

# Topics in Analysis and Linear Algebra

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# Chapter 1. Topics in calculus



Georg Cantor (1845- 1918)  
– the founder of modern set theory

This part is mostly based on Chapter 1 of

*J. McDonald and N. Weiss, **A course in real analysis**, Academic Press,  
2005.*

# Chapter 1. Topics in calculus

§ 2.1 Basic definitions and properties

§ 2.2 Functions and sets

§ 2.3 Equivalence of sets and countability

# Chapter 1. Topics in calculus

## § 2.1 Basic definitions and properties

## § 2.2 Functions and sets

## § 2.3 Equivalence of sets and countability

Def. A **set** is a collection of elements.

If  $A$  is a set and  $x$  is an element of  $A$ , we write  $x \in A$ .

$x \notin A$  means  $x$  is not an element of  $A$ .

A set contains no elements is called an **empty set**, denoted as  $\emptyset$ .

Def. Let  $A$  and  $B$  be sets.

If every element of  $A$  is an element of  $B$ , then  $A$  is said to be a **subset** of  $B$ , denoted  $A \subset B$  or  $B \supset A$ .

Two sets  $A$  and  $B$  are **equal**, denoted  $A = B$ , if and only if  $A \subset B$  and  $A \supset B$ .

If  $A \subset B$  but  $B \not\subset A$ , then we say that  $A$  is a **proper subset** of  $B$ .

E.g. Let

$\mathbb{C}$  = collection of complex numbers

$\mathbb{R}$  = collection of real numbers

$\mathbb{Q}$  = collection of rational numbers

$\mathbb{Z}$  = collection of integers

$\mathbb{N}$  = collection of natural numbers, i.e., positive integers

Then we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Assume all sets under consideration are subsets of some fixed set  $\Omega$ , commonly referred as the **universal set**.

The set of all subsets of  $\Omega$  is called the **power set** of  $\Omega$ , denoted  $\mathcal{P}(\Omega)$ .

Hence,  $A \subset \Omega$  iff  $A \in \mathcal{P}(\Omega)$ .

Remark  $\Omega^c = \emptyset$  and  $\emptyset^c = \Omega$ .



Def. Let  $A$  and  $B$  be subsets of  $\Omega$ .

The **complement** of  $A$ , denoted  $A^c$ , is the set of elements of  $\Omega$  that do not belong to  $A$ , namely,

$$A^c := \{x \in \Omega : x \notin A\}.$$

The **complement of  $A$  relative to  $B$** , denoted  $B \setminus A$ , is the set of all elements in  $B$  that do not belong to  $A$ , namely,

$$B \setminus A := \{x \in B : x \notin A\}.$$

The **intersection** of  $A$  and  $B$ , denoted  $A \cap B$ , is the set of elements of  $\Omega$  that belong to both  $A$  and  $B$ , namely,

$$A \cap B := \{x \in \Omega : x \in A \text{ and } x \in B\}.$$

The **union** of  $A$  and  $B$ , denoted  $A \cup B$ , is the set of elements of  $\Omega$  that belong to either  $A$  or  $B$ , namely,

$$A \cup B := \{x \in \Omega : x \in A \text{ or } x \in B\}.$$

### *Commutative Laws*

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

### *Idempotent Laws*

$$A \cup A = A$$

$$A \cap A = A$$

### *Associative Laws*

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

### *Domination Laws*

$$A \cup \Omega = \Omega$$

$$A \cap \emptyset = \emptyset$$

### *Distributive Laws*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

### *Absorption Laws*

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

### *De Morgan's Laws*

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

### *Various Identities*

$$A \cap A^c = \emptyset$$

$$A \cup A^c = \Omega$$

$$\emptyset^c = \Omega$$

$$\Omega^c = \emptyset$$

$$(A^c)^c = A$$

Def. Let  $\mathcal{C}$  be a collection of subsets of  $\Omega$ , that is,  $\mathcal{C} \subset \mathcal{P}(\Omega)$ .

- a) The **intersection of  $\mathcal{C}$** , denoted  $\cap_{A \in \mathcal{C}} A$ , is the set of elements of  $\Omega$  that belong to each set in the collection of  $\mathcal{C}$ , namely,

$$\bigcap_{A \in \mathcal{C}} A := \{x \in \Omega : x \in A \text{ for all } A \in \mathcal{C}\}.$$

- b) The **union of  $\mathcal{C}$** , denoted  $\cup_{A \in \mathcal{C}} A$ , is the set of elements of  $\Omega$  that belong to at least one of the sets in the collection of  $\mathcal{C}$ , namely,

$$\bigcup_{A \in \mathcal{C}} A := \{x \in \Omega : x \in A \text{ for some } A \in \mathcal{C}\}.$$

Set operations still work in this case, e.g.,

### *De Morgan's Laws*

$$\left( \bigcup_{A \in \mathcal{C}} A \right)^c = \bigcap_{A \in \mathcal{C}} A^c$$

$$\left( \bigcap_{A \in \mathcal{C}} A \right)^c = \bigcup_{A \in \mathcal{C}} A^c$$

$$C \setminus \left( \bigcup_{A \in \mathcal{C}} A \right) = \bigcap_{A \in \mathcal{C}} (C \setminus A)$$

$$C \setminus \left( \bigcap_{A \in \mathcal{C}} A \right) = \bigcup_{A \in \mathcal{C}} (C \setminus A)$$

### *Distributive Laws*

$$B \cap \left( \bigcup_{A \in \mathcal{C}} A \right) = \bigcup_{A \in \mathcal{C}} (B \cap A)$$

$$B \cup \left( \bigcap_{A \in \mathcal{C}} A \right) = \bigcap_{A \in \mathcal{C}} (B \cup A)$$

E.g. Let  $\Omega = \mathbb{R}$  and  $\mathcal{C} = \{[0, 1/n] : n \in \mathbb{N}\}$ . Show that

$$\bigcap_{A \in \mathcal{C}} A = \{0\} \quad \text{and} \quad \bigcup_{A \in \mathcal{C}} A = [0, 1].$$

Remark Equivalently, one can write  $A_n = [0, 1/n]$  for  $n \in \mathbb{N}$  and then

$$\bigcap_{n=1}^{\infty} A_n = \{0\} \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = [0, 1].$$

Proof. ...



In general, we have:

E.g.' Show that

$A_n$	$\bigcap_{i=1}^n A_n$	$\bigcup_{i=1}^n A_n$
$(0, 1/n)$	$\emptyset$	$(0, 1)$
$(0, 1/n]$	$\emptyset$	$(0, 1]$
$[0, 1/n)$	$\{0\}$	$[0, 1)$
$[0, 1/n]$	$\{0\}$	$[0, 1]$

Def. Two subsets,  $A$  and  $B$ , of  $\Omega$  are said to be **disjoint** if  $A \cap B = \emptyset$ .



Ex. 1.8, 1.13.

# Chapter 1. Topics in calculus

§ 2.1 Basic definitions and properties

§ 2.2 Functions and sets

§ 2.3 Equivalence of sets and countability

Def. Suppose that  $\Omega$  and  $\Lambda$  are sets. A **function** (or **mapping**, **transformation**) from  $\Omega$  to  $\Lambda$  is a rule that assigns each element  $x \in \Omega$  a **unique** element  $f(x) \in \Lambda$ .

We call  $f(x)$  the **value** of  $f$  at  $x$ , or the **image** of  $x$  under  $f$ .

A function  $f$  from  $\Omega$  to  $\Lambda$  is often denoted  $f : \Omega \rightarrow \Lambda$ .

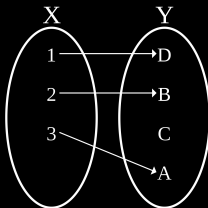
The set  $\Omega$  is called the **domain** of  $f$ .

The set  $\{f(x) : x \in \Omega\}$  is called the **range** of  $f$ .

Def. Let  $f$  be a function from  $\omega$  to  $\lambda$ .

- a)  $f$  is said to be **one-to-one** or **injective** if distinct elements of  $\omega$  have distinct images; that is,

$$\forall x_1, x_2 \in \omega, \quad f(x_1) = f(x_2) \quad \rightarrow \quad x_1 = x_2.$$

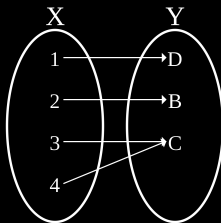


Def. Let  $f$  be a function from  $\omega$  to  $\lambda$ .

- b)  $f$  is said to be **onto** or **surjective** if each element of  $\lambda$  is the image of some element of  $\omega$ ; that is,

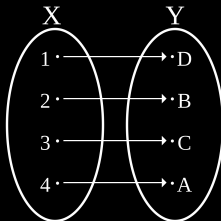
$$\forall y \in \lambda, \exists x \in \omega, \text{ s.t. } y = f(x).$$

or  $f$  is onto iff the range of  $f$  equals  $\lambda$ .



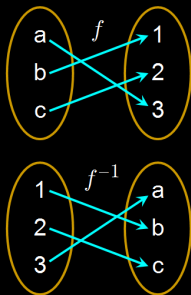
Def. Let  $f$  be a function from  $\omega$  to  $\lambda$ .

- c)  $f$  is said to be **1-1 correspondence** or **bijective** if  $f$  is both surjective (onto) and injective (one-to-one).



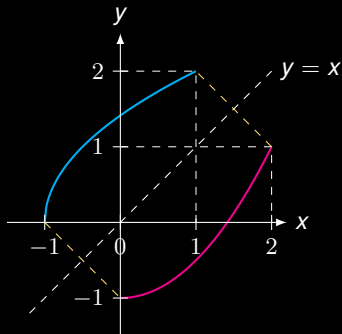
Def. Suppose that  $f : \Omega \rightarrow \Lambda$  is both **one-to-one** and **onto**. For any  $y \in \Lambda$ , let  $f^{-1}(y)$  be the unique  $x \in \Omega$  such that  $y = f(x)$ .

The function  $f^{-1} : \Lambda \rightarrow \Omega$  defined in this way is called the **inverse** of the function  $f$ .



E.g. Let  $f : [0, 2] \rightarrow [-1, 1]$  be defined as  $f(x) = \frac{1}{2}x^2 - 1$ .

The inverse function is  $f^{-1} : [-1, 1] \rightarrow [0, 2]$  with  $f^{-1}(x) = \sqrt{2x + 2}$ .





Def. Let  $f : \Omega \rightarrow \Lambda$  and  $g : \Lambda \rightarrow \Gamma$ . Then the **composition** of  $g$  with  $f$ , denoted  $g \circ f$ , is the function  $g \circ f : \Omega \rightarrow \Gamma$  defined by

$$(g \circ f)(x) = g(f(x)).$$

Def. Let  $f : \Omega \rightarrow \Lambda$  and  $A \subset \Omega$ . The **restriction** of  $f$  to  $A$ , denoted  $f|_A$ , is defined to be a function  $A \rightarrow \Lambda$  such that

$$f|_A(x) = f(x), \quad \text{for all } x \in A.$$

# Infinite and finite sequences

Infinite sequences such as

- ▶  $\{1, 2, 4, 8, 16, \dots\}$
- ▶  $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$
- ▶  $\{1, -1, 1, -1, 1, -1, \dots\}$
- ▶  $\{1, 1, 2, 3, 5, 8, 13, \dots\}$

are nothing but functions defined on  $\mathbb{N}$ .

We use  $\{s_n : n \in \mathbb{N}\}$  or  $\{s_n\}_{n=1}^{\infty}$  to denote an infinite sequence.

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Finite sequence of length  $n$  such as

- ▶  $\{a_1, a_2, \dots, a_n\}$

is nothing but a function defined on  $\{1, \dots, n\}$ .

## Images and inverse images

Def. Let  $f : \Omega \rightarrow \Lambda$ .

If  $A \subset \Omega$ , then define

$$f(A) := \{f(x) : x \in A\},$$

which is called the **image of  $A$**   
**under  $f$ .**

If  $B \in \Lambda$ , then define

$$f^{-1}(B) := \{x \in \Omega : f(x) \in B\},$$

called the **inverse image of  $B$**   
**under  $f$ .**

Thm Let  $f : \Omega \rightarrow \Lambda$ ,  $A \subset \Omega$ , and  $\{A_i\}_{i \in I}$  an indexed collection of subsets of  $\Omega$ . Then

$$\text{a) } f \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i)$$

$$\text{b) } f \left( \bigcap_{i \in I} A_i \right) \subset \bigcap_{i \in I} f(A_i) \text{ and}$$

$$f \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} f(A_i) \text{ provided } f$$

is one-to-one.

$$\text{c) } f(A^c) \subset (f(A))^c \text{ and}$$

$$f(A^c) = (f(A))^c \text{ provided that } f$$

is one-to-one.

Thm Let  $f : \Omega \rightarrow \Lambda$ ,  $B \subset \Omega$ , and  $\{B_i\}_{i \in I}$  an indexed collection of subsets of  $\Omega$ . Then

$$\text{a) } f^{-1} \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f^{-1}(A_i)$$

$$\text{b) } f^{-1} \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} f^{-1}(A_i)$$

$$\text{c) } f^{-1}(A^c) = (f^{-1}(A))^c$$

Proof. ...

□

## Cartesian Products

Def. Let  $A$  and  $B$  be two sets. Then the **Cartesian product** of  $A$  and  $B$  (in that order), denoted  $A \times B$ , is the set of all **ordered pairs**  $(a, b)$  such that  $a \in A$  and  $b \in B$ , namely,

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

Similarly, if  $A_1, A_2, \dots, A_n$  are sets, then the **Cartesian product** of those  $n$  sets, denoted  $A_1 \times A_2 \times \dots \times A_n$  or  $\bigtimes_{k=1}^n A_k$ , is the set of all **ordered  $n$ -tuples**  $(a_1, \dots, a_n)$  such that  $a_k \in A_k$  for  $k = 1, \dots, n$ , namely,





















































$$\bigtimes_{k=1}^n A_k := \{(a_1, \dots, a_n) : a_k \in A_k, 1 \leq k \leq n\}.$$

E.g. 1. The standard 52-card deck is  $A \times B$  with

$$A = \{\text{Ace}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{Jack}, \text{Queen}, \text{King}\}$$

$$B = \{\text{Club}, \text{Diamond}, \text{Heart}, \text{Spade}\}$$

$$\Omega = A \cup B$$

	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades													

2.  $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$ : the Euclidean  $n$ -space.

Remark If at least one of  $A$  and  $B$  are empty, then so is  $A \times B$ .

Def. Let  $\{A_i\}_{i \in I}$  be an indexed collection of sets. The **Cartesian product** of the collection, denoted  $\prod_{i \in I} A_i$ , is the set of **all functions  $x$  on  $I$**  such that  $x(i) \in A_i$  for each  $i \in I$ , namely,

$$\prod_{i \in I} A_i := \left\{ x : I \rightarrow \bigcup_{i \in I} A_i : x(i) \in A_i, \forall i \in I \right\}.$$

Remark 1.

2. If  $A_i = \emptyset$  for some  $i \in I$ , then  $\prod_{i \in I} A_i = \emptyset$ .
3. On the other hand, if  $A_i \neq \emptyset$  for all  $i \in I$ , then  $\prod_{i \in I} A_i \neq \emptyset$ <sup>1</sup>.

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<sup>1</sup>Thanks to the *Axiom of Choice*.



# Notation and examples

When	$\prod_{i \in I} A_i$
$I = \{1, \dots, n\}$	$\prod_{i=1}^n A_i$
$A_i = A, \forall i \in I$	$A^I$
$I = \{1, \dots, n\}$ and $A_i = A, \forall i \in I$	write $A^n$ instead of $A^{\{1, \dots, n\}}$ or $\prod_{i=1}^n A$
$I = \mathbb{N}$	write $A^\infty$ instead of $A^{\{1, 2, \dots\}}$ or $A^\mathbb{N}$
$I = [0, 1]$ and $A_i = \mathbb{R}, \forall i \in I$	$A^{[0, 1]}$ is the set of all functions on $[0, 1]$ .

Remark Infinite sequence  $\{a_1, a_2, \dots\}$  can be viewed as either

1. a function on  $\mathbb{N}$  or
2. Cartesian product with  $I = \mathbb{N}$ , namely,  $A^\infty$ .

HW Ex. 1.14, 1.21, 1.23.

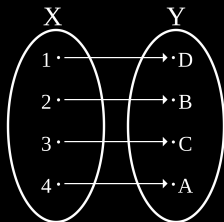
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Recall if  $f$  is both **one-to-one (injective)** and **onto (surjective)**, then  $f$  is **one-to-one correspondence (bijective)**.



Def. For two sets  $X$  and  $Y$ , if there exists a bijective function between  $X$  and  $Y$ , then we say that  $X$  and  $Y$  are **equivalent**, denoted  $X \sim Y$ .

Equivalent sets satisfy the following properties:

Reflexive	$A \sim A$
Symmetric	$A \sim B \Rightarrow B \sim A$
Transitive	$A \sim B \wedge B \sim C \Rightarrow A \sim C$

Remark Any relation that satisfies reflexive, symmetric, transitive properties is an equivalence relation.

E.g. 1. Let  $X = \{1, 2, 3, 4\}$  and  $Y = \{A, B, C, D\}$ . Then  $X \sim Y$  because one can find a bijective function between  $X$  and  $Y$ .

2. Let  $X = \{1, 2, 3, 4, 5\}$  and  $Y = \{A, B, C, D\}$ . Does  $X \sim Y$ ? Why?

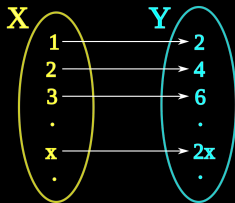
**Remark** For sets of finite element, in order to be equivalent, they have to have the same number of elements.

E.g. 3. Let  $X = \mathbb{N} = \{1, 2, 3, \dots\}$  and  $Y = \{2, 4, 6, 8, \dots\}$  (even integers).

Does  $X \sim Y$ ?

Do they have the same number of elements?

Sol. Here is one apparent solution<sup>2</sup>:  $f : X \rightarrow Y$  defined as  $f(x) = 2x$ :



This is a bijective function (why?). Hence,  $X \sim Y$ . They have the same number of elements (infinite many, which is called countably infinite).

□

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<sup>2</sup>There are other constructions. Can you give another bijection between  $X$  and  $Y$ ?

## Size of sets

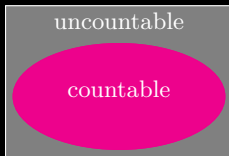
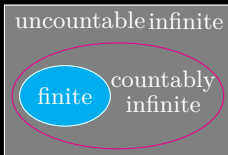
Def. Let  $A$  be a set. We say that

- a)  $A$  is *finite* if it is either empty or equivalent to the first  $N$  positive integers for some  $N \in \mathbb{N}$ .

In the former case,  $A$  is said to consist of 0 elements and, in the latter case,  $N$  elements.

- b)  $A$  is *infinite* if  $A$  is not finite.
- c)  $A$  is *countably infinite* if it is equivalent to  $\mathbb{N}$ .
- d)  $A$  is *countable* if it is either finite or countably infinite.
- e)  $A$  is *uncountable* if it is not countable.





E.g.1. Show that the set of  $\mathbb{Z}$  is countably infinite.

Sol. Construct the bijection

$$f : \mathbb{N} \rightarrow \mathbb{Z}:$$

$$1 \rightarrow 0$$

$$2 \rightarrow 1$$

$$3 \rightarrow -1$$

$$4 \rightarrow 2$$

$$5 \rightarrow -2$$

$$6 \rightarrow 3$$

$$7 \rightarrow -3$$

$$\vdots$$

It is not hard to see that

$$f(n) = \begin{cases} n/2 & n \text{ is even} \\ -(n-1)/2 & n \text{ is odd} \end{cases}$$

□

Sol'. Construct the bijection

$$f : \mathbb{N} \rightarrow \mathbb{Z}:$$

$$1 \rightarrow 0$$

$$2 \rightarrow -1$$

$$3 \rightarrow 1$$

$$4 \rightarrow -2$$

$$5 \rightarrow 2$$

$$6 \rightarrow -3$$

$$7 \rightarrow 3$$

$$\vdots$$

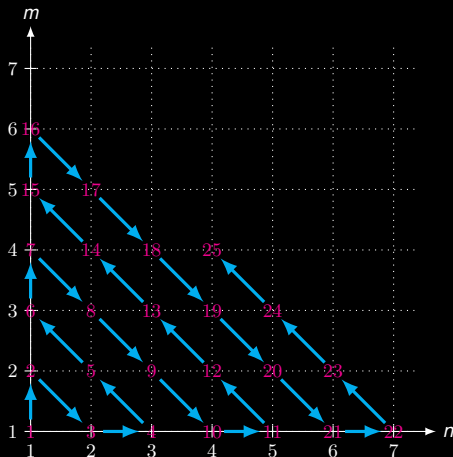
With enough patience, one can determine the formula:

$$f(n) = \frac{n(-1)^{n-1}}{2} - \frac{1 - (-1)^n}{4}$$

□

E.g. 2. Show that  $\mathbb{N}^2$  is countably infinite.

Sol. Construct a bijection  $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ :



Can you find a formula for this bijection?



Sol'. We claim that  $f : \mathbb{N}^2 \rightarrow \mathbb{N}$  defined below is a bijection:

$$f(m, n) := 2^{m-1}(2n-1).$$

a)  $f$  is one-to-one (injective): It suffices to show that

$$f(m_1, n_1) = f(m_2, n_2) \implies m_1 = m_2 \text{ and } n_1 = n_2.$$

Without loss of generality, suppose  $m_1 \geq m_2$ . Notice that

$$2^{m_1-m_2} = \frac{2n_1-1}{2n_2-1}. \quad (\star)$$

The LHS is an even integer unless  $m_1 = m_2$ . The RHS is a fraction unless  $n_1 = n_2$ . Hence, in order to make  $(\star)$  valid, one has to have both sides equal to 1. Hence,  $m_1 = m_2$  and  $n_1 = n_2$ .

b)  $f$  is onto (surjective). For any integer  $k \in \mathbb{N}$ , one has to find  $m$  and  $n$  such that  $f(m, n) = k$ . One can keep dividing  $k$  by 2 until it becomes an odd function. In this way, one easily find out  $m$  and  $n$ .  $\square$

Thm 1. A nonempty set is countable if and only if it is the range of an infinite sequence.

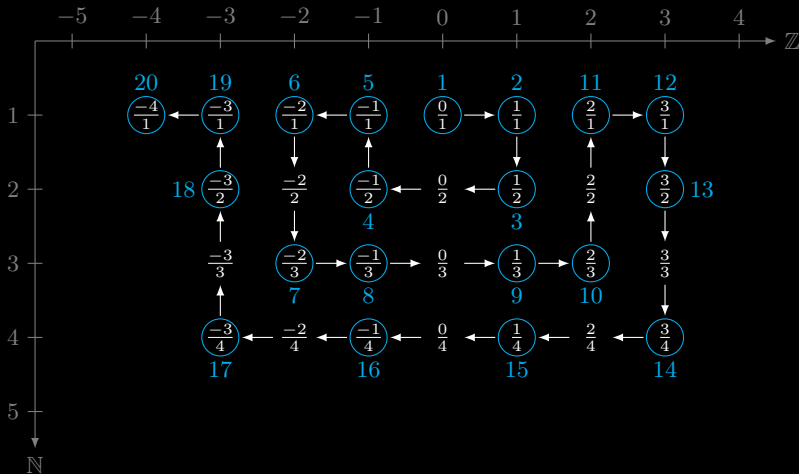
2. A subset of a countable set is countable.
3. The image of a countable set is countable.
4. A countable union of countable sets is countable.
5. The Cartesian product of two countable sets is countable.

E.g. 3. Show that the set of rationals  $\mathbb{Q}$  is countably infinite.

Recall that rationals are numbers that can be written as a ratio of two integers, i.e.,

$$q \in \mathbb{Q} \quad \Rightarrow \quad \exists m, n \in \mathbb{Z} \quad \text{such that} \quad q = \frac{m}{n}.$$

Sol. Let's construct the bijection from  $\mathbb{N}$  to  $\mathbb{Q}$ :



Can you find out the explicit formula for the bijection  $f : \mathbb{N} \rightarrow \mathbb{Q}$ ?



Sol'. Alternatively, one can define:

$$f(z, n) = \frac{z}{n}, \quad z \in \mathbb{Z}, n \in \mathbb{N}.$$

1.  $f : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$  is onto (surjective) but not one-to-one (injective).

Hence,  $f(\mathbb{Z} \times \mathbb{N}) \supset \mathbb{Q}$ .

(Figure above)

2.  $\mathbb{Z}$  is countable. (E.g. 1)
3.  $\mathbb{Z} \times \mathbb{N}$  is countable. (Thm. 5)
4.  $f(\mathbb{Z} \times \mathbb{N})$ , as the image of  $\mathbb{Z} \times \mathbb{N}$  under  $f$ , is countable. (Thm. 3)
5. Therefore,  $\mathbb{Q}$ , as a subset of the countable set  $f(\mathbb{Z} \times \mathbb{N})$ ,  
has to be countable. (Thm. 2)

□



HW Prove Thm's 1-5, which are Propositions 1.7 – 1.11 of the book.