Topics in Analysis and Linear Algebra

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Last updated on

July 26, 2021

Summer Bootcamp for
Emory Biostatistics and Bioinformatics
PhD Program

July 22 - 28, 2021

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https://github.com/chenle02/2021_Summer_Bio_bootcamp_slides/

Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

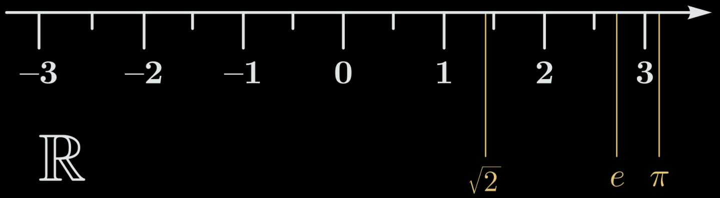
§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

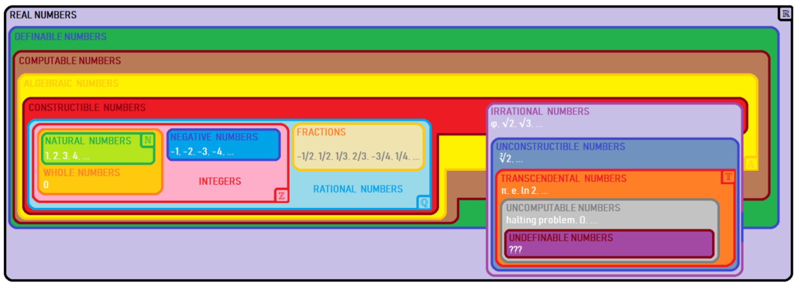
§ 3.6 Some techniques in calculus

What is a real number?



[1]

[1] Image from Wikipedia.



[2]

[2] Image from
https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exit $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \qquad \text{(Identities)}$$

(F5) For each $x \in \mathbb{R}$, there exits a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exits an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

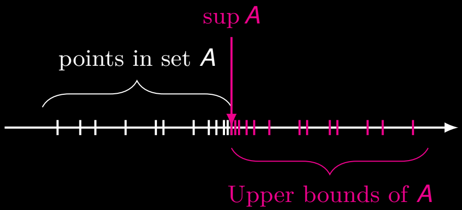
(O3) $x < y$ and $z > 0$ implies that $xz < yz$.

(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

Completeness Axiom

Axiom A nonempty subset of real numbers that is bounded above has a least upper bound, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$



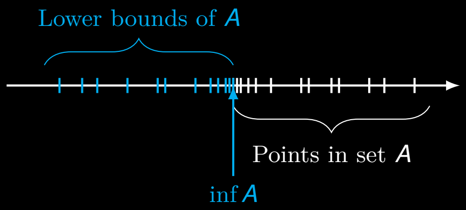
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$
\begin{aligned}
\sup_n x_n \leq a \quad &\iff \quad \forall n, \;\; x_n \leq a \\
\sup_n x_n < a \quad &\iff \quad \forall n, \;\; x_n < a \\
a < \sup_n x_n \quad &\iff \quad \exists n, \;\; x_n > a
\end{aligned}
$$

$$
a \leq \sup_n x_n \quad \Longleftarrow \quad \exists n, \;\; x_n \geq a
$$

Corr. A nonempty subset of real numbers that is bounded below has a greatest lower bound, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$



Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$
\begin{aligned}
a \leq \inf_n x_n \quad &\iff \quad \forall n,\ \ x_n \geq a \\
a < \inf_n x_n \quad &\iff \quad \forall n,\ \ x_n > a \\
\inf_n x_n < a \quad &\iff \quad \exists n,\ \ x_n < a
\end{aligned}
$$

$$a \leq \inf_n x_n \leq a \quad \Longleftarrow \quad \exists n,\ \ x_n \leq a$$

E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

Properties

1. Archimedean principle

   For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

   Between any two real numbers there is an irrational number.

3. Density of the rational numbers

   Between any two real numbers there is an rational number.

Proof As exercises. $\square$

Extended Real Number System

Def. The *extended real numbers* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
$x < 0$	$x \cdot \infty = \infty \cdot x = -\infty$	$x \cdot (-\infty) = (-\infty) \cdot x = \infty$
$x = \infty$	$\infty + \infty = \infty$	$(-\infty) + (-\infty) = -\infty$
	$\infty \cdot \infty = \infty$	$(-\infty) \cdot (-\infty) = \infty$
	$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$	

$\infty - \infty$ cannot be defined (HW).

Def. Let a and b be extended real numbers such that $a < b$. Then the *intervals on* $\mathbb{R}^*$ with endpoints a and b are as follows:

$$(a,b) = \{x \in \mathbb{R}^* : a < x < b\}$$
$$[a,b) = \{x \in \mathbb{R}^* : a \leq x < b\}$$
$$(a,b] = \{x \in \mathbb{R}^* : a < x \leq b\}$$
$$[a,b] = \{x \in \mathbb{R}^* : a \leq x \leq b\}$$

If both a and b are in $\mathbb{R}$, these intervals are the *bounded intervals* in $\mathbb{R}$.

Otherwise, if either $a = -\infty$ or $b = \infty$, then these intervals are *unbounded intervals*.

Thm Every subset A of $\mathbb{R}^*$ has both a least upper bound and greatest lower bound. Moreover,

a) If $A = \emptyset$, then $\sup A = -\infty$ and $\inf A = \infty$.

b) If A is bounded above in $\mathbb{R}$, then $\sup A \in \mathbb{R}$; otherwise, $\sup A = \infty$.

c) If A is bounded below in $\mathbb{R}$, then $\inf A \in \mathbb{R}$; otherwise, $\inf A = -\infty$.

E.g.

a) $\inf \mathbb{N} = 1$ and $\sup \mathbb{N} = \infty$.

b) $\inf \mathbb{Z} = -\infty$ and $\sup \mathbb{Z} = \infty$.

c) If I is an interval in $\mathbb{R}^*$ with endpoints a and b, $a \leq b$. Then $\inf I = a$ and $\sup I = b$.

HW Ex. 2.10 and 2.11 on p. 43.

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence[3], namely,

$$\lim_{n\to\infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n\to\infty} x_n = L \iff \forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\; s.t.\; (n \geq N \to |x_n - L| < \epsilon)$$

[3] In this case, we say that $\{x_n\}_{n=1}^{\infty}$ is *convergent*. Otherwise, we say that $\{x_n\}_{n=1}^{\infty}$ is *divergent*.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(n-1)/n = 1$.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is divergent.

$\left\{n^2\right\}_{n=1}^{\infty}$ is divergent.

Recall that $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ is the *extended real line*.

Def. A sequence of real numbers $\{a_1, a_2, \cdots\}$ is said to *converge in $\mathbb{R}^*$* if one of the following three conditions hold:

(i) The sequence converges to a finite real number as in the previous definition.

(ii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

(iii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

(i) We say that the sequence converges in $\mathbb{R}$ or the limit exits and is finite.

(ii) We say the sequence converges to ∞ and write $\lim_{n\to\infty} x_n = \infty$.

(iii) We say the sequence converges to $-\infty$ and write $\lim_{n\to\infty} x_n = -\infty$.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}^*$.

$\left\{n^2\right\}_{n=1}^{\infty}$ converges in $\mathbb{R}^*$ and $\lim_{n\to\infty} n^2 = \infty$.

Monotone sequence

Def. If $x_1 \le x_2 \le \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \ge x_2 \ge \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is not monotone.

$\left\{n^2\right\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$. Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n\to\infty} x_n = \sup\{x_n :\ n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded above and is ∞ otherwise.

b) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

$$\lim_{n\to\infty} x_n = \inf\{x_n :\ n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded below and is $-\infty$ otherwise.

Proof. We will prove the case when $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. The nonincreasing case can be proved in a similar way.

As $\sup_n x_n$ always exists in $\mathbb{R}^*$, we need to consider two cases:

Case I: $\sup_n x_n \in \mathbb{R}$

Case II: $\sup_n x_n = \infty$

Let's prove Case I here. Let $x = \sup_n x_n$. In order to show that $\lim_n x_n = x$, by monotonicity, we need to show that

$$\forall \epsilon > 0 \, \exists N \, \forall n \text{ s.t. } (n \geq N) \to (x - a_n \leq \epsilon).$$

Proof. (Continued) Otherwise,

$$\exists \epsilon > 0 \, \forall N \, \exists n \text{ s.t. } (n \geq N) \wedge (x - a_n > \epsilon).$$

Hence, there are infinitely many terms falling below $x - \epsilon$.

Since $\{x_n\}$ is nondecreasing, this implies all a_n fall below $x - \epsilon$, i.e.,

$$a_n < x - \epsilon, \quad \text{for all } n \geq 1.$$

which is equivalent to $\sup_n x_n < x - \epsilon$. This contradicts with the fact that $\sup_n x_n = x$.

Therefore,

$$\lim_n x_n = x = \sup_n x_n.$$

□

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is nondecreasing and converges in $\mathbb{R}$. It is bounded above.

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ is not monotone.

$\left\{n^2\right\}_{n=1}^{\infty}$ is nondecreasing, does not converge in $\mathbb{R}$, converges in $\mathbb{R}^*$.

Cluster points

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.

b) *∞ is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n > M$.

c) *$-\infty$ is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n < M$.

---

a) $x \in \mathbb{R}$ is a cluster point of $\{x_n\}$ if

$$\forall \epsilon \, \forall N \, \exists n \quad (n \geq N) \rightarrow (|x - x_n| < \epsilon). \tag{1}$$

E.g.1 $\{(n-1)/n\}_{n=1}^{\infty}$ has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$ has two cluster points: -1 and $+1$.

$\{n^2\}_{n=1}^{\infty}$ has one cluster point: $+\infty$.

E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞.

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Thm A convergent sequence has exactly one cluster point, namely, its limit. Thus, a sequence having more than one cluster point cannot converge.

Proof Suppose that $\{x_n\}$ is a convergent sequence and let x be its limit, namely, $\lim_{n\to\infty} x_n = x$. We need to prove:

(1) x is a cluster point.

(2) x is the only cluster point of $\{x_n\}$.

We also need to consider two cases:

Case I: $x \in \mathbb{R}$.

Case II: $x = \infty$ or $-\infty$.

We will focus on Case I only.

Now we first prove (1).

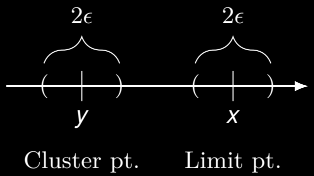
$$\lim_n x_n = x \qquad \forall\epsilon\ \overbrace{\exists N\ \forall n}^{\text{Once for all rest}}\quad (n \geq N) \to (|x_n - x| < \epsilon)$$

$$\Downarrow\ ??$$

$$x \text{ is a cluster point of } \{x_n\} \qquad \forall\epsilon\ \underbrace{\forall\widetilde{N}\ \exists\widetilde{n}}_{\text{Infinite games}}\quad (\widetilde{n} \geq \widetilde{N}) \wedge (|x_{\widetilde{n}} - x| < \epsilon)$$

(1) is proved by choosing any $\widetilde{n} \geq \max\left(\widetilde{N}, N\right)$.

As for (2), suppose y is another cluster point.



By choosing any $\epsilon < |x - y|/2$, we see that

1. In the ϵ-neighborhood of y, there are infinitely many terms.
2. In the ϵ-neighborhood of x, all but finite many terms are here.

Contradiction!

Therefore, there exits only one cluster point. $\square$

A few more properties

1. A sequence is convergent iff each subsequence is convergent.

2. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \to L$ and $b_n \to L$, then $c_n \to L$.

Limit superior and limit inferior

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n\to\infty} x_n := \sup_n \left(\inf_{m\geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\limsup_{n\to\infty} x_n := \inf_n \left(\sup_{m\geq n} a_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

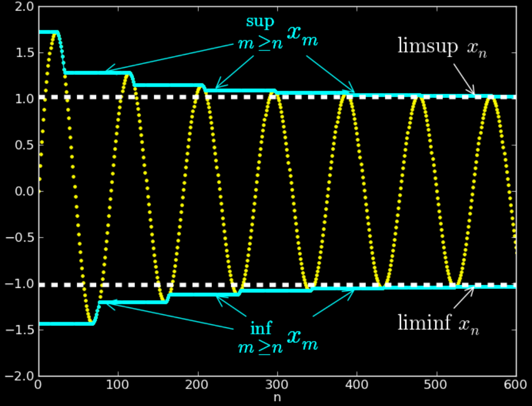
are, respectively, nonincreasing and nondecreasing, we see that

$$\inf_n y_n = \inf_n \sup_{m \geq n} x_m = \limsup_{x \to \infty} x_n$$

$$\sup_n z_n = \sup_n \inf_{m \geq n} x_m = \liminf_{x \to \infty} x_n.$$

Hence,

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{m \geq n} x_m.$$



[4]

[4] Image from Wikipedia.

Characterization of the limsup and liminf.

Thm Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

a) $\limsup x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,

i) there is an $N \in \mathbb{N}$ such that $x_n \leq x + \epsilon$ for all $n \geq N$, and

ii) for each $n \in \mathbb{N}$, there is an $m \geq n$ such that $x_m > x - \epsilon$.

b) $\limsup x_n = \infty$ iff for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there is an $n \geq N$ such that $x_n > M$; in other words, iff the sequence is unbounded from above.

c) Similarly, $\limsup x_n = -\infty$ if and only $\lim x_n = -\infty$.

---

a) $\limsup x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,

i) the ϵ-neighborhood of x has been visited infinitely many times; and

ii) only finite many terms are greater than $x + \epsilon$.

Proof. (Sketch) We only prove part (a). Let $y_n = \sup_{k \geq n} x_k$.

$$\limsup x_n = x \iff \lim_{n \to \infty} \sup_{k \geq n} x_k = x$$

$$\iff \forall \epsilon\, \exists N\, \forall n\ \ (n \geq N) \to \left(\sup_{k \geq n} x_k \in (x - \epsilon, x + \epsilon) \right)$$

$$\sup_{k \geq n} x_k < x + \epsilon \iff \text{all terms starting from } n \text{ fall below } x + \epsilon$$

$$\sup_{k \geq n} x_k > x - \epsilon \iff \exists k \geq n \text{ s.t. } x_k > x + \epsilon$$

$\square$

Thm Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

a) $\liminf x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,

i) there is an $N \in \mathbb{N}$ such that $x_n \geq x - \epsilon$ for all $n \geq N$, and

ii) for each $n \in \mathbb{N}$, there is an $m \geq n$ such that $x_m < x + \epsilon$.

b) $\liminf x_n = -\infty$ iff for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there is an $n \geq N$ such that $x_n < M$; in other words, iff the sequence is unbounded from below.

c) Similarly, $\liminf x_n = \infty$ if and only $\lim x_n = \infty$.

Still other characterization (as an exercise):

Thm $\limsup_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the smallest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

$\liminf_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the largest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n > x + \epsilon$ for all $n > N$.

Proof. HW for motivated students. □

E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1, amount which

$$\liminf_{n\to\infty} x_n = -1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi)\frac{n^2}{n^2+8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞, among which

$$\liminf_{n\to\infty} x_n = 1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

$$\liminf_{n\to\infty} r_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

a) $\limsup x_n$ is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$.

b) $\liminf x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.

Proof. We only prove (a). (b) can be proved in a similar way.

Let $x = \limsup x_n$. We have seen that x is a cluster point.

It remains to prove that x is the largest cluster point. The case when $x = \pm\infty$ is left for the motivated students.

Now assume that $x \in \mathbb{R}$.

Only finite many terms exceed $x + 1$, hence, ∞ is not a cluster point.

Let $y \in \mathbb{R}$ s.t. $x < y$. Set $\epsilon = (y - x)/2$.

Only finite many terms exceed $x + \epsilon$.

Since $y - \epsilon = x + \epsilon$, the ϵ-neighborhood of y has been visited only finitely many times.

Hence, y cannot be a cluster point. $\square$

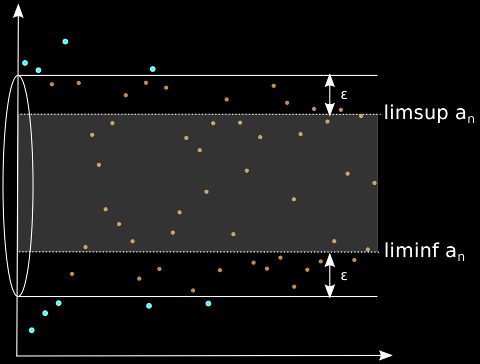
Properties

1.

$$\inf_n x_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = c \quad \Longleftrightarrow \quad \lim_{n\to\infty} x_n = c.$$



[5]

E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

[5] Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2}\left(\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n\right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right).$$

Cauchy criterion

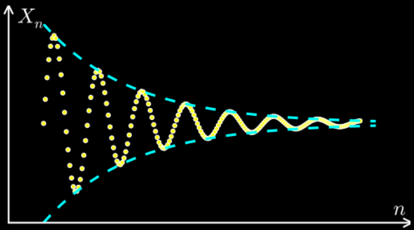
As we have seen that

A sequence of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point.

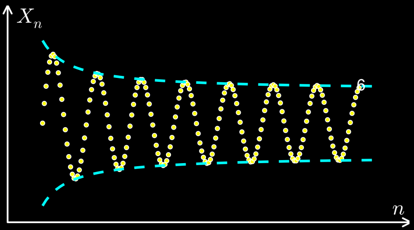
There is another famous criterion for a sequence to converge in $\mathbb{R}$:

Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



[6] Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in $\mathbb{R}$ iff it is Cauchy.

Proof. "$\Rightarrow$" Easy!

"$\Leftarrow$": ... $\square$

E.g.1 Let $a_n = \sqrt{n}$. Show that

(i) The consecutive terms become arbitrarily close to each other as $n \to \infty$.

(ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg \left(\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}\right) \\ \iff \quad & \exists \epsilon > 0\, \forall N \in \mathbb{N}\, \exists m, n \geq N \, \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose $m = N$ and $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

$\square$

E.g.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

Variation

$$\lim_{n\to\infty}\left(1+\mathrm{Small}\right)^{\mathrm{Large}}=e^{\lim_{n\to\infty}\mathrm{Small}\times\mathrm{Large}}$$

HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exits an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of $\mathbb{R}$. Then K cannot be an open set. For example, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ are not open sets.

$\mathbb{Q}^c$ – the set of irrational numbers – is not an open set.

Properties

1. $\mathbb{R}$ and $\emptyset$ are open sets.

2. If A and B are open sets, so is $A \cap B$. (finite intersection)

3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open.

(arbitrary union)

Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

The set of all limit point of E, denoted $\overline{E}$, is called the *closure* of E.

E.g. $\overline{\mathbb{R}} = \mathbb{R}$ and $\overline{\emptyset} = \emptyset$.

Let $a, b \in \mathbb{R}$ such that $a < b$. Then

$$\overline{(a, b)} = \overline{(a, b]} = \overline{[a, b)} = \overline{[a, b]} = [a, b].$$

$\overline{\mathbb{N}} = \mathbb{N}$ and $\overline{\mathbb{Z}} = \mathbb{Z}$.

$\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{Q}^c} = \mathbb{R}$.

If A is a finite subset of $\mathbb{R}$, then $\overline{A} = A$.

Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

E.g. $\mathbb{R}$ and $\emptyset$ are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

$\mathbb{N}$ and $\mathbb{Z}$ are closed sets.

The sets of rationals $\mathbb{Q}$ and irrationals $\mathbb{Q}^c$ are neither open nor close.

If A is a finite subset of $\mathbb{R}$, then A is a close set.

Properties

0. A set is open if and only if its complement is closed.

$$\Downarrow$$

1. $\mathbb{R}$ and $\emptyset$ are closed sets.

2. If A and B are closed sets, so is $A \cup B$. (finite union)

3. If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

(arbitrary intersection)

Def. Let $G \subset D \subset \mathbb{R}$.

(a) G is said to be open in D if for each $x \in G$, there is an $r > 0$ such that

$$(x - r, x + r) \cap D \subset G.$$

(b) G is said to be *closed in D* if $D \setminus E$ is open in D.

E.g.

D	G	Is G open in $\mathbb{R}$	Is G open in D
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
$\mathbb{N}$	$A \subset \mathbb{N}$	closed	open

Def. A *real-valued function* is a function whose range is a subset of $\mathbb{R}$. If $f : \Omega \to \mathbb{R}$, we say that f is a *real-valued function on* Ω.

Def. *Algebraic operations:* Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

$$
\begin{aligned}
(f+g)(x) &:= f(x) + g(x) \\
(\alpha f)(x) &:= \alpha f(x) \\
(f \cdot g)(x) &:= f(x)g(x)
\end{aligned}
$$

(Local) Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is *continuous at a point c* if the limit of $f(x)$, as x approaches c, exits and is equal to $f(c)$.

Def'. (Epsion-delta definition) The function f is *continuous at a point c* if

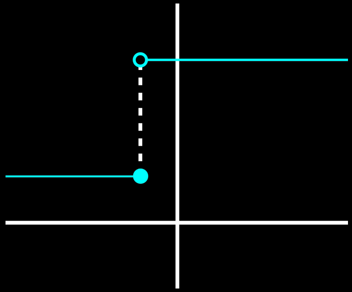
$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in \mathbb{R} \, \{|x - c| \leq \delta \to |f(x) - f(c)| \leq \epsilon\}$$

Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in $\mathbb{R}$.

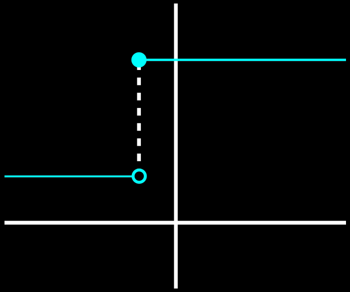
Def. f is *left-continuous at c* if

$$\lim_{x \to c+} f(x) = f(c)$$



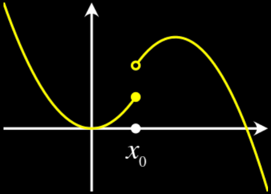
Def. f is *right-continuous at c* if

$$\lim_{x \to c-} f(x) = f(c)$$



Def. f is *lower semi-continuous at x_0* if

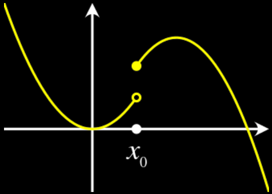
$$\liminf_{x \to x_0} f(x) \geq f(x_0)$$



$f(x_0)$ can be all points at or below the blue point.

f is *upper semi-continuous at x_0* if

$$\limsup_{x \to x_0} f(x) \leq f(x_0)$$



$f(x_0)$ can be all points at or above the blue point.

(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let I be an interval of $\mathbb{R}$. Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exits a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Def.' f is *uniformly continuous over I* if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in I \, \forall y \in I \, \{|x - y| < \delta \to |f(x) - f(y)| < \epsilon\}$$

f is continuous at $x_0 \in I$ iff

$$\forall \epsilon > 0 \, \forall x \in I \, \exists \delta > 0 \; \{|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon\}$$

Π_2-form

---

f is uniformly continuous over I iff

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in I \, \forall y \in I \; \{|x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon\}$$

Π_3-form

Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I, we need to show

$$\neg\left(\forall\epsilon > 0\, \exists\delta > 0\, \forall x \in I\, \forall y \in I\ \{|x-y| < \delta \to |f(x) - f(y)| < \epsilon\}\right)$$

$$\Leftrightarrow \quad \exists\epsilon > 0\, \forall\delta > 0\, \exists x \in I\, \exists y \in I\, \neg\{|x-y| < \delta \to |f(x) - f(y)| < \epsilon\}$$

$$\Leftrightarrow \quad \exists\epsilon > 0\, \forall\delta > 0\, \exists x \in I\, \exists y \in I\, \neg\{\neg\{|x-y| < \delta\} \vee |f(x) - f(y)| < \epsilon\}$$

$$\Leftrightarrow \quad \exists\epsilon > 0\, \forall\delta > 0\, \exists x \in I\, \exists y \in I\ \{|x-y| < \delta \wedge |f(x) - f(y)| \geq \epsilon\}$$

Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned}
|f(x) - f(y)| &= |x^3 - y^3| \\
&= |x - y| \times |x^2 + xy + y^2| \\
&\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\
&= 1 = \epsilon.
\end{aligned}$$

$\square$

Prop. 2 If I is compact [7] set such as $I = [a, b]$, then

$$f \text{ is continuous at all points in } I \quad \Longleftrightarrow \quad f \text{ is uniformly continuous on } I.$$

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.
$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on $\mathbb{R}$ nor on $[0, \infty)$.

[7] namely, bounded and closed

Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

Notation For $D \subset \mathbb{R}$, let $C(D)$ denote the set of continuous functions defined on D.

Thm (*Algebra of $C(D)$*) Let $D \subset \mathbb{R}$. Then the collection $C(D)$ of continuous functions on D is an algebra of functions, that is, for all $f, g \in C(D)$ and $\alpha \in \mathbb{R}$,

$$f + g \in C(D)$$
$$\alpha f \in C(D)$$
$$f \cdot g \in C(D)$$

Remark Can one add one more operation in this algebra: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(D)$, under what condition the limit $f_n \to f$ is closed?

Def. $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on Ω, namely, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$.

We say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on Ω* if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}$.

E.g.

(a) $f_n \in C(\mathbb{R})$ defined as $f_n = (1 + x/n)^n$. Then f_n converges pointwise on $\mathbb{R}$ to $f(x) = e^x$. It is clear that $f \in C(\mathbb{R})$.

(b) Let $D = [0, 1]$ and $f_n \in C(D)$ be defined as $f_n = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

It is clear that $f \notin C(D)$.

Def. Let $\mathcal{F}$ be a collection of real-valued functions on Ω. We say that $\mathcal{F}$ is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \to f$ pointwise on Ω, then $f \in \mathcal{F}$.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω.
We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω, if

$$\boxed{\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N}\ \forall x \in \Omega\ \ |f_n(x) - f(x)| < \epsilon,$$

written as $f_n \to f$ *uniformly*.

Recall $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if

$$\forall x \in \Omega\ \boxed{\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N}\ |f_n(x) - f(x)| < \epsilon,$$

Prop. Let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \to f$ uniformly. Then $f \in C(D)$.

Proof.

Therefore, the collection $C(D)$ of real-valued continuous functions is closed under: $+$, $\cdot$, scalar multiplication, and uniform convergence.

Some part of subsection is taken from Chapter 1 Section 4 of

P. Billingsley, **Probability and Measure**, Wiley, 1995.

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Remark Both $\limsup_n A_n$ and $\liminf_n A_n$ are sets.

Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$
\begin{aligned}
\omega \in \limsup_n A_n \quad &\iff \quad \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
&\iff \quad (\forall n \geq 1)\,(\exists k \geq n)\, \omega \in A_k \\
&\iff \quad \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n
\end{aligned}
$$

Notation

$$\limsup_n A_n = [\, A_n \quad \text{i.o.}\,]$$

Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$\omega \in \liminf_n A_n \iff \omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$$\iff (\exists n \geq 1)\ (\forall k \geq n)\ \omega \in A_k$$

$$\iff \omega \text{ lies in } \textit{all but finitely many} \text{ of the } A_n$$

Notation

$$\liminf_n A_n = [\, A_n \quad \text{all but finitely many} \,]$$

Def. If both $\limsup_n A_n$ and $\liminf_n A_n$ exist and are equal, then the *limit set* of the sequence $\{A_n\}$ is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

which is also often written as $A_n \to A$.

Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$
\begin{array}{ccccc}
\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \liminf_{n} A_n & \Longrightarrow & \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k = \liminf_{n} A_n \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & \\
A_n & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & \\
\left(\bigcup_{k=n}^{\infty} A_k\right) & \downarrow & \left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = \limsup_{n} A_n & \Longrightarrow & \lim_{n\to\infty}\bigcup_{k=n}^{\infty} A_k = \limsup_{n} A_n
\end{array}
$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$
\begin{array}{cccl}
\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) & = \mathbb{P}\left(\liminf_n A_n\right) \\
 & & \rotatebox[origin=c]{-90}{$\leq$} & \\
\rotatebox[origin=c]{-90}{$\leq$} & & \liminf_{n\to\infty} \mathbb{P}(A_n) & \\
\mathbb{P}\left(A_n\right) & & \rotatebox[origin=c]{-90}{$\leq$} & \\
\rotatebox[origin=c]{-90}{$\leq$} & & \limsup_{n\to\infty} \mathbb{P}(A_n) & \\
 & & \rotatebox[origin=c]{-90}{$\leq$} & \\
\mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) & \downarrow & \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) & = \mathbb{P}\left(\limsup_n A_n\right)
\end{array}
$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned}
1 \geq \mathbb{P}\left(A_n^c \text{ all but finitely many}\right) &= 1 - \mathbb{P}\left(\left\{A_n^c \text{ all but finitely many}\right\}^c\right) \\
&= 1 - \mathbb{P}\left(A_n \text{ i.o.}\right) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \\
&\geq 1 - \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\
&= 1 - 0 = 1.
\end{aligned}$$

□

Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$\begin{aligned} A_1 &= (-1, 0] \\ A_2 &= \left(-\frac{1}{2}, \frac{1}{2}\right] \\ A_3 &= \left(-\frac{1}{3}, \frac{2}{3}\right] \\ A_4 &= \left(-\frac{1}{4}, \frac{3}{4}\right] \\ \vdots \quad & \quad \vdots \\ A_{100} &= \left(-\frac{1}{100}, \frac{99}{100}\right] \\ \vdots \quad & \quad \vdots \end{aligned}$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n}\right] = [0, 1)$$

and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right) = [0, 1).$$

Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

$\square$

Exercise

(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right]$:

$$
\begin{array}{ll}
A_1 = (-1, 2] & A_2 = \left(\frac{1}{2}, \frac{1}{2}\right] \\
A_3 = \left(-\frac{1}{3}, \frac{4}{3}\right] & A_4 = \left(\frac{1}{4}, \frac{3}{4}\right] \\
A_5 = \left(-\frac{1}{5}, \frac{6}{5}\right] & A_6 = \left(\frac{1}{6}, \frac{5}{6}\right] \\
\vdots \quad \vdots & \vdots \quad \vdots \\
A_{99} = \left(-\frac{1}{99}, \frac{100}{99}\right] & A_{100} = \left(\frac{1}{100}, \frac{99}{100}\right] \\
\vdots \quad \vdots & \vdots \quad \vdots
\end{array}
$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

Sol.

$$
\begin{aligned}
&\liminf_n A_n \\
=&\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right] \\
=&\left\{\bigcup_{n=1,3,5}^{\infty}\bigcap_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\}\cup\left\{\bigcup_{n=2,4,6}^{\infty}\bigcap_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\} \\
=&\left\{\bigcup_{n=1,3,5}^{\infty}\left(\frac{1}{n+1},\frac{n}{n+1}\right]\right\}\cup\left\{\bigcup_{n=2,4,6}^{\infty}\left(\frac{1}{n},\frac{n-1}{n}\right]\right\} \\
=&(0,1)\cup(0,1) \\
=&(0,1)
\end{aligned}
$$

Sol. (continued) Similarly,

$$
\begin{aligned}
&\limsup_n A_n \\
=&\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right] \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\} \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\left(-\frac{1}{n+1},\frac{n+2}{n+1}\right]\right\} \\
=&[0,1]\cap[0,1] \\
=&[0,1]
\end{aligned}
$$

$\square$

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \dfrac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \dfrac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$.

Solution:

$$\liminf_{n\to\infty} A_n = \{0\} \subset [0, 1] = \limsup_{n\to\infty} A_n$$

Examples

1. $\int_0^1 \tan^{-1}(x)dx$
2. $\int_0^x t^2 e^t dt$
3. $\int e^x \sin(x)dx$

more to come ...

Taylor expansions

Examples

1. e^x
2. $\sin(x)$
3. e^{x^2}

more to come ...