

Topics in Analysis and Linear Algebra

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Chapter 4. Topics in linear algebra

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§ 4.1 Abstract vector spaces

§ 4.2 Geometric Interpretation of Eigenvalues and Eigenvectors

§ 4.3 Determinant

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§ 4.1 Abstract vector spaces

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§ 4.3 Determinant

What is a vector space?

1. \mathbb{R}^n

2. Polynomials of order at most n :

$$\{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}, i = 1, \dots, n\}$$

3. The set of $m \times n$ matrices.

4. The set of continuous functions on $[0, 1]$, i.e., $C([0, 1])$.

5. The set of functions on $[0, 1]$ having n th continuous derivatives, i.e., $C^n([0, 1])$.

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Def. Let V be a nonempty set of objects with two operations: vector addition and scalar multiplication.

Then V is called a *vector space* if it satisfies the following:

Axioms of Addition
and
Axioms of Scalar Multiplication.

The elements of V are called *vectors*.

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Axioms of addition

A1. V is closed under addition.

$$\vec{v}, \vec{w} \in V \implies \vec{u} + \vec{v} \in V$$

A2. Addition is commutative.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \text{ for all } \vec{u}, \vec{v} \in V.$$

A3. Addition is associative.

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \text{ for all } \vec{u}, \vec{v}, \vec{w} \in V.$$

A4. Existence of an additive identity.

$$\text{There exists an element } \vec{0} \text{ in } V \text{ so that } \vec{u} + \vec{0} = \vec{u} \text{ for all } \vec{u} \in V.$$

A5. Existence of an additive inverse.

$$\text{For each } \vec{u} \in V \text{ there exists an element } -\vec{u} \in V \text{ so that } \vec{u} + (-\vec{u}) = \vec{0}.$$

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- S1. V is closed under scalar multiplication.
 $\vec{v} \in V$ and $k \in \mathbb{R}$, $\implies k\vec{v} \in V$.
- S2. Scalar multiplication distributes over vector addition.
 $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ for all $a \in \mathbb{R}$ and $\vec{u}, \vec{v} \in V$.
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Def. Let V be a vector space and $\vec{u}, \vec{v} \in V$. The *difference* of \vec{u} and \vec{v} is defined as

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

(where $-\vec{v}$ is the additive inverse of \vec{v}).

Thm Let V be a vector space, $\vec{u}, \vec{v}, \vec{w} \in V$, and $a \in \mathbb{R}$.

1. If $\vec{u} + \vec{v} = \vec{u} + \vec{w}$, then $\vec{v} = \vec{w}$.
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Examples

E.g.1 Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$, with addition \oplus and scalar multiplication \odot defined as follows:

For $(x_1, y_1), (x_2, y_2) \in V$, and $a, b \in \mathbb{R}$:

Addition: $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 1)$.

Scalar multiplication: $a \odot (x_1, y_1) = (ax_1, ay_1 + a - 1)$.

Show that V , with addition and scalar multiplication as defined, is a vector space.

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Show that V , with addition and scalar multiplication as defined, is a vector space.

Sol. It is clear that V is closed under \oplus and \odot , since both operations produce ordered pairs of real numbers.

1. It is routine to verify that \oplus is commutative and associative.
2. What is the additive identity?
3. What is the additive inverse of $(x, y) \in V$?
4. Verify that $(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$.
5. Verify that $a \odot ((x_1, y_1) \oplus (x_2, y_2)) = (a \odot (x_1, y_1)) \oplus (a \odot (x_2, y_2))$.
6. Verify that $a \odot (b \odot (x_1, y_1)) = (ab) \odot (x_1, y_1)$.
7. Verify that $1 \odot (x, y) = (x, y)$. □

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Sol. It is clear that V is closed under \oplus and \odot , since both operations produce ordered pairs of real numbers.

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E.g.2 Let \mathbb{R}_+ be the set of positive reals.

Let the addition \oplus and the scalar multiplication \odot defined as follows:

For $x, y \in \mathbb{R}_+$, and $a \in \mathbb{R}$:

Addition: $x \oplus y = xy$.

Scalar multiplication: $a \odot x = x^a$.

Prove that \mathbb{R}_+ equipped with \oplus and \odot is a vector space.

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E.g.3 Let $\mathcal{C}([0, 1])$ be the set of continuous functions defined on $[0, 1]$ equipped with usual addition and scalar multiplication. Prove that $\mathcal{C}([0, 1])$ is a vector space.

E.g.4 Let $\mathcal{C}^n([0, 1])$ be the set of functions that have continuous n th derivatives ($n \geq 0$) defined on $[0, 1]$, equipped with usual addition and scalar multiplication. Prove that $\mathcal{C}^n([0, 1])$ is a vector space.

E.g.5 The set of $m \times n$ matrices M_{mn} .

E.g.6 Polynomials of degree n .

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Chapter 4. Topics in linear algebra

§ 4.1 Abstract vector spaces

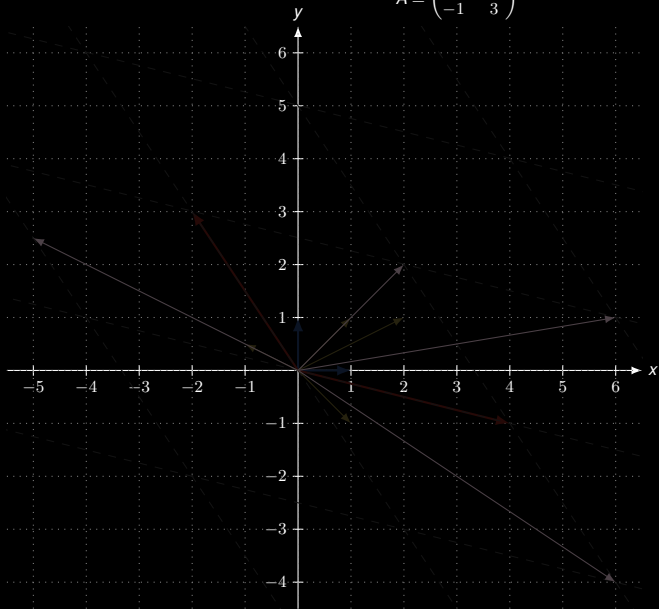
§ 4.2 Geometric Interpretation of Eigenvalues and Eigenvectors

§ 4.3 Determinant

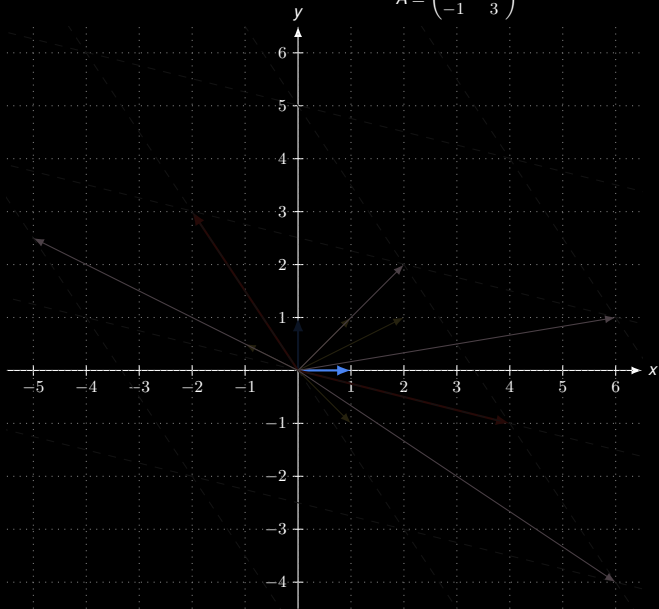
E.g. $A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$ has two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 5$ with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

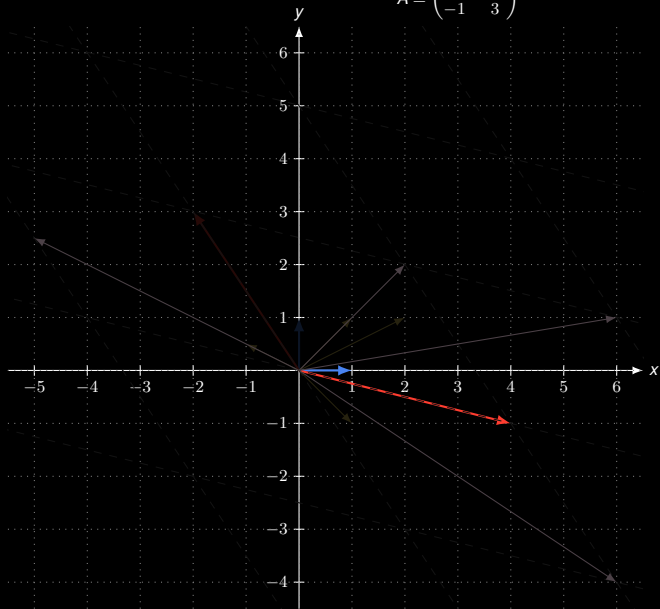
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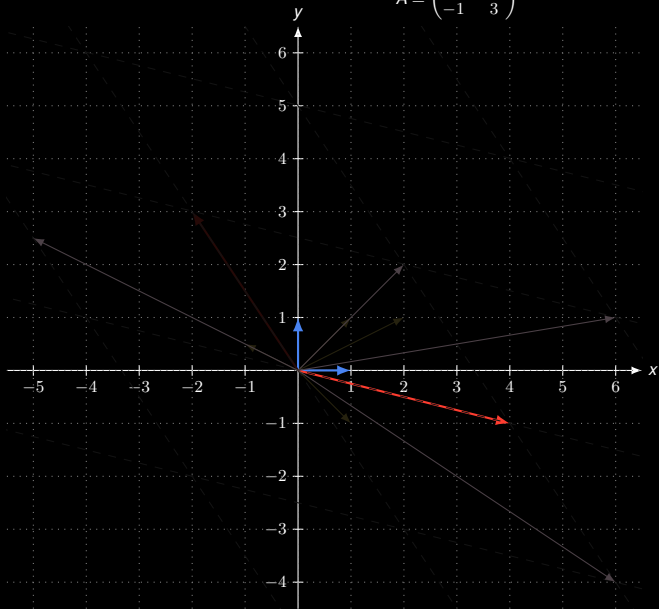
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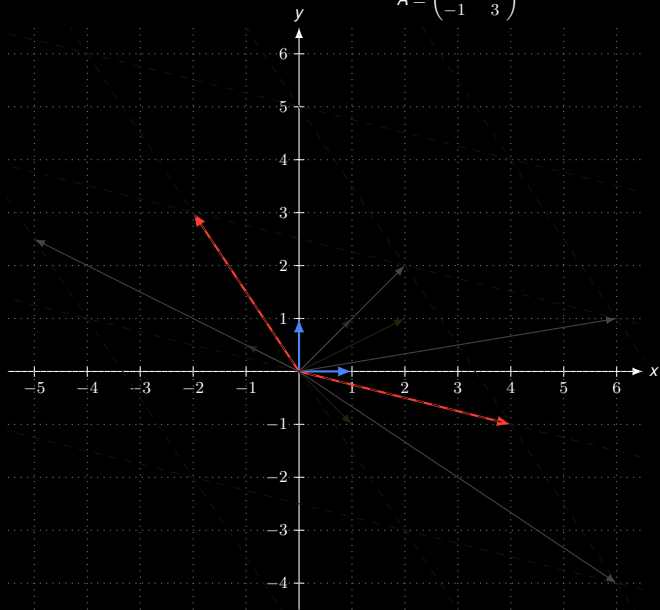
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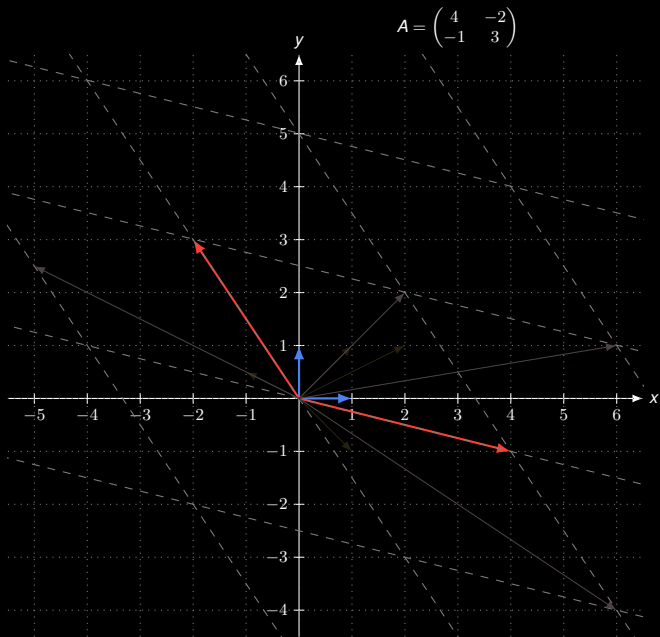


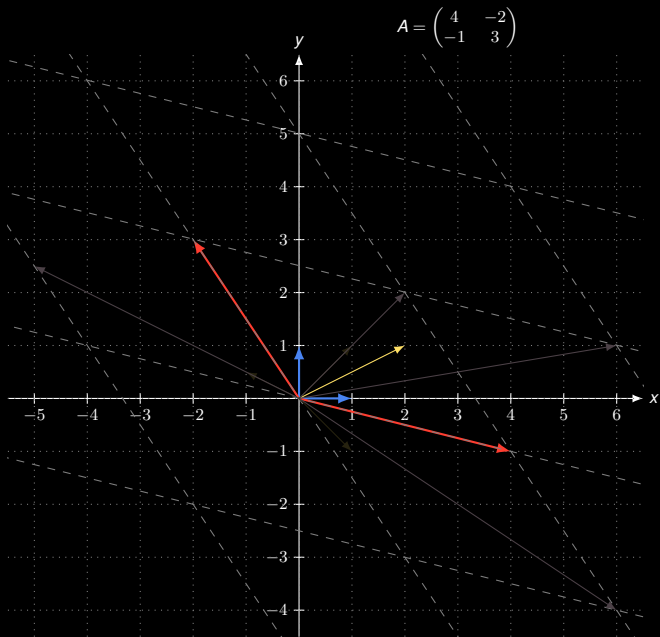
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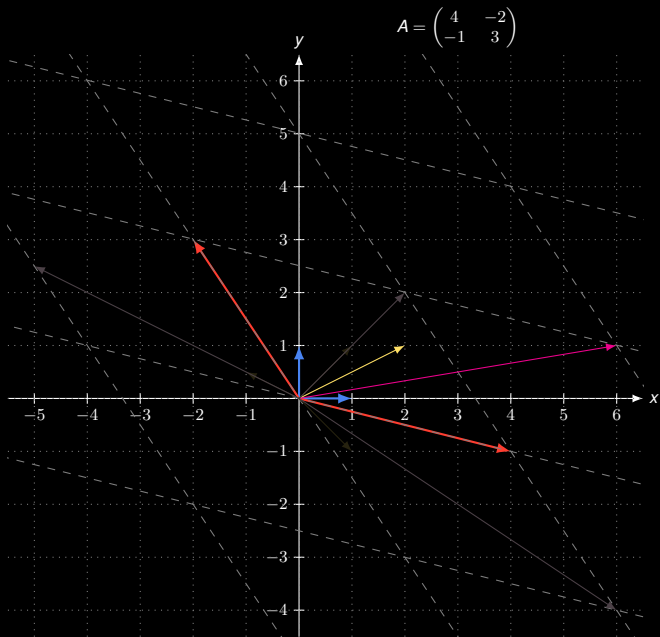


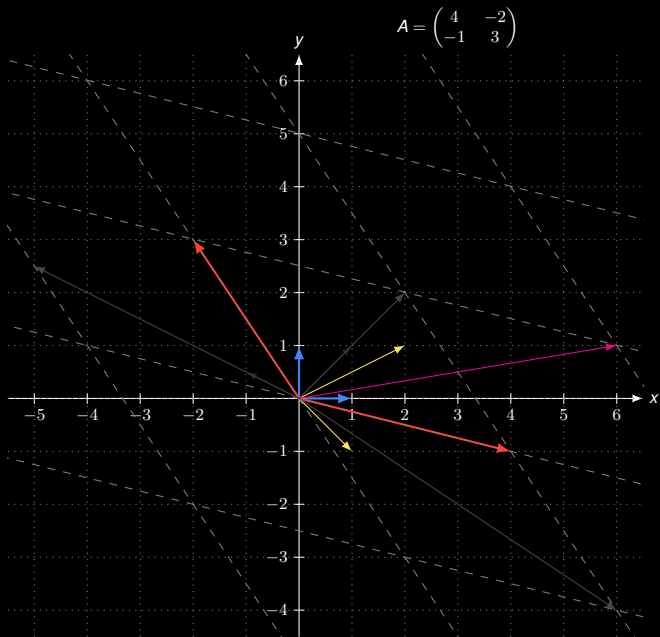
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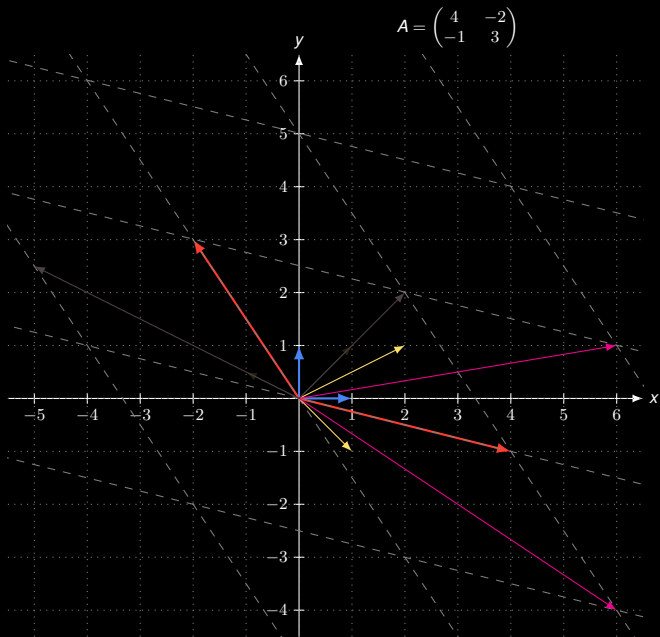


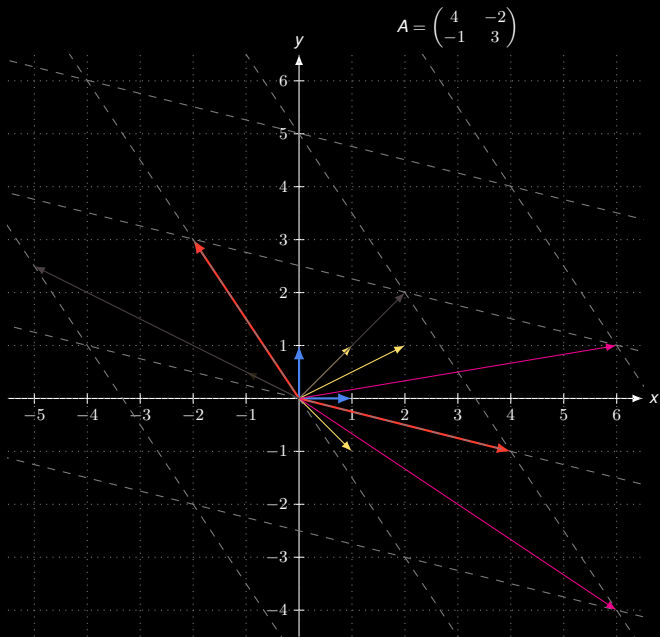


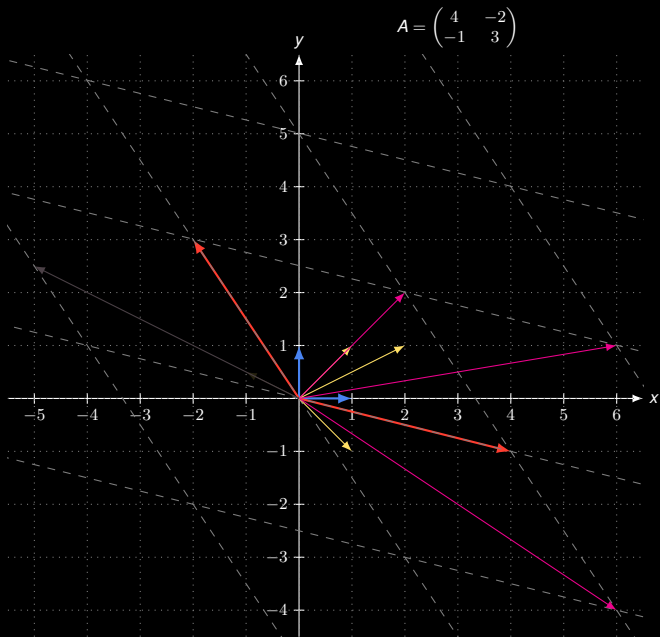


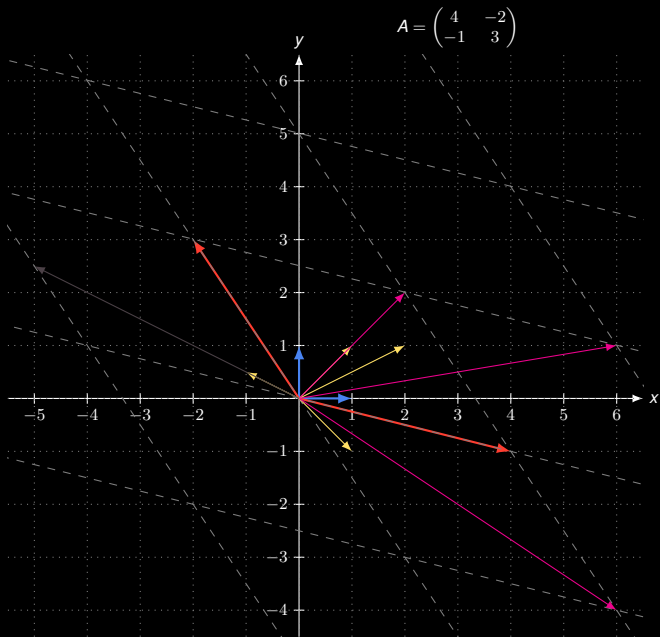


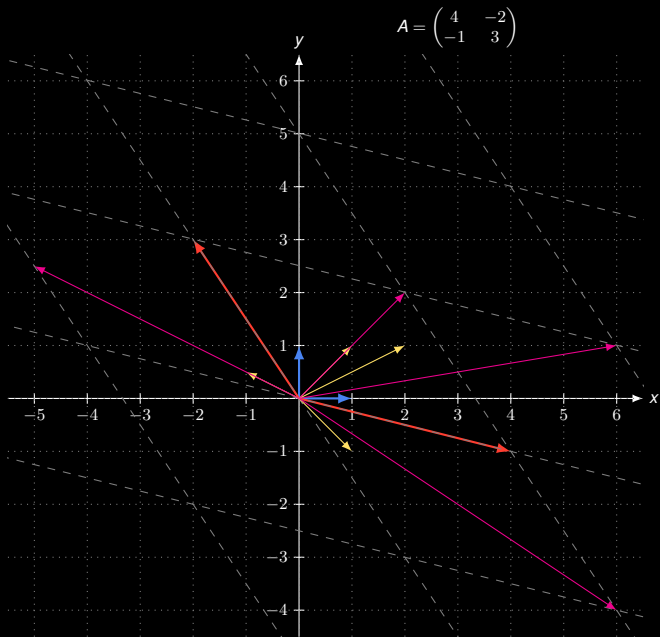












Def. Let A be a 2×2 matrix and L a line in \mathbb{R}^2 through the origin. Then L is said to be **A-invariant** if the vector $A\vec{x}$ lies in L whenever \vec{x} lies in L ,

i.e., $A\vec{x}$ is a scalar multiple of \vec{x} ,

i.e., $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda \in \mathbb{R}$,

i.e., \vec{x} is an eigenvector of A .

Def. Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a nonzero vector in \mathbb{R}^2 . Then $L_{\vec{v}}$ is the set of all scalar multiples of \vec{v} , i.e.,

$$L_{\vec{v}} = \mathbb{R}\vec{v} = \{t\vec{v} \mid t \in \mathbb{R}\}.$$

Thm Let A be a 2×2 matrix and let $\vec{v} \neq 0$ be a vector in \mathbb{R}^2 . Then $L_{\vec{v}}$ is A -invariant if and only if \vec{v} is an eigenvector of A .

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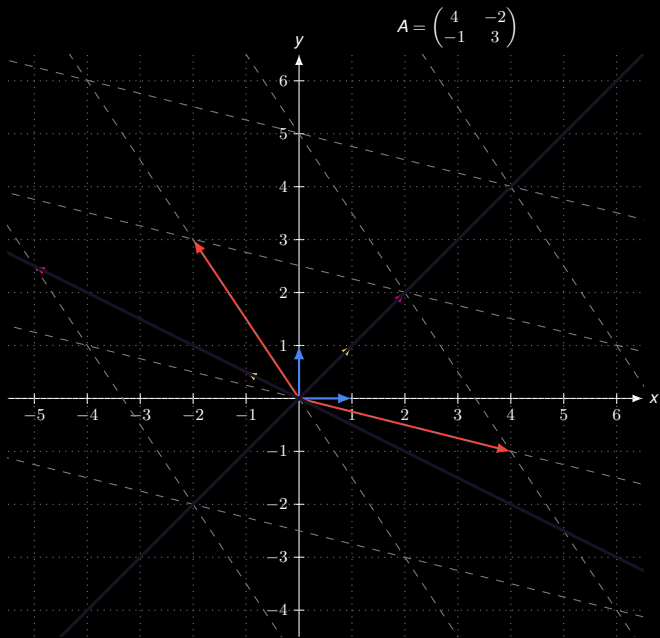
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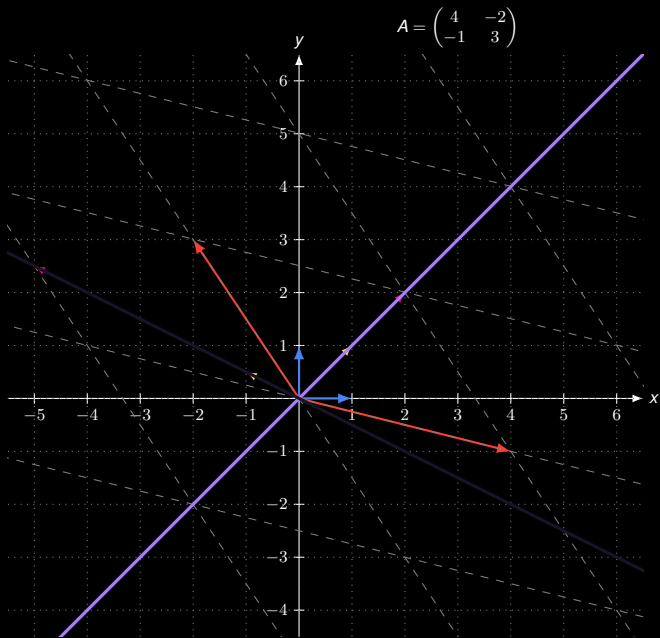
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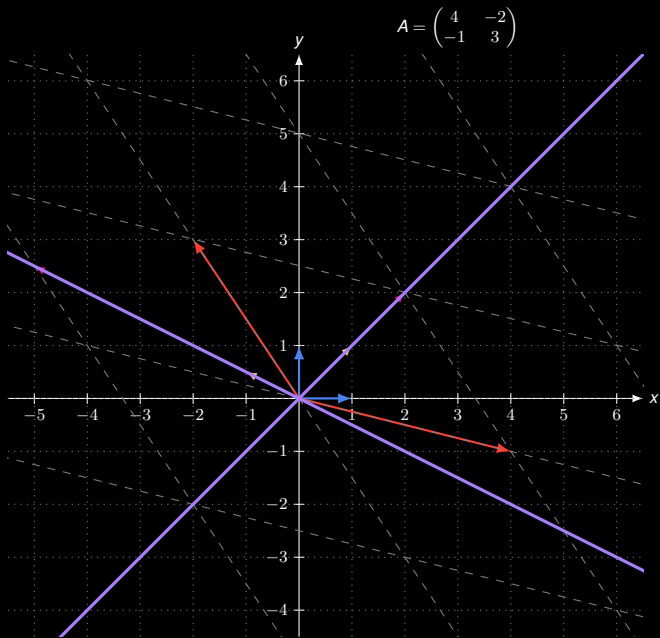
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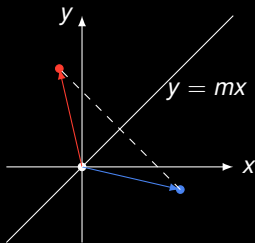
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E.g. Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, i.e., reflection in the line $y = mx$.

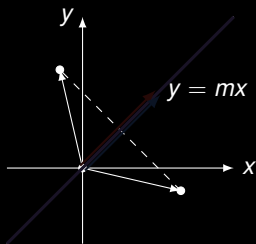


Recall that this is a matrix transformation induced by

$$A = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Find the lines that pass through origin and are A -invariant. Determine corresponding eigenvalues.

Sol.

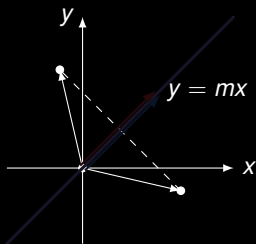


Let $\vec{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$. Then $L_{\vec{x}_1}$ is A -invariant, that is, \vec{x}_1 is an eigenvector.

Since the vector won't change, its eigenvalue should be 1. Indeed, one can verify that

$$A\vec{x}_1 = \frac{1}{1+m^2} \begin{bmatrix} 1+m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} = \vec{x}_1.$$

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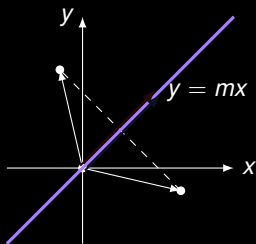


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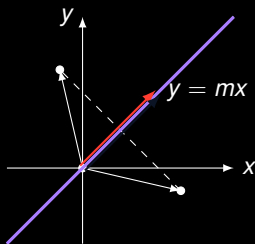


Let $\vec{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$. Then L_m is A -invariant, that is, \vec{x}_1 is an eigenvector.

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$$A\vec{x}_1 = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} = \vec{x}_1.$$

Sol.

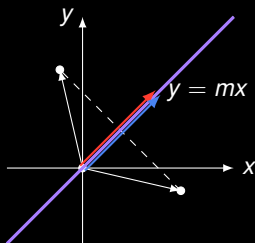


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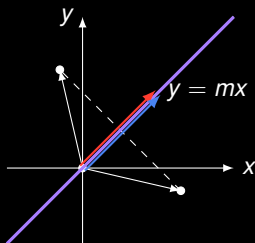


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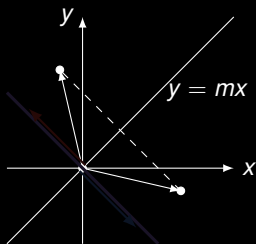


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Sol. (Continued)



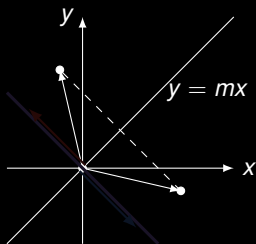
Let $x_2 = \begin{bmatrix} -m \\ 1 \end{bmatrix}$. Then $L_{y=mx}$ is A -invariant, that is, x_2 is an eigenvector.

Since the vector won't change the size, only flip the direction, its eigenvalue should be -1 . Indeed, one can verify that

$$Ax_2 = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{pmatrix} -m \\ 1 \end{pmatrix} = \dots = \begin{pmatrix} m \\ -1 \end{pmatrix} = -x_2.$$

□

Sol. (Continued)



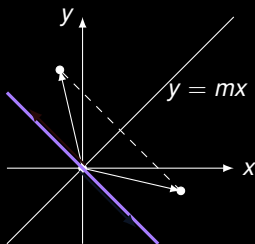
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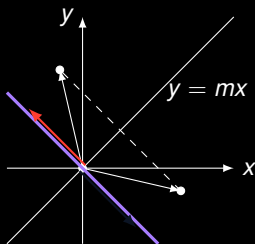
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Sol. (Continued)



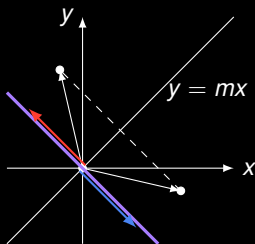
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□

Sol. (Continued)



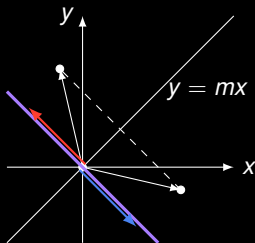
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$$A\bar{x}_2 = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{pmatrix} -m \\ 1 \end{pmatrix} = \dots = \begin{pmatrix} m \\ -1 \end{pmatrix} = -\bar{x}_2.$$

□

Sol. (Continued)



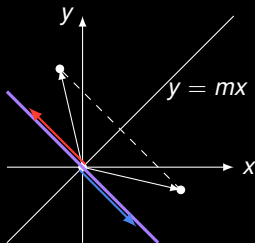
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E.g. Let θ be a real number, and $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotation through an angle of θ , induced by the matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Claim A has no real eigenvalues unless θ is an integer multiple of π , i.e., $\pm\pi, \pm2\pi, \pm3\pi$, etc.

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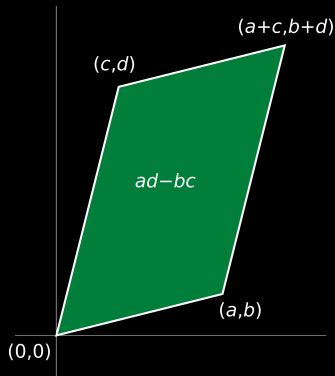
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Chapter 4. Topics in linear algebra

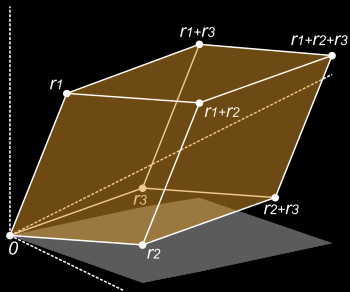
§ 4.1 Abstract vector spaces

§ 4.2 Geometric Interpretation of Eigenvalues and Eigenvectors

§ 4.3 Determinant



$$\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \text{signed area of parallelogram}$$



$\det \begin{pmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{pmatrix} = \text{signed volume of the parallelepiped}$

Cofactor and cofactor expansion

Def. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- The **sign** of the (i, j) position is $(-1)^{i+j}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \Rightarrow \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

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- Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j . The (i,j) -cofactor of A is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

- The determinant of A is defined as

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$$

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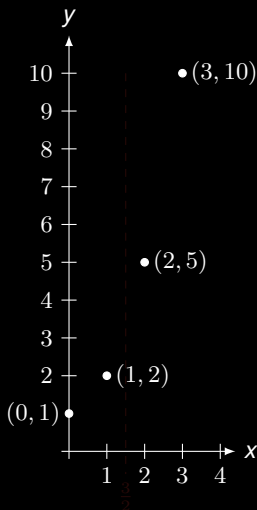
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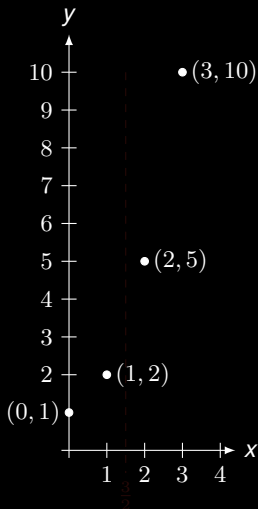
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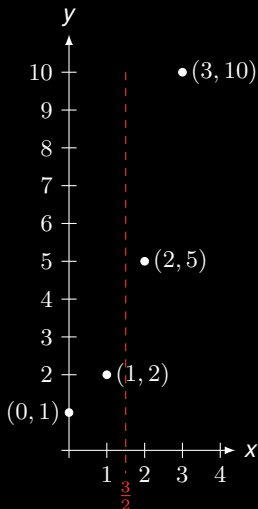
Problem Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = 3/2$.



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Sol. We want to find the coefficients r_0 , r_1 , r_2 and r_3 of

$$p(x) = r_0 + r_1x + r_2x^2 + r_3x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$p(0) = r_0 = 1$$

$$p(1) = r_0 + r_1 + r_2 + r_3 = 2$$

$$p(2) = r_0 + 2r_1 + 4r_2 + 8r_3 = 5$$

$$p(3) = r_0 + 3r_1 + 9r_2 + 27r_3 = 10$$

Solve this system of four equations in the four variables r_0 , r_1 , r_2 and r_3 .

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

Finally, the estimate is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4}.$$

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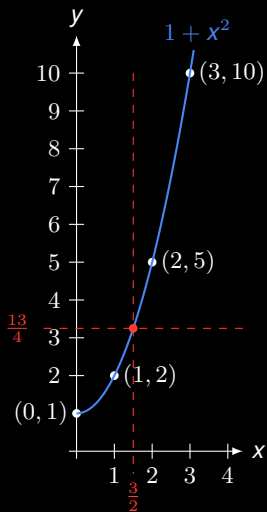
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Thm (Polynomial Interpolation)

Given n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1x + r_2x^2 + \dots + r_{n-1}x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \dots, n$.

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To find $p(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-1}x^{n-1}$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \dots, r_{n-1}$.

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The coefficient matrix for this system is

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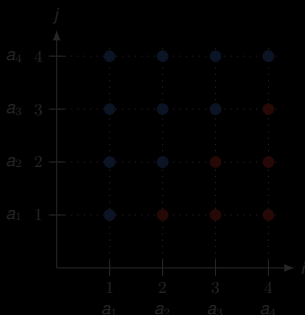
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Let a_1, a_2, \dots, a_n be real numbers, $n \geq 2$. The corresponding Vandermonde determinant is

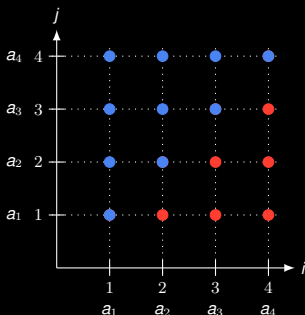
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Proof. We will prove this by induction. It is clear that when $n = 2$,

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Because $p(a_1) = \cdots = p(a_{n-1}) = 0$ (why?), $p(x)$ has to take the following form:

$$p(x) = c(x - a_1)(x - a_2) \cdots (x - a_{n-1}).$$

To identify the constant c , notice that c is the coefficient for x^{n-1} . By cofactor expansion of the determinant along the last row,

$$\begin{aligned} c &= (-1)^{n+n} \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{bmatrix} \\ &= \prod_{1 \leq j < l \leq n-1} (a_j - a_l). \end{aligned}$$

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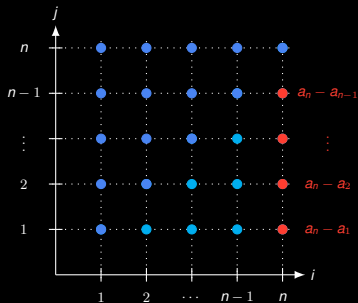
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Hence,

$$p(a_n) = \left(\prod_{1 \leq j < i \leq n-1} (a_i - a_j) \right) \times (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})$$



$$p(a_n) = \prod_{1 \leq j < i \leq n} (a_i - a_j).$$

□

E.g. In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3$$

giving us the *Vandermonde* determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

According to the previous theorem, this determinant is equal to

$$\begin{aligned} & (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ &= (1 - 0)(2 - 0)(2 - 1)(3 - 0)(3 - 1)(3 - 2) \\ &= 2 \times 3 \times 2 \\ &= 12. \end{aligned}$$

E.g. In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3$$

giving us the *Vandermonde* determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

According to the previous theorem, this determinant is equal to

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Corr. The Vandermonde determinant is nonzero if a_1, a_2, \dots, a_n are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with distinct x_i , then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-1}x^{n-1}.$$

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$$p(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-1}x^{n-1}.$$