

Topics in Analysis and Linear Algebra

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Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

*J. McDonald and N. Weiss, **A course in real analysis***, Academic Press, 2005.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

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What is a real number?



1

¹Image from Wikipedia.



2

²Image from

<https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/>

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

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Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exist $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$x + 0 = x$ and $x \cdot 1 = x$. (Identities)

(F5) For each $x \in \mathbb{R}$, there exists a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exists an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

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Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

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Completeness Axiom

Axiom A nonempty subset of real numbers that is **bounded above** has a **least upper bound**, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

Corr. A nonempty subset of real numbers that is bounded below has a greatest lower bound, which is denoted as

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E.g. $\sup[0, 1) = 1$ and $\inf[0, 1) = 0$.

\mathbb{N} has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

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Properties

1. Archimedean principle

For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

Between any two real numbers there is an irrational number.

3. Density of the rational numbers

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Proof As exercises.



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Extended Real Number System

HW Prove the three Properties.

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Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence³, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n \rightarrow \infty} (n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

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Def. Let \mathbb{R}^* denote *the extended real line*, namely, $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$.

Def. A sequence of real numbers $\{a_1, a_2, \dots\}$ is said to *converge in \mathbb{R}^** if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
 - (ii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.
 - (iii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.
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$\{(-1)^n\}_{n=1}^{\infty}$ does not converge in \mathbb{R}^* .

$\{n^2\}_{n=1}^{\infty}$ converges in \mathbb{R}^* and $\lim_{n \rightarrow \infty} n^2 = \infty$.

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Def. If $x_1 \leq x_2 \leq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

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Axiom Let A be a nonempty subset of real numbers that is **bounded above**.
Then *the least upper bound* of A exists, which is denoted by

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Prop. Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R}^* .

Moreover,

(a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

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In particular, if $\{x_n\}_{n=1}^{\infty}$ converges in \mathbb{R} , it is bounded above.

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Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

- (a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.
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$$\{x_n\}_{n=1}^{\infty} \text{ converges in } \mathbb{R}^* \iff \{x_n\}_{n=1}^{\infty} \text{ has exactly one cluster point in } \mathbb{R}^*.$$

A few more properties

1. If a sequence is **bounded and monotonic**, then it is convergent.
2. A sequence is convergent iff each subsequence is convergent.
3. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \rightarrow L$ and $b_n \rightarrow L$, then $c_n \rightarrow L$.

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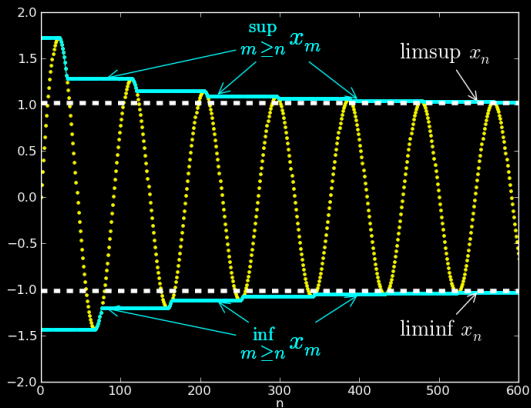
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4

⁴Image from Wikipedia.

Can we use the first-order logic to formulate the definitions?

Def'. $\limsup_{n \rightarrow \infty} x_n = x \in \mathbb{R}$ if and only if x is the smallest real number such that for any positive real number $\epsilon > 0$, there exists a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

Set

$$A(b) : \quad \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \{n \geq N \rightarrow x_n < b + \epsilon\}$$

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Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

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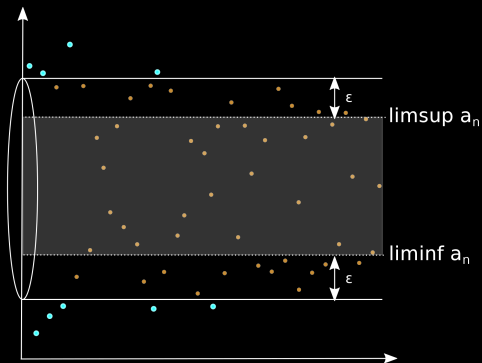
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E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

⁵Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2} \left(\limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

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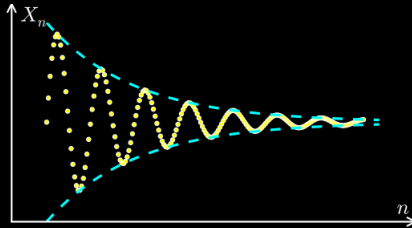
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A sequence of real numbers **converges in \mathbb{R}^*** if and only if it has exactly one cluster point.

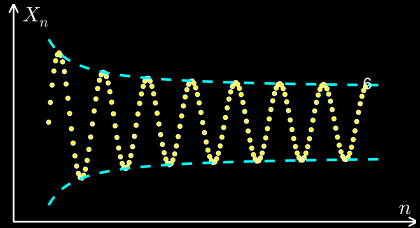
There is another famous criterion for a sequence to **converge in \mathbb{R}** :

Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



⁶Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{ |x_n - x_m| < \epsilon \}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in \mathbb{R} iff it is Cauchy.

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Thm (Cauchy Criterion)

A sequence of real numbers *converges in \mathbb{R}* iff it is Cauchy.

E.g.1 Let $a_n = \sqrt{n}$. Show that

- (i) The consecutive terms become arbitrarily close to each other as $n \rightarrow \infty$.
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

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Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

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Indeed, let's choose $m = N$ and $n = 4N$

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$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

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$$\lim_{n \rightarrow \infty} \left(1 + \text{Small}\right)^{\text{Large}} = e^{\lim_{n \rightarrow \infty} \text{Small} \times \text{Large}}$$

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Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exists an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of \mathbb{R} . Then K cannot be an open set. For example, \mathbb{N} , \mathbb{Q} , \mathbb{Z} are not open sets.

\mathbb{Q}^c – the set of irrational numbers – is not an open set.

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Properties

1. \mathbb{R} and \emptyset are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open. (arbitrary union)

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Let $a, b \in \mathbb{R}$ such that $a < b$. Then

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Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

E.g. \mathbb{R} and \emptyset are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

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0. A set is open if and only if its complement is closed.



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Def. Let $G \subset D \subset \mathbb{R}$.

(a) G is said to be open in D if for each $x \in G$, there is an $r > 0$ such that

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E.g.

D	G	Is G open in \mathbb{R}	Is G open in D
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
\mathbb{N}	$A \subset \mathbb{N}$	closed	open

Chapter 3. Real Number System and Calculus

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Def. A *real-valued function* is a function whose range is a subset of \mathbb{R} . If $f : \Omega \rightarrow \mathbb{R}$, we say that f is a *real-valued function on Ω* .

Def. *Algebraic operations*: Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

$$(f + g)(x) := f(x) + g(x)$$

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(Local) Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. f is *continuous at a point c* if the limit of $f(x)$, as x approaches c , exists and is equal to $f(c)$.

Def'. (Epsilon-delta definition) The function f is *continuous at a point c* if

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Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in \mathbb{R} .

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Def. f is *left-continuous at c* if

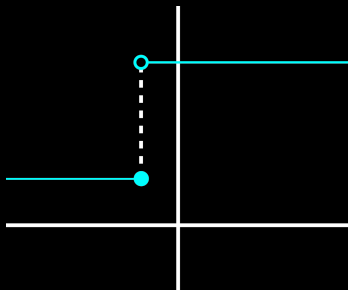
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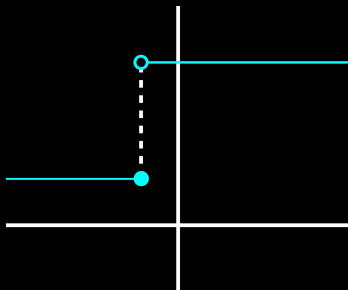


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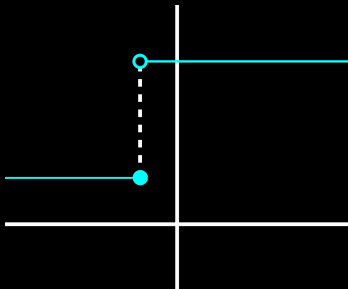


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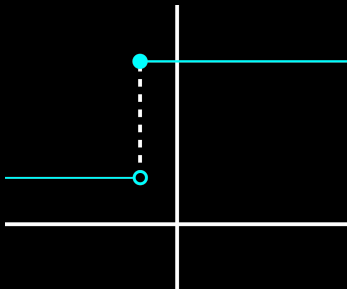
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Def. f is *lower semi-continuous at x_0* if

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$f(x_0)$ can be all points
at or below the blue point.

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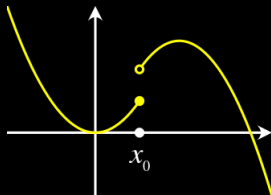
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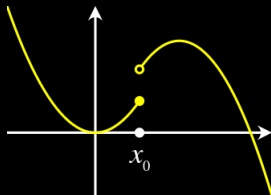
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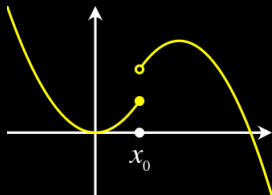
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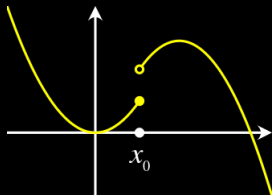
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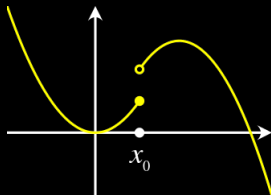
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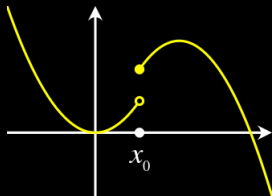
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(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let I be an interval of \mathbb{R} . Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

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Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I , we need to show

$$\begin{aligned} & \neg \left(\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \right) \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ \neg \{ |x - y| < \delta \} \vee |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \{ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon \} \end{aligned}$$

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Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

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$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on \mathbb{R} nor on $[0, \infty)$.

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Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

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Notation For $D \subset \mathbb{R}$, let $C(D)$ denote the set of continuous functions defined on D .

Thm (*Algebra of $C(D)$*) Let $D \subset \mathbb{R}$. Then the collection $C(D)$ of continuous functions on D is an algebra of functions, that is, for all $f, g \in C(D)$ and $\alpha \in \mathbb{R}$,

$$f + g \in C(D)$$

$$\alpha f \in C(D)$$

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Remark Can one add one more operation in this algebra: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(D)$, under what condition the limit $f_n \rightarrow f$ is closed?

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Def. $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on Ω , namely, $f_n : \Omega \rightarrow \mathbb{R}$ for each $n \in \mathbb{N}$.

We say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on Ω* if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R} .

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(a) $f_n \in C(\mathbb{R})$ defined as $f_n = (1 + x/n)^n$. Then f_n converges pointwise on \mathbb{R} to $f(x) = e^x$. It is clear that $f \in C(\mathbb{R})$.

(b) Let $D = [0, 1]$ and $f_n \in C(D)$ be defined as $f_n = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

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Def. Let \mathcal{F} be a collection of real-valued functions on Ω . We say that \mathcal{F} is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \rightarrow f$ pointwise on Ω , then $f \in \mathcal{F}$.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω .

We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω , if

$$\boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad \forall x \in \Omega \quad |f_n(x) - f(x)| < \epsilon,$$

written as $f_n \rightarrow f$ *uniformly*.

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Therefore, the collection $\mathcal{C}(D)$ of real-valued continuous functions is closed under: $+$, \cdot , scalar multiplication, and uniform convergence.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Some part of subsection is taken from Chapter 1 Section 4 of

*P. Billingsley, **Probability and Measure**, Wiley, 1995.*

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

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Use the relation:

set	logic
\cap	\forall
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$$\begin{aligned}
 \omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
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$$\limsup_n A_n = [A_n \text{ i.o.}]$$

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Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$\begin{array}{ccc}
 \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \liminf_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
 \cap & & \cap \\
 A_n & & \\
 \cap & & \\
 \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \limsup_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n
 \end{array}$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space (Ω, \mathbb{P})

(i) The above Property (ii) can be translated to a probability statement:

$$\begin{array}{ccc}
 \mathbb{P} \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \mathbb{P} \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\liminf_n A_n \right) \\
 & & \uparrow \wedge \\
 & & \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 \mathbb{P}(A_n) & & \uparrow \wedge \\
 & & \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 & & \uparrow \wedge \\
 \mathbb{P} \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\limsup_n A_n \right)
 \end{array}$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned} 1 &\geq \mathbb{P}(A_n^c \text{ all but finitely many}) = 1 - \mathbb{P}(\{A_n^c \text{ all but finitely many}\}^c) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \\ &\geq 1 - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

□

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□

Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$A_1 = (-1, 0]$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{2}{3}\right]$$

$$A_4 = \left(-\frac{1}{4}, \frac{3}{4}\right]$$

$$\vdots \quad \vdots$$

$$A_{100} = \left(-\frac{1}{100}, \frac{99}{100}\right]$$

$$\vdots \quad \vdots$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n} \right] = [0, 1)$$

and

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Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Exercise

(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$:

$$A_1 = (-1, 2]$$

$$A_2 = \left(\frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3} \right]$$

$$A_4 = \left(\frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5} \right]$$

$$A_6 = \left(\frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \quad \vdots$$

$$\vdots \quad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99} \right]$$

$$A_{100} = \left(\frac{1}{100}, \frac{99}{100} \right]$$

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Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

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Sol.

$$\begin{aligned}& \liminf_n A_n \\&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\&= (0, 1) \cup (0, 1) \\&= (0, 1)\end{aligned}$$

Sol. (continued) Similarly,

$$\begin{aligned}
 & \limsup_n A_n \\
 &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \left(-\frac{1}{n}, \frac{n+1}{n} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \left(-\frac{1}{n+1}, \frac{n+2}{n+1} \right] \right\} \\
 &= [0, 1] \cap [0, 1] \\
 &= [0, 1]
 \end{aligned}$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \frac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \frac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$.

Solution:

$$\liminf_{n \rightarrow \infty} A_n = \{0\} \subset [0, 1] = \limsup_{n \rightarrow \infty} A_n$$

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Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Integration by parts

Examples

1. $\int_0^1 \tan^{-1}(x) dx$

2. $\int_0^x t^2 e^t dt$

3. $\int e^x \sin(x) dx$

more to come ...

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Examples

1. e^x

2. $\sin(x)$

3. e^{x^2}

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