

Topics in Analysis and Linear Algebra

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Chapter 2. Set Theory



Georg Cantor (1845- 1918)
– the founder of modern set theory

This part is mostly based on Chapter 1 of

*J. McDonald and N. Weiss, **A course in real analysis**, Academic Press,
2005.*

Chapter 2. Set Theory

§ 2.1 Basic definitions and properties

§ 2.2 Functions and sets

§ 2.3 Equivalence of sets and countability

Chapter 2. Set Theory

§ 2.1 Basic definitions and properties

§ 2.2 Functions and sets

§ 2.3 Equivalence of sets and countability

Def. A **set** is a collection of elements.

If A is a set and x is an element of A , we write $x \in A$.

$x \notin A$ means x is not an element of A .

A set contains no elements is called an **empty set**, denoted as \emptyset .

Def. Let A and B be sets.

If every element of A is an element of B , then A is said to be a **subset** of B , denoted $A \subset B$ or $B \supset A$.

Two sets A and B are **equal**, denoted $A = B$, if and only if $A \subset B$ and $A \supset B$.

If $A \subset B$ but $B \not\subset A$, then we say that A is a **proper subset** of B .

E.g. Let

\mathbb{C} = collection of complex numbers

\mathbb{R} = collection of real numbers

\mathbb{Q} = collection of rational numbers

\mathbb{Z} = collection of integers

\mathbb{N} = collection of natural numbers, i.e., positive integers

Then we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Assume all sets under consideration are subsets of some fixed set Ω , commonly referred as the **universal set**.

The set of all subsets of Ω is called the **power set** of Ω , denoted $\mathcal{P}(\Omega)$.

Hence, $A \subset \Omega$ iff $A \in \mathcal{P}(\Omega)$.

Remark $\Omega^c = \emptyset$ and $\emptyset^c = \Omega$.

Def. Let A and B be subsets of Ω .

The **complement** of A , denoted A^c , is the set of elements of Ω that do not belong to A , namely,

$$A^c := \{x \in \Omega : x \notin A\}.$$

The **complement of A relative to B** , denoted $B \setminus A$, is the set of all elements in B that do not belong to A , namely,

$$B \setminus A := \{x \in B : x \notin A\}.$$

The **intersection** of A and B , denoted $A \cap B$, is the set of elements of Ω that belong to both A and B , namely,

$$A \cap B := \{x \in \Omega : x \in A \text{ and } x \in B\}.$$

The **union** of A and B , denoted $A \cup B$, is the set of elements of Ω that belong to either A or B , namely,

$$A \cup B := \{x \in \Omega : x \in A \text{ or } x \in B\}.$$

Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Idempotent Laws

$$A \cup A = A$$

$$A \cap A = A$$

Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Domination Laws

$$A \cup \Omega = \Omega$$

$$A \cap \emptyset = \emptyset$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

De Morgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

Various Identities

$$A \cap A^c = \emptyset$$

$$A \cup A^c = \Omega$$

$$\emptyset^c = \Omega$$

$$\Omega^c = \emptyset$$

$$(A^c)^c = A$$

Def. Let \mathcal{C} be a collection of subsets of Ω , that is, $\mathcal{C} \subset \mathcal{P}(\Omega)$.

- a) The **intersection of \mathcal{C}** , denoted $\cap_{A \in \mathcal{C}} A$, is the set of elements of Ω that belong to each set in the collection of \mathcal{C} , namely,

$$\bigcap_{A \in \mathcal{C}} A := \{x \in \Omega : x \in A \text{ for all } A \in \mathcal{C}\}.$$

- b) The **union of \mathcal{C}** , denoted $\cup_{A \in \mathcal{C}} A$, is the set of elements of Ω that belong to at least one of the sets in the collection of \mathcal{C} , namely,

$$\bigcup_{A \in \mathcal{C}} A := \{x \in \Omega : x \in A \text{ for some } A \in \mathcal{C}\}.$$

Set operations still work in this case, e.g.,

De Morgan's Laws

$$\left(\bigcup_{A \in \mathcal{C}} A \right)^c = \bigcap_{A \in \mathcal{C}} A^c$$

$$\left(\bigcap_{A \in \mathcal{C}} A \right)^c = \bigcup_{A \in \mathcal{C}} A^c$$

$$C \setminus \left(\bigcup_{A \in \mathcal{C}} A \right) = \bigcap_{A \in \mathcal{C}} (C \setminus A)$$

$$C \setminus \left(\bigcap_{A \in \mathcal{C}} A \right) = \bigcup_{A \in \mathcal{C}} (C \setminus A)$$

Distributive Laws

$$B \cap \left(\bigcup_{A \in \mathcal{C}} A \right) = \bigcup_{A \in \mathcal{C}} (B \cap A)$$

$$B \cup \left(\bigcap_{A \in \mathcal{C}} A \right) = \bigcap_{A \in \mathcal{C}} (B \cup A)$$

E.g. Let $\Omega = \mathbb{R}$ and $\mathcal{C} = \{[0, 1/n] : n \in \mathbb{N}\}$. Show that

$$\bigcap_{A \in \mathcal{C}} A = \{0\} \quad \text{and} \quad \bigcup_{A \in \mathcal{C}} A = [0, 1].$$

Remark Equivalently, one can write $A_n = [0, 1/n]$ for $n \in \mathbb{N}$ and then

$$\bigcap_{n=1}^{\infty} A_n = \{0\} \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = [0, 1].$$

Proof. ...



In general, we have:

E.g.' Show that

A_n	$\bigcap_{i=1}^n A_n$	$\bigcup_{i=1}^n A_n$
$(0, 1/n)$	\emptyset	$(0, 1)$
$(0, 1/n]$	\emptyset	$(0, 1]$
$[0, 1/n)$	$\{0\}$	$[0, 1)$
$[0, 1/n]$	$\{0\}$	$[0, 1]$

Def. Two subsets, A and B , of Ω are said to be **disjoint** if $A \cap B = \emptyset$.

Ex. 1.8, 1.13.

Chapter 2. Set Theory

§ 2.1 Basic definitions and properties

§ 2.2 Functions and sets

§ 2.3 Equivalence of sets and countability

Def. Suppose that Ω and Λ are sets. A **function** (or **mapping**, **transformation**) from Ω to Λ is a rule that assigns each element $x \in \Omega$ a **unique** element $f(x) \in \Lambda$.

We call $f(x)$ the **value** of f at x , or the **image** of x under f .

A function f from Ω to Λ is often denoted $f : \Omega \rightarrow \Lambda$.

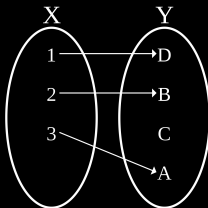
The set Ω is called the **domain** of f .

The set $\{f(x) : x \in \Omega\}$ is called the **range** of f .

Def. Let f be a function from Ω to Λ .

- a) f is said to be **one-to-one** or **injective** if distinct elements of Ω have distinct images; that is,

$$\forall x_1, x_2 \in \Omega, \quad f(x_1) = f(x_2) \quad \rightarrow \quad x_1 = x_2.$$

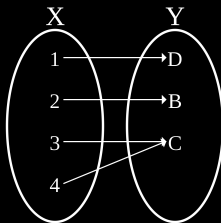


Def. Let f be a function from Ω to Λ .

- b) f is said to be **onto** or **surjective** if each element of Λ is the image of some element of Ω ; that is,

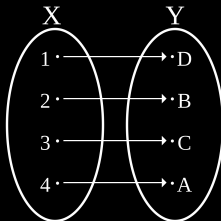
$$\forall y \in \Lambda, \exists x \in \Omega, \text{ s.t. } y = f(x).$$

or f is onto iff the range of f equals Λ .



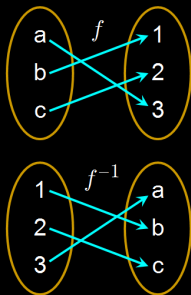
Def. Let f be a function from Ω to Λ .

- c) f is said to be **1-1 correspondence** or **bijective** if f is both surjective (onto) and injective (one-to-one).



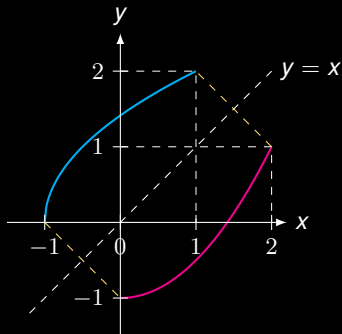
Def. Suppose that $f : \Omega \rightarrow \Lambda$ is both **one-to-one** and **onto**. For any $y \in \Lambda$, let $f^{-1}(y)$ be the unique $x \in \Omega$ such that $y = f(x)$.

The function $f^{-1} : \Lambda \rightarrow \Omega$ defined in this way is called the **inverse** of the function f .



E.g. Let $f : [0, 2] \rightarrow [-1, 1]$ be defined as $f(x) = \frac{1}{2}x^2 - 1$.

The inverse function is $f^{-1} : [-1, 1] \rightarrow [0, 2]$ with $f^{-1}(x) = \sqrt{2x + 2}$.



Def. Let $f : \Omega \rightarrow \Lambda$ and $g : \Lambda \rightarrow \Gamma$. Then the **composition** of g with f , denoted $g \circ f$, is the function $g \circ f : \Omega \rightarrow \Gamma$ defined by

$$(g \circ f)(x) = g(f(x)).$$

Def. Let $f : \Omega \rightarrow \Lambda$ and $A \subset \Omega$. The **restriction** of f to A , denoted $f|_A$, is defined to be a function $A \rightarrow \Lambda$ such that

$$f|_A(x) = f(x), \quad \text{for all } x \in A.$$

Infinite and finite sequences

Infinite sequences such as

- ▶ $\{1, 2, 4, 8, 16, \dots\}$
- ▶ $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$
- ▶ $\{1, -1, 1, -1, 1, -1, \dots\}$
- ▶ $\{1, 1, 2, 3, 5, 8, 13, \dots\}$

are nothing but functions defined on \mathbb{N} .

We use $\{s_n : n \in \mathbb{N}\}$ or $\{s_n\}_{n=1}^{\infty}$ to denote an infinite sequence.

Finite sequence of length n such as

- ▶ $\{a_1, a_2, \dots, a_n\}$

is nothing but a function defined on $\{1, \dots, n\}$.

Images and inverse images

Def. Let $f : \Omega \rightarrow \Lambda$.

If $A \subset \Omega$, then define

$$f(A) := \{f(x) : x \in A\},$$

which is called the **image of A**
under f .

If $B \in \Lambda$, then define

$$f^{-1}(B) := \{x \in \Omega : f(x) \in B\},$$

called the **inverse image of B**
under f .

Thm Let $f : \Omega \rightarrow \Lambda$, $A \subset \Omega$, and $\{A_i\}_{i \in I}$ an indexed collection of subsets of Ω . Then

$$\text{a) } f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$$

$$\text{b) } f\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} f(A_i) \text{ and}$$

$$f\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f(A_i) \text{ provided } f$$

is one-to-one.

$$\text{c) } f(A^c) \subset (f(A))^c \text{ and}$$

$$f(A^c) = (f(A))^c \text{ provided that } f$$

is one-to-one.

Thm Let $f : \Omega \rightarrow \Lambda$, $B \subset \Omega$, and $\{B_i\}_{i \in I}$ an indexed collection of subsets of Ω . Then

$$\text{a) } f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f^{-1}(A_i)$$

$$\text{b) } f^{-1}\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f^{-1}(A_i)$$

$$\text{c) } f^{-1}(A^c) = (f^{-1}(A))^c$$

Proof. ...

□

Cartesian Products

Def. Let A and B be two sets. Then the **Cartesian product** of A and B (in that order), denoted $A \times B$, is the set of all **ordered pairs** (a, b) such that $a \in A$ and $b \in B$, namely,

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

Similarly, if A_1, A_2, \dots, A_n are sets, then the **Cartesian product** of those n sets, denoted $A_1 \times A_2 \times \dots \times A_n$ or $\bigtimes_{k=1}^n A_k$, is the set of all **ordered n -tuples** (a_1, \dots, a_n) such that $a_k \in A_k$ for $k = 1, \dots, n$, namely,





















































$$\bigtimes_{k=1}^n A_k := \{(a_1, \dots, a_n) : a_k \in A_k, 1 \leq k \leq n\}.$$

E.g. 1. The standard 52-card deck is $A \times B$ with

$$A = \{\text{Ace}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{Jack}, \text{Queen}, \text{King}\}$$

$$B = \{\text{Club}, \text{Diamond}, \text{Heart}, \text{Spade}\}$$

$$\Omega = A \cup B$$

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

2. $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$: the Euclidean n -space.

Remark If at least one of A and B are empty, then so is $A \times B$.

Def. Let $\{A_i\}_{i \in I}$ be an indexed collection of sets. The **Cartesian product** of the collection, denoted $\prod_{i \in I} A_i$, is the set of **all functions x on I** such that $x(i) \in A_i$ for each $i \in I$, namely,

$$\prod_{i \in I} A_i := \left\{ x : I \rightarrow \bigcup_{i \in I} A_i : x(i) \in A_i, \forall i \in I \right\}.$$

Remark 1.

2. If $A_i = \emptyset$ for some $i \in I$, then $\prod_{i \in I} A_i = \emptyset$.
3. On the other hand, if $A_i \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} A_i \neq \emptyset$ ¹.

¹Thanks to the *Axiom of Choice*.

Notation and examples

When	$\prod_{i \in I} A_i$
$I = \{1, \dots, n\}$	$\prod_{i=1}^n A_i$
$A_i = A, \forall i \in I$	A^I
$I = \{1, \dots, n\}$ and $A_i = A, \forall i \in I$	write A^n instead of $A^{\{1, \dots, n\}}$ or $\prod_{i=1}^n A$
$I = \mathbb{N}$	write A^∞ instead of $A^{\{1, 2, \dots\}}$ or $A^\mathbb{N}$
$I = [0, 1]$ and $A_i = \mathbb{R}, \forall i \in I$	$A^{[0, 1]}$ is the set of all functions on $[0, 1]$.

Remark Infinite sequence $\{a_1, a_2, \dots\}$ can be viewed as either

1. a function on \mathbb{N} or
2. Cartesian product with $I = \mathbb{N}$, namely, A^∞ .

HW Ex. 1.14, 1.21, 1.23.

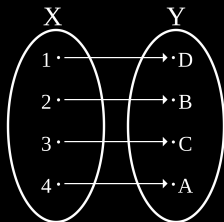
Chapter 2. Set Theory

§ 2.1 Basic definitions and properties

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Recall if f is both **one-to-one (injective)** and **onto (surjective)**, then f is **one-to-one correspondence (bijective)**.



Def. For two sets X and Y , if there exists a bijective function between X and Y , then we say that X and Y are **equivalent**, denoted $X \sim Y$.

Equivalent sets satisfy the following properties:

Reflexive	$A \sim A$
Symmetric	$A \sim B \Rightarrow B \sim A$
Transitive	$A \sim B \wedge B \sim C \Rightarrow A \sim C$

Remark Any relation that satisfies reflexive, symmetric, transitive properties is an equivalence relation.

E.g. 1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{A, B, C, D\}$. Then $X \sim Y$ because one can find a bijective function between X and Y .

2. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{A, B, C, D\}$. Does $X \sim Y$? Why?

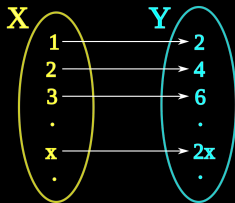
Remark For sets of finite element, in order to be equivalent, they have to have the same number of elements.

E.g. 3. Let $X = \mathbb{N} = \{1, 2, 3, \dots\}$ and $Y = \{2, 4, 6, 8, \dots\}$ (even integers).

Does $X \sim Y$?

Do they have the same number of elements?

Sol. Here is one apparent solution²: $f : X \rightarrow Y$ defined as $f(x) = 2x$:



This is a bijective function (why?). Hence, $X \sim Y$. They have the same number of elements (infinite many, which is called countably infinite).

□

²There are other constructions. Can you give another bijection between X and Y ?

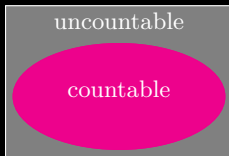
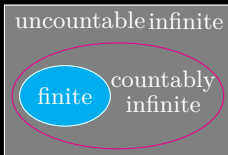
Size of sets

Def. Let A be a set. We say that

- a) A is *finite* if it is either empty or equivalent to the first N positive integers for some $N \in \mathbb{N}$.

In the former case, A is said to consist of 0 elements and, in the latter case, N elements.

- b) A is *infinite* if A is not finite.
- c) A is *countably infinite* if it is equivalent to \mathbb{N} .
- d) A is *countable* if it is either finite or countably infinite.
- e) A is *uncountable* if it is not countable.



E.g.1. Show that the set of \mathbb{Z} is countably infinite.

Sol. Construct the bijection

$$f : \mathbb{N} \rightarrow \mathbb{Z}:$$

$$1 \rightarrow 0$$

$$2 \rightarrow 1$$

$$3 \rightarrow -1$$

$$4 \rightarrow 2$$

$$5 \rightarrow -2$$

$$6 \rightarrow 3$$

$$7 \rightarrow -3$$

$$\vdots$$

It is not hard to see that

$$f(n) = \begin{cases} n/2 & n \text{ is even} \\ -(n-1)/2 & n \text{ is odd} \end{cases}$$

□

Sol'. Construct the bijection

$$f : \mathbb{N} \rightarrow \mathbb{Z}:$$

$$1 \rightarrow 0$$

$$2 \rightarrow -1$$

$$3 \rightarrow 1$$

$$4 \rightarrow -2$$

$$5 \rightarrow 2$$

$$6 \rightarrow -3$$

$$7 \rightarrow 3$$

$$\vdots$$

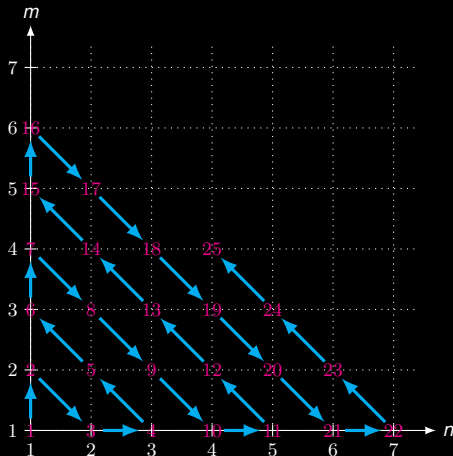
With enough patience, one can determine the formula:

$$f(n) = \frac{n(-1)^{n-1}}{2} - \frac{1 - (-1)^n}{4}$$

□

E.g. 2. Show that \mathbb{N}^2 is countably infinite.

Sol. Construct a bijection $f : \mathbb{N}^2 \rightarrow \mathbb{N}$:



Can you find a formula for this bijection?



Sol'. We claim that $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined below is a bijection:

$$f(m, n) := 2^{m-1}(2n-1).$$

a) f is one-to-one (injective): It suffices to show that

$$f(m_1, n_1) = f(m_2, n_2) \implies m_1 = m_2 \text{ and } n_1 = n_2.$$

Without loss of generality, suppose $m_1 \geq m_2$. Notice that

$$2^{m_1-m_2} = \frac{2n_1-1}{2n_2-1}. \quad (\star)$$

The LHS is an even integer unless $m_1 = m_2$. The RHS is a fraction unless $n_1 = n_2$. Hence, in order to make (\star) valid, one has to have both sides equal to 1. Hence, $m_1 = m_2$ and $n_1 = n_2$.

b) f is onto (surjective). For any integer $k \in \mathbb{N}$, one has to find m and n such that $f(m, n) = k$. One can keep dividing k by 2 until it becomes an odd function. In this way, one easily find out m and n . \square

Thm 1. A nonempty set is countable if and only if it is the range of an infinite sequence.

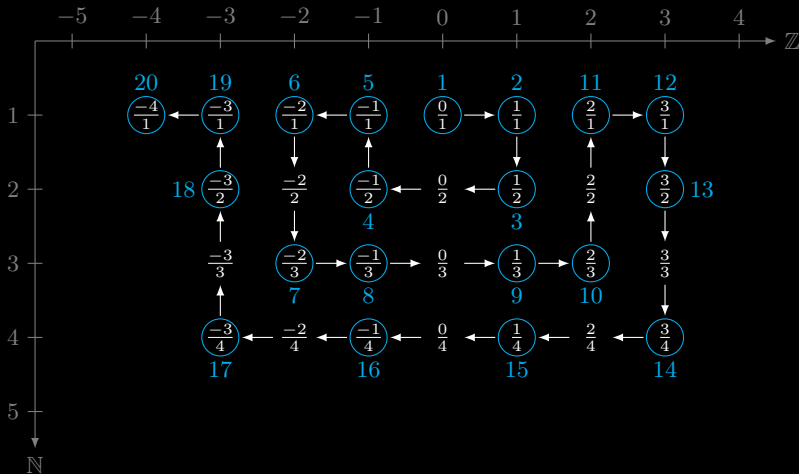
2. A subset of a countable set is countable.
3. The image of a countable set is countable.
4. A countable union of countable sets is countable.
5. The Cartesian product of two countable sets is countable.

E.g. 3. Show that the set of rationals \mathbb{Q} is countably infinite.

Recall that rationals are numbers that can be written as a ratio of two integers, i.e.,

$$q \in \mathbb{Q} \quad \Rightarrow \quad \exists m, n \in \mathbb{Z} \quad \text{such that } q = \frac{m}{n}.$$

Sol. Let's construct the bijection from \mathbb{N} to \mathbb{Q} :



Can you find out the explicit formula for the bijection $f : \mathbb{N} \rightarrow \mathbb{Q}$?



Sol'. Alternatively, one can define:

$$f(z, n) = \frac{z}{n}, \quad z \in \mathbb{Z}, n \in \mathbb{N}.$$

1. $f : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ is onto (surjective) but not one-to-one (injective).

Hence, $f(\mathbb{Z} \times \mathbb{N}) \supset \mathbb{Q}$.

(Figure above)

2. \mathbb{Z} is countable. (E.g. 1)
3. $\mathbb{Z} \times \mathbb{N}$ is countable. (Thm. 5)
4. $f(\mathbb{Z} \times \mathbb{N})$, as the image of $\mathbb{Z} \times \mathbb{N}$ under f , is countable. (Thm. 3)
5. Therefore, \mathbb{Q} , as a subset of the countable set $f(\mathbb{Z} \times \mathbb{N})$,
has to be countable. (Thm. 2)

□

HW Prove Thm's 1-5, which are Propositions 1.7 – 1.11 of the book.