Topics in Analysis and Linear Algebra

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https://github.com/chenle02/2021_Summer_Bio_bootcamp_slides/

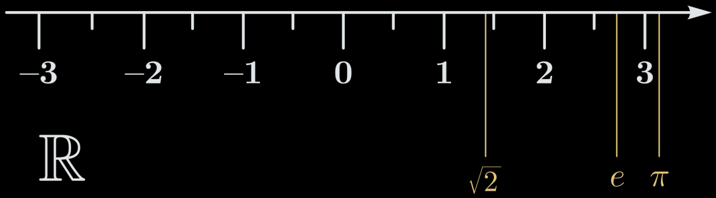
Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.

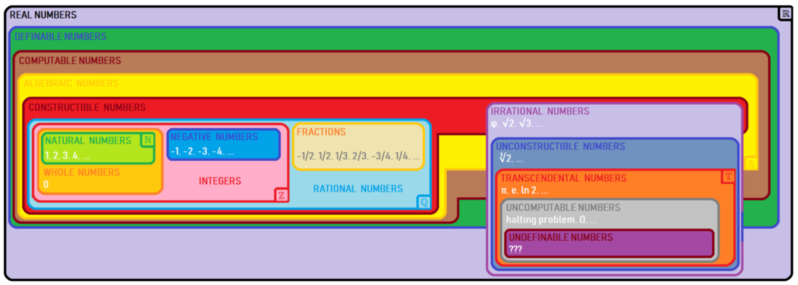
Chapter 3. Real Number System and Calculus

What is a real number?



[1]

[1] Image from Wikipedia.



[2]

Image from
https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exit $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \qquad \text{(Identities)}$$

(F5) For each $x \in \mathbb{R}$, there exits a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exits an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

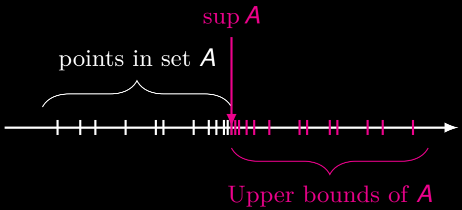
(O3) $x < y$ and $z > 0$ implies that $xz < yz$.

(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

Completeness Axiom

Axiom A nonempty subset of real numbers that is bounded above has a least upper bound, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$



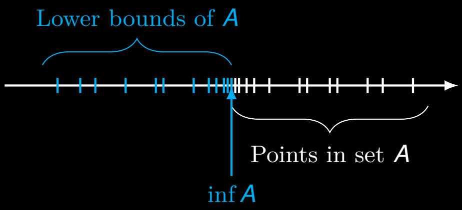
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$\begin{aligned}
\sup_n x_n \leq a \quad &\iff \quad \forall n,\ \ x_n \leq a \\
\sup_n x_n < a \quad &\iff \quad \forall n,\ \ x_n < a \\
a < \sup_n x_n \quad &\iff \quad \exists n,\ \ x_n > a
\end{aligned}$$

$$a \leq \sup_n x_n \quad \Longleftarrow \quad \exists n,\ \ x_n \geq a$$

Corr. A nonempty subset of real numbers that is bounded below has a greatest lower bound, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$



Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$
\begin{aligned}
a \leq \inf_n x_n \quad &\iff \quad \forall n,\ \ x_n \geq a \\
a < \inf_n x_n \quad &\iff \quad \forall n,\ \ x_n > a \\
\inf_n x_n < a \quad &\iff \quad \exists n,\ \ x_n < a
\end{aligned}
$$

$$\inf_n x_n \leq a \quad \Longleftarrow \quad \exists n,\ \ x_n \leq a$$

E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

Properties

1. Archimedean principle

   For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

   Between any two real numbers there is an irrational number.

3. Density of the rational numbers

   Between any two real numbers there is an rational number.

Proof As exercises. $\square$

Extended Real Number System

Def. The *extended real numbers* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
$x < 0$	$x \cdot \infty = \infty \cdot x = -\infty$	$x \cdot (-\infty) = (-\infty) \cdot x = \infty$
$x = \infty$	$\infty + \infty = \infty$	$(-\infty) + (-\infty) = -\infty$
	$\infty \cdot \infty = \infty$	$(-\infty) \cdot (-\infty) = \infty$
	$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$	

$\infty - \infty$ cannot be defined (HW).

Def. Let a and b be extended real numbers such that $a < b$. Then the *intervals on $\mathbb{R}^*$* with endpoints a and b are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : \; a < x < b\}$$
$$[a, b) = \{x \in \mathbb{R}^* : \; a \leq x < b\}$$
$$(a, b] = \{x \in \mathbb{R}^* : \; a < x \leq b\}$$
$$[a, b] = \{x \in \mathbb{R}^* : \; a \leq x \leq b\}$$

If both a and b are in $\mathbb{R}$, these intervals are the *bounded intervals* in $\mathbb{R}$.

Otherwise, if either $a = -\infty$ or $b = \infty$, then these intervals are *unbounded intervals*.

Thm Every subset A of $\mathbb{R}^*$ has both a least upper bound and greatest lower bound. Moreover,

a) If $A = \emptyset$, then $\sup A = -\infty$ and $\inf A = \infty$.

b) If A is bounded above in $\mathbb{R}$, then $\sup A \in \mathbb{R}$; otherwise, $\sup A = \infty$.

c) If A is bounded below in $\mathbb{R}$, then $\inf A \in \mathbb{R}$; otherwise, $\inf A = -\infty$.

E.g.

a) $\inf \mathbb{N} = 1$ and $\sup \mathbb{N} = \infty$.

b) $\inf \mathbb{Z} = -\infty$ and $\sup \mathbb{Z} = \infty$.

c) If I is an interval in $\mathbb{R}^*$ with endpoints a and b, $a \leq b$. Then $\inf I = a$ and $\sup I = b$.

HW Ex. 2.10 and 2.11 on p. 43.

Chapter 3. Real Number System and Calculus

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence[3], namely,

$$\lim_{n\to\infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n\to\infty} x_n = L \quad \Longleftrightarrow \quad \forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \in \mathbb{N} \; s.t. \; (n \geq N \to |x_n - L| < \epsilon)$$

[3] In this case, we say that $\{x_n\}_{n=1}^{\infty}$ is *convergent*. Otherwise, we say that $\{x_n\}_{n=1}^{\infty}$ is *divergent*.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(n-1)/n = 1$.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is divergent.

$\left\{n^2\right\}_{n=1}^{\infty}$ is divergent.

Recall that $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ is the *extended real line*.

Def. A sequence of real numbers $\{a_1, a_2, \cdots\}$ is said to *converge in $\mathbb{R}^*$* if one of the following three conditions hold:

(i) The sequence converges to a finite real number as in the previous definition.

(ii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

(iii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

(i) We say that the sequence converges in $\mathbb{R}$ or the limit exits and is finite.

(ii) We say the sequence converges to ∞ and write $\lim_{n\to\infty} x_n = \infty$.

(iii) We say the sequence converges to $-\infty$ and write $\lim_{n\to\infty} x_n = -\infty$.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}^*$.

$\left\{n^2\right\}_{n=1}^{\infty}$ converges in $\mathbb{R}^*$ and $\lim_{n\to\infty} n^2 = \infty$.

Monotone sequence

Def. If $x_1 \leq x_2 \leq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is not monotone.

$\left\{n^2\right\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$. Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded above and is ∞ otherwise.

b) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded below and is $-\infty$ otherwise.

Proof. We will prove the case when $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. The nonincreasing case can be proved in a similar way.

As $\sup_n x_n$ always exists in $\mathbb{R}^*$, we need to consider two cases:

Case I: $\sup_n x_n \in \mathbb{R}$

Case II: $\sup_n x_n = \infty$

Let's prove Case I here. Let $x = \sup_n x_n$. In order to show that $\lim_n x_n = x$, by monotonicity, we need to show that

$$\forall \epsilon > 0\, \exists N\, \forall n \text{ s.t. } (n \geq N) \rightarrow (x - a_n \leq \epsilon).$$

Proof. (Continued) Otherwise,

$$\exists \epsilon > 0 \, \forall N \, \exists n \text{ s.t. } (n \geq N) \wedge (x - a_n > \epsilon).$$

Hence, there are infinitely many terms falling below $x - \epsilon$.

Since $\{x_n\}$ is nondecreasing, this implies all a_n fall below $x - \epsilon$, i.e.,

$$a_n < x - \epsilon, \quad \text{for all } n \geq 1.$$

which is equivalent to $\sup_n x_n < x - \epsilon$. This contradicts with the fact that $\sup_n x_n = x$.

Therefore,

$$\lim_n x_n = x = \sup_n x_n.$$

$\square$

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is nondecreasing and converges in $\mathbb{R}$. It is bounded above.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotone.

$\{n^2\}_{n=1}^{\infty}$ is nondecreasing, does not converge in $\mathbb{R}$, converges in $\mathbb{R}^*$.

Cluster points

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.

b) *∞ is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n > M$.

c) *$-\infty$ is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n < M$.

---

a) $x \in \mathbb{R}$ is a cluster point of $\{x_n\}$ if

$$\forall \epsilon \, \forall N \, \exists n \quad (n \geq N) \rightarrow (|x - x_n| < \epsilon). \tag{1}$$

E.g.1 $\{(n-1)/n\}_{n=1}^{\infty}$ has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$ has two cluster points: -1 and $+1$.

$\{n^2\}_{n=1}^{\infty}$ has one cluster point: $+\infty$.

E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞.

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Thm A convergent sequence has exactly one cluster point, namely, its limit. Thus, a sequence having more than one cluster point cannot converge.

Proof Suppose that $\{x_n\}$ is a convergent sequence and let x be its limit, namely, $\lim_{n\to\infty} x_n = x$. We need to prove:

(1) x is a cluster point.

(2) x is the only cluster point of $\{x_n\}$.

We also need to consider two cases:

Case I: $x \in \mathbb{R}$.

Case II: $x = \infty$ or $-\infty$.

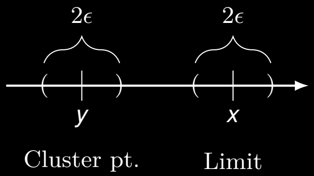
We will focus on Case I only.

Now we first prove (1).

$$
\begin{array}{ll}
\lim_n x_n = x & \forall\epsilon\ \overbrace{\exists N\ \forall n}^{\text{Once for all rest}}\quad (n \geq N) \to (|x_n - x| < \epsilon) \\
\Downarrow ?? & \\
x \text{ is a cluster point of } \{x_n\} & \forall\epsilon\ \underbrace{\forall\widetilde{N}\ \exists\widetilde{n}}_{\text{Infinite games}}\quad (\widetilde{n} \geq \widetilde{N}) \wedge (|x_{\widetilde{n}} - x| < \epsilon)
\end{array}
$$

(1) is proved by choosing any $\widetilde{n} \geq \max\left(\widetilde{N}, N\right)$.

As for (2), suppose y is another cluster point.



By choosing any $\epsilon < |x - y|/2$, we see that

1. In the ϵ-neighborhood of y, there are infinitely many terms.
2. In the ϵ-neighborhood of x, all but finite many terms are here.

Contradiction!

Therefore, there exits only one cluster point. $\square$

A few more properties

1. A sequence is convergent iff each subsequence is convergent.

2. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \to L$ and $b_n \to L$, then $c_n \to L$.

Limit superior and limit inferior

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n\to\infty} x_n := \sup_n \left(\inf_{m\geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\limsup_{n\to\infty} x_n := \inf_n \left(\sup_{m\geq n} a_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

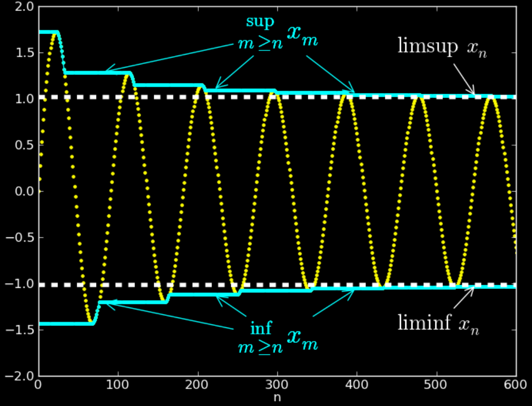
are, respectively, nonincreasing and nondecreasing, we see that

$$\inf_n y_n = \inf_n \sup_{m \geq n} x_m = \limsup_{x \to \infty} x_n$$

$$\sup_n z_n = \sup_n \inf_{m \geq n} x_m = \liminf_{x \to \infty} x_n.$$

Hence,

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{m \geq n} x_m.$$



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[4] Image from Wikipedia.

Characterization of the limsup and liminf.

Thm Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

a) $\limsup x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,

i) there is an $N \in \mathbb{N}$ such that $x_n \leq x + \epsilon$ for all $n \geq N$, and

ii) for each $n \in \mathbb{N}$, there is an $m \geq n$ such that $x_m > x - \epsilon$.

b) $\limsup x_n = \infty$ iff for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there is an $n \geq N$ such that $x_n > M$; in other words, iff the sequence is unbounded from above.

c) Similarly, $\limsup x_n = -\infty$ if and only $\lim x_n = -\infty$.

---

a) $\limsup x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,

i) the ϵ-neighborhood of x has been visited infinitely many times; and

ii) only finite many terms are greater than $x + \epsilon$.

Proof. (Sketch) We only prove part (a). Let $y_n = \sup_{k \geq n} x_k$.

$$\limsup x_n = x \iff \lim_{n\to\infty} \sup_{k \geq n} x_k = x$$

$$\iff \forall \epsilon \, \exists N \, \forall n \ (n \geq N) \to \left(\sup_{k \geq n} x_k \in (x - \epsilon, x + \epsilon) \right)$$

$$\sup_{k \geq n} x_k < x + \epsilon \iff \text{all terms starting from } n \text{ fall below } x + \epsilon$$

$$\sup_{k \geq n} x_k > x - \epsilon \iff \exists k \geq n \text{ s.t. } x_k > x + \epsilon$$

$\square$

Thm Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

a) $\liminf x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,

   i) there is an $N \in \mathbb{N}$ such that $x_n \geq x - \epsilon$ for all $n \geq N$, and

   ii) for each $n \in \mathbb{N}$, there is an $m \geq n$ such that $x_m < x + \epsilon$.

b) $\liminf x_n = -\infty$ iff for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there is an $n \geq N$ such that $x_n < M$; in other words, iff the sequence is unbounded from below.

c) Similarly, $\liminf x_n = \infty$ if and only $\lim x_n = \infty$.

Still other characterization (as an exercise):

Thm $\limsup_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the smallest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

$\liminf_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the largest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n > x + \epsilon$ for all $n > N$.

Proof. HW for motivated students. $\square$

E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1, amount which

$$\liminf_{n\to\infty} x_n = -1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi)\frac{n^2}{n^2+8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞, among which

$$\liminf_{n\to\infty} x_n = 1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

$$\liminf_{n\to\infty} r_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

a) $\limsup x_n$ is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$.

b) $\liminf x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.

Proof. We only prove (a). (b) can be proved in a similar way.

Let $x = \limsup x_n$. We have seen that x is a cluster point.

It remains to prove that x is the largest cluster point. The case when $x = \pm\infty$ is left for the motivated students.

Now assume that $x \in \mathbb{R}$.

Only finite many terms exceed $x + 1$, hence, ∞ is not a cluster point.

Let $y \in \mathbb{R}$ s.t. $x < y$. Set $\epsilon = (y - x)/2$.

Only finite many terms exceed $x + \epsilon$.

Since $y - \epsilon = x + \epsilon$, the ϵ-neighborhood of y has been visited only finitely many times.

Hence, y cannot be a cluster point. $\square$

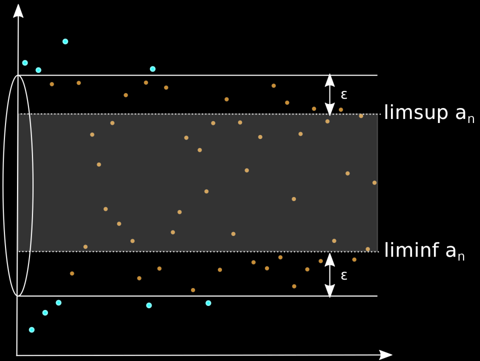
Properties

1.

$$\inf_n x_n \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = c \quad \Longleftrightarrow \quad \lim_{n\to\infty} x_n = c.$$



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E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

[5] Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2}\left(\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n\right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n\to\infty} x_n + \epsilon,\ \limsup_{n\to\infty} x_n - \epsilon\right).$$

Cauchy criterion

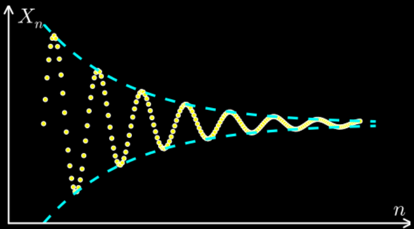
As we have seen that

A sequence of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point.

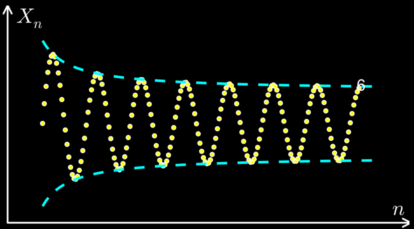
There is another famous criterion for a sequence to converge in $\mathbb{R}$:

Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



[6] Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in $\mathbb{R}$ iff it is Cauchy.

Proof. "$\Rightarrow$" Easy!

"$\Leftarrow$": ... $\square$

E.g.1 Let $a_n = \sqrt{n}$. Show that

(i) The consecutive terms become arbitrarily close to each other as $n \to \infty$.

(ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg\left(\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N\, \{|x_n - x_m| < \epsilon\}\right) \\ \iff & \quad \exists \epsilon > 0\, \forall N \in \mathbb{N}\, \exists m, n \geq N\, \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose $m = N$ and $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

$\square$

HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

Open sets

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exits an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of $\mathbb{R}$. Then K cannot be an open set. For example, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ are not open sets.

$\mathbb{Q}^c$ – the set of irrational numbers – is not an open set.

Properties of open sets

1. $\mathbb{R}$ and $\emptyset$ are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open. (arbitrary union)

Proof. Exercise. $\square$

Let $Q_n = (-1/n, 1/n)$. Then $\bigcap_{i \in \mathbb{N}} Q_n = \{0\}$ is not an open set.

Def. For $a, b \in \mathbb{R}^*$ with $a < b$, (a, b) is an open set, which is called an *open interval*.

Thm. Each open set O is a countable union of disjoint open intervals.

Moreover, the representation is unique in the sense that if $\mathcal{C}$ and $\mathcal{D}$ are two pairwise disjoint collections of open intervals whose union is O, then $\mathcal{C} = \mathcal{D}$.

Closed sets

Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

The set of all limit point of E, denoted $\overline{E}$, is called the *closure* of E.

E.g. $\overline{\mathbb{R}} = \mathbb{R}$ and $\overline{\emptyset} = \emptyset$.

Let $a, b \in \mathbb{R}$ such that $a < b$. Then

$$\overline{(a,b)} = \overline{(a,b]} = \overline{[a,b)} = \overline{[a,b]} = [a,b].$$

$\overline{\mathbb{N}} = \mathbb{N}$ and $\overline{\mathbb{Z}} = \mathbb{Z}$.

$\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{Q}^c} = \mathbb{R}$.

If A is a finite subset of $\mathbb{R}$, then $\overline{A} = A$.

Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

E.g. $\mathbb{R}$ and $\emptyset$ are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

$\mathbb{N}$ and $\mathbb{Z}$ are closed sets.

The sets of rationals $\mathbb{Q}$ and irrationals $\mathbb{Q}^c$ are neither open nor close.

If A is a finite subset of $\mathbb{R}$, then A is a close set.

Thm. A set is open if and only if its complement is closed.

Or equivalently, a set is closed if and only if its complement is open.

Properties of closed sets

1. $\mathbb{R}$ and $\emptyset$ are closed sets.
2. If A and B are closed sets, so is $A \cup B$. (finite union)
3. If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed. (arbitrary intersection)

Let $Q_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$. Then $\bigcup_{i \in \mathbb{N}} Q_n = (0, 1)$ is an open set.

or

$\bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q}$ is neither open nor closed.

Relative open and closed sets

Def. Let $G \subset D \subset \mathbb{R}$.

a) G is said to be open in D if for each $x \in G$, there is an $r > 0$ such that

$$(x - r, x + r) \cap D \subset G.$$

b) G is said to be *closed in D* if $D \setminus G$ is open in D.

E.g.

D	G	Is G open in $\mathbb{R}$	Is G open in D
$[0,2]$	$[0,1)$	Neither open nor closed	open
$[0,2]$	$[0,1]$	closed	closed
$\mathbb{N}$	$A \subset \mathbb{N}$	closed	open

Thm. Let $D \subset \mathbb{R}$. A set $G \subset D$ is open in D if and only if there is an open set O of $\mathbb{R}$ such that $G = D \cap O$.

In other words, the open sets in D are precisely the open sets of $\mathbb{R}$ intersected with D.

HW Ex. 2.38, 2.46, 2.47, 2.49, 2.52 on p. 63 – 64.

Def. A *real-valued function* is a function whose range is a subset of $\mathbb{R}$. If $f : \Omega \to \mathbb{R}$, we say that f is a *real-valued function on* Ω.

Algebraic operations

Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

$$(f+g)(x) := f(x) + g(x)$$
$$(\alpha f)(x) := \alpha f(x)$$
$$(f \cdot g)(x) := f(x)g(x)$$

Continuous functions

Def. Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $x_0 \in D$. We say that f is *continuous at x_0* if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in D$ and $|x - x_0| < \delta$.

We say that f is *continuous on D* if it is continuous on every point of D.

We use $C(D)$ to denote the collection of all continuous functions on D.

If f is not continuous at x_0, then we say that f is *discontinuous at x_0* or that x_0 is a *point of discontinuity* of f.

$$f \text{ is continuous at } x_0$$

$$\Updownarrow$$

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\, \{|x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon\}$$

$$f \in C(D)$$

$$\Updownarrow$$

$$\forall x \in D\, \boxed{f \text{ is continuous at } x}$$

$$\Updownarrow$$

$$\forall x \in D\, \boxed{\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\, \{|x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon\}}$$

E.g.

a) Let $D = (0, \infty)$ and define $f(x) = 1/x$. Then f is continuous function on D.

b) Let $D = \mathbb{R}$ and define $f(0) = 0$ and $f(x) = \sin(1/x)$ for $x \neq 0$. Then f is a continuous function except at 0.

c) Let $D = \mathbb{R}$ and define $f(x) = \lfloor x \rfloor$. Then f is continuous except at points of $\mathbb{Z}$.

d) Every function is continuous on $\mathbb{N}$. Or in other words, any infinite series $\{a_n\}_{n \in \mathbb{N}}$, when viewed as a function $a : \mathbb{N} \to \mathbb{R}$, is a continuous function.

Thm. Let $D \subset \mathbb{R}$. The collection $C(D)$ of continuous functions on D is an algebra of functions, i.e., if $f, g \in C(D)$ and $a \in \mathbb{R}$, then

a) $f + g \in C(D)$.

b) $\alpha f \in C(D)$.

c) $f \cdot g \in C(D)$.

Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in $\mathbb{R}$, i.e., the preimage of each open set in $\mathbb{R}$ is open in D.

Corr. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(O)$ is open (in $\mathbb{R}$) whenever O is open (in $\mathbb{R}$).

Proof. “$\Longrightarrow$”

Suppose that f is continuous on D. Let O be an arbitrary open set in $\mathbb{R}$. We need to show that $f^{-1}(O)$ is open in D.

Hence, we need to show that for any $x_0 \in f^{-1}(O)$, one can find $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O)$.

Notice that

$$x_0 \in f^{-1}(O) \Longleftrightarrow f(x_0) \in O.$$

Since O is open, one can find $r > 0$ such that r-neighborhood of $f(x_0)$ is in O.

By the continuity of f at x_0

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\, |x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon$$

applied with $\epsilon = r$, we can find some $\delta > 0$ such that

$$\forall x \left\{ \left(x \in (x_0 - \delta, x_0 + \delta) \cap D \right) \to \left(f(x) \in (f(x_0) - r, f(x_0) + r) \right) \right\}.$$

Proof. (Continued)

Because

$$f(x) \in (f(x_0) - r, f(x_0) + r) \subset O$$

we have that

$$f(x) \in O,$$

or equivalently,

$$x \in f^{-1}(O).$$

Hence, if $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $x \in f^{-1}(O)$, i.e.,

$$(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O).$$

Proof. "$\Longleftarrow$"

Suppose $f^{-1}(O)$ is open in D for each open set $O \subset \mathbb{R}$.

For each $x_0 \in D$, we need to prove f is continuous at x_0, namely,

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\, |x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon$$

Now fix arbitrary $x_0 \in D$ and arbitrary $\epsilon > 0$.

Let $O = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, which is an open interval.

By the assumption, $f^{-1}(O)$ is open in D. Hence, there is $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O).$$

In other words,

$$\forall x \in D \left(|x - x_0| < \delta \to x \in f^{-1}(O) \right).$$

Finally, notice that

$$x \in f^{-1}(O) \iff f(x) \in O \iff |f(x) - f(x_0)| < \epsilon.$$

$\square$

Pointwise limits

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω, that is, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$. Then we say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on* Ω if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}$.

If $\{f_n\}_{n=1}^{\infty}$ converges pointwise in Ω, then we define

$$f : \Omega \to \mathbb{R}$$

by

$$f(x) := \lim_{n\to\infty} f_n(x),$$

which is called the *pointwise limit of the sequence of functions* $\{f_n\}_{n=1}^{\infty}$. In this case, we also call the sequence of functions $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* to f, denoted as $f_n \to f$ pointwise.

E.g.

a) Let $f_n : \mathbb{R} \to \mathbb{R}$ defined as $f_n(x) = (1 + x/n)^n$. Then $f_n \to f$ pointwise on $\mathbb{R}$ with $f(x) = e^x$.

b) Let $f_n : D \to \mathbb{R}$ defined as $f_n(x) = x^n$.

(i) If $D = [0, 1]$, then $f_n \to f$ pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(ii) If $D = [-1, 1]$, $\{f_n\}_{n=1}^{\infty}$ fails to converge pointwise because the sequence $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

(iii) If $D = [0, 3]$, $\{f_n\}_{n=1}^{\infty}$ fails to converge pointwise because the sequence $\{3^n\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}$.

c) Let $f_n : \mathbb{R} \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} n^2 x & \text{if } |x| < \frac{1}{n} \\ 1/x & \text{otherwise} \end{cases}$$

Then $f_n \to f$ pointwise on $\mathbb{R}$ with

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{otherwise} \end{cases}$$

d) Let $D \subset \mathbb{R}$ and define $f_n(x) = x/n$. Then $f_n \to 0$ pointwise on D.

Def. Let $\mathcal{F}$ be a collection of real-valued functions on Ω. We say that $\mathcal{F}$ is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \to f$ pointwise on Ω, then $f \in \mathcal{F}$.

E.g.

a) If $\mathcal{F}$ is the collection of all real-valued functions, then $\mathcal{F}$ is closed under pointwise limits.

b) If $\mathcal{F} = C(D)$, then it is not closed under pointwise limit.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω, that is, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$. Then we say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω, if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in \Omega$. We write $f_n \to f$ uniformly.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω.

$f_n \to f$ pointwise on Ω iff

$$\forall x \in \Omega \; \boxed{\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N}} \left(n \geq N \to |f_n(x) - f(x)| < \epsilon \right)$$

$f_n \to f$ uniformly on Ω iff

$$\boxed{\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N}} \; \forall x \in \Omega \left(n \geq N \to |f_n(x) - f(x)| < \epsilon \right)$$

Thm. $C(D)$ is closed under uniform limits.

More precisely, let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \to f$ uniformly. Then $f \in C(D)$.

Proof. In order to show $f \in C(D)$, we need to show that

$$\forall x_0 \in D\, \forall \epsilon > 0\, \exists \delta > 0 \left(|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon \right).$$

Let's fix arbitrary $x_0 \in D$ and $\epsilon > 0$.

$f_n \to f$ uniformly implies that for some $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < \epsilon/3 \qquad \text{for all } x \in D.$$

Because f_N is continuous on D, and hence, at x_0, we can find $\delta > 0$ such that

$$|f_N(x) - f(x)| < \epsilon/3 \qquad \text{whenever } x \in D \text{ and } |x - x_0| < \delta.$$

Hence, whenever $x \in D$ and $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x) - f(x_0)| + |f(x) - f(x_0)| + |f(x) - f(x_0)| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore, $f \in C(D)$. $\square$

E.g. (Revisit the previous examples)

a) Let $f_n : \mathbb{R} \to \mathbb{R}$ defined as $f_n(x) = (1 + x/n)^n$. Then $f_n \to f$ pointwise on $\mathbb{R}$ with $f(x) = e^x$. Does $f_n \to f$ uniformly?

b) Let $f_n : D \to \mathbb{R}$ defined as $f_n(x) = x^n$ with $D = [0, 1]$. Then $f_n \to f$ pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since $f \notin C(D)$, this convergence cannot be uniform.

c) We have seen that

$$f_n(x) = \begin{cases} n^2 x & \text{if } |x| < \frac{1}{n} \\ 1/x & \text{otherwise} \end{cases} \to f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{otherwise} \end{cases} \qquad \text{pointwise.}$$

Because $f \notin C(\mathbb{R})$, this convergence cannot be uniform.

d) Let $D \subset \mathbb{R}$ and define $f_n(x) = x/n$. Then $f_n \to 0$ pointwise on D. However,

(i) If $D = [a, b]$ with $a, b \in \mathbb{R}$, then the converge is uniform.

(ii) If $D = \mathbb{R}$, this convergence cannot be uniform (why?).

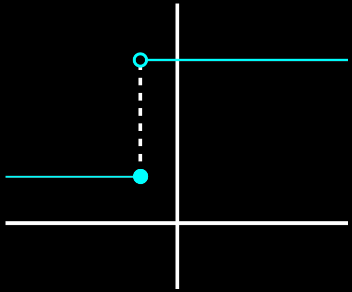
Finally, the collection $C(D)$ of real-valued continuous functions is closed under: $+$, $\cdot$, scalar multiplication, and uniform convergence.

HW Ex. 3.59, 2.60, 2.61, 2.66, on p. 71 – 73.

- Left (right)-continuity
- Lower (upper) semi-continuity
- Uniform continuity

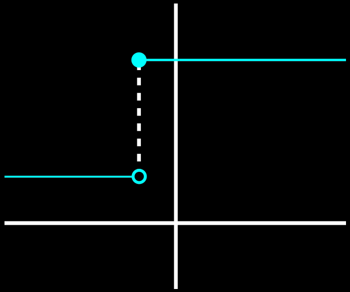
Def. f is *left-continuous at c* if

$$\lim_{x \to c+} f(x) = f(c)$$



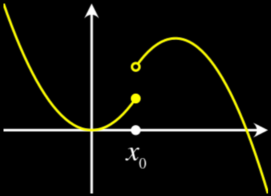
Def. f is *right-continuous at c* if

$$\lim_{x \to c-} f(x) = f(c)$$



Def. f is *lower semi-continuous at x_0* if

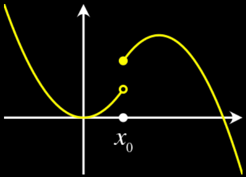
$$\liminf_{x\to x_0} f(x) \geq f(x_0)$$



$f(x_0)$ can be all points
at or below the yellow point.

f is *upper semi-continuous at x_0* if

$$\limsup_{x\to x_0} f(x) \leq f(x_0)$$



$f(x_0)$ can be all points
at or above the yellow point.

(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let I be an interval of $\mathbb{R}$. Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exits a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Def.' f is *uniformly continuous over I* if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in I \, \forall y \in I \; \{|x - y| < \delta \to |f(x) - f(y)| < \epsilon\}$$

f is continuous at $x_0 \in I$ iff

$$\forall \epsilon > 0 \, \forall x \in I \, \exists \delta > 0 \; \{|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon\}$$

Π_2-form

---

f is uniformly continuous over I iff

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in I \, \forall y \in I \; \{|x - y| < \delta \to |f(x) - f(y)| < \epsilon\}$$

Π_3-form

Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I, we need to show

$$\neg\Big(\forall\epsilon > 0\,\exists\delta > 0\,\forall x \in I\,\forall y \in I\ \{|x-y| < \delta \rightarrow |f(x)-f(y)| < \epsilon\}\Big)$$

$$\Leftrightarrow \quad \exists\epsilon > 0\,\forall\delta > 0\,\exists x \in I\,\exists y \in I\,\neg\{|x-y| < \delta \rightarrow |f(x)-f(y)| < \epsilon\}$$

$$\Leftrightarrow \quad \exists\epsilon > 0\,\forall\delta > 0\,\exists x \in I\,\exists y \in I\,\neg\{\neg\{|x-y| < \delta\} \vee |f(x)-f(y)| < \epsilon\}$$

$$\Leftrightarrow \quad \exists\epsilon > 0\,\forall\delta > 0\,\exists x \in I\,\exists y \in I\ \{|x-y| < \delta \wedge |f(x)-f(y)| \geq \epsilon\}$$

Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

□

Prop. 2 If I is compact [7] set such as $I = [a, b]$, then

$$f \text{ is continuous at all points in } I \iff f \text{ is uniformly continuous on } I.$$

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.
$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on $\mathbb{R}$ nor on $[0, \infty)$.

[7] namely, bounded and closed

Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

Chapter 3. Real Number System and Calculus

Some part of subsection is taken from Chapter 1 Section 4 of

P. Billingsley, **Probability and Measure**, Wiley, 1995.

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Remark Both $\limsup_n A_n$ and $\liminf_n A_n$ are sets.

Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$\omega \in \limsup_n A_n \iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$\iff (\forall n \geq 1)\,(\exists k \geq n)\ \omega \in A_k$$

$$\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n$$

Notation

$$\limsup_n A_n = [\, A_n \quad \text{i.o.}\,]$$

Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$\omega \in \liminf_n A_n \iff \omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$$\iff (\exists n \geq 1)\ (\forall k \geq n)\ \omega \in A_k$$

$$\iff \omega \text{ lies in } \textit{all but finitely many} \text{ of the } A_n$$

Notation

$$\liminf_n A_n = [\, A_n \quad \text{all but finitely many} \,]$$

Def. If both $\limsup_n A_n$ and $\liminf_n A_n$ exist and are equal, then the *limit set* of the sequence $\{A_n\}$ is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

which is also often written as $A_n \to A$.

Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$
\begin{array}{ccccccc}
\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \liminf_n A_n & \Longrightarrow & \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & \\
A_n & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & \\
\left(\bigcup_{k=n}^{\infty} A_k\right) & \downarrow & \left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = \limsup_n A_n & \Longrightarrow & \lim_{n\to\infty}\bigcup_{k=n}^{\infty} A_k = \limsup_n A_n
\end{array}
$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$
\begin{array}{ccc}
\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \mathbb{P}\left(\liminf_{n} A_n\right) \\
 & & \rotatebox{-90}{$\leq$} \\
\rotatebox{-90}{$\leq$} & & \liminf_{n\to\infty} \mathbb{P}(A_n) \\
\mathbb{P}\left(A_n\right) & & \rotatebox{-90}{$\leq$} \\
\rotatebox{-90}{$\leq$} & & \limsup_{n\to\infty} \mathbb{P}(A_n) \\
 & & \rotatebox{-90}{$\leq$} \\
\mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) & \downarrow & \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = \mathbb{P}\left(\limsup_{n} A_n\right)
\end{array}
$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned}
1 \geq \mathbb{P}\left(A_n^c \text{ all but finitely many}\right) &= 1 - \mathbb{P}\left(\left\{A_n^c \text{ all but finitely many}\right\}^c\right) \\
&= 1 - \mathbb{P}\left(A_n \text{ i.o.}\right) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \\
&\geq 1 - \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\
&= 1 - 0 = 1.
\end{aligned}$$

$\square$

Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$\begin{aligned}
A_1 &= (-1, 0] \\
A_2 &= \left(-\frac{1}{2}, \frac{1}{2}\right] \\
A_3 &= \left(-\frac{1}{3}, \frac{2}{3}\right] \\
A_4 &= \left(-\frac{1}{4}, \frac{3}{4}\right] \\
\vdots \quad & \quad \vdots \\
A_{100} &= \left(-\frac{1}{100}, \frac{99}{100}\right] \\
\vdots \quad & \quad \vdots
\end{aligned}$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n}\right] = [0, 1)$$

and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right) = [0, 1).$$

Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

$\square$

Exercise

(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3} \right] \qquad A_4 = \left(\frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100} \right]$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

Sol.

$$
\begin{aligned}
&\liminf_n A_n \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
&= (0,1) \cup (0,1) \\
&= (0,1)
\end{aligned}
$$

Sol. (continued) Similarly,

$$
\begin{aligned}
&\limsup_n A_n \\
=&\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right] \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\} \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\left(-\frac{1}{n+1},\frac{n+2}{n+1}\right]\right\} \\
=&[0,1]\cap[0,1] \\
=&[0,1]
\end{aligned}
$$

$\square$

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \dfrac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\[2ex] \left[-1 - \dfrac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$.

Solution:

$$\liminf_{n\to\infty} A_n = \{0\} \subset [0, 1] = \limsup_{n\to\infty} A_n$$