

Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

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Auburn University

Acknwolegement

NSF 2246850, NSF 2443823, & Simons Foundation Travel Grant (2022-2027)

Talk available at: github.com/chenle02

Emerging Synergies between Stochastic Analysis and Statistical Mechanics
Banff, Alberta, Canada
October 28, 2025

Outreach

- ▶ Auburn Summer Science Institute (AU-SSI): **2024, 2025**
Selected talented high school students
- ▶ Destination STEM: **2023, 2024**
Junior middle school students
- ▶ Graduate Student Seminars (Mathematics), Auburn: **2022–2025**

Teaching

- ▶ **Math 7820/7830: Applied Stochastic Processes**
Course project, **2023/24**

Most materials are available at

github.com/chenle02

Math 7820/30: Applied Stochastic Processes (2023/24):



Mauricio Montes and Ian Ruau



Image is generated by OpenAI's *DALL-E*

Plan

Introduction to growth model and SPDE

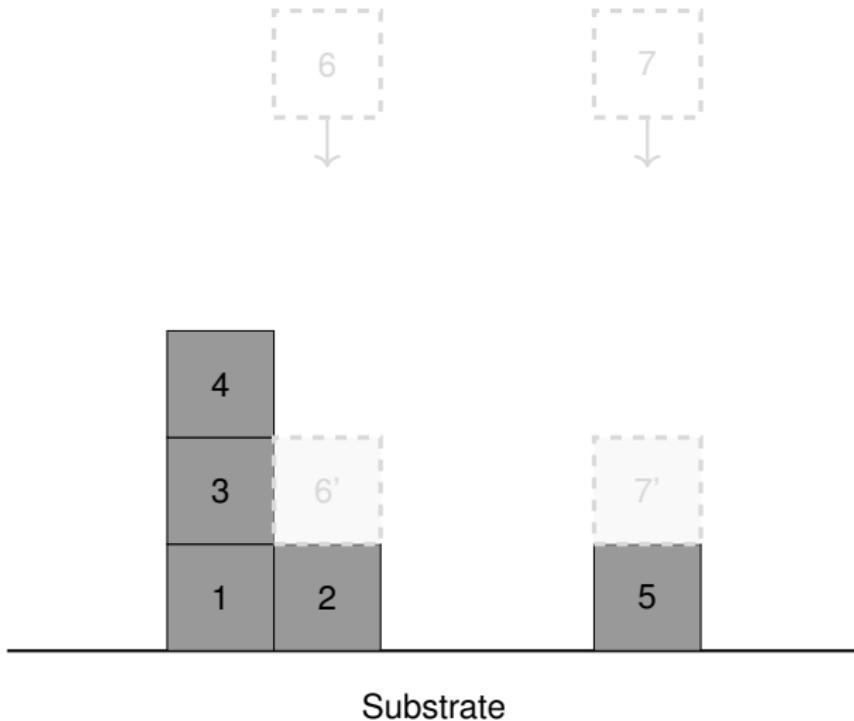
Tetromino Pieces

Plan

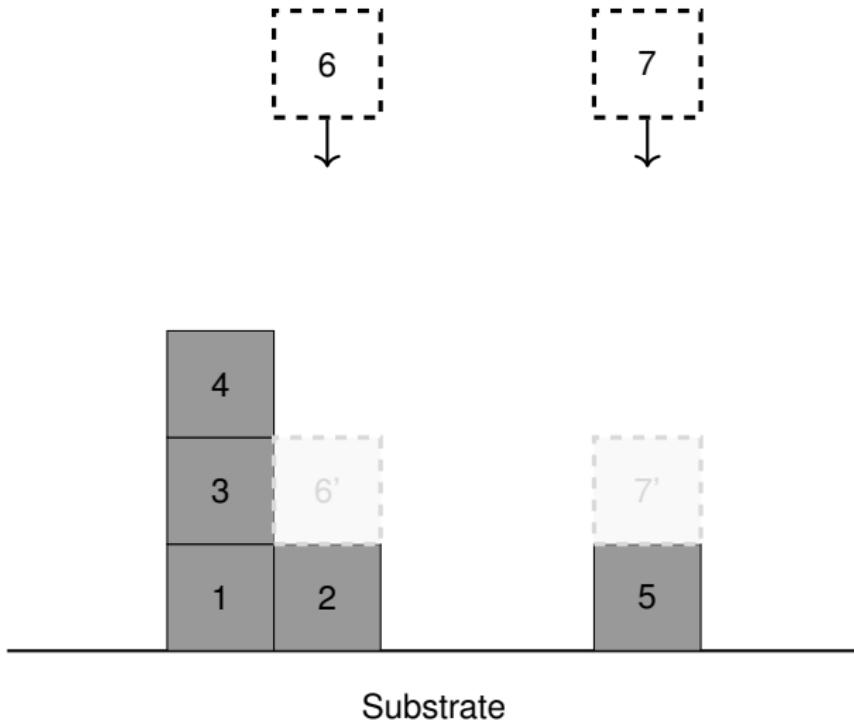
Introduction to growth model and SPDE

Tetromino Pieces

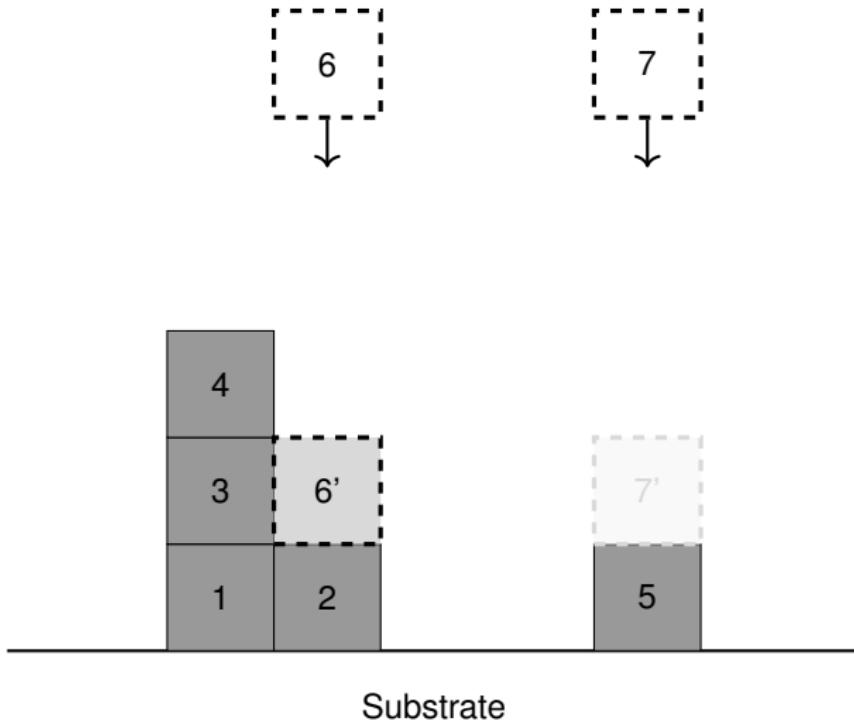
Random deposition



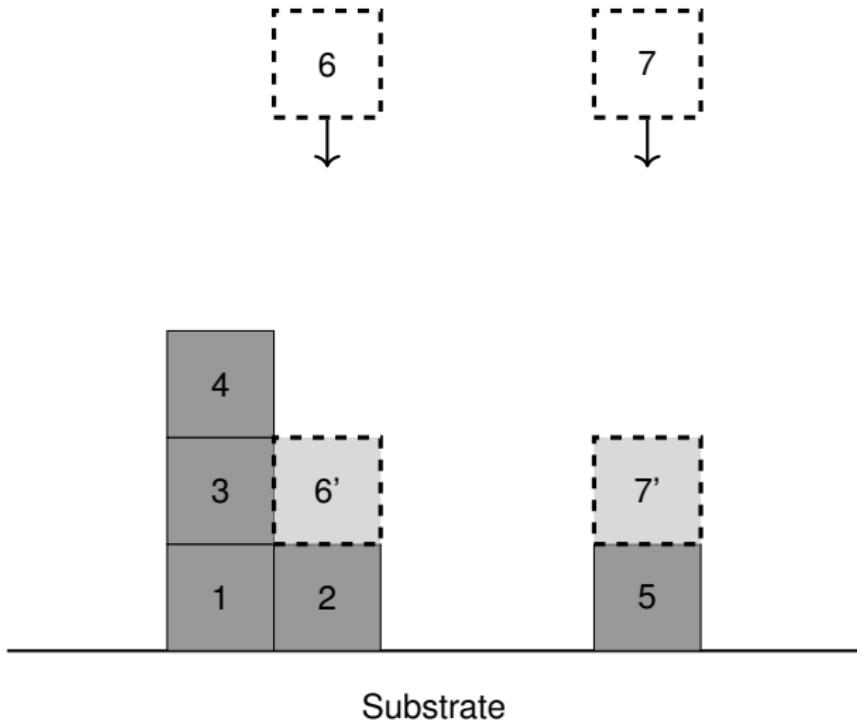
Random deposition



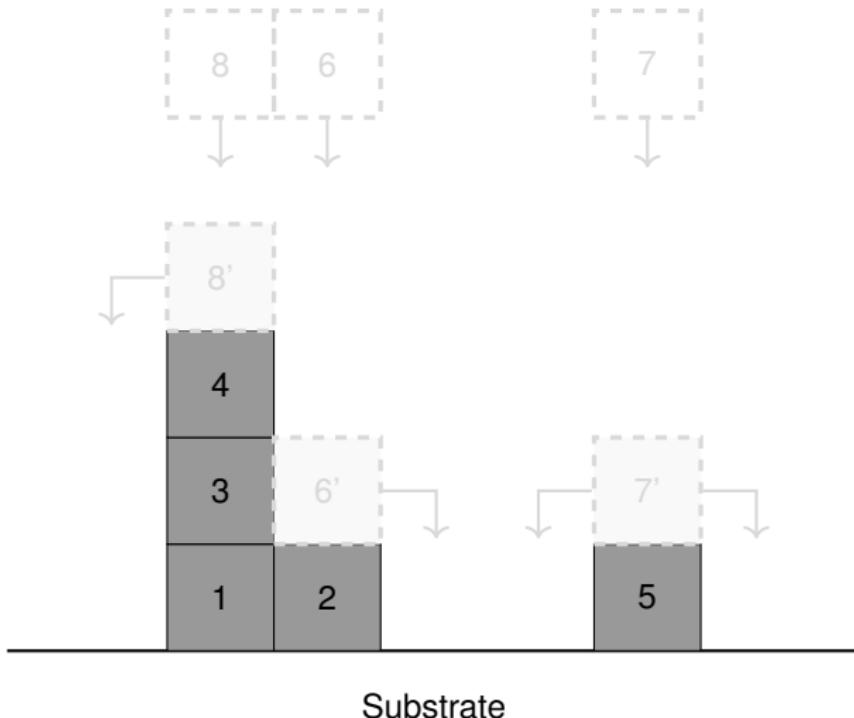
Random deposition



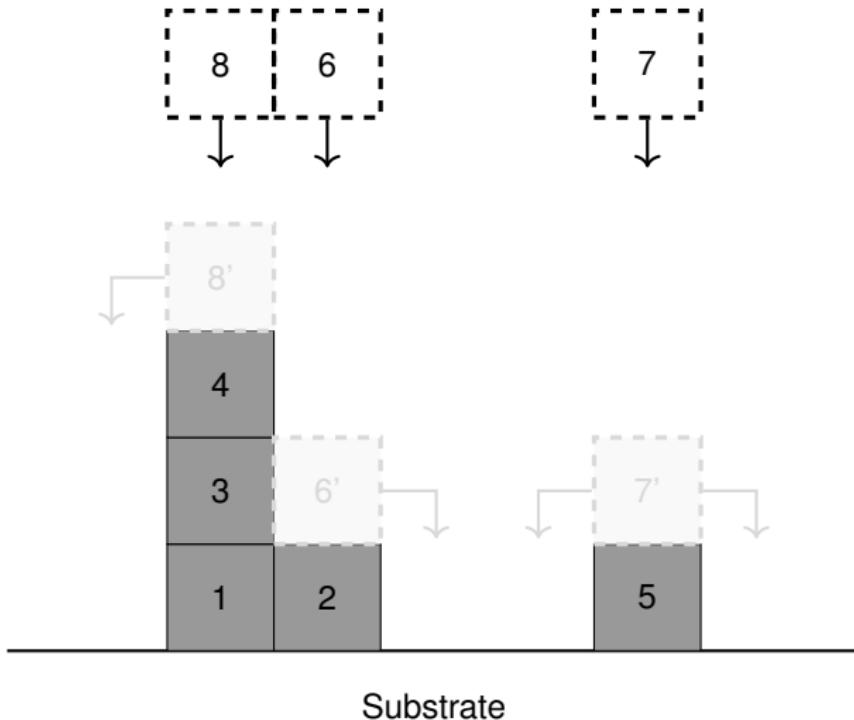
Random deposition



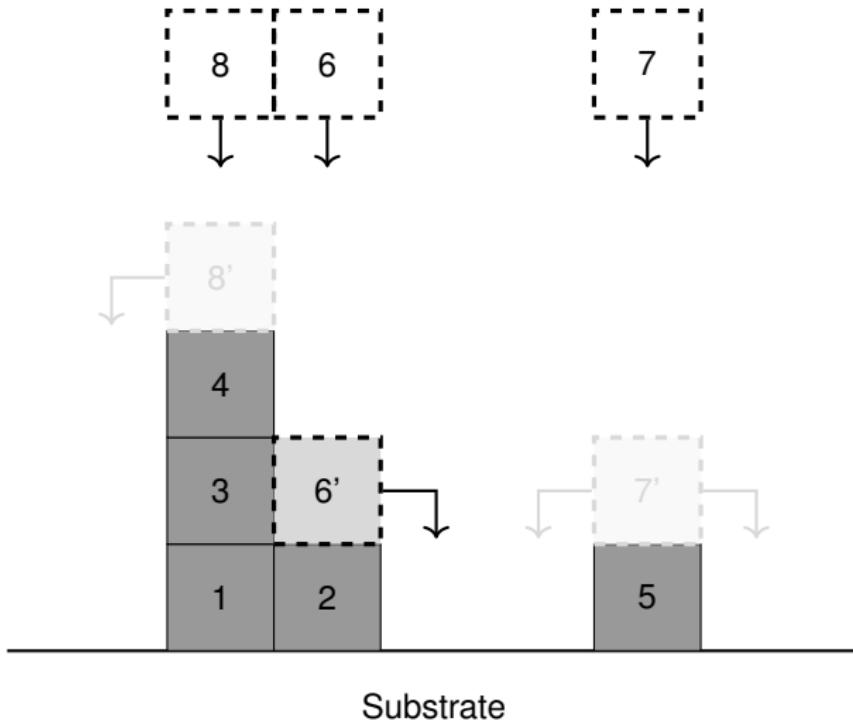
Random deposition with surface relaxation



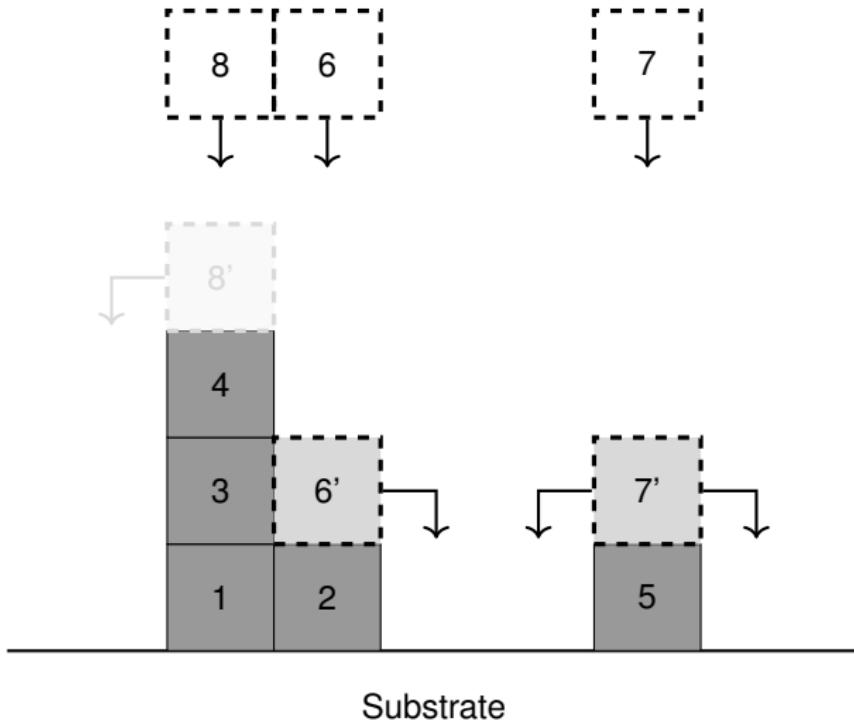
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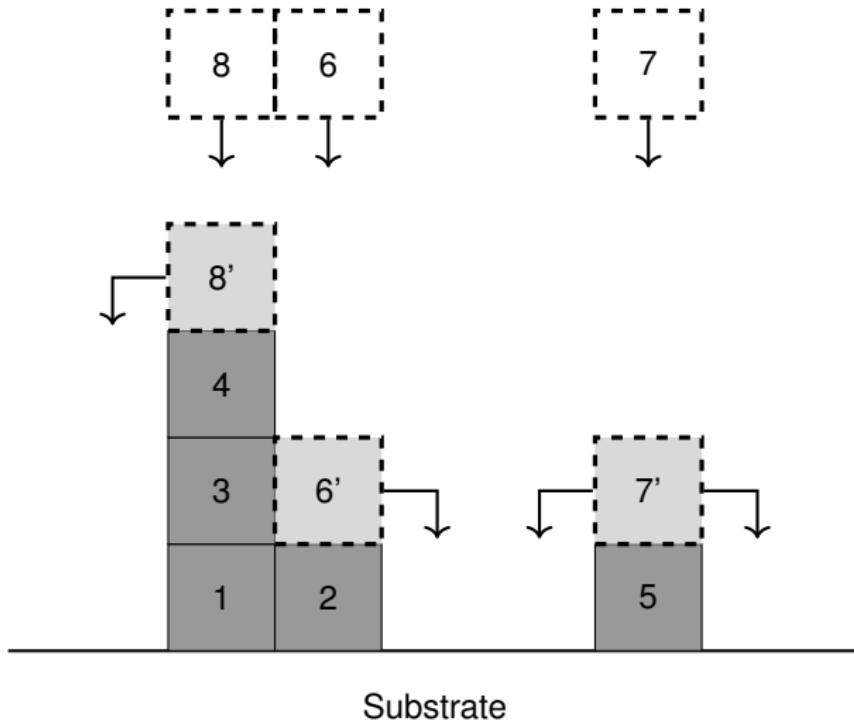
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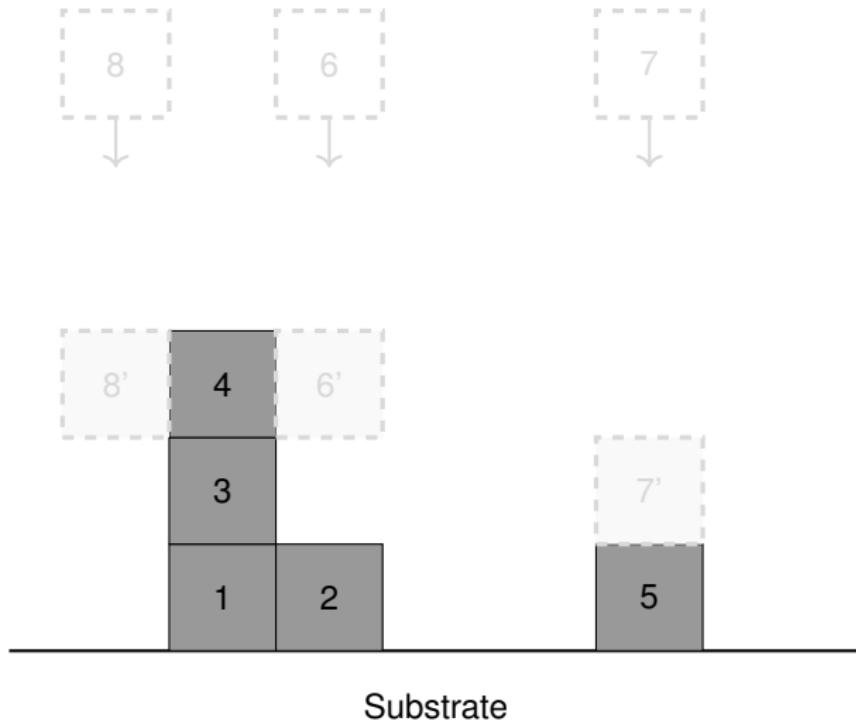
Random deposition with surface relaxation



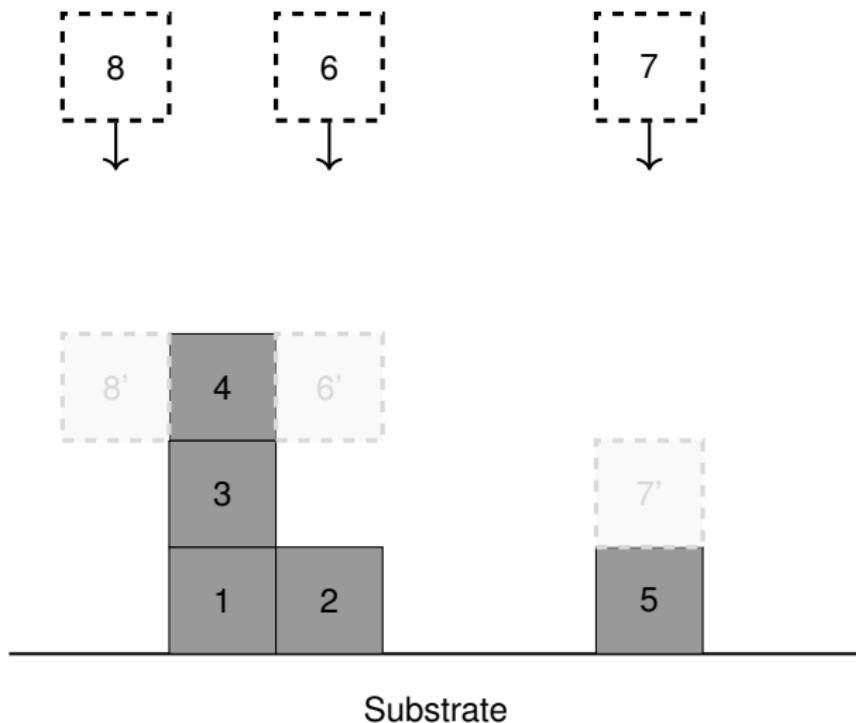
Random deposition with surface relaxation



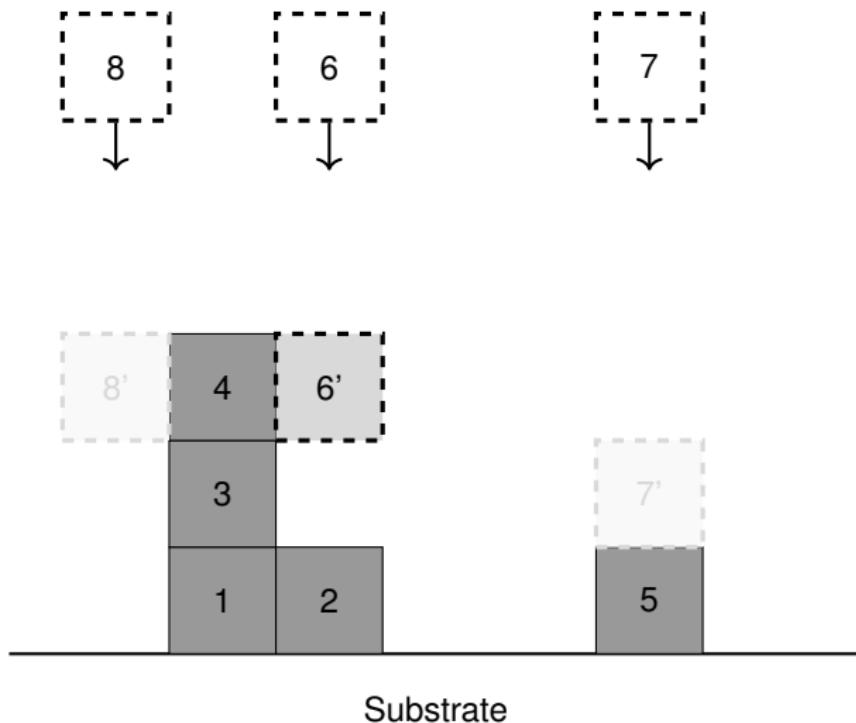
Ballistic deposition



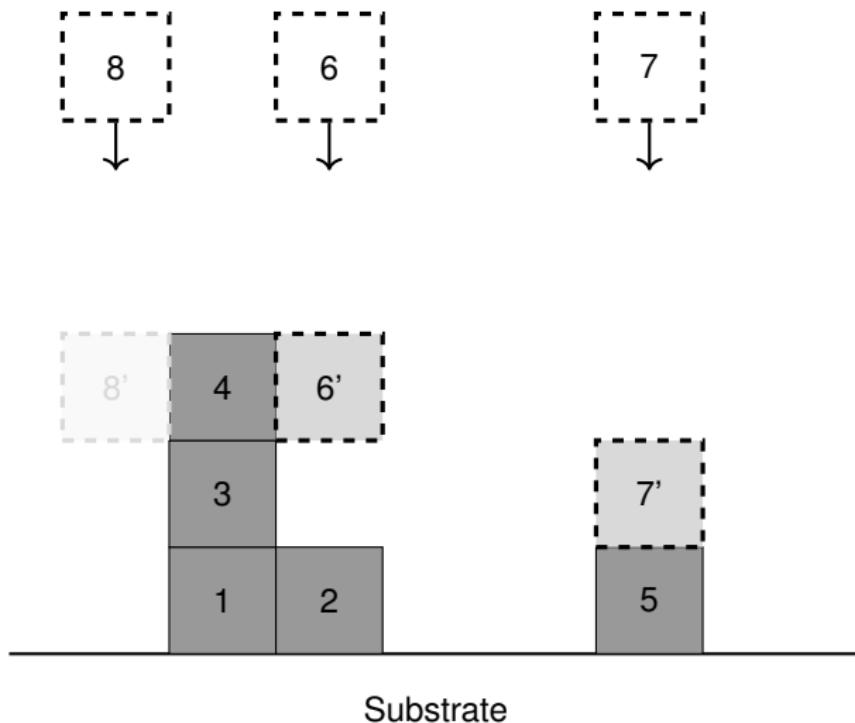
Ballistic deposition



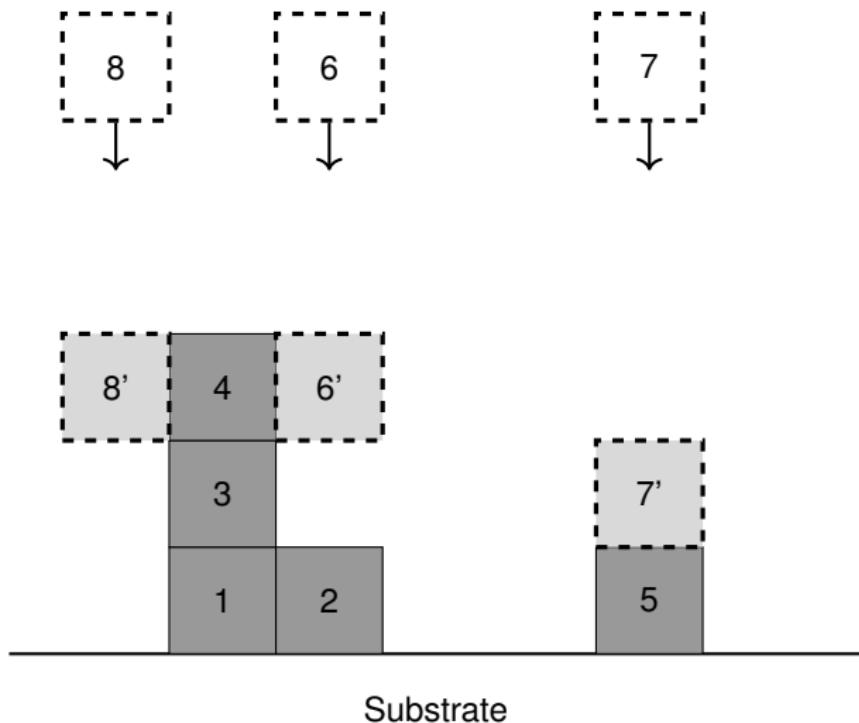
Ballistic deposition



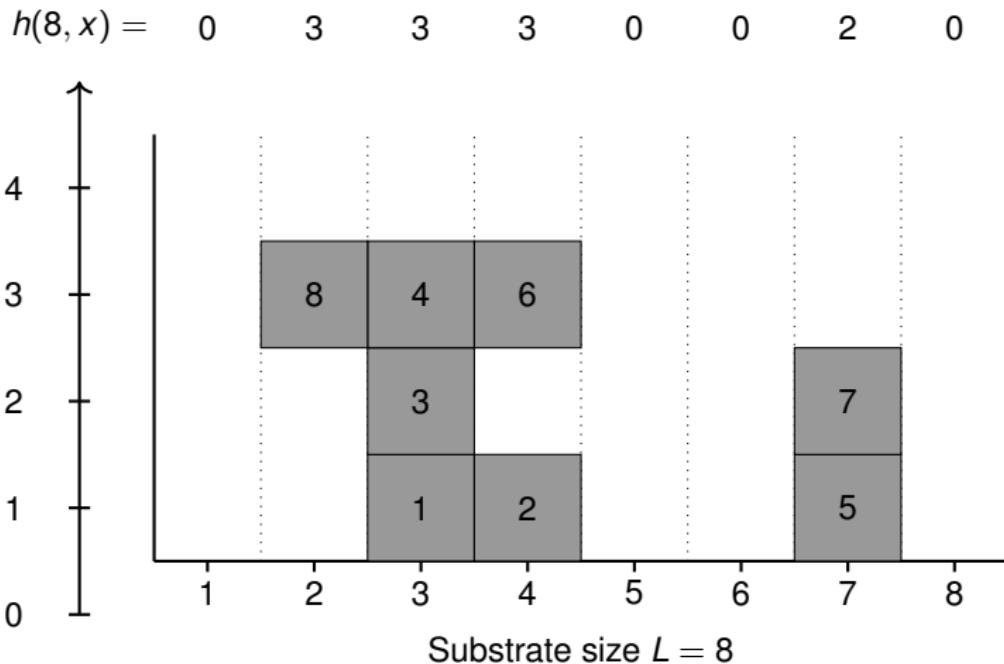
Ballistic deposition



Ballistic deposition



Average height and fluctuation

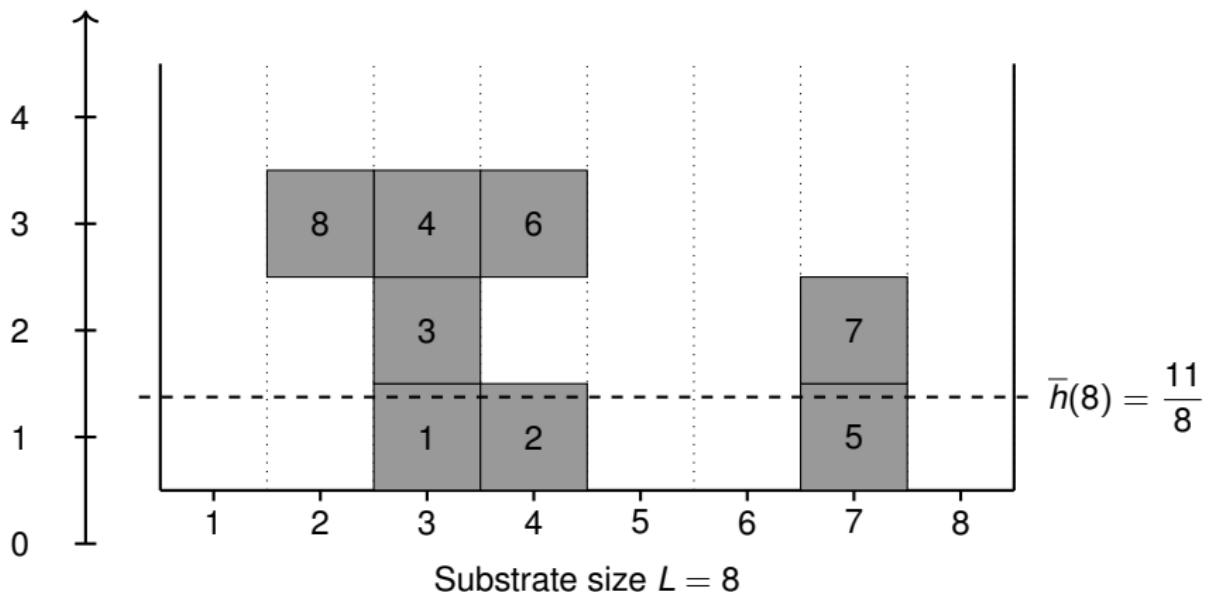


Average height and fluctuation

$$\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x)$$

$$\text{Fluctuation } W(L, t) = \sqrt{\frac{1}{L} \sum_{x=1}^L [h(t, x) - \bar{h}(t)]^2}$$

$$h(8, x) = \begin{array}{cccccccc} 0 & 3 & 3 & 3 & 0 & 0 & 2 & 0 \end{array}$$

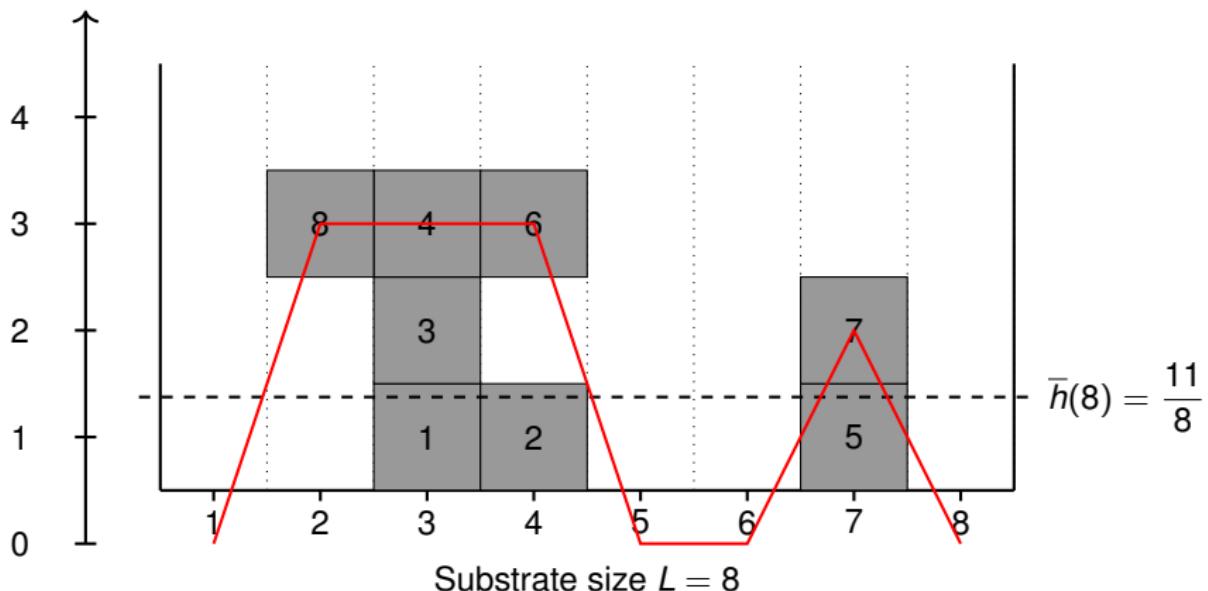


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Random Deposition (independent columns, nonsticky)

Model. L independent columns. At each integer time $t = 1, 2, \dots$, drop *one* particle on a uniformly random column. Heights $h(t, x)$, mean $\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x) = \frac{t}{L}$, width

$$W^2(L, t) = \frac{1}{L} \sum_{x=1}^L (h(t, x) - \bar{h}(t))^2.$$

Single-column law: After t drops total,

$$h(t, x) \sim \text{Binomial}\left(t, \frac{1}{L}\right), \quad \mathbb{E}[h(t, x)] = \frac{t}{L}, \quad \text{Var}(h(t, x)) = t \frac{1}{L} \left(1 - \frac{1}{L}\right).$$

Fluctuation: By i.i.d. columns,

$$\mathbb{E}[W^2(L, t)] = \frac{1}{L} \sum_{x=1}^L \mathbb{E}[h(t, x)^2] - \mathbb{E}[\bar{h}^2(t)] = \mathbb{E}[h(t, 1)^2] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right) \text{Var}(h(t, 1)).$$

Hence

$$\boxed{\mathbb{E}[W^2(L, t)] = \left(1 - \frac{1}{L}\right) t \frac{1}{L} \left(1 - \frac{1}{L}\right) = \frac{t}{L} \left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L, t) \simeq \left(1 - \frac{1}{L}\right) \left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \frac{1}{2}$.

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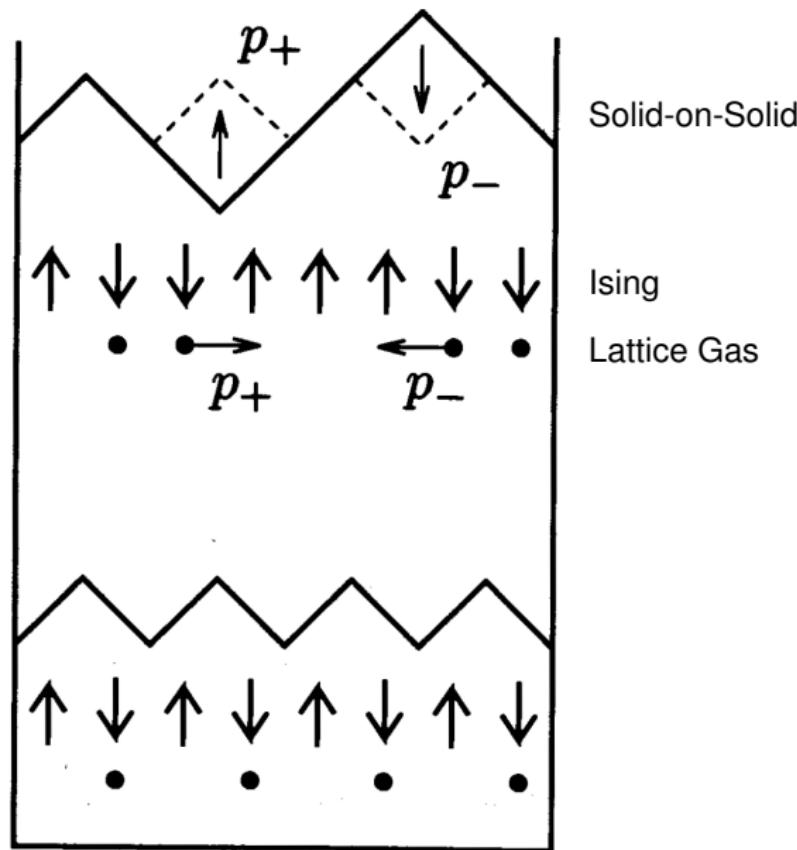
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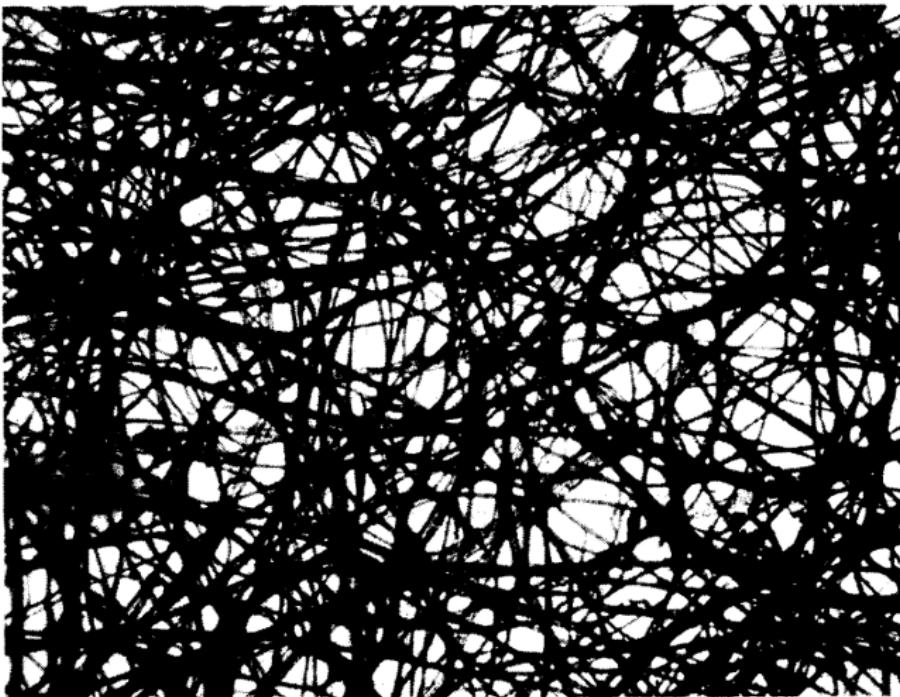
For ballistic deposition (sticky), lateral interactions appear, can we view it as a perturbation of random deposition (nonsticky)?

Simulations on
Random deposition vs. Ballistic decomposition

More models? Even more simpler?

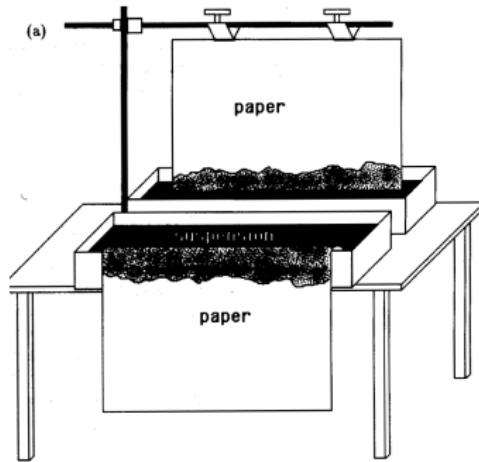


Paper – a random environment



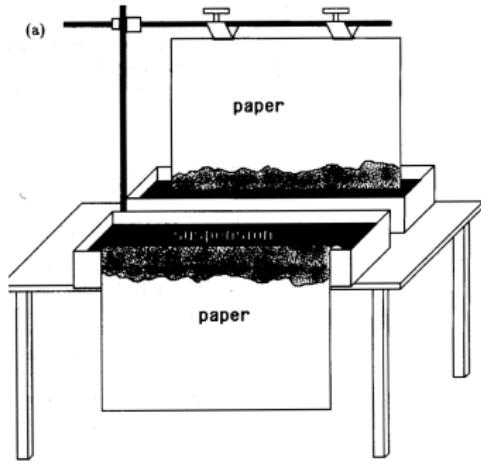
Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper wetting experiment



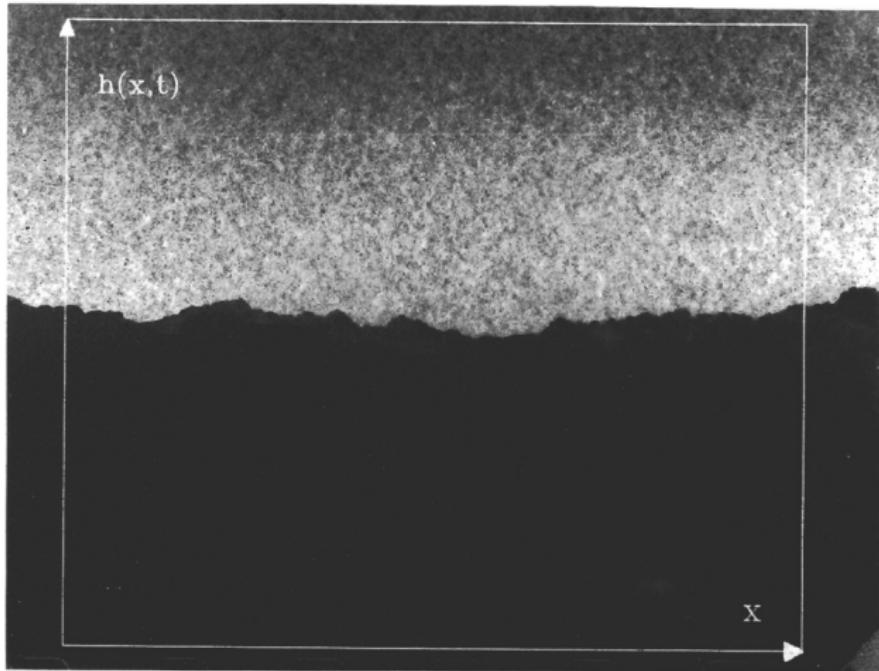
Barabási, A.-L., Stanley, H. E., 1995

Paper wetting experiment



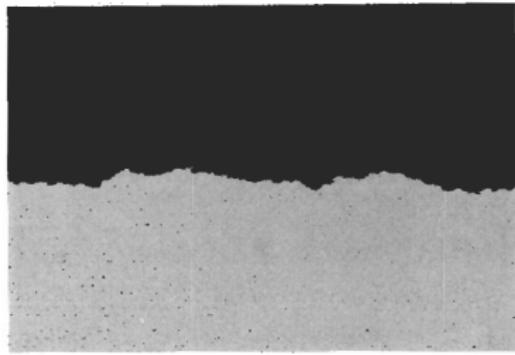
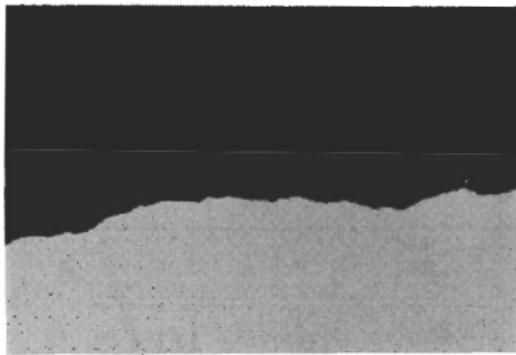
Barabási, A.-L., Stanley, H. E., 1995

Paper burning experiment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper rupture experiment



Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

Study of growing interfaces in a thin film

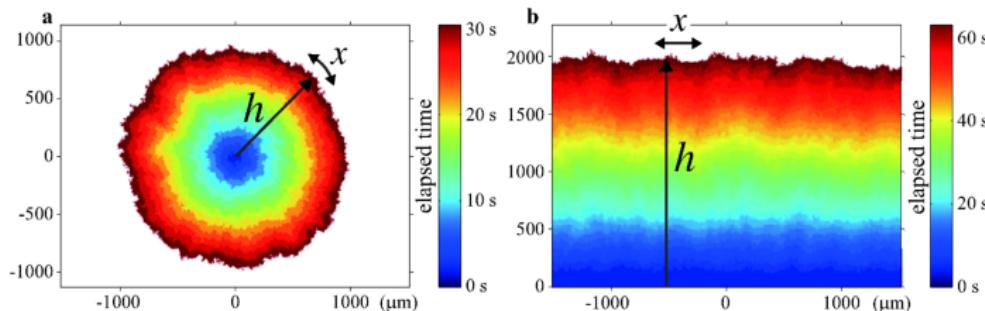
— Convection of nematic liquid crystal*

Show movies !

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*



Prediction from KPZ equation:

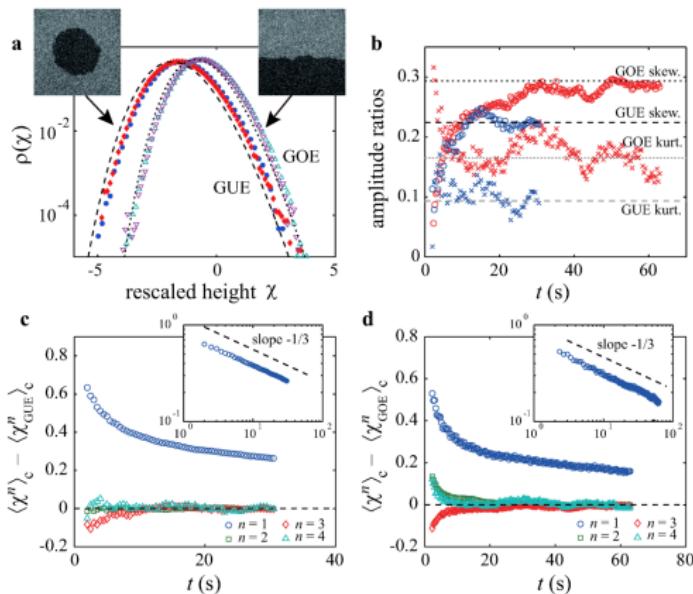
$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

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Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$



KPZ Equation '86

$$\frac{\partial}{\partial t} h(t, x) = \frac{1}{2} \Delta h(t, x) + \frac{\lambda}{2} (\nabla h)^2 + \dot{W}(t, x) \quad (\text{KPZ})$$



Mehran Kardar (1957 –) Giorgio Parisi (1948 –)



Yicheng Zhang

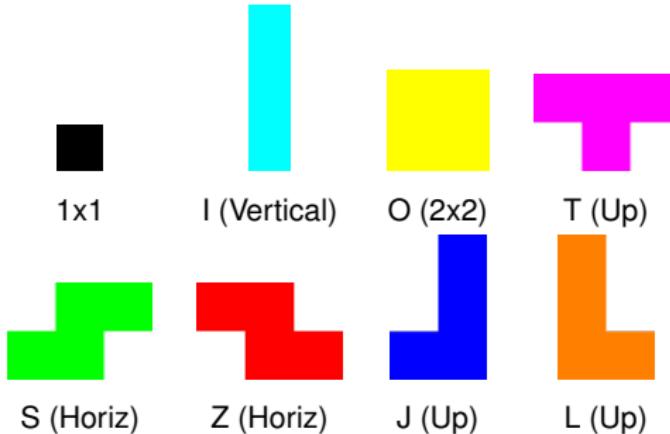
Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

Plan

Introduction to growth model and SPDE

Tetromino Pieces

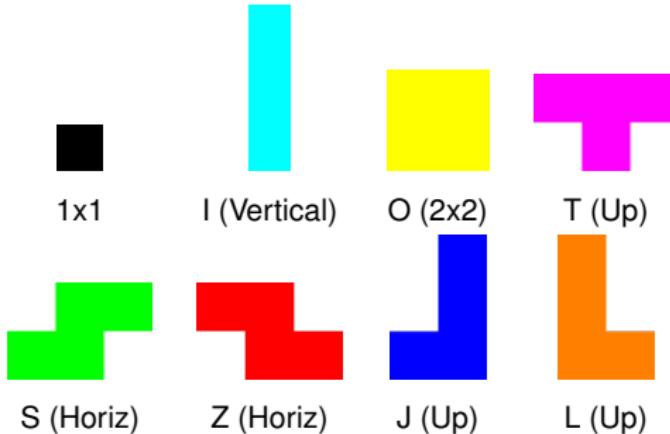
Tetrominoes



- ▶ “1x1”: Single (extra single-site particle)
- ▶ “I”: Horizontal, Vertical
- ▶ “J, L, T”: Up, Right, Down, Left
- ▶ “S, Z”: Horizontal, Vertical
- ▶ “O”: Single (2x2 square)
- ▶ Sticky
- ▶ Nonsticky

$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

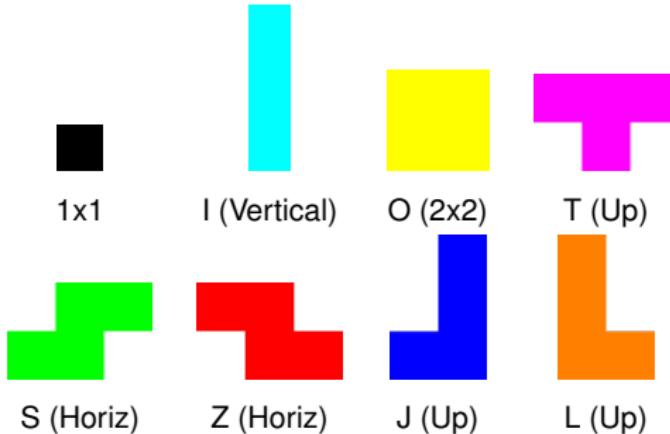
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Configure files

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [20, 0]  
Piece-01: [20, 0]  
Piece-02: [20, 0]  
Piece-03: [20, 0]  
Piece-04: [20, 0]  
Piece-05: [20, 0]  
Piece-06: [20, 0]  
Piece-07: [20, 0]  
Piece-08: [20, 0]  
Piece-09: [20, 0]  
Piece-10: [20, 0]  
Piece-11: [20, 0]  
Piece-12: [20, 0]  
Piece-13: [20, 0]  
Piece-14: [20, 0]  
Piece-15: [20, 0]  
Piece-16: [20, 0]  
Piece-17: [20, 0]  
Piece-18: [20, 0]  
Piece-19: [20, 0]
```

All nonsticky pieces
with equal prob.

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [0, 20]  
Piece-01: [0, 20]  
Piece-02: [0, 20]  
Piece-03: [0, 20]  
Piece-04: [0, 20]  
Piece-05: [0, 20]  
Piece-06: [0, 20]  
Piece-07: [0, 20]  
Piece-08: [0, 20]  
Piece-09: [0, 20]  
Piece-10: [0, 20]  
Piece-11: [0, 20]  
Piece-12: [0, 20]  
Piece-13: [0, 20]  
Piece-14: [0, 20]  
Piece-15: [0, 20]  
Piece-16: [0, 20]  
Piece-17: [0, 20]  
Piece-18: [0, 20]  
Piece-19: [0, 20]
```

All sticky pieces
with equal prob.

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [0, 0]  
Piece-01: [0, 0]  
Piece-02: [0, 0]  
Piece-03: [0, 0]  
Piece-04: [0, 0]  
Piece-05: [0, 0]  
Piece-06: [0, 0]  
Piece-07: [0, 0]  
Piece-08: [0, 0]  
Piece-09: [0, 0]  
Piece-10: [0, 0]  
Piece-11: [0, 0]  
Piece-12: [0, 0]  
Piece-13: [0, 0]  
Piece-14: [0, 0]  
Piece-15: [0, 0]  
Piece-16: [0, 0]  
Piece-17: [0, 0]  
Piece-18: [0, 0]  
Piece-19: [20, 80]
```

20% nonsticky
+ 80% sticky
of 1x1 piece

Main References:

- Barabási, A.-L., & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press, Cambridge.
- Family, F., & Vicsek, T. (1985). Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model. *Journal of Physics A: Mathematical and General*, 18(2), L75.
- Kardar, M., Parisi, G., & Zhang, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56(9), 889.
- Kertész, J., Horváth, V. k., & Weber, F. (1993). Self-affine rupture lines in paper sheets. *Fractals*, 01(01), 67–74.
- Takeuchi, K. A., Sano, M., Sasamoto, T., & Spohn, H. (2011). Growing interfaces uncover universal fluctuations behind scale invariance. *Sci. Rep.*, 1(1), 1–5.
- Zhang, J., Zhang, Y.-C., Alstrøm, P., & Levinsen, M. (1992). Modeling forest fire by a paper-burning experiment, a realization of the interface growth mechanism. *Phys. A: Stat. Mech. Appl.*, 189(3), 383–389.

Thank you!

Questions?