Topics in Analysis and Linear Algebra

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Chapter 3. Topics in linear algebra

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

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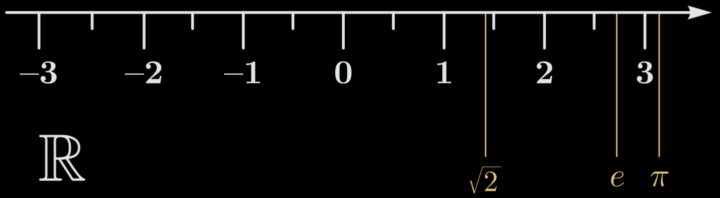
§ 3.2 Sequences of real numbers

What is a real number?



[1]

[1] Image from Wikipedia.

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exit $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \qquad \text{(Identities)}$$

(F5) For each $x \in \mathbb{R}$, there exits a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exits an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

(O3) $x < y$ and $z > 0$ implies that $xz < yz$.

(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

Completeness Axiom

Axiom A nonempty subset of real numbers that is bounded above has a least upper bound, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

Corr. A nonempty subset of real numbers that is bounded below has a greatest lower bound, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$

E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

Properties

1. Archimedean principle

   For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

   Between any two real numbers there is an irrational number.

3. Density of the rational numbers

   Between any two real numbers there is an rational number.

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence[2], namely,

$$\lim_{n\to\infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n\to\infty} x_n = L \iff \forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\; s.t.\; (n \geq N \to |x_n - L| < \epsilon)$$

[2] In this case, we say that $\{x_n\}_{n=1}^{\infty}$ is *convergent*. Otherwise, we say that $\{x_n\}_{n=1}^{\infty}$ is *divergent*.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(n-1)/n = 1$.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is divergent.

$\left\{n^2\right\}_{n=1}^{\infty}$ is divergent.

Def. Let $\mathbb{R}^*$ denote *the extended real line*, namely, $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$.

Def. A sequence of real numbers $\{a_1, a_2, \cdots\}$ is said to *converge in $\mathbb{R}^*$* if one of the following three conditions hold:

(i) The sequence converges to a finite real number as in the previous definition.

(ii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

(iii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

(i) We say that the sequence converges in $\mathbb{R}$.

(ii) Denoted as $\lim_{n\to\infty} x_n = \infty$

(iii) Denoted as $\lim_{n\to\infty} x_n = -\infty$

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}^*$.

$\left\{n^2\right\}_{n=1}^{\infty}$ converges in $\mathbb{R}^*$ and $\lim_{n\to\infty} n^2 = \infty$.

Def. If $x_1 \leq x_2 \leq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is not monotone.

$\left\{n^2\right\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

Axiom Let A be a nonempty subset of real numbers that is bounded above. Then *the least upper bound* of A exits, which is denoted by

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

Axiom Let A be a nonempty subset of real numbers that is bounded below. Then *the greatest lower bound* of A exits, which is denoted by

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$

Prop. Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$.

Moreover,

(a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, if $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$, it is bounded above.

(a) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

In particular, if $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$, it is bounded below.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is nondecreasing and converges in $\mathbb{R}$. It is bounded above.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is not monotone.

$\left\{n^2\right\}_{n=1}^{\infty}$ is nondecreasing, does not converge in $\mathbb{R}$, converges in $\mathbb{R}^*$.

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

(a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.

(b) ∞ *is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n > M$.

(c) $-\infty$ *is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n < M$.

E.g.1 $\{(n-1)/n\}_{n=1}^{\infty}$ has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$ has two cluster points: -1 and $+1$.

$\{n^2\}_{n=1}^{\infty}$ has one cluster point: $+\infty$.

E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞.

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Prop.

$$\{x_n\}_{n=1}^{\infty} \text{ converges in } \mathbb{R}^* \quad \Longleftrightarrow \quad \{x_n\}_{n=1}^{\infty} \text{ has exactly one cluster point in } \mathbb{R}^*.$$

A few more properties

1. If a sequence is bounded and monotonic, then it is convergent.

2. A sequence is convergent iff each subsequence is convergent.

3. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \to L$ and $b_n \to L$, then $c_n \to L$.

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n\to\infty} x_n := \sup_n \left(\inf_{m\geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\limsup_{n\to\infty} x_n := \inf_n \left(\sup_{m\geq n} a_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

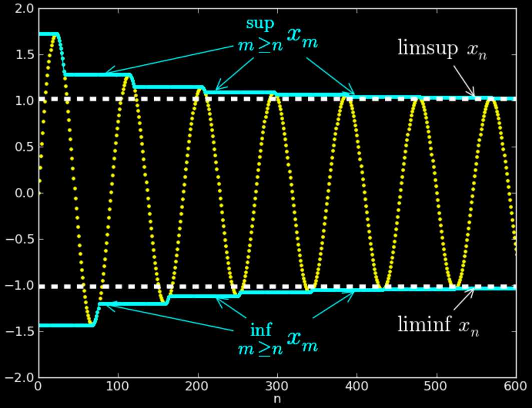
are, respectively, nonincreasing and nondecreasing, we see that

$$\inf_n y_n = \inf_n \sup_{m \geq n} x_m = \limsup_{x \to \infty} x_n$$

$$\sup_n z_n = \sup_n \inf_{m \geq n} x_m = \liminf_{x \to \infty} x_n.$$

Hence,

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{m \geq n} x_m.$$



[3]

[3] Image from Wikipedia.

Can we use the first-order logic to formulate the definitions?

Def'. $\limsup_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the smallest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

Set

$$A(b): \quad \forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\ \{n \geq N \to x_n < b + \epsilon\}$$

Then

$$\limsup_{n\to\infty} x_n = x \quad \Longleftrightarrow \quad \forall c \in \mathbb{R}\, \{c \geq x \leftrightarrow A(c)\}$$

Def'. $\liminf_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the largest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n > x + \epsilon$ for all $n > N$.

Set

$$B(b): \quad \forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\ \{n \geq N \to x_n > b + \epsilon\}$$

Then

$$\liminf_{n\to\infty} x_n = x \quad \Longleftrightarrow \quad \forall c \in \mathbb{R}\, \{c \leq x \leftrightarrow B(c)\}$$

E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1, amount which

$$\liminf_{n\to\infty} x_n = -1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi) \frac{n^2}{n^2+8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞, among which

$$\liminf_{n\to\infty} x_n = 1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

$$\liminf_{n\to\infty} r_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

(a) $\liminf x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.

(b) $\limsup x_n$ is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$.

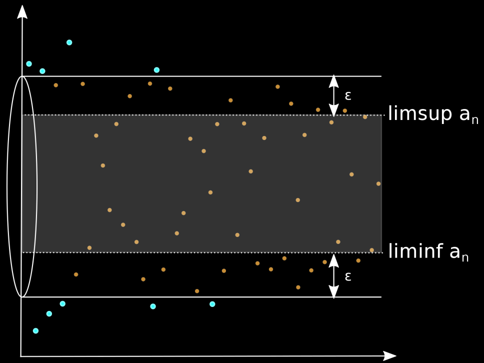
Properties

1.

$$\inf_n x_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = c \quad \Longleftrightarrow \quad \lim_{n\to\infty} x_n = c.$$



[4]

E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

[4] Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2}\left(\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n\right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right).$$

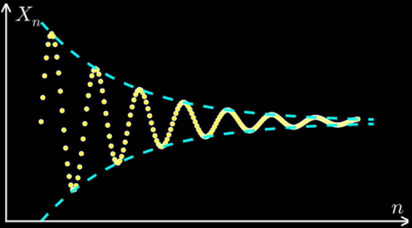
As we have seen that

A sequence of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point.

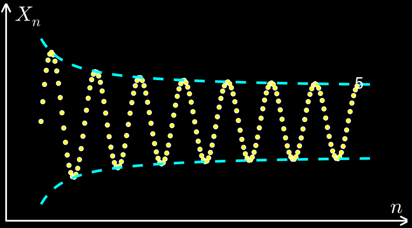
There is another famous criterion for a sequence to converge in $\mathbb{R}$:

Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



[5] Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in $\mathbb{R}$ iff it is Cauchy.

E.g.1 Let $a_n = \sqrt{n}$. Show that

(i) The consecutive terms become arbitrarily close to each other as $n \to \infty$.

(ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg \left(\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}\right) \\ \iff \quad & \exists \epsilon > 0 \, \forall N \in \mathbb{N} \, \exists m, n \geq N \, \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose $m = N$ and $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

$\square$

E.g.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

Variation

$$\lim_{n\to\infty}\Big(1+\mathrm{Small}\Big)^{\mathrm{Large}}=e^{\lim_{n\to\infty}\mathrm{Small}\times\mathrm{Large}}$$