

# Topics in Analysis and Linear Algebra

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Summer Bootcamp for  
Emory Biostatistics and Bioinformatics  
PhD Program

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## Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

*J. McDonald and N. Weiss, **A course in real analysis***, Academic Press, 2005.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

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What is a real number?



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<sup>1</sup>Image from Wikipedia.



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<sup>2</sup>Image from

<https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/>

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms



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(C) Completeness Axioms

## Field Axioms

Let  $x, y, z \in \mathbb{R}$ . Then we have that

(F1)  $x + y = y + x$  and  $xy = yx$ . (Commutative)

(F2)  $(x + y) + z = x + (y + z)$  and  $(xy)z = x(yz)$ . (Associative)

(F3)  $x(y + z) = xy + xz$ . (Distributive)

(F4) There exist  $0, 1 \in \mathbb{R}$  with  $0 \neq 1$  such that for all  $x \in \mathbb{R}$

$x + 0 = x$  and  $x \cdot 1 = x$ . (Identities)

(F5) For each  $x \in \mathbb{R}$ , there exists a  $-x \in \mathbb{R}$  such that  $x + (-x) = 0$  and, if  $x \neq 0$ , there exists an  $x^{-1} \in \mathbb{R}$  such that  $xx^{-1} = 1$ . (Inverses)

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## Order Axioms

Let  $x, y, z \in \mathbb{R}$ . Then we have that

(O1)  $x < y$  and  $y < z$  implies that  $x < z$ . (Transitive)

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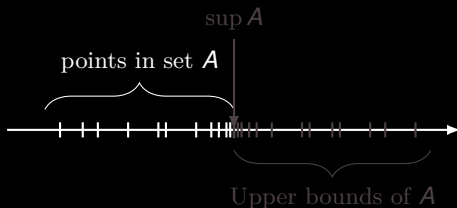
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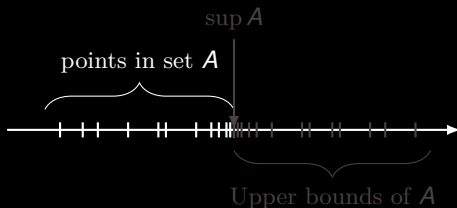
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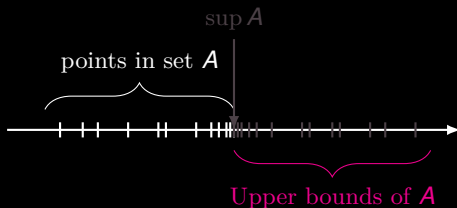
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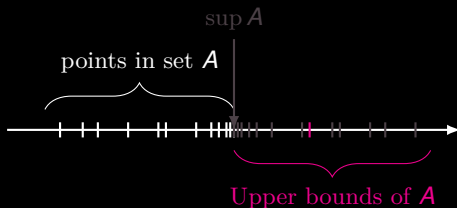
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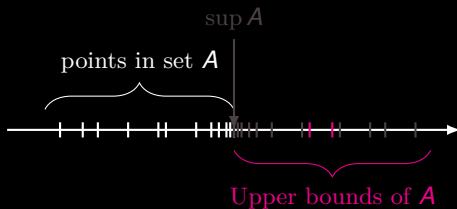
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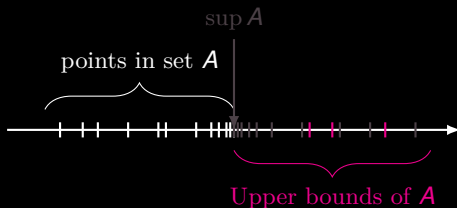




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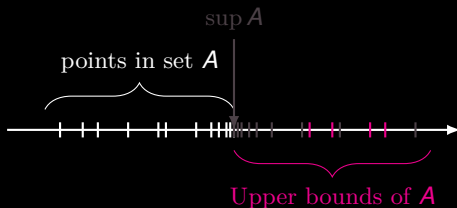
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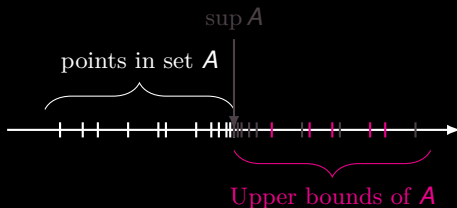
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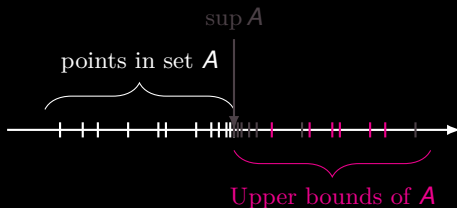
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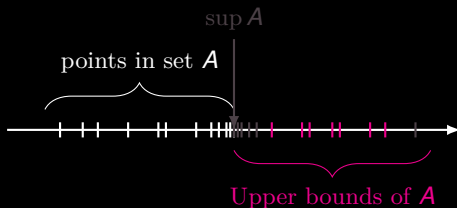
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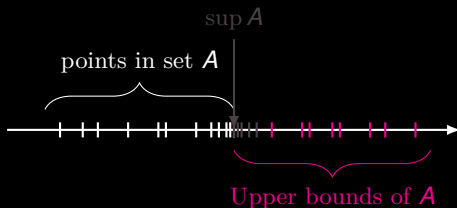
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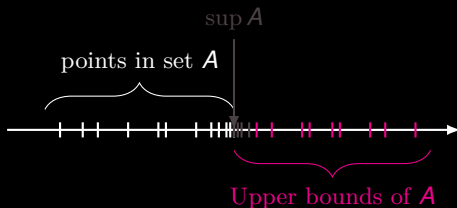
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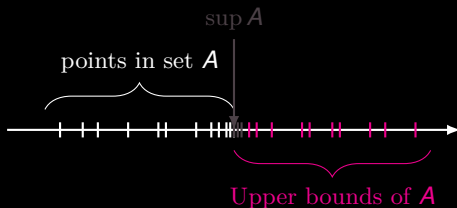
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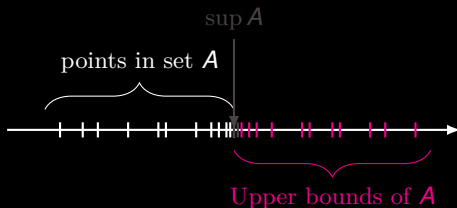




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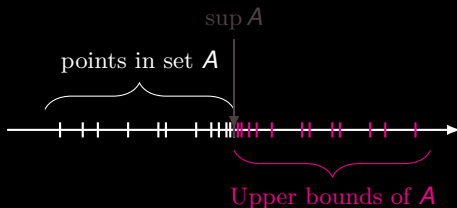
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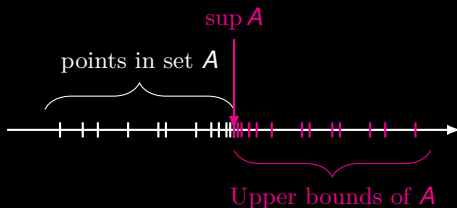
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Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence and  $a \in \mathbb{R}$ . Then

$$\sup_n x_n \leq a \iff \forall n, x_n \leq a$$

$$\sup_n x_n < a \iff \forall n, x_n < a$$

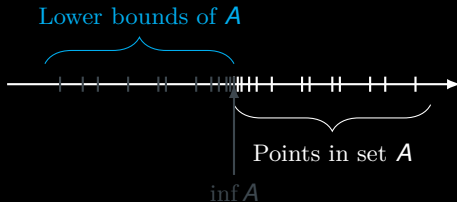
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$$a \leq \sup_n x_n \iff \exists n, x_n \geq a$$

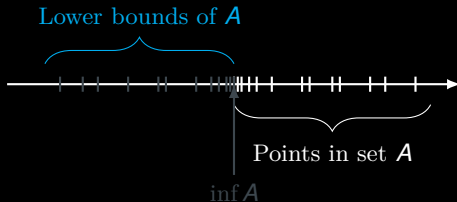
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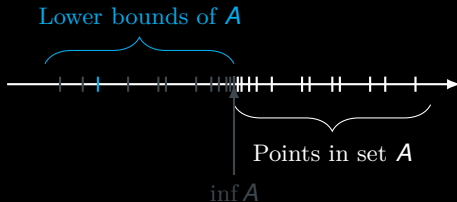
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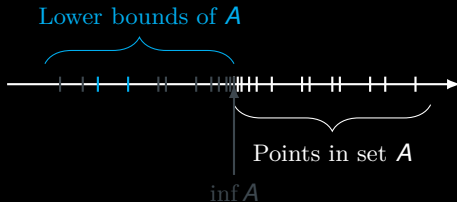
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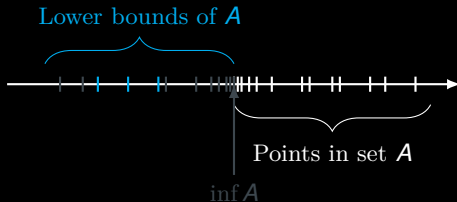
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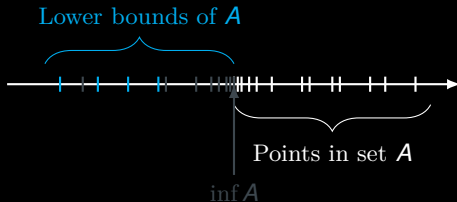
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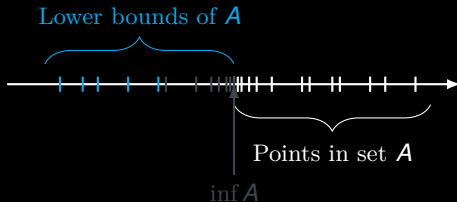
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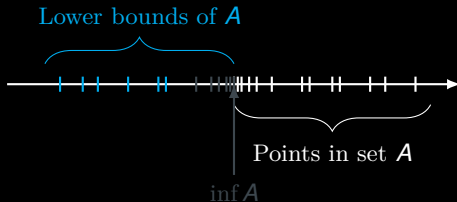
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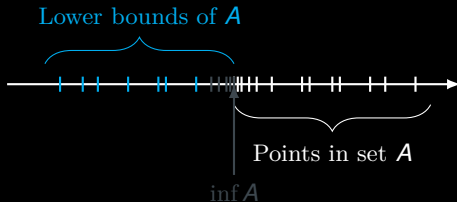
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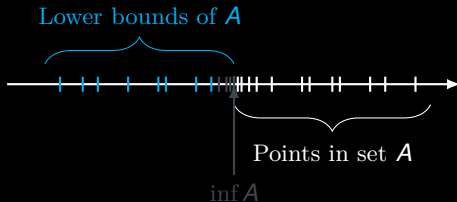
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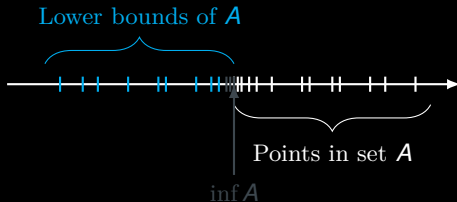
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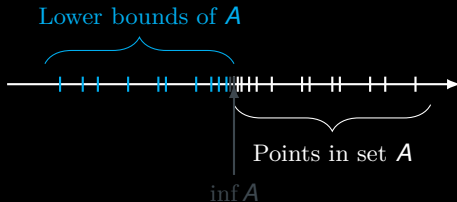
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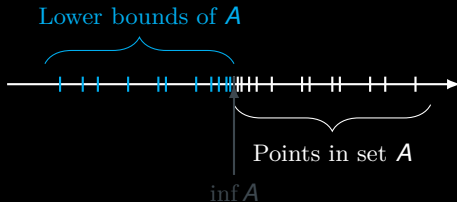
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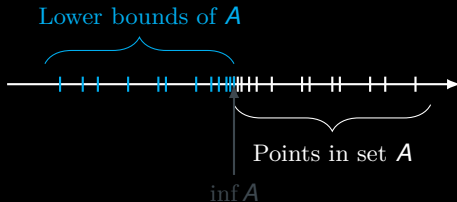
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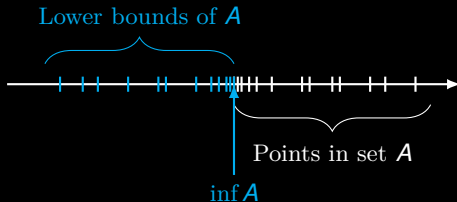
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E.g.  $\sup[0, 1) = 1$  and  $\inf[0, 1) = 0$ .

$\mathbb{N}$  has no least upper bound, but  $\inf \mathbb{N} = 1$ .

Let  $A = \{x : x^2 < 3\}$ . Then

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## Properties

### 1. Archimedean principle

For each  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{N}$  such that  $n > x$ .

### 2. Density of the irrational numbers

Between any two real numbers there is an irrational number.

### 3. Density of the rational numbers

Between any two real numbers there is a rational number.

Proof As exercises.





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# Extended Real Number System

Def. The *extended real numbers*  $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ .

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
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Def. Let  $a$  and  $b$  be extended real numbers such that  $a < b$ . Then the *intervals on  $\mathbb{R}^*$*  with *endpoints*  $a$  and  $b$  are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R}^* : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R}^* : a < x \leq b\}$$

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If both  $a$  and  $b$  are in  $\mathbb{R}$ , these intervals are the *bounded intervals* in  $\mathbb{R}$ .

Otherwise, if either  $a = -\infty$  or  $b = \infty$ , then these intervals are *unbounded intervals*.

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**Thm** Every subset  $A$  of  $\mathbb{R}^*$  has both a least upper bound and greatest lower bound. Moreover,

- a) If  $A = \emptyset$ , then  $\sup A = -\infty$  and  $\inf A = \infty$ .
- b) If  $A$  is bounded above in  $\mathbb{R}$ , then  $\sup A \in \mathbb{R}$ ; otherwise,  $\sup A = \infty$ .
- c) If  $A$  is bounded below in  $\mathbb{R}$ , then  $\inf A \in \mathbb{R}$ ; otherwise,  $\inf A = -\infty$ .

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E.g.

a)  $\inf \mathbb{N} = 1$  and  $\sup \mathbb{N} = \infty$ .

b)  $\inf \mathbb{Z} = -\infty$  and  $\sup \mathbb{Z} = \infty$ .

c) If  $I$  is an interval in  $\mathbb{R}^*$  with endpoints  $a$  and  $b$ ,  $a \leq b$ . Then  $\inf I = a$  and  $\sup I = b$ .

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HW Ex. 2.10 and 2.11 on p. 43.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. We say that the real number  $L \in \mathbb{R}$  is the *limit* of this sequence<sup>3</sup>, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number  $\epsilon > 0$ , there exists a natural number  $N$  such that for all  $n > N$ , we have  $|x_n - L| < \epsilon$ .

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

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<sup>3</sup>In this case, we say that  $\{x_n\}_{n=1}^{\infty}$  is *convergent*. Otherwise, we say that  $\{x_n\}_{n=1}^{\infty}$  is *divergent*.

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E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is convergent and  $\lim_{n \rightarrow \infty} (n-1)/n = 1$ .

$\{(-1)^n\}_{n=1}^{\infty}$  is divergent.

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Def. Let  $\mathbb{R}^*$  denote *the extended real line*, namely,  $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ .

Def. A sequence of real numbers  $\{a_1, a_2, \dots\}$  is said to *converge in  $\mathbb{R}^*$*  if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
  - (ii) For each  $M \in \mathbb{R}$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n > M$ .
  - (iii) For each  $M \in \mathbb{R}$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n < M$ .
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- (i) We say that the sequence converges in  $\mathbb{R}$ .
- (ii) Denoted as  $\lim_{n \rightarrow \infty} x_n = \infty$
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$\{(-1)^n\}_{n=1}^{\infty}$  does not converge in  $\mathbb{R}^*$ .

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If  $x_1 \geq x_2 \geq \cdots$ , then  $\{x_n\}_{n=1}^{\infty}$  is said to be *nonincreasing*.

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E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is monotone and it is nondecreasing.

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**Prop.** Any monotone sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers converges in  $\mathbb{R}^*$ .

Moreover,

(a) If  $\{x_n\}_{n=1}^{\infty}$  is nondecreasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, if  $\{x_n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$ , it is bounded above.

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$\{x_n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}^*$   $\iff$   $\{x_n\}_{n=1}^{\infty}$  has exactly one cluster point in  $\mathbb{R}^*$ .

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2. Sandwich theorem: If  $x_n \leq c_n \leq b_n$  for all  $n > N$  and  $x_n \rightarrow L$  and  $b_n \rightarrow L$ , then  $c_n \rightarrow L$ .

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Remark Since the sequences  $\{y_n\}_{n=1}^{\infty}$  and  $\{z_n\}_{n=1}^{\infty}$  defined as

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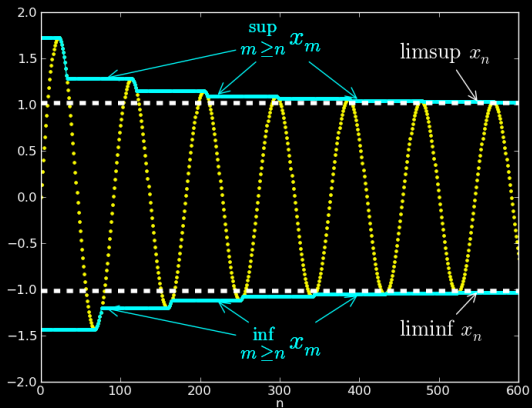
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<sup>4</sup>Image from Wikipedia.



Can we use the first-order logic to formulate the definitions?

Def'.  $\limsup_{n \rightarrow \infty} x_n = x \in \mathbb{R}$  if and only if  $x$  is the smallest real number such that for any positive real number  $\epsilon > 0$ , there exists a natural number  $N$  such that  $x_n < x + \epsilon$  for all  $n > N$ .

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$$x_n = (-1)^n \frac{n+5}{n}$$

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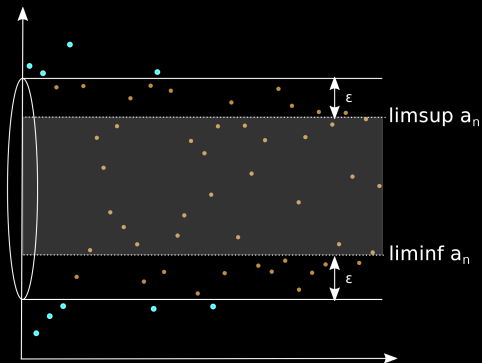
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2. A sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers **converges in  $\mathbb{R}^*$**  if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = c \iff \lim_{n \rightarrow \infty} x_n = c.$$



5

E.g. For all  $\epsilon > 0$ , the interval

$$\left( \liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in  $\{x_n\}$ .

---

<sup>5</sup>Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all  $\epsilon$  with

$$0 < \epsilon < \frac{1}{2} \left( \limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in  $\{x_n\}$  fall outside of the interval

$$\left( \liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right).$$

As we have seen that

A sequence of real numbers converges in  $\mathbb{R}^*$  if and only if it has exactly one cluster point.

There is another famous criterion for a sequence to converge in  $\mathbb{R}$ :

### Cauchy Criterion



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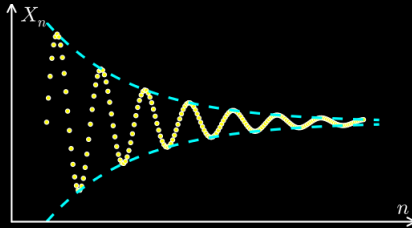
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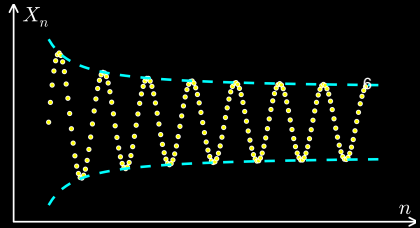
There is another famous criterion for a sequence to **converge in  $\mathbb{R}$** :

### **Cauchy Criterion**

Cauchy sequence



Non-Cauchy sequence



---

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Def. A sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers is called a *Cauchy sequence* if for each  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have  $|x_n - x_m| < \epsilon$ .

Def.' A sequence  $\{x_n\}_{n=1}^{\infty}$  is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{ |x_n - x_m| < \epsilon \}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in  $\mathbb{R}$  iff it is Cauchy.

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Thm (Cauchy Criterion)

A sequence of real numbers *converges in  $\mathbb{R}$*  iff it is Cauchy.

E.g.1 Let  $a_n = \sqrt{n}$ . Show that

- (i) The consecutive terms become arbitrarily close to each other as  $n \rightarrow \infty$ .
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show  $\{a_n\}_{n=1}^{\infty}$  is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$



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Sol. (Continued) Let's choose  $\epsilon = 1$ . For any  $N \in \mathbb{N}$ , we need to find  $m, n \geq N$  such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose  $m = N$  and  $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

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E.g.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Variation

$$\lim_{n \rightarrow \infty} \left(1 + \text{Small}\right)^{\text{Large}} = e^{\lim_{n \rightarrow \infty} \text{Small} \times \text{Large}}$$

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# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. A subset  $O \subset \mathbb{R}$  is said to be an *open set* if for each  $x \in O$ , there exists an  $r > 0$  such that  $(x - r, x + r) \subset O$ .

E.g.  $(a, b)$  with  $-\infty \leq a < b \leq \infty$  is an open set, which are called *open interval intervals*.

$(0, 1]$  is not an open set.

Let  $K$  be a nonempty countable subset of  $\mathbb{R}$ . Then  $K$  cannot be an open set. For example,  $\mathbb{N}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}$  are not open sets.

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## Properties

1.  $\mathbb{R}$  and  $\emptyset$  are open sets.
2. If  $A$  and  $B$  are open sets, so is  $A \cap B$ . (finite intersection)
3. If  $\{O_i\}_{i \in I}$  is a collection of open sets, then  $\bigcup_{i \in I} O_i$  is open.  
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Def. Let  $E \subset \mathbb{R}$ . A real number  $x$  is called a *limit point* of  $E$  if for each  $\epsilon > 0$ , there is a  $y \in E$  such that  $|y - x| < \epsilon$ .

The set of all limit point of  $E$ , denoted  $\overline{E}$ , is called the *closure* of  $E$ .

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Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then

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Def. A subset  $F \subset \mathbb{R}$  is said to be a *closed set* if  $\overline{F} = F$ , i.e,  $F$  contains all its limit points.

E.g.  $\mathbb{R}$  and  $\emptyset$  are both open and closed.

Intervals such as  $[a, b]$ ,  $[a, \infty)$ ,  $(-\infty, b]$  with  $a, b \in \mathbb{R}$  are closed sets. They are called *closed intervals*.

$\mathbb{N}$  and  $\mathbb{Z}$  are closed sets.

The sets of rationals  $\mathbb{Q}$  and irrationals  $\mathbb{Q}^c$  are neither open nor close.

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## Properties

0. A set is open if and only if its complement is closed.



1.  $\mathbb{R}$  and  $\emptyset$  are closed sets.

2. If  $A$  and  $B$  are closed sets, so is  $A \cup B$ . (finite union)

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**Def.** Let  $G \subset D \subset \mathbb{R}$ .

(a)  $G$  is said to be open in  $D$  if for each  $x \in G$ , there is an  $r > 0$  such that

$$(x - r, x + r) \cap D \subset G.$$

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E.g.

$D$	$G$	Is $G$ open in $\mathbb{R}$	Is $G$ open in $D$
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
$\mathbb{N}$	$A \subset \mathbb{N}$	closed	open

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. A *real-valued function* is a function whose range is a subset of  $\mathbb{R}$ . If  $f : \Omega \rightarrow \mathbb{R}$ , we say that  $f$  is a *real-valued function on  $\Omega$* .

Def. *Algebraic operations*: Let  $f, g$  be real-valued functions on  $\Omega$  and let  $\alpha \in \mathbb{R}$ . Then for all  $x \in \Omega$ ,

$$(f + g)(x) := f(x) + g(x)$$

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## (Local) Continuity

Def. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function.  $f$  is *continuous at a point  $c$*  if the limit of  $f(x)$ , as  $x$  approaches  $c$ , exists and is equal to  $f(c)$ .

Def'. (Epsilon-delta definition) The function  $f$  is *continuous at a point  $c$*  if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} \{ |x - c| \leq \delta \rightarrow |f(x) - f(c)| \leq \epsilon \}$$

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Def. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function.  $f$  is *continuous at a point  $c$*  if the limit of  $f(x)$ , as  $x$  approaches  $c$ , exists and is equal to  $f(c)$ .

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$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} \{ |x - c| \leq \delta \rightarrow |f(x) - f(c)| \leq \epsilon \}$$

Here is a more abstract definition of continuous functions:

Thm let  $D \subset \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ . Then  $f$  is continuous on  $D$  if and only if  $f^{-1}(O)$  is open in  $D$  for each open set  $O$  in  $\mathbb{R}$ .

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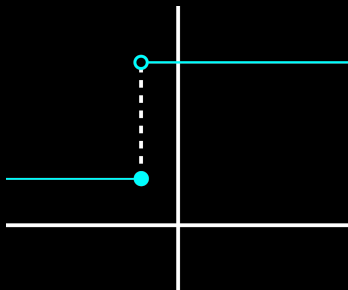
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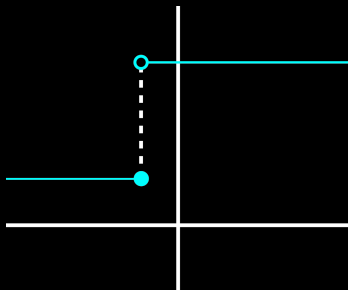


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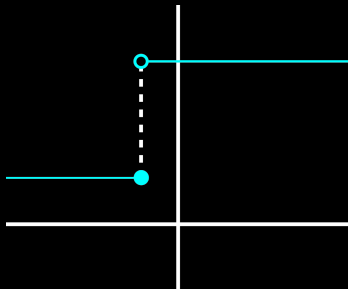
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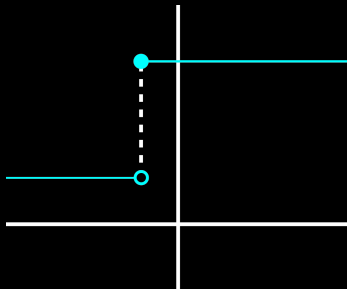
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Def.  $f$  is *lower semi-continuous at  $x_0$*  if

$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$$

$f(x_0)$  can be all points  
at or below the blue point.

$f$  is *upper semi-continuous at  $x_0$*  if

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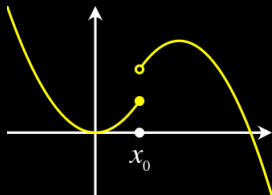
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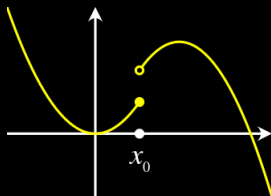
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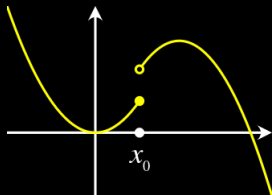
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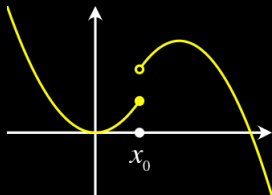
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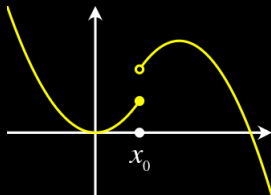
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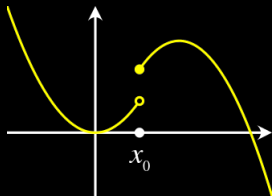
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## (Global) Uniform Continuity

Def. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $I$  be an interval of  $\mathbb{R}$ . Then  $f$  is *uniformly continuous over  $I$*  if for every real number  $\epsilon > 0$ , there exists a real number  $\delta > 0$  such that for every  $x, y \in I$  with  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .

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## Properties

**Prop. 1** Every uniformly continuous function is continuous, but the converse does not hold.

Ex.  $f(x) = x^3$  is a continuous functions on  $I = \mathbb{R}$ . Show that  $f$  is not uniformly continuous on  $I = \mathbb{R}$ .

Sol. In order to show  $f$  is not uniformly continuous on  $I$ , we need to show

$$\begin{aligned} & \neg \left( \forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \right) \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ \neg \{ |x - y| < \delta \} \vee |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \{ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon \} \end{aligned}$$

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Let's choose  $\epsilon = 1$ . For any  $\delta > 0$ , we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

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$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left( \frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

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Prop. 2 If  $I$  is compact<sup>7</sup> set such as  $I = [a, b]$ , then

$f$  is continuous at all points in  $I \iff f$  is uniformly continuous on  $I$ .

E.g.  $f(x) = 1/x$  is not uniformly continuous on  $(0, 1)$ .

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Why does it matter at all?

**Answer:** Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

**Thm** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a uniformly continuous function and let  $\{x_n\}_{n=1}^{\infty}$  be a Cauchy sequence. Then  $\{f(x_n)\}_{n=1}^{\infty}$  is also a Cauchy sequence.

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**Notation** For  $D \subset \mathbb{R}$ , let  $C(D)$  denote the set of continuous functions defined on  $D$ .

**Thm (*Algebra of  $C(D)$* )** Let  $D \subset \mathbb{R}$ . Then the collection  $C(D)$  of continuous functions on  $D$  is an algebra of functions, that is, for all  $f, g \in C(D)$  and  $\alpha \in \mathbb{R}$ ,

$$f + g \in C(D)$$

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**Remark** Can one add one more operation in this algebra: Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of functions in  $C(D)$ , under what condition the limit  $f_n \rightarrow f$  is closed?

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E.g.

(a)  $f_n \in C(\mathbb{R})$  defined as  $f_n = (1 + x/n)^n$ . Then  $f_n$  converges pointwise on  $\mathbb{R}$  to  $f(x) = e^x$ . It is clear that  $f \in C(\mathbb{R})$ .

(b) Let  $D = [0, 1]$  and  $f_n \in C(D)$  be defined as  $f_n = x^n$ . Then  $f_n$  converges pointwise to

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Def. Let  $\mathcal{F}$  be a collection of real-valued functions on  $\Omega$ . We say that  $\mathcal{F}$  is *closed under pointwise limits* if whenever  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$  and  $f_n \rightarrow f$  pointwise on  $\Omega$ , then  $f \in \mathcal{F}$ .

**Def.** Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of real-valued functions on a set  $\Omega$ .

We say that  $\{f_n\}_{n=1}^{\infty}$  *converges uniformly* to the real-valued function  $f$  on  $\Omega$ , if

$$\boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad \forall x \in \Omega \quad |f_n(x) - f(x)| < \epsilon,$$

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Recall  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to  $f$  if

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Recall  $\{f_n\}_{n=1}^{\infty}$  converges pointwise to  $f$  if

$$\forall x \in \Omega \quad \boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad |f_n(x) - f(x)| < \epsilon,$$

Def. Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of real-valued functions on a set  $\Omega$ .

We say that  $\{f_n\}_{n=1}^{\infty}$  *converges uniformly* to the real-valued function  $f$  on  $\Omega$ , if

$$\boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad \forall x \in \Omega \quad |f_n(x) - f(x)| < \epsilon,$$

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**Prop.** Let  $D \subset \mathbb{R}$ . Suppose that  $\{f_n\}_{n=1}^{\infty} \subset C(D)$  and that  $f_n \rightarrow f$  uniformly.  
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**Proof.**



Therefore, the collection  $\mathcal{C}(D)$  of real-valued continuous functions is closed under:  $+$ ,  $-$ , scalar multiplication, and uniform convergence.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Some part of subsection is taken from Chapter 1 Section 4 of

*P. Billingsley, **Probability and Measure**, Wiley, 1995.*

Def. For a sequence  $A_1, A_2 \cdots$  of sets, define the *limits superior and inferior* of the sequence  $\{A_n\}$  as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

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Remark Both  $\limsup_n A_n$  and  $\liminf_n A_n$  are sets.

Use the relation:

set	logic
$\cap$	$\forall$
$\cup$	$\exists$

$$\begin{aligned}
 \omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
 &\iff (\forall n \geq 1) (\exists k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\limsup_n A_n = [A_n \text{ i.o.}]$$

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Def. If both  $\limsup_n A_n$  and  $\liminf_n A_n$  exist and are equal, then the *limit set* of the sequence  $\{A_n\}$  is defined to be

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## Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left( \limsup_n A_n^c \right)^c$$

## Properties

(ii) Monotone increasing and decreasing sets:

$$\begin{array}{ccc}
 \left( \bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \liminf_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
 \cap & & \cap \\
 A_n & & \\
 \cap & & \\
 \left( \bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \limsup_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n
 \end{array}$$

## Interpretation Under a Probability Space

Suppose that  $\{A_n\}$  are events from a probability space  $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$\begin{array}{ccc}
 \mathbb{P} \left( \bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \mathbb{P} \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \mathbb{P} \left( \liminf_n A_n \right) \\
 & & \quad \quad \quad \uparrow \wedge \\
 & & \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 \mathbb{P}(A_n) & & \quad \quad \quad \uparrow \wedge \\
 & & \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 & & \quad \quad \quad \uparrow \wedge \\
 \mathbb{P} \left( \bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \mathbb{P} \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \mathbb{P} \left( \limsup_n A_n \right)
 \end{array}$$



(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned} 1 &\geq \mathbb{P}(A_n^c \text{ all but finitely many}) = 1 - \mathbb{P}(\{A_n^c \text{ all but finitely many}\}^c) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \\ &\geq 1 - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

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### Exercise

(i) Let  $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$ :

$$A_1 = (-1, 0]$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{2}{3}\right]$$

$$A_4 = \left(-\frac{1}{4}, \frac{3}{4}\right]$$

$$\vdots \quad \vdots$$

$$A_{100} = \left(-\frac{1}{100}, \frac{99}{100}\right]$$

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Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left( -\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[ 0, \frac{n-1}{n} \right] = [0, 1)$$

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(ii) Let  $A_n = \left( \frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$ :

$$A_1 = (-1, 2] \qquad A_2 = \left( \frac{1}{2}, \frac{1}{2} \right]$$

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$$A_5 = \left( -\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left( \frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$A_{99} = \left( -\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left( \frac{1}{100}, \frac{99}{100} \right]$$

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Show that  $\lim_n A_n$  doesn't exist by demonstrating that

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Sol.

$$\begin{aligned}& \liminf_n A_n \\&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left( \frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left( \frac{1}{n}, \frac{n-1}{n} \right] \right\} \\&= (0, 1) \cup (0, 1) \\&= (0, 1)\end{aligned}$$

Sol. (continued) Similarly,

$$\begin{aligned}
 & \limsup_n A_n \\
 &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \bigcup_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \bigcup_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \left( -\frac{1}{n}, \frac{n+1}{n} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \left( -\frac{1}{n+1}, \frac{n+2}{n+1} \right] \right\} \\
 &= [0, 1] \cap [0, 1] \\
 &= [0, 1]
 \end{aligned}$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \frac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \frac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine  $\liminf_{n \rightarrow \infty} A_n$  and  $\limsup_{n \rightarrow \infty} A_n$ .

Solution:

$$\liminf_{n \rightarrow \infty} A_n = \{0\} \subset [0, 1] = \limsup_{n \rightarrow \infty} A_n$$

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# Integration by parts

## Examples

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3.  $\int e^x \sin(x) dx$

more to come ...

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