

# Topics in Analysis and Linear Algebra

Le Chen

le.chen@emory.edu

Emory University

Atlanta GA

Last updated on

July 29, 2021

Summer Bootcamp for  
Emory Biostatistics and Bioinformatics  
PhD Program

July 22 - 28, 2021



## Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

*J. McDonald and N. Weiss, **A course in real analysis***, Academic Press, 2005.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

What is a real number?



1

---

<sup>1</sup>Image from Wikipedia.



2

<sup>2</sup>Image from

<https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/>

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms



## Field Axioms

Let  $x, y, z \in \mathbb{R}$ . Then we have that

$$(F1) \quad x + y = y + x \text{ and } xy = yx. \quad (\text{Commutative})$$

$$(F2) \quad (x + y) + z = x + (y + z) \text{ and } (xy)z = x(yz). \quad (\text{Associative})$$

$$(F3) \quad x(y + z) = xy + xz. \quad (\text{Distributive})$$

$$(F4) \quad \text{There exist } 0, 1 \in \mathbb{R} \text{ with } 0 \neq 1 \text{ such that for all } x \in \mathbb{R}$$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \quad (\text{Identities})$$

$$(F5) \quad \text{For each } x \in \mathbb{R}, \text{ there exists a } -x \in \mathbb{R} \text{ such that } x + (-x) = 0 \text{ and, if } x \neq 0, \text{ there exists an } x^{-1} \in \mathbb{R} \text{ such that } xx^{-1} = 1. \quad (\text{Inverses})$$

## Order Axioms

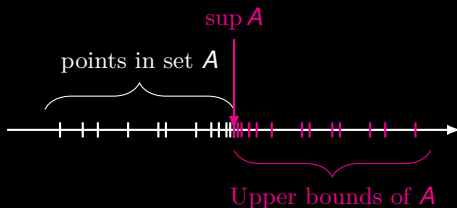
Let  $x, y, z \in \mathbb{R}$ . Then we have that

- (O1)  $x < y$  and  $y < z$  implies that  $x < z$ . (Transitive)
- (O2)  $x < y$  implies that  $x + z < y + z$ .
- (O3)  $x < y$  and  $z > 0$  implies that  $xz < yz$ .
- (O4) Exactly one of  $x = y$ ,  $x < y$ , and  $x > y$  holds. (Trichotomous)

## Completeness Axiom

Axiom A nonempty subset of real numbers that is **bounded above** has a **least upper bound**, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$



Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence and  $a \in \mathbb{R}$ . Then

$$\sup_n x_n \leq a \iff \forall n, x_n \leq a$$

$$\sup_n x_n < a \iff \forall n, x_n < a$$

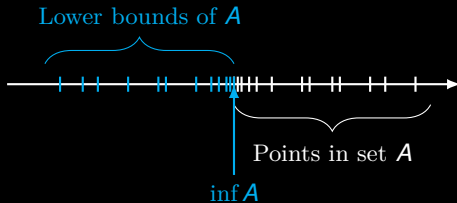
$$a < \sup_n x_n \iff \exists n, x_n > a$$

---

$$a \leq \sup_n x_n \iff \exists n, x_n \geq a$$

Corr. A nonempty subset of real numbers that is **bounded below** has a **greatest lower bound**, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$



Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence and  $a \in \mathbb{R}$ . Then

$$a \leq \inf_n x_n \iff \forall n, x_n \geq a$$

$$a < \inf_n x_n \iff \forall n, x_n > a$$

$$\inf_n x_n < a \iff \exists n, x_n < a$$

---

$$\inf_n x_n \leq a \iff \exists n, x_n \leq a$$

E.g.  $\sup[0, 1) = 1$  and  $\inf[0, 1) = 0$ .

$\mathbb{N}$  has no least upper bound, but  $\inf \mathbb{N} = 1$ .

Let  $A = \{x : x^2 < 3\}$ . Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

## Properties

### 1. Archimedean principle

For each  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{N}$  such that  $n > x$ .

### 2. Density of the irrational numbers

Between any two real numbers there is an **irrational** number.

### 3. Density of the rational numbers

Between any two real numbers there is an **rational** number.

**Proof** As exercises.





# Extended Real Number System

Def. The *extended real numbers*  $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ .

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
$x < 0$	$x \cdot \infty = \infty \cdot x = -\infty$	$x \cdot (-\infty) = (-\infty) \cdot x = \infty$
$x = \infty$	$\infty + \infty = \infty$ $\infty \cdot \infty = \infty$	$(-\infty) + (-\infty) = -\infty$ $(-\infty) \cdot (-\infty) = \infty$
	$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$	

$\infty - \infty$  cannot be defined (HW).

Def. Let  $a$  and  $b$  be extended real numbers such that  $a < b$ . Then the *intervals on  $\mathbb{R}^*$*  with *endpoints*  $a$  and  $b$  are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R}^* : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R}^* : a < x \leq b\}$$

$$[a, b] = \{x \in \mathbb{R}^* : a \leq x \leq b\}$$

If both  $a$  and  $b$  are in  $\mathbb{R}$ , these intervals are the *bounded intervals* in  $\mathbb{R}$ .

Otherwise, if either  $a = -\infty$  or  $b = \infty$ , then these intervals are *unbounded intervals*.

Thm Every subset  $A$  of  $\mathbb{R}^*$  has both a least upper bound and greatest lower bound. Moreover,

- a) If  $A = \emptyset$ , then  $\sup A = -\infty$  and  $\inf A = \infty$ .
- b) If  $A$  is bounded above in  $\mathbb{R}$ , then  $\sup A \in \mathbb{R}$ ; otherwise,  $\sup A = \infty$ .
- c) If  $A$  is bounded below in  $\mathbb{R}$ , then  $\inf A \in \mathbb{R}$ ; otherwise,  $\inf A = -\infty$ .

E.g.

- a)  $\inf \mathbb{N} = 1$  and  $\sup \mathbb{N} = \infty$ .
- b)  $\inf \mathbb{Z} = -\infty$  and  $\sup \mathbb{Z} = \infty$ .
- c) If  $I$  is an interval in  $\mathbb{R}^*$  with endpoints  $a$  and  $b$ ,  $a \leq b$ . Then  $\inf I = a$  and  $\sup I = b$ .

HW Ex. 2.10 and 2.11 on p. 43.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

Def. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. We say that the real number  $L \in \mathbb{R}$  is the *limit* of this sequence<sup>3</sup>, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number  $\epsilon > 0$ , there exists a natural number  $N$  such that for all  $n > N$ , we have  $|x_n - L| < \epsilon$ .

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

---

<sup>3</sup>In this case, we say that  $\{x_n\}_{n=1}^{\infty}$  is *convergent*. Otherwise, we say that  $\{x_n\}_{n=1}^{\infty}$  is *divergent*.

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is convergent and  $\lim_{n \rightarrow \infty} (n-1)/n = 1$ .

$\{(-1)^n\}_{n=1}^{\infty}$  is divergent.

$\{n^2\}_{n=1}^{\infty}$  is divergent.



Recall that  $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$  is the *extended real line*.

**Def.** A sequence of real numbers  $\{a_1, a_2, \dots\}$  is said to *converge in  $\mathbb{R}^*$*  if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
  - (ii) For each  $M \in \mathbb{R}$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n > M$ .
  - (iii) For each  $M \in \mathbb{R}$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n < M$ .
- 

- (i) We say that the sequence converges in  $\mathbb{R}$  or the limit exists and is finite.
- (ii) We say the sequence converges to  $\infty$  and write  $\lim_{n \rightarrow \infty} x_n = \infty$ .
- (iii) We say the sequence converges to  $-\infty$  and write  $\lim_{n \rightarrow \infty} x_n = -\infty$ .

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$ .

$\{(-1)^n\}_{n=1}^{\infty}$  does not converge in  $\mathbb{R}^*$ .

$\{n^2\}_{n=1}^{\infty}$  converges in  $\mathbb{R}^*$  and  $\lim_{n \rightarrow \infty} n^2 = \infty$ .

## Monotone sequence

Def. If  $x_1 \leq x_2 \leq \dots$ , then  $\{x_n\}_{n=1}^{\infty}$  is said to be *nondecreasing*.

If  $x_1 \geq x_2 \geq \dots$ , then  $\{x_n\}_{n=1}^{\infty}$  is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$  is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is monotone and it is nondecreasing.

$\{(-1)^n\}_{n=1}^{\infty}$  is not monotone.

$\{n^2\}_{n=1}^{\infty}$  is monotone and it is nondecreasing.

Thm Any monotone sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers converges in  $\mathbb{R}^*$ .

Moreover, we have the following:

a) If  $\{x_n\}_{n=1}^{\infty}$  is nondecreasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular,  $\{x_n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$  if it is **bounded above** and is  $\infty$  otherwise.

b) If  $\{x_n\}_{n=1}^{\infty}$  is nonincreasing, then

$$\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

In particular,  $\{x_n\}_{n=1}^{\infty}$  converges in  $\mathbb{R}$  if it is **bounded below** and is  $-\infty$  otherwise.

**Proof.** We will prove the case when  $\{x_n\}_{n=1}^{\infty}$  is nondecreasing. The nonincreasing case can be proved in a similar way.

As  $\sup_n x_n$  always exists in  $\mathbb{R}^*$ , we need to consider two cases:

Case I:  $\sup_n x_n \in \mathbb{R}$

Case II:  $\sup_n x_n = \infty$

Let's prove Case I here. Let  $x = \sup_n x_n$ . In order to show that  $\lim_n x_n = x$ , by monotonicity, we need to show that

$$\forall \epsilon > 0 \exists N \forall n \text{ s.t. } (n \geq N) \rightarrow (x - a_n \leq \epsilon).$$

Proof. (Continued) Otherwise,

$$\exists \epsilon > 0 \forall N \exists n \text{ s.t. } (n \geq N) \wedge (x - a_n > \epsilon).$$

Hence, there are infinitely many terms falling below  $x - \epsilon$ .

Since  $\{x_n\}$  is nondecreasing, this implies all  $a_n$  fall below  $x - \epsilon$ , i.e.,

$$a_n < x - \epsilon, \quad \text{for all } n \geq 1.$$

which is equivalent to  $\sup_n x_n < x - \epsilon$ . This contradicts with the fact that  $\sup_n x_n = x$ .

Therefore,

$$\lim_n x_n = x = \sup_n x_n.$$

□

E.g.  $\{(n-1)/n\}_{n=1}^{\infty}$  is nondecreasing and converges in  $\mathbb{R}$ . It is bounded above.

$\{(-1)^n\}_{n=1}^{\infty}$  is not monotone.

$\{n^2\}_{n=1}^{\infty}$  is nondecreasing, does not converge in  $\mathbb{R}$ , converges in  $\mathbb{R}^*$ .



## Cluster points

Def. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers.

- a) A real number  $x$  is said to be a **cluster point** of  $\{x_n\}_{n=1}^{\infty}$  if for each  $\epsilon > 0$  and  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $|x - x_n| < \epsilon$ .
  - b)  $\infty$  **is a cluster point** of  $\{x_n\}_{n=1}^{\infty}$  if for each  $M \in \mathbb{R}$  and  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $x_n > M$ .
  - c)  $-\infty$  **is a cluster point** of  $\{x_n\}_{n=1}^{\infty}$  if for each  $M \in \mathbb{R}$  and  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $x_n < M$ .
- 

a)  $x \in \mathbb{R}$  is a cluster point of  $\{x_n\}$  if

$$\forall \epsilon \forall N \exists n \quad (n \geq N) \rightarrow (|x - x_n| < \epsilon). \quad (1)$$

E.g.1  $\{(n-1)/n\}_{n=1}^{\infty}$  has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$  has two cluster points:  $-1$  and  $+1$ .

$\{n^2\}_{n=1}^{\infty}$  has one cluster point:  $+\infty$ .

E.g.2 Consider the sequence  $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$ , that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2,  $\infty$ .

**E.g.3** Let  $\{r_n\}_{n=1}^{\infty}$  be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence  $\{r_n\}_{n=1}^{\infty}$ .

**Thm** A convergent sequence has exactly one cluster point, namely, its limit.  
Thus, a sequence having more than one cluster point cannot converge.

**Proof** Suppose that  $\{x_n\}$  is a convergent sequence and let  $x$  be its limit, namely,  $\lim_{n \rightarrow \infty} x_n = x$ . We need to prove:

- (1)  $x$  is a cluster point.
- (2)  $x$  is the only cluster point of  $\{x_n\}$ .

We also need to consider two cases:

Case I:  $x \in \mathbb{R}$ .

Case II:  $x = \infty$  or  $-\infty$ .

We will focus on Case I only.

Now we first prove (1).

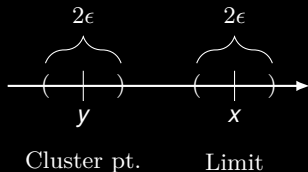
---

	Once for all rest
	$\underbrace{\hspace{1cm}}$
$\lim_n x_n = x$	$\forall \epsilon \exists N \forall n \ (n \geq N) \rightarrow ( x_n - x  < \epsilon)$
$\Downarrow ??$	
$x$ is a cluster point of $\{x_n\}$	$\forall \epsilon \forall \tilde{N} \exists \tilde{n} \ (\tilde{n} \geq \tilde{N}) \wedge ( x_{\tilde{n}} - x  < \epsilon)$
	$\underbrace{\hspace{1cm}}$
	Infinite games

---

(1) is proved by choosing any  $\tilde{n} \geq \max(\tilde{N}, N)$ .

As for (2), suppose  $y$  is another cluster point.



By choosing any  $\epsilon < |x - y|/2$ , we see that

1. In the  $\epsilon$ -neighborhood of  $y$ , there are infinitely many terms.
2. In the  $\epsilon$ -neighborhood of  $x$ , all but finite many terms are here.

Contradiction!

Therefore, there exists only one cluster point.

□

## A few more properties

1. A sequence is convergent iff each subsequence is convergent.
2. **Sandwich theorem:** If  $x_n \leq c_n \leq b_n$  for all  $n > N$  and  $x_n \rightarrow L$  and  $b_n \rightarrow L$ , then  $c_n \rightarrow L$ .



## Limit superior and limit inferior

Def. The *limit inferior* of a sequence  $\{x_n\}_{n=1}^{\infty}$  is defined as

$$\liminf_{n \rightarrow \infty} x_n := \sup_n \left( \inf_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of  $\{x_n\}_{n=1}^{\infty}$  is defined as

$$\limsup_{n \rightarrow \infty} x_n := \inf_n \left( \sup_{m \geq n} a_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences  $\{y_n\}_{n=1}^{\infty}$  and  $\{z_n\}_{n=1}^{\infty}$  defined as

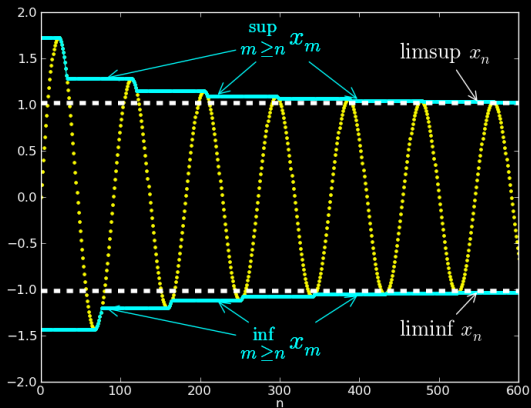
$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

$$\begin{aligned} \inf_n y_n &= \inf_n \sup_{m \geq n} x_m = \limsup_{x \rightarrow \infty} x_n \\ \sup_n z_n &= \sup_n \inf_{m \geq n} x_m = \liminf_{x \rightarrow \infty} x_n. \end{aligned}$$

Hence,

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} x_m.$$



4

<sup>4</sup>Image from Wikipedia.

## Characterization of the limsup and liminf.

Thm Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. Then

- a)  $\limsup x_n = x \in \mathbb{R}$  iff for each  $\epsilon > 0$ ,
    - i) there is an  $N \in \mathbb{N}$  such that  $x_n \leq x + \epsilon$  for all  $n \geq N$ , and
    - ii) for each  $n \in \mathbb{N}$ , there is an  $m \geq n$  such that  $x_m > x - \epsilon$ .
  - b)  $\limsup x_n = \infty$  iff for each  $M \in \mathbb{R}$  and  $N \in \mathbb{N}$ , there is an  $n \geq N$  such that  $x_n > M$ ; in other words, iff the sequence is unbounded from above.
  - c) Similarly,  $\limsup x_n = -\infty$  if and only if  $\lim x_n = -\infty$ .
- 

- a)  $\limsup x_n = x \in \mathbb{R}$  iff for each  $\epsilon > 0$ ,
  - i) the  $\epsilon$ -neighborhood of  $x$  has been visited infinitely many times; and
  - ii) only finite many terms are greater than  $x + \epsilon$ .

**Proof.** (Sketch) We only prove part (a). Let  $y_n = \sup_{k \geq n} x_k$ .

$$\begin{aligned} \limsup x_n = x &\iff \lim_{n \rightarrow \infty} \sup_{k \geq n} x_k = x \\ &\iff \forall \epsilon \exists N \forall n \ (n \geq N) \rightarrow \left( \sup_{k \geq n} x_k \in (x - \epsilon, x + \epsilon) \right) \end{aligned}$$

$$\sup_{k \geq n} x_k < x + \epsilon \iff \text{all terms starting from } n \text{ fall below } x + \epsilon$$

$$\sup_{k \geq n} x_k > x - \epsilon \iff \exists k \geq n \text{ s.t. } x_k > x - \epsilon$$

□

Thm Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. Then

- a)  $\liminf x_n = x \in \mathbb{R}$  iff for each  $\epsilon > 0$ ,
  - i) there is an  $N \in \mathbb{N}$  such that  $x_n \geq x - \epsilon$  for all  $n \geq N$ , and
  - ii) for each  $n \in \mathbb{N}$ , there is an  $m \geq n$  such that  $x_m < x + \epsilon$ .
- b)  $\liminf x_n = -\infty$  iff for each  $M \in \mathbb{R}$  and  $N \in \mathbb{N}$ , there is an  $n \geq N$  such that  $x_n < M$ ; in other words, iff the sequence is unbounded from below.
- c) Similarly,  $\liminf x_n = \infty$  if and only if  $\lim x_n = \infty$ .

Still other characterization (as an exercise):

Thm  $\limsup_{n \rightarrow \infty} x_n = x \in \mathbb{R}$  if and only if  $x$  is the **smallest** real number such that for any positive real number  $\epsilon > 0$ , there exists a natural number  $N$  such that  $x_n < x + \epsilon$  for all  $n > N$ .

$\liminf_{n \rightarrow \infty} x_n = x \in \mathbb{R}$  if and only if  $x$  is the **largest** real number such that for any positive real number  $\epsilon > 0$ , there exists a natural number  $N$  such that  $x_n > x - \epsilon$  for all  $n > N$ .

Proof. HW for motivated students.



E.g.1 Let  $x_n = (-1)^n$ . Then  $\{x_n\}_{n=1}^{\infty}$  has two cluster points:  $\pm 1$ , amount which

$$\liminf_{n \rightarrow \infty} x_n = -1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi) \frac{n^2}{n^2 + 8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$



E.g.2 Consider the sequence  $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$ , that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2,  $\infty$ , among which

$$\liminf_{n \rightarrow \infty} x_n = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

E.g.3 Let  $\{r_n\}_{n=1}^{\infty}$  be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence  $\{r_n\}_{n=1}^{\infty}$ , amount which

$$\liminf_{n \rightarrow \infty} r_n = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of real numbers. Then

- a)  $\limsup x_n$  is the largest cluster point of  $\{x_n\}_{n=1}^{\infty}$ .
- b)  $\liminf x_n$  is the smallest cluster point of  $\{x_n\}_{n=1}^{\infty}$ .

**Proof.** We only prove (a). (b) can be proved in a similar way.

Let  $x = \limsup x_n$ . We have seen that  $x$  is a cluster point.

It remains to prove that  $x$  is the largest cluster point. The case when  $x = \pm\infty$  is left for the motivated students.

Now assume that  $x \in \mathbb{R}$ .

Only finite many terms exceed  $x + 1$ , hence,  $\infty$  is not a cluster point.

Let  $y \in \mathbb{R}$  s.t.  $x < y$ . Set  $\epsilon = (y - x)/2$ .

Only finite many terms exceed  $x + \epsilon$ .

Since  $y - \epsilon = x + \epsilon$ , the  $\epsilon$ -neighborhood of  $y$  has been visited only finitely many times.

Hence,  $y$  cannot be a cluster point. □

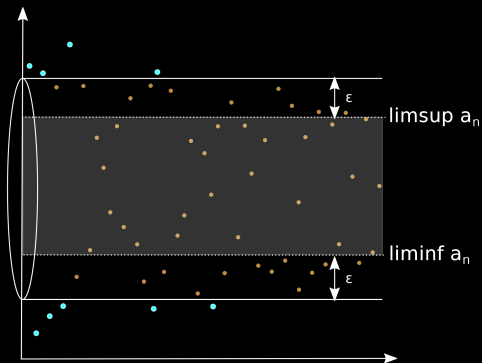
## Properties

1.

$$\inf_n x_n \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq \sup_n x_n$$

2. A sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers **converges in  $\mathbb{R}^*$**  if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = c \iff \lim_{n \rightarrow \infty} x_n = c.$$



5

E.g. For all  $\epsilon > 0$ , the interval

$$\left( \liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in  $\{x_n\}$ .

---

<sup>5</sup>Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all  $\epsilon$  with

$$0 < \epsilon < \frac{1}{2} \left( \limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in  $\{x_n\}$  fall outside of the interval

$$\left( \liminf_{n \rightarrow \infty} x_n + \epsilon, \limsup_{n \rightarrow \infty} x_n - \epsilon \right).$$

## Cauchy criterion

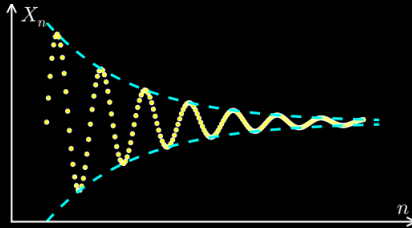
As we have seen that

A sequence of real numbers **converges in  $\mathbb{R}^*$**  if and only if it has exactly one cluster point.

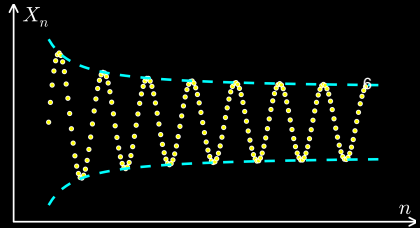
There is another famous criterion for a sequence to **converge in  $\mathbb{R}$** :

### Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



---

<sup>6</sup>Images from Wikipedia.



Def. A sequence  $\{x_n\}_{n=1}^{\infty}$  of real numbers is called a *Cauchy sequence* if for each  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have  $|x_n - x_m| < \epsilon$ .

Def.' A sequence  $\{x_n\}_{n=1}^{\infty}$  is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers *converges in  $\mathbb{R}$*  iff it is Cauchy.

Proof. " $\Rightarrow$ " Easy!

" $\Leftarrow$ ": ...



E.g.1 Let  $a_n = \sqrt{n}$ . Show that

- (i) The consecutive terms become arbitrarily close to each other as  $n \rightarrow \infty$ .
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show  $\{a_n\}_{n=1}^{\infty}$  is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose  $\epsilon = 1$ . For any  $N \in \mathbb{N}$ , we need to find  $m, n \geq N$  such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose  $m = N$  and  $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

□

HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

## Open sets

Def. A subset  $O \subset \mathbb{R}$  is said to be an *open set* if for each  $x \in O$ , there exists an  $r > 0$  such that  $(x - r, x + r) \subset O$ .

E.g.  $(a, b)$  with  $-\infty \leq a < b \leq \infty$  is an open set, which are called *open interval intervals*.

$(0, 1]$  is not an open set.

Let  $K$  be a nonempty countable subset of  $\mathbb{R}$ . Then  $K$  cannot be an open set. For example,  $\mathbb{N}$ ,  $\mathbb{Q}$ ,  $\mathbb{Z}$  are not open sets.

$\mathbb{Q}^c$  – the set of irrational numbers – is not an open set.



## Properties of open sets

1.  $\mathbb{R}$  and  $\emptyset$  are open sets.
2. If  $A$  and  $B$  are open sets, so is  $A \cap B$ . (finite intersection)
3. If  $\{O_i\}_{i \in I}$  is a collection of open sets, then  $\bigcup_{i \in I} O_i$  is open. (arbitrary union)

Proof. Exercise.



---

Let  $Q_n = (-1/n, 1/n)$ . Then  $\bigcap_{n \in \mathbb{N}} Q_n = \{0\}$  is not an open set.

Def. For  $a, b \in \mathbb{R}^*$  with  $a < b$ ,  $(a, b)$  is an open set, which is called an *open interval*.

Thm. Each open set  $O$  is a countable union of disjoint open intervals.

Moreover, the representation is unique in the sense that if  $\mathcal{C}$  and  $\mathcal{D}$  are two pairwise disjoint collections of open intervals whose union is  $O$ , then  $\mathcal{C} = \mathcal{D}$ .

## Closed sets

Def. Let  $E \subset \mathbb{R}$ . A real number  $x$  is called a *limit point* of  $E$  if for each  $\epsilon > 0$ , there is a  $y \in E$  such that  $|y - x| < \epsilon$ .

The set of all limit point of  $E$ , denoted  $\bar{E}$ , is called the *closure* of  $E$ .

E.g.  $\overline{\mathbb{R}} = \mathbb{R}$  and  $\overline{\emptyset} = \emptyset$ .

Let  $a, b \in \mathbb{R}$  such that  $a < b$ . Then

$$\overline{(a, b)} = \overline{(a, b]} = \overline{[a, b)} = \overline{[a, b]} = [a, b].$$

$$\overline{\mathbb{N}} = \mathbb{N} \text{ and } \overline{\mathbb{Z}} = \mathbb{Z}.$$

$$\overline{\mathbb{Q}} = \mathbb{R} \text{ and } \overline{\mathbb{Q}^c} = \mathbb{R}.$$

If  $A$  is a finite subset of  $\mathbb{R}$ , then  $\overline{A} = A$ .

Def. A subset  $F \subset \mathbb{R}$  is said to be a *closed set* if  $\overline{F} = F$ , i.e,  $F$  contains all its limit points.

E.g.  $\mathbb{R}$  and  $\emptyset$  are both open and closed.

Intervals such as  $[a, b]$ ,  $[a, \infty)$ ,  $(-\infty, b]$  with  $a, b \in \mathbb{R}$  are closed sets. They are called *closed intervals*.

$\mathbb{N}$  and  $\mathbb{Z}$  are closed sets.

The sets of rationals  $\mathbb{Q}$  and irrationals  $\mathbb{Q}^c$  are neither open nor close.

If  $A$  is a finite subset of  $\mathbb{R}$ , then  $A$  is a close set.

Thm. A set is open if and only if its complement is closed.

Or equivalently, a set is closed if and only if its complement is open.

## Properties of closed sets

1.  $\mathbb{R}$  and  $\emptyset$  are closed sets.
2. If  $A$  and  $B$  are closed sets, so is  $A \cup B$ . (finite union)
3. If  $\{F_i\}_{i \in I}$  is a collection of closed sets, then  $\bigcap_{i \in I} F_i$  is closed. (arbitrary intersection)

---

Let  $Q_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$ . Then  $\bigcup_{i \in \mathbb{N}} Q_n = (0, 1)$  is an open set.

or

$\bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q}$  is neither open nor closed.



## Relative open and closed sets

Def. Let  $G \subset D \subset \mathbb{R}$ .

a)  $G$  is said to be **open in  $D$**  if for each  $x \in G$ , there is an  $r > 0$  such that

$$(x - r, x + r) \cap D \subset G.$$

b)  $G$  is said to be **closed in  $D$**  if  $D \setminus G$  is open in  $D$ .

E.g.

$D$	$G$	Is $G$ open in $\mathbb{R}$	Is $G$ open in $D$
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
$\mathbb{N}$	$A \subset \mathbb{N}$	closed	open

Thm. Let  $D \subset \mathbb{R}$ . A set  $G \subset D$  is open in  $D$  if and only if there is an open set  $O$  of  $\mathbb{R}$  such that  $G = D \cap O$ .

In other words, the open sets in  $D$  are precisely the open sets of  $\mathbb{R}$  intersected with  $D$ .

HW Ex. 2.38, 2.46, 2.47, 2.49, 2.52 on p. 63 – 64.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

**§ 3.4 Real-valued functions**

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

Def. A *real-valued function* is a function whose range is a subset of  $\mathbb{R}$ . If  $f : \Omega \rightarrow \mathbb{R}$ , we say that  $f$  is a *real-valued function on  $\Omega$* .

### *Algebraic operations*

Let  $f, g$  be real-valued functions on  $\Omega$  and let  $\alpha \in \mathbb{R}$ . Then for all  $x \in \Omega$ ,

$$(f + g)(x) := f(x) + g(x)$$

$$(\alpha f)(x) := \alpha f(x)$$

$$(f \cdot g)(x) := f(x)g(x)$$

## Continuous functions

Def. Let  $D \subset \mathbb{R}$ ,  $f : D \rightarrow \mathbb{R}$ , and  $x_0 \in D$ . We say that  $f$  is *continuous at  $x_0$*  if for each  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|f(x) - f(x_0)| < \epsilon$  whenever  $x \in D$  and  $|x - x_0| < \delta$ .

We say that  $f$  is *continuous on  $D$*  if it is continuous on every point of  $D$ .

We use  $C(D)$  to denote the collection of all continuous functions on  $D$ .

If  $f$  is not continuous at  $x_0$ , then we say that  $f$  is *discontinuous at  $x_0$*  or that  $x_0$  is a *point of discontinuity* of  $f$ .

$f$  is continuous at  $x_0$

$\Updownarrow$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \{ |x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon \}$$

$f \in C(D)$

$\Updownarrow$

$$\forall x \in D \boxed{f \text{ is continuous at } x}$$

$\Updownarrow$

$$\forall x \in D \boxed{\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \{ |x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon \}}$$



E.g.

- a) Let  $D = (0, \infty)$  and define  $f(x) = 1/x$ . Then  $f$  is continuous function on  $D$ .
- b) Let  $D = \mathbb{R}$  and define  $f(0) = 0$  and  $f(x) = \sin(1/x)$  for  $x \neq 0$ . Then  $f$  is a continuous function except at 0.
- c) Let  $D = \mathbb{R}$  and define  $f(x) = \lfloor x \rfloor$ . Then  $f$  is continuous except at points of  $\mathbb{Z}$ .
- d) Every function is continuous on  $\mathbb{N}$ . Or in other words, any infinite series  $\{a_n\}_{n \in \mathbb{N}}$ , when viewed as a function  $a : \mathbb{N} \rightarrow \mathbb{R}$ , is a continuous function.

Thm. Let  $D \subset \mathbb{R}$ . The collection  $C(D)$  of continuous functions on  $D$  is an algebra of functions, i.e., if  $f, g \in C(D)$  and  $a \in \mathbb{R}$ , then

a)  $f + g \in C(D)$ .

b)  $\alpha f \in C(D)$ .

c)  $f \cdot g \in C(D)$ .

Here is a more abstract definition of continuous functions:

**Thm** let  $D \subset \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ . Then  $f$  is continuous on  $D$  if and only if  $f^{-1}(O)$  is open in  $D$  for each open set  $O$  in  $\mathbb{R}$ , i.e., the preimage of each open set in  $\mathbb{R}$  is open in  $D$ .

**Corr.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous if and only if  $f^{-1}(O)$  is open (in  $\mathbb{R}$ ) whenever  $O$  is open (in  $\mathbb{R}$ ).

Proof. “ $\implies$ ”

Suppose that  $f$  is continuous on  $D$ . Let  $O$  be an arbitrary open set in  $\mathbb{R}$ . We need to show that  $f^{-1}(O)$  is open in  $D$ .

Hence, we need to show that for any  $x_0 \in f^{-1}(O)$ , one can find  $\delta > 0$  such that  $(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O)$ .

Notice that

$$x_0 \in f^{-1}(O) \iff f(x_0) \in O.$$

Since  $O$  is open, one can find  $r > 0$  such that  $r$ -neighborhood of  $f(x_0)$  is in  $O$ .

By the continuity of  $f$  at  $x_0$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \mid x - x_0 \mid < \delta \rightarrow \mid f(x) - f(x_0) \mid < \epsilon$$

applied with  $\epsilon = r$ , we can find some  $\delta > 0$  such that

$$\forall x \left\{ \left( x \in (x_0 - \delta, x_0 + \delta) \cap D \right) \rightarrow \left( f(x) \in (f(x_0) - r, f(x_0) + r) \right) \right\}.$$

Proof. (Continued)

Because

$$f(x) \in (f(x_0) - r, f(x_0) + r) \subset O$$

we have that

$$f(x) \in O,$$

or equivalently,

$$x \in f^{-1}(O).$$

Hence, if  $x \in (x_0 - \delta, x_0 + \delta) \cap D$ , then  $x \in f^{-1}(O)$ , i.e.,

$$(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O).$$

Proof. “ $\Leftarrow$ ”

Suppose  $f^{-1}(O)$  is open in  $D$  for each open set  $O \subset \mathbb{R}$ .

For each  $x_0 \in D$ , we need to prove  $f$  is continuous at  $x_0$ , namely,

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon$$

Now fix arbitrary  $x_0 \in D$  and arbitrary  $\epsilon > 0$ .

Let  $O = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ , which is an open interval.

By the assumption,  $f^{-1}(O)$  is open in  $D$ . Hence, there is  $\delta > 0$  such that

$$(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O).$$

In other words,

$$\forall x \in D \left( |x - x_0| < \delta \rightarrow x \in f^{-1}(O) \right).$$

Finally, notice that

$$x \in f^{-1}(O) \iff f(x) \in O \iff |f(x) - f(x_0)| < \epsilon.$$

□

## Pointwise limits

Def. Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of real-valued functions on a set  $\Omega$ , that is,  $f_n : \Omega \rightarrow \mathbb{R}$  for each  $n \in \mathbb{N}$ . Then we say that  $\{f_n\}_{n=1}^{\infty}$  **converges pointwise on  $\Omega$**  if for each  $x \in \Omega$ , the sequence  $\{f_n(x)\}_{n=1}^{\infty}$  of real numbers converges in  $\mathbb{R}$ .

If  $\{f_n\}_{n=1}^{\infty}$  converges pointwise in  $\Omega$ , then we define

$$f : \Omega \rightarrow \mathbb{R}$$

by

$$f(x) := \lim_{n \rightarrow \infty} f_n(x),$$

which is called the **pointwise limit of the sequence of functions  $\{f_n\}_{n=1}^{\infty}$** . In this case, we also call the sequence of functions  $\{f_n\}_{n=1}^{\infty}$  **converges pointwise** to  $f$ , denoted as  $f_n \rightarrow f$  pointwise.

E.g.

- a) Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f_n(x) = (1 + x/n)^n$ . Then  $f_n \rightarrow f$  pointwise on  $\mathbb{R}$  with  $f(x) = e^x$ .
- b) Let  $f_n : D \rightarrow \mathbb{R}$  defined as  $f_n(x) = x^n$ .
- (i) If  $D = [0, 1]$ , then  $f_n \rightarrow f$  pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

- (ii) If  $D = [-1, 1]$ ,  $\{f_n\}_{n=1}^\infty$  fails to converge pointwise because the sequence  $\{(-1)^n\}_{n=1}^\infty$  does not converge.
- (iii) If  $D = [0, 3]$ ,  $\{f_n\}_{n=1}^\infty$  fails to converge pointwise because the sequence  $\{3^n\}_{n=1}^\infty$  does not converge in  $\mathbb{R}$ .



c) Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$f_n(x) = \begin{cases} n^2 x & \text{if } |x| < \frac{1}{n} \\ 1/x & \text{otherwise} \end{cases}$$

Then  $f_n \rightarrow f$  pointwise on  $\mathbb{R}$  with

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{otherwise} \end{cases}$$

d) Let  $D \subset \mathbb{R}$  and define  $f_n(x) = x/n$ . Then  $f_n \rightarrow 0$  pointwise on  $D$ .

Def. Let  $\mathcal{F}$  be a collection of real-valued functions on  $\Omega$ . We say that  $\mathcal{F}$  is *closed under pointwise limits* if whenever  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$  and  $f_n \rightarrow f$  pointwise on  $\Omega$ , then  $f \in \mathcal{F}$ .

E.g.

- a) If  $\mathcal{F}$  is the collection of all **real-valued functions**, then  $\mathcal{F}$  is closed under pointwise limits.
- b) If  $\mathcal{F} = C(D)$ , then it is not closed under pointwise limit.

Def. Let  $\{f_n\}_{n=1}^\infty$  be a sequence of real-valued functions on a set  $\Omega$ , that is,  $f_n : \Omega \rightarrow \mathbb{R}$  for each  $n \in \mathbb{N}$ . Then we say that  $\{f_n\}_{n=1}^\infty$  *converges uniformly* to the real-valued function  $f$  on  $\Omega$ , if for each  $\epsilon > 0$ , there is an  $N \in \mathbb{N}$  such that whenever  $n \geq N$ ,  $|f_n(x) - f(x)| < \epsilon$  for all  $x \in \Omega$ . We write  $f_n \rightarrow f$  uniformly.

Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of real-valued functions on a set  $\Omega$ .

---

$f_n \rightarrow f$  **pointwise** on  $\Omega$  iff

$$\forall x \in \Omega \left[ \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \left( n \geq N \rightarrow |f_n(x) - f(x)| < \epsilon \right) \right]$$

---

$f_n \rightarrow f$  **uniformly** on  $\Omega$  iff

$$\left[ \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \right] \forall x \in \Omega \left( n \geq N \rightarrow |f_n(x) - f(x)| < \epsilon \right)$$

Thm.  $C(D)$  is closed under uniform limits.

More precisely, let  $D \subset \mathbb{R}$ . Suppose that  $\{f_n\}_{n=1}^{\infty} \subset C(D)$  and that  $f_n \rightarrow f$  uniformly. Then  $f \in C(D)$ .

**Proof.** In order to show  $f \in C(D)$ , we need to show that

$$\forall x_0 \in D \forall \epsilon > 0 \exists \delta > 0 \left( |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon \right).$$

Let's fix arbitrary  $x_0 \in D$  and  $\epsilon > 0$ .

$f_n \rightarrow f$  uniformly implies that for some  $N \in \mathbb{N}$  such that

$$|f_N(x) - f(x)| < \epsilon/3 \quad \text{for all } x \in D.$$

Because  $f_N$  is continuous on  $D$ , and hence, at  $x_0$ , we can find  $\delta > 0$  such that

$$|f_N(x) - f_N(x_0)| < \epsilon/3 \quad \text{whenever } x \in D \text{ and } |x - x_0| < \delta.$$

Hence, whenever  $x \in D$  and  $|x - x_0| < \delta$ ,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore,  $f \in C(D)$ . □

E.g. (Revisit the previous examples)

- a) Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f_n(x) = (1 + x/n)^n$ . Then  $f_n \rightarrow f$  pointwise on  $\mathbb{R}$  with  $f(x) = e^x$ . Does  $f_n \rightarrow f$  uniformly?
- b) Let  $f_n : D \rightarrow \mathbb{R}$  defined as  $f_n(x) = x^n$  with  $D = [0, 1]$ . Then  $f_n \rightarrow f$  pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since  $f \notin C(D)$ , this convergence cannot be uniform.



c) We have seen that

$$f_n(x) = \begin{cases} n^2 x & \text{if } |x| < \frac{1}{n} \\ 1/x & \text{otherwise} \end{cases} \rightarrow f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{otherwise} \end{cases} \quad \text{pointwise.}$$

Because  $f \notin C(\mathbb{R})$ , this convergence cannot be uniform.

d) Let  $D \subset \mathbb{R}$  and define  $f_n(x) = x/n$ . Then  $f_n \rightarrow 0$  pointwise on  $D$ .  
However,

- (i) If  $D = [a, b]$  with  $a, b \in \mathbb{R}$ , then the convergence is uniform.
- (ii) If  $D = \mathbb{R}$ , this convergence cannot be uniform (why?).

Finally, the collection  $\mathcal{C}(D)$  of real-valued continuous functions is closed under:  $+$ ,  $\cdot$ , scalar multiplication, and uniform convergence.

HW Ex. 3.59, 2.60, 2.61, 2.66, on p. 71 – 73.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

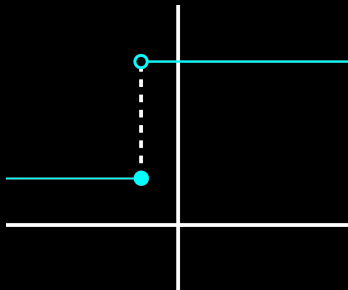
§ 3.7 Some techniques in calculus

1. Left (right)-continuity
  2. Lower (upper) semi-continuity
  3. Uniform continuity
- 

This part is gathered from various sources such as Wikipedia.

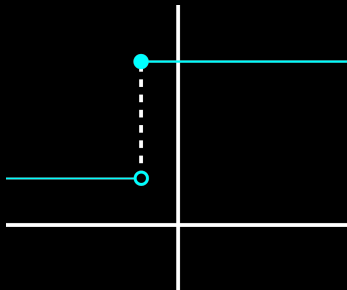
Def.  $f$  is *left-continuous at  $c$*  if

$$\lim_{x \rightarrow c^+} f(x) = f(c)$$



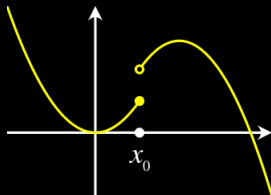
Def.  $f$  is *right-continuous at  $c$*  if

$$\lim_{x \rightarrow c^-} f(x) = f(c)$$



Def.  $f$  is *lower semi-continuous at  $x_0$*  if

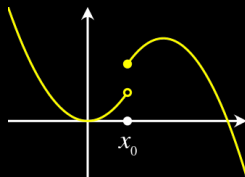
$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$$



$f(x_0)$  can be all points  
at or below the yellow point.

$f$  is *upper semi-continuous at  $x_0$*   
if

$$\limsup_{x \rightarrow x_0} f(x) \leq f(x_0)$$



$f(x_0)$  can be all points  
at or above the yellow point.

## (Global) Uniform Continuity

Def. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $I$  be an interval of  $\mathbb{R}$ . Then  $f$  is *uniformly continuous over  $I$*  if for every real number  $\epsilon > 0$ , there exists a real number  $\delta > 0$  such that for every  $x, y \in I$  with  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .

Def.'  $f$  is *uniformly continuous over  $I$*  if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \}$$



$f$  is continuous at  $x_0 \in I$  iff

$$\forall \epsilon > 0 \forall x \in I \exists \delta > 0 \{ |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon \}$$

$\Pi_2$ -form

---

$f$  is uniformly continuous over  $I$  iff

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \}$$

$\Pi_3$ -form

## Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex.  $f(x) = x^3$  is a continuous functions on  $I = \mathbb{R}$ . Show that  $f$  is not uniformly continuous on  $I = \mathbb{R}$ .

Sol. In order to show  $f$  is not uniformly continuous on  $I$ , we need to show

$$\begin{aligned} & \neg \left( \forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \right) \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ \neg \{ |x - y| < \delta \} \vee |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \{ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon \} \end{aligned}$$

Sol. (Continued) In other words, we need to find  $\epsilon > 0$  such that no matter how small  $\delta > 0$  is chosen, we can always find out  $x, y \in I$  with  $|x - y| < \delta$  and  $|f(x) - f(y)| \geq \epsilon$ .

Let's choose  $\epsilon = 1$ . For any  $\delta > 0$ , we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left( \frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

□

Prop. 2 If  $I$  is compact<sup>7</sup> set such as  $I = [a, b]$ , then

$f$  is continuous at all points in  $I \iff f$  is uniformly continuous on  $I$ .

E.g.  $f(x) = 1/x$  is not uniformly continuous on  $(0, 1)$ .

$f(x) = x^3$  is uniformly continuous on  $[-1, 1]$  but neither on  $\mathbb{R}$  nor on  $[0, \infty)$ .

---

<sup>7</sup>namely, bounded and closed

Why does it matter at all?

**Answer:** Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

**Thm** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a uniformly continuous function and let  $\{x_n\}_{n=1}^{\infty}$  be a Cauchy sequence. Then  $\{f(x_n)\}_{n=1}^{\infty}$  is also a Cauchy sequence.

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

Some part of subsection is taken from Chapter 1 Section 4 of

*P. Billingsley, **Probability and Measure**, Wiley, 1995.*

Def. For a sequence  $A_1, A_2 \cdots$  of sets, define the *limits superior and inferior* of the sequence  $\{A_n\}$  as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Remark Both  $\limsup_n A_n$  and  $\liminf_n A_n$  are sets.



Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$\begin{aligned}
 \omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
 &\iff (\forall n \geq 1) (\exists k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\limsup_n A_n = [A_n \text{ i.o.}]$$

Use the relation:

set	logic
$\bigcap$	$\forall$
$\bigcup$	$\exists$

$$\begin{aligned}
 \omega \in \liminf_n A_n &\iff \omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\
 &\iff (\exists n \geq 1) (\forall k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{all but finitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\liminf_n A_n = [A_n \text{ all but finitely many}]$$

Def. If both  $\limsup_n A_n$  and  $\liminf_n A_n$  exist and are equal, then the *limit set* of the sequence  $\{A_n\}$  is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

which is also often written as  $A_n \rightarrow A$ .

## Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left( \limsup_n A_n^c \right)^c$$

## Properties

(ii) Monotone increasing and decreasing sets:

$$\begin{array}{ccc}
 \left( \bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \liminf_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
 \cap & & \cap \\
 A_n & & \\
 \cap & & \\
 \left( \bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \limsup_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n
 \end{array}$$

## Interpretation Under a Probability Space

Suppose that  $\{A_n\}$  are events from a probability space  $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$\begin{array}{ccc}
 \mathbb{P} \left( \bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \mathbb{P} \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \mathbb{P} \left( \liminf_n A_n \right) \\
 & & \uparrow \wedge \\
 & & \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 \mathbb{P}(A_n) & & \uparrow \wedge \\
 & & \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 & & \uparrow \wedge \\
 \mathbb{P} \left( \bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \mathbb{P} \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \mathbb{P} \left( \limsup_n A_n \right)
 \end{array}$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned} 1 &\geq \mathbb{P}(A_n^c \text{ all but finitely many}) = 1 - \mathbb{P}(\{A_n^c \text{ all but finitely many}\}^c) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \\ &\geq 1 - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

□

### Exercise

(i) Let  $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$ :

$$A_1 = (-1, 0]$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{2}{3}\right]$$

$$A_4 = \left(-\frac{1}{4}, \frac{3}{4}\right]$$

$$\vdots \quad \vdots$$

$$A_{100} = \left(-\frac{1}{100}, \frac{99}{100}\right]$$

$$\vdots \quad \vdots$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$



Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left( -\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[ 0, \frac{n-1}{n} \right] = [0, 1)$$

and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left( -\frac{1}{k}, \frac{k-1}{k} \right] = \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, 1 \right) = (-1, 1].$$

Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

□

### Exercise

(ii) Let  $A_n = \left( \frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$ :

$$A_1 = (-1, 2] \qquad A_2 = \left( \frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left( -\frac{1}{3}, \frac{4}{3} \right] \qquad A_4 = \left( \frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left( -\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left( \frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$A_{99} = \left( -\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left( \frac{1}{100}, \frac{99}{100} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

Show that  $\lim_n A_n$  doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

Sol.

$$\begin{aligned}
 & \liminf_n A_n \\
 &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcup_{n=1,3,5}^{\infty} \left( \frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left( \frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
 &= (0, 1) \cup (0, 1) \\
 &= (0, 1)
 \end{aligned}$$

Sol. (continued) Similarly,

$$\begin{aligned}
 & \limsup_n A_n \\
 &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \bigcup_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \bigcup_{k=n}^{\infty} \left( \frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \left( -\frac{1}{n}, \frac{n+1}{n} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \left( -\frac{1}{n+1}, \frac{n+2}{n+1} \right] \right\} \\
 &= [0, 1] \cap [0, 1] \\
 &= [0, 1]
 \end{aligned}$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \frac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \frac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine  $\liminf_{n \rightarrow \infty} A_n$  and  $\limsup_{n \rightarrow \infty} A_n$ .

Solution:

$$\liminf_{n \rightarrow \infty} A_n = \{0\} \subset [0, 1] = \limsup_{n \rightarrow \infty} A_n$$

# Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

## Examples

1.  $\int_0^1 \tan^{-1}(x) dx$
2.  $\int_0^x t^2 e^t dt$
3.  $\int e^x \sin(x) dx$
4.  $\int_0^1 (x^2 + 1) e^{-x} dx$
5.  $\int_4^9 \frac{\ln y}{y} dy$
6.  $\int_1^3 r^3 \ln r dr$
7.  $\int_1^2 (\ln x)^2 dx$
8.  $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) d\theta$
9.  $\int_0^{\pi/2} \sin^{2n+1}(x) dx = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}$
10.  $\int_0^{\pi/2} \sin^{2n}(x) dx = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$

## Examples

1.  $e^x$

2.  $\sin(x)$

3.  $e^{x^2}$