Broadcast 1

Alex Chen

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Abstract

Let N be a finite set and z be a real-valued function defined on the set of subsets of N that satisfied

$$z(S) + z(T) \geq z(S \cup T) + z(S \cap T), \forall S, T \subseteq N$$

. Such a function is called submodular. We consider the problem

$$\max_{S \subseteq N} \{z(S) : |S| \leq K, \text{z(S) is submodular}\}$$

.

The problem of finding a maximum weight independent set in a matroid, when the elements of the matroid are colored and the elements of the independent set can have no more than K colors, can be posed in this framework.

Then we analyze the greedy heuristic for submodular set functions, when z(S) is nondecreasing and $z(\emptyset) = 0$, the greedy heuristic always produces a solution whose value is at least $1 - [(K-1)/K]^K$ times the optimal value. This bound can be achieved for each K and has a limiting value of $(1 - 1/e) \approx 63\%$

1 The greedy heuristic for submodular set functions

Here is a submodular problem

$$\max_{S \subseteq N} \{z(S) : |S| \leq K, z(S) submoluar.\} \tag{1}$$

A natural solution to this problem is to start from the null set and add elements one at a time, taking at each step that element which increases z the most. We define this solution as greedy heuristic formally. we let $\rho_j(S) = z(S \cup \{j\}) - z(S)$

Algorithm 1 The greedy heuristic

Initialize a null set $S = \emptyset$ and a finite set N ;
Every iteration, add a element to S, $|S| \leq K$
Iteration t.
while $t \leq K$ **do**
  Select $\max i \in N^{t-1}$
  For which $\rho_i(S^{t-1}) = z(S^{t-1} \cup \{i\}) - z(S^{t-1})$
  Set $\rho_{t-1} = \rho_i(S^{t-1})$
  if $\rho > 0$ **then**
    Select$\{i\} \in N^{t-1}$
    Then update $S^t = S^{t-1} \cup \{i(t)\}$ and $N^t = N^{t-1} - i(t)$
  else
    return $S^{K^*}, K^* = t - 1 < K$
  end if
  $t \to t + 1$
end while$K^* = t = K$

Note that

$$Z^G = Z(S^{K^*}) = z(\emptyset) + \rho_0 + ... + \rho_{K^*-1}, K^* \leq K \tag{2}$$

is the solution generated by greedy heuristic.
And $z(S)$ is the optimal solution.
We have the theorem:

Theorem 1

$$\frac{\textit{value of greedy approximation}}{\textit{value of optimal solution}} = \frac{Z^G - z(\emptyset)}{Z - z(\emptyset)} \geq 1 - (\frac{K-1}{K})^K \tag{3}$$

Proof 1.1 *Before we proof Eq.(3), We will proof other equivalence that needed:*
Let N be a finite set and $\forall A, B \in N$ that satisfy:

$$z(A) + z(B) \geq z(A \cup B) + z(A \cap B) \tag{4}$$

The function z defined on the set of subsets of N is called submodular function.
Take $A = S \cup \{j\}$ and $B = T$, $S \subseteq T$, $j \notin T$:
We have

$$z(S \cup \{j\}) + z(T) \geq z(T \cup \{j\}) + z(S) \tag{5}$$

Or

$$\rho_j(S) \geq \rho_j(T) \tag{6}$$

For arbitrary S and T with $T - S = \{j1, ..., jr\}$ and $S - T = \{k1, ..., kq\}$, we have

$$\begin{aligned} z(S \cup T) - z(S) &= z(S \cup T) - Z(S \cup \{j_1, ..., j_{r-1}\}) \\ &+ Z(S \cup \{j_1, ..., j_{r-1}\}) - Z(S \cup \{j_1, ..., j_{r-2}\}) \\ &+ ... - ... \\ &+ Z(S \cup \{j_1, j_2\}) - Z(S \cup \{j_1\}) \\ &+ Z(S \cup \{j_1\}) - Z(S) \\ &= \sum_{t=1}^{r} \rho_{j_t}(S \cup \{j_1, ..., j_{t-1}\}) \\ &\leq \sum_{t=1}^{r} \rho_{j_t}(S) \\ &= \sum_{j \in T-S} \rho_j(S) \end{aligned} \tag{7}$$

Similarly:

$$\begin{aligned} z(S \cup T) - z(T) &= \sum_{t=1}^{q} \rho_{k_t}(T \cup \{k_1, ..., k_{t-1}\}) \\ &= \sum_{t=1}^{q} \rho_{k_t}(T \cup \{k_1, ..., k_t\} - \{k_t\}) \\ &\geq \sum_{t=1}^{q} \rho_{k_t}(T \cup S - \{k_t\}) \\ &= \sum_{j \in S-T} \rho_j(S \cup T - \{j\}) \end{aligned} \tag{8}$$

With Eq.(7) and (8)

$$z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S) - \sum_{j \in S-T} \rho_j(S \cup T - \{j\}), \forall S, T \subseteq E \tag{9}$$

If z is a submodular set function on E with $-\theta \leq \rho_j(S) \leq \psi, \forall S \subseteq E, j \in E - S$, then:

$$z(T) \leq z(S) + \sum_{j \in T-S} \rho_j(S) + |S - T|\theta \tag{10}$$

Taking T to be an optimal solution, S to be the set S^t which is generated after t iterations of the algorithm[1]. Using Eq. (10)

$$Z = z(T), \rho_j(S^t) \leq \rho_t, \rho_t \geq 0,$$

$$|S^t - T| \leq t, \theta \geq 0, |T - S^t| \leq K, t \leq K$$

and

$$z(S^t) = z(\emptyset) + \sum_{i=0}^{t-1} \rho_i$$

we obtain

$$\begin{aligned} Z &\leq z(S^t) + K\rho_t + t\theta \\ &= z(\emptyset) + \sum_{i=0}^{t-1} \rho_i + K\rho_t + t\theta \end{aligned} \tag{11}$$

when t=0, we obtain $Z - z(\emptyset) \leq K\rho_0 \leq K(Z^{G_0} - z(\emptyset))$, therefore

$$\frac{Z - Z^{G_0}}{Z - z(\emptyset)} \leq \frac{K-1}{K} \tag{12}$$

when $\theta = 0$,

$$\begin{aligned} Z - z(\emptyset) &\leq \sum_{i=0}^{t-1} \rho_i + K\rho_t \\ &= \rho_0 + ... + \rho_{t-1} + K\rho_t \\ &Eq.(2) \\ &= Z^{G_{t+1}} - z(\emptyset) + (K-1)\rho_t \\ &= Z^{G_{t+1}} - z(\emptyset) + (K-1)(Z^{G_{t+1}} - Z^{G_t}) \end{aligned} \tag{13}$$

We obtain

$$\frac{Z - Z^{G_{t+1}}}{Z - Z^{G_t}} \leq \frac{K-1}{K}$$

$$\begin{aligned} \frac{Z - Z^G}{Z - z(\emptyset)} &= \frac{Z - Z^{G_K}}{Z - Z^{G_{K-1}}} \frac{Z - Z^{G_{K-1}}}{Z - Z^{G_{K-2}}} ... \frac{Z - Z^{G_t}}{Z - Z^{G_{t-1}}} ... \frac{Z - Z^{G_0}}{Z - Z(\emptyset)} \\ &\leq (\frac{K-1}{K})^K \end{aligned} \tag{14}$$

so we have

$$\frac{\textit{value of greedy approximation}}{\textit{value of optimal solution}} = \frac{Z^G - z(\emptyset)}{Z - z(\emptyset)} \geq 1 - (\frac{K-1}{K})^K \geq 1 - \frac{1}{e} \approx 63\% \tag{15}$$