Topics in Analysis and Linear Algebra

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Summer Bootcamp for
Emory Biostatistics and Bioinformatics
PhD Program

July 22 - 28, 2021

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https://github.com/chenle02/2021_Summer_Bio_bootcamp_slides/

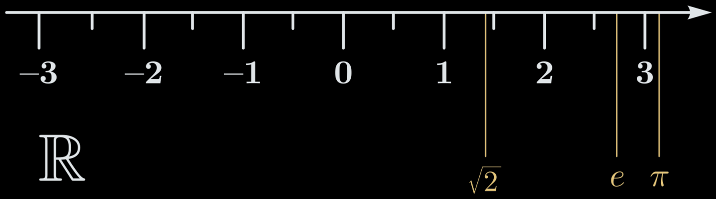
Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.

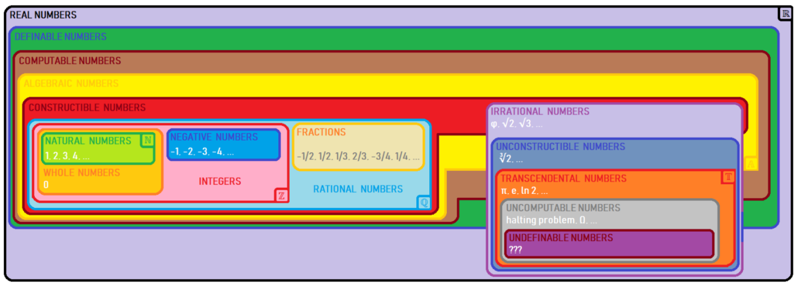
Chapter 3. Real Number System and Calculus

What is a real number?



[1]

[1] Image from Wikipedia.



[2]

[2] Image from
https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

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Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exit $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \qquad \text{(Identities)}$$

(F5) For each $x \in \mathbb{R}$, there exits a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exits an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

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Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

(O3) $x < y$ and $z > 0$ implies that $xz < yz$.

(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

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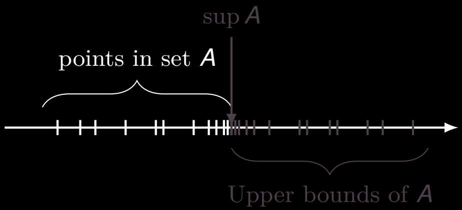
Completeness Axiom

Axiom A nonempty subset of real numbers that is bounded above has a least upper bound, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$



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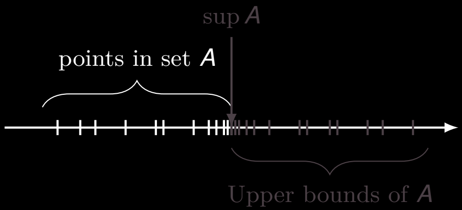

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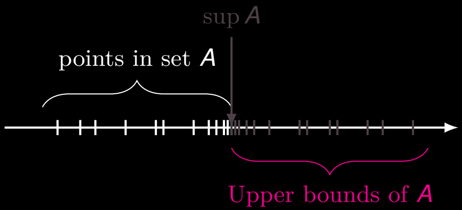

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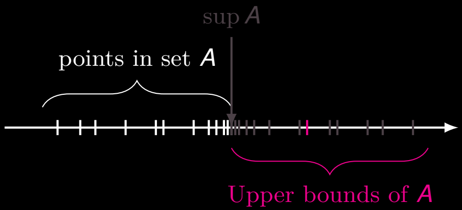

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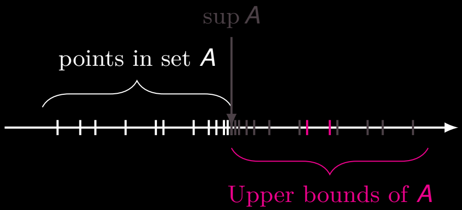

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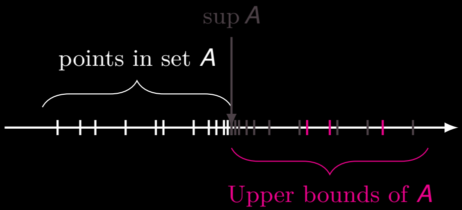

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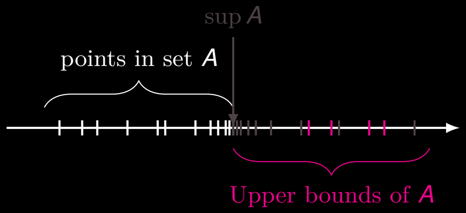

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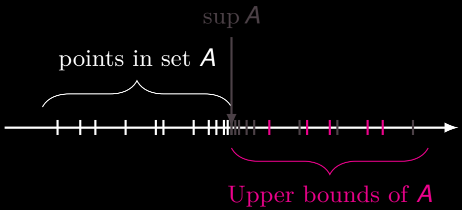

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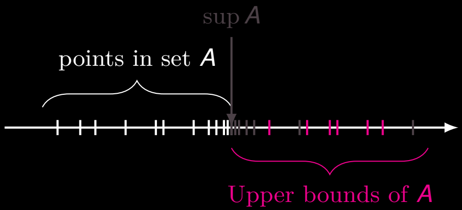

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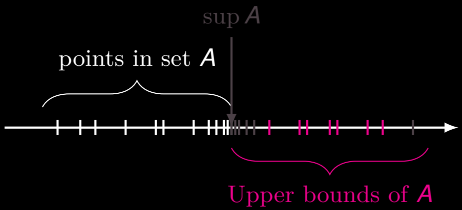

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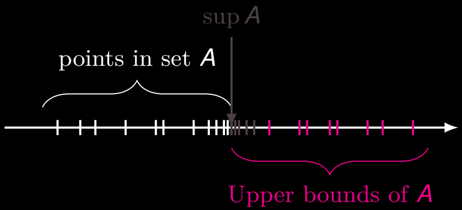

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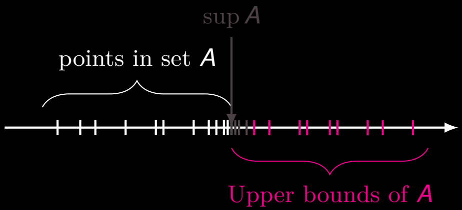

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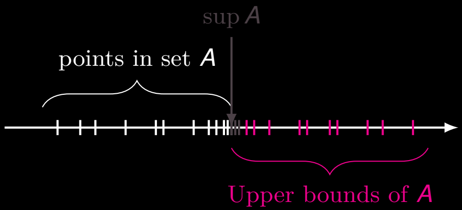

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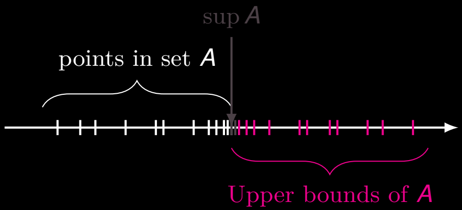

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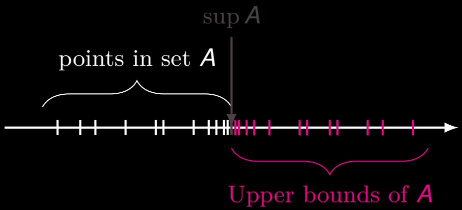

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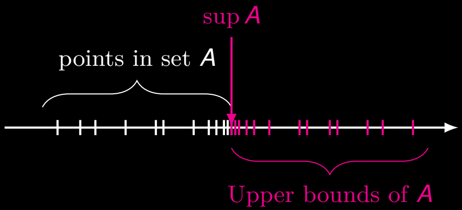

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$\begin{aligned}
\sup_n x_n \leq a &\iff \forall n,\ \ x_n \leq a \\
\sup_n x_n < a &\iff \forall n,\ \ x_n < a \\
a < \sup_n x_n &\iff \exists n,\ \ x_n > a
\end{aligned}$$

$$a \leq \sup_n x_n \iff \exists n,\ \ x_n \geq a$$

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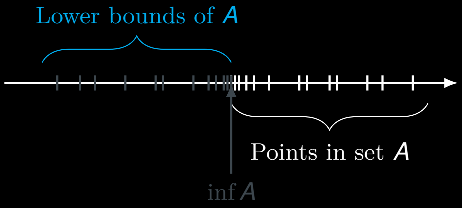

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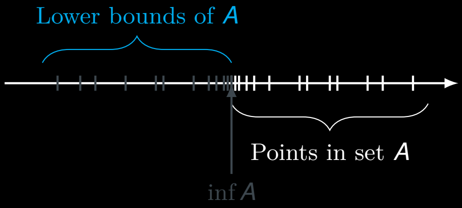

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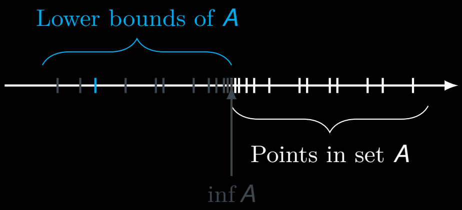

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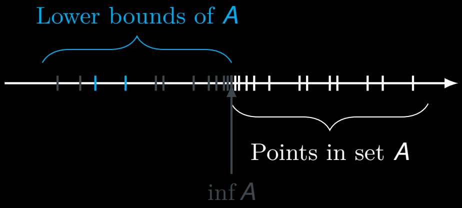

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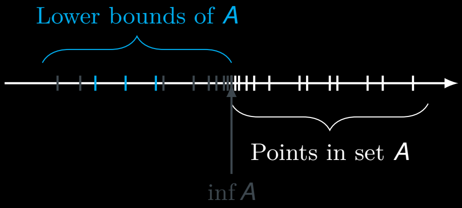

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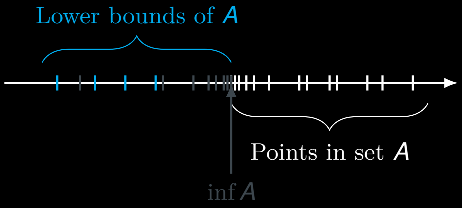

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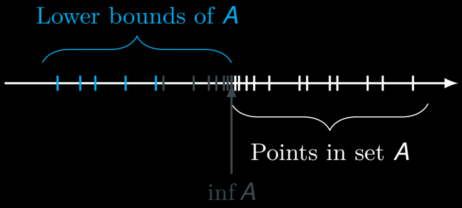

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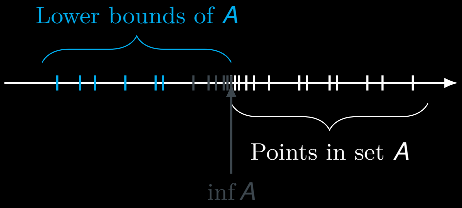

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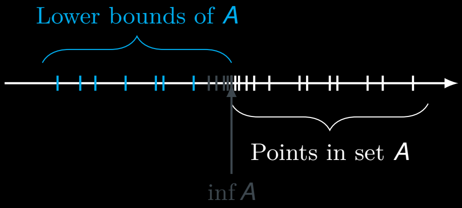

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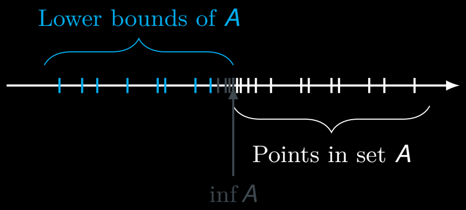

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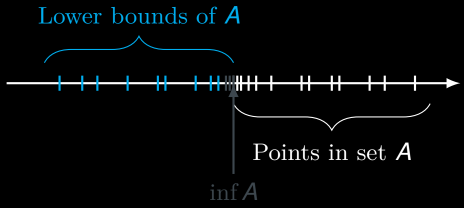

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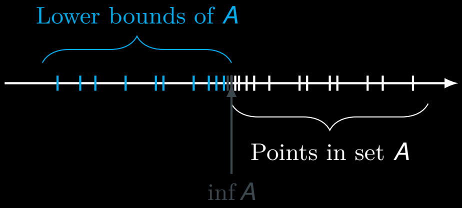

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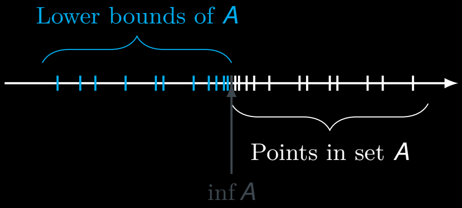

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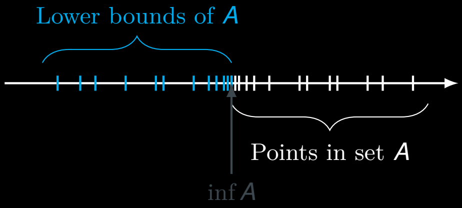

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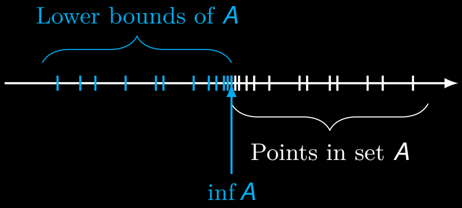

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

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\end{aligned}
$$

$$a \le \inf_n x_n \le a \quad \Longleftarrow \quad \exists n,\ \ x_n \le a$$

E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

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Properties

1. Archimedean principle

   For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

   Between any two real numbers there is an irrational number.

3. Density of the rational numbers

   Between any two real numbers there is an rational number.

Proof As exercises. $\square$

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Extended Real Number System

Def. The *extended real numbers* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
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$\infty - \infty$ cannot be defined (HW).

Def. Let a and b be extended real numbers such that $a < b$. Then the *intervals on $\mathbb{R}^*$* with endpoints a and b are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : \ a < x < b\}$$
$$[a, b) = \{x \in \mathbb{R}^* : \ a \leq x < b\}$$
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If both a and b are in $\mathbb{R}$, these intervals are the *bounded intervals* in $\mathbb{R}$.

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Otherwise, if either $a = -\infty$ or $b = \infty$, then these intervals are *unbounded intervals*.

Thm Every subset A of $\mathbb{R}^*$ has both a least upper bound and greatest lower bound. Moreover,

a) If $A = \emptyset$, then $\sup A = -\infty$ and $\inf A = \infty$.

b) If A is bounded above in $\mathbb{R}$, then $\sup A \in \mathbb{R}$; otherwise, $\sup A = \infty$.

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E.g.

a) $\inf \mathbb{N} = 1$ and $\sup \mathbb{N} = \infty$.

b) $\inf \mathbb{Z} = -\infty$ and $\sup \mathbb{Z} = \infty$.

c) If I is an interval in $\mathbb{R}^*$ with endpoints a and b, $a \leq b$. Then $\inf I = a$ and $\sup I = b$.

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HW Ex. 2.10 and 2.11 on p. 43.

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence[3], namely,

$$\lim_{n\to\infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n\to\infty} x_n = L \iff \forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \in \mathbb{N} \; s.t. \; (n \geq N \to |x_n - L| < \epsilon)$$

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

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Recall that $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ is the *extended real line*.

Def. A sequence of real numbers $\{a_1, a_2, \cdots\}$ is said to *converge in* $\mathbb{R}^*$ if one of the following three conditions hold:

(i) The sequence converges to a finite real number as in the previous definition.

(ii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

(iii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

(i) We say that the sequence converges in $\mathbb{R}$ or the limit exits and is finite.

(ii) We say the sequence converges to ∞ and write $\lim_{n\to\infty} x_n = \infty$.

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$\{(-1)^n\}_{n=1}^{\infty}$ does not converge in $\mathbb{R}^*$.

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Monotone sequence

Def. If $x_1 \leq x_2 \leq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \cdots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotone.

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Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$.

Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n\to\infty} x_n = \sup\{x_n :\ n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$ if it is bounded above and is ∞ otherwise.

b) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

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Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$. Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

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Proof. We will prove the case when $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. The nonincreasing case can be proved in a similar way.

As $\sup_n x_n$ always exists in $\mathbb{R}^*$, we need to consider two cases:

Case I: $\sup_n x_n \in \mathbb{R}$

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Let's prove Case I here. Let $x = \sup_n x_n$. In order to show that $\lim_n x_n = x$, by monotonicity, we need to show that

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Hence, there are infinitely many terms falling below $x - \epsilon$.

Since $\{x_n\}$ is nondecreasing, this implies all a_n fall below $x - \epsilon$, i.e.,

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which is equivalent to $\sup_n x_n < x - \epsilon$. This contradicts with the fact that $\sup_n x_n = x$.

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Cluster points

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.

b) ∞ *is a cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exits an $n \geq N$ such that $x_n > M$.

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E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞.

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Thm A convergent sequence has exactly one cluster point, namely, its limit. Thus, a sequence having more than noe cluster point cannot converge.

Proof Suppose that $\{x_n\}$ is a convergent sequence and let x be its limit, namely, $\lim_{n\to\infty} x_n = x$. We need to prove:

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We also need to consider two cases:

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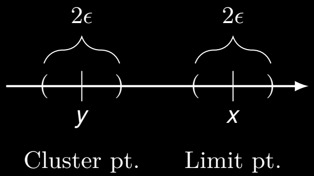
As for (2), suppose y is another cluster point.



By choosing any $\epsilon < |x - y|/2$, we see that

1. In the ϵ-neighborhood of y, there are infinitely many terms.
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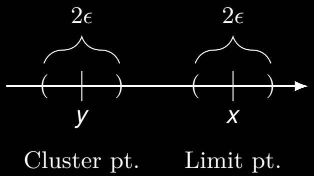

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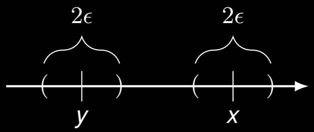

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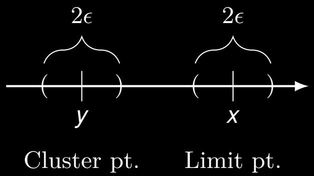

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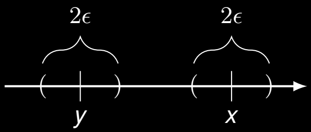

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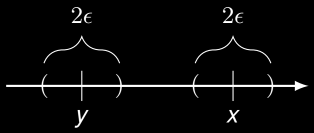
2. In the ϵ-neighborhood of x, all but finite many terms are here.

Contradiction!

Therefore, there exits only one cluster point. $\square$

As for (2), suppose y is another cluster point.



Cluster pt. Limit pt.

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A few more properties

1. A sequence is convergent iff each subsequence is convergent.

2. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \to L$ and $b_n \to L$, then $c_n \to L$.

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Limit superior and limit inferior

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

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Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

$$\inf_n y_n = \inf_n \sup_{m \geq n} x_m = \limsup_{x \to \infty} x_n$$

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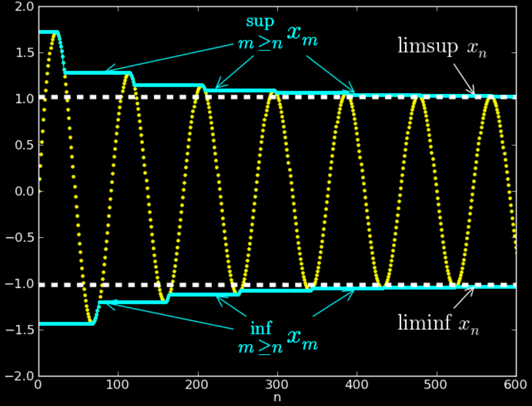

[4]

[4] Image from Wikipedia.

Other characterizations:

Thm $\limsup_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the smallest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

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E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1, amount which

$$\liminf_{n\to\infty} x_n = -1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi)\frac{n^2}{n^2+8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

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E.g.2 Consider the sequence $\left\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \cdots\right\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \\ (n+1)/3 & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

It has three cluster points: 1, 2, ∞, among which

$$\liminf_{n\to\infty} x_n = 1 \quad \text{and} \quad \limsup_{n\to\infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

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The above examples suggest that

Prop. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

(a) $\liminf x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.

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Properties

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$$\inf_n x_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

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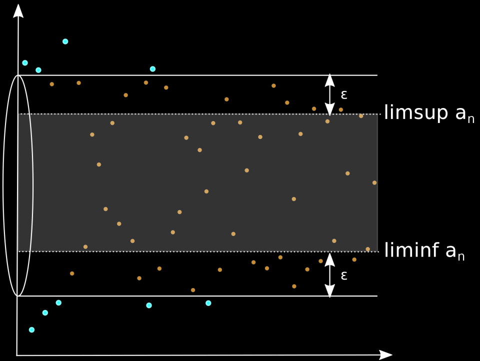

[5]

E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

[5] Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2}\left(\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n\right),$$

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As we have seen that

A sequence of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point.

There is another famous criterion for a sequence to converge in $\mathbb{R}$:

Cauchy Criterion

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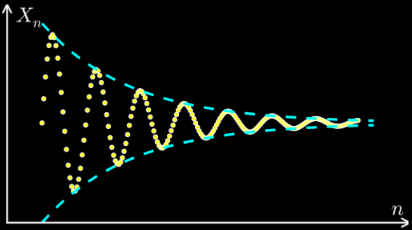
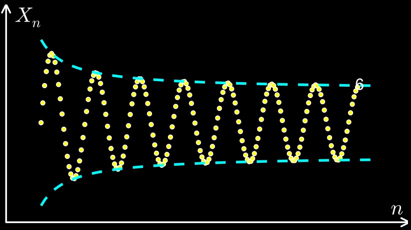
Cauchy sequence



Non-Cauchy sequence



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[6] Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall m, n \geq N \, \{|x_n - x_m| < \epsilon\}.$$

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A sequence of real numbers converges in $\mathbb{R}$ iff it is Cauchy.

E.g.1 Let $a_n = \sqrt{n}$. Show that

(i) The consecutive terms become arbitrarily close to each other as $n \to \infty$.

(ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg\left(\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall m, n \geq N \left\{|x_n - x_m| < \epsilon\right\}\right) \\ \iff \quad & \exists \epsilon > 0\, \forall N \in \mathbb{N}\, \exists m, n \geq N \left\{|x_n - x_m| > \epsilon\right\} \end{aligned}$$

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Indeed, let's choose $m = N$ and $n = 4N$

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$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e.$$

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HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exits an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of $\mathbb{R}$. Then K cannot be an open set. For example, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ are not open sets.

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E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

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Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

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Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

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Def. Let $G \subset D \subset \mathbb{R}$.

(a) G is said to be **open in D** if for each $x \in G$, there is an $r > 0$ such that

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E.g.

D	G	Is G open in $\mathbb{R}$	Is G open in D
$[0,2]$	$[0,1)$	Neither open nor closed	open
$[0,2]$	$[0,1]$	closed	closed
$\mathbb{N}$	$A \subset \mathbb{N}$	closed	open

Def. A *real-valued function* is a function whose range is a subset of $\mathbb{R}$. If $f : \Omega \to \mathbb{R}$, we say that f is a *real-valued function on* Ω.

Def. *Algebraic operations:* Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

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(f+g)(x) &:= f(x) + g(x) \\
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(Local) Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is *continuous at a point c* if the limit of $f(x)$, as x approaches c, exits and is equal to $f(c)$.

Def'. (Epsion-delta definition) The function f is *continuous at a point c* if

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Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in $\mathbb{R}$.

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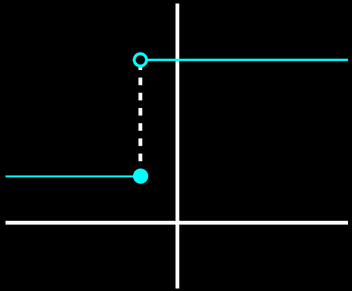

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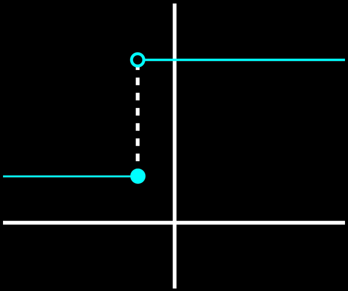

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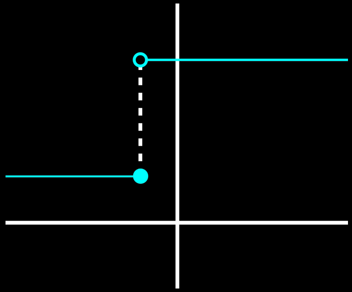

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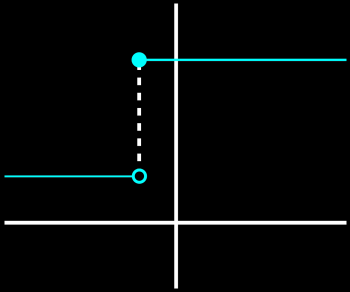

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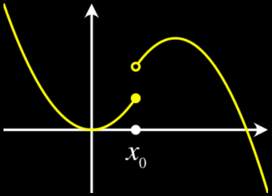

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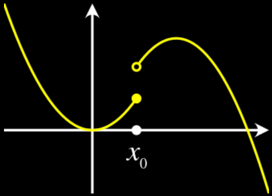

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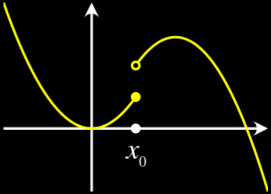

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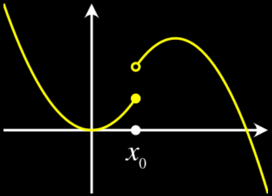

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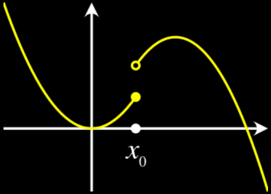

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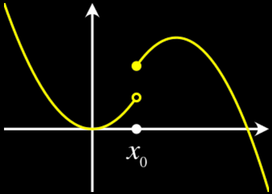

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(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let I be an interval of $\mathbb{R}$. Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exits a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

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Π_2-form

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Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I, we need to show

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Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

$\square$

Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

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Prop. 2 If I is compact [7] set such as $I = [a, b]$, then

$$f \text{ is continuous at all points in } I \iff f \text{ is uniformly continuous on } I.$$

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.
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Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

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Notation For $D \subset \mathbb{R}$, let $C(D)$ denote the set of continuous functions defined on D.

Thm (*Algebra of $C(D)$*) Let $D \subset \mathbb{R}$. Then the collection $C(D)$ of continuous functions on D is an algebra of functions, that is, for all $f, g \in C(D)$ and $\alpha \in \mathbb{R}$,

$$f + g \in C(D)$$
$$\alpha f \in C(D)$$
$$f \cdot g \in C(D)$$

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Def. $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on Ω, namely, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$.

We say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on* Ω if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}$.

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(a) $f_n \in C(\mathbb{R})$ defined as $f_n = (1 + x/n)^n$. Then f_n converges pointwise on $\mathbb{R}$ to $f(x) = e^x$. It is clear that $f \in C(\mathbb{R})$.

(b) Let $D = [0, 1]$ and $f_n \in C(D)$ be defined as $f_n = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

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Def. Let $\mathcal{F}$ be a collection of real-valued functions on Ω. We say that $\mathcal{F}$ is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \to f$ pointwise on Ω, then $f \in \mathcal{F}$.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω.

We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω, if

$$\boxed{\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \geq N}\, \forall x \in \Omega\ |f_n(x) - f(x)| < \epsilon,$$

written as $f_n \to f$ *uniformly*.

Recall $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if

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Prop. Let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \to f$ uniformly. Then $f \in C(D)$.

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Therefore, the collection $C(D)$ of real-valued continuous functions is closed under: $+$, $\cdot$, scalar multiplication, and uniform convergence.

Some part of subsection is taken from Chapter 1 Section 4 of

P. Billingsley, **Probability and Measure**, Wiley, 1995.

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

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Use the relation:

set	logic
$\bigcap$	$\forall$
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$$\omega \in \limsup_n A_n \iff \omega \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$$

$$\iff (\forall n \geq 1)\,(\exists k \geq n)\,\omega \in A_k$$

$$\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n$$

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Def. If both $\limsup_n A_n$ and $\liminf_n A_n$ exist and are equal, then the *limit set* of the sequence $\{A_n\}$ is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

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Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$
\begin{array}{ccccccc}
\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \liminf_n A_n & \Longrightarrow & \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
\subseteq & & & & \\
A_n & & & & \subseteq \\
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\end{array}
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Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$
\begin{array}{ccccc}
\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) & = & \mathbb{P}\left(\liminf_n A_n\right) \\
 & & \rotatebox{-90}{$\leq$} & & \\
\rotatebox{-90}{$\leq$} & & \liminf_{n\to\infty} \mathbb{P}(A_n) & & \\
\mathbb{P}\left(A_n\right) & & \rotatebox{-90}{$\leq$} & & \\
\rotatebox{-90}{$\leq$} & & \limsup_{n\to\infty} \mathbb{P}(A_n) & & \\
 & & \rotatebox{-90}{$\leq$} & & \\
\mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) & \downarrow & \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) & = & \mathbb{P}\left(\limsup_n A_n\right)
\end{array}
$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned}
1 \geq \mathbb{P}\left(A_n^c \text{ all but finitely many}\right) &= 1 - \mathbb{P}\left(\left\{A_n^c \text{ all but finitely many}\right\}^c\right) \\
&= 1 - \mathbb{P}\left(A_n \text{ i.o.}\right) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \\
&\geq 1 - \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\
&= 1 - 0 = 1.
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Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

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\begin{aligned}
A_1 &= (-1, 0] \\
A_2 &= \left(-\frac{1}{2}, \frac{1}{2}\right] \\
A_3 &= \left(-\frac{1}{3}, \frac{2}{3}\right] \\
A_4 &= \left(-\frac{1}{4}, \frac{3}{4}\right] \\
\vdots &\quad \vdots \\
A_{100} &= \left(-\frac{1}{100}, \frac{99}{100}\right] \\
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\end{aligned}
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Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n}\right] = [0, 1)$$

and

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Finally,

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(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3} \right] \qquad A_4 = \left(\frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100} \right]$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

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Sol.

$$
\begin{aligned}
&\liminf_n A_n \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
&= (0,1) \cup (0,1) \\
&= (0,1)
\end{aligned}
$$

Sol. (continued) Similarly,

$$
\begin{aligned}
&\limsup_n A_n \\
=&\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right] \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\} \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\left(-\frac{1}{n+1},\frac{n+2}{n+1}\right]\right\} \\
=&[0,1]\cap[0,1] \\
=&[0,1]
\end{aligned}
$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \dfrac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\[2ex] \left[-1 - \dfrac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$.

Solution:

$$\liminf_{n\to\infty} A_n = \{0\} \subset [0, 1] = \limsup_{n\to\infty} A_n$$

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Chapter 3. Real Number System and Calculus

Examples

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2. $\int_0^x t^2 e^t dt$
3. $\int e^x \sin(x)dx$

more to come ...

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2. $\sin(x)$
3. e^{x^2}

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Taylor expansions

Examples

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more to come ...