1
0
1
2


A = 1.2, r = 3.3
x = 2.1
2
1
0
-1
-1
0
1
2



Question 3:

$S_t = I_t$
$\dfrac{S_t}{Y_t} = \alpha$
$\dfrac{I_{t+1}}{\Delta Y_t} = \dfrac{I_{t+1}}{Y_{t+1} - Y_t} = \beta$

Part A)

Assume $\beta > \alpha > 0$
$S_t = \alpha Y_t$
Solving for Y_t:
$Y_{t+1} - Y_t = \dfrac{I_{t+1}}{\beta} \implies Y_{t+1} = \dfrac{I_{t+1}}{\beta} + Y_t$
$Y_{t+1} = \dfrac{\alpha Y_{t+1}}{\beta} + Y_t \implies Y_{t+1} - \dfrac{\alpha Y_{t+1}}{\beta} = Y_t \implies Y_{t+1}(1 - \dfrac{\alpha}{\beta}) = Y_t \implies Y_{t+1}\dfrac{\beta - \alpha}{\beta} = Y_t \implies Y_{t+1} = Y_t \dfrac{\beta}{\beta - \alpha}$
Therefore, $Y_t = \left(\dfrac{\beta}{\beta - \alpha}\right)^t Y_0$

Solving for I_t:
$Y_{t+1} = \dfrac{\alpha Y_{t+1}}{\beta} + Y_t$:
$\dfrac{I_{t+1}}{\alpha} = \dfrac{I_{t+1}}{\beta} + \dfrac{I_t}{\alpha} \implies \dfrac{I_{t+1}}{\alpha} - \dfrac{I_{t+1}}{\beta} = \dfrac{I_t}{\alpha} \implies I_{t+1}\left(\dfrac{1}{\alpha} - \dfrac{1}{\beta}\right) = \dfrac{I_t}{\alpha} \implies I_{t+1}\left(\dfrac{\beta - \alpha}{\alpha\beta}\right) = \dfrac{I_t}{\alpha}$
$\implies I_{t+1} = \left(\dfrac{\beta}{\beta - \alpha}\right) I_t$

Therefore, $I_t = \left(\frac{\beta}{\beta - \alpha}\right)^t I_0$

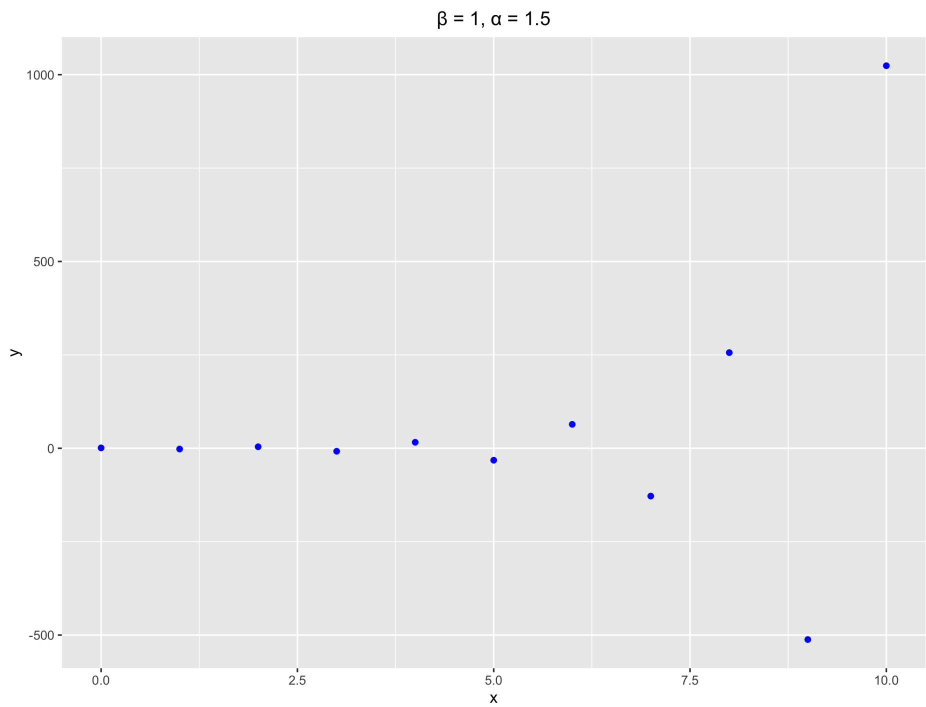
This checks out with our previous result for Y_t.

Part B)

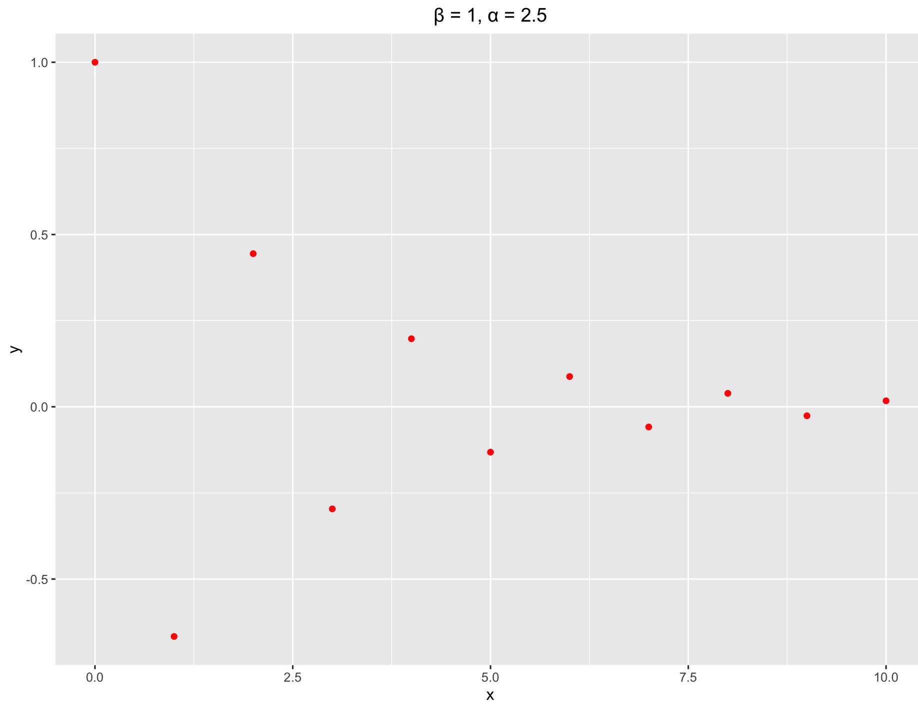
The national income grows exponentially by a ratio of $\frac{\beta}{\beta - \alpha}$

If $\alpha > \beta > 0$, then $\frac{\beta}{\beta - \alpha}$ is a negative number.

Here is an example of $-1 < \frac{\beta}{\beta - \alpha} < 0$. It oscillates and diverges.



Here is an example of $\frac{\beta}{\beta - \alpha} < -1$. It oscillates and converges to 0.



Note: If $\dfrac{\beta}{\beta - \alpha} = -1$, it just oscillates between Y_0 and $-Y_0$.