An Invitation to Probability

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Science Center Auditorium

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Summer Science Institute 2022
Auburn University

What is chance

How to measure chance

Birthday problem

Rolling three dices

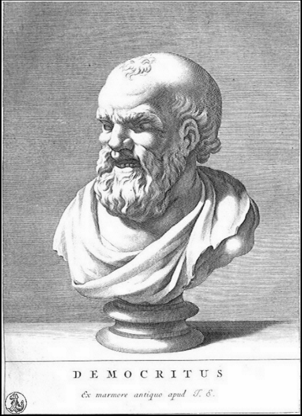
Poker

What is CHANCE?

Tyche – The Greek goddess of chance

Fortuna – The goddess of chance in Roman religion

Democritus (460 – 370 BC)
— Father of modern science



Atomic theory of the universe:

A physical chance affecting all the atoms
that made up the universe.

Games of chance
— using knucklebones or dice

Known to Egyptians, Babylonians, Romans, ...

There was no qualitative theory of chance in these times.

How to measure chance?

How about measure length?

The determination of a “right and lawful rood” or rod in the early sixteenth century in Germany by measuring an essentially random selection of 16 men as they leave church [1].

[1] Stephen Stigler (1996). Statistics and the Question of Standards, *Journal of Research of the National Institute of Standards and Technology*, vol. 101.

The same for chance

To measure probability,

1. we first find or make equally probable cases,
2. then we count.

The probability of an event A, denoted by $\mathbb{P}(A)$, is then

$$\mathbb{P}(A) = \frac{\text{no. of cases in which } A \text{ occurs}}{\text{total no. of cases}}.$$

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$$\mathbb{P}(A) = \frac{\text{no. of cases in which } A \text{ occurs}}{\text{total no. of cases}}.$$

Now we see that probability has to satisfy the following properties:

1. Probability should be never negative.
2. If A occurs in all cases, then $\mathbb{P}(A) = 1$.
3. If A and B never occur in the same case, then

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

In particular, the probability of an event not occurring is equal to

$$\mathbb{P}(\text{not } A) = 1 - \mathbb{P}(A).$$

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Birthday Problem

Question: How likely do two students have the same birth day if there are

1. 2
2. 5
3. 15
4. 23
5. 46
6. 64
7. 366

students in the class?

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students in the class?

- Suppose $n = 5$.
- Let A be the **event** that two students have the same birth day.
- Let (A) denote the probability that this event A will happen.
- It is not easy to compute (A) directly.
- However, one can compute (A'), – the complement of event A, namely,

  the probability that no two students have the same birthday, or

  all students have different birthday,

  as follows:

$$\mathbb{P}(A') = \frac{365}{365}\frac{364}{365}\frac{363}{365}\frac{362}{365}\frac{361}{365}$$

- Hence,

$$\mathbb{P}(A) = 1 - (A') = 1 - \frac{365}{365}\frac{364}{365}\frac{363}{365}\frac{362}{365}\frac{361}{365} \approx 0.027.$$

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$$\mathbb{P}(A) = 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}$$

n	2	5	15	23	46	64	366
(A)							1.0

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(A)	0.0027						1.0

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n	2	5	15	23	46	64	366
(A)	0.0027	0.0271					1.0

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n	2	5	15	23	46	64	366
(A)	0.0027	0.0271	0.2529				1.0

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n	2	5	15	23	46	64	366
(A)	0.0027	0.0271	0.2529	0.5073			1.0

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n	2	5	15	23	46	64	366
(A)	0.0027	0.0271	0.2529	0.5073	0.9483		1.0

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n	2	5	15	23	46	64	366
(A)	0.0027	0.0271	0.2529	0.5073	0.9483	0.9972	1.0

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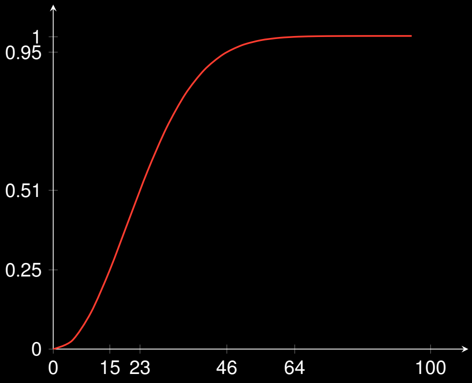
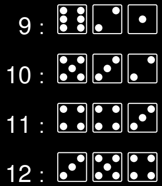
(A)	0.0027	0.0271	0.2529	0.5073	0.9483	0.9972	1.0



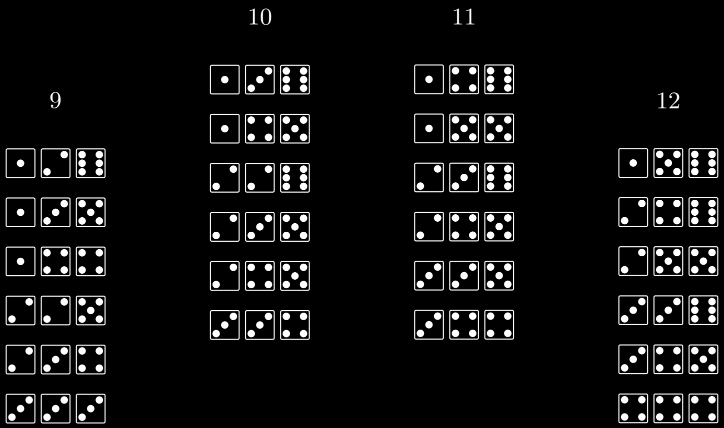
When time permits, we will try some simulations !

A question asked by Grand Duke of Tuscany to Galileo in early seventh century

Three dice are thrown, such as



Counting combinations of numbers, 10 and 11 can be made in 6 ways, as can 9 and 12. Yet it is known that long observation has made dice-players consider 10 and 11 to be more advantageous than 9 and 12. How can this be?


9
10
11
12

3 = ⚀ + ⚀ + ⚀

4 = ⚀ + ⚀ + ⚁

5 = ⚀ + ⚀ + ⚂ = ⚀ + ⚁ + ⚁

6 = ⚀ + ⚀ + ⚃ = ⚀ + ⚁ + ⚂ = ⚁ + ⚁ + ⚁

7 = ⚀ + ⚀ + ⚄ = ⚀ + ⚁ + ⚃ = ⚁ + ⚁ + ⚂ = ⚂ + ⚂ + ⚀

8 = ⚀ + ⚀ + ⚅ = ⚀ + ⚁ + ⚄ = ⚀ + ⚂ + ⚃ = ⚁ + ⚁ + ⚃ = ⚁ + ⚂ + ⚂

9 = ⚀ + ⚁ + ⚅ = ⚀ + ⚂ + ⚄ = ⚀ + ⚃ + ⚃ = ⚁ + ⚁ + ⚄ = ⚁ + ⚂ + ⚃ = ⚂ + ⚂ + ⚂

10 = ⚀ + ⚂ + ⚅ = ⚀ + ⚃ + ⚄ = ⚁ + ⚁ + ⚅ = ⚁ + ⚂ + ⚄ = ⚁ + ⚃ + ⚄ = ⚂ + ⚂ + ⚃

11 = ⚀ + ⚃ + ⚅ = ⚀ + ⚄ + ⚄ = ⚁ + ⚂ + ⚅ = ⚁ + ⚃ + ⚄ = ⚂ + ⚂ + ⚄ = ⚂ + ⚃ + ⚃

12 = ⚀ + ⚄ + ⚅ = ⚁ + ⚃ + ⚅ = ⚁ + ⚄ + ⚄ = ⚂ + ⚂ + ⚅ = ⚂ + ⚃ + ⚄ = ⚃ + ⚃ + ⚃

13 = ⚀ + ⚅ + ⚅ = ⚁ + ⚄ + ⚅ = ⚂ + ⚃ + ⚅ = ⚂ + ⚄ + ⚄ = ⚃ + ⚃ + ⚄

14 = ⚁ + ⚅ + ⚅ = ⚂ + ⚄ + ⚅ = ⚃ + ⚃ + ⚅ = ⚃ + ⚄ + ⚄

15 = ⚂ + ⚅ + ⚅ = ⚃ + ⚄ + ⚅ = ⚄ + ⚄ + ⚄

16 = ⚃ + ⚅ + ⚅ = ⚄ + ⚄ + ⚅

17 = ⚄ + ⚅ + ⚅

18 = ⚅ + ⚅ + ⚅

$$
\begin{aligned}
3 &= 1+1+1 \\
4 &= 1+1+2 \\
5 &= 1+1+3 = 1+2+2 \\
6 &= 1+1+4 = 1+2+3 = 2+2+2 \\
7 &= 1+1+5 = 1+2+4 = 2+2+3 = 3+3+1 \\
8 &= 1+1+6 = 1+2+5 = 1+3+4 = 2+2+4 = 2+3+3 \\
9 &= 1+2+6 = 1+3+5 = 1+4+4 = 2+2+5 = 2+3+4 = 3+3+3 \\
10 &= 1+3+6 = 1+4+5 = 2+2+6 = 2+3+5 = 2+4+4 = 3+3+4 \\
11 &= 1+4+6 = 1+5+5 = 2+3+6 = 2+4+5 = 3+3+5 = 3+4+4 \\
12 &= 1+5+6 = 2+4+6 = 2+5+5 = 3+3+6 = 3+4+5 = 4+4+4 \\
13 &= 1+6+6 = 2+5+6 = 3+4+6 = 3+5+5 = 4+4+5 \\
14 &= 2+6+6 = 3+5+6 = 4+4+6 = 4+5+5 \\
15 &= 3+6+6 = 4+5+6 = 5+5+5 \\
16 &= 4+6+6 = 5+5+6 \\
17 &= 5+6+6 \\
18 &= 6+6+6
\end{aligned}
$$

k	Probability of a sum of k	$\approx$
3	1/216	0.5%
4	3/216	1.4%
5	6/216	2.8%
6	10/216	4.6%
7	15/216	7.0%
8	21/216	9.7%
9	25/216	11.6%
10	27/216	12.5%
11	27/216	12.5%
12	25/216	11.6%
13	21/216	9.7%
14	15/216	7.0%
15	10/216	4.6%
16	6/216	2.8%
17	3/216	1.4%
18	1/216	0.5%


POKER
HAND RANKINGS
#1 ROYAL FLUSH
#6 STRAIGHT
#2 STRAIGHT FLUSH
#7 THREE OF A KIND
#3 FOUR OF A KIND
#8 TWO PAIR
#4 FULL HOUSE
#9 ONE PAIR
#5 FLUSH
#10 HIGH CARD

Table 10-1. Ways to Deal Five-Card Poker Hands

Hand	Number of ways	Probability that opponent has a better hand[a] when the number of opposing hands is:					
		1	2	3	4	5	6
Straight flush	40	0	0	0	0	0	0
Four of a kind	624	.00002	.00003	.00005	.00006	.00008	.00009
Full house	3,744	.00026	.00051	.00078	.00102	.00128	.00153
Flush	5,108	.00170	.00339	.00509	.00677	.00847	.01015
Straight	10,200	.00366	.00731	.01094	.01457	.01817	.02177
Three of a kind	54,912	.00759	.01511	.02259	.03000	.03736	.04466
Two pairs:							
Aces high	19,008	.02871	.05660	.08369	.11001	.13556	.16038
Kings	17,424	.03603	.07076	.10424	.13651	.16762	.19761
Queens	15,840	.04273	.08364	.12280	.16028	.19617	.23052
Jacks	14,256	.04883	.09527	.13945	.18146	.22143	.25945
Tens	12,672	.05431	.10568	.15425	.20019	.24362	.28470
Nines	11,088	.05919	.11487	.16726	.21655	.26292	.30655
Eights	9,504	.06345	.12288	.17854	.23067	.27948	.32520
Sevens	7,920	.06711	.12972	.18812	.24261	.29344	.34086
Sixes	6,336	.07016	.13540	.19606	.25246	.30491	.35367
Fives	4,752	.07260	.13992	.20236	.26027	.31397	.36378
Fours	3,168	.07443	.14331	.20707	.26608	.32071	.37126
Threes	1,584	.07564	.14556	.21019	.26993	.32515	.37620

Table 10-1. *continued*

Hand	Number of ways	Probability that opponent has a better hand[a] when the number of opposing hands is:					
		1	2	3	4	5	6
One pair:							
Aces	84,480	.07625	.14669	.21176	.27187	.32739	.37868
Kings	"	.10876	.20569	.29208	.36907	.43769	.49885
Queens	"	.14126	.26257	.36674	.45620	.53302	.59899
Jacks	"	.17377	.31734	.43597	.53398	.61496	.68187
Tens	"	.20627	.37000	.49995	.60310	.68497	.74995
Nines	"	.23878	.42054	.55891	.66423	.74441	.80544
Eights	"	.27129	.46898	.61303	.71801	.79451	.85026
Sevens	"	.30379	.51529	.66254	.76506	.83643	.88612
Sixes	"	.33630	.55950	.70764	.80596	.87121	.91452
Fives	"	.36880	.60159	.74852	.84127	.89981	.93676
Fours	"	.40131	.64157	.78541	.87153	.92309	.95395
Threes	"	.43381	.67943	.81850	.89724	.94182	.96706
Deuces	"	.46632	.71518	.84800	.91888	.95671	.97690
No pair	1,302,540	.49882	.74882	.87411	.93691	.96838	.98415
All hands	2,598,960						

a. Assuming that you have the best hand of its type—for example, ace-high if you have no pair.