Topics in Analysis and Linear Algebra

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Summer Bootcamp for
Emory Biostatistics and Bioinformatics
PhD Program

July 22 - 28, 2021

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https://github.com/chenle02/2021_Summer_Bio_bootcamp_slides/

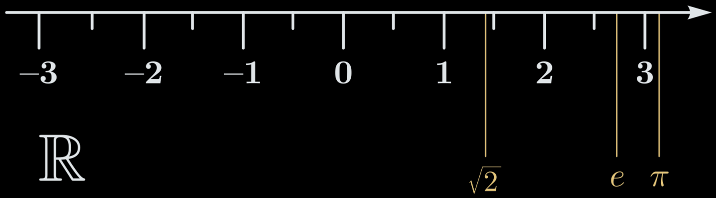
Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.

Chapter 3. Real Number System and Calculus

What is a real number?



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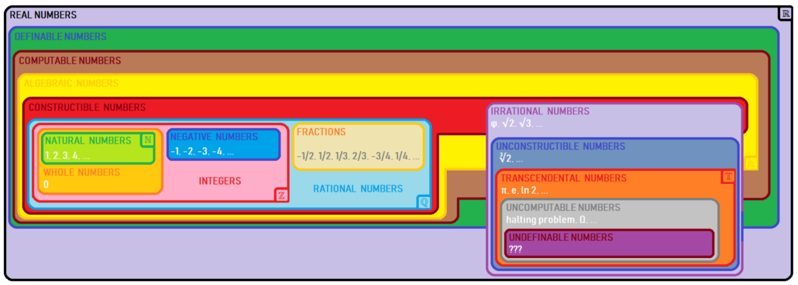
[1] Image from Wikipedia.



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[2] Image from
https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

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Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exit $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \qquad \text{(Identities)}$$

(F5) For each $x \in \mathbb{R}$, there exits a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exits an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

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Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

(O3) $x < y$ and $z > 0$ implies that $xz < yz$.

(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

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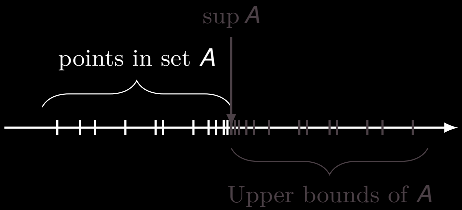
Completeness Axiom

Axiom A nonempty subset of real numbers that is bounded above has a least upper bound, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$



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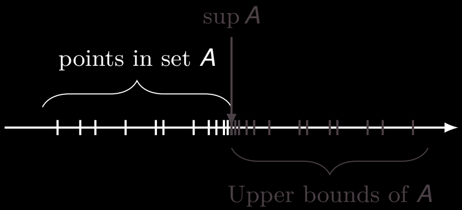

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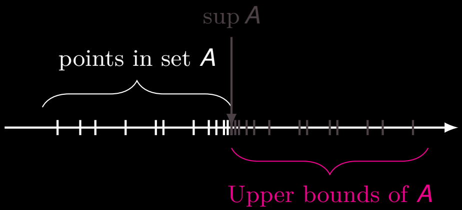

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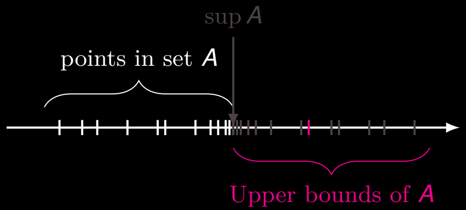

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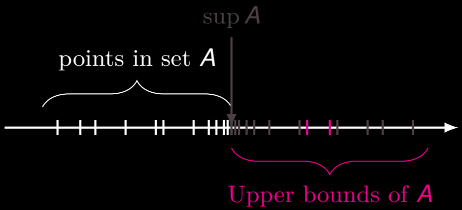

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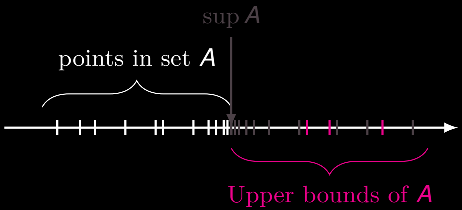

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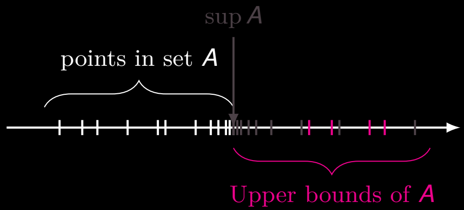

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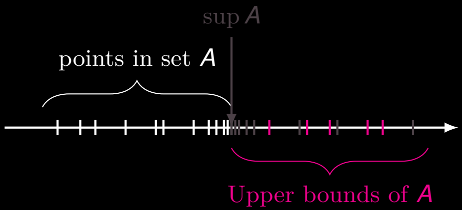

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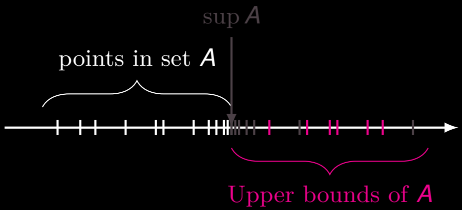

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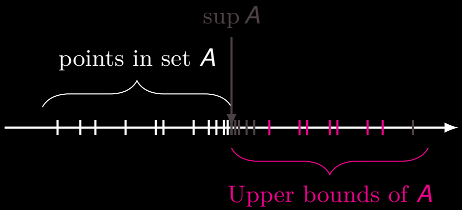

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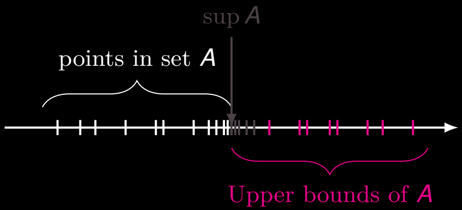

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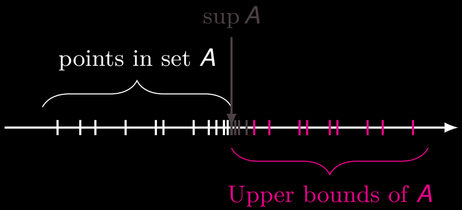

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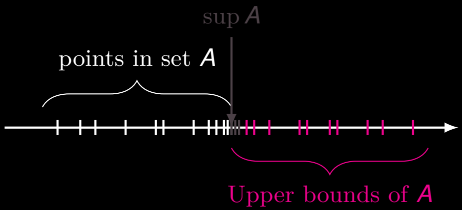

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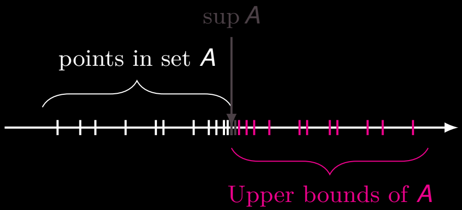

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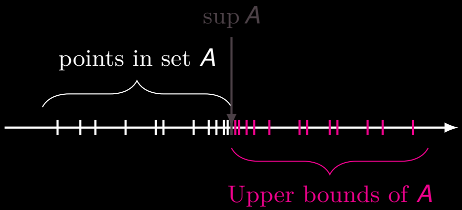

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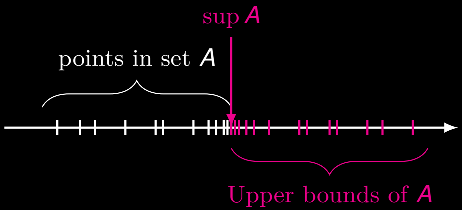

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$\sup_n x_n \le a \iff \forall n,\ \ x_n \le a$$

$$\sup_n x_n < a \iff \forall n,\ \ x_n < a$$

$$a < \sup_n x_n \iff \exists n,\ \ x_n > a$$

$$a \le \sup_n x_n \impliedby \exists n,\ \ x_n \ge a$$

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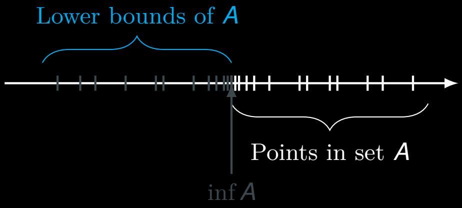

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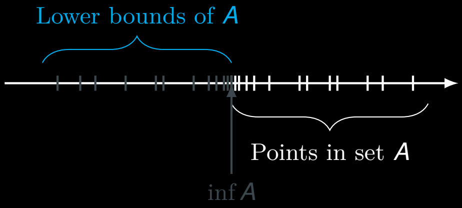

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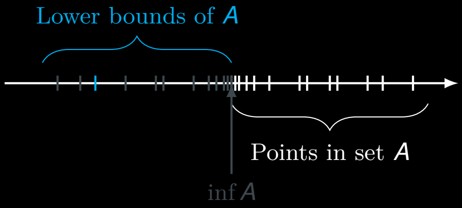

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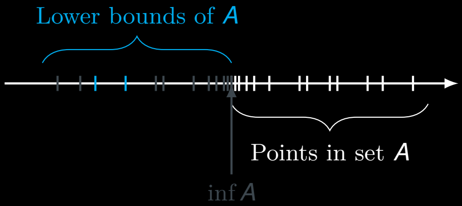

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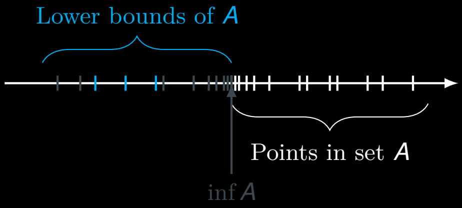

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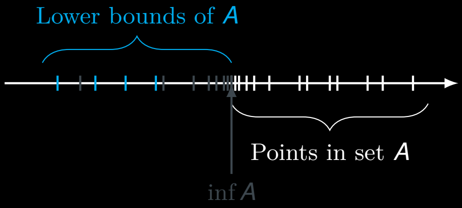

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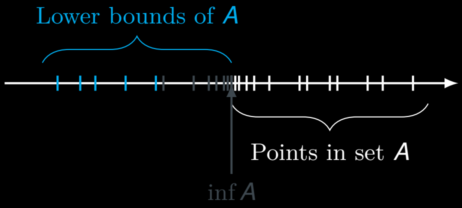

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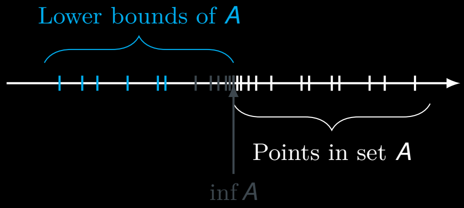

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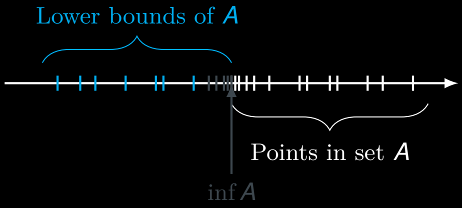

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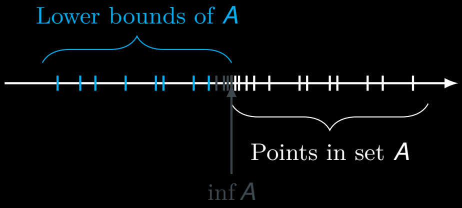

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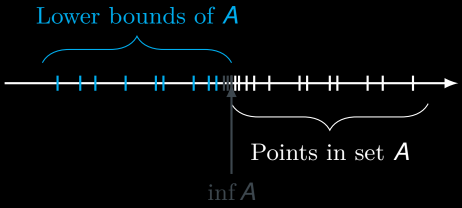

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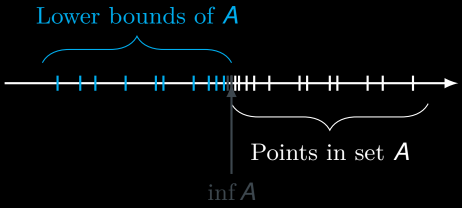

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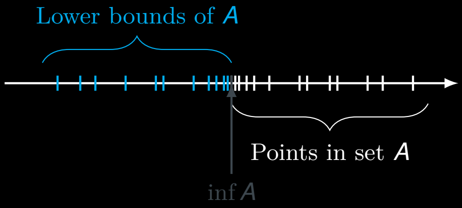

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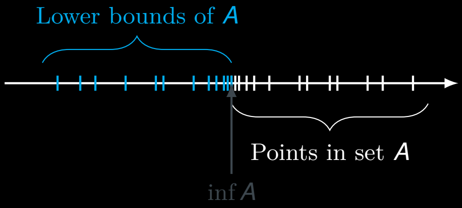

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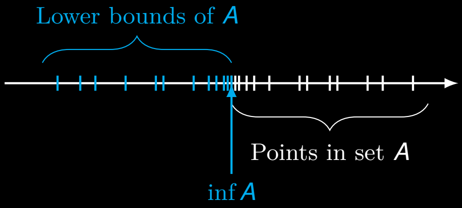

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$\begin{aligned} a \leq \inf_n x_n \quad &\iff \quad \forall n,\ \ x_n \geq a \\ a < \inf_n x_n \quad &\iff \quad \forall n,\ \ x_n > a \\ \inf_n x_n < a \quad &\iff \quad \exists n,\ \ x_n < a \end{aligned}$$

$$\inf_n x_n \leq a \quad \Longleftarrow \quad \exists n,\ \ x_n \leq a$$

E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

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$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

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Properties

1. Archimedean principle

   For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

   Between any two real numbers there is an irrational number.

3. Density of the rational numbers

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Proof As exercises. □

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Extended Real Number System

Def. The *extended real numbers* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
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Def. Let a and b be extended real numbers such that $a < b$. Then the *intervals on* $\mathbb{R}^*$ with endpoints a and b are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : \ a < x < b\}$$
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If both a and b are in $\mathbb{R}$, these intervals are the *bounded intervals* in $\mathbb{R}$.

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Thm Every subset A of $\mathbb{R}^*$ has both a least upper bound and greatest lower bound. Moreover,

a) If $A = \emptyset$, then $\sup A = -\infty$ and $\inf A = \infty$.
b) If A is bounded above in $\mathbb{R}$, then $\sup A \in \mathbb{R}$; otherwise, $\sup A = \infty$.
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E.g.

a) $\inf \mathbb{N} = 1$ and $\sup \mathbb{N} = \infty$.

b) $\inf \mathbb{Z} = -\infty$ and $\sup \mathbb{Z} = \infty$.

c) If I is an interval in $\mathbb{R}^*$ with endpoints a and b, $a \leq b$. Then $\inf I = a$ and $\sup I = b$.

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HW Ex. 2.10 and 2.11 on p. 43.

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence[3], namely,

$$\lim_{n\to\infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

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$$\lim_{n\to\infty} x_n = L \iff \forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\ s.t.\ (n \geq N \to |x_n - L| < \epsilon)$$

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

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Recall that $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ is the *extended real line*.

Def. A sequence of real numbers $\{a_1, a_2, \cdots\}$ is said to *converge in* $\mathbb{R}^*$ if one of the following three conditions hold:

(i) The sequence converges to a finite real number as in the previous definition.

(ii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

(iii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

(i) We say that the sequence converges in $\mathbb{R}$ or the limit exits and is finite.

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Monotone sequence

Def. If $x_1 \le x_2 \le \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \ge x_2 \ge \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

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Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$.

Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded above and is ∞ otherwise.

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In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded above and is ∞ otherwise.

b) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

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In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded below and is $-\infty$ otherwise.

Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$. Moreover, we have the following:

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Proof. We will prove the case when $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. The nonincreasing case can be proved in a similar way.

As $\sup_n x_n$ always exists in $\mathbb{R}^*$, we need to consider two cases:

Case I: $\sup_n x_n \in \mathbb{R}$

Case II: $\sup_n x_n = \infty$

Let's prove Case I here. Let $x = \sup_n x_n$. In order to show that $\lim_n x_n = x$, by monotonicity, we need to show that

$$\forall \epsilon > 0\, \exists N\, \forall n \text{ s.t. } (n \geq N) \to (x - a_n \leq \epsilon).$$

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Hence, there are infinitely many terms falling below $x - \epsilon$.

Since $\{x_n\}$ is nondecreasing, this implies all a_n fall below $x - \epsilon$, i.e.,

$$a_n < x - \epsilon, \quad \text{for all } n \geq 1.$$

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Cluster points

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.

b) ∞ *is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n > M$.

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E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞.

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Thm A convergent sequence has exactly one cluster point, namely, its limit. Thus, a sequence having more than one cluster point cannot converge.

Proof Suppose that $\{x_n\}$ is a convergent sequence and let x be its limit, namely, $\lim_{n\to\infty} x_n = x$. We need to prove:

(1) x is a cluster point.

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We also need to consider two cases:

Case I: $x \in \mathbb{R}$.

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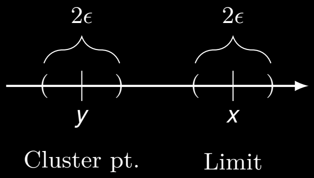
As for (2), suppose y is another cluster point.



By choosing any $\epsilon < |x - y|/2$, we see that

1. In the ϵ-neighborhood of y, there are infinitely many terms.
2. In the ϵ-neighborhood of x, all but finite many terms are here.

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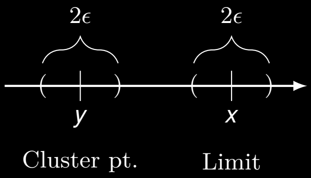

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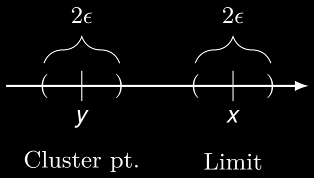

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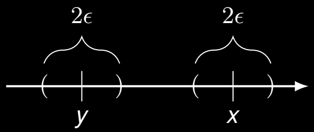

Cluster pt. Limit

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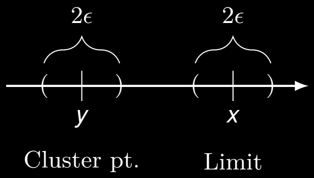

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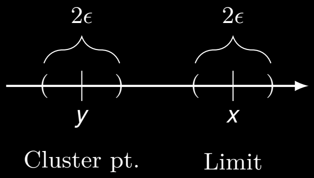
1. In the ϵ-neighborhood of y, there are infinitely many terms.
2. In the ϵ-neighborhood of x, all but finite many terms are here.

Contradiction!

Therefore, there exits only one cluster point. $\square$

As for (2), suppose y is another cluster point.



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A few more properties

1. A sequence is convergent iff each subsequence is convergent.

2. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \to L$ and $b_n \to L$, then $c_n \to L$.

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Limit superior and limit inferior

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n\to\infty} x_n := \sup_n \left(\inf_{m\geq n} x_m \right) \in \mathbb{R}^*.$$

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Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

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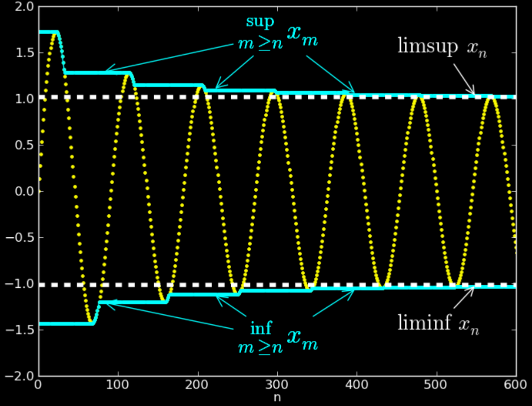

[4]

[4] Image from Wikipedia.

Characterization of the limsup and liminf.

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Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi)\frac{n^2}{n^2+8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

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$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi)\frac{n^2}{n^2+8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1, amount which

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E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞, among which

$$\liminf_{n\to\infty} x_n = 1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

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Proof. We only prove (a). (b) can be proved in a similar way.

Let $x = \limsup x_n$. We have seen that x is a cluster point.

It remains to prove that x is the largest cluster point. The case when $x = \pm\infty$ is left for the motivated students.

Now assume that $x \in \mathbb{R}$.

Only finite many terms exceed $x + 1$, hence, ∞ is not a cluster point.

Let $y \in \mathbb{R}$ s.t. $x < y$. Set $\epsilon = (y - x)/2$.

Only finite many terms exceed $x + \epsilon$.

Since $y - \epsilon = x + \epsilon$, the ϵ-neighborhood of y has been visited only finitely many times.

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Properties

1. $$\inf_n x_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = c \quad \Longleftrightarrow \quad \lim_{n\to\infty} x_n = c.$$

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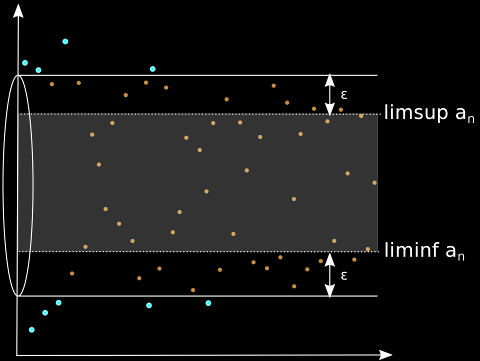

[5]

E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

[5] Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2}\left(\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n\right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n\to\infty} x_n + \epsilon,\ \limsup_{n\to\infty} x_n - \epsilon\right).$$

Cauchy criterion

As we have seen that

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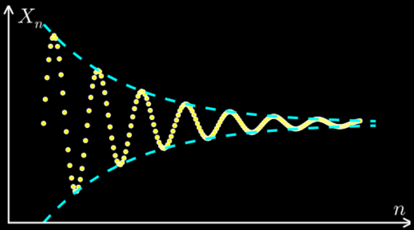
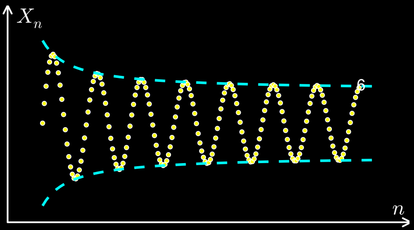


Non-Cauchy sequence[6]



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Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in $\mathbb{R}$ iff it is Cauchy.

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E.g.1 Let $a_n = \sqrt{n}$. Show that

(i) The consecutive terms become arbitrarily close to each other as $n \to \infty$.

(ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg \left(\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N \{|x_n - x_m| < \epsilon\} \right) \\ \iff \quad & \exists \epsilon > 0\, \forall N \in \mathbb{N}\, \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

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Indeed, let's choose $m = N$ and $n = 4N$

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HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

Open sets

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exits an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of $\mathbb{R}$. Then K cannot be an open set. For example, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ are not open sets.

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Properties of open sets

1. $\mathbb{R}$ and $\emptyset$ are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open. (arbitrary union)

Proof. Exercise. □

Let $Q_n = (-1/n, 1/n)$. Then $\bigcap_{i \in \mathbb{N}} Q_n = \{0\}$ is not an open set.

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Thm. Each open set O is a countable union of disjoint open intervals.

Moreover, the representation is unique in the sense that if $\mathcal{C}$ and $\mathcal{D}$ are two pairwise disjoint collections of open intervals whose union is O, then $\mathcal{C} = \mathcal{D}$.

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Closed sets

Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

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If A is a finite subset of $\mathbb{R}$, then $\overline{A} = A$.

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Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

E.g. $\mathbb{R}$ and $\emptyset$ are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

$\mathbb{N}$ and $\mathbb{Z}$ are closed sets.

The sets of rationals $\mathbb{Q}$ and irrationals $\mathbb{Q}^c$ are neither open nor close.

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Thm. A set is open if and only if its complement is closed.

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2. If A and B are closed sets, so is $A \cup B$. (finite union)
3. If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed. (arbitrary intersection)

Let $Q_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$. Then $\bigcup_{i \in \mathbb{N}} Q_n = (0, 1)$ is an open set.

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Relative open and closed sets

Def. Let $G \subset D \subset \mathbb{R}$.

a) G is said to be open in D if for each $x \in G$, there is an $r > 0$ such that

$$(x - r, x + r) \cap D \subset G.$$

b) G is said to be *closed in D* if $D \setminus G$ is open in D.

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D	G	Is G open in $\mathbb{R}$	Is G open in D
$[0,2]$	$[0,1)$		
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Thm. Let $D \subset \mathbb{R}$. A set $G \subset D$ is open in D if and only if there is an open set O of $\mathbb{R}$ such that $G = D \cap O$.

In other words, the open sets in D are precisely the open sets of $\mathbb{R}$ intersected with D.

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HW Ex. 2.38, 2.46, 2.47, 2.49, 2.52 on p. 63 – 64.

Def. A *real-valued function* is a function whose range is a subset of $\mathbb{R}$. If $f : \Omega \to \mathbb{R}$, we say that f is a *real-valued function on* Ω.

Algebraic operations

Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

$$(f+g)(x) := f(x) + g(x)$$
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Continuous functions

Def. Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and $x_0 \in D$. We say that f is *continuous at x_0* if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in D$ and $|x - x_0| < \delta$.

We say that f is *continuous on D* if it is continuous on every point of D.

We use $C(D)$ to denote the collection of all continuous functions on D.

If f is not continuous at x_0, then we say that f is *discontinuous at x_0* or that x_0 is a *point of discontinuity* of f.

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$$\Updownarrow$$

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in D\, \{|x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon\}$$

$$f \in C(D)$$

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E.g.

a) Let $D = (0, \infty)$ and define $f(x) = 1/x$. Then f is continuous function on D.

b) Let $D = \mathbb{R}$ and define $f(0) = 0$ and $f(x) = \sin(1/x)$ for $x \neq 0$. Then f is a continuous function except at 0.

c) Let $D = \mathbb{R}$ and define $f(x) = \lfloor x \rfloor$. Then f is continuous except at points of $\mathbb{Z}$.

d) Every function is continuous on $\mathbb{N}$. Or in other words, any infinite series $\{a_n\}_{n \in \mathbb{N}}$, when viewed as a function $a : \mathbb{N} \to \mathbb{R}$, is a continuous function.

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Thm. Let $D \subset \mathbb{R}$. The collection $C(D)$ of continuous functions on D is an algebra of functions, i.e., if $f, g \in C(D)$ and $a \in \mathbb{R}$, then

a) $f + g \in C(D)$.

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Pointwise limits

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω, that is, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$. Then we say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on* Ω if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}$.

If $\{f_n\}_{n=1}^{\infty}$ converges pointwise in Ω, then we define

$$f : \Omega \to \mathbb{R}$$

by

$$f(x) := \lim_{n\to\infty} f_n(x),$$

which is called the *pointwise limit of the sequence of functions* $\{f_n\}_{n=1}^{\infty}$. In this case, we also call the sequence of functions $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* to f, denoted as $f_n \to f$ pointwise.

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E.g.

a) Let $f_n : \mathbb{R} \to \mathbb{R}$ defined as $f_n(x) = (1 + x/n)^n$. Then $f_n \to f$ pointwise on $\mathbb{R}$ with $f(x) = e^x$.

b) Let $f_n : D \to \mathbb{R}$ defined as $f_n(x) = x^n$.

(i) If $D = [0, 1]$, then $f_n \to f$ pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(ii) If $D = [-1, 1]$, $\{f_n\}_{n=1}^{\infty}$ fails to converge pointwise because the sequence $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

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Def. Let $\mathcal{F}$ be a collection of real-valued functions on Ω. We say that $\mathcal{F}$ is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \to f$ pointwise on Ω, then $f \in \mathcal{F}$.

E.g.

a) If $\mathcal{F}$ is the collection of all real-valued functions, then $\mathcal{F}$ is closed under pointwise limits.

b) If $\mathcal{F} = C(D)$, then it is not closed under pointwise limit.

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Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω, that is, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$. Then we say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω, if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in \Omega$. We write $f_n \to f$ uniformly.

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Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω.

$f_n \to f$ pointwise on Ω iff

$$\forall x \in \Omega \boxed{\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \in \mathbb{N}} \left(n \geq N \to |f_n(x) - f(x)| < \epsilon \right)$$

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Thm. $C(D)$ is closed under uniform limits.

More precisely, let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \to f$ uniformly. Then $f \in C(D)$.

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Proof. In order to show $f \in C(D)$, we need to show that

$$\forall x_0 \in D\, \forall \epsilon > 0\, \exists \delta > 0 \left(|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon \right).$$

Let's fix arbitrary $x_0 \in D$ and $\epsilon > 0$.

$f_n \to f$ uniformly implies that for some $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < \epsilon/3 \qquad \text{for all } x \in D.$$

Because f_N is continuous on D, and hence, at x_0, we can find $\delta > 0$ such that

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Because $f \notin C(\mathbb{R})$, this convergence cannot be uniform.

d) Let $D \subset \mathbb{R}$ and define $f_n(x) = x/n$. Then $f_n \to 0$ pointwise on D. However,

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(ii) If $D = \mathbb{R}$, this convergence cannot be uniform (why?).

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Finally, the collection $C(D)$ of real-valued continuous functions is closed under: $+$, $\cdot$, scalar multiplication, and uniform convergence.

HW Ex. 3.59, 2.60, 2.61, 2.66, on p. 71 – 73.

- Left (right)-continuity
- Lower (upper) semi-continuity
- Uniform continuity

- Left (right)-continuity
- Lower (upper) semi-continuity
- Uniform continuity

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Def. f is *left-continuous at c* if

$$\lim_{x \to c+} f(x) = f(c)$$

Def. f is *right-continuous at c* if

$$\lim_{x \to c-} f(x) = f(c)$$

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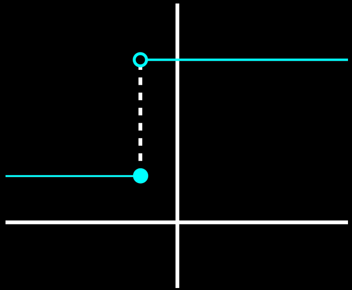

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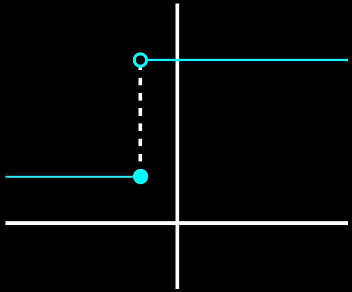

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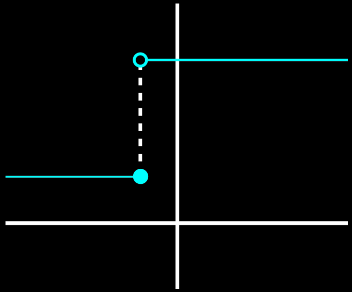

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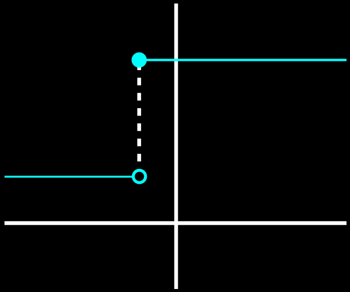

Def. f is *lower semi-continuous at x_0* if

$$\liminf_{x \to x_0} f(x) \geq f(x_0)$$

f is *upper semi-continuous at x_0* if

$$\limsup_{x \to x_0} f(x) \leq f(x_0)$$

$f(x_0)$ can be all points at or below the yellow point.

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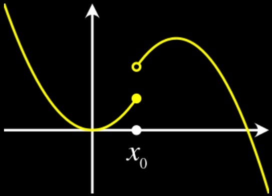

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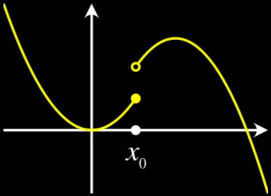

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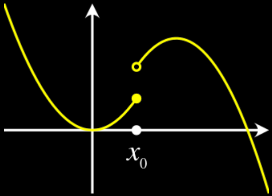

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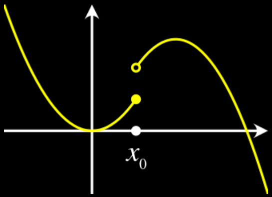

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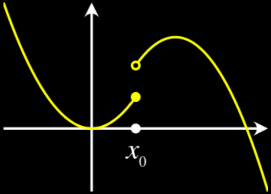

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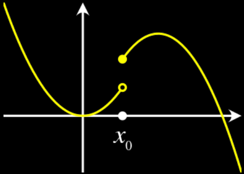

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(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let I be an interval of $\mathbb{R}$. Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exits a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

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Π_2-form

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Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I, we need to show

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&\neg\left(\forall\epsilon > 0\,\exists\delta > 0\,\forall x \in I\,\forall y \in I\ \{|x-y| < \delta \to |f(x)-f(y)| < \epsilon\}\right)\\
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Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

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Prop. 2 If I is compact [7] set such as $I = [a, b]$, then

$$f \text{ is continuous at all points in } I \iff f \text{ is uniformly continuous on } I.$$

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on $\mathbb{R}$ nor on $[0, \infty)$.

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Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

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Some part of subsection is taken from Chapter 1 Section 4 of

P. Billingsley, **Probability and Measure**, Wiley, 1995.

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

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Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$\begin{aligned}\omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \\ &\iff (\forall n \geq 1)\,(\exists k \geq n)\; \omega \in A_k \\ &\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n\end{aligned}$$

Notation

$$\limsup_n A_n = [\, A_n \quad \text{i.o.} \,]$$

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Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$
\begin{array}{ccccc}
\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \liminf_n A_n & \Longrightarrow & \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
\rotatebox{90}{$\subseteq$} & & & & \\
A_n & & & & \rotatebox{90}{$\subseteq$} \\
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\end{array}
$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$
\begin{array}{ccccl}
\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) & = & \mathbb{P}\left(\liminf_n A_n\right) \\
 & & \rotatebox{90}{$\leq$} & & \\
\rotatebox{90}{$\leq$} & & \liminf_{n\to\infty} \mathbb{P}(A_n) & & \\
\mathbb{P}\left(A_n\right) & & \rotatebox{90}{$\leq$} & & \\
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\end{array}
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(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned}
1 \geq \mathbb{P}\left(A_n^c \text{ all but finitely many}\right) &= 1 - \mathbb{P}\left(\left\{A_n^c \text{ all but finitely many}\right\}^c\right) \\
&= 1 - \mathbb{P}\left(A_n \text{ i.o.}\right) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \\
&\geq 1 - \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\
&= 1 - 0 = 1.
\end{aligned}$$

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Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$\begin{aligned} A_1 &= (-1, 0] \\ A_2 &= \left(-\frac{1}{2}, \frac{1}{2}\right] \\ A_3 &= \left(-\frac{1}{3}, \frac{2}{3}\right] \\ A_4 &= \left(-\frac{1}{4}, \frac{3}{4}\right] \\ \vdots \quad & \quad \vdots \\ A_{100} &= \left(-\frac{1}{100}, \frac{99}{100}\right] \\ \vdots \quad & \quad \vdots \end{aligned}$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n}\right] = [0, 1)$$

and

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Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Exercise

(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3}\right] \qquad A_4 = \left(\frac{1}{4}, \frac{3}{4}\right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5}\right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6}\right]$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99}\right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100}\right]$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

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Sol.

$$
\begin{aligned}
&\liminf_n A_n \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
&= (0,1) \cup (0,1) \\
&= (0,1)
\end{aligned}
$$

Sol. (continued) Similarly,

$$
\begin{aligned}
&\limsup_n A_n \\
=&\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right] \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\} \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\left(-\frac{1}{n+1},\frac{n+2}{n+1}\right]\right\} \\
=&[0,1]\cap[0,1] \\
=&[0,1]
\end{aligned}
$$

$\square$

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \dfrac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\[2ex] \left[-1 - \dfrac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$.

Solution:

$$\liminf_{n\to\infty} A_n = \{0\} \subset [0,1] = \limsup_{n\to\infty} A_n$$

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