Graded Problem Set 5

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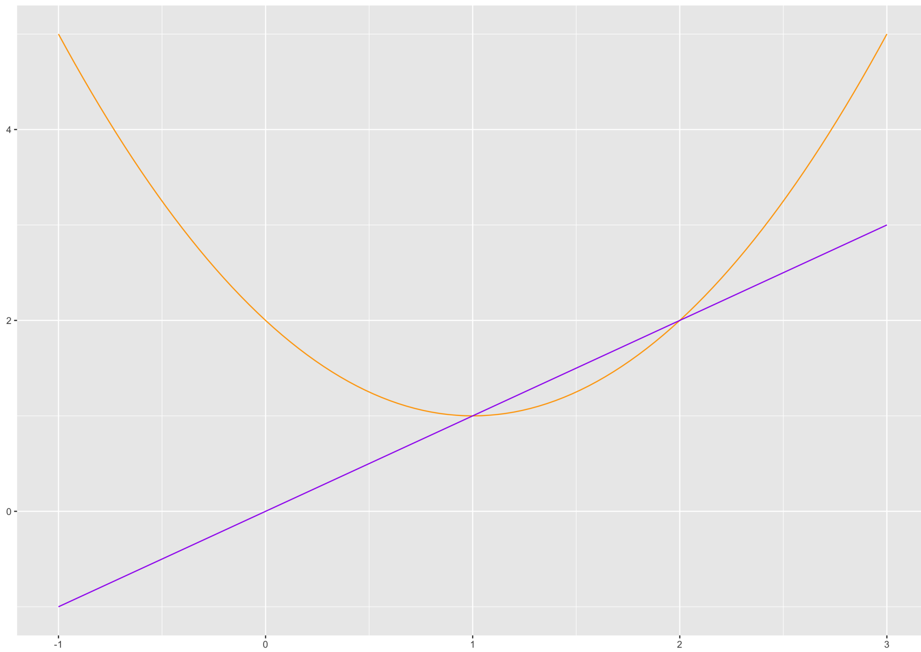
Question 1: Mickens 2.22

$y_{k+1} = y_k^2 - 2y_k + 2$

Below is the graph: $f(x) = x^2 - 2x + 2$ and $y = x$. It is clear that the points of intersection are at $(1, 1)$ and $(2, 2)$



$y^* = {y^*}^2 - 2y^* + 2 \Rightarrow 0 = {y^*}^2 - 3y^* + 2 = (y^* - 1)(y^* - 2)$
y^* is fixed at 1 and 2.

$f'(y) = 2y - 2$
$0 = 2y - 2 \Rightarrow y = 1$
Y is stable.

$f'(2) = 2(2) - 2 = 2$ This is greater than 1 so $y = 2$ is an unstable fixed points.

Prove $y_k = 1 + c^{2^k}$

Base Case:
Let $y_0 = 1 + c$
$y_1 = y_0^2 - 2y_0 + 2 = (1 + c)^2 - 2(1 + c) + 2 = c^2 + 2c + 1 - 2 - 2c + 2 = 1 + c^2 \checkmark$

Induction Hypothesis: For some $k \in Z^+$, all $k_i \leq k$ satisfy:$y_k = y_{k-1}^2 - 2y_{k-1} + 2 = 1 + c^{2^k}$.

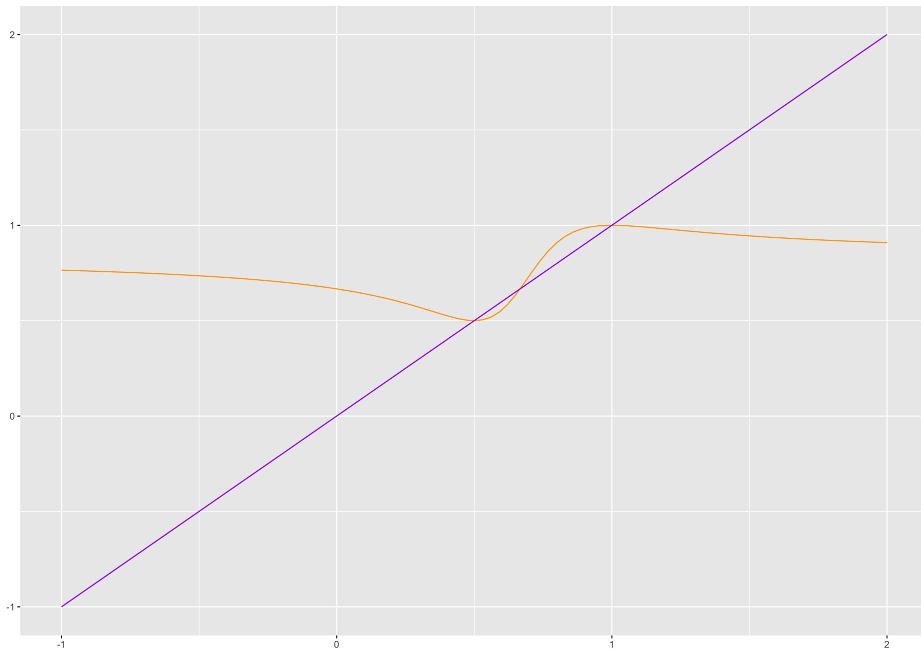
Induction Step: Want to prove: $y_{k+1} = y_k^2 - 2y_k + 2 = 1 + c^{2^{k+1}}$ $y_{k+1} = y_k^2 - 2y_k + 2 = (1 + c^{2^k})^2 - 2(1 + c^{2^k}) + 2 = 1 + 2c^{2^k} + c^{{2^k}^2} - 2c^{2^k} - 2 + 2 = 1 + c^{{2^k}^2} = 1 + c^{2^{k+1}} \checkmark$

Q.E.D.

Determine the interval of initial values of y_k that correspond, respectively, to $|c| > 1$ and $|c| < 1$.
$y_k = 1 + c^{2^k} \Rightarrow y_0 = 1 + c^{2^0} \Rightarrow y_0 = 1 + c$
If $|c| < 1$, $1 - 1 < y_0 < 1 + 1 \Rightarrow 0 < y_0 < 2$
If $|c| > 1$, $y_0 < 1 - 1$ or $y_0 > 1 + 1 \Rightarrow y_0 < 0$ or $y_0 > 2$

Question 2: Mickens 2.23

Part A:
$y_{k+1} = \dfrac{5y_k^2 - 6y_k + 2}{6y_k^2 - 8y_k + 3}$



There are three apparent solutions with rational solutions apparent at $(\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$

$y^* = \dfrac{5y^{*^2} - 6y^* + 2}{6y^{*^2} - 8y^* + 3} \Rightarrow 6y^{*^3} - 8y^{*^2} + 3y^* = 5y^{*^2} - 6y^* + 2 \Rightarrow 6y^{*^3} - 13y^{*^2} + 9y^* - 2 = 0$

Using a cubic calculator, there are solutions at: $y = \frac{1}{2}, \frac{2}{3}, 1$
These are the fixed points.

$f'(y) = \dfrac{(5y^2 - 6y + 2)(12y - 8) - (6y^2 - 8y + 3)(10y - 6)}{(6y^2 - 8y + 3)^2}$

```r
y<-c(1/2,2/3,1)
deriv <- function(y){
   ((5*y^2-6*y+2)*(12*y-8)-(6*y^2-8*y+3)*(10*y-6))/((6*y^2-8*y+3)^2)
}
x <- deriv(y)
x
```

```
## [1]  0 -2  0
```

```r
!(abs(x) > 1)
```

```
## [1]  TRUE FALSE  TRUE
```

Therefore, only $\dfrac{1}{2}$ and 1 are stable fixed points. $\dfrac{2}{3}$ is not.

Prove $y_{k+1} = \dfrac{1+c^{2^k}}{2+c^{2^k}}$

Base Case:
Let $y_0 = \dfrac{1+c}{2+c}$

$$y_1 = \frac{5y_0^2 - 6y_0 + 2}{6y_0^2 - 8y_0 + 3} = \frac{5\left(\dfrac{1+c}{2+c}\right)^2 - 6\dfrac{1+c}{2+c} + 2}{6\left(\dfrac{1+c}{2+c}\right)^2 - 8\dfrac{1+c}{2+c} + 3} = \frac{5(1+c)^2 - 6(1+c)(2+c) + 2(2+c)^2}{6(1+c)^2 - 8(1+c)(2+c) + 3(2+c)^2} = \frac{5(1+2c+c^2) - 6(2+3c+c^2) +}{6(1+2c+c^2) - 8(2+3c+c^2) +}$$

$$\frac{1+c^2}{2+c^2}\checkmark$$

Induction Hypothesis: For some $k \in Z^+$, all $k_i \leq k$ satisfy:$y_k = \dfrac{5y_{k-1}^2 - 6y_{k-1} + 2}{6y_{k-1}^2 - 8y_{k-1} + 3} = \dfrac{1+c^{2^{k-1}}}{2+c^{2^{k-1}}}$

Induction Step: Given $y_{k+1} = \dfrac{5y_k^2 - 6y_k + 2}{6y_k^2 - 8y_k + 3}$, prove $y_{k+1} = \dfrac{1+c^{2^{k+1}}}{2+c^{2^{k+1}}}$

$$y_k + 1 = \frac{5y_k^2 - 6y_k + 2}{6y_k^2 - 8y_k + 3} = \frac{5\left(\dfrac{1+c^{2^k}}{2+c^{2^k}}\right)^2 - 6\dfrac{1+c^{2^k}}{2+c^{2^k}} + 2}{6\left(\dfrac{1+c^{2^k}}{2+c^{2^k}}\right)^2 - 8\dfrac{1+c^{2^k}}{2+c^{2^k}} + 3} = \frac{5(1+c^{2^k})^2 - 6(1+c^{2^k})(2+c^{2^k}) + 2(2+c^{2^k})^2}{6(1+c^{2^k})^2 - 8(1+c^{2^k})(2+c^{2^k}) + 3(2+c^{2^k})^2}$$

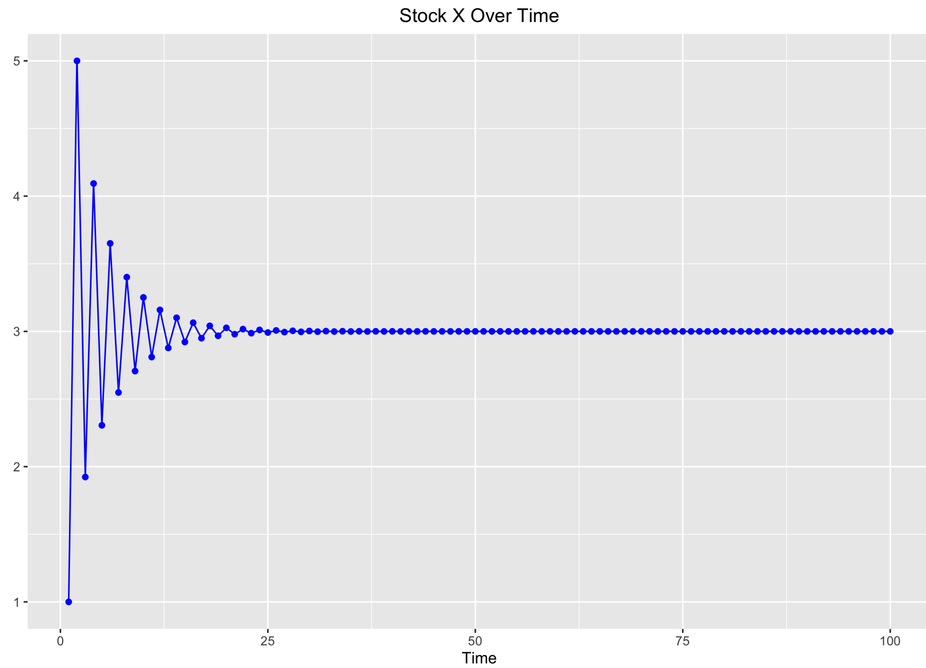
$$= \frac{5(1+2c^{2^k} + c^{2^{k^2}}) - 6(2+3c^{2^k} + c^{2^{k^2}}) + 2(4+4c^{2^k} + c^{2^{k^2}})}{6(1+2c^{2^k} + c^{2^{k^2}}) - 8(2+3c^{2^k} + c^{2^{k^2}}) + 3(4+4c^{2^k} + c^{2^{k^2}})} = \frac{1+c^{2^{k^2}}}{2+c^{2^{k^2}}} = \frac{1+c^{2^{k+1}}}{2+c^{2^{k+1}}}\checkmark$$

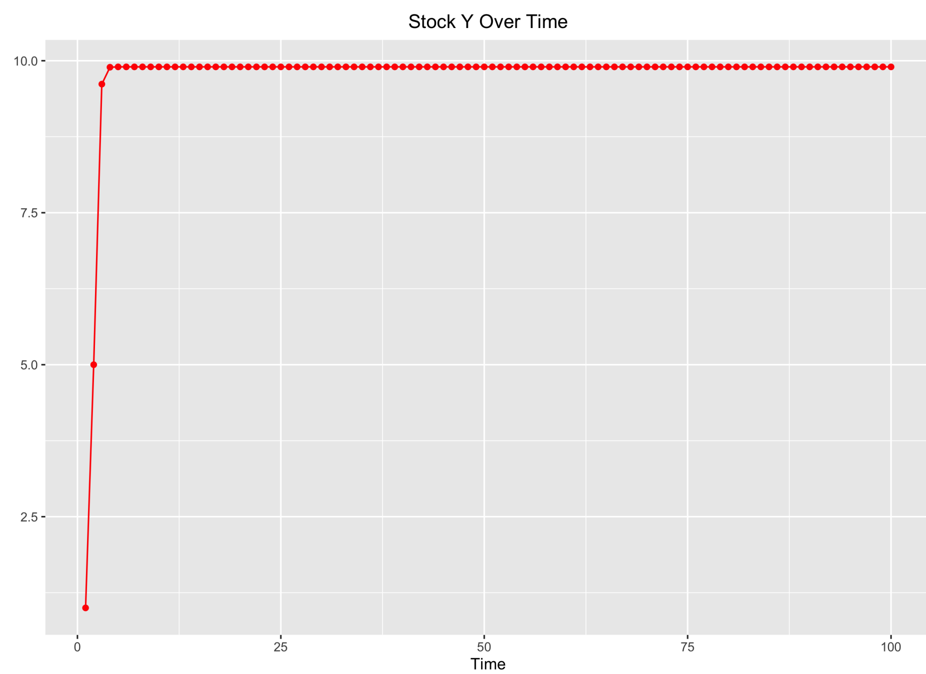
Question 3: Stock Market Investment Decisions

Datasets for this problem can be found here.

Part 1

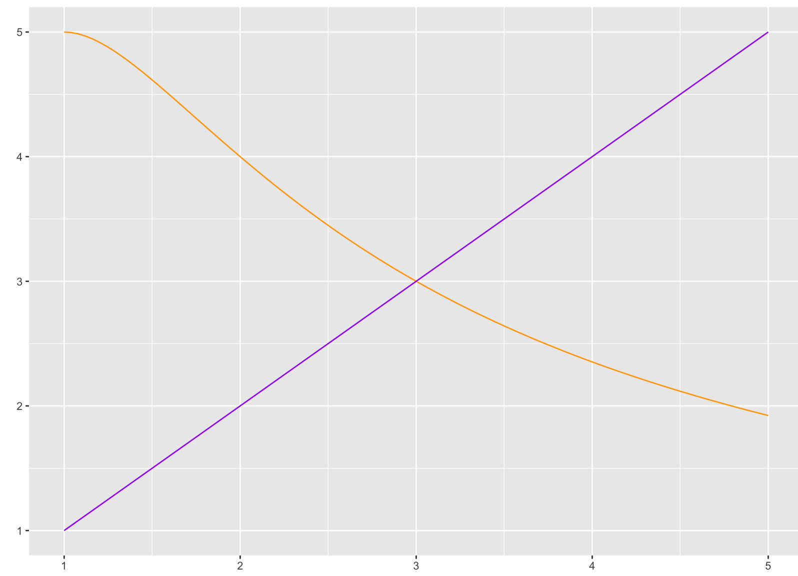
As seen in the graphs below, only stock X shows oscillatory behavior. However, stock X's oscilatory behavior is finite as it converges.





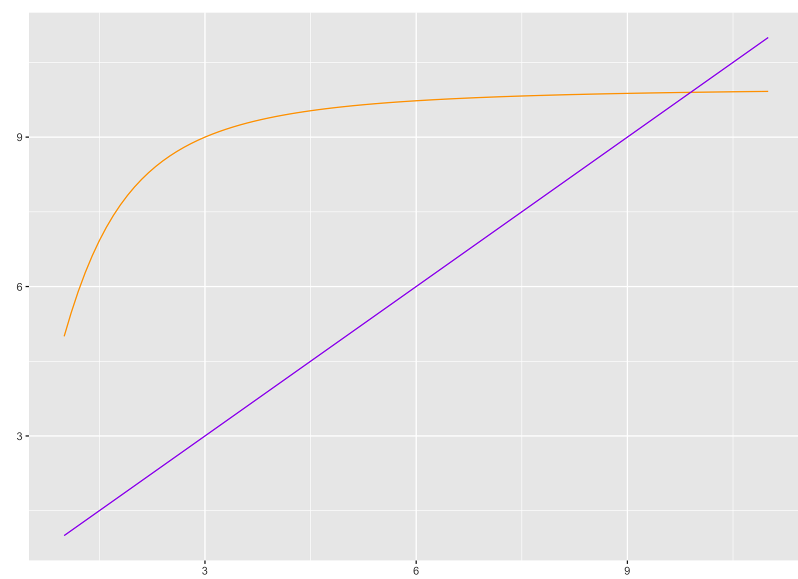
Part 2

For stock X, the graph below show the $f(x) = \dfrac{10x}{1+x^2}$ and $g(x) = x$



They clearly converge at $x = 3$ and this can easily be verifies. $f(3) = \dfrac{10(3)}{1+3^2} = \dfrac{30}{10} = 3$ and $g(3) = 3$

For stock Y, the graph below show the $f(y) = \dfrac{10y^2}{1+y^2}$ and $g(y) = y$



To find the point of intersection:

$\dfrac{10(y^2)}{1+y^2} - y = 0 \Rightarrow \dfrac{10(y^2)}{1+y^2} = y \Rightarrow 10(y)1 + y^2 = 1 \Rightarrow 10y = 1 + y^2 \Rightarrow y^2 - 10y + 1 = 0$

```
a <- 1
b <- -10
c <- 1
(-b + sqrt(b^2 - 4 * a * c))/(2*a)
```

```
## [1] 9.898979
```

Looking at the data:

```
data <- readr::read_csv("question3/stock_data.csv")
```

```
## Warning: Missing column names filled in: 'X1' [1]
```

```
data[69,"x"]
```

```
## # A tibble: 1 x 1
##       x
##   <dbl>
## 1  3.00
```

```
data[7,"y"]
```

```
## # A tibble: 1 x 1
##       y
##   <dbl>
## 1  9.90
```

By Brouwer's Theorem, we can conclude that stock X converges to 3, starting from $t = 69$ and stock Y converges to 9.898979 starting from $t = 7$

Part 3:

Over time, Y is the better better stock as stock Y converges to a higher price than stock X thought they start at the same price.

Question 4: Arms Race Model

Datasets for this problem can be found here.

Part A)

```
arms_data <- readr::read_csv("question4/arms_data.csv")
```

```
## Warning: Missing column names filled in: 'X1' [1]
```

```
firstten <- head(arms_data, 10)
max(firstten$`ut/st`)
```

```
## [1] 3.337716
```

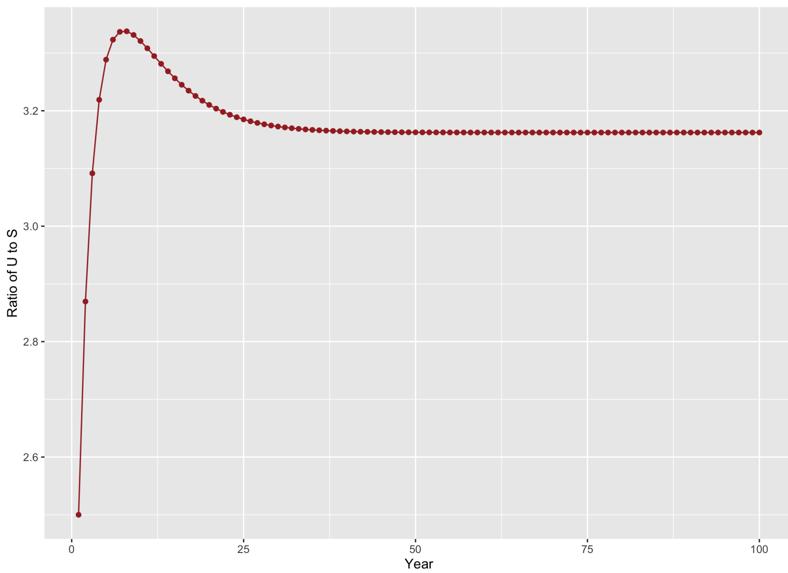
```
which.max(firstten$`ut/st`)
```

```
## [1] 8
```

As seen above, in year 8, the ratio between country U's spending and country X's spending is maximized. This ratio is 3.337716.

Part B)

Country U wins the arms race in the long wrong in terms of spending. If we continue the chart, we see the ratio convergesto around 3.162278. Because the ratio is over 1, this means that country U will continually be spending more money than country S.


Ratio of U to S
3.2
3.0
2.8
2.6
0
25
50
75
100
Year