

Topics in Analysis and Linear Algebra

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Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

*J. McDonald and N. Weiss, **A course in real analysis***, Academic Press, 2005.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

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What is a real number?



1

¹Image from Wikipedia.



2

²Image from

<https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/>

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

$$(F1) \quad x + y = y + x \text{ and } xy = yx. \quad (\text{Commutative})$$

$$(F2) \quad (x + y) + z = x + (y + z) \text{ and } (xy)z = x(yz). \quad (\text{Associative})$$

$$(F3) \quad x(y + z) = xy + xz. \quad (\text{Distributive})$$

$$(F4) \quad \text{There exist } 0, 1 \in \mathbb{R} \text{ with } 0 \neq 1 \text{ such that for all } x \in \mathbb{R}$$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \quad (\text{Identities})$$

$$(F5) \quad \text{For each } x \in \mathbb{R}, \text{ there exists a } -x \in \mathbb{R} \text{ such that } x + (-x) = 0 \text{ and, if } x \neq 0, \text{ there exists an } x^{-1} \in \mathbb{R} \text{ such that } xx^{-1} = 1. \quad (\text{Inverses})$$

Order Axioms

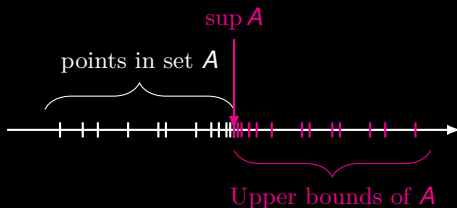
Let $x, y, z \in \mathbb{R}$. Then we have that

- (O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)
- (O2) $x < y$ implies that $x + z < y + z$.
- (O3) $x < y$ and $z > 0$ implies that $xz < yz$.
- (O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

Completeness Axiom

Axiom A nonempty subset of real numbers that is **bounded above** has a **least upper bound**, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$



Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$\sup_n x_n \leq a \iff \forall n, x_n \leq a$$

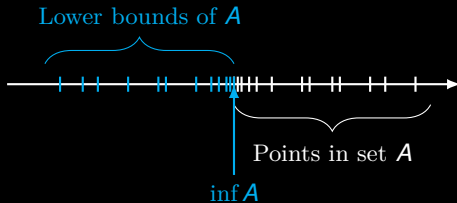
$$\sup_n x_n < a \iff \forall n, x_n < a$$

$$a < \sup_n x_n \iff \exists n, x_n > a$$

$$a \leq \sup_n x_n \iff \exists n, x_n \geq a$$

Corr. A nonempty subset of real numbers that is **bounded below** has a **greatest lower bound**, which is denoted as

$$\inf A, \quad \inf_{x \in A} x, \quad \text{or} \quad \inf\{x : x \in A\}.$$



Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence and $a \in \mathbb{R}$. Then

$$a \leq \inf_n x_n \iff \forall n, x_n \geq a$$

$$a < \inf_n x_n \iff \forall n, x_n > a$$

$$\inf_n x_n < a \iff \exists n, x_n < a$$

$$\inf_n x_n \leq a \iff \exists n, x_n \leq a$$

E.g. $\sup[0, 1) = 1$ and $\inf[0, 1) = 0$.

\mathbb{N} has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

Properties

1. Archimedean principle

For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

Between any two real numbers there is an **irrational** number.

3. Density of the rational numbers

Between any two real numbers there is an **rational** number.

Proof As exercises.



Extended Real Number System

Def. The *extended real numbers* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
$x < 0$	$x \cdot \infty = \infty \cdot x = -\infty$	$x \cdot (-\infty) = (-\infty) \cdot x = \infty$
$x = \infty$	$\infty + \infty = \infty$ $\infty \cdot \infty = \infty$	$(-\infty) + (-\infty) = -\infty$ $(-\infty) \cdot (-\infty) = \infty$
	$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$	

$\infty - \infty$ cannot be defined (HW).

Def. Let a and b be extended real numbers such that $a < b$. Then the *intervals on \mathbb{R}^** with *endpoints* a and b are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R}^* : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R}^* : a < x \leq b\}$$

$$[a, b] = \{x \in \mathbb{R}^* : a \leq x \leq b\}$$

If both a and b are in \mathbb{R} , these intervals are the *bounded intervals* in \mathbb{R} .

Otherwise, if either $a = -\infty$ or $b = \infty$, then these intervals are *unbounded intervals*.

Thm Every subset A of \mathbb{R}^* has both a least upper bound and greatest lower bound. Moreover,

- a) If $A = \emptyset$, then $\sup A = -\infty$ and $\inf A = \infty$.
- b) If A is bounded above in \mathbb{R} , then $\sup A \in \mathbb{R}$; otherwise, $\sup A = \infty$.
- c) If A is bounded below in \mathbb{R} , then $\inf A \in \mathbb{R}$; otherwise, $\inf A = -\infty$.

E.g.

- a) $\inf \mathbb{N} = 1$ and $\sup \mathbb{N} = \infty$.
- b) $\inf \mathbb{Z} = -\infty$ and $\sup \mathbb{Z} = \infty$.
- c) If I is an interval in \mathbb{R}^* with endpoints a and b , $a \leq b$. Then $\inf I = a$ and $\sup I = b$.

HW Ex. 2.10 and 2.11 on p. 43.

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Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence³, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

³In this case, we say that $\{x_n\}_{n=1}^{\infty}$ is *convergent*. Otherwise, we say that $\{x_n\}_{n=1}^{\infty}$ is *divergent*.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n \rightarrow \infty} (n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

$\{n^2\}_{n=1}^{\infty}$ is divergent.

Recall that $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ is the *extended real line*.

Def. A sequence of real numbers $\{a_1, a_2, \dots\}$ is said to *converge in \mathbb{R}^** if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
 - (ii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.
 - (iii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.
-

- (i) We say that the sequence converges in \mathbb{R} or the limit exists and is finite.
- (ii) We say the sequence converges to ∞ and write $\lim_{n \rightarrow \infty} x_n = \infty$.
- (iii) We say the sequence converges to $-\infty$ and write $\lim_{n \rightarrow \infty} x_n = -\infty$.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ converges in \mathbb{R} .

$\{(-1)^n\}_{n=1}^{\infty}$ does not converge in \mathbb{R}^* .

$\{n^2\}_{n=1}^{\infty}$ converges in \mathbb{R}^* and $\lim_{n \rightarrow \infty} n^2 = \infty$.

Monotone sequence

Def. If $x_1 \leq x_2 \leq \dots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \dots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotone.

$\{n^2\}_{n=1}^{\infty}$ is monotone and it is nondecreasing.

Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R}^* .

Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in \mathbb{R} if it is **bounded above** and is ∞ otherwise.

b) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

$$\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in \mathbb{R} if it is **bounded below** and is $-\infty$ otherwise.

Proof. We will prove the case when $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. The nonincreasing case can be proved in a similar way.

As $\sup_n x_n$ always exists in \mathbb{R}^* , we need to consider two cases:

Case I: $\sup_n x_n \in \mathbb{R}$

Case II: $\sup_n x_n = \infty$

Let's prove Case I here. Let $x = \sup_n x_n$. In order to show that $\lim_n x_n = x$, by monotonicity, we need to show that

$$\forall \epsilon > 0 \exists N \forall n \text{ s.t. } (n \geq N) \rightarrow (x - a_n \leq \epsilon).$$

Proof. (Continued) Otherwise,

$$\exists \epsilon > 0 \forall N \exists n \text{ s.t. } (n \geq N) \wedge (x - a_n > \epsilon).$$

Hence, there are infinitely many terms falling below $x - \epsilon$.

Since $\{x_n\}$ is nondecreasing, this implies all a_n fall below $x - \epsilon$, i.e.,

$$a_n < x - \epsilon, \quad \text{for all } n \geq 1.$$

which is equivalent to $\sup_n x_n < x - \epsilon$. This contradicts with the fact that $\sup_n x_n = x$.

Therefore,

$$\lim_n x_n = x = \sup_n x_n.$$

□

E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is nondecreasing and converges in \mathbb{R} . It is bounded above.

$\{(-1)^n\}_{n=1}^{\infty}$ is not monotone.

$\{n^2\}_{n=1}^{\infty}$ is nondecreasing, does not converge in \mathbb{R} , converges in \mathbb{R}^* .

Cluster points

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

- a) A real number x is said to be a **cluster point** of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.
 - b) ∞ **is a cluster point** of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n > M$.
 - c) $-\infty$ **is a cluster point** of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n < M$.
-

a) $x \in \mathbb{R}$ is a cluster point of $\{x_n\}$ if

$$\forall \epsilon \forall N \exists n \quad (n \geq N) \rightarrow (|x - x_n| < \epsilon). \quad (1)$$

E.g.1 $\{(n-1)/n\}_{n=1}^{\infty}$ has one cluster point: 1.

$\{(-1)^n\}_{n=1}^{\infty}$ has two cluster points: -1 and $+1$.

$\{n^2\}_{n=1}^{\infty}$ has one cluster point: $+\infty$.

E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ .

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Thm A convergent sequence has exactly one cluster point, namely, its limit.
Thus, a sequence having more than one cluster point cannot converge.

Proof Suppose that $\{x_n\}$ is a convergent sequence and let x be its limit, namely, $\lim_{n \rightarrow \infty} x_n = x$. We need to prove:

- (1) x is a cluster point.
- (2) x is the only cluster point of $\{x_n\}$.

We also need to consider two cases:

Case I: $x \in \mathbb{R}$.

Case II: $x = \infty$ or $-\infty$.

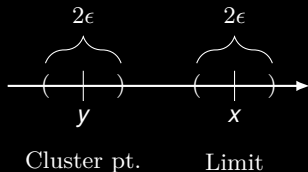
We will focus on Case I only.

Now we first prove (1).

	Once for all rest
	$\underbrace{\hspace{1cm}}$
$\lim_n x_n = x$	$\forall \epsilon \exists N \forall n \ (n \geq N) \rightarrow (x_n - x < \epsilon)$
$\Downarrow ??$	
x is a cluster point of $\{x_n\}$	$\forall \epsilon \forall \tilde{N} \exists \tilde{n} \ (\tilde{n} \geq \tilde{N}) \wedge (x_{\tilde{n}} - x < \epsilon)$
	$\underbrace{\hspace{1cm}}$
	Infinite games

(1) is proved by choosing any $\tilde{n} \geq \max(\tilde{N}, N)$.

As for (2), suppose y is another cluster point.



By choosing any $\epsilon < |x - y|/2$, we see that

1. In the ϵ -neighborhood of y , there are infinitely many terms.
2. In the ϵ -neighborhood of x , all but finite many terms are here.

Contradiction!

Therefore, there exists only one cluster point.

□

A few more properties

1. A sequence is convergent iff each subsequence is convergent.
2. **Sandwich theorem:** If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \rightarrow L$ and $b_n \rightarrow L$, then $c_n \rightarrow L$.

Limit superior and limit inferior

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n \rightarrow \infty} x_n := \sup_n \left(\inf_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\limsup_{n \rightarrow \infty} x_n := \inf_n \left(\sup_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

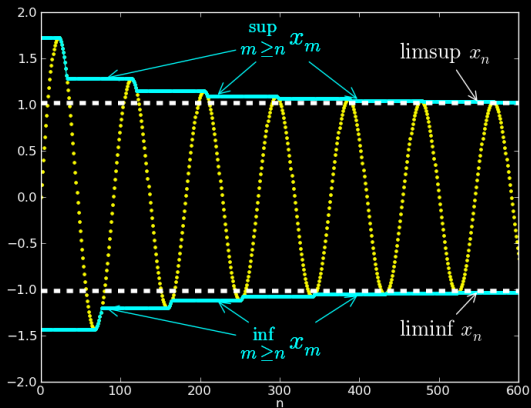
$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

$$\begin{aligned} \inf_n y_n &= \inf_n \sup_{m \geq n} x_m = \limsup_{x \rightarrow \infty} x_n \\ \sup_n z_n &= \sup_n \inf_{m \geq n} x_m = \liminf_{x \rightarrow \infty} x_n. \end{aligned}$$

Hence,

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} x_m.$$



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⁴Image from Wikipedia.

Characterization of the limsup and liminf.

Thm Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

- a) $\limsup x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,
 - i) there is an $N \in \mathbb{N}$ such that $x_n \leq x + \epsilon$ for all $n \geq N$, and
 - ii) for each $n \in \mathbb{N}$, there is an $m \geq n$ such that $x_m > x - \epsilon$.
 - b) $\limsup x_n = \infty$ iff for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there is an $n \geq N$ such that $x_n > M$; in other words, iff the sequence is unbounded from above.
 - c) Similarly, $\limsup x_n = -\infty$ if and only if $\lim x_n = -\infty$.
-

- a) $\limsup x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,
 - i) the ϵ -neighborhood of x has been visited infinitely many times; and
 - ii) only finite many terms are greater than $x + \epsilon$.

Proof. (Sketch) We only prove part (a). Let $y_n = \sup_{k \geq n} x_k$.

$$\begin{aligned} \limsup x_n = x &\iff \lim_{n \rightarrow \infty} \sup_{k \geq n} x_k = x \\ &\iff \forall \epsilon \exists N \forall n \ (n \geq N) \rightarrow \left(\sup_{k \geq n} x_k \in (x - \epsilon, x + \epsilon) \right) \end{aligned}$$

$$\sup_{k \geq n} x_k < x + \epsilon \iff \text{all terms starting from } n \text{ fall below } x + \epsilon$$

$$\sup_{k \geq n} x_k > x - \epsilon \iff \exists k \geq n \text{ s.t. } x_k > x - \epsilon$$

□

Thm Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

- a) $\liminf x_n = x \in \mathbb{R}$ iff for each $\epsilon > 0$,
 - i) there is an $N \in \mathbb{N}$ such that $x_n \geq x - \epsilon$ for all $n \geq N$, and
 - ii) for each $n \in \mathbb{N}$, there is an $m \geq n$ such that $x_m < x + \epsilon$.
- b) $\liminf x_n = -\infty$ iff for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there is an $n \geq N$ such that $x_n < M$; in other words, iff the sequence is unbounded from below.
- c) Similarly, $\liminf x_n = \infty$ if and only if $\lim x_n = \infty$.

Still other characterization (as an exercise):

Thm $\limsup_{n \rightarrow \infty} x_n = x \in \mathbb{R}$ if and only if x is the **smallest** real number such that for any positive real number $\epsilon > 0$, there exists a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

$\liminf_{n \rightarrow \infty} x_n = x \in \mathbb{R}$ if and only if x is the **largest** real number such that for any positive real number $\epsilon > 0$, there exists a natural number N such that $x_n > x - \epsilon$ for all $n > N$.

Proof. HW for motivated students.



E.g.1 Let $x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ has two cluster points: ± 1 , amount which

$$\liminf_{n \rightarrow \infty} x_n = -1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi) \frac{n^2}{n^2 + 8}$$

Similar examples

$$x_n = \sin\left(\frac{n\pi}{3}\right)$$

E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ , among which

$$\liminf_{n \rightarrow \infty} x_n = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

$$\liminf_{n \rightarrow \infty} r_n = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} r_n = +\infty.$$

The above examples suggest that

Prop. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then

- a) $\limsup x_n$ is the largest cluster point of $\{x_n\}_{n=1}^{\infty}$.
- b) $\liminf x_n$ is the smallest cluster point of $\{x_n\}_{n=1}^{\infty}$.

Proof. We only prove (a). (b) can be proved in a similar way.

Let $x = \limsup x_n$. We have seen that x is a cluster point.

It remains to prove that x is the largest cluster point. The case when $x = \pm\infty$ is left for the motivated students.

Now assume that $x \in \mathbb{R}$.

Only finite many terms exceed $x + 1$, hence, ∞ is not a cluster point.

Let $y \in \mathbb{R}$ s.t. $x < y$. Set $\epsilon = (y - x)/2$.

Only finite many terms exceed $x + \epsilon$.

Since $y - \epsilon = x + \epsilon$, the ϵ -neighborhood of y has been visited only finitely many times.

Hence, y cannot be a cluster point. □

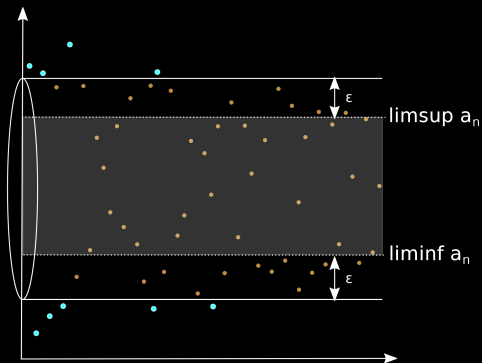
Properties

1.

$$\inf_n x_n \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers **converges in \mathbb{R}^*** if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = c \iff \lim_{n \rightarrow \infty} x_n = c.$$



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E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

⁵Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2} \left(\limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n \rightarrow \infty} x_n + \epsilon, \limsup_{n \rightarrow \infty} x_n - \epsilon \right).$$

Cauchy criterion

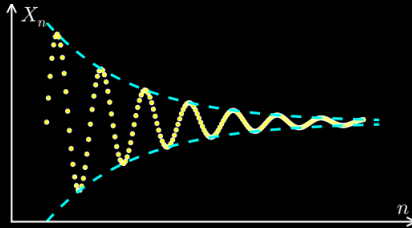
As we have seen that

A sequence of real numbers **converges in \mathbb{R}^*** if and only if it has exactly one cluster point.

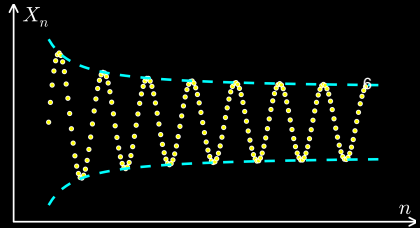
There is another famous criterion for a sequence to **converge in \mathbb{R}** :

Cauchy Criterion

Cauchy sequence



Non-Cauchy sequence



⁶Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}.$$

Thm (Cauchy Criterion)

A sequence of real numbers *converges in \mathbb{R}* iff it is Cauchy.

Proof. " \Rightarrow " Easy!

" \Leftarrow ": ...



E.g.1 Let $a_n = \sqrt{n}$. Show that

- (i) The consecutive terms become arbitrarily close to each other as $n \rightarrow \infty$.
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

$$|a_n - a_m| \geq 1.$$

Indeed, let's choose $m = N$ and $n = 4N$

$$|a_n - a_m| = \sqrt{4N} - \sqrt{N} = \sqrt{N}(\sqrt{4} - 1) = \sqrt{N} \geq 1 = \epsilon.$$

□

HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

Open sets

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exists an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of \mathbb{R} . Then K cannot be an open set. For example, \mathbb{N} , \mathbb{Q} , \mathbb{Z} are not open sets.

\mathbb{Q}^c – the set of irrational numbers – is not an open set.

Properties of open sets

1. \mathbb{R} and \emptyset are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open. (arbitrary union)

Proof. Exercise.



Let $Q_n = (-1/n, 1/n)$. Then $\bigcap_{n \in \mathbb{N}} Q_n = \{0\}$ is not an open set.

Def. For $a, b \in \mathbb{R}^*$ with $a < b$, (a, b) is an open set, which is called an *open interval*.

Thm. Each open set O is a countable union of disjoint open intervals.

Moreover, the representation is unique in the sense that if \mathcal{C} and \mathcal{D} are two pairwise disjoint collections of open intervals whose union is O , then $\mathcal{C} = \mathcal{D}$.

Closed sets

Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

The set of all limit point of E , denoted \bar{E} , is called the *closure* of E .

E.g. $\overline{\mathbb{R}} = \mathbb{R}$ and $\overline{\emptyset} = \emptyset$.

Let $a, b \in \mathbb{R}$ such that $a < b$. Then

$$\overline{(a, b)} = \overline{(a, b]} = \overline{[a, b)} = \overline{[a, b]} = [a, b].$$

$$\overline{\mathbb{N}} = \mathbb{N} \text{ and } \overline{\mathbb{Z}} = \mathbb{Z}.$$

$$\overline{\mathbb{Q}} = \mathbb{R} \text{ and } \overline{\mathbb{Q}^c} = \mathbb{R}.$$

If A is a finite subset of \mathbb{R} , then $\overline{A} = A$.

Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

E.g. \mathbb{R} and \emptyset are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

\mathbb{N} and \mathbb{Z} are closed sets.

The sets of rationals \mathbb{Q} and irrationals \mathbb{Q}^c are neither open nor close.

If A is a finite subset of \mathbb{R} , then A is a close set.

Thm. A set is open if and only if its complement is closed.

Or equivalently, a set is closed if and only if its complement is open.

Properties of closed sets

1. \mathbb{R} and \emptyset are closed sets.
2. If A and B are closed sets, so is $A \cup B$. (finite union)
3. If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed. (arbitrary intersection)

Let $Q_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$. Then $\bigcup_{i \in \mathbb{N}} Q_n = (0, 1)$ is an open set.

or

$\bigcup_{r \in \mathbb{Q}} \{r\} = \mathbb{Q}$ is neither open nor closed.

Relative open and closed sets

Def. Let $G \subset D \subset \mathbb{R}$.

a) G is said to be **open in D** if for each $x \in G$, there is an $r > 0$ such that

$$(x - r, x + r) \cap D \subset G.$$

b) G is said to be **closed in D** if $D \setminus G$ is open in D .

E.g.

D	G	Is G open in \mathbb{R}	Is G open in D
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
\mathbb{N}	$A \subset \mathbb{N}$	closed	open

Thm. Let $D \subset \mathbb{R}$. A set $G \subset D$ is open in D if and only if there is an open set O of \mathbb{R} such that $G = D \cap O$.

In other words, the open sets in D are precisely the open sets of \mathbb{R} intersected with D .

HW Ex. 2.38, 2.46, 2.47, 2.49, 2.52 on p. 63 – 64.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

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§ 3.7 Some techniques in calculus

Def. A *real-valued function* is a function whose range is a subset of \mathbb{R} . If $f : \Omega \rightarrow \mathbb{R}$, we say that f is a *real-valued function on Ω* .

Algebraic operations

Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

$$(f + g)(x) := f(x) + g(x)$$

$$(\alpha f)(x) := \alpha f(x)$$

$$(f \cdot g)(x) := f(x)g(x)$$

Continuous functions

Def. Let $D \subset \mathbb{R}$, $f : D \rightarrow \mathbb{R}$, and $x_0 \in D$. We say that f is *continuous at x_0* if for each $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in D$ and $|x - x_0| < \delta$.

We say that f is *continuous on D* if it is continuous on every point of D .

We use $C(D)$ to denote the collection of all continuous functions on D .

If f is not continuous at x_0 , then we say that f is *discontinuous at x_0* or that x_0 is a *point of discontinuity* of f .

f is continuous at x_0

\Updownarrow

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \{ |x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon \}$$

$f \in C(D)$

\Updownarrow

$$\forall x \in D \boxed{f \text{ is continuous at } x}$$

\Updownarrow

$$\forall x \in D \boxed{\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \{ |x - x_0| \leq \delta \rightarrow |f(x) - f(x_0)| \leq \epsilon \}}$$

E.g.

- a) Let $D = (0, \infty)$ and define $f(x) = 1/x$. Then f is continuous function on D .
- b) Let $D = \mathbb{R}$ and define $f(0) = 0$ and $f(x) = \sin(1/x)$ for $x \neq 0$. Then f is a continuous function except at 0.
- c) Let $D = \mathbb{R}$ and define $f(x) = \lfloor x \rfloor$. Then f is continuous except at points of \mathbb{Z} .
- d) Every function is continuous on \mathbb{N} . Or in other words, any infinite series $\{a_n\}_{n \in \mathbb{N}}$, when viewed as a function $a : \mathbb{N} \rightarrow \mathbb{R}$, is a continuous function.

Thm. Let $D \subset \mathbb{R}$. The collection $C(D)$ of continuous functions on D is an algebra of functions, i.e., if $f, g \in C(D)$ and $a \in \mathbb{R}$, then

a) $f + g \in C(D)$.

b) $\alpha f \in C(D)$.

c) $f \cdot g \in C(D)$.

Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in \mathbb{R} , i.e., the preimage of each open set in \mathbb{R} is open in D .

Corr. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous if and only if $f^{-1}(O)$ is open (in \mathbb{R}) whenever O is open (in \mathbb{R}).

Proof. “ \implies ”

Suppose that f is continuous on D . Let O be an arbitrary open set in \mathbb{R} . We need to show that $f^{-1}(O)$ is open in D .

Hence, we need to show that for any $x_0 \in f^{-1}(O)$, one can find $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O)$.

Notice that

$$x_0 \in f^{-1}(O) \iff f(x_0) \in O.$$

Since O is open, one can find $r > 0$ such that r -neighborhood of $f(x_0)$ is in O .

By the continuity of f at x_0

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D \mid x - x_0 \mid < \delta \rightarrow \mid f(x) - f(x_0) \mid < \epsilon$$

applied with $\epsilon = r$, we can find some $\delta > 0$ such that

$$\forall x \left\{ \left(x \in (x_0 - \delta, x_0 + \delta) \cap D \right) \rightarrow \left(f(x) \in (f(x_0) - r, f(x_0) + r) \right) \right\}.$$

Proof. (Continued)

Because

$$f(x) \in (f(x_0) - r, f(x_0) + r) \subset O$$

we have that

$$f(x) \in O,$$

or equivalently,

$$x \in f^{-1}(O).$$

Hence, if $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $x \in f^{-1}(O)$, i.e.,

$$(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O).$$

Proof. “ \Leftarrow ”

Suppose $f^{-1}(O)$ is open in D for each open set $O \subset \mathbb{R}$.

For each $x_0 \in D$, we need to prove f is continuous at x_0 , namely,

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon$$

Now fix arbitrary $x_0 \in D$ and arbitrary $\epsilon > 0$.

Let $O = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, which is an open interval.

By the assumption, $f^{-1}(O)$ is open in D . Hence, there is $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \cap D \subset f^{-1}(O).$$

In other words,

$$\forall x \in D \left(|x - x_0| < \delta \rightarrow x \in f^{-1}(O) \right).$$

Finally, notice that

$$x \in f^{-1}(O) \iff f(x) \in O \iff |f(x) - f(x_0)| < \epsilon.$$

□

Pointwise limits

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω , that is, $f_n : \Omega \rightarrow \mathbb{R}$ for each $n \in \mathbb{N}$. Then we say that $\{f_n\}_{n=1}^{\infty}$ **converges pointwise on Ω** if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R} .

If $\{f_n\}_{n=1}^{\infty}$ converges pointwise in Ω , then we define

$$f : \Omega \rightarrow \mathbb{R}$$

by

$$f(x) := \lim_{n \rightarrow \infty} f_n(x),$$

which is called the **pointwise limit of the sequence of functions $\{f_n\}_{n=1}^{\infty}$** . In this case, we also call the sequence of functions $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** to f , denoted as $f_n \rightarrow f$ pointwise.

E.g.

- a) Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f_n(x) = (1 + x/n)^n$. Then $f_n \rightarrow f$ pointwise on \mathbb{R} with $f(x) = e^x$.
- b) Let $f_n : D \rightarrow \mathbb{R}$ defined as $f_n(x) = x^n$.
- (i) If $D = [0, 1]$, then $f_n \rightarrow f$ pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

- (ii) If $D = [-1, 1]$, $\{f_n\}_{n=1}^\infty$ fails to converge pointwise because the sequence $\{(-1)^n\}_{n=1}^\infty$ does not converge.
- (iii) If $D = [0, 3]$, $\{f_n\}_{n=1}^\infty$ fails to converge pointwise because the sequence $\{3^n\}_{n=1}^\infty$ does not converge in \mathbb{R} .

c) Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} n^2 x & \text{if } |x| < \frac{1}{n} \\ 1/x & \text{otherwise} \end{cases}$$

Then $f_n \rightarrow f$ pointwise on \mathbb{R} with

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{otherwise} \end{cases}$$

d) Let $D \subset \mathbb{R}$ and define $f_n(x) = x/n$. Then $f_n \rightarrow 0$ pointwise on D .

Def. Let \mathcal{F} be a collection of real-valued functions on Ω . We say that \mathcal{F} is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \rightarrow f$ pointwise on Ω , then $f \in \mathcal{F}$.

E.g.

- a) If \mathcal{F} is the collection of all real-valued functions, then \mathcal{F} is closed under pointwise limits.
- b) If $\mathcal{F} = C(D)$, then it is not closed under pointwise limit.

Def. Let $\{f_n\}_{n=1}^\infty$ be a sequence of real-valued functions on a set Ω , that is, $f_n : \Omega \rightarrow \mathbb{R}$ for each $n \in \mathbb{N}$. Then we say that $\{f_n\}_{n=1}^\infty$ *converges uniformly* to the real-valued function f on Ω , if for each $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for all $x \in \Omega$. We write $f_n \rightarrow f$ uniformly.

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω .

$f_n \rightarrow f$ **pointwise** on Ω iff

$$\forall x \in \Omega \left[\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \left(n \geq N \rightarrow |f_n(x) - f(x)| < \epsilon \right) \right]$$

$f_n \rightarrow f$ **uniformly** on Ω iff

$$\left[\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \right] \forall x \in \Omega \left(n \geq N \rightarrow |f_n(x) - f(x)| < \epsilon \right)$$

Thm. $C(D)$ is closed under uniform limits.

More precisely, let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \rightarrow f$ uniformly. Then $f \in C(D)$.

Proof. In order to show $f \in C(D)$, we need to show that

$$\forall x_0 \in D \forall \epsilon > 0 \exists \delta > 0 \left(|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon \right).$$

Let's fix arbitrary $x_0 \in D$ and $\epsilon > 0$.

$f_n \rightarrow f$ uniformly implies that for some $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < \epsilon/3 \quad \text{for all } x \in D.$$

Because f_N is continuous on D , and hence, at x_0 , we can find $\delta > 0$ such that

$$|f_N(x) - f(x)| < \epsilon/3 \quad \text{whenever } x \in D \text{ and } |x - x_0| < \delta.$$

Hence, whenever $x \in D$ and $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f(x_0)| + |f(x) - f(x_0)| + |f(x) - f(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore, $f \in C(D)$. □

E.g. (Revisit the previous examples)

- a) Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f_n(x) = (1 + x/n)^n$. Then $f_n \rightarrow f$ pointwise on \mathbb{R} with $f(x) = e^x$. Does $f_n \rightarrow f$ uniformly?
- b) Let $f_n : D \rightarrow \mathbb{R}$ defined as $f_n(x) = x^n$ with $D = [0, 1]$. Then $f_n \rightarrow f$ pointwise with

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since $f \notin C(D)$, this convergence cannot be uniform.

c) We have seen that

$$f_n(x) = \begin{cases} n^2 x & \text{if } |x| < \frac{1}{n} \\ 1/x & \text{otherwise} \end{cases} \rightarrow f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{otherwise} \end{cases} \quad \text{pointwise.}$$

Because $f \notin C(\mathbb{R})$, this convergence cannot be uniform.

d) Let $D \subset \mathbb{R}$ and define $f_n(x) = x/n$. Then $f_n \rightarrow 0$ pointwise on D .
However,

(i) If $D = [a, b]$ with $a, b \in \mathbb{R}$, then the convergence is uniform.

(ii) If $D = \mathbb{R}$, this convergence cannot be uniform (why?).

Finally, the collection $\mathcal{C}(D)$ of real-valued continuous functions is closed under: $+$, \cdot , scalar multiplication, and uniform convergence.

HW Ex. 3.59, 2.60, 2.61, 2.66, on p. 71 – 73.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

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§ 3.5 Various other types of continuity

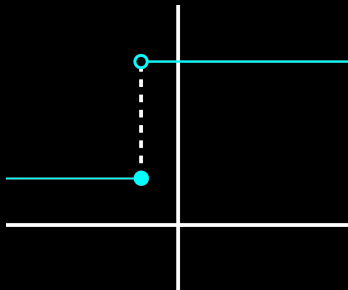
§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

- ▶ Left (right)-continuity
- ▶ Lower (upper) semi-continuity
- ▶ Uniform continuity

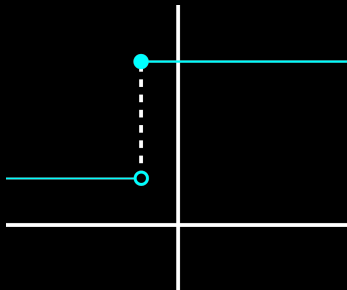
Def. f is *left-continuous at c* if

$$\lim_{x \rightarrow c^+} f(x) = f(c)$$



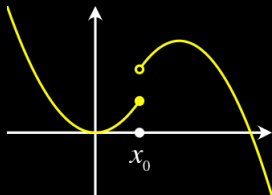
Def. f is *right-continuous at c* if

$$\lim_{x \rightarrow c^-} f(x) = f(c)$$



Def. f is *lower semi-continuous at x_0* if

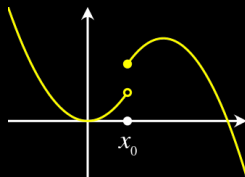
$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$$



$f(x_0)$ can be all points
at or below the yellow point.

f is *upper semi-continuous at x_0*
if

$$\limsup_{x \rightarrow x_0} f(x) \leq f(x_0)$$



$f(x_0)$ can be all points
at or above the yellow point.

(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let I be an interval of \mathbb{R} . Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Def.' f is *uniformly continuous over I* if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \}$$

f is continuous at $x_0 \in I$ iff

$$\forall \epsilon > 0 \forall x \in I \exists \delta > 0 \{ |x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \epsilon \}$$

Π_2 -form

f is uniformly continuous over I iff

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \}$$

Π_3 -form

Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I , we need to show

$$\begin{aligned} & \neg \left(\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \right) \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ \neg \{ |x - y| < \delta \} \vee |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \{ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon \} \end{aligned}$$

Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

□

Prop. 2 If I is compact⁷ set such as $I = [a, b]$, then

f is continuous at all points in $I \iff f$ is uniformly continuous on I .

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on \mathbb{R} nor on $[0, \infty)$.

⁷namely, bounded and closed

Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

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Some part of subsection is taken from Chapter 1 Section 4 of

*P. Billingsley, **Probability and Measure**, Wiley, 1995.*

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Remark Both $\limsup_n A_n$ and $\liminf_n A_n$ are sets.

Use the relation:

set	logic
\bigcap	\forall
\bigcup	\exists

$$\begin{aligned}
 \omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
 &\iff (\forall n \geq 1) (\exists k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\limsup_n A_n = [A_n \text{ i.o.}]$$

Use the relation:

set	logic
\bigcap	\forall
\bigcup	\exists

$$\begin{aligned}
 \omega \in \liminf_n A_n &\iff \omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \\
 &\iff (\exists n \geq 1) (\forall k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{all but finitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\liminf_n A_n = [A_n \text{ all but finitely many}]$$

Def. If both $\limsup_n A_n$ and $\liminf_n A_n$ exist and are equal, then the *limit set* of the sequence $\{A_n\}$ is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

which is also often written as $A_n \rightarrow A$.

Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$\begin{array}{ccc}
 \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \liminf_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
 \cap & & \cap \\
 A_n & & \\
 \cap & & \\
 \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \limsup_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n
 \end{array}$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space (Ω, \mathbb{P})

(i) The above Property (ii) can be translated to a probability statement:

$$\begin{array}{ccc}
 \mathbb{P} \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \mathbb{P} \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\liminf_n A_n \right) \\
 & & \uparrow \wedge \\
 & & \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 \mathbb{P}(A_n) & & \uparrow \wedge \\
 & & \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 & & \uparrow \wedge \\
 \mathbb{P} \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\limsup_n A_n \right)
 \end{array}$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned} 1 &\geq \mathbb{P}(A_n^c \text{ all but finitely many}) = 1 - \mathbb{P}(\{A_n^c \text{ all but finitely many}\}^c) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \\ &\geq 1 - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

□

Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$A_1 = (-1, 0]$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{2}{3}\right]$$

$$A_4 = \left(-\frac{1}{4}, \frac{3}{4}\right]$$

$$\vdots \quad \vdots$$

$$A_{100} = \left(-\frac{1}{100}, \frac{99}{100}\right]$$

$$\vdots \quad \vdots$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n} \right] = [0, 1)$$

and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 \right) = (-1, 1].$$

Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

□

Exercise

(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3} \right] \qquad A_4 = \left(\frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

Sol.

$$\begin{aligned}
 & \liminf_n A_n \\
 &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
 &= (0, 1) \cup (0, 1) \\
 &= (0, 1)
 \end{aligned}$$

Sol. (continued) Similarly,

$$\begin{aligned}
 & \limsup_n A_n \\
 &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \left(-\frac{1}{n}, \frac{n+1}{n} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \left(-\frac{1}{n+1}, \frac{n+2}{n+1} \right] \right\} \\
 &= [0, 1] \cap [0, 1] \\
 &= [0, 1]
 \end{aligned}$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \frac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \frac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$.

Solution:

$$\liminf_{n \rightarrow \infty} A_n = \{0\} \subset [0, 1] = \limsup_{n \rightarrow \infty} A_n$$

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Various other types of continuity

§ 3.6 Liminf and limsup of sets

§ 3.7 Some techniques in calculus

Examples

1. $\int_0^1 \tan^{-1}(x) dx$
2. $\int_0^x t^2 e^t dt$
3. $\int e^x \sin(x) dx$
4. $\int_0^1 (x^2 + 1) e^{-x} dx$
5. $\int_4^9 \frac{\ln y}{y} dy$
6. $\int_1^3 r^3 \ln r dr$
7. $\int_1^2 (\ln x)^2 dx$
8. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) d\theta$
9. $\int_0^{\pi/2} \sin^{2n+1}(x) dx = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}$
10. $\int_0^{\pi/2} \sin^{2n}(x) dx = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$

Examples

1. e^x

2. $\sin(x)$

3. e^{x^2}