

# Topics in Analysis and Linear Algebra

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## Chapter 3. Topics in linear algebra

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## § 3.1 Real number system

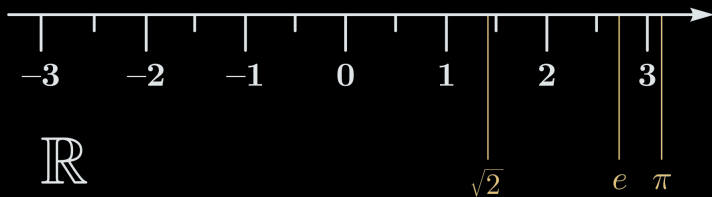
## § 3.2 Sequences of real numbers

# Chapter 3. Topics in linear algebra

## § 3.1 Real number system

## § 3.2 Sequences of real numbers

What is a real number?



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<sup>1</sup>Image from Wikipedia.

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

## Field Axioms

Let  $x, y, z \in \mathbb{R}$ . Then we have that

$$(F1) \quad x + y = y + x \text{ and } xy = yx. \quad (\text{Commutative})$$

$$(F2) \quad (x + y) + z = x + (y + z) \text{ and } (xy)z = x(yz). \quad (\text{Associative})$$

$$(F3) \quad x(y + z) = xy + xz. \quad (\text{Distributive})$$

$$(F4) \quad \text{There exist } 0, 1 \in \mathbb{R} \text{ with } 0 \neq 1 \text{ such that for all } x \in \mathbb{R}$$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \quad (\text{Identities})$$

$$(F5) \quad \text{For each } x \in \mathbb{R}, \text{ there exists a } -x \in \mathbb{R} \text{ such that } x + (-x) = 0 \text{ and, if } x \neq 0, \text{ there exists an } x^{-1} \in \mathbb{R} \text{ such that } xx^{-1} = 1. \quad (\text{Inverses})$$

## Order Axioms

Let  $x, y, z \in \mathbb{R}$ . Then we have that

- (O1)  $x < y$  and  $y < z$  implies that  $x < z$ . (Transitive)
- (O2)  $x < y$  implies that  $x + z < y + z$ .
- (O3)  $x < y$  and  $z > 0$  implies that  $xz < yz$ .
- (O4) Exactly one of  $x = y$ ,  $x < y$ , and  $x > y$  holds. (Trichotomous)



## Completeness Axiom

**Axiom** A nonempty subset of real numbers that is **bounded above** has a **least upper bound**, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

**Corr.** A nonempty subset of real numbers that is **bounded below** has a **greatest lower bound**, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$

E.g.  $\sup[0, 1) = 1$  and  $\inf[0, 1) = 0$ .

$\mathbb{N}$  has no least upper bound, but  $\inf \mathbb{N} = 1$ .

Let  $A = \{x : x^2 < 3\}$ . Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

## Properties

### 1. Archimedean principle

For each  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{N}$  such that  $n > x$ .

### 2. Density of the irrational numbers

Between any two real numbers there is an **irrational** number.

### 3. Density of the rational numbers

Between any two real numbers there is an **rational** number.

## Chapter 3. Topics in linear algebra

### § 3.1 Real number system

### § 3.2 Sequences of real numbers

Def. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. We say that the real number  $L \in \mathbb{R}$  is the *limit* of this sequence<sup>2</sup>, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number  $\epsilon > 0$ , there exists a natural number  $N$  such that for all  $n > N$ , we have  $|x_n - L| < \epsilon$ .

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

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<sup>2</sup>In this case, we say that  $\{x_n\}_{n=1}^{\infty}$  is *convergent*. Otherwise, we say that  $\{x_n\}_{n=1}^{\infty}$  is *divergent*.

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is convergent and  $\lim_{n \rightarrow \infty} (n-1)/n = 1$ .

$\{(-1)^n\}_{n=1}^{\infty}$  is divergent.

$\{n^2\}_{n=1}^{\infty}$  is divergent.

Def. Let  $\mathbb{R}^*$  denote *the extended real line*, namely,  $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ .

Def. A sequence of real numbers  $\{a_1, a_2, \dots\}$  is said to *converge in  $\mathbb{R}^*$*  if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
  - (ii) For each  $M \in \mathbb{R}$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n > M$ .
  - (iii) For each  $M \in \mathbb{R}$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n < M$ .
- 

- (i) We say that the sequence converges in  $\mathbb{R}$ .
- (ii) Denoted as  $\lim_{n \rightarrow \infty} x_n = \infty$
- (iii) Denoted as  $\lim_{n \rightarrow \infty} x_n = -\infty$

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$ .

$\{(-1)^n\}_{n=1}^{\infty}$  does not converge in  $\mathbb{R}^*$ .

$\{n^2\}_{n=1}^{\infty}$  converges in  $\mathbb{R}^*$  and  $\lim_{n \rightarrow \infty} n^2 = \infty$ .



Def. If  $x_1 \leq x_2 \leq \dots$ , then  $\{x_n\}_{n=1}^{\infty}$  is said to be *nondecreasing*.

If  $x_1 \geq x_2 \geq \dots$ , then  $\{x_n\}_{n=1}^{\infty}$  is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$  is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is monotone and it is nondecreasing.

$\{(-1)^n\}_{n=1}^{\infty}$  is not monotone.

$\{n^2\}_{n=1}^{\infty}$  is monotone and it is nondecreasing.

**Axiom** Let  $A$  be a nonempty subset of real numbers that is **bounded above**.  
Then *the least upper bound* of  $A$  exists, which is denoted by

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

**Axiom** Let  $A$  be a nonempty subset of real numbers that is **bounded below**.  
Then *the greatest lower bound* of  $A$  exists, which is denoted by

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$

**Prop.** Any monotone sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers converges in  $\mathbb{R}^*$ .

Moreover,

(a) If  $\{x_n\}_{n=1}^{\infty}$  is nondecreasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, if  $\{x_n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$ , it is **bounded above**.

(a) If  $\{x_n\}_{n=1}^{\infty}$  is nonincreasing, then

$$\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

In particular, if  $\{x_n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$ , it is **bounded below**.

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is nondecreasing and converges in  $\mathbb{R}$ . It is bounded above.

$\{(-1)^n\}_{n=1}^{\infty}$  is not monotone.

$\{n^2\}_{n=1}^{\infty}$  is nondecreasing, does not converge in  $\mathbb{R}$ , converges in  $\mathbb{R}^*$ .

Def. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers.

- (a) A real number  $x$  is said to be a *cluster point* of  $\{x_n\}_{n=1}^{\infty}$  if for each  $\epsilon > 0$  and  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $|x - x_n| < \epsilon$ .
- (b)  $\infty$  is a *cluster point* of  $\{x_n\}_{n=1}^{\infty}$  if for each  $M \in \mathbb{R}$  and  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $x_n > M$ .
- (c)  $-\infty$  is a *cluster point* of  $\{x_n\}_{n=1}^{\infty}$  if for each  $M \in \mathbb{R}$  and  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $x_n < M$ .

E.g.1  $\{(n-1)/n\}_{n=1}^{\infty}$  has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$  has two cluster points:  $-1$  and  $+1$ .

$\{n^2\}_{n=1}^{\infty}$  has one cluster point:  $+\infty$ .

E.g.2 Consider the sequence  $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$ , that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2,  $\infty$ .

E.g.3 Let  $\{r_n\}_{n=1}^{\infty}$  be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence  $\{r_n\}_{n=1}^{\infty}$ .



Prop.

$$\{x_n\}_{n=1}^{\infty} \text{ converges in } \mathbb{R}^* \iff \{x_n\}_{n=1}^{\infty} \text{ has exactly one cluster point in } \mathbb{R}^*.$$

## A few more properties

1. If a sequence is **bounded and monotonic**, then it is convergent.
2. A sequence is convergent iff each subsequence is convergent.
3. **Sandwich theorem**: If  $x_n \leq c_n \leq b_n$  for all  $n > N$  and  $x_n \rightarrow L$  and  $b_n \rightarrow L$ , then  $c_n \rightarrow L$ .

Def. The *limit inferior* of a sequence  $\{x_n\}_{n=1}^{\infty}$  is defined as

$$\liminf_{n \rightarrow \infty} x_n := \sup_n \left( \inf_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of  $\{x_n\}_{n=1}^{\infty}$  is defined as

$$\limsup_{n \rightarrow \infty} x_n := \inf_n \left( \sup_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences  $\{y_n\}_{n=1}^{\infty}$  and  $\{z_n\}_{n=1}^{\infty}$  defined as

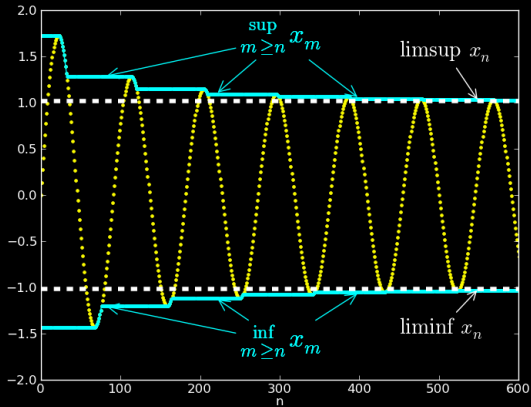
$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

$$\begin{aligned} \inf_n y_n &= \inf_n \sup_{m \geq n} x_m = \limsup_{x \rightarrow \infty} x_n \\ \sup_n z_n &= \sup_n \inf_{m \geq n} x_m = \liminf_{x \rightarrow \infty} x_n. \end{aligned}$$

Hence,

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} x_m.$$



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Can we use the first-order logic to formulate the definitions?

Def'.  $\limsup_{n \rightarrow \infty} x_n = x \in \mathbb{R}$  if and only if  $x$  is the **smallest** real number such that for any positive real number  $\epsilon > 0$ , there exists a natural number  $N$  such that  $x_n < x + \epsilon$  for all  $n > N$ .

Set

$$A(b) : \quad \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \{n \geq N \rightarrow x_n < b + \epsilon\}$$

Then

$$\limsup_{n \rightarrow \infty} x_n = x \iff \forall c \in \mathbb{R} \{c \geq x \leftrightarrow A(c)\}$$

Def'.  $\liminf_{n \rightarrow \infty} x_n = x \in \mathbb{R}$  if and only if  $x$  is the largest real number such that for any positive real number  $\epsilon > 0$ , there exists a natural number  $N$  such that  $x_n > x - \epsilon$  for all  $n > N$ .

Set

$$B(b) : \quad \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \{n \geq N \rightarrow x_n > b - \epsilon\}$$

Then

$$\liminf_{n \rightarrow \infty} x_n = x \iff \forall c \in \mathbb{R} \{c \leq x \leftrightarrow B(c)\}$$

E.g.1 Let  $x_n = (-1)^n$ . Then  $\{x_n\}_{n=1}^{\infty}$  has two cluster points:  $\pm 1$ , amount which

$$\liminf_{n \rightarrow \infty} x_n = -1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi) \frac{n^2}{n^2 + 8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$



E.g.2 Consider the sequence  $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$ , that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2,  $\infty$ , among which

$$\liminf_{n \rightarrow \infty} x_n = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

E.g.3 Let  $\{r_n\}_{n=1}^{\infty}$  be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence  $\{r_n\}_{n=1}^{\infty}$ , amount which

$$\liminf_{n \rightarrow \infty} r_n = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. Then

- (a)  $\liminf x_n$  is the smallest cluster point of  $\{x_n\}_{n=1}^{\infty}$ .
- (b)  $\limsup x_n$  is the largest cluster point of  $\{x_n\}_{n=1}^{\infty}$ .

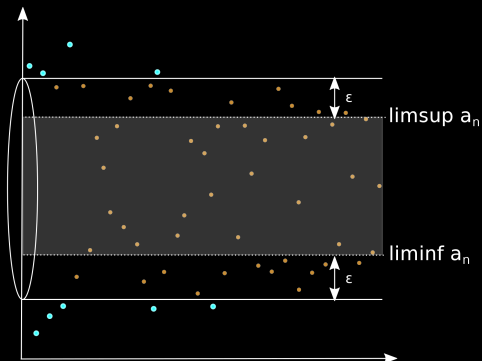
## Properties

1.

$$\inf_n x_n \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq \sup_n x_n$$

2. A sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers **converges in  $\mathbb{R}^*$**  if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = c \iff \lim_{n \rightarrow \infty} x_n = c.$$



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E.g. For all  $\epsilon > 0$ , the interval

$$\left( \liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in  $\{x_n\}$ .

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<sup>4</sup>Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all  $\epsilon$  with

$$0 < \epsilon < \frac{1}{2} \left( \limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in  $\{x_n\}$  fall outside of the interval

$$\left( \liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right).$$

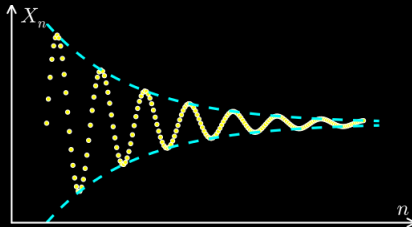
As we have seen that

A sequence of real numbers **converges in  $\mathbb{R}^*$**  if and only if it has exactly one cluster point.

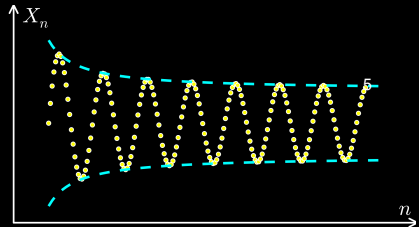
There is another famous criterion for a sequence to **converge in  $\mathbb{R}$** :

### **Cauchy Criterion**

Cauchy sequence



Non-Cauchy sequence



Def. A sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers is called a *Cauchy sequence* if for each  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have  $|x_n - x_m| < \epsilon$ .

Def.' A sequence  $\{x_n\}_{n=1}^{\infty}$  is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers *converges in  $\mathbb{R}$*  iff it is Cauchy.



E.g.1 Let  $a_n = \sqrt{n}$ . Show that

- (i) The consecutive terms become arbitrarily close to each other as  $n \rightarrow \infty$ .
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show  $\{a_n\}_{n=1}^{\infty}$  is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose  $\epsilon = 1$ . For any  $N \in \mathbb{N}$ , we need to find  $m, n \geq N$  such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose  $m = N$  and  $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

□

E.g.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Variation

$$\lim_{n \rightarrow \infty} \left(1 + \text{Small}\right)^{\text{Large}} = e^{\lim_{n \rightarrow \infty} \text{Small} \times \text{Large}}$$