Topics in Analysis and Linear Algebra

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Chapter 2. Topics in real analysis

This part is mostly based on Chapter 2 of

J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.

Chapter 2. Topics in real analysis

Def. A set is a collection of elements.

If A is a set and x is an element of A, we write $x \in A$.

$x \notin A$ means x is not an element of A.

A set contains no elements is called an empty set, denoted as $\emptyset$.

Def. Let A and B be sets.

If very element of A is an element of B, then A is said to be a subset of B, denoted $A \subset B$ or $B \supset A$.

Two sets A and B are equal, denoted $A = B$, if and only if $A \subset B$ and $A \supset B$.

If $A \subset B$ but $B \not\subset A$, then we say that A is a proper subset of B.

E.g. Let

$$\mathbb{C} = \text{collection of complex numbers}$$
$$\mathbb{R} = \text{collection of real numbers}$$
$$\mathbb{Q} = \text{collection of rational numbers}$$
$$\mathbb{Z} = \text{collection of integers}$$
$$\mathbb{N} = \text{collection of natural numbers, i.e., positive integers}$$

Then we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Assume all sets under consideration are subsets of some fixed set Ω, commonly referred as the universal set.

The set of all subsets of Ω is called the power set of Ω, denoted $\mathcal{P}(\Omega)$.

Hence, $A \subset \Omega$ iff $A \in \mathcal{P}(\Omega)$.

Remark $\Omega^c = \emptyset$ and $\emptyset^c = \Omega$.

Def. Let A and B be subsets of Ω.

The complement of A, denoted A^c, is the set of elements of Ω that do not belong to A, namely,

$$A^c := \{x \in \Omega : x \notin A\}.$$

The complement of A relative to B, denoted $B \setminus A$, is the set of all elements in B that do not belong to A, namely,

$$B \setminus A := \{x \in B : x \notin A\}.$$

The intersection of A and B, denoted $A \cap B$, is the set of elements of Ω that belong to both A and B, namely,

$$A \cap B := \{x \in \Omega : x \in A \text{ and } x \in B\}.$$

The union of A and B, denoted $A \cup B$, is the set of elements of Ω that belong to either A or B, namely,

$$A \cup B := \{x \in \Omega : x \in A \text{ or } x \in B\}.$$

Commutative Laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap B)$$
$$A \cup (B \cup C) = (A \cup B) \cap (A \cup B)$$

Idempotent Laws

$$A \cup A = A$$
$$A \cap A = A$$

Domination Laws

$$A \cup \Omega = \Omega$$
$$A \cap \emptyset = \emptyset$$

Absorption Laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

De Morgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$
$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

Various Identities

$$A \cap A^c = \emptyset$$
$$A \cup A^c = \Omega$$

$$\emptyset^c = \Omega$$
$$\Omega^c = \emptyset$$

$$(A^c)^c = A$$

Def. Let $\mathcal{C}$ be a collection of subsets of Ω, that is, $\mathcal{C} \subset \mathcal{P}(\Omega)$.

a) The intersection of $\mathcal{C}$, denoted $\cap_{A \in \mathcal{C}} A$, is the set of elements of Ω that belong to each set in the collection of $\mathcal{C}$, namely,

$$\bigcap_{A \in \mathcal{C}} A := \{x \in \Omega : x \in A \text{ for all } A \in \mathcal{C}\}.$$

b) The union of $\mathcal{C}$, denoted $\cup_{A \in \mathcal{C}} A$, is the set of elements of Ω that belong to at least one of the sets in the collection of $\mathcal{C}$, namely,

$$\bigcup_{A \in \mathcal{C}} A := \{x \in \Omega : x \in A \text{ for some } A \in \mathcal{C}\}.$$

Set operations still work in this case, e.g.,

De Morgan's Laws

$$\left(\bigcup_{A\in\mathcal{C}} A\right)^c = \bigcap_{A\in\mathcal{C}} A^c$$

$$\left(\bigcap_{A\in\mathcal{C}} A\right)^c = \bigcup_{A\in\mathcal{C}} A^c$$

$$C \setminus \left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcap_{A\in\mathcal{C}} (C \setminus A)$$

$$C \setminus \left(\bigcap_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} (C \setminus A)$$

Distributive Laws

$$B \cap \left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} (B\cap A)$$

$$B \cup \left(\bigcap_{A\in\mathcal{C}} A\right) = \bigcap_{A\in\mathcal{C}} (B\cup A)$$

E.g. Let $\Omega = \mathbb{R}$ and $\mathcal{C} = \{[0, 1/n] : n \in \mathbb{N}\}$. Show that

$$\bigcap_{A \in \mathcal{C}} A = \{0\} \quad \text{and} \quad \bigcup_{A \in \mathcal{C}} A = [0, 1].$$

Remark Equivalently, one can write $A_n = [0, 1/n]$ for $n \in \mathbb{N}$ and then

$$\bigcap_{n=1}^{\infty} A_n = \{0\} \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = [0, 1].$$

Proof. ... $\square$

In general, we have:

E.g.' Show that

A_n	$\bigcap_{i=1}^{n} A_n$	$\bigcup_{i=1}^{n} A_n$
$(0, 1/n)$	$\emptyset$	$(0, 1)$
$(0, 1/n]$	$\emptyset$	$(0, 1]$
$[0, 1/n)$	$\{0\}$	$[0, 1)$
$[0, 1/n]$	$\{0\}$	$[0, 1]$

Def. Two subsets, A and B, of Ω are said to be disjoint if $A \cap B = \emptyset$.

Ex. 1.8, 1.13.

Def. Suppose that Ω and Λ are sets. A function (or mapping, transformation) from Ω to Λ is a rule that assigns each element $x \in \Omega$ a unique element $f(x) \in \Lambda$.

We call $f(x)$ the value of f at x, or the image of x under f.

A function f from Ω to Λ is often denoted $f : \Omega \to \Lambda$.

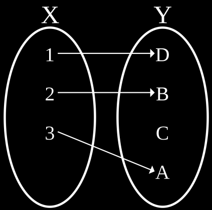
The set Ω is called the domain of f.

The set $\{f(x) : x \in \Omega\}$ is called the range of f.

Def. Let f be a function from ω to λ.

a) f is said to be one-to-one or injective if distinct elements of ω have distinct images; that is,

$$\forall x_1, x_2 \in \omega, \quad f(x_1) = f(x_2) \quad \rightarrow \quad x_1 = x_2.$$

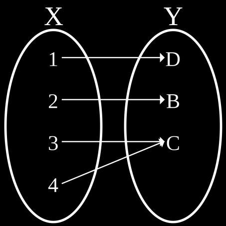


Def. Let f be a function from ω to λ.

b) f is said to be onto or surjective if each element of λ is the image of some element of ω; that is,

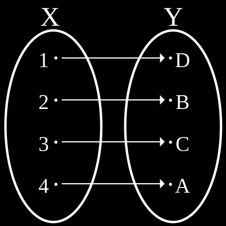
$$\forall y \in \lambda, \exists x \in \omega, \text{ s.t. } \quad y = f(x).$$

or f is onto iff the range of f equals λ.



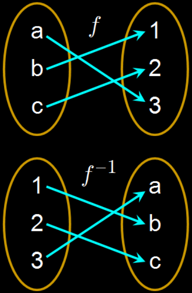
Def. Let f be a function from ω to λ.

c) f is said to be 1-1 correspondence or bijective if f is both surjective (onto) and injective (one-to-one).



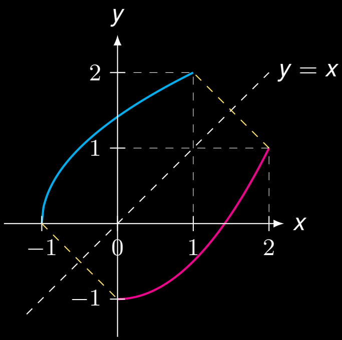
Def. Suppose that $f : \Omega \to \Lambda$ is both one-to-one and onto. For any $y \in \Lambda$, let $f^{-1}(y)$ be the unique $x \in \Omega$ such that $y = f(x)$.

The function $f^{-1} : \Lambda \to \Omega$ defined in this way is called the inverse of the function f.



E.g. Let $f : [0, 2] \to [-1, 1]$ be defined as $f(x) = \frac{1}{2}x^2 - 1$.

The inverse function is $f^{-1} : [-1, 1] \to [0, 2]$ with $f^{-1}(x) = \sqrt{2x + 2}$.



Def. Let $f : \Omega \to \Lambda$ and $g : \Lambda \to \Gamma$. Then the composition of g with f, denoted $g \circ f$, is the function $g \circ f : \Omega \to \Gamma$ defined by

$$(g \circ f)(x) = g\left(f(x)\right).$$

Def. Let $f : \Omega \to \Lambda$ and $A \subset \Omega$. The restriction of f to A, denoted $f_{|A}$, is defined to be a function $A \to \Lambda$ such that

$$f_{|A}(x) = f(x), \qquad \text{for all } x \in A.$$

Infinite and finite sequences

Infinite sequences such as

- $\{1, 2, 4, 8, 16, \cdots\}$
- $\{1, 1/2, 1/3, 1/4, 1/5, \cdots\}$
- $\{1, -1, 1, -1, 1, -1, \cdots\}$
- $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$

are nothing but functions defined on $\mathbb{N}$.

We use $\{s_n : n \in \mathbb{N}\}$ or $\{s_n\}_{n=1}^{\infty}$ to denote an infinite sequence.

Finite sequence of length n such as

- $\{a_1, a_2, \cdots, a_n\}$

is nothing but a function defined on $\{1, \cdots, n\}$.

Images and inverse images

Def. Let $f : \Omega \to \Lambda$.

If $A \subset \Omega$, then define

$$f(A) := \{f(x) : x \in A\}\,,$$

which is called the image of A under f.

If $B \in \Lambda$, then define

$$f^{-1}(B) := \{x \in \Omega : f(x) \in B\}\,,$$

called the inverse image of B under f.

Thm Let $f : \Omega \to \Lambda$, $A \subset \Omega$, and $\{A_i\}_{i \in I}$ an indexed collection of subsets of Ω. Then

a) $$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$$

b) $$f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$$ and

$$f\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$$ provided f is one-to-one.

c) $f(A^c) \subset (f(A))^c$ and

$f(A^c) = (f(A))^c$ provided that f is one-to-one.

Thm Let $f : \Omega \to \Lambda$, $B \subset \Omega$, and $\{B_i\}_{i \in I}$ an indexed collection of subsets of Ω. Then

a) $$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i)$$

b) $$f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i)$$

c) $$f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$$

Proof. ... $\square$

Cartesian Products

Def. Let A and B be two sets. Then the Cartesian product of A and B (in that order), denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, namely,

$$A \times B := \{(a, b) : a \in A,\ b \in B\}.$$

Similarly, if $A_1, A_2, \cdots, A_n$ are sets, then the Cartesian product of those n sets, denoted $A_1 \times A_2 \times \cdots \times A_n$ or $\bigtimes_{k=1}^{n} A_k$, is the set of all ordered n-tuples $(a_1, \cdots, a_n)$ such that $a_k \in A_k$ for $k = 1, \cdots, n$, namely,

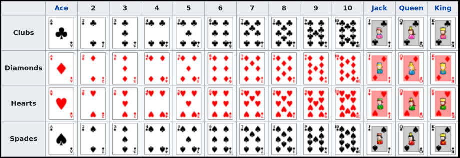
$$\bigtimes_{k=1}^{n} A_k := \{(a_1, \cdots, a_n) : a_k \in A_k, 1 \leq k \leq n\}.$$

E.g. 1. The standard 52-card deck is $A \times B$ with

$$A = \{\text{Ace}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{Jack}, \text{Queen}, \text{King}\}$$

$$B = \{\text{Club}, \text{Diamond}, \text{Heart}, \text{Spade}\}$$

$$\Omega = A \cup B$$



2. $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$: the Euclidean n-space.

Remark If at least one of A and B are empty, then so is $A \times B$.

Def. Let $\{A_i\}_{i \in I}$ be an indexed collection of sets. The Cartesian product of the collection, denoted $\bigtimes_{i \in I} A_i$, is the set of all functions x on I such that $x(i) \in A_i$ for each $i \in I$, namely,

$$\bigtimes_{i \in I} A_i := \left\{ x : I \to \bigcup_{i \in I} A_i : x(i) \in A_i, \forall i \in I \right\}.$$

Remark 1.

2. If $A_i = \emptyset$ for some $i \in I$, then $\times_{i \in I} A_i = \emptyset$.
3. On the other hand, if $A_i \neq \emptyset$ for all $i \in I$, then $\times_{i \in I} A_i \neq \emptyset$ [1].

[1] Thanks to the *Axiom of Choice*.

Notation and examples

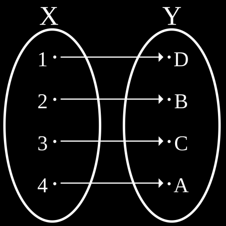
When	$\bigtimes_{i\in I} A_i$
$I=\{1,\cdots,n\}$	$\times_{i=1}^{n} A_i$
$A_i = A, \forall i \in I$	A^I
$I=\{1,\cdots,n\}$ and $A_i = A, \forall i \in I$	write A^n instead of $A^{\{1,\cdots,n\}}$ or $\bigtimes_{i=1}^{n} A$
$I=\mathbb{N}$	write A^∞ instead of $A^{\{1,2,\cdots\}}$ or $A^{\mathbb{N}}$
$I=[0,1]$ and $A_i=\mathbb{R}, \forall i \in I$	$A^{[0,1]}$ is the set of all functions on $[0,1]$.

Remark Infinite sequence $\{a_1, a_2, \cdots\}$ can be viewed as either

1. a function on $\mathbb{N}$ or
2. Cartesian product with $I=\mathbb{N}$, namely, A^∞.

HW Ex. 1.14, 1.21, 1.23.

Recall if f is both one-to-one (injective) and onto (surjective), then f is one-to-one correspondence (bijective).



Def. For two sets X and Y, if there exists a bijective function between X and Y, then we say that X and Y are equivalent, denoted $X \sim Y$.

Equivalent sets satisfy the following properties:

Reflexive	$A \sim A$
Symmetric	$A \sim B \quad \Rightarrow \quad B \sim A$
Transitive	$A \sim B \wedge B \sim C \Rightarrow A \sim C$

Remark Any relation that satisfies reflexive, symmetric, transitive properties is an equivalence relation.

E.g. 1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{A, B, C, D\}$. Then $X \sim Y$ because one can find a bijective function between X and Y.

2. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{A, B, C, D\}$. Does $X \sim Y$? Why?

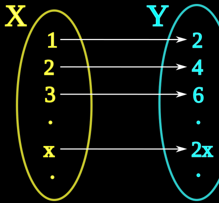
Remark For sets of finite element, in order to be equivalent, they have to have the same number of elements.

E.g. 3. Let $X = \mathbb{N} = \{1, 2, 3, \cdots\}$ and $Y = \{2, 4, 6, 8, \cdots\}$ (even integers).

Does $X \sim Y$?

Do they have the same number of elements?

Sol. Here is one apparent solution[2]: $f : X \to Y$ defined as $f(x) = 2x$:



This is a bijective function (why?). Hence, $X \sim Y$. They have the same number of elements (infinite many, which is called countably infinite).

$\square$

[2] There are other constructions. Can you give another bijection between X and Y?

Size of sets

Def. Let A be a set. We say that

a) A is *finite* if it is either empty or equivalent to the first N positive integers for some $N \in \mathbb{N}$.

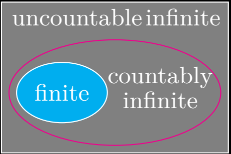
   In the former case, A is said to consist of 0 elements and, in the latter case, N elements.

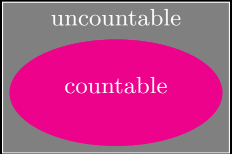
b) A is *infinite* if A is not finite.

c) A is *countably infinite* if it is equivalent to $\mathbb{N}$.

d) A is *countable* if it either finite or countably infinite.

e) A is *uncountable* if it is not countable.


uncountable infinite
finite
countably infinite


uncountable
countable

E.g.1. Show that the set of $\mathbb{Z}$ is countably infinite.

Sol. Construct the bijection $f : \mathbb{N} \to \mathbb{Z}$:

$$
\begin{aligned}
1 &\to \ \ 0\\
2 &\to \ \ 1\\
3 &\to -1\\
4 &\to \ \ 2\\
5 &\to -2\\
6 &\to \ \ 3\\
7 &\to -3\\
&\vdots
\end{aligned}
$$

It is not hard to see that

$$f(n) = \begin{cases} n/2 & n \text{ is even} \\ -(n-1)/2 & n \text{ is odd} \end{cases}$$

$\square$

Sol'. Construct the bijection $f : \mathbb{N} \to \mathbb{Z}$:

$$
\begin{aligned}
1 &\to \ \ 0\\
2 &\to -1\\
3 &\to \ \ 1\\
4 &\to -2\\
5 &\to \ \ 2\\
6 &\to -3\\
7 &\to \ \ 3\\
&\vdots
\end{aligned}
$$

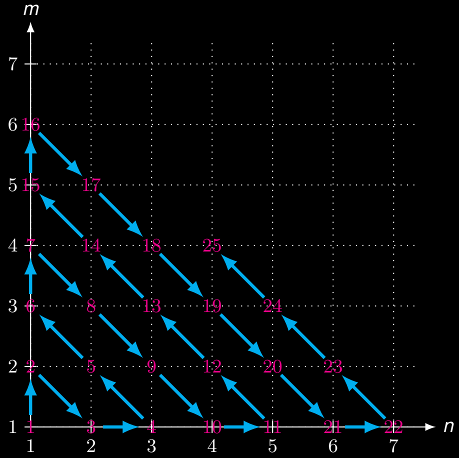
With enough patience, one can determine the formula:

$$f(n) = \frac{n(-1)^{n-1}}{2} - \frac{1-(-1)^n}{4}$$

$\square$

E.g. 2. Show that $\mathbb{N}^2$ is countably infinite.

Sol. Construct a bijection $f : \mathbb{N}^2 \to \mathbb{N}$:



Can you find a formula for this bijection?

$\square$

Sol'. We claim that $f : \mathbb{N}^2 \to \mathbb{N}$ defined below is a bijection:

$$f(m, n) := 2^{m-1}(2n-1).$$

a) f is one-to-one (injective): It suffices to show that

$$f(m_1, n_1) = f(m_2, n_2) \quad \Longrightarrow \quad m_1 = m_2 \quad \text{and} \quad n_1 = n_2.$$

Without loss of generality, suppose $m_1 \geq m_2$. Notice that

$$2^{m_1 - m_2} = \frac{2n_1 - 1}{2n_2 - 1}. \tag{$\star$}$$

The LHS is an even integer unless $m_1 = m_2$. The RHS is a fraction unless $n_1 = n_2$. Hence, in order to make $(\star)$ valid, one has to have both sides equal to 1. Hence, $m_1 = m_2$ and $n_1 = n_2$.

b) f is onto (surjective). For any integer $k \in \mathbb{N}$, one has to find m and n such that $f(m, n) = k$. One can keep dividing k by 2 until it becomes an odd function. In this way, one easily find out m and n. $\square$

Thm 1. A nonempty set is countable if and only if it is the range of an infinite sequence.

2. A subset of a countable set is countable.

3. The image of a countable set is countable.

4. A countable union of countable sets is countable.

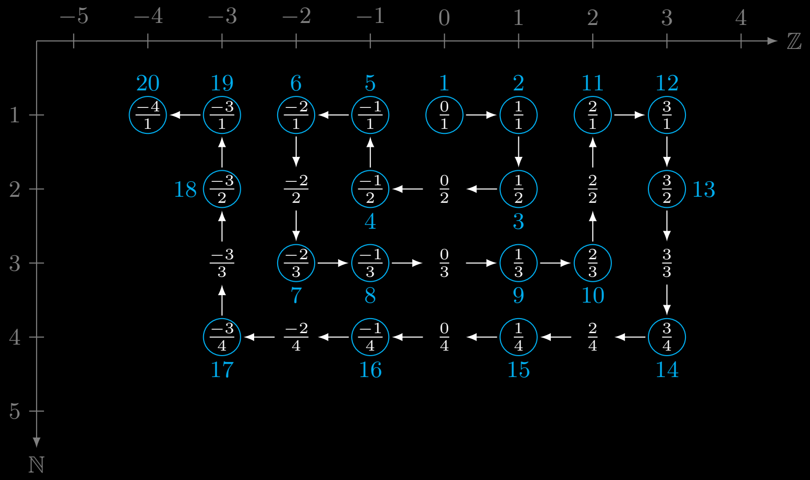
5. The Cartesian product of two countable sets is countable.

E.g. 3. Show that the set of rationals $\mathbb{Q}$ is countably infinite.

Recall that rationals are numbers that can be written as a ratio of two integers, i.e.,

$$q \in \mathbb{Q} \quad \Rightarrow \quad \exists m, n \in \mathbb{Z} \quad \text{such that } q = \frac{m}{n}.$$

Sol. Let's construct the bijection from $\mathbb{N}$ to $\mathbb{Q}$:



Can you find out the explicit formula for the bijection $f : \mathbb{N} \to \mathbb{Q}$?

$\square$

Sol'. Alternatively, one can define:

$$f(z,n) = \frac{z}{n}, \quad z \in \mathbb{Z},\ n \in \mathbb{N}.$$

1. $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ is onto (surjective) but not one-to-one (injective).
   Hence, $f(\mathbb{Z} \times \mathbb{N}) \supset \mathbb{Q}$. (Figure above)
2. $\mathbb{Z}$ is countable. (E.g. 1)
3. $\mathbb{Z} \times \mathbb{N}$ is countable. (Thm. 5)
4. $f(\mathbb{Z} \times \mathbb{N})$, as the image of $\mathbb{Z} \times \mathbb{N}$ under f, is countable. (Thm. 3)
5. Therefore, $\mathbb{Q}$, as a subset of the countable set $f(\mathbb{Z} \times \mathbb{N})$,
   has to be countable. (Thm. 2)

$\square$

HW Prove Thm's 1-5, which are Propositions 1.7 – 1.11 of the book.