Chapter1:The Physics of Waves

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1 One-dim wave equation

All classical mechanical waves can be described by the same equation,the *differential wave equation:*

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2} \tag{1.1}$$

1.1 Example

Show that

$$\Psi(x,t) = (x+vt)^2$$

is a solution to the differential wave equation.Assume that v is a constant.

Solution

We must show that $\Psi(x,t)$ solves Equation (1.1):

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

To do this,we must take the indicated derivatives.Begin with the partial derivative with respect to x:

$$\frac{\partial \Psi(x,t)}{\partial x} = \frac{\partial}{\partial x}(x+vt)^2 = 2(x+vt)\frac{\partial}{\partial x}(x+vt) = 2(x+vt)(1) = 2(x+vt)$$

Now take the second partial derivative with respect to x:

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}[\frac{\partial \Psi}{\partial x}] = \frac{\partial}{\partial x}[2(x+vt)] = 2$$

Next,take the partial derivatives of $\Psi(x,t)$ with respect to t:

$$\frac{\partial \Psi}{\partial t} = 2(x+vt)\frac{\partial}{\partial t}(x+vt) = 2(x+vt)(v) = 2v(x+vt)$$

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial}{\partial t}[2v(x+vt)] = \frac{\partial}{\partial t}[2vx+2v^2t] = 2v^2$$

Substitute these results into the differential wave equation.

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = 2$$

$$\frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} = \frac{1}{v^2}(2v^2) = 2$$

Thus

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2}$$

$\Psi(x,t)$ solves the differential wave equation,and thus we have shown that it is a traveling wave.

2 General solutions to the one-dim wave equation

We will show below that the most general solutions to Equation(1.1) may be expressed as follows:

$$\Psi(x,t) = f(x - vt) \tag{2.1}$$

$$\Psi(x,t) = g(x + vt) \tag{2.2}$$

where the function f and g represent any function that has finite second derivatives,and where the parameters x,v and t all occur explicitly within the function as $x - vt$ or $x + vt$.

As an example,consider the function

$$\Psi(x,t) = \frac{A}{1 + (x - vt)^2} \tag{2.3}$$

where A is a constant.Equation(2.3) represents a peaked function whose maximum is located at points given by $x = vt$. A plot of this wave function at two different times is shown in Figure 1.

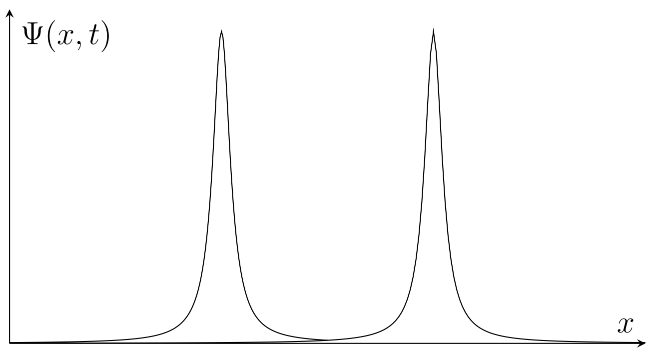


Figure 1: The traveling pulse of Equation (2.3), shown at two different times $t_1 < t_2$.

Velocity v is determined,x depends on time t. While $vt_1 = x$ or $vt_2 = x$, $\Psi(x,t) = A$, x is a coordinates of wave occurs pulse during traveling.

While v has a plus sign,such as $+v$,we have $x-(+v)t = x-vt$ in Equation(2.1),wave *traveling forward* and v has a minus sign, such as $-v$,$x-(-v)t = x+vt$ in Equation(2.2),*wave traveling backward.*

2.1 Example

Show that

$$\Psi(x,t) = Ae^{-(a^2x^2+b^2t^2+2abxt)}$$

is a traveling wave,and find the wave speed and direction of propagation.Assume that A,a and b are all constants,and that a and b have units that make the quantity in the exponential function unitless.

Solution

$$\begin{aligned}\Psi(x,t) &= Ae^{-(a^2x^2+b^2t^2+2abxt)} \\ &= Ae^{-(ax+bt)^2}\end{aligned}$$

While $ax+bt=0$, thus $x = -\frac{bt}{a}$ and $v = -\frac{b}{a}$,this wave traveling backward.

3 Harmonic traveling waves

According to the results of the previous section,any function described by Equation(2.1) or (2.2) represents a traveling wave. In particular,*harmonic function*(i.e.,sines and consines) with the appropriate arguments solve the differential wave equation.Thus,the following function represents a traveling wave:

$$\Psi(x,t) = Asin\frac{2\pi}{\lambda}(x \mp vt) \tag{3.1}$$

The $\Psi(x,t)$ given in Equation(3.1) is periodic in both space and time coordinates.The term λ represents the *spatial period*:

$$\Psi(x+\lambda,t) = Asin\frac{2\pi}{\lambda}(x+\lambda \mp vt) = Asin[\frac{2\pi}{\lambda}(x \mp vt) + 2\pi] = \Psi(x,t)$$

Let T represent the *temporal period*;the time required for one cycle.Since Equation(3.1) is periodic in T,we have

$$\Psi(x,t+T) = Asin\frac{2\pi}{\lambda}[x \mp v(t+T)] = Asin[\frac{2\pi}{\lambda}(x \mp vt) \mp \frac{2\pi}{\lambda}vT]$$

T represents the temporal period provided that

$$\frac{vT}{\lambda} = 1 \tag{3.2}$$

or

$$v = \frac{\lambda}{T} \tag{3.3}$$

Thus,a periodic classical wave travels one wavelength λ in one temporal period T.It is customary to define the wave*frequency* as

$$f = \frac{1}{T} \tag{3.4}$$

In terms of frequency,(3.3) becomes

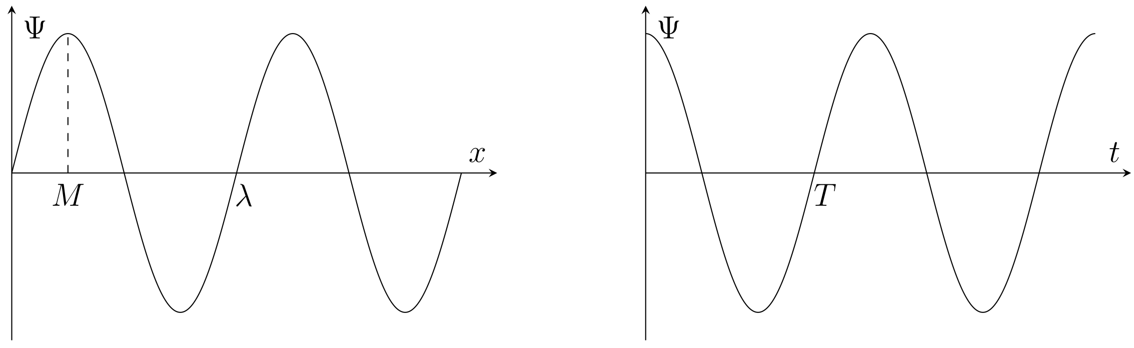
$$f\lambda = v \tag{3.5}$$

A plot of $\Psi(x,t)$ vs. x is shown in Figure 2a. Figure 2b shows a plot of $\Psi(x,t)$ vs. t. It is customary to define the *propagation constant* as follows:

$$k = \frac{2\pi}{\lambda} \tag{3.6}$$

This quantity is also sometimes referred to as the *wave number*.Since k converts meters to radians,the units are r ad/m.We may rewrite Equation (3.1) as

$$\Psi(x,t) = Asink(x \mp vt)$$



(a) (b)

Figure 2: Plots of harmonic wavefunction. (a) A plot of $\Psi(x,t)$ vs. position x. (b) A plot of $\Psi(x,t)$ vs. time using data recorded by a single measuring device located at M in Figure (a)

Similarly, we can define the *angular frequency*

$$\omega = kv = \frac{2\pi}{\lambda}v = 2\pi f \tag{3.7}$$

Since ω converts time to radians,the units are r ad/s. Note that according to the last result.

$$\frac{\omega}{k} = v \tag{3.8}$$

In terms of k and ω, Equation (3.1) becomes

$$\Psi(x,t) = Asin(kx \mp \omega t)$$

Collectively, the terms $kx \mp \omega t$ are called the *phase* of the harmonic traveling wave. We may also include an explicit value of the phase when x and t are zero by specifying the *initial phase* ϕ:

$$\Psi(x,t) = Asin(kx \mp \omega t + \phi) \tag{3.9}$$

3.1 Example

A harmonic traveling wave is given by $\Psi(z,t) = Asin(50z + 3000t)$.Find the wave speed,frequency,angular frequency,and direction of propagation.

Solution

$$\Psi(z,t) = Asin(50z + 3000t) = Asin(kz + \omega t)$$

$$k = \frac{2\pi}{\lambda} = 50$$

$$\omega = kv = 2\pi f = 3000$$

wave speed:

$$v = \frac{3000}{k} = 60$$

frequency:

$$f = \frac{3000}{2\pi} = \frac{1500}{\pi}$$

angular frequency:

$$\omega = 3000$$

direction of propagation:backward

4 The three-dimensional wave equation

In Cartesian coordinates,the extension of Equation (1.1) to include three dimensions is made in the obvious way:

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \tag{4.1}$$

where $\Psi = \Psi(x, y, z, t)$ is now understood to be a function of time and all three spatial coordinates.

Equation(4.1) can be written in a more general form using the Laplacian operator:

$$\nabla^2 \Psi = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2} \tag{4.2}$$

where in Cartesian coordinates,the *Laplacian* is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{4.3}$$

Equation (4.2) is the *coordinates independent* representation of the three-dimensional wave equation.

4.1 Three-Dimensional Plane Waves

Consider an acoustic wave traveling through air.In a three-dimensional *plane wave*,the properties of the medium are constant over any *plane* oriented normal to the direction of propagation. To describe such a wave,we define the propagation vector $\vec{k}$ with magnitude $\frac{2\pi}{\lambda}$ and direction given by the wave propagation.The corresponding plane wave is given by

$$\Psi(x,y,z,t) = Asin(\vec{k}\cdot\vec{r} - \omega t + \varphi) \tag{4.4}$$

A plane has Cartesian symmetry,and in Cartesian coordinates,$\vec{k}$ is given by

$$\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k} \tag{4.5}$$

The position vector in Cartesian coordinates is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{4.6}$$

For example, in a plane harmonic sound wave traveling in the positive x-direction,the value of the air pressure is constant over any plane that is parallel to the y-z plane:

$$\Psi(x,y,z,t) = Asin(kx - \omega t + \varphi) \tag{4.7}$$

In Figure it is seen that $\vec{k}$ is normal to planes defined by $\vec{k}\cdot\vec{r} = const.$It is convenient to define *wavefronts* located at points where the phase of Equation (4.4) is equal to integer multiples of 2π. Thus,a three-dimensional plane wave may be visualized as a train of wavefronts separated by one wavelength λ and moving with the wave speed v.

In the complex representation,harmonic plane waves are given by

$$\Psi(x,y,z,t) = Ae^{i(\vec{k}\cdot\vec{r} - \omega t + \varphi)} \tag{4.8}$$

4.2 Spherical Waves

Spherical waves are waves with wavefronts that are spherical in shape. Examples include waves that emanate from an isotropic point source,or are converging to a point. An isotropic source emits waves symmetrically in all directions.

In spherical coordinates,spherical waves have no dependence on the angular coordinates θ and ϕ,so the derivatives with respect to these coordinates are zero,this means that the

$$\Psi(\vec{r},t) = \Psi(r,t)$$

In otherwords,$\Psi(r,t)$ depends only on the spherical coordinates r and time t.Derivatives of such a function with respect to θ and ϕ given zero,leading to a simplified version of the Laplacian for spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) \tag{4.9}$$

Using this Laplacian, we obtain the *spherical symmetric differential wave equation*:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Psi(r,t)}{\partial r}) = \frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2} \tag{4.10}$$

An important solution to this equation is the *harmonic spherical wave*

$$\Psi(r,t) = \frac{A}{r}sin(kr \mp \omega t + \varphi) \tag{4.11}$$

If the minus sign is chosen,the wave emanate from an isotropic source located at $r = 0$. A plus sign represents a wave that is converging to the point $r = 0$.The amplitude $\frac{A}{r}$ of the wave is not constant,since it depends on radial coordinate r.For a wave emanating from a point, the amplitude decreases as the wave travels. Waves converging to a point have amplitudes that increase as time increases.

In the complex representation,Equation (4.11) becomes

$$\Psi(r,t) = \frac{A}{r}e^{i(kr \mp \omega t + \varphi)} \tag{4.12}$$

A Appendix:Complex Numbers and the Complex Representation

Complex numbers can provide algebraic shortcuts that will prove very convenient as we continue our discussion of classical optics. In this section,we provide a quick overview of the properties of complex numbers,and a few of their algebraic features that we will find most useful.

Complex numbers include the concept of the *imaginary number* i:

$$i = \sqrt{-1} \tag{A.1}$$

Clearly,the square root of a negative number has no counterpart within the set of all *real* numbers.A complex number z has both a real part and an imaginary part:

$$z = x + iy \tag{A.2}$$

where x is the *real* part of z and y is the *imaginary* part of z.

In the *Cartesian representation*, a complex number is plotted with coordinates (x,y); thus the horizontal axis is called the *real axis*, and the vertical axis is called *imaginary axis*.

Many of the features of complex numbers that we will find most useful result from the *Euler relation*:

$$e^{i\theta} = cos\theta + isin\theta \tag{A.3}$$

We may use the Euler relation to express any complex number z in polar form.Let

$$x = rcos\theta \tag{A.4}$$

$$y = rsin\theta \tag{A.5}$$

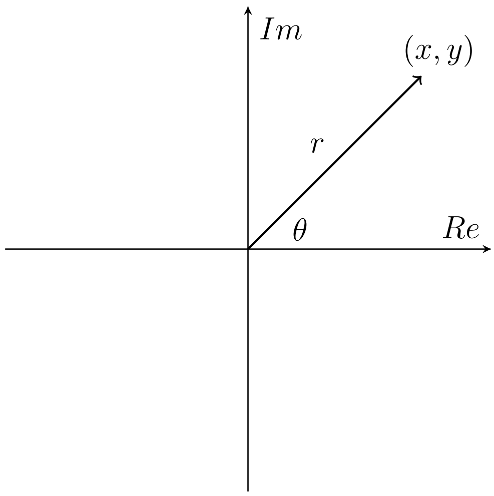


Figure 3: The comple plane

where

$$r = \sqrt{x^2 + y^2} \tag{A.6}$$

$$\theta = tan^{-1}(\frac{y}{x}) \tag{A.7}$$

Using these relation,we may represent z as

$$z = x + iy = rcos\theta + i(rsin\theta) \tag{A.8}$$

Thus,according to Equation (A.3),

$$z = re^{i\theta} \tag{A.9}$$

In thr polar representation,r is called the *magnitude* of z,and θ is called the *phase* of z.
The following examples are often useful:

$$e^{\pm i(2\pi)} = 1; e^{\pm i(\pi)} = -1; e^{i\frac{\pi}{2}} = i; e^{i\frac{3\pi}{2}} = -i \tag{A.10}$$

The *complex conjugate* z^* of a complex number is z is obtained ny inverting the sign on each occurrence of i.For example, in the Cartesian representation where $z = x + iy$,z^* is given by

$$z^* = x - iy \tag{A.11}$$

In the polar representation where $z = re^{i\theta}$, z^* is given by

$$z^* = re^{-i\theta} \tag{A.12}$$

Let $Re[z]$ and $Im[z]$ denote the real and imaginary parts of z. In the Cartesian form,Re[z]=x and Im[z]=y. In the polar form,$Re[z] = rcos\theta$ and $Im[z] = rsin\theta$.In either case,

$$Re[z] = \frac{z + z^*}{2} \tag{A.13}$$

$$Im[z] = \frac{z - z^*}{2i} \tag{A.14}$$

According to the Euler relation,

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{A.15}$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{A.16}$$

Multiplication of a complex number by its complex conjugate gives the square of the magnitude.In the Cartesian form,

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 \tag{A.17}$$

and in the polar form,

$$zz^* = (re^{i\theta})(re^{-i\theta}) = r^2 \tag{A.18}$$

which,according to Equation (A.6),agrees with the previous result.We will also refer to the magnitude of z as $|z|$:

$$zz^* = |z|^2 \tag{A.19}$$