

Topics in Analysis and Linear Algebra

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Summer Bootcamp for
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PhD Program

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Chapter 3. Real Number System and Calculus

This part is mostly based on Chapter 2 of

*J. McDonald and N. Weiss, **A course in real analysis**, Academic Press, 2005.*

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

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What is a real number?



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¹Image from Wikipedia.



2

²Image from

<https://geeksoutofthebox.com/2019/03/15/simons-real-numbers-diagram/>

Real number system can be formulated in three groups of axioms

(F) Field Axioms

(O) Order Axioms

(C) Completeness Axioms

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Field Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(F1) $x + y = y + x$ and $xy = yx$. (Commutative)

(F2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$. (Associative)

(F3) $x(y + z) = xy + xz$. (Distributive)

(F4) There exist $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that for all $x \in \mathbb{R}$

$x + 0 = x$ and $x \cdot 1 = x$. (Identities)

(F5) For each $x \in \mathbb{R}$, there exists a $-x \in \mathbb{R}$ such that $x + (-x) = 0$ and, if $x \neq 0$, there exists an $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. (Inverses)

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Order Axioms

Let $x, y, z \in \mathbb{R}$. Then we have that

(O1) $x < y$ and $y < z$ implies that $x < z$. (Transitive)

(O2) $x < y$ implies that $x + z < y + z$.

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(O4) Exactly one of $x = y$, $x < y$, and $x > y$ holds. (Trichotomous)

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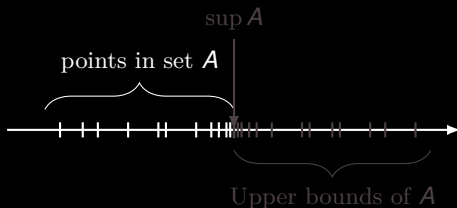
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Axiom A nonempty subset of real numbers that is **bounded above** has a **least upper bound**, which is denoted as

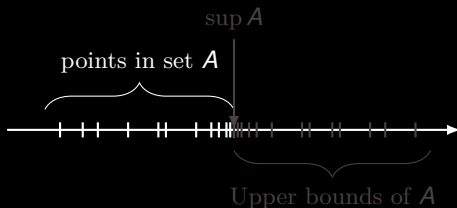
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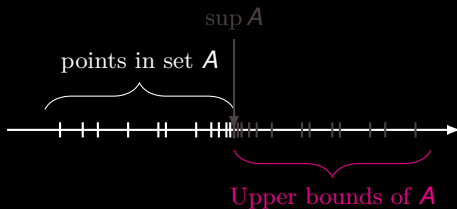
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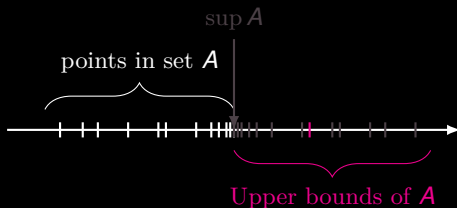
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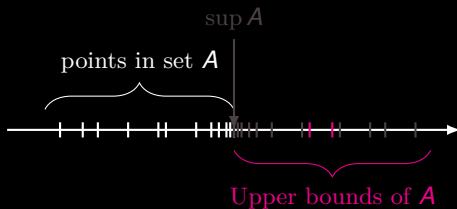
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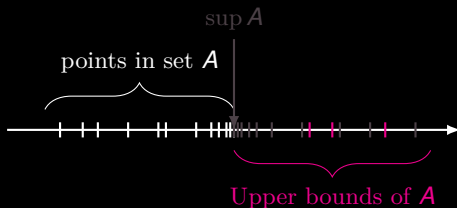
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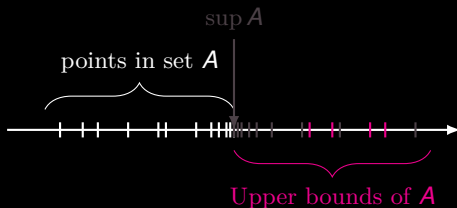
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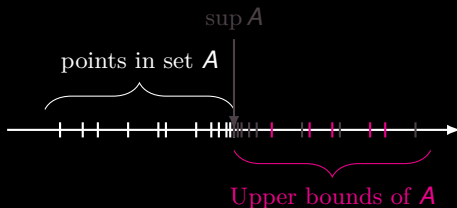
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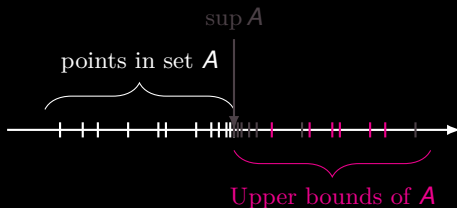
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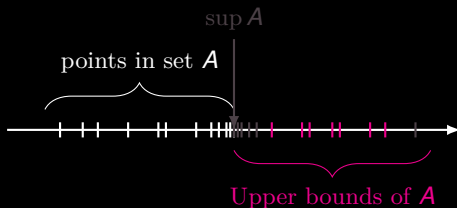
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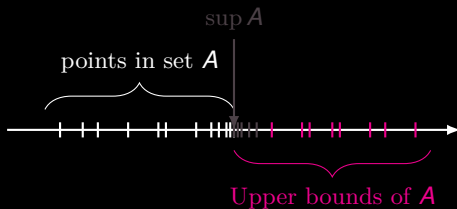
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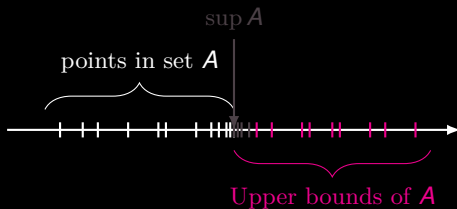
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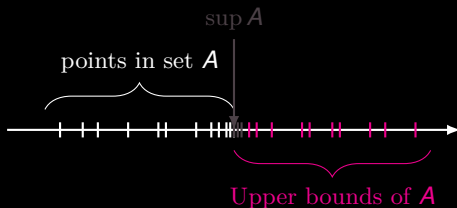
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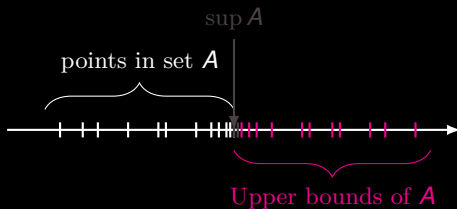
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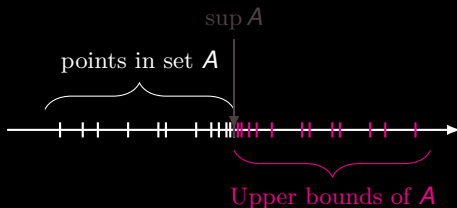
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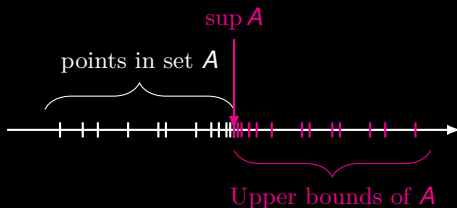
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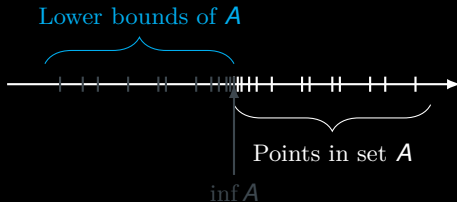
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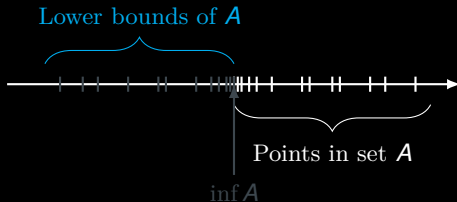
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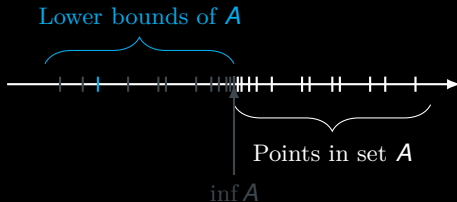
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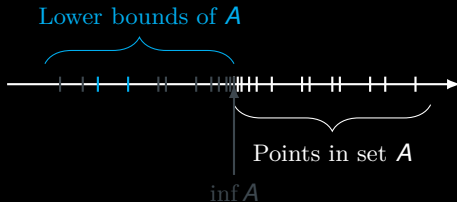
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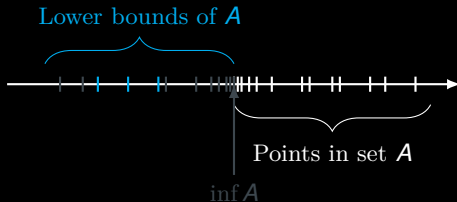
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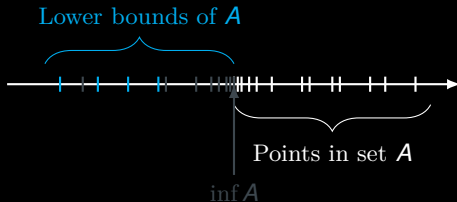
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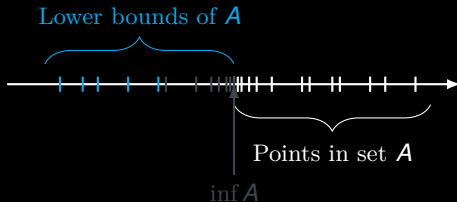
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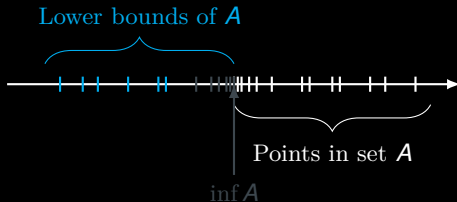
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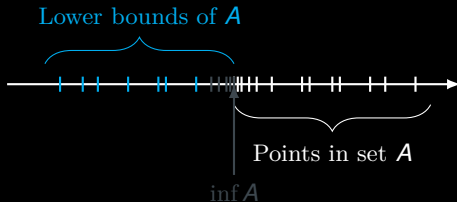
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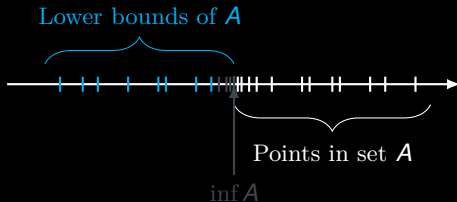
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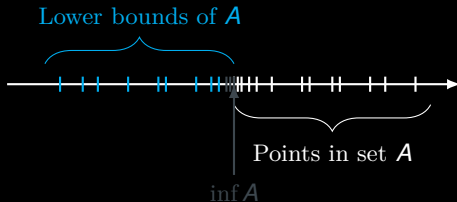
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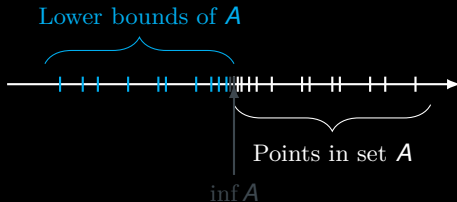
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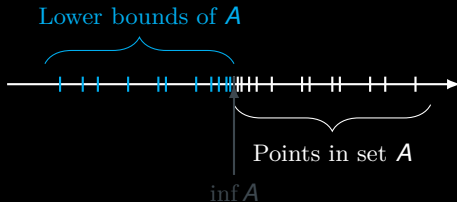
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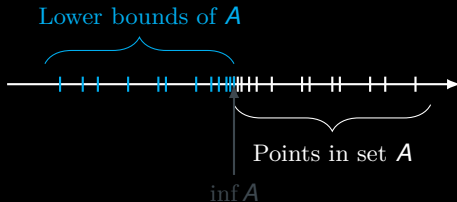
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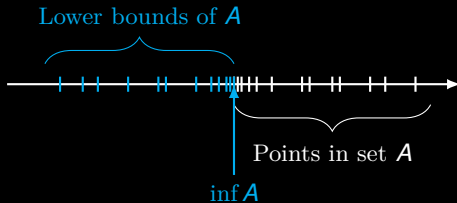
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Properties

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For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

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Between any two real numbers there is an irrational number.

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Between any two real numbers there is a rational number.

Proof As exercises.



Properties

1. Archimedean principle

For each $x \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $n > x$.

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Extended Real Number System

Def. The *extended real numbers* $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$.

$x \in \mathbb{R}$	$x + \infty = \infty + x = \infty$	$x - \infty = -\infty + x = -\infty$
$x > 0$	$x \cdot \infty = \infty \cdot x = \infty$	$x \cdot (-\infty) = (-\infty) \cdot x = -\infty$
$x = 0$	$0 \cdot \infty = \infty \cdot 0 = 0$	$0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$
$x < 0$	$x \cdot \infty = \infty \cdot x = -\infty$	$x \cdot (-\infty) = (-\infty) \cdot x = \infty$
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$\infty - \infty$ cannot be defined (HW).

Def. Let a and b be extended real numbers such that $a < b$. Then the *intervals on \mathbb{R}^** with *endpoints* a and b are as follows:

$$(a, b) = \{x \in \mathbb{R}^* : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R}^* : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R}^* : a < x \leq b\}$$

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If both a and b are in \mathbb{R} , these intervals are the *bounded intervals* in \mathbb{R} .

Otherwise, if either $a = -\infty$ or $b = \infty$, then these intervals are *unbounded intervals*.

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Thm Every subset A of \mathbb{R}^* has both a least upper bound and greatest lower bound. Moreover,

- a) If $A = \emptyset$, then $\sup A = -\infty$ and $\inf A = \infty$.
- b) If A is bounded above in \mathbb{R} , then $\sup A \in \mathbb{R}$; otherwise, $\sup A = \infty$.
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E.g.

a) $\inf \mathbb{N} = 1$ and $\sup \mathbb{N} = \infty$.

b) $\inf \mathbb{Z} = -\infty$ and $\sup \mathbb{Z} = \infty$.

c) If I is an interval in \mathbb{R}^* with endpoints a and b , $a \leq b$. Then $\inf I = a$ and $\sup I = b$.

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HW Ex. 2.10 and 2.11 on p. 43.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that the real number $L \in \mathbb{R}$ is the *limit* of this sequence³, namely,

$$\lim_{n \rightarrow \infty} x_n = L$$

if and only if for every real number $\epsilon > 0$, there exists a natural number N such that for all $n > N$, we have $|x_n - L| < \epsilon$.

Def'.

$$\lim_{n \rightarrow \infty} x_n = L \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} \text{ s.t. } (n \geq N \rightarrow |x_n - L| < \epsilon)$$

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is convergent and $\lim_{n \rightarrow \infty} (n-1)/n = 1$.

$\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

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Recall that $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ is the *extended real line*.

Def. A sequence of real numbers $\{a_1, a_2, \dots\}$ is said to *converge in* \mathbb{R}^* if one of the following three conditions hold:

- (i) The sequence converges to a finite real number as in the previous definition.
- (ii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.
- (iii) For each $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < M$.

- (i) We say that the sequence converges in \mathbb{R} or the limit exists and is finite.
- (ii) We say the sequence converges to ∞ and write $\lim_{n \rightarrow \infty} x_n = \infty$.
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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ converges in \mathbb{R} .

$\{(-1)^n\}_{n=1}^{\infty}$ does not converge in \mathbb{R}^* .

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Monotone sequence

Def. If $x_1 \leq x_2 \leq \dots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nondecreasing*.

If $x_1 \geq x_2 \geq \dots$, then $\{x_n\}_{n=1}^{\infty}$ is said to be *nonincreasing*.

$\{x_n\}_{n=1}^{\infty}$ is said to be *monotone* if it is either nondecreasing or nonincreasing.

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Thm Any monotone sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R}^* .

Moreover, we have the following:

a) If $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, then

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

In particular, $\{x_n\}_{n=1}^{\infty}$ converges in \mathbb{R} if it is bounded above and is ∞ otherwise.

b) If $\{x_n\}_{n=1}^{\infty}$ is nonincreasing, then

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Proof. We will prove the case when $\{x_n\}_{n=1}^{\infty}$ is nondecreasing. The nonincreasing case can be proved in a similar way.

As $\sup_n x_n$ always exists in \mathbb{R}^* , we need to consider two cases:

Case I: $\sup_n x_n \in \mathbb{R}$

Case II: $\sup_n x_n = \infty$

Let's prove Case I here. Let $x = \sup_n x_n$. In order to show that $\lim_n x_n = x$, by monotonicity, we need to show that

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Proof. (Continued) Otherwise,

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Hence, there are infinitely many terms falling below $x - \epsilon$.

Since $\{x_n\}$ is nondecreasing, this implies all a_n fall below $x - \epsilon$, i.e.,

$$a_n < x - \epsilon, \quad \text{for all } n \geq 1.$$

which is equivalent to $\sup_n x_n < x - \epsilon$. This contradicts with the fact that $\sup_n x_n = x$.

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E.g. $\{(n-1)/n\}_{n=1}^{\infty}$ is nondecreasing and converges in \mathbb{R} . It is bounded above.

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Cluster points

Def. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

- a) A real number x is said to be a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $|x - x_n| < \epsilon$.
- b) ∞ is a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n > M$.
- c) $-\infty$ is a *cluster point* of $\{x_n\}_{n=1}^{\infty}$ if for each $M \in \mathbb{R}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n < M$.

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E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ .

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$.

Thm A convergent sequence has exactly one cluster point, namely, its limit.
Thus, a sequence having more than one cluster point cannot converge.

Proof Suppose that $\{x_n\}$ is a convergent sequence and let x be its limit, namely, $\lim_{n \rightarrow \infty} x_n = x$. We need to prove:

- (1) x is a cluster point.
- (2) x is the only cluster point of $\{x_n\}$.

We also need to consider two cases:

Case I: $x \in \mathbb{R}$.

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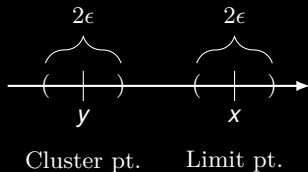
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As for (2), suppose y is another cluster point.



By choosing any $\epsilon < |x - y|/2$, we see that

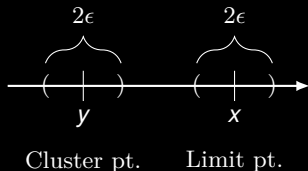
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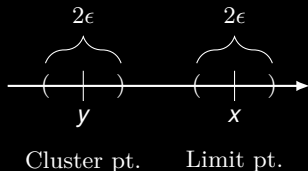
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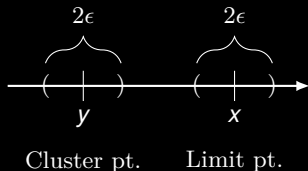
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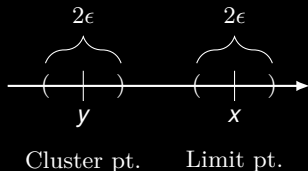
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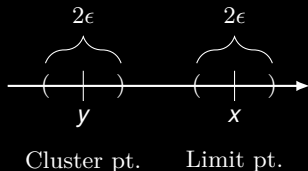
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A few more properties

1. A sequence is convergent iff each subsequence is convergent.
2. Sandwich theorem: If $x_n \leq c_n \leq b_n$ for all $n > N$ and $x_n \rightarrow L$ and $b_n \rightarrow L$, then $c_n \rightarrow L$.

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Limit superior and limit inferior

Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n \rightarrow \infty} x_n := \sup_n \left(\inf_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

The *limit superior* of $\{x_n\}_{n=1}^{\infty}$ is defined as

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Def. The *limit inferior* of a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as

$$\liminf_{n \rightarrow \infty} x_n := \sup_n \left(\inf_{m \geq n} x_m \right) \in \mathbb{R}^*.$$

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Remark Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ defined as

$$y_n := \sup_{m \geq n} x_m \quad \text{and} \quad z_n := \inf_{m \geq n} x_m,$$

are, respectively, nonincreasing and nondecreasing, we see that

$$\begin{aligned} \inf_n y_n &= \inf_n \sup_{m \geq n} x_m = \limsup_{x \rightarrow \infty} x_n \\ \sup_n z_n &= \sup_n \inf_{m \geq n} x_m = \liminf_{x \rightarrow \infty} x_n. \end{aligned}$$

Hence,

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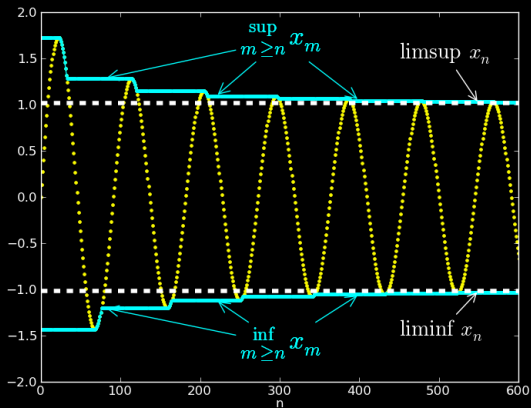
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4

⁴Image from Wikipedia.

Characterization of the limsup and liminf.

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Still other characterization (as an exercise):

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Proof. HW for motivated students.

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$$\liminf_{n \rightarrow \infty} x_n = -1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = 1.$$

Variations

$$x_n = (-1)^n \frac{n+5}{n}$$

$$x_n = 3 + \sin(n\pi) \frac{n^2}{n^2 + 8}$$

Similar examples

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E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ , among which

$$\liminf_{n \rightarrow \infty} x_n = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

E.g.3 Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers. By the density of the rational numbers, every extended real number is a cluster point of the sequence $\{r_n\}_{n=1}^{\infty}$, amount which

$$\liminf_{n \rightarrow \infty} r_n = -\infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} r_n = +\infty.$$

E.g.2 Consider the sequence $\{2, 1, 0, 2, 2, \frac{1}{2}, 2, 3, \frac{2}{3}, 2, 4, \frac{3}{4}, \dots\}$, that is,

$$x_n = \begin{cases} (n-3)/n & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3} \\ (n+1)/3 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

It has three cluster points: 1, 2, ∞ , among which

$$\liminf_{n \rightarrow \infty} x_n = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} x_n = \infty.$$

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Proof. We only prove (a). (b) can be proved in a similar way.

Let $x = \limsup x_n$. We have seen that x is a cluster point.

It remains to prove that x is the largest cluster point. The case when $x = \pm\infty$ is left for the motivated students.

Now assume that $x \in \mathbb{R}$.

Only finite many terms exceed $x + 1$, hence, ∞ is not a cluster point.

Let $y \in \mathbb{R}$ s.t. $x < y$. Set $\epsilon = (y - x)/2$.

Only finite many terms exceed $x + \epsilon$.

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Hence, y cannot be a cluster point. □

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Properties

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$$\inf_n x_n \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq \sup_n x_n$$

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers converges in \mathbb{R}^* if and only if it has exactly one cluster point. In such cases, the limit of the sequence is the unique cluster point. In other words,

$$\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = c \iff \lim_{n \rightarrow \infty} x_n = c.$$

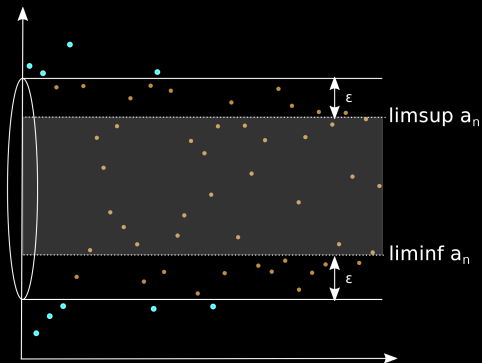
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5

E.g. For all $\epsilon > 0$, the interval

$$\left(\liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right)$$

contains *all but finitely many* numbers in $\{x_n\}$.

⁵Image is from Wikipedia.

E.g. (Continued) Similarly, if

$$\liminf_{n \rightarrow \infty} x_n < \limsup_{n \rightarrow \infty} x_n,$$

then for all ϵ with

$$0 < \epsilon < \frac{1}{2} \left(\limsup_{n \rightarrow \infty} x_n - \liminf_{n \rightarrow \infty} x_n \right),$$

infinitely many numbers in $\{x_n\}$ fall outside of the interval

$$\left(\liminf_{n \rightarrow \infty} x_n - \epsilon, \limsup_{n \rightarrow \infty} x_n + \epsilon \right).$$

Cauchy criterion

As we have seen that

A sequence of real numbers converges in \mathbb{R}^* if and only if it has exactly one cluster point.

There is another famous criterion for a sequence to converge in \mathbb{R} :

Cauchy Criterion

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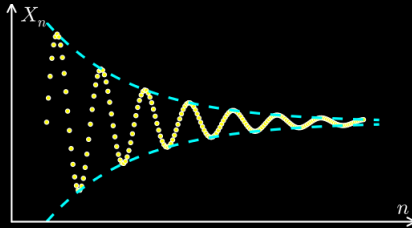
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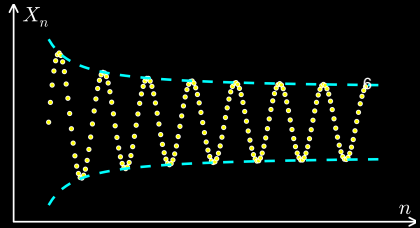
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Cauchy sequence



Non-Cauchy sequence



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Def. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is called a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_n - x_m| < \epsilon$.

Def.' A sequence $\{x_n\}_{n=1}^{\infty}$ is *Cauchy* iff

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{ |x_n - x_m| < \epsilon \}.$$

Thm (Cauchy Criterion)

A sequence of real numbers converges in \mathbb{R} iff it is Cauchy.

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Proof. “ \Rightarrow ” Easy!

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E.g.1 Let $a_n = \sqrt{n}$. Show that

- (i) The consecutive terms become arbitrarily close to each other as $n \rightarrow \infty$.
- (ii) The sequence is not Cauchy.

Sol. (i) This is because

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(ii) To show $\{a_n\}_{n=1}^{\infty}$ is not Cauchy, let's negate the statement as follows:

$$\begin{aligned} & \neg (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \{|x_n - x_m| < \epsilon\}) \\ \iff & \exists \epsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N \{|x_n - x_m| > \epsilon\} \end{aligned}$$

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Sol. (Continued) Let's choose $\epsilon = 1$. For any $N \in \mathbb{N}$, we need to find $m, n \geq N$ such that

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Indeed, let's choose $m = N$ and $n = 4N$

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Variation

$$\lim_{n \rightarrow \infty} \left(1 + \text{Small}\right)^{\text{Large}} = e^{\lim_{n \rightarrow \infty} \text{Small} \times \text{Large}}$$

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HW Ex. 2.15, 2.17 (a), 2.18, 2.19, 2.22, 2.29.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. A subset $O \subset \mathbb{R}$ is said to be an *open set* if for each $x \in O$, there exists an $r > 0$ such that $(x - r, x + r) \subset O$.

E.g. (a, b) with $-\infty \leq a < b \leq \infty$ is an open set, which are called *open interval intervals*.

$(0, 1]$ is not an open set.

Let K be a nonempty countable subset of \mathbb{R} . Then K cannot be an open set. For example, \mathbb{N} , \mathbb{Q} , \mathbb{Z} are not open sets.

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Properties

1. \mathbb{R} and \emptyset are open sets.
2. If A and B are open sets, so is $A \cap B$. (finite intersection)
3. If $\{O_i\}_{i \in I}$ is a collection of open sets, then $\bigcup_{i \in I} O_i$ is open. (arbitrary union)

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Def. Let $E \subset \mathbb{R}$. A real number x is called a *limit point* of E if for each $\epsilon > 0$, there is a $y \in E$ such that $|y - x| < \epsilon$.

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Let $a, b \in \mathbb{R}$ such that $a < b$. Then

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E.g. \mathbb{R} and \emptyset are both open and closed.

Intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$ with $a, b \in \mathbb{R}$ are closed sets. They are called *closed intervals*.

\mathbb{N} and \mathbb{Z} are closed sets.

The sets of rationals \mathbb{Q} and irrationals \mathbb{Q}^c are neither open nor close.

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Properties

0. A set is open if and only if its complement is closed.



1. \mathbb{R} and \emptyset are closed sets.

2. If A and B are closed sets, so is $A \cup B$. (finite union)

3. If $\{F_i\}_{i \in I}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

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Def. Let $G \subset D \subset \mathbb{R}$.

(a) G is said to be open in D if for each $x \in G$, there is an $r > 0$ such that

$$(x - r, x + r) \cap D \subset G.$$

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E.g.

D	G	Is G open in \mathbb{R}	Is G open in D
$[0, 2]$	$[0, 1)$	Neither open nor closed	open
$[0, 2]$	$[0, 1]$	closed	closed
\mathbb{N}	$A \subset \mathbb{N}$	closed	open

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Def. A *real-valued function* is a function whose range is a subset of \mathbb{R} . If $f : \Omega \rightarrow \mathbb{R}$, we say that f is a *real-valued function on Ω* .

Def. *Algebraic operations*: Let f, g be real-valued functions on Ω and let $\alpha \in \mathbb{R}$. Then for all $x \in \Omega$,

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(Local) Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. f is *continuous at a point c* if the limit of $f(x)$, as x approaches c , exists and is equal to $f(c)$.

Def'. (Epsilon-delta definition) The function f is *continuous at a point c* if

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Thm let $D \subset \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. Then f is continuous on D if and only if $f^{-1}(O)$ is open in D for each open set O in \mathbb{R} .

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Def. f is *left-continuous at c* if

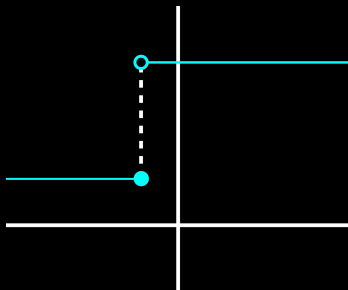
$$\lim_{x \rightarrow c+} f(x) = f(c)$$

Def. f is *right-continuous at c* if

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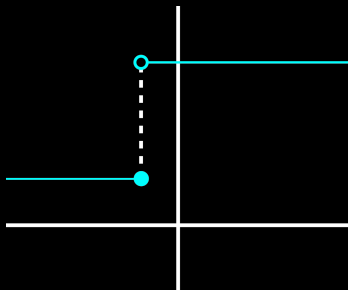


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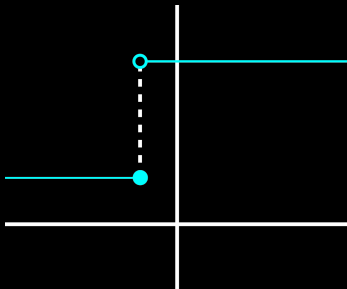


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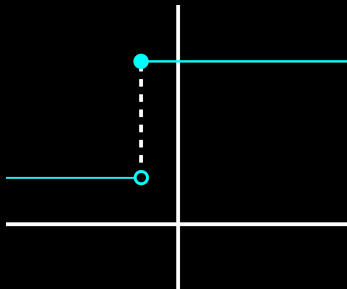
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Def. f is *lower semi-continuous at x_0* if

$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$$

$f(x_0)$ can be all points
at or below the blue point.

f is *upper semi-continuous at x_0* if

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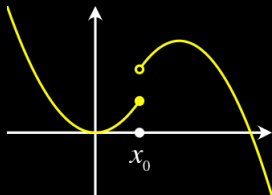
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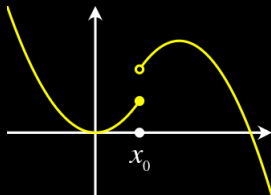
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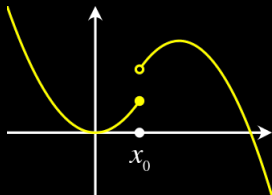
if f is lower semi-continuous at x_0 and f is continuous at x_0 then f is upper semi-continuous at x_0 .

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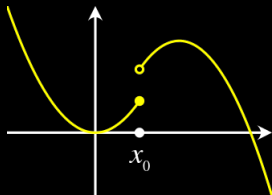
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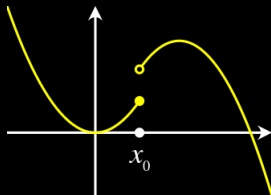
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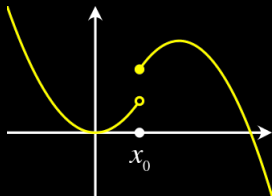
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(Global) Uniform Continuity

Def. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let I be an interval of \mathbb{R} . Then f is *uniformly continuous over I* if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every $x, y \in I$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

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Properties

Prop. 1 Every uniformly continuous function is continuous, but the converse does not hold.

Ex. $f(x) = x^3$ is a continuous functions on $I = \mathbb{R}$. Show that f is not uniformly continuous on $I = \mathbb{R}$.

Sol. In order to show f is not uniformly continuous on I , we need to show

$$\begin{aligned} & \neg \left(\forall \epsilon > 0 \exists \delta > 0 \forall x \in I \forall y \in I \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \right) \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \neg \{ \neg \{ |x - y| < \delta \} \vee |f(x) - f(y)| < \epsilon \} \\ \Leftrightarrow & \exists \epsilon > 0 \forall \delta > 0 \exists x \in I \exists y \in I \{ |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon \} \end{aligned}$$

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Let's choose $\epsilon = 1$. For any $\delta > 0$, we can choose

$$x = \frac{1}{\sqrt{\delta}} \quad \text{and} \quad y = \frac{1}{\sqrt{\delta}} + \frac{\delta}{3}.$$

Then we see that

$$|x - y| \leq \delta$$

and

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f is continuous at all points in $I \iff f$ is uniformly continuous on I .

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Why does it matter at all?

Answer: Uniformly continuous functions map Cauchy sequences to Cauchy sequences.

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Remark Can one add one more operation in this algebra: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $C(D)$, under what condition the limit $f_n \rightarrow f$ is closed?

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E.g.

(a) $f_n \in C(\mathbb{R})$ defined as $f_n = (1 + x/n)^n$. Then f_n converges pointwise on \mathbb{R} to $f(x) = e^x$. It is clear that $f \in C(\mathbb{R})$.

(b) Let $D = [0, 1]$ and $f_n \in C(D)$ be defined as $f_n = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

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Def. Let \mathcal{F} be a collection of real-valued functions on Ω . We say that \mathcal{F} is *closed under pointwise limits* if whenever $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ and $f_n \rightarrow f$ pointwise on Ω , then $f \in \mathcal{F}$.

Def. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set Ω .

We say that $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* to the real-valued function f on Ω , if

$$\boxed{\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N} \quad \forall x \in \Omega \quad |f_n(x) - f(x)| < \epsilon,$$

written as $f_n \rightarrow f$ *uniformly*.

Recall $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f if

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Prop. Let $D \subset \mathbb{R}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset C(D)$ and that $f_n \rightarrow f$ uniformly.
Then $f \in C(D)$.

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Therefore, the collection $\mathcal{C}(D)$ of real-valued continuous functions is closed under: $+$, $-$, scalar multiplication, and uniform convergence.

Chapter 3. Real Number System and Calculus

§ 3.1 Real number system

§ 3.2 Sequences of real numbers

§ 3.3 Open and closed sets

§ 3.4 Real-valued functions

§ 3.5 Liminf and limsup of sets

§ 3.6 Some techniques in calculus

Some part of subsection is taken from Chapter 1 Section 4 of

*P. Billingsley, **Probability and Measure**, Wiley, 1995.*

Def. For a sequence $A_1, A_2 \cdots$ of sets, define the *limits superior and inferior* of the sequence $\{A_n\}$ as

$$\limsup_n A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \text{and} \quad \liminf_n A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

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Remark Both $\limsup_n A_n$ and $\liminf_n A_n$ are sets.

Use the relation:

set	logic
\cap	\forall
\cup	\exists

$$\begin{aligned}
 \omega \in \limsup_n A_n &\iff \omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\
 &\iff (\forall n \geq 1) (\exists k \geq n) \omega \in A_k \\
 &\iff \omega \text{ lies in } \textit{infinitely many} \text{ of the } A_n
 \end{aligned}$$

Notation

$$\limsup_n A_n = [A_n \text{ i.o.}]$$

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Def. If both $\limsup_n A_n$ and $\liminf_n A_n$ exist and are equal, then the *limit set* of the sequence $\{A_n\}$ is defined to be

$$\lim_n A_n := \limsup_n A_n = \liminf_n A_n,$$

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Properties

(i) By De Morgan's law,

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c = \left(\limsup_n A_n^c \right)^c$$

Properties

(ii) Monotone increasing and decreasing sets:

$$\begin{array}{ccc}
 \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \liminf_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_n A_n \\
 \cap & & \cap \\
 A_n & & \\
 \cap & & \\
 \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \limsup_n A_n \quad \Longrightarrow \quad \lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_n A_n \\
 \cup & & \cup
 \end{array}$$

Interpretation Under a Probability Space

Suppose that $\{A_n\}$ are events from a probability space (Ω, \mathbb{P})

(i) The above Property (ii) can be translated to a probability statement:

$$\begin{array}{ccc}
 \mathbb{P} \left(\bigcap_{k=n}^{\infty} A_k \right) & \uparrow & \mathbb{P} \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\liminf_n A_n \right) \\
 & & \uparrow \wedge \\
 & & \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 \mathbb{P}(A_n) & & \uparrow \wedge \\
 & & \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \\
 & & \uparrow \wedge \\
 \mathbb{P} \left(\bigcup_{k=n}^{\infty} A_k \right) & \downarrow & \mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) = \mathbb{P} \left(\limsup_n A_n \right)
 \end{array}$$

(ii) *Borel Cantelli lemma*

$$\sum_n \mathbb{P}(A_n) \text{ converges} \quad \Rightarrow \quad \mathbb{P}(A_n \text{ i.o.}) = 0.$$

Proof.

$$\begin{aligned} 1 &\geq \mathbb{P}(A_n^c \text{ all but finitely many}) = 1 - \mathbb{P}(\{A_n^c \text{ all but finitely many}\}^c) \\ &= 1 - \mathbb{P}(A_n \text{ i.o.}) \\ &= 1 - \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} A_k\right) \\ &\geq 1 - \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 1 - 0 = 1. \end{aligned}$$

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Exercise

(i) Let $A_n = \left(-\frac{1}{n}, 1 - \frac{1}{n}\right]$:

$$A_1 = (-1, 0]$$

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{2}{3}\right]$$

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$$\vdots \quad \vdots$$

$$A_{100} = \left(-\frac{1}{100}, \frac{99}{100}\right]$$

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Show that

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

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Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n} \right] = [0, 1)$$

and

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 \right) = [0, 1).$$

Finally,

$$\limsup_n A_n = \liminf_n A_n = [0, 1).$$

□

Sol.

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(-\frac{1}{k}, \frac{k-1}{k} \right] = \bigcup_{n=1}^{\infty} \left[0, \frac{n-1}{n} \right] = [0, 1)$$

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(ii) Let $A_n = \left(\frac{(-1)^n}{n}, 1 - \frac{(-1)^n}{n} \right]$:

$$A_1 = (-1, 2] \qquad A_2 = \left(\frac{1}{2}, \frac{1}{2} \right]$$

$$A_3 = \left(-\frac{1}{3}, \frac{4}{3} \right] \qquad A_4 = \left(\frac{1}{4}, \frac{3}{4} \right]$$

$$A_5 = \left(-\frac{1}{5}, \frac{6}{5} \right] \qquad A_6 = \left(\frac{1}{6}, \frac{5}{6} \right]$$

$$\vdots \qquad \vdots \qquad \qquad \qquad \vdots \qquad \vdots$$

$$A_{99} = \left(-\frac{1}{99}, \frac{100}{99} \right] \qquad A_{100} = \left(\frac{1}{100}, \frac{99}{100} \right]$$

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Show that $\lim_n A_n$ doesn't exist by demonstrating that

$$\liminf_n A_n = (0, 1) \subset [0, 1] = \limsup_n A_n.$$

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Sol.

$$\begin{aligned}
 & \liminf_n A_n \\
 &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcup_{n=1,3,5}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \bigcap_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcup_{n=1,3,5}^{\infty} \left(\frac{1}{n+1}, \frac{n}{n+1} \right] \right\} \cup \left\{ \bigcup_{n=2,4,6}^{\infty} \left(\frac{1}{n}, \frac{n-1}{n} \right] \right\} \\
 &= (0, 1) \cup (0, 1) \\
 &= (0, 1)
 \end{aligned}$$

Sol. (continued) Similarly,

$$\begin{aligned}
 & \limsup_n A_n \\
 &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \bigcup_{k=n}^{\infty} \left(\frac{(-1)^k}{k}, \frac{k - (-1)^k}{k} \right] \right\} \\
 &= \left\{ \bigcap_{n=1,3,5}^{\infty} \left(-\frac{1}{n}, \frac{n+1}{n} \right] \right\} \cap \left\{ \bigcap_{n=2,4,6}^{\infty} \left(-\frac{1}{n+1}, \frac{n+2}{n+1} \right] \right\} \\
 &= [0, 1] \cap [0, 1] \\
 &= [0, 1]
 \end{aligned}$$

□

HW (Ex. 1.8 (c) on p. 11 of McDonald and Weiss) Let

$$A_n = \begin{cases} \left[0, 1 + \frac{1}{n}\right] & \text{if } n \text{ is an even integer,} \\ \left[-1 - \frac{1}{n}, 0\right] & \text{if } n \text{ is an odd integer.} \end{cases}$$

Determine $\liminf_{n \rightarrow \infty} A_n$ and $\limsup_{n \rightarrow \infty} A_n$.

Solution:

$$\liminf_{n \rightarrow \infty} A_n = \{0\} \subset [0, 1] = \limsup_{n \rightarrow \infty} A_n$$

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Integration by parts

Examples

1. $\int_0^1 \tan^{-1}(x) dx$

2. $\int_0^x t^2 e^t dt$

3. $\int e^x \sin(x) dx$

more to come ...

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Examples

1. e^x

2. $\sin(x)$

3. e^{x^2}

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Taylor expansions

Examples

1. e^x

2. $\sin(x)$

3. e^{x^2}

more to come ...