Physica A 490 (2018) 653–663

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A nonlinear merging protocol for consensus in multi-agent systems on signed and weighted graphs

Shasha Feng [a], Li Wang [a,b], Yijia Li [c,*], Shiwen Sun [a,b], Chengyi Xia [a,b,**]

[a] *Tianjin Key Laboratory of Intelligence Computing and Novel Software Technology, Tianjin University of Technology, Tianjin, 300384, China*
[b] *Key Laboratory of Computer Vision and System (Ministry of Education), Tianjin University of Technology, Tianjin 300384, China*
[c] *School of Management, Tianjin University of Technology, Tianjin 300384, China*

HIGHLIGHTS

- A novel nonlinear merging consensus protocol is proposed by introducing the parameter r to guarantee the states of all agents to converge to the same state.
- The signed graph and negative-weight graph are considered.
- If the undirected graph is connected, then the convergence rate of agents is faster than most other protocols.
- If the undirected graph is unconnected, then the states of all agents can still converge to the same state. However, it is irrealizable for the former protocols to drive all agents to agree on the same state.
- By increasing the parameter r, the convergence rate of agents can be sped up.

ARTICLE INFO

Article history:
Received 15 May 2017
Received in revised form 12 August 2017
Available online 1 September 2017

Keywords:
Signed graph
Negative-weight graph
Unconnected graph
Connected subgraphs
Merging consensus
Multi-agent systems

ABSTRACT

In this paper, we investigate the multi-agent consensus for networks with undirected graphs which are not connected, especially for the signed graph in which some edge weights are positive and some edges have negative weights, and the negative-weight graph whose edge weights are negative. We propose a novel nonlinear merging consensus protocol to drive the states of all agents to converge to the same state zero which is not dependent upon the initial states of agents. If the undirected graph whose edge weights are positive is connected, then the states of all agents converge to the same state more quickly when compared to most other protocols. While the undirected graph whose edge weights might be positive or negative is unconnected, the states of all agents can still converge to the same state zero under the premise that the undirected graph can be divided into several connected subgraphs with more than one node. Furthermore, we also discuss the impact of parameter r presented in our protocol. Current results can further deepen the understanding of consensus processes for multi-agent systems.

1. Introduction

Over the past few years, the researches of complex systems and social networks have captured a great deal of concern from various research communities [1–10]. Among them, multi-agent systems are a typical class of complex systems which

* Corresponding author.
** Corresponding author at: Tianjin Key Laboratory of Intelligence Computing and Novel Software Technology, Tianjin University of Technology, Tianjin, 300384, China.
E-mail addresses: liyijia_tjut@126.com (Y. Li), xialooking@163.com (C. Xia).

http://dx.doi.org/10.1016/j.physa.2017.08.054

consist of agents communicating with each other. Moreover, the applications of multi-agent systems cover an extensive range, which usually require the individual agents to cooperate with each other so as to accomplish the missions beyond capability of single agent [11–13], such as sensor networks [14], traffic control [15], formation control [16], rendezvous [17], coordination control [18] and so on. In particular, during the process of the coordination or formation control of multiple agents, we often make use of the local or nearest neighbor information of agents to design the consensus protocol and it is often considered that consensus is one of the most fundamentally critical problems in coordination control for multi-agent systems, which has attracted substantial attention in the field of information science and great progresses have also been made. For instance, in the pioneering work [19], Jadbabaie et al. successfully explained the consensus on the heading angles observed by Vicsek [20]. Subsequently, Savkin et al. gave another qualitative analysis for the Vicsek model in [21]. In many multi-agent systems, there usually exist some time-delays in practical communication. Hence, it is very important to discuss the consensus problems with time-delays. Xiao et al. introduced the state consensus protocol with time-delays and firstly extended the model of networks of dynamical agents to the case with multiple time-delays in [22]. As a further step, Olfati-Saber et al. further presented a new approach to consensus problems for multiple second-order agents with time-delays in [23]. Till now, the consensus problems with more complex dynamics have been considered, including general linear systems [24–26], general Lipschitz nonlinear multi-agent systems [27], high-order nonlinear systems [28–30], discrete-time systems [31] and so on.

For the consensus protocol design of multi-agent systems, a particularly important problem is to guarantee that all agents can achieve agreement within a finite time [32], and one important reason is that the finite-time consensus can provide much better performance for multi-agent systems under disturbances or uncertain environments. In fact, the convergence rate has also been recognized as a key index to judge the performance of protocol design in the research and development of consensus issues [33,34]. A linear consensus protocol was given in [35] and it was shown that the second smallest eigenvalue of interaction graph Laplacian, named after algebraic connectivity of graph, quantifies the convergence rate of consensus protocol. Xiao and Boyd [36] found that the fastest converging linear literation can be cast as a semi-definite program and increased the algebraic connectivity via the semi-definite convex program.

Recently, most prior works on finite-time consensus results within multi-agent systems often deal with graphs whose edge weights are positive. However, systems associated with undirected graphs whose edge weights might be positive or negative are very common in practice, such as the bidirectional flying of unmanned air vehicles [37]. As an example, Valcher et al. [38] provided a bipartite consensus in high-order multi-agent systems associated with the signed graph; Meng et al. [39] presented the finite-time bipartite consensus results which took the signed graph into consideration. But in their work, the signed graph must be connected and all agents agree on a common quantity only when the signed graph is structurally unbalanced. In [40], the reverse group consensus of multi-agent systems with signed graph was firstly studied without in-degree balance condition and the novel concept of mirror graph was introduced, but it was required that the mirror graph was strongly connected. In the real scenarios, it is too difficult to satisfy such a strong constraint condition in networks. Accordingly, it is a very challenging task to design a merging consensus protocol which can drive all agents converge to the same state under a weaker constraint condition on networks for multi-agent systems. So, the merging consensus protocol can be applied more easily in the real circumstance.

However, during the process of consensus design of multi agent systems, fewer works are performed to investigate the merging consensus problems at present, which is a very significant topic and to be explored further. Thus, we here take into account the multi-agent consensus problems taking place upon undirected graphs which are not connected, in particular for the signed graph and negative-weight graph, and propose a novel nonlinear merging consensus protocol, which just requires local or nearest neighbor information of agents to render all agents reach agreement on a common quantity under a weaker constraint condition on networks for multi-agent systems.

The key features are given in Introduction section: Compared with the existing protocols mentioned in Section 4, the key features of our protocol are as follows: 1. If the undirected graph whose edge weights are positive is connected, then the states of all agents converge to the same state more quickly. 2. If the undirected graph whose edge weights might be positive or negative is unconnected, our protocol can still render all agents to converge to the same state under the condition that undirected graph can be divided into several connected subgraphs with more than one node. While other protocols mentioned in Section 4 cannot do it. By numerical simulations performed in three cases for undirected graphs, we illustrate the effectiveness of our proposed protocol and demonstrate that our obtained consensus results can be viewed as a further extension of those existing consensus protocols which require a strong constraint condition on networks.

The rest of this paper is structured as follows. After introducing some notational preliminaries, the problem addressed in this paper is described in Section 2. Then, we propose a nonlinear distributed protocol and demonstrate the corresponding merging consensus results in Section 3. The numerical simulations are carried out in Section 4 to validate the theoretical analysis. Finally, some concluding remarks are drawn in the last section.

2. Problem descriptions

2.1. Notations

At first, we introduce two notations, $I_n = \{1, 2, \ldots, n\}$ denotes the set of n autonomous agents labeled from 1 to n and $\mathrm{sgn}(x)$ indicates the sign function of any $x \in R$ such that

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases} \tag{1}$$

2.2. Undirected graphs, signed graphs and negative-weight graph

Here we introduce the undirected graph, especially for the signed graph and negative-weight graph. The signed graph belongs to a special undirected graph whose edge weights contain positive weights and negative weights. And the negative-weight graph is a particular signed graph whose edge weights are negative.

Let a graph $G = (V, \varepsilon, A)$ be an undirected graph of order n, with a vertex set $V = \{v_1, v_2, \ldots, v_n\}$, an edge set $\varepsilon \subseteq (v_i, v_j) : v_i, v_j \in V$, and a weighted adjacency matrix $A = [w_{ij}] \in R^{n\times n}$ with weights $w_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \varepsilon \Leftrightarrow (v_i, v_j) \in \varepsilon$. And $(v_i, v_i) \notin \varepsilon$, i.e. $w_{ii} = 0$, that is, there is no self-loops or multiple links in G (i.e., G is a simple graph). The set of neighbors of vertex v_i is denoted by $N_i = \{v_j : (v_j, v_i) \in \varepsilon\}$, $N_{i_1} = \{v_j : (v_j, v_i) \in \varepsilon, w_{ij} > 0\}$, $N_{i_2} = \{v_j : (v_j, v_i) \in \varepsilon, w_{ij} < 0\}$.

If G is a classical graph, then $w_{ij} \geq 0$ and the corresponding Laplacian matrix $L = (l_{ij}) \in R^{n\times n}$ is defined by [41] as follows:

$$l_{ij} = \begin{cases} \sum_{k\in N_i} w_{ik}, & j = i \\ -w_{ij}, & j \neq i. \end{cases} \tag{2}$$

If G is a signed graph with $w_{ij} \neq 0$ or a negative-weight graph with $w_{ij} < 0$, then the corresponding Laplacian matrix is defined by [42] in the form of

$$l'_{ij} = \begin{cases} \sum_{k\in N_i} |w_{ik}|, & j = i \\ -w_{ij}, & j \neq i. \end{cases} \tag{3}$$

If there exists a path between v_i and v_j, then it is connected between the two vertexes. And if any two vertexes in G can be connected by paths, then G is connected.

2.3. Some lemmas

Lemma 1 (*[43]*). *Matrix $A = \{w_{ij}\} \in C^{n\times n}$, then all of the eigenvalues of matrix A which are labeled $\lambda_1, \lambda_2, \ldots, \lambda_n$ fall in the complex plane $\bigcup D_i(A) = \bigcup \{z | z - w_{ii}| \leq P_i\}$, where $i \in \{1, 2, \ldots, n\}$ and $P_i = \sum_{j=1, i\neq j}^{n} |w_{ij}|$.*

Lemma 2 (*[44]*). *Let I be an interval having the point a as a limit point. Let f, g, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have: $g(x) \leq f(x) \leq h(x)$ and also suppose that: $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$*

3. Merging consensus

For any agent labeled i, $i \in I_n$. The state space of all agents are R and the state equations of the system are given by,

$$\dot{x}_i = u_i, \qquad i \in I_n. \tag{4}$$

We consider a distributed nonlinear protocol as follows,

$$u_i = \sum_{j\in N_i} w_{ij}\,\mathrm{sgn}(x_j - r\,\mathrm{sgn}(w_{ij})x_i)|x_j - r\,\mathrm{sgn}(w_{ij})x_i|^{\alpha} \tag{5}$$

where $0 < \alpha < 1$, $r > 1$.

For the protocol of each agent, we adopt the interaction information between neighboring agents in (5). It is obvious that each agent considers individual state r times as much as the states of neighboring agents. The parameter $r > 1$ indicates that each agent regards individual state more than the neighboring agents states. And if the neighboring agents v_i and v_j have the cooperative interaction, then they trust each other and use the true state for information interaction, i.e., $x_j - r\,\mathrm{sgn}(w_{ij})x_i = x_j - rx_i$. Otherwise, if they have the antagonistic interaction, then they do not trust each other and utilize the opposite state for information interaction. i.e., $x_j - r\,\mathrm{sgn}(w_{ij})x_i = x_j - r(-x_i)$. For a particular situation $r = 1$, then (5) changes into

$$u_i = \sum_{j\in N_i} w_{ij}\,\mathrm{sgn}(x_j - \mathrm{sgn}(w_{ij})x_i)|x_j - \mathrm{sgn}(w_{ij})x_i|^{\alpha} \tag{6}$$

which is a distributed nonlinear protocol proposed in [39]. By designing a parameter $r > 1$, we will prove that merging consensus protocol (5) can guarantee the agents associated with an undirected graph G whose edge weights might be positive or negative to achieve merging consensus which can drive all agents agree on a common state zero even though G is not connected but can be divided into several connected subgraphs with more than one node. i.e. $x_i(t)$ satisfies,

$$\lim_{t\to\infty} x_i(t) = 0, \qquad i \in I_n. \tag{7}$$

Before the former result (7) is derived, an assumption is needed.

Assumption 1. Assume that the undirected graph G is not connected but can be partitioned into several connected subgraphs with more than one node, and the weight matrix A of G is divided into two parts A_1 and A_2 according to the sign of w_{ij}, i.e., $A_1 = (w_{ij_1})_{n\times n}$, where

$$wij_1 = \begin{cases} w_{ij}, & w_{ij} \geqslant 0 \\ 0, & w_{ij} < 0 \end{cases}$$

and $A_2 = (w_{ij_2})_{n\times n}$, where

$$wij_2 = \begin{cases} w_{ij}, & w_{ij} \leqslant 0 \\ 0, & w_{ij} > 0 \end{cases}$$

Theorem 1. *For system (4), under Assumption 1, if the decimal part of $\alpha \in (0, 1)$ is an even number and $r > 1$, then the common state zero is reached.*

Proof. On basis of the sign of w_{ij}, (5) collapses into

$$\dot{x}_i(t) = \sum_{j\in N_{i_1}} w_{ij_1} \mathrm{sgn}(x_j - rx_i) \times (x_j - rx_i)^\alpha + \sum_{j\in N_{i_2}} w_{ij_2} \mathrm{sgn}(x_j + rx_i) \times (x_j + rx_i)^\alpha \tag{8}$$

let

$$\dot{x}_{i_1}(t) = \sum_{j\in N_{i_1}} w_{ij_1} \mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha$$

$$\dot{x}_{i_2}(t) = \sum_{j\in N_{i_2}} w_{ij_2} \mathrm{sgn}(x_j + rx_i)(x_j + rx_i|)^\alpha$$

then

$$\dot{x}_i(t) = \dot{x}_{i_1}(t) + \dot{x}_{i_2}(t).$$

Among them, $\dot{x}_{i_1}(t)$ can be divided into four components as follows,

$$\dot{x}_{i_1}(t) = \dot{x}_{i_{11}}(t) + \dot{x}_{i_{12}}(t) + \dot{x}_{i_{13}}(t) + \dot{x}_{i_{14}}(t) \tag{9}$$

where

$$\dot{x}_{i_{11}}(t) = \sum_{j\in N_{i_1}, x_j - rx_i \leqslant -1} w_{ij_1} \mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha$$

$$\dot{x}_{i_{12}}(t) = \sum_{j\in N_{i_1}, -1 < x_j - rx_i \leqslant 0} w_{ij_1} \mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha$$

$$\dot{x}_{i_{13}}(t) = \sum_{j\in N_{i_1}, 0 < x_j - rx_i < 1} w_{ij_1} \mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha$$

$$\dot{x}_{i_{14}}(t) = \sum_{j\in N_{i_1}, x_j - rx_i \geqslant 1} w_{ij_1} \mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha.$$

We will demonstrate $\lim_{t\to\infty} x_{i_{11}}(t) = 0$ where $i \in I_n$ at first. In virtue of the deduced processes of $x_{i_{12}}(t)$, $x_{i_{13}}(t)$ and $x_{i_{14}}(t)$ are similar to $x_{i_{11}}(t)$, we omit them here.

For $x_{i_{11}}(t)$,

$$\begin{pmatrix} \sum\limits_{j\in N_{1_1}, x_j - rx_1 \leqslant -1} w_{ij_1}(x_j - rx_1) \\ \sum\limits_{j\in N_{2_1}, x_j - rx_2 \leqslant -1} w_{ij_1}(x_j - rx_2) \\ \vdots \\ \sum\limits_{j\in N_{n_1}, x_j - rx_n \leqslant -1} w_{ij_1}(x_j - rx_n) \end{pmatrix} = [-L_1 - (r-1)D_1] \begin{pmatrix} x_{1_{11}}(t) \\ x_{2_{11}}(t) \\ \vdots \\ x_{n_{11}}(t) \end{pmatrix} \tag{10}$$

where L_1, D_1 are the corresponding Laplacian and degree matrices of A_1 respectively.

According to Gerschgorin disc theorem which is introduced in Lemma 1, we can get $\lambda[-L_1-(r-1)D_1]<0$ and in virtue of $\sum_{j\in N_{i_1},x_j-rx_1\leqslant -1} w_{ij_1}(x_j-rx_i)\leqslant 0$, we can obtain

$$x_{i_{11}}(t)\geqslant 0 \tag{11}$$

And then, taking the limit on both sides of the Ineq. (11) as t goes to infinity, we acquire

$$\lim_{t\to\infty} x_{i_{11}}(t)\geqslant 0. \tag{12}$$

Meanwhile,

$$\begin{aligned}\dot{x}_{i_{11}}(t) &= \sum_{j\in N_{i_1},x_j-rx_i\leqslant -1} w_{ij_1}\operatorname{sgn}(x_j-rx_i)(x_j-rx_i)^{\alpha}\\ &= \sum_{j\in N_{i_1},x_j-rx_i\leqslant -1} -w_{ij_1}(x_j-rx_i)^{\alpha}.\end{aligned}$$

Due to $(x_j-rx_i)^{\alpha}\geqslant 1$ when $x_j-rx_i\leqslant -1$,
then

$$\dot{x}_{i_{11}}(t)\leqslant -\sum_{j\in N_{i_1},x_j-rx_i\leqslant -1} w_{ij_1}. \tag{13}$$

By solving the differential Ineq. (13), we can obtain the following inequality,

$$x_{i_{11}}(t)\leqslant x_{i_{11}}(0)-\sum_{j\in N_{i_1},x_j-rx_i\leqslant -1} w_{ij_1}\times t \tag{14}$$

Taking the limit on both sides of the Ineq. (14) as t goes to infinity,
we have

$$\lim_{t\to\infty} x_{i_{11}}(t)\leqslant \lim_{t\to\infty}(x_{i_{11}}(0)-\sum_{j\in N_{i_1},x_j-rx_i\leqslant -1} w_{ij_1}\times t)\leqslant 0. \tag{15}$$

By Ineq. (12) and Ineq. (15), we can gain $\lim_{t\to\infty}x_{i_{11}}(t)=0$ on the basis of Squeeze theorem given in Lemma 2.
Likewise, we can prove $\lim_{t\to\infty}x_{i_{12}}(t)=0$, $\lim_{t\to\infty}x_{i_{13}}(t)=0$ and $\lim_{t\to\infty}x_{i_{14}}(t)=0$.
Hence, we can get

$$\lim_{t\to\infty} x_{i_1}(t)=\lim_{t\to\infty}(x_{i_{11}}(t)+x_{i_{12}}(t)+x_{i_{13}}(t)+x_{i_{14}}(t))=0. \tag{16}$$

Let

$$\dot{x}_{i_2}(t)=\dot{x}_{i_{21}}(t)+\dot{x}_{i_{22}}(t)+\dot{x}_{i_{23}}(t)+\dot{x}_{i_{24}}(t) \tag{17}$$

where

$$\dot{x}_{i_{21}}(t)=\sum_{j\in N_{i_2},x_j+rx_i\leqslant -1} w_{ij_2}\operatorname{sgn}(x_j+rx_i)(x_j+rx_i)^{\alpha}$$

$$\dot{x}_{i_{22}}(t)=\sum_{j\in N_{i_2},-1<x_j+rx_i\leqslant 0} w_{ij_2}\operatorname{sgn}(x_j+rx_i)(x_j+rx_i)^{\alpha}$$

$$\dot{x}_{i_{23}}(t)=\sum_{j\in N_{i_2},0<x_j+rx_i<1} w_{ij_2}\operatorname{sgn}(x_j+rx_i)(x_j+rx_i)^{\alpha}$$

$$\dot{x}_{i_{24}}(t)=\sum_{j\in N_{i_2},x_j+rx_i\geqslant 1} w_{ij_2}\operatorname{sgn}(x_j+rx_i)(x_j+rx_i)^{\alpha}.$$

In the same method as $x_{i_1}(t)$ to prove that $\lim_{t\to\infty}x_{i_2}(t)$ is zero, and thus we can obtain the following equality

$$\lim_{t\to\infty} x_i(t)=\lim_{t\to\infty}(x_{i_1}(t)+x_{i_2}(t))=0. \tag{18}$$

From Eq. (18), Theorem 1 can be proved.

After that, the similar result can be derived for system (4) with decimal part of $\alpha \in (0, 1)$ is an odd number as follows.

Theorem 2. *For system (4), under Assumption 1, if the decimal part of $\alpha \in (0, 1)$ is an odd number and $r > 1$, then the common state zero is achieved.*

Proof. (1) If $\alpha \in (0, 1)$ is an odd number and $\alpha \neq 0.5$, the proof of $\lim_{t\to\infty} x_i(t) = 0$ is similar to Theorem 1, so we omit them here.

(2) If $\alpha = 0.5$, $(x_j - r\mathrm{sgn}(w_{ij}))^\alpha$ is non-existent when $x_j - r\mathrm{sgn}(w_{ij}) < 0$. According to the sign of w_{ij}, (4) can be described by

$$\dot{x}_i(t) = \sum_{j\in N_{i_1}} w_{ij_1}\mathrm{sgn}(x_j - rx_i) \times (x_j - rx_i)^\alpha + \sum_{j\in N_{i_2}} w_{ij_2}\mathrm{sgn}(x_j + rx_i) \times (x_j + rx_i)^\alpha \tag{19}$$

let

$$\dot{x}_{i_1}(t) = \sum_{j\in N_{i_1}} w_{ij_1}\mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha$$

$$\dot{x}_{i_2}(t) = \sum_{j\in N_{i_2}} w_{ij_2}\mathrm{sgn}(x_j + rx_i)(x_j + rx_i|)^\alpha$$

then

$$\dot{x}_i(t) = \dot{x}_{i_1}(t) + \dot{x}_{i_2}(t).$$

Among them, $\dot{x}_{i_1}(t)$ can be divided into two ingredients as follows,

$$\dot{x}_{i_1}(t) = \dot{x}_{i_{11}}(t) + \dot{x}_{i_{12}}(t) \tag{20}$$

where

$$\dot{x}_{i_{11}}(t) = \sum_{j\in N_{i_1}, 0\leqslant x_j - rx_i < 1} w_{ij_1}\mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha$$

$$\dot{x}_{i_{12}}(t) = \sum_{j\in N_{i_1}, x_j - rx_i \geqslant 1} w_{ij_1}\mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha.$$

Next, we will prove $\lim_{t\to\infty} x_{i_{11}}(t) = 0$.
For $x_{i_{11}}(t)$,

$$\begin{pmatrix} \sum_{j\in N_{1_1}, 0\leqslant x_j - rx_1 < 1} w_{ij_1}(x_j - rx_1) \\ \sum_{j\in N_{2_1}, 0\leqslant x_j - rx_2 < 1} w_{ij_1}(x_j - rx_2) \\ \vdots \\ \sum_{j\in N_{n_1}, 0\leqslant x_j - rx_n < 1} w_{ij_1}(x_j - rx_n) \end{pmatrix} = [-L_1 - (r-1)D_1] \begin{pmatrix} x_{1_{11}}(t) \\ x_{2_{11}}(t) \\ \vdots \\ x_{n_{11}}(t) \end{pmatrix} \tag{21}$$

where L_1, D_1 are the corresponding Laplacian and degree matrices of A_1 respectively.
On the basis of Gerschgorin disc theorem which is introduced in Lemma 1, we can get $\lambda[-L_1 - (r-1)D_1] < 0$ and because of $\sum_{j\in N_{i_1}, 0\leqslant x_j - rx_1 < 1} w_{ij_1}(x_j - rx_i) \geqslant 0$, we can obtain

$$x_{i_{11}}(t) \leqslant 0. \tag{22}$$

And then, taking the limit on both sides of the InEq. (22) as t goes to infinity, we can get

$$\lim_{t\to\infty} x_{i_{11}}(t) \leqslant 0 \tag{23}$$

Meanwhile,

$$\begin{aligned} \dot{x}_{i_{11}}(t) &= \sum_{j\in N_{i_1}, 0\leqslant x_j - rx_i < 1} w_{ij_1}\mathrm{sgn}(x_j - rx_i)(x_j - rx_i)^\alpha \\ &= \sum_{j\in N_{i_1}, 0\leqslant x_j - rx_i < 1} w_{ij_1}(x_j - rx_i)^\alpha. \end{aligned}$$

Table 1
Estimated Settling Time of protocols.

T	$x(0) = [8, 12, 0, 14, 5, 8]^T$
Our protocol (5)	0.8822
Protocol in [32]	10.5755
Protocol in [39]	6.0399

Due to $(x_j - rx_i)^\alpha \geqslant x_j - rx_i$ when $0 \leqslant x_j - rx_i < 1$,
then

$$\dot{x}_{i_{11}}(t) \geqslant \sum_{j \in N_{i_1}, 0 \leqslant x_j - rx_i < 1} w_{ij_1}(x_j - rx_i) = [-L_1 - (r-1)D_1]x_{11}(t)(i) \tag{24}$$

where $[-L_1 - (r-1)D_1]x_{11}(t)(i)$ denotes the ith element of the vector $[-L_1 - (r-1)D_1]x_{11}(t)$.
By solving the differential Ineq. (24), we can acquire the following inequality,

$$x_{i_{11}}(t) \geqslant x_{i_{11}}(0) \times e^{t \times \lambda[-L_1 - (r-1)D_1]}. \tag{25}$$

Because of $\lambda[-L_1 - (r-1)D_1] < 0$ and taking the limit on both sides of the Ineq. (25) as t goes to infinity, we have

$$\lim_{t \to \infty} x_{i_{11}}(t) \geqslant 0. \tag{26}$$

By Ineq. (23) and Ineq. (26), we can gain $\lim_{t\to\infty} x_{i_{11}}(t) = 0$ on the basis of Squeeze theorem given in Lemma 2. Similarly, we can prove $\lim_{t\to\infty} x_{i_{12}}(t) = 0$. Hence, we can get

$$\lim_{t \to \infty} x_{i_1}(t) = \lim_{t \to \infty} (x_{i_{11}}(t) + x_{i_{12}}(t)) = 0. \tag{27}$$

Let

$$\dot{x}_{i_2}(t) = \dot{x}_{i_{21}}(t) + \dot{x}_{i_{22}}(t) \tag{28}$$

where

$$\dot{x}_{i_{21}}(t) = \sum_{j \in N_{i_2}, 0 \leqslant x_j + rx_i < 1} w_{ij_2} \operatorname{sgn}(x_j + rx_i)(x_j + rx_i)^\alpha$$

$$\dot{x}_{i_{22}}(t) = \sum_{j \in N_{i_2}, x_j + rx_i \geqslant 1} w_{ij_2} \operatorname{sgn}(x_j + rx_i)(x_j + rx_i)^\alpha.$$

In the same way as $x_{i_1}(t)$ to demonstrate that $\lim_{t\to\infty} x_{i_2}(t)$ is zero, and thus we can obtain the following equality

$$\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} (x_{i_1}(t) + x_{i_2}(t)) = 0. \tag{29}$$

To combine situation (1) and (2), Theorem 2 can be proved.

Thus, we can get the common state zero can be reached when $\alpha \in (0, 1)$ and $r > 1$ by combining the results of Theorem 1 and Theorem 2.

4. Simulation results

4.1. Example 1

In order to compare the convergence rate of agents when we use our protocol (5) with that when we utilize protocols mentioned in [32] and [39], we perform simulations in Fig. 2(a), Fig. 2(b) and Fig. 2(c) respectively using our protocol (5), protocols proposed in [32] and [39] under the same undirected graph in Fig. 1 whose edge weights are positive.

For the simulation tests, we choose $\alpha = 0.9$, $r = 3$ and the initial states of agents are $x(0) = [8, 12, 0, 14, 5, 8]^T$ in our protocol and protocols in [32] and [39].

When we utilize our protocol (5), protocols in [32] and [39] to guarantee all agents to agree on a common state under the same undirected graph in Fig. 1, it is not difficult to see that the convergence rate of agents with our protocol is the quickest, which is demonstrated in Table 1 that shows the estimated settling time of agents with protocol (5) and protocols in [32] and [39]. In Table 1, the estimated settling time of agents with our protocol (5) is 0.8822 which is the quickest, that with protocol in [32] is 10.5755 and that with protocol in [39] is 6.0399.

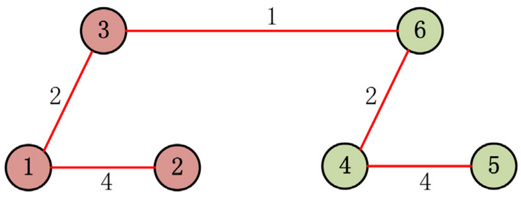

Fig. 1. Connected graph whose edge weights are positive.

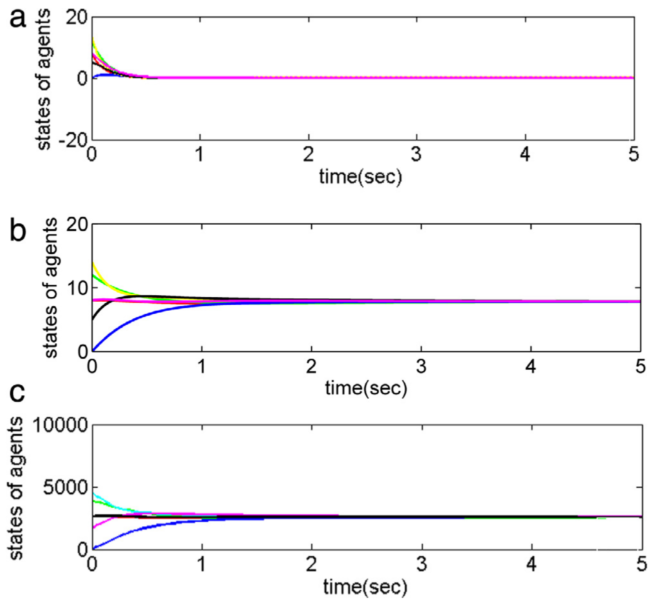

Fig. 2. Fig. 2(a), Fig. 2(b) and Fig. 2(c) correspond to the network associated with graph in Fig. 1 respectively using our protocol (5), protocols in [32] and [39] to control agents. And we can see clearly that the convergence rate of agents with our protocol is the quickest.

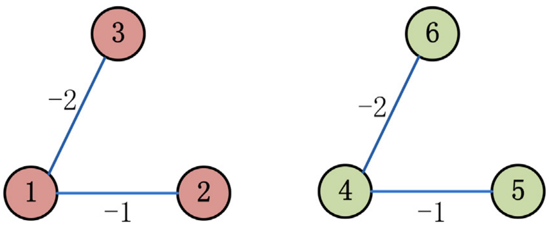

Fig. 3. Negative-weight graph which is not connected but can be divided into two connected subgraphs.

4.2. Example 2

In this example, we consider multi-agent network associated with the negative-weight graph which is not connected but can be divided into several connected subgraphs in Fig. 3. We use our protocol and the protocol mentioned in [39] respectively to control agents.

For the simulation tests, we choose $\alpha = 0.9$, $r = 3$ and the initial states of agents are $x(0) = [8, 12, 0, 14, 5, 8]^T$ in our protocol and protocol in [39], and the corresponding simulation results are shown in Fig. 4(a) and Fig. 4(b). It is obvious that our protocol can guarantee all agents to agree on a common state zero, whereas the protocol in [39] cannot render all agents reach agreement, but make them divide into four groups.

4.3. Example 3

So as to indicate that the convergence rate of agents becomes faster and faster with the increasing of parameter r in our protocol, we select $r = 1.01$, $r = 1.50$, $r = 3.01$, $r = 5.01$ and $r = 7.00$ respectively under the same signed graph in Fig. 3.

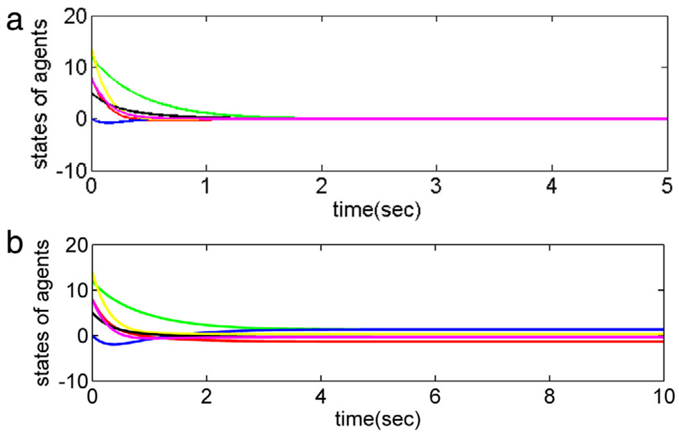

Fig. 4. Fig. 4(a) and Fig. 4(b) correspond to the network associated with negative-weight graph in Fig. 3 respectively using our protocol (5) and the protocol in [39] to control agents. And we it is obvious that our protocol can make all agents arrive at the same state, which the protocol in [39] cannot do.

Table 2
Estimated Settling Time of our protocol with $r = 1.01$, $r = 1.50$, $r = 3.01$, $r = 5.01$ and $r = 7.00$.

T	$x(0) = [8, 12, 0, 14, 5, 8]^T$
$r = 1.01$	140.2427
$r = 1.50$	5.0112
$r = 3.01$	2.4330
$r = 5.01$	1.1513
$r = 7.00$	0.8617

For the simulation tests, we choose $\alpha = 0.9$ and the initial states of agents are $x(0) = [8, 12, 0, 14, 5, 8]^T$ in our protocol, and the corresponding simulation tests when $r = 1.01$, $r = 1.50$, $r = 3.01$, $r = 5.01$ and $r = 7.00$ are represented in Fig. 5(a), Fig. 5(b), Fig. 5(c), Fig. 5(d) and Fig. 5(e). It is evident that the convergence rate of agents with our protocol gets faster and faster with the increasing of parameter r, which is exhibited in Table 2 that shows the estimated settling time of agents with our protocol when $r = 1.01$, $r = 1.50$, $r = 3.01$, $r = 5.01$ and $r = 7.00$. In Table 2, the estimated settling time of agents with our protocol when $r = 1.01$ is 140.2427, that when $r = 1.5$ is 5.0112, that when $r = 3.01$ is 2.4330, that when $r = 5.01$ is 1.1513 and that when $r = 7.00$ is 0.8617.

5. Conclusions

In summary, we have investigated the merging consensus problems on multi-agent networks associated with undirected networks through a novel nonlinear protocol. It has been shown that the merging consensus problems can be solved regardless of whether the associated undirected graph is connected or not. If the undirected graph whose edge weights are positive is connected, then the states of all agents converge to the same state more quickly than that with the former protocols in [32] and [39]. And this phenomenon can be clearly shown in the simulation results of example 1. Also, if the undirected graph whose edge weights might be positive or negative is unconnected, all the agents can still reach agreement on a common state under the premise that the undirected graph can be divided into several connected subgraphs in excess of one node. However, it is irrealizable for the former protocol in [39] to drive all agents to agree on a common quantity when the undirected graph is not connected. And this phenomenon can also be exhibited in the simulations of example 2 and example 3. In example 4, it is easy to see that the convergence rate of agents gets faster with the increasing of parameter r. Moreover, the final consensus state of agents can be demonstrated to be zero which is not dependent on the initial states of agents. Our obtained consensus results bring new insights into consensus studies on multi-agent networks which are not connected. Furthermore, we illustrate the effectiveness of our proposed protocol by numerical simulations performed in three examples for undirected graphs.

In addition, there still remains a deficiency of the established merging consensus results in this paper. We cannot demonstrate the settling time for the proposed protocol due to the limitation of our mathematical experience, only illustrate it via numerical simulations. This leaves us a further study of the results in this paper on how to further deduce the settling time for our proposed protocol.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China under Grant Nos. 61403280, 61374169, and 61773286. LW acknowledge the support from Excellent Young Teachers Program of Tianjin. LW, SWS and

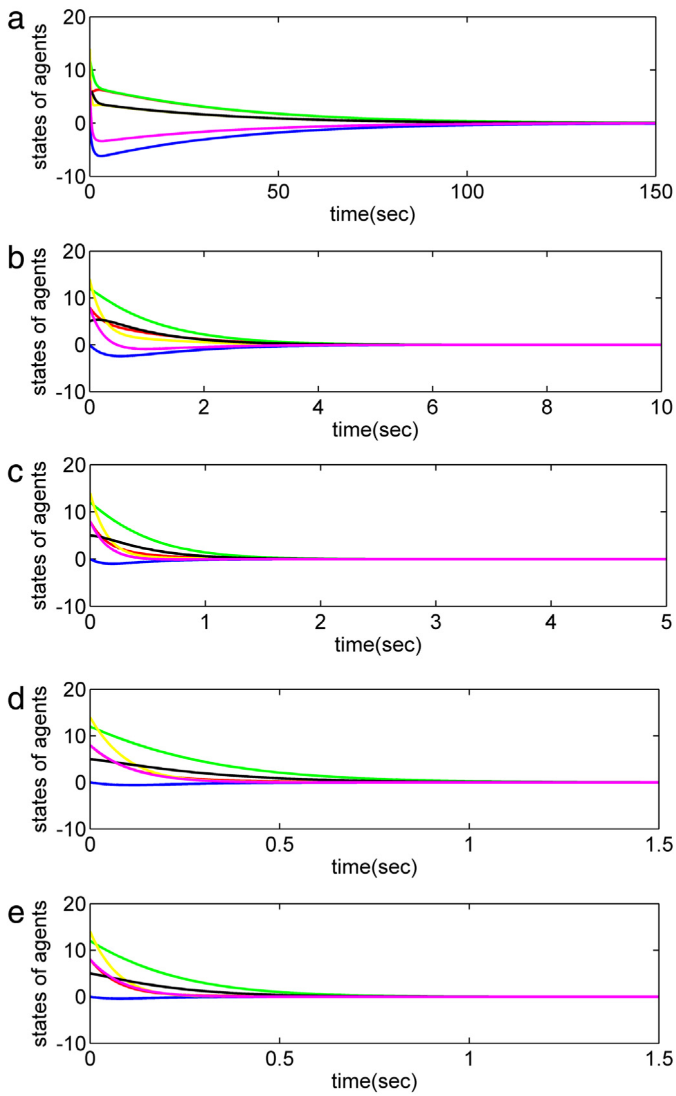

Fig. 5. Fig. 5(a), Fig. 5(b), Fig. 5(c), Fig. 5(d) and Fig. 5(e) correspond to the network associated with signed graph in Fig. 3 using our protocol with $r = 1.01$, $r = 1.50$, $r = 3.01$, $r = 5.01$ and $r = 7.00$ respectively. And it is evident that the convergence rate of agents with our protocol gets faster and faster with the increasing of parameter r.

CYX acknowledge the support from 131 Innovative Talents Program of Tianjin. CYX also acknowledges the support from the Scientific Research Foundation for the Returned Overseas Chinese Scholars (Ministry of Education).

References

[1] C.B. Keating, P.F. Katina, J.M. Bradley, Complex system governance: concept, challenges, and emerging research, Int. J. Syst. Syst. Eng. 5 (3) (2014) 263–288.
[2] Z. Wang, C.T. Bauch, S. Bhattacharyya, Statistical physics of vaccination, Phys. Rep. 664 (9) (2016) 1–113.
[3] M.R. D'Orsogna, M. Perc, Statistical physics of crime: A review, Phys. Life Rev. 12 (2015) 1–21.
[4] D. Helbing, D. Brockmann, T. Chadefaux, Saving human lives: What complexity science and information systems can contribute, J. Stat. Phys. 158 (3) (2015) 735–781.
[5] M. Perc, The matthew effect in empirical data, J. R. Soc. 11 (98) (2014) 20140378.
[6] M. Perc, Phase transitions in models of human cooperation, Phys. Lett. A 380 (36) (2016) 2803–2808.
[7] Z. Wang, L. Wang, A. Szolnoki, Evolutionary games on multilayer networks: A colloquium, Eur. Phys. J. B 88 (5) (2015) 124.
[8] C.J. Wang, L. Wang, J. Wang, S.W. Sun, C.Y. Xia, Inferring the reputation enhances the cooperation in the public goods game on interdependent lattices, Appl. Math. Comput. 293 (2017) 18–29.
[9] M.H. Chen, L. Wang, S.W. Sun, J. Wang, C.Y. Xia, Evolution of cooperation in the spatial public goods game with adaptive reputation assortment, Phys. Lett. A 380 (2016) 40–47.

[10] M.H. Chen, L. Wang, J. Wang, S.W. Sun, C.Y. Xia, Impact of individual response strategy on the spatial public goods game within mobile agents, Appl. Math. Comput. 251 (2015) 192–202.
[11] R. Olfati-Saber, Flocking for multi-agent dynamic systems: algorithms and theory, IEEE Trans. Automat. Control 51 (3) (2006) 401–420.
[12] M. Perc, A. Szolnoki, Coevolutionary games - A mini review, BioSystems 99 (2010) 109–125.
[13] L. Cheng, Z.G. Hou, M. Tan, Y.Z. Lin, W.J. Zhang, Neural-network-based adaptive leader-following control for multiagent systems with uncertainties, IEEE Trans. Neural Netw. 21 (8) (2010) 1351–1358.
[14] R. Olfati-Saber, J.S. Shamma, Consensus filters for sensor networks and distributed sensor fusion, in: Procceedings of 44th IEEE Conference on Decision and Control, and the European Control Conference, 2005, pp. 6698–6703.
[15] J.A. Fax, R.M. Murray, Information flow and cooperative control of vehicle formations, IEEE Trans. Automat. Control 49 (9) (2004) 1465–1476.
[16] Z. Lin, B. Francis, M. Maggiore, Necessary and sufficient graphical conditions for formation control of unicycles, IEEE Trans. Automat. Control 50 (1) (2005) 121–127.
[17] J. Lin, A.S. Morse, B.D.O. Anderson, The multi-agent rendezvous problem, in: Proceedings of 42nd IEEE Conference on Decision and Control, vol. 2, 2003, pp. 1508–1513.
[18] G.J. Shen, Y.Y. Yang, A dynamic signal coordination control method for urban arterial roads and its application, Front. Inf. Technol. Electr. Eng. 17 (9) (2016) 907–918.
[19] A. Jadbabaie, J. Lin, A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Trans. Automat. Control 48 (6) (2007) 943–948.
[20] T. Vicsek, A. Czirok, E.B. Jacob, I. Cohen, O. Schochet, Novel type of phase transition in a system of self-driven particles, Phys. Rev. Lett. 75 (6) (1995) 1226–1229.
[21] A.V. Savkin, Coordinated collective motion of groups of autonomous mobile robots: Analysis of Vicsek's model, IEEE Trans. Automat. Control 49 (6) (2004) 981–983.
[22] F. Xiao, L. Wang, State consensus for multi-agent systems with switching topologies and time-varying delays, Internat. J. Control 79 (10) (2006) 1277–1284.
[23] J.P. Hu, Y.G. Hong, Leader-following coordination of multi-agent systems with coupling time delays, Physica A 374 (2) (2007) 853–863.
[24] H. Zhang, F.L. Lewis, A. Das, Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback, IEEE Trans. Automat. Control 56 (8) (2011) 1948–1952.
[25] L. Gao, B. Xu, J. Li, H. Zhang, Distributed reduced-order observer-based approach to consensus problems for linear multi-agent systems, IET Control Theory Appl. 9 (5) (2015) 784–792.
[26] Q. Ma, G.Y. Miao, Distributed containment control of linear multi-agent systems, Neurocomputing 133 (10) (2014) 399–403.
[27] Z. Li, W. Ren, X. Liu, M. Fu, Consensus of multiagent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols, IEEE Trans. Automat. Control 58 (7) (2013) 1786–1791.
[28] L. Cheng, Y.P. Wang, W. Ren, Containment control of multi-agent systems with dynamic leaders based on a pin-type approach, IEEE Trans. Cybern. 46 (12) (2016) 3004–3017.
[29] L. Cheng, Y.P. Wang, W. Ren, On convergence rate of leader-following consensus of linear multi-agent systems with communication noises, IEEE Trans. Automat. Control 6 (11) (2016) 3586–3592.
[30] Y.P. Wang, L. Cheng, W. Ren, Seeking consensus in networks of linear agents: Communication noises and Markovian switching topologies, IEEE Trans. Automat. Control 60 (5) (2015) 1374–1379.
[31] X.L. Xu, S.Y. Chen, W. Huang, L.X. Gao, Leader-following consensus of discrete-time multi-agent systems with observer-based protocols, Neurocomputing 118 (22) (2013) 334–341.
[32] L. Wang, Z.Q. Chen, Z.X. Liu, Z.Z. Yuan, Finite time agreement protocol design of multi-agent systems with communication delays, Asian J. Control 11 (3) (2009) 281–286.
[33] S.H. Li, H.B. Du, X.Z. Lin, Finite-time consensus algorithm for multi-agent systems with double integrator dynamics, Automatica 47 (8) (2011) 1706–1712.
[34] X.L. Wang, Y.G. Hong, Finite-time consensus for multi-agent networks with second-order agent dynamics, IFAC Proc. 41 (2) (2008) 15185–15190.
[35] R. Olfati-Saber, R.M. Murray, Consensus problems in networks of agents with switching topology and time delays, IEEE Trans. Automat. Control 49 (9) (2004) 1520–1533.
[36] L. Xiao, S. Boyd, Fast linear iterations for distributed averaging, Systems Control Lett. 53 (1) (2004) 65–78.
[37] R. Ghabcheloo, A. Pascoal, C. Silvestre, I. Kaminer, Non-linear coordinated path following control of multiple wheeled robots with bidirectional communication constraints, Internat. J. Adapt. Control Signal Process. 21 (2–3) (2007) 133–157.
[38] M.E. Valcher, P. Misra, On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions, Systems Control Lett. 66 (2014) 94–103.
[39] D.Y. Meng, Y.M. Jia, J.P. Du, Nonlinear finite-time bipartite consensus protocol for multi-agent systems associated with signed graphs, Internat. J. Control 88 (10) (2015) 2074–2085.
[40] H.X. Hu, W.W. Yu, G.H. Wen, Q. Xuan, J.D. Cao, Reverse group consensus of multi-agent systems in the cooperation-competition network, IEEE Trans. Circuits Syst. 63 (11) (2016) 2036–2047.
[41] T. Balch, R.C. Arkin, Behavior-based formation control for multirobot teams, IEEE Trans. Robot. Autom. 14 (6) (1998) 926–939.
[42] J.N. Wang, M. Xin, Integrated optimal formation control of multiple unmanned aerial vehicles, IEEE Trans. Control Syst. Technol. 21 (5) (2012) 1731–1744.
[43] R. Piziak, D. Turner, Exploring Gerschgorin circles and Cassini ovals, Math. Ed. 3 (3) (1994) 13–21.
[44] A.S. Lewis, R.E. Lucchetti, Nonsmooth duality, sandwich, and squeeze theorems, SIAM J. Control Optim. 38 (2) (2000) 613–626.