Topics in Analysis and Linear Algebra

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Summer Bootcamp for
Emory Biostatistics and Bioinformatics
PhD Program

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https://github.com/chenle02/2021_Summer_Bio_bootcamp_slides/

Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

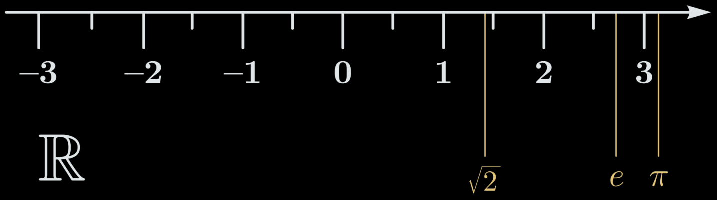
§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

What is a real number?



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[1] Image from Wikipedia.

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

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Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exit $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$$x + 0 = x \quad \text{and} \quad x \cdot 1 = x. \qquad \text{(Identities)}$$

(F5) For each $x \in \mathbb{R}$, there exits a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exits an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

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Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

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(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

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Completeness Axiom

Axiom A nonempty subset of real numbers that is bounded above has a least upper bound, which is denoted as

$$\sup A, \quad \sup_{x \in A} x, \quad \text{or} \quad \sup\{x : x \in A\}.$$

Corr. A nonempty subset of real numbers that is bounded below has a greatest lower bound, which is denoted as

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E.g. $\sup[0,1) = 1$ and $\inf[0,1) = 0$.

$\mathbb{N}$ has no least upper bound, but $\inf \mathbb{N} = 1$.

Let $A = \{x : x^2 < 3\}$. Then

$$\sup_{x \in A} x = \sqrt{3} \quad \text{and} \quad \inf_{x \in A} x = -\sqrt{3}.$$

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Properties

1. Archimedean principle

   For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

2. Density of the irrational numbers

   Between any two real numbers there is an irrational number.

3. Density of the rational numbers

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Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence[2], namely,

$$\lim_{n\to\infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n\to\infty} x_n = L \quad \Longleftrightarrow \quad \forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \in \mathbb{N} \; s.t. \; (n \geq N \to |x_n - L| < \epsilon)$$

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

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Def. Let $\mathbb{R}^*$ denote *the extended real line*, namely, $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$.

Def. A sequence of real numbers $\{a_1, a_2, \cdots\}$ is said to *converge in* $\mathbb{R}^*$ if one of the following three conditions hold:

(i) The sequence converges to a finite real number as in the previous definition.

(ii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

(iii) For each $M \in \mathbb{R}$, there exits an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

(i) We say that the sequence converges in $\mathbb{R}$.

(ii) Denoted as $\lim_{n\to\infty} x_n = \infty$

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Axiom Let A be a nonempty subset of real numbers that is bounded above. Then *the least upper bound* of A exits, which is denoted by

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Axiom Let A be a nonempty subset of real numbers that is bounded below. Then *the greatest lower bound* of A exits, which is denoted by

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Prop. Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$.

Moreover,

(a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : \ n \in \mathbb{N}\}.$$

In particular, if $\{x_n\}_{n=1}^{\infty}$ converges in $\mathbb{R}$, it is bounded above.

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(a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.

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1. If a sequence is bounded and monotonic, then it is convergent.

2. A sequence is convergent iff each subsequence is convergent.

3. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \to L$ and $b_n \to L$, then $c_n \to L$.

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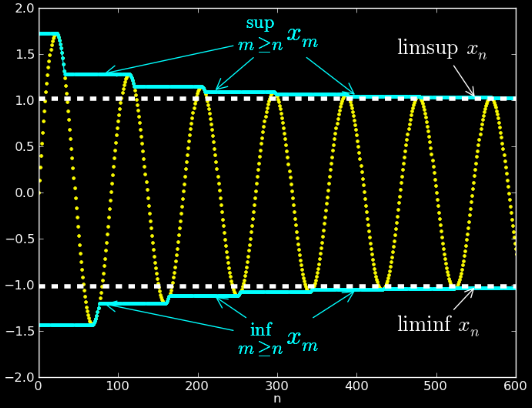

[3]

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Can we use the first-order logic to formulate the definitions?

Def'. $\limsup_{n\to\infty} x_n = x \in \mathbb{R}$ if and only if x is the smallest real number such that for any positiver real number $\epsilon > 0$, there exits a natural number N such that $x_n < x + \epsilon$ for all $n > N$.

Set

$$A(b): \quad \forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \in \mathbb{N}\ \{n \geq N \to x_n < b + \epsilon\}$$

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Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi)\frac{n^2}{n^2+8}$$

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2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}^*$ if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

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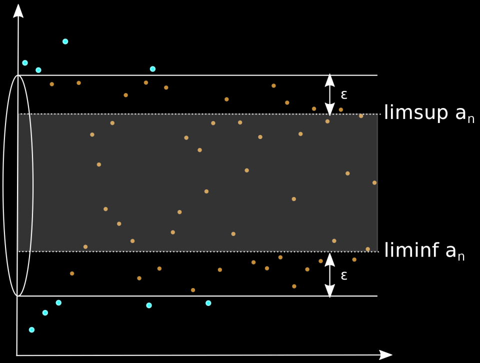

[4]

E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n\to\infty} x_n - \epsilon,\ \limsup_{n\to\infty} x_n + \epsilon\right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

[4] Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2}\left(\limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n\right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

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As we have seen that

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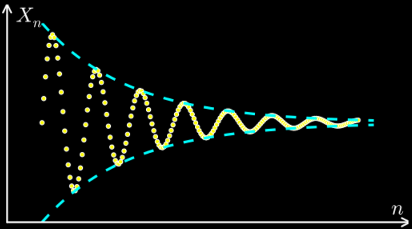
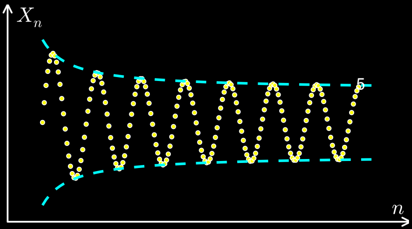


Non-Cauchy sequence



[5] Images from Wikipedia.

Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

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A sequence of real numbers converges in $\mathbb{R}$ iff it is Cauchy.

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E.g.1 Let $a_n = \sqrt{n}$. Show that

(i) The consecutive terms become arbitrarily close to each other as $n \to \infty$.

(ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

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$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of $\mathbb{R}$. Then K cannot be an open set. For example, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ are not open sets.

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Properties

1. $\mathbb{R}$ and $\emptyset$ are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
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Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

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Def. A subset $F \subset \mathbb{R}$ is said to be a *closed set* if $\overline{F} = F$, i.e, F contains all its limit points.

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Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

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Def. Let $G \subset D \subset \mathbb{R}$.

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D	G	Is G open in $\mathbb{R}$	Is G open in D
$[0,2]$	$[0,1)$	Neither open nor closed	open
$[0,2]$	$[0,1]$	closed	closed
$\mathbb{N}$	$A \subset \mathbb{N}$	closed	open

Def. A *real-valued function* is a function whose range is a subset of $\mathbb{R}$. If $f : \Omega \to \mathbb{R}$, we say that f is a *real-valued function on* Ω.

Def. *Algebraic operations:* Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

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(Local) Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. f is *continuous at a point c* if the limit of $f(x)$, as x approaches c, exits and is equal to $f(c)$.

Def'. (Epsion-delta definition) The function f is *continuous at a point c* if

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Here is a more abstract definition of continuous functions:

Thm let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in $\mathbb{R}$.

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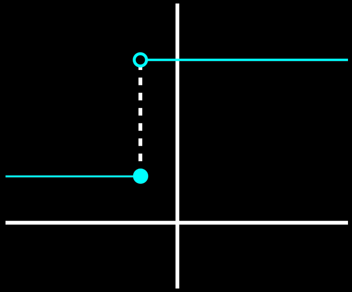

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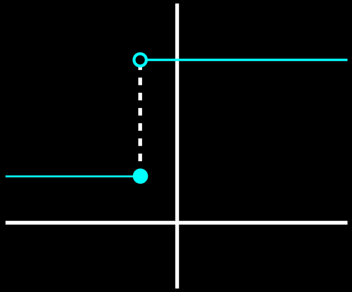

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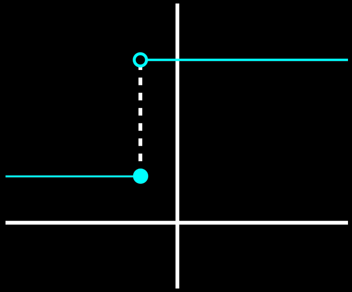

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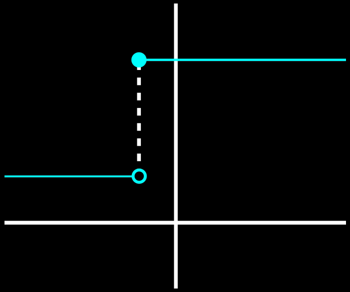

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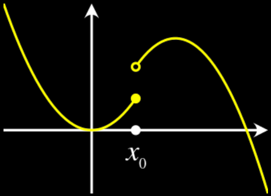

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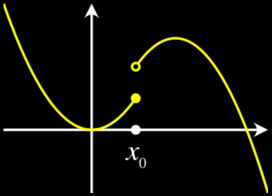

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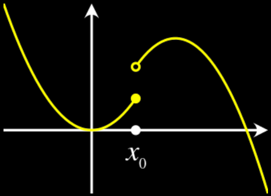

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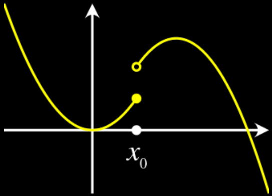

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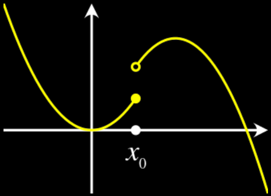

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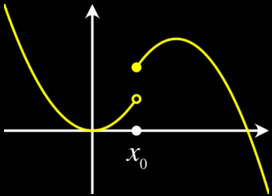

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(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let I be an interval of $\mathbb{R}$. Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exits a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

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Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I, we need to show

$$\neg\left(\forall\epsilon > 0\, \exists\delta > 0\, \forall x \in I\, \forall y \in I\, \{|x-y| < \delta \to |f(x) - f(y)| < \epsilon\}\right)$$

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Sol. (Continued) In other words, we need to find $\epsilon > 0$ such that no matter how small $\delta > 0$ is chosen, we can always find out $x, y \in I$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

$$\begin{aligned} |f(x) - f(y)| &= |x^3 - y^3| \\ &= |x - y| \times |x^2 + xy + y^2| \\ &\geq \frac{\delta}{3} \times \left(\frac{1}{(\sqrt{\delta})^2} + \frac{1}{\sqrt{\delta}} \times \frac{1}{\sqrt{\delta}} + \frac{1}{(\sqrt{\delta})^2} \right) \\ &= 1 = \epsilon. \end{aligned}$$

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Prop. 2 If I is compact [6] set such as $I = [a, b]$, then

$$f \text{ is continuous at all points in } I \iff f \text{ is uniformly continuous on } I.$$

E.g. $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.
$f(x) = x^3$ is uniformly continuous on $[-1, 1]$ but neither on $\mathbb{R}$ nor on $[0, \infty)$.

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Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

Thm Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then $\{f(x_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

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Notation For $D \subset \mathbb{R}$, let $C(D)$ denote the set of continuous functions defined on D.

Thm (*Algebra of $C(D)$*) Let $D \subset \mathbb{R}$. Then the collection $C(D)$ of continuous functions on D is an algebra of functions, that is, for all $f, g \in C(D)$ and $\alpha \in \mathbb{R}$,

$$f + g \in C(D)$$
$$\alpha f \in C(D)$$
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Remark Can one add one more operation in this algebra: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(D)$, under what condition the limit $f_n \to f$ is closed?

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Def. $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on Ω, namely, $f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$.

We say that $\{f_n\}_{n=1}^{\infty}$ *converges pointwise on* Ω if for each $x \in \Omega$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers converges in $\mathbb{R}$.

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E.g.

(a) $f_n \in C(\mathbb{R})$ defined as $f_n = (1 + x/n)^n$. Then f_n converges pointwise on $\mathbb{R}$ to $f(x) = e^x$. It is clear that $f \in C(\mathbb{R})$.

(b) Let $D = [0, 1]$ and $f_n \in C(D)$ be defined as $f_n = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

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Def. Let $\mathcal{F}$ be a collection of real-valued functions on Ω. We say that $\mathcal{F}$ is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \to f$ pointwise on Ω, then $f \in \mathcal{F}$.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω.
We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω, if

$$\boxed{\forall \epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \geq N}\, \forall x \in \Omega \;\; |f_n(x) - f(x)| < \epsilon,$$

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Recall $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if

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Prop. Let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \to f$ uniformly. Then $f \in C(D)$.

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Proof.

Therefore, the collection $C(D)$ of real-valued continuous functions is closed under: $+$, $\cdot$, scalar multiplication, and uniform convergence.

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

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Use the relation:

set	logic
$\bigcap$	$\forall$
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$$\omega \in \limsup_n A_n \iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

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Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right)^c = \left(\limsup_n A_n \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$
\begin{array}{ccccc}
\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \liminf_n A_n & \Longrightarrow & \lim_{n\to\infty}\bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & \\
A_n & & & & \rotatebox[origin=c]{-90}{$\subseteq$} \\
\rotatebox[origin=c]{-90}{$\subseteq$} & & & & \\
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\end{array}
$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space $(\Omega, \mathbb{P})$

(i) The above Property (ii) can be translated to a probability statement:

$$
\begin{array}{ccc}
\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k\right) & \uparrow & \mathbb{P}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \mathbb{P}\left(\liminf_{n} A_n\right) \\
 & & \rotatebox{-90}{$\leq$} \\
\rotatebox{-90}{$\leq$} & & \liminf_{n\to\infty} \mathbb{P}(A_n) \\
\mathbb{P}\left(A_n\right) & & \rotatebox{-90}{$\leq$} \\
\rotatebox{-90}{$\leq$} & & \limsup_{n\to\infty} \mathbb{P}(A_n) \\
 & & \rotatebox{-90}{$\leq$} \\
\mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) & \downarrow & \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = \mathbb{P}\left(\limsup_{n} A_n\right)
\end{array}
$$

(ii) *Borel Cantelli lemma*

$$\sum_n P(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ all but finitely many}) = 1.$$

Proof.

$$\begin{aligned} 1 \geq \mathbb{P}\left(A_n \text{ all but finitely many}\right) &= 1 - \mathbb{P}\left(\{A_n \text{ all but finitely many}\}^c\right) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \\ &\geq 1 - \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

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□

Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$\begin{aligned}
A_1 &= (-1, 0] \\
A_2 &= \left(-\frac{1}{2}, \frac{1}{2}\right] \\
A_3 &= \left(-\frac{1}{3}, \frac{2}{3}\right] \\
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&\vdots \quad \vdots \\
A_{100} &= \left(-\frac{1}{100}, \frac{99}{100}\right] \\
&\vdots \quad \vdots
\end{aligned}$$

Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k}\right] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right) = [0, 1).$$

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(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n}\right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2}\right]$$

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$$A_5 = \left(-\frac{1}{5}, \frac{6}{5}\right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6}\right]$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99}\right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100}\right]$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$

Show that $\lim_n A_n$ doesn't exist by demonstrating that

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Sol.

$$
\begin{aligned}
&\liminf_n A_n \\
&= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k-(-1)^k}{k} \right] \right\} \\
&= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
&= (0,1) \cup (0,1) \\
&= (0,1)
\end{aligned}
$$

Sol. (continued) Similarly,

$$\begin{aligned}
&\limsup_n A_n \\
=&\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right] \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\bigcup_{k=n}^{\infty}\left(\frac{(-1)^k}{k},\frac{k-(-1)^k}{k}\right]\right\} \\
=&\left\{\bigcap_{n=1,3,5}^{\infty}\left(-\frac{1}{n},\frac{n+1}{n}\right]\right\}\bigcap\left\{\bigcap_{n=2,4,6}^{\infty}\left(-\frac{1}{n+1},\frac{n+2}{n+1}\right]\right\} \\
=&[0,1]\cap[0,1] \\
=&[0,1]
\end{aligned}$$

$\square$

Chapter 3. Real Number System and Calculus

Examples

1. $\int_0^1 \tan^{-1}(x)dx$
2. $\int_0^x t^2 e^t dt$
3. $\int e^x \sin(x)dx$

more to come ...

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Taylor expansions

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more to come ...