Topics in Analysis and Linear Algebra

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https://github.com/chenle02/2021_Summer_Bio_bootcamp_slides/

Chapter 1. Mathematical Logics

The language and grammar of mathematics.

This part is mostly based on Chapters 1, 3, 4 of

Hamilton, A. G., **Logic for mathematicians**. 2^{nd} ed., Cambridge Univ. Press, 1988.

This section is based on Chapter 1 of

Hamilton, A. G., **Logic for mathematicians**. 2nd ed., Cambridge Univ. Press, 1988.

- **Simple sentences** : subject +predicate

  Napoleon is dead

  John owes James two pounds

  All eggs which are not square are round

- **Compound sentences** : subject + predicate with connectives

  Napoleon is dead and the world is rejoicing

  If all eggs are not square then all eggs are round

  If the barometer falls then either it will rain or it will snow

Simple/compound sentences $\rightarrow$ Simple/compound statements.

Basic assumption: All simple statements are either true (T) or false (F).

$A, B, C, \cdots$: simple statements.

$p, q, r, \cdots$: statements variables.

Connectives

not A	$\neg A$	negation
A and B	$A \wedge B$	conjunction
A or B	$A \vee B$	disjunction
if A then B	$A \rightarrow B$	conditional
A if and only if B	$A \leftrightarrow B$	biconditional

Napoleon is dead and the world is rejoicing	$A \wedge B$
If all eggs are not square then all eggs are round	$C \rightarrow D$
If the barometer falls then either it will rain or it will snow	$E \rightarrow (F \vee G)$

HW Ex. 1 (a) – (d).

Negation

p	$\neg p$
T	F
F	T

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Def. A statement form is an expression involving statement variables and connectives, which can be formed using the rules:

(i) Any statement variable is a statement form.

(ii) If $\mathcal{A}$ and $\mathcal{B}$ are statement forms, then $(\neg\mathcal{A})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \to \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are statement forms.

E.g. Construct the truth table of the statement form $p \to (q \vee r)$

Sol.

p	q	r	$q \vee r$	$p \to (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	T

$\square$

Def. A statement form is a tautology if it only takes true value T.

A statement form is a contradiction if it only takes false value F.

E.g. $p \vee (\neg p)$ is a tautology

$p \wedge (\neg p)$ is a contradiction

Sol.

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

$\square$

Def. If $\mathcal{A}$ and $\mathcal{B}$ are statement forms, then

$\mathcal{A}$ logically implies $\mathcal{B}$, denoted as $\mathcal{A} \Rightarrow \mathcal{B}$, if $\mathcal{A} \to \mathcal{B}$ is a tautology.

$\mathcal{A}$ is logically equivalent to $\mathcal{B}$, denoted as $\mathcal{A} \Leftrightarrow \mathcal{B}$, if $\mathcal{A} \leftrightarrow \mathcal{B}$ is a tautology.

E.g. $p \wedge q \Rightarrow p$

$\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

$\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

Sol. Check whether the truth tables of the following statement forms only produce T, i.e., tautology:

$$p \wedge q \to p$$

$$\neg(p \wedge q) \leftrightarrow (\neg p) \vee (\neg q)$$

$$\neg(p \vee q) \leftrightarrow (\neg p) \wedge (\neg q)$$

Sol. (continued) Let's check $p \wedge q \to p$:

p	q	$p \wedge q$	$p \wedge q \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

$\square$

HW Ex. 4, 6 (a) – (b), 7.

- $A \Leftrightarrow \neg\neg A$

- $A \Leftrightarrow A \wedge A$
- $A \Leftrightarrow A \vee A$

- $A \wedge B \Leftrightarrow B \wedge A$
- $A \vee B \Leftrightarrow B \vee A$

- $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$
- $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$

- $A \wedge (B \vee C) \Leftrightarrow (A \wedge C) \vee (A \wedge B)$
- $A \vee (B \wedge C) \Leftrightarrow (A \vee C) \wedge (A \vee B)$

- $\neg (A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$ $\quad \neg \left(\bigwedge_{i=1}^{n} A_i\right) = \bigvee_{i=1}^{n} (\neg A_i)$
- $\neg (A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$ $\quad \neg \left(\bigvee_{i=1}^{n} A_i\right) = \bigwedge_{i=1}^{n} (\neg A_i)$

- $A \vee (A \wedge B) \Leftrightarrow A$
- $A \wedge (A \vee B) \Leftrightarrow A$

- $A \vee T \Leftrightarrow T$
- $A \wedge F \Leftrightarrow F$

- $A \wedge T \Leftrightarrow A$
- $A \vee F \Leftrightarrow A$

- $A \vee (\neg A) \Leftrightarrow T$
- $A \wedge (\neg A) \Leftrightarrow F$

- $A \rightarrow B \Leftrightarrow (\neg A) \vee B$
- $A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$

- $A \rightarrow B \Leftrightarrow (\neg B) \rightarrow (\neg A)$
- $A \leftrightarrow B \Leftrightarrow (\neg B) \leftrightarrow (\neg A)$

HW Ex. 11 (a), (d), namely, using what we have learnt in this subsection to show that

(a)

$$((\neg (p \vee (\neg q)))) \to (q \to r) \quad \Longleftrightarrow \quad (\neg(q \to p)) \to ((\neg q) \vee r)$$

(d)

$$((\neg (p \vee (\neg q)))) \to (q \to r) \quad \Longleftrightarrow \quad q \to (p \vee r)$$

Def. Disjunctive normal form: $\bigvee_{i=1}^{m}\left(\bigwedge_{j=1}^{n} O_{ij}\right)$

Conjunctive normal form: $\bigwedge_{i=1}^{m}\left(\bigvee_{j=1}^{n} O_{ij}\right)$

where O_{ij} is either a statement variable or the negation of a statement variable.

Thm. Every statement form which is not a contradiction can be write as disjunctive normal form.

Every statement form which is not a tautology can be write as conjunctive normal form.

E.g. 1 Transform the following truth table to disjunctive normal form.

p	q	r	$f(p,q,r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

Sol. Find the entries with "T" and combine them with $\vee$.

Sol. (Continued)

p	q	r	$f(p,q,r)$	
T	T	T	T	$p \wedge q \wedge r$
T	T	F	T	$p \wedge q \wedge \neg r$
T	F	T	F	
T	F	F	F	
F	T	T	F	
F	T	F	F	
F	F	T	F	
F	F	F	T	$\neg p \wedge \neg q \wedge \neg r$

Hence,

$$f(p,q,r) \iff (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

$\square$

E.g.2 Find a conjunctive normal form for $((\neg p)\vee q) \to r$.

Sol. Construct the form by the truth table:

p	q	r	$\neg p$	$\neg p \vee q$	$(\neg p \vee q) \to r$	
T	T	T	F	T	T	$p \vee q \vee r$
T	T	F	F	T	F	
T	F	T	F	F	T	$p \vee \neg q \vee r$
T	F	F	F	F	T	$p \vee \neg q \vee \neg r$
F	T	T	T	T	T	$\neg p \vee q \vee r$
F	T	F	T	T	F	
F	F	T	T	T	T	$\neg p \vee \neg q \vee r$
F	F	F	T	T	F	

Therefore,

$$((\neg p)\vee q) \to r$$
$$\Updownarrow$$
$$(p\vee q\vee r) \wedge (p\vee\neg q\vee r) \wedge (p\vee\neg q\vee\neg r) \wedge (\neg p\vee q\vee r) \wedge (\neg p\vee\neg q\vee r)$$

$\square$

HW Ex. 12 (a) and 13 (b), namely, write $p \leftrightarrow q$ in both disjunctive and conjunctive forms.

Def. An adequate set of connectives is a set such that every truth function can be represented by a statement form containing only connectives from that set.

Remark $\{\wedge, \vee, \neg\}$ is an adequate set.

For example,

$$\begin{aligned} A \to B &\quad\Leftrightarrow\quad \neg B \vee A \\ A \leftrightarrow B &\quad\Leftrightarrow\quad (A \to B) \wedge (B \to A). \end{aligned}$$

Thm. The pairs

$$\{\neg, \wedge\}, \quad \{\neg, \vee\} \quad \text{and} \quad \{\neg, \rightarrow\}$$

are adequate sets of connectives.

Proof We only show the case $\{\neg, \wedge\}$. This is true because

$$A \vee B \quad \Leftrightarrow \quad \neg(\neg A \wedge \neg B).$$

$\square$

Nor

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Nand

p	q	$p \mid q$
T	T	F
T	F	T
F	T	T
F	F	T

Thm. The singleton sets

$$\{\downarrow\} \quad \text{and} \quad \{|\}$$

are adequate sets of connectives.

Proof. Show this as an exercise. $\square$

Simplest **argument forms**:

$$\begin{array}{rl} & p \to q \\ & p \\ \therefore & q \end{array}$$

In general, a argument form takes the following form:

$$A_1, A_2, \cdots, A_n; \qquad \therefore A$$

Premisses Conclusion

Sometimes, the above argument form is also written as

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n \Rightarrow A$$

Def. The argument form

$$A_1, A_2, \cdots, A_n; \quad \therefore A$$

is valid if the statement form

$$(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \to A \quad \text{is tautology;}$$

otherwise, the argument form is invalid.

E.g.1 Check the validity of the argument form:

$$p \to q, \quad (\neg q) \to r, \quad r; \quad \therefore p.$$

Sol. Let's simplify the following expression:

$$(p \to q) \wedge ((\neg q) \to r) \wedge r \to p$$
$$\Updownarrow$$
$$\neg\left[(p \to q) \wedge ((\neg q) \to r) \wedge r\right] \vee p$$
$$\Updownarrow$$
$$\neg\left[(\neg p \vee q) \wedge ((\neg\neg q) \vee r) \wedge r\right] \vee p$$
$$\Updownarrow$$
$$(p \wedge \neg q) \vee (\neg q \wedge \neg r) \vee (\neg r) \vee p$$
$$\Updownarrow$$
$$(\neg r) \vee p$$

which is not a tautology. Hence, the argument form is invalid. $\square$

E.g.2 Check the validity of the following argument form:

$$p_1 \to (p_2 \to p_3), \quad p_2; \quad \therefore p_1 \to p_3$$

Sol. Let's simplify the following expression:

$$
\begin{gathered}
[p_1 \to (p_2 \to p_3)] \wedge p_2 \to (p_1 \to p_3) \\
\Updownarrow \\
\neg[p_1 \to (p_2 \to p_3)] \vee \neg p_2 \vee (p_1 \to p_3) \\
\Updownarrow \\
\neg[\neg p_1 \vee (\neg p_2 \vee p_3)] \vee \neg p_2 \vee (\neg p_1 \vee p_3) \\
\Updownarrow \\
[p_1 \wedge p_2 \wedge \neg p_3] \vee \neg p_2 \vee \neg p_1 \vee p_3 \\
\Updownarrow \\
[p_1 \wedge p_2 \wedge \neg p_3] \vee \neg(p_1 \wedge p_2 \wedge \neg p_3) \\
\Updownarrow \\
T,
\end{gathered}
$$

which is a tautology. Hence, the argument form is valid. $\square$

HW Ex. 22.

Method 1. Proof by contrapositive

$$A \to B \quad \Leftrightarrow \quad (\neg B) \to (\neg A)$$

Proof.

$$\begin{aligned} A \to B \quad &\Leftrightarrow \quad (\neg A) \vee B \\ &\Leftrightarrow \quad (\neg\neg B) \vee (\neg A) \\ &\Leftrightarrow \quad (\neg B) \to (\neg A). \end{aligned}$$

□

E.g. Show by contradiction that an even integer has to be an integer.

Method 2. Proof by contradiction

$$A \to B \quad \Leftrightarrow \quad \neg\,((\neg B) \wedge A)$$

Proof.

$$\begin{aligned} A \to B \quad &\Leftrightarrow \quad (\neg A) \vee B \\ &\Leftrightarrow \quad \neg\,((\neg A) \vee B) \end{aligned}$$

□

Method 3. Proof by cases

$$(A_1 \vee A_2 \vee \cdots \vee A_n) \to B \quad \Leftrightarrow \quad (A_1 \to B) \wedge (A_2 \to B) \wedge \cdots \wedge (A_n \to B).$$

Proof.

$$\begin{aligned} \text{LHS} \quad & \Leftrightarrow \quad \neg(A_1 \vee A_2 \vee \cdots \vee A_n) \vee B \\ & \Leftrightarrow \quad (\neg A_1 \wedge \neg A_2 \wedge \cdots \wedge \neg A_n) \vee B \\ & \Leftrightarrow \quad (\neg A_1 \vee B) \wedge (\neg A_2 \vee B) \wedge \cdots \wedge (\neg A_n \vee B) \\ & \Leftrightarrow \quad \text{RHS}. \end{aligned}$$

$\square$

E.g. Suppose n is either an even integer or an odd one. Then n is an integer.

Method 4. Proof by exportation/importation

$$(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \rightarrow (A \rightarrow B) \quad \Leftrightarrow \quad (A_1 \wedge A_2 \wedge \cdots \wedge A_n \wedge A) \rightarrow B.$$

Proof.

$$\begin{aligned}
\text{LHS} \quad & \Leftrightarrow \quad \neg(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \vee (\neg A \vee B) \\
& \Leftrightarrow \quad (\neg A_1 \vee \neg A_2 \vee \cdots \vee \neg A_n) \vee (\neg A \vee B) \\
& \Leftrightarrow \quad (\neg A_1 \vee \neg A_2 \vee \cdots \vee \neg A_n \vee \neg A) \vee B \\
& \Leftrightarrow \quad \neg(A_1 \wedge A_2 \wedge \cdots \wedge A_n \wedge A) \vee B \\
& \Leftrightarrow \quad \text{RHS}.
\end{aligned}$$

$\square$

Method 5. Proof by chain argument

$$(A \to B) \land (B \to C) \quad \Rightarrow \quad A \to C.$$

Proof.

$[(A$	$\to$	$B)$	$\land$	$(B$	$\to$	$C)]$	$\to$	$(A$	$\to$	$C)$
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T	T	F	F
T	F	F	F	F	T	T	T	T	T	T
T	F	F	F	F	T	F	T	T	F	F
F	T	T	T	T	T	T	T	F	T	T
F	T	T	F	T	F	F	T	F	T	F
F	T	F	T	F	T	T	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F

$\square$

This section is based on Chapters 3 and 4 of

Hamilton, A. G., **Logic for mathematicians**. 2^{nd} ed., Cambridge Univ. Press, 1988.

$$\begin{array}{lll} & \text{All men are mortal} & \\ & \text{You are a man} & \\ \therefore & \text{You are mortal} & \end{array} \qquad\qquad \begin{array}{ll} & A \\ & B \\ \therefore & C \end{array}$$

We need quantifiers to make sense of the above arguments.

Chapters 1-2	Chapters 3-4
Statement logic	Predicate logic
Statement calculus	Predicate calculus
Propositional logic	Quantification logic
zero-order logic	first-order logic
zero-order language	first-order language

First-order logic is the extension of the zero-order logic

by including

quantifiers

Def. Universal quantifier: "for all x", denoted as $\forall x$,

Existential quantifier: "there exists a x", denoted as $\exists x$,

where x is called the bound variable.

Examples:

1. Not all birds can fly:

$$\neg\Big((\forall x)\big(B(x) \to F(x)\big) \Big)$$

2. Some people are stupid:

$$(\exists y)(P(y) \wedge S(y))$$

3. There is an integer which is greater than very other integer:

$$(\exists x)\Big(I(x) \wedge \big[(\forall y)\,(I(y) \to \{x \geq y\})\big] \Big)$$

HW Ex. 1 (a) – (b), 2 (a) – (b).

The alphabet of symbols of a first order language $\mathcal{L}$

1. variables $x_1, x_2, \cdots,$
2. some (possibly none) of the individual constants $a_1, a_2, \cdots,$
3. some (possibly none) of the predicate letters A_i^n,
4. some (possibly none) of the function letters f_i^n,
5. the punctuation symbols: "(", ")", ",",
6. the connectives: $\neg$ and $\rightarrow$,
7. the quantifier: $\forall$.

Remarks

- Recall that $\{\neg, \rightarrow\}$ is an adequate set of connectives because

$$\begin{aligned} A \wedge B &\quad\Leftrightarrow\quad \neg\,(A \rightarrow \neg B) \\ A \vee B &\quad\Leftrightarrow\quad (\neg A) \rightarrow \neg B \end{aligned}$$

- Existential quantifier $\exists$ can be represented using universal quantifier $\forall$ because

$$(\exists x)A(x) \quad\Leftrightarrow\quad \neg(\forall x)(\neg A(x))$$

- A_i^n is the i-th predicate that takes n arguments.

  f_i^n is the i-th function that takes n arguments.

E.g. Let $\mathcal{L}$ be the language for the arithmetic of natural numbers. Let

symbols	stands for
a_1	0
A_1^2	$=$
f_1^2	$+$
f_2^2	$\times$

Then, the following statement:

For any integer x there exists
an integer y such that $x + y = xy$

can be stated as

$$(\forall x)(\exists y)\, A_1^2\left(f_1^2(x, y), f_2^2(x, y)\right)$$

Def. Let $\mathcal{L}$ be a first order language. A *term* in $\mathcal{L}$ is defined as follows:

(i) Variables and individual constants are terms.

(ii) If $t_1, \cdots, t_n$ are terms, then $f_i^n(t_1, \cdots, t_n)$ is a term.

(iii) All possible terms are generated as in (i) and (ii).

E.g. $x_1,\ f_4^1(x_3),\ f_2^3(x_3, x_5, f_2^1(x_2))$ are terms.

Def. If $t_1, \cdots, t_n$ are terms in $\mathcal{L}$, then $A_i^n(t_1, \cdots, t_n)$ is an *atomic formula* in $\mathcal{L}$.

E.g. $A_1^2(x_3, x_5)$ and $A_2^3\left(x_2, f_2^1(x_4), f_3^2(x_1, x_4, x_5)\right)$ are atomic formulas.

Def. A *well-formed formula*, or *wf.*, of $\mathcal{L}$ is defined by

(i) Every atomic formula is a *wf.*.

(ii) If A and B are *wfs.* of $\mathcal{L}$, so are $\neg A$, $A \to B$ and $(\forall x)A$.

(iii) The set of all *wfs.* of $\mathcal{L}$ is generated as in (i) and (ii).

E.g. Here are some *wfs.*:

$$\forall x_1 A_1^1(x_1) \quad \text{and} \quad \forall x_1 \left(A_2^1(x_1) \to \neg\forall x_2 A_1^2(x_1, x_2)\right).$$

Def. In the *wf.* $(\forall x_i)A$, we say that A is the ***scope*** of the quantifier.

More generally, when $((\forall x_i)A)$ occurs as a subformula in a bigger *wf.* B, we say that the scope of this quantifier in B is A.

An occurrence of the variable x_i in the *wf.* is said to to be ***bound*** if it occurs within the scope of a $(\forall x_i)$ in the *wf.* or it is the x_i in a $(\forall x_i)$. Otherwise, it is called ***free***.

E.g.

wfs.	$(\forall x_1)A_1^1(x_2)$	$(\forall x_1)(\forall x_2)(A_1^2(x_1, x_2) \to A_1^1(x_2))$
x_1 free or bound	bound	bound
x_1 scope	$A_1^1(x_2)$	$(\forall x_2)(A_1^2(x_1, x_2) \to A_1^1(x_2))$
x_2 free or bound	free	bound
x_2 scope	–	$A_1^2(x_1, x_2) \to A_1^1(x_2)$

□

HW Ex. 7.

Def. An *interpretation I* of $\mathcal{L}$ consists of four parts:

(1) A non-empty set D_I — the domain of the interpretation I;

(2) A collection of distinguished elements: $\{\overline{a}_1, \overline{a}_2, \cdots\}$;

(3) A collection of functions on D_I: $\{\overline{f}_i^n : i, n \geq 1\}$;

(4) A collection of relations on D_I: $\{\overline{A}_i^n : i, n \geq 1\}$.

Remark The meaning of a *wf.* is given by the interpretation.

Truth or falsity of a particular *wf.* depends on the interpretation.

E.g.1 Interpret the following *wf.*

$$\forall x_1 \forall x_2 \neg \forall x_3 \left(\neg A_1^2 \left(f_1^2(x_1, x_3), x_2\right)\right)$$

with

$$D_N = \{0, 1, 2, \cdots\}$$
$$\overline{a}_1 = 0$$
$$\overline{f}_1^2(x, y) = x + y, \quad \overline{f}_2^2(x, y) = xy$$
$$\overline{A}_1^2(x, y): \quad x = y, \quad x, y \in D_N.$$

Sol. For all integers x and y, it is not true that all integer z satisfies that $x + z \neq y$.

Or equivalently

For all integers x and y, there is some integer z such that $x + z = y$. $\square$

Remark Apparently, this is a false statement.

E.g.2 Interpret the following *wf.*

$$\forall x_1 \forall x_2 \neg \forall x_3 \left(\neg A_1^2 \left(f_1^2(x_1, x_3), x_2 \right) \right)$$

with

$$D_N = \text{the set of positive rational nubmers}$$
$$\overline{a}_1 = 1$$
$$\overline{f}_1^2(x, y) = xy, \quad \overline{f}_2^2(x, y) = x/y$$
$$\overline{A}_1^2(x, y): \quad x = y, \quad x, y \in D_N.$$

Sol. For all positive rationals x and y, it is not true that all positive rational z satisfies that $xz \neq y$.

Or equivalently

For all positive rationals x and y, there is some positive rational z such that $xz = y$. □

Remark Apparently, this is a true statement.

HW Ex. 11 on p. 59.

Def. For any two *wfs.* A and B in $\mathcal{L}$,

$$A \Rightarrow B \quad \text{if and only if } A \rightarrow B \text{ is logically valid.}$$

Thm. *Distribution of quantifiers*

Suppose that x_i occurs free in both $A(x_i)$ and $B(x_i)$. Then

$$\forall x_i A(x_i) \vee \forall x_i B(x_i) \Rightarrow \forall x_i \Big(A(x_i) \vee B(x_i) \Big)$$

$$\forall x_i A(x_i) \wedge \forall x_i B(x_i) \Rightarrow \forall x_i \Big(A(x_i) \wedge B(x_i) \Big)$$

and

$$\forall x_i \Big(A(x_i) \to B(x_i) \Big) \Rightarrow \forall x_i A(x_i) \to \forall x_i B(x_i)$$

$$\exists x_i \Big(A(x_i) \to B(x_i) \Big) \Rightarrow \exists x_i A(x_i) \to \exists x_i B(x_i)$$

Thm. *Increasing, decreasing and switching quantifiers*

$$\forall x \forall y\, A(x,y) \Rightarrow \forall x\, A(x,x)$$
$$\exists x\, A(x,x) \Rightarrow \exists x \exists y\, A(x,y)$$
$$\exists x \forall y\, A(x,y) \Rightarrow \forall y \exists x\, A(x,y)$$

E.g. (the last one) Uniform continuity $\Rightarrow$ continuity.

Def. Let A and B be two *wfs.*. We say A and B are *provably equivalent* if

$$A \leftrightarrow B \quad \text{is logically valid,}$$

which is denoted as $A \Leftrightarrow B$.

That is,

$$A \Leftrightarrow B \quad \text{if and only if } A \leftrightarrow B \text{ is logically valid.}$$

Thm. *Negation of quantifiers*

$$\begin{aligned} \neg\forall x\, A \quad &\Leftrightarrow \quad \exists x \neg\, A \\ \neg\exists x\, A \quad &\Leftrightarrow \quad \forall x \neg\, A \end{aligned}$$

E.g.

$$\neg\forall x \exists y \forall z\, F(x, y, z) \quad \Leftrightarrow \quad \exists x \forall y \exists z\, \neg F(x, y, z)$$

Thm. *Enlarging and shrinking the scope of quantifiers*

Let A and B be two *wfs.* Suppose x_i occurs free in $A(x_i)$ but not in B. Then

$$\begin{aligned}
\forall x_i \, (A(x_i) \vee B) &\Leftrightarrow (\forall x_i A(x_i)) \vee B \\
\forall x_i \, (A(x_i) \wedge B) &\Leftrightarrow (\forall x_i A(x_i)) \wedge B \\
\forall x_i \left(A(x_i) \to B \right) &\Leftrightarrow (\exists x_i A(x_i)) \to B \\
\forall x_i \left(B \to A(x_i) \right) &\Leftrightarrow B \to \forall x_i A(x_i)
\end{aligned}$$

and

$$\begin{aligned}
\exists x_i \, (A(x_i) \vee B) &\Leftrightarrow (\exists x_i A(x_i)) \vee B \\
\exists x_i \, (A(x_i) \wedge B) &\Leftrightarrow (\exists x_i A(x_i)) \wedge B \\
\exists x_i \left(A(x_i) \to B \right) &\Leftrightarrow (\forall x_i A(x_i)) \to B \\
\exists x_i \left(B \to A(x_i) \right) &\Leftrightarrow B \to \exists x_i A(x_i)
\end{aligned}$$

E.g. Show that

$$\forall x \forall y \Big(F(x) \to G(y) \Big) \iff \exists x F(x) \to \forall y G(y).$$

Proof

$$\begin{aligned} \forall x \forall y \Big(F(x) \to G(y) \Big) &\iff \forall x \Big(F(x) \to \forall y G(y) \Big) \\ &\iff \Big(\exists x F(x) \to \forall y G(y) \Big) \end{aligned}$$

$\square$

Thm. *Distribution of quantifiers*

$$\forall x\Big(A(x)\wedge B(x)\Big) \quad \Leftrightarrow \quad \forall x A(x)\wedge\forall x B(x)$$

$$\exists x\Big(A(x)\vee B(x)\Big) \quad \Leftrightarrow \quad \exists x A(x)\vee\exists x B(x)$$

Thm. *Substitution*

If x_i occurs free in $A(x_i)$ and x_j is a variable which does not occur, free or bound, in $A(x_i)$, then

$$\forall x_i\, A(x_i) \Leftrightarrow \forall x_j\, A(x_j).$$

HW Show that

$$\forall x \forall y \Big(F(x) \leftrightarrow G(y) \Big) \quad \Longrightarrow \quad \forall x F(x) \leftrightarrow \forall x G(x).$$

Recall that the normal forms for the statement calculus are

Disjunctive normal form: $\vee_{i=1}^{m} \wedge_{j=1}^{n} O_{ij}$

Conjunctive normal form: $\wedge_{i=1}^{m} \vee_{j=1}^{n} O_{ij}$

What about the predicate calculus?

Def. A *wf.* of $\mathcal{L}$ is said to be in *prenex form* if it is of the form

$$Q_1 x_1 Q_2 x_2 \cdots Q_k x_k\, B$$

where Q_i $(1 \leq i \leq k)$ is either $\forall$ or $\exists$, and B is a *wf.* with no quantifiers.

E.g. Find the prenex form for

$$\left(\forall x_1 A_1^2(x_1, x_2) \to \exists x_2 A_1^1(x_2)\right) \to \neg\forall x_1 \forall x_2 A_2^2(x_1, x_2) \qquad (\star)$$

Sol.

$$\begin{aligned}
(\star) &\Leftrightarrow \left(\forall x_1 A_1^2(x_1, x_2) \to \exists x_3 A_1^1(x_3)\right) \to \neg\forall x_4 \forall x_5 A_2^2(x_4, x_5) \\
&\Leftrightarrow \left(\forall x_1 A_1^2(x_1, x_2) \to \exists x_3 A_1^1(x_3)\right) \to \exists x_4 \exists x_5 \neg A_2^2(x_4, x_5) \\
&\Leftrightarrow \exists x_1 \exists x_3 \left(A_1^2(x_1, x_2) \to A_1^1(x_3)\right) \to \exists x_4 \exists x_5 \neg A_2^2(x_4, x_5) \\
&\Leftrightarrow \forall x_1 \forall x_3 \Big(\left(A_1^2(x_1, x_2) \to A_1^1(x_3)\right) \to \exists x_4 \exists x_5 \neg A_2^2(x_4, x_5) \Big) \\
&\Leftrightarrow \forall x_1 \forall x_3 \exists x_4 \exists x_5 \left[\left(A_1^2(x_1, x_2) \to A_1^1(x_3)\right) \to \neg A_2^2(x_4, x_5)\right]
\end{aligned}$$

$\square$

Remark The prenex form is not unique.

$$(\star) \Leftrightarrow \exists x_4 \exists x_5 \forall x_1 \forall x_3 \Big(\left(A_1^2(x_1, x_2) \to A_1^1(x_3)\right) \to \neg A_2^2(x_4, x_5) \Big).$$

Prenex form gives a way to measure the complexity of *wfs.*

E.g. Which one of the following two *wfs.* is more complicated?

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 \, A_1^2 \left(f_1^2(x_1, x_2), f_1^2(x_3, x_4) \right)$$

$$\forall x_1 \exists x_2 \forall x_3 \exists x_4 \, A_1^2 \left(f_1^2(x_1, x_2), f_1^2(x_3, x_4) \right)$$

E.g.' Uniform continuity implies continuity: $\exists x \forall y A(x, y) \Rightarrow \forall y \exists x A(x, y)$.

Def. Let $n \geq 1$.

A *wf.* in prenex form is a Π_n-*form* if it starts with a universal quantifier and has $n-1$ alternations of quantifiers.

A *wf.* in prenex form is a Σ_n-*form* if it starts with an existential quantifier and has $n-1$ alternations of quantifiers.

E.g.

$$
\begin{array}{l|l}
\forall x_1 \forall x_2 \forall x_3 \forall x_4 \; A_1^4(x_1, x_2, x_3, x_4) & \Pi_1 \\
\forall x_1 \exists x_2 \forall x_3 \exists x_4 \; A_1^4(x_1, x_2, x_3, x_4) & \Pi_4 \\
\exists x_1 \forall x_2 \forall x_3 \exists x_4 \; A_1^4(x_1, x_2, x_3, x_4) & \Sigma_3
\end{array}
$$

HW Ex. 8 (a) – (b) on p. 92.

E.g. *Convergence in probability*

X_n converges to X in probability if for all $\epsilon > 0$

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0, \quad \text{as } n \to \infty.$$

Translate this definition into a *wf.* in the prenex form.

Sol. We first translate the limit $\lim_{n\to\infty} \mathbb{P}(\cdots) = 0$ as follows:

$$\forall \epsilon' \exists N \forall n \left[(\epsilon' > 0) \wedge (N \geq 1) \wedge (n \geq N) \to \left(\mathbb{P}(\cdots) \leq \epsilon' \right) \right].$$

Then put back the quantifier $\forall \epsilon$ to see that

$$X_n \to X \text{ in probability}$$

$$\Updownarrow$$

$$\forall \epsilon \forall \epsilon' \exists N \forall n \left[(\epsilon > 0) \wedge (\epsilon' > 0) \wedge (N \geq 1) \wedge (n \geq N) \to \left(\mathbb{P}\left(|X_n - X| > \epsilon \right) \leq \epsilon' \right) \right],$$

which is Π_3-form. $\square$

Problem How to show that X_n does not converge to X in probability?

Sol. We only need to make the negation of the above *wf.*:

$$\neg\forall\epsilon\forall\epsilon'\exists N\forall n\left[(\epsilon>0)\wedge(\epsilon'>0)\wedge(N\geq 1)\wedge(n\geq N)\rightarrow\left(\mathbb{P}\left(|X_n-X|>\epsilon\right)\leq\epsilon'\right)\right]$$

$$\Updownarrow$$

$$\exists\epsilon\exists\epsilon'\forall N\exists n\,\neg\left[\neg\left\{(\epsilon>0)\wedge(\epsilon'>0)\wedge(N\geq 1)\wedge(n\geq N)\right\}\vee\left(\mathbb{P}\left(\cdots\right)\leq\epsilon'\right)\right]$$

$$\Updownarrow$$

$$\exists\epsilon\exists\epsilon'\forall N\exists n\,\left[\left\{(\epsilon>0)\wedge(\epsilon'>0)\wedge(N\geq 1)\wedge(n\geq N)\right\}\wedge\neg\left(\mathbb{P}\left(\cdots\right)\leq\epsilon'\right)\right]$$

$$\Updownarrow$$

$$\exists\epsilon\exists\epsilon'\forall N\exists n\,\left[(\epsilon>0)\wedge(\epsilon'>0)\wedge(N\geq 1)\wedge(n\geq N)\wedge\left(\mathbb{P}\left(|X_n-X|>\epsilon\right)>\epsilon'\right)\right]$$

$\square$

HW Suppose that $Y_1, \cdots, Y_n$ is a random sample from the exponential pdf, $f_Y(y) = \lambda e^{-\lambda y}$, $\lambda > 0, y > 0$.

Show that $\Lambda_n := \sum_{i=1}^{n} Y_i$ does not converges to λ in probability.

(i.e., Λ_n is not a consistent estimator for λ.)

(*Hints*: Prove first the case $\Lambda_n := Y_1$.)