

Topics in Analysis and Linear Algebra

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Last updated on

July 23, 2021

Summer Bootcamp for
Emory Biostatistics and Bioinformatics
PhD Program

July 22 - 28, 2021

Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

*J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.*

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Chapter 3. Real Number System and Calculus

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What is a real number?



1

¹Image from Wikipedia.



2

²Image from

<https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/>

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

$$(F1) \quad x + y = y + x \text{ and } xy = yx. \quad (\text{Commutative})$$

$$(F2) \quad (x + y) + z = x + (y + z) \text{ and } (xy)z = x(yz). \quad (\text{Associative})$$

$$(F3) \quad x(y + z) = xy + xz. \quad (\text{Distributive})$$

$$(F4) \quad \text{There exist } 0, 1 \in \mathbb{R} \text{ with } 0 \neq 1 \text{ such that for all } x \in \mathbb{R}$$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \quad (\text{Identities})$$

$$(F5) \quad \text{For each } x \in \mathbb{R}, \text{ there exists a } -x \in \mathbb{R} \text{ such that } x + (-x) = 0 \text{ and, if } x \neq 0, \text{ there exists an } x^{-1} \in \mathbb{R} \text{ such that } xx^{-1} = 1. \quad (\text{Inverses})$$

Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

- (O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)
- (O2) $x < y$ implies that $x + z < y + z$.
- (O3) $x < y$ and $z > 0$ implies that $xz < yz$.
- (O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

Completeness Axiom

Axiom A nonempty subset of real numbers that is **bounded above** has a **least upper bound**, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

Corr. A nonempty subset of real numbers that is **bounded below** has a **greatest lower bound**, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$

E.g. $\sup[0, 1) = 1$ and $\inf[0, 1) = 0$.

\mathbb{N} has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

Properties

1. Archimedean principle

For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

Between any two real numbers there is an **irrational** number.

3. Density of the rational numbers

Between any two real numbers there is an **rational** number.

Proof As exercises.



Extended Real Number System

HW Prove the three Properties.

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Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence³, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

³In this case, we say that $\{x_n\}_{n=1}^{\infty}$ is *convergent*. Otherwise, we say that $\{x_n\}_{n=1}^{\infty}$ is *divergent*.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n \rightarrow \infty} (n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

$\{n^2\}_{n=1}^{\infty}$ is divergent.

Def. Let \mathbb{R}^* denote *the extended real line*, namely, $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$.

Def. A sequence of real numbers $\{a_1, a_2, \dots\}$ is said to *converge in \mathbb{R}^** if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
 - (ii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.
 - (iii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.
-

- (i) We say that the sequence converges in \mathbb{R} .
- (ii) Denoted as $\lim_{n \rightarrow \infty} x_n = \infty$
- (iii) Denoted as $\lim_{n \rightarrow \infty} x_n = -\infty$

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ converges in \mathbb{R} .

$\{(-1)^n\}_{n=1}^{\infty}$ does not converge in \mathbb{R}^* .

$\{n^2\}_{n=1}^{\infty}$ converges in \mathbb{R}^* and $\lim_{n \rightarrow \infty} n^2 = \infty$.

Def. If $x_1 \leq x_2 \leq \dots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \dots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotone.

$\{n^2\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

Axiom Let A be a nonempty subset of real numbers that is **bounded above**.
Then *the least upper bound* of A exists, which is denoted by

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

Axiom Let A be a nonempty subset of real numbers that is **bounded below**.
Then *the greatest lower bound* of A exists, which is denoted by

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$

Prop. Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R}^* .

Moreover,

(a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, if $\{x_n\}_{n=1}^{\infty}$ converges in \mathbb{R} , it is **bounded above**.

(a) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

$$\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

In particular, if $\{x_n\}_{n=1}^{\infty}$ converges in \mathbb{R} , it is **bounded below**.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is nondecreasing and converges in \mathbb{R} . It is bounded above.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotone.

$\{n^2\}_{n=1}^{\infty}$ is nondecreasing, does not converge in \mathbb{R} , converges in \mathbb{R}^* .

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

- (a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.
- (b) ∞ *is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n > M$.
- (c) $-\infty$ *is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n < M$.

E.g.1 $\{(n-1)/n\}_{n=1}^{\infty}$ has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$ has two cluster points: -1 and $+1$.

$\{n^2\}_{n=1}^{\infty}$ has one cluster point: $+\infty$.

E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ .

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Prop.

$$\{x_n\}_{n=1}^{\infty} \text{ converges in } \mathbb{R}^* \iff \{x_n\}_{n=1}^{\infty} \text{ has exactly one cluster point in } \mathbb{R}^*.$$

A few more properties

1. If a sequence is **bounded and monotonic**, then it is convergent.
2. A sequence is convergent iff each subsequence is convergent.
3. **Sandwich theorem**: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \rightarrow L$ and $b_n \rightarrow L$, then $c_n \rightarrow L$.

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n \rightarrow \infty} x_n := \sup_n \left(\inf_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\limsup_{n \rightarrow \infty} x_n := \inf_n \left(\sup_{m \geq n} a_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

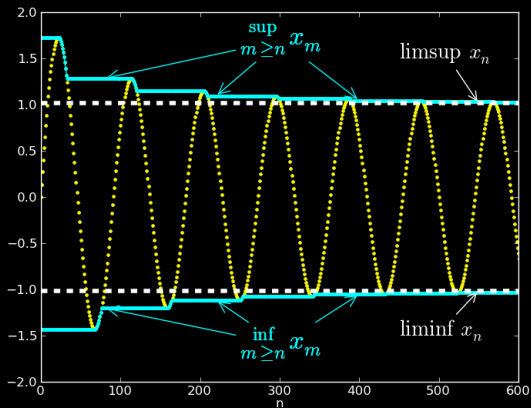
$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

$$\begin{aligned} \inf_n y_n &= \inf_n \sup_{m \geq n} x_m = \limsup_{x \rightarrow \infty} x_n \\ \sup_n z_n &= \sup_n \inf_{m \geq n} x_m = \liminf_{x \rightarrow \infty} x_n. \end{aligned}$$

Hence,

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} x_m.$$



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⁴Image from Wikipedia.

Can we use the first-order logic to formulate the definitions?

Def'. $\limsup_{n \rightarrow \infty} x_n = x \in \mathbb{R}$ if and only if x is the **smallest** real number such that for any positive real number $\epsilon > 0$, there exists a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

Set

$$A(b) : \quad \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \{n \geq N \rightarrow x_n < b + \epsilon\}$$

Then

$$\limsup_{n \rightarrow \infty} x_n = x \iff \forall c \in \mathbb{R} \{c \geq x \leftrightarrow A(c)\}$$

Def'. $\liminf_{n \rightarrow \infty} x_n = x \in \mathbb{R}$ if and only if x is the **largest** real number such that for any positive real number $\epsilon > 0$, there exists a natural number N such that $x_n > x - \epsilon$ for all $n > N$.

Set

$$B(b) : \quad \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \{n \geq N \rightarrow x_n > b - \epsilon\}$$

Then

$$\liminf_{n \rightarrow \infty} x_n = x \iff \forall c \in \mathbb{R} \{c \leq x \leftrightarrow B(c)\}$$

E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1 , amount which

$$\liminf_{n \rightarrow \infty} x_n = -1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi) \frac{n^2}{n^2 + 8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ , among which

$$\liminf_{n \rightarrow \infty} x_n = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

$$\liminf_{n \rightarrow \infty} r_n = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

- (a) $\liminf x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.
- (b) $\limsup x_n$ is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$.

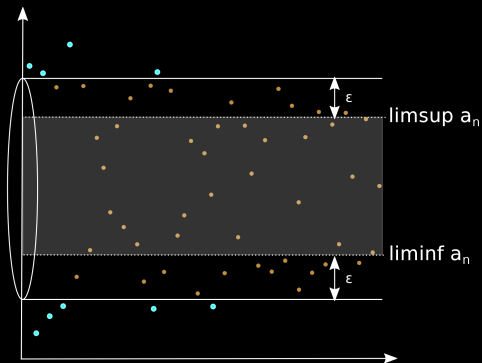
Properties

1.

$$\inf_n x_n \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers **converges in \mathbb{R}^*** if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = c \iff \lim_{n \rightarrow \infty} x_n = c.$$



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E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

⁵Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2} \left(\limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right).$$

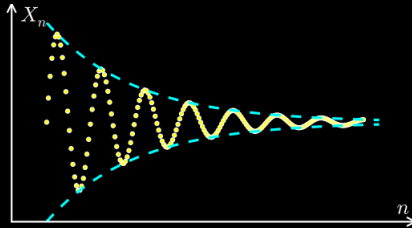
As we have seen that

A sequence of real numbers **converges in \mathbb{R}^*** if and only if it has exactly one cluster point.

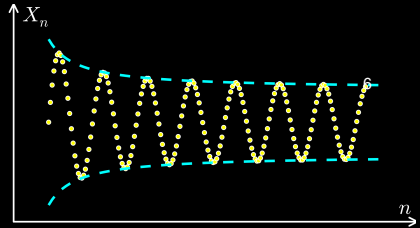
There is another famous criterion for a sequence to **converge in \mathbb{R}** :

Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



⁶Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers *converges in \mathbb{R}* iff it is Cauchy.

E.g.1 Let $a_n = \sqrt{n}$. Show that

- (i) The consecutive terms become arbitrarily close to each other as $n \rightarrow \infty$.
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose $m = N$ and $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$



E.g.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Variation

$$\lim_{n \rightarrow \infty} \left(1 + \text{Small}\right)^{\text{Large}} = e^{\lim_{n \rightarrow \infty} \text{Small} \times \text{Large}}$$

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Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exists an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of \mathbb{R} . Then K cannot be an open set. For example, \mathbb{N} , \mathbb{Q} , \mathbb{Z} are not open sets.

\mathbb{Q}^c – the set of irrational numbers – is not an open set.

Properties

1. \mathbb{R} and \emptyset are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open. (arbitrary union)

Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

The set of all limit point of E , denoted \overline{E} , is called the *closure* of E .

E.g. $\overline{\mathbb{R}} = \mathbb{R}$ and $\overline{\emptyset} = \emptyset$.

Let $a, b \in \mathbb{R}$ such that $a < b$. Then

$$\overline{(a, b)} = \overline{(a, b]} = \overline{[a, b)} = \overline{[a, b]} = [a, b].$$

$$\overline{\mathbb{N}} = \mathbb{N} \text{ and } \overline{\mathbb{Z}} = \mathbb{Z}.$$

$$\overline{\mathbb{Q}} = \mathbb{R} \text{ and } \overline{\mathbb{Q}^c} = \mathbb{R}.$$

If A is a finite subset of \mathbb{R} , then $\overline{A} = A$.

Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

E.g. \mathbb{R} and \emptyset are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

\mathbb{N} and \mathbb{Z} are closed sets.

The sets of rationals \mathbb{Q} and irrationals \mathbb{Q}^c are neither open nor close.

If A is a finite subset of \mathbb{R} , then A is a close set.

Properties

0. A set is open if and only if its complement is closed.



1. \mathbb{R} and \emptyset are closed sets.

2. If A and B are closed sets, so is $A \cup B$. (finite union)

3. If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

(arbitrary intersection)

Def. Let $G \subset D \subset \mathbb{R}$.

(a) G is said to be **open in D** if for each $x \in G$, there is an $r > 0$ such that

$$(x - r, x + r) \cap D \subset G.$$

(b) G is said to be **closed in D** if $D \setminus G$ is open in D .

E.g.

D	G	Is G open in \mathbb{R}	Is G open in D
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
\mathbb{N}	$A \subset \mathbb{N}$	closed	open

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Def. A *real-valued function* is a function whose range is a subset of \mathbb{R} . If $f : \Omega \rightarrow \mathbb{R}$, we say that f is a *real-valued function on Ω* .

Def. *Algebraic operations*: Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

$$(f + g)(x) := f(x) + g(x)$$

$$(\alpha f)(x) := \alpha f(x)$$

$$(f \cdot g)(x) := f(x)g(x)$$

(Local) Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. f is *continuous at a point c* if the limit of $f(x)$, as x approaches c , exists and is equal to $f(c)$.

Def'. (Epsilon-delta definition) The function f is *continuous at a point c* if

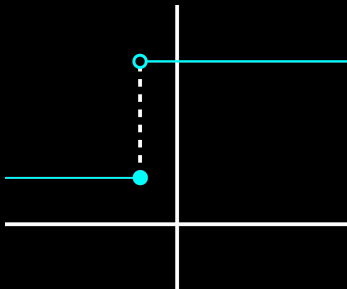
$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} \{ |x - c| \leq \delta \rightarrow |f(x) - f(c)| \leq \epsilon \}$$

Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in \mathbb{R} .

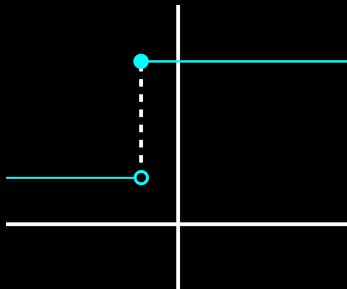
Def. f is *left-continuous at c* if

$$\lim_{x \rightarrow c+} f(x) = f(c)$$



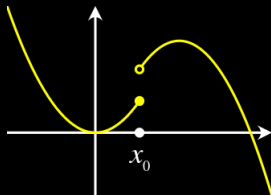
Def. f is *right-continuous at c* if

$$\lim_{x \rightarrow c-} f(x) = f(c)$$



Def. f is *lower semi-continuous at x_0* if

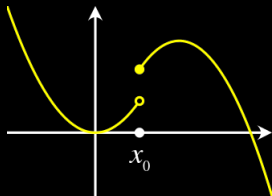
$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$$



$f(x_0)$ can be all points
at or below the blue point.

f is *upper semi-continuous at x_0*
if

$$\limsup_{x \rightarrow x_0} f(x) \leq f(x_0)$$



$f(x_0)$ can be all points
at or above the blue point.

(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let I be an interval of \mathbb{R} . Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Def.' f is *uniformly continuous over I* if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \}$$

f is continuous at $x_0 \in I$ iff

$$\forall \epsilon > 0 \forall x \in I \exists \delta > 0 \{ |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon \}$$

Π_2 -form

f is uniformly continuous over I iff

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \}$$

Π_3 -form

Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I , we need to show

$$\begin{aligned} & \neg \left(\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \right) \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ \neg \{ |x - y| < \delta \} \vee |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \{ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon \} \end{aligned}$$

Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

□

Prop. 2 If I is compact⁷ set such as $I = [a, b]$, then

f is continuous at all points in $I \iff f$ is uniformly continuous on I .

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on \mathbb{R} nor on $[0, \infty)$.

⁷namely, bounded and closed

Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

Notation For $D \subset \mathbb{R}$, let $C(D)$ denote the set of continuous functions defined on D .

Thm (*Algebra of $C(D)$*) Let $D \subset \mathbb{R}$. Then the collection $C(D)$ of continuous functions on D is an algebra of functions, that is, for all $f, g \in C(D)$ and $\alpha \in \mathbb{R}$,

$$f + g \in C(D)$$

$$\alpha f \in C(D)$$

$$f \cdot g \in C(D)$$

Remark Can one add one more operation in this algebra: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(D)$, under what condition the limit $f_n \rightarrow f$ is closed?

Def. $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on Ω , namely, $f_n : \Omega \rightarrow \mathbb{R}$ for each $n \in \mathbb{N}$.

We say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on Ω* if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R} .

E.g.

(a) $f_n \in C(\mathbb{R})$ defined as $f_n = (1 + x/n)^n$. Then f_n converges pointwise on \mathbb{R} to $f(x) = e^x$. It is clear that $f \in C(\mathbb{R})$.

(b) Let $D = [0, 1]$ and $f_n \in C(D)$ be defined as $f_n = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

It is clear that $f \notin C(D)$.

Def. Let \mathcal{F} be a collection of real-valued functions on Ω . We say that \mathcal{F} is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \rightarrow f$ pointwise on Ω , then $f \in \mathcal{F}$.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω .

We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω , if

$$\boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad \forall x \in \Omega \quad |f_n(x) - f(x)| < \epsilon,$$

written as $f_n \rightarrow f$ *uniformly*.

Recall $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* to f if

$$\forall x \in \Omega \quad \boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad |f_n(x) - f(x)| < \epsilon,$$

Prop. Let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \rightarrow f$ uniformly.
Then $f \in C(D)$.

Proof.

Therefore, the collection $\mathcal{C}(D)$ of real-valued continuous functions is closed under: $+$, \cdot , scalar multiplication, and uniform convergence.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Some part of subsection is taken from Chapter 1 Section 4 of

*P. Billingsley, **Probability and Measure**, Wiley, 1995.*

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Remark Both $\limsup_n A_n$ and $\liminf_n A_n$ are sets.

Use the relation:

set	logic
\bigcap	\forall
\bigcup	\exists

$$\begin{aligned}
 \omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
 &\iff (\forall n \geq 1) (\exists k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\limsup_n A_n = [A_n \text{ i.o.}]$$

Use the relation:

set	logic
\bigcap	\forall
\bigcup	\exists

$$\begin{aligned}
 \omega \in \liminf_n A_n &\iff \omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\
 &\iff (\exists n \geq 1) (\forall k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{all but finitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\liminf_n A_n = [A_n \text{ all but finitely many}]$$

Def. If both $\limsup_n A_n$ and $\liminf_n A_n$ exist and are equal, then the *limit set* of the sequence $\{A_n\}$ is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

which is also often written as $A_n \rightarrow A$.

Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$\begin{array}{ccc}
 \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \liminf_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
 \cap & & \cap \\
 A_n & & \\
 \cap & & \\
 \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \limsup_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n \\
 \cup & & \cup
 \end{array}$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space (Ω, \mathbb{P})

(i) The above Property (ii) can be translated to a probability statement:

$$\begin{array}{ccc}
 \mathbb{P} \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \mathbb{P} \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\liminf_n A_n \right) \\
 & & \uparrow \wedge \\
 & & \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 \mathbb{P}(A_n) & & \uparrow \wedge \\
 & & \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 & & \uparrow \wedge \\
 \mathbb{P} \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\limsup_n A_n \right)
 \end{array}$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned} 1 &\geq \mathbb{P}(A_n^c \text{ all but finitely many}) = 1 - \mathbb{P}(\{A_n^c \text{ all but finitely many}\}^c) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \\ &\geq 1 - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

□

Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$A_1 = (-1, 0]$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{2}{3}\right]$$

$$A_4 = \left(-\frac{1}{4}, \frac{3}{4}\right]$$

$$\vdots \quad \vdots$$

$$A_{100} = \left(-\frac{1}{100}, \frac{99}{100}\right]$$

$$\vdots \quad \vdots$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n} \right] = [0, 1)$$

and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 \right) = (-1, 1).$$

Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

□

Exercise

(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3} \right] \qquad A_4 = \left(\frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

Sol.

$$\begin{aligned}& \liminf_n A_n \\&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\&= (0, 1) \cup (0, 1) \\&= (0, 1)\end{aligned}$$

Sol. (continued) Similarly,

$$\begin{aligned}
 & \limsup_n A_n \\
 &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \left(-\frac{1}{n}, \frac{n+1}{n} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \left(-\frac{1}{n+1}, \frac{n+2}{n+1} \right] \right\} \\
 &= [0, 1] \cap [0, 1] \\
 &= [0, 1]
 \end{aligned}$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \frac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \frac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$.

Solution:

$$\liminf_{n \rightarrow \infty} A_n = \{0\} \subset [0, 1] = \limsup_{n \rightarrow \infty} A_n$$

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Integration by parts

Examples

1. $\int_0^1 \tan^{-1}(x) dx$

2. $\int_0^x t^2 e^t dt$

3. $\int e^x \sin(x) dx$

more to come ...

Examples

1. e^x

2. $\sin(x)$

3. e^{x^2}

more to come ...