

Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

Le Chen
Auburn University

Acknwolegement

NSF 2246850, NSF 2443823, & Simons Foundation Travel Grant (2022-2027)

Talk available at: github.com/chenle02

Emerging Synergies between Stochastic Analysis and Statistical Mechanics
Banff, Alberta, Canada
October 28, 2025

Integration of Research, Education, and Outreach

Outreach

- ▶ Auburn Summer Science Institute (AU-SSI): **2024, 2025**
Selected talented high school students
- ▶ Destination STEM: **2023, 2024**
Junior middle school students

Education

- ▶ Graduate Student Seminars (Mathematics), Auburn: **2022–2025**
- ▶ Math 7820/7830: Applied Stochastic Processes Course project, **2023/24**

Research

- ▶ Simulation and modeling packages (open source)
- ▶ Forming conjectures and validating results

Most materials are available at

github.com/chenle02

Math 7820/30: Applied Stochastic Processes (2023/24):



Mauricio Montes and Ian Ruau

Simulation package:

https://github.com/chenle02/Simulations_on_Some_Surface_Growth_Models

```
pip install tetris-ballistic
```



Image is generated by OpenAI's *DALL-E*

Random deposition



Ballistic deposition

Image is generated by OpenAI's *DALL-E*

Plan

Introduction to growth model and SPDE

Family-Vicsek scaling and experiments

More examples

Tetromino Pieces

Plan

Introduction to growth model and SPDE

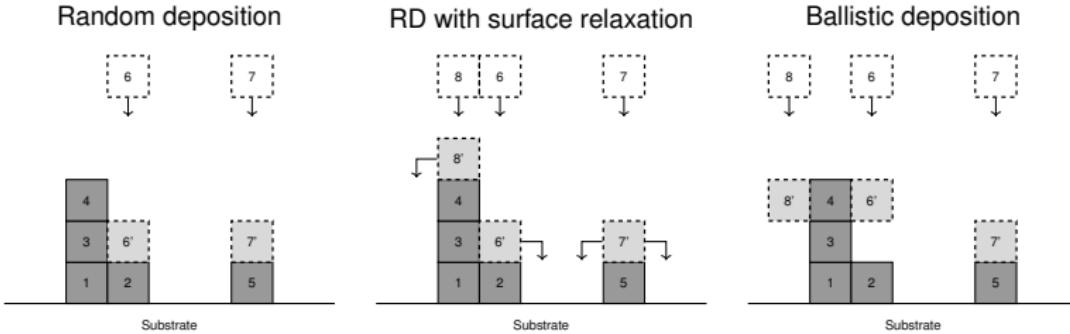
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More examples

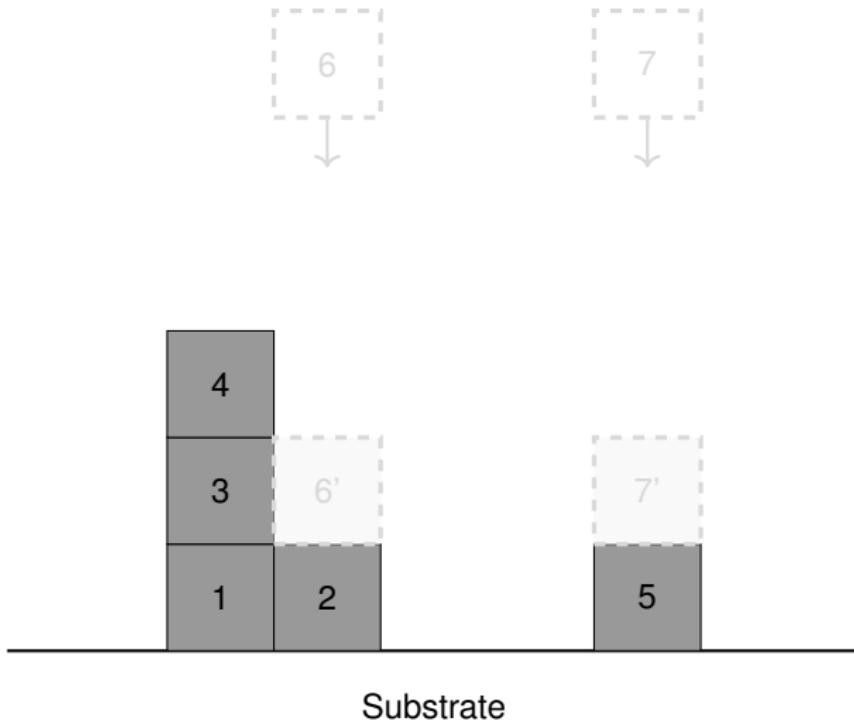
Tetromino Pieces

How do surfaces grow?

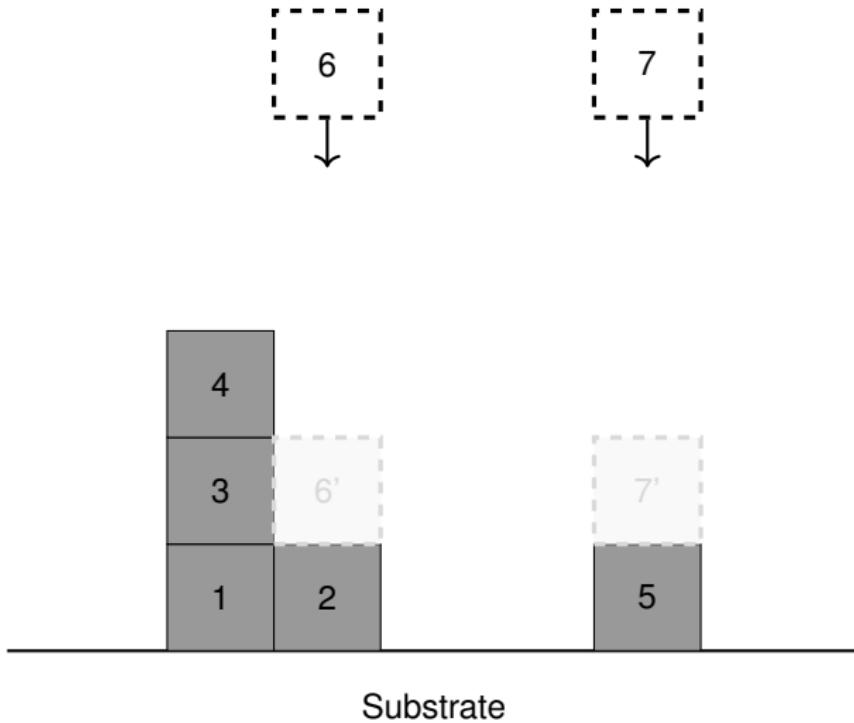
Three models



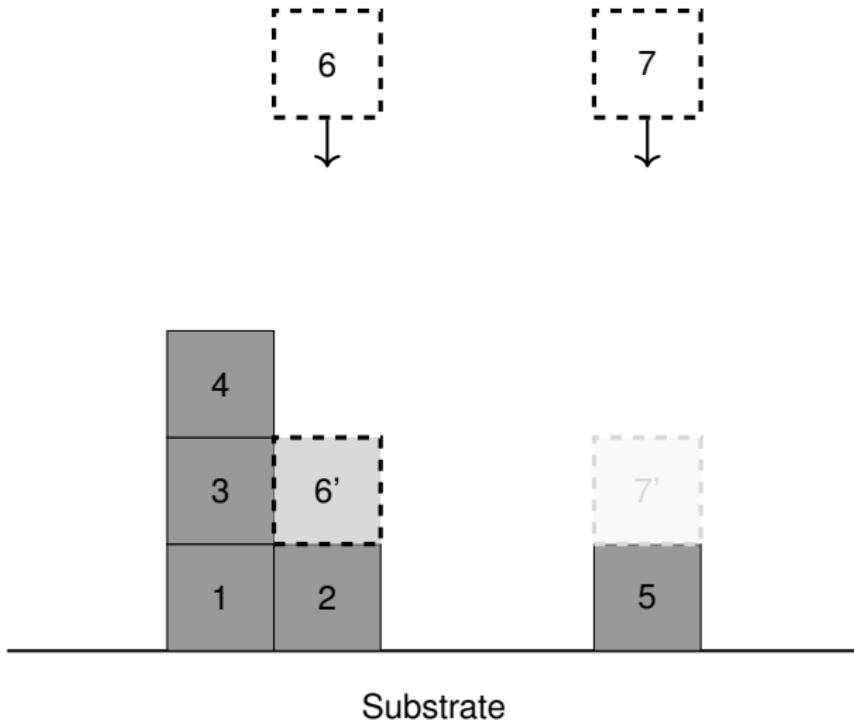
Random deposition



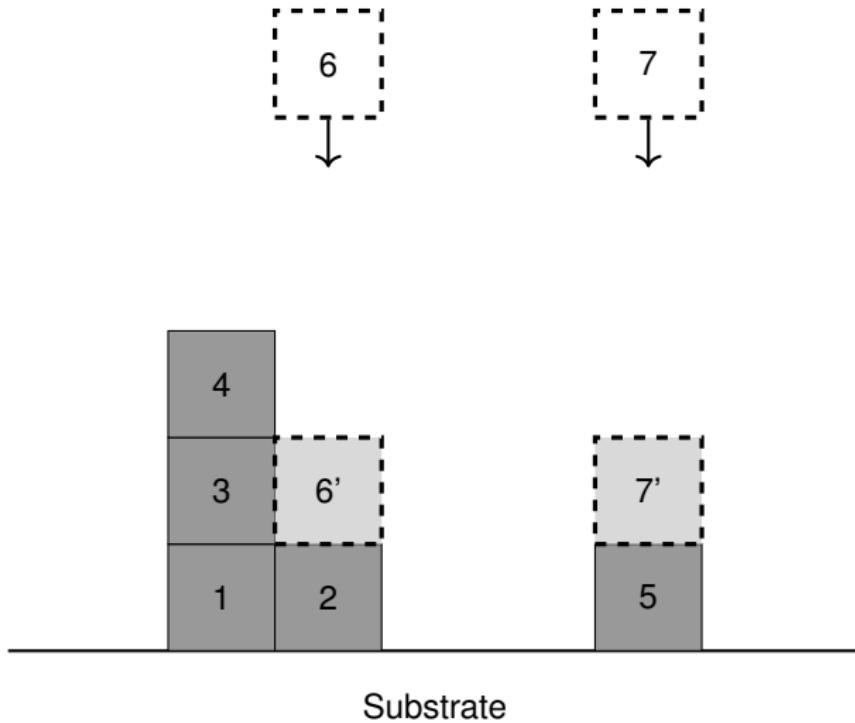
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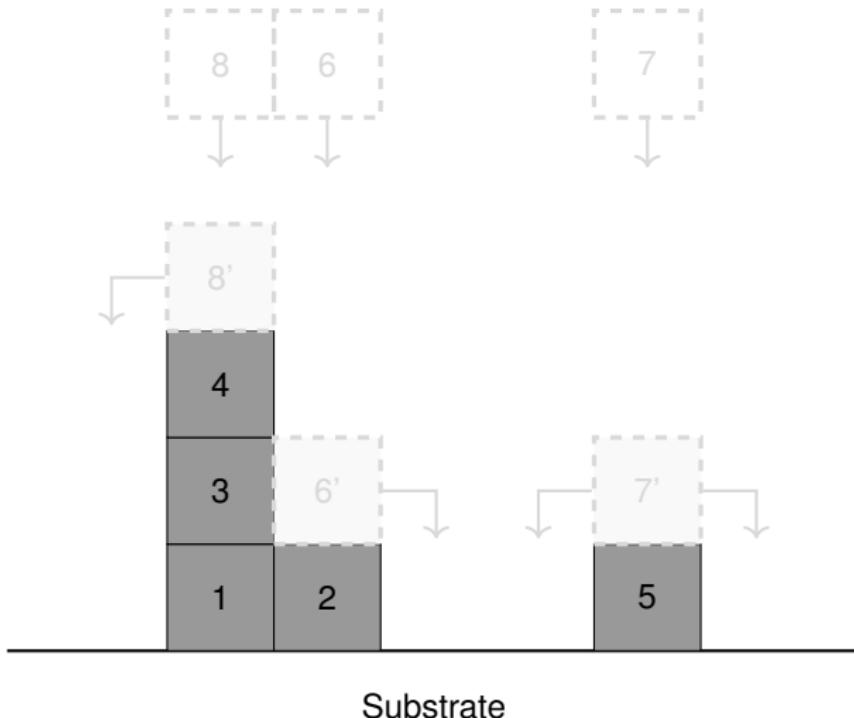
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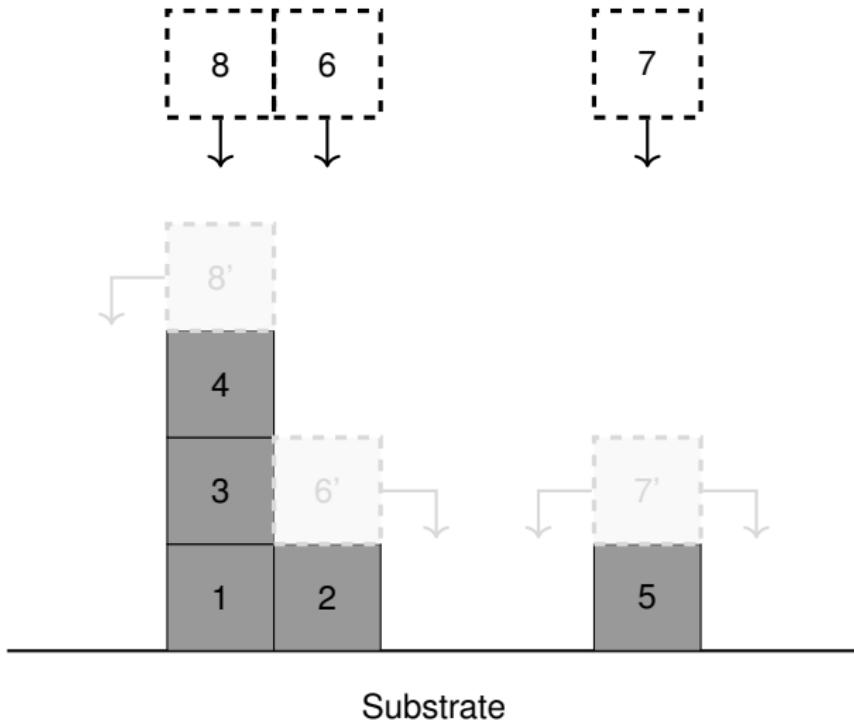
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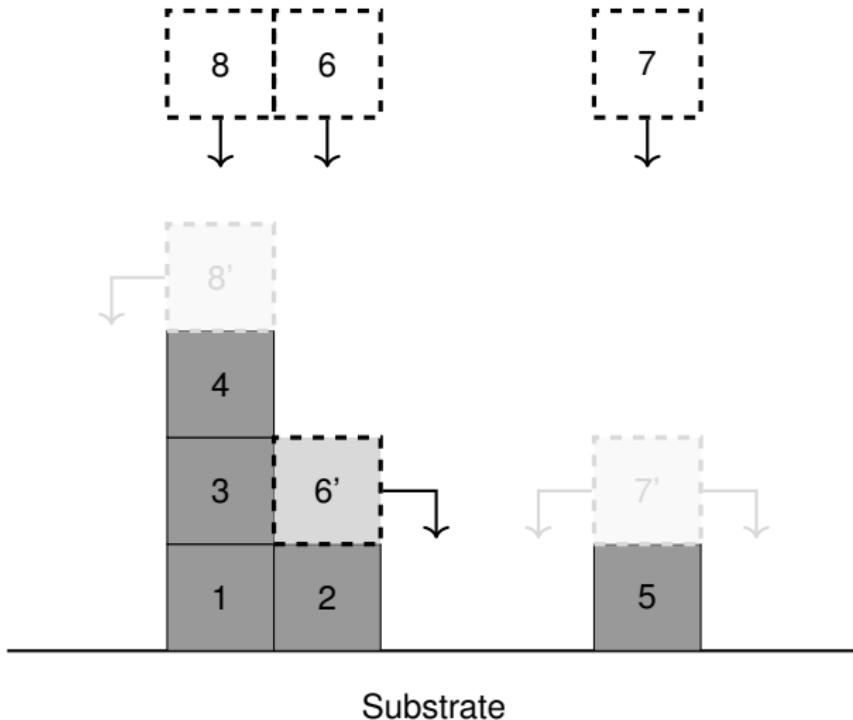
Random deposition with surface relaxation



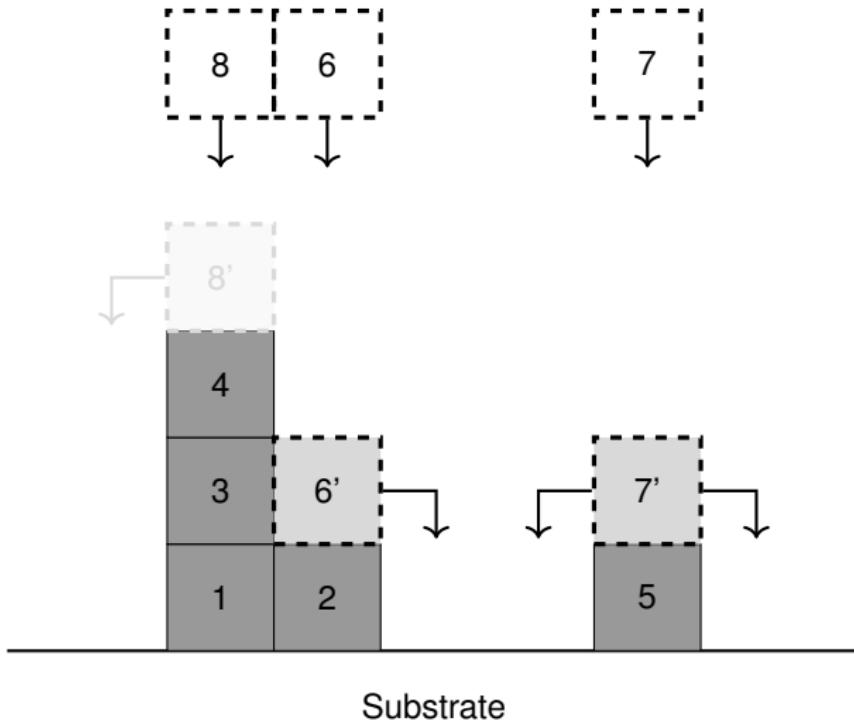
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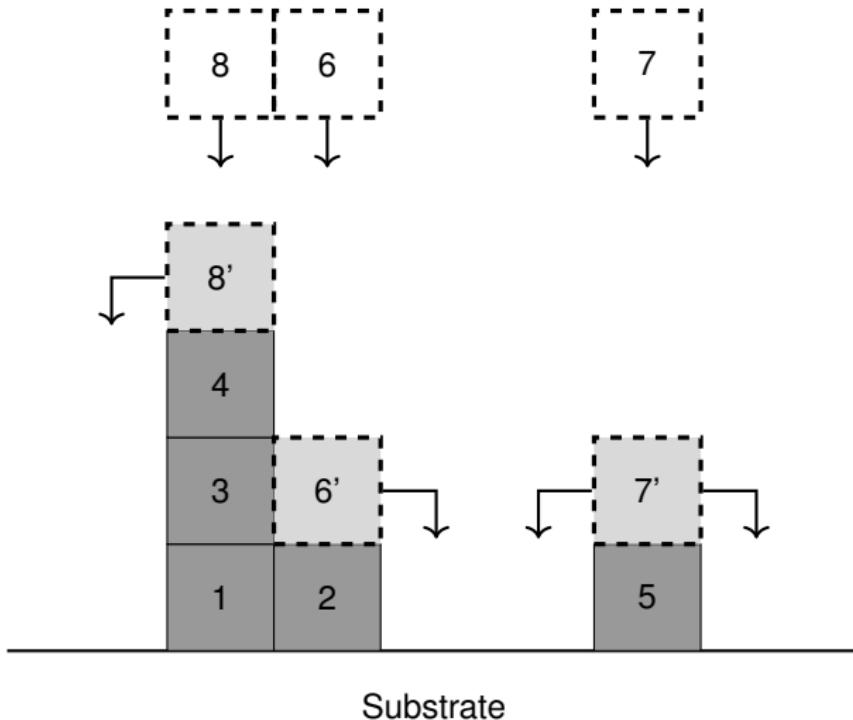
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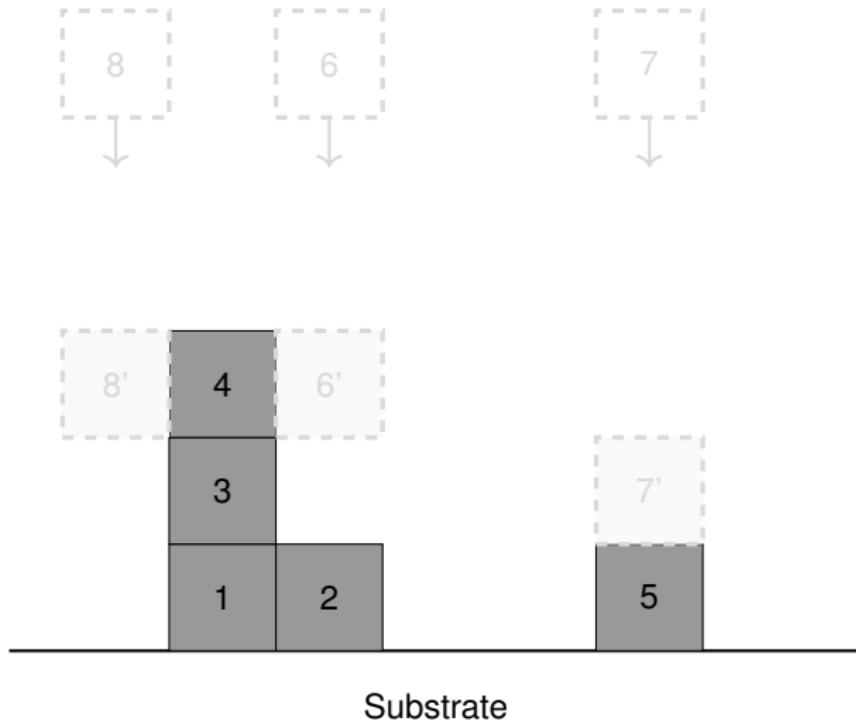
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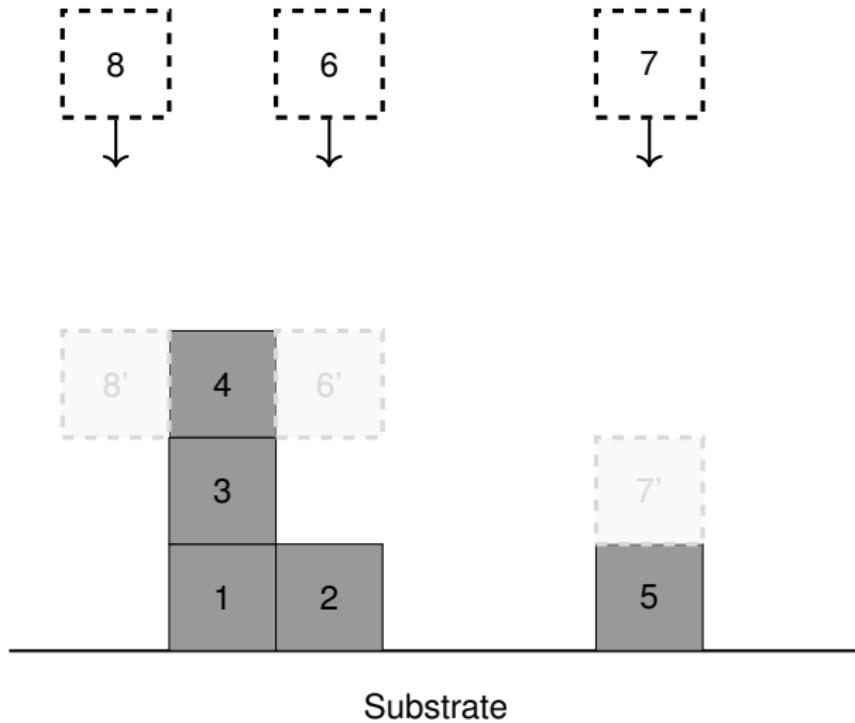
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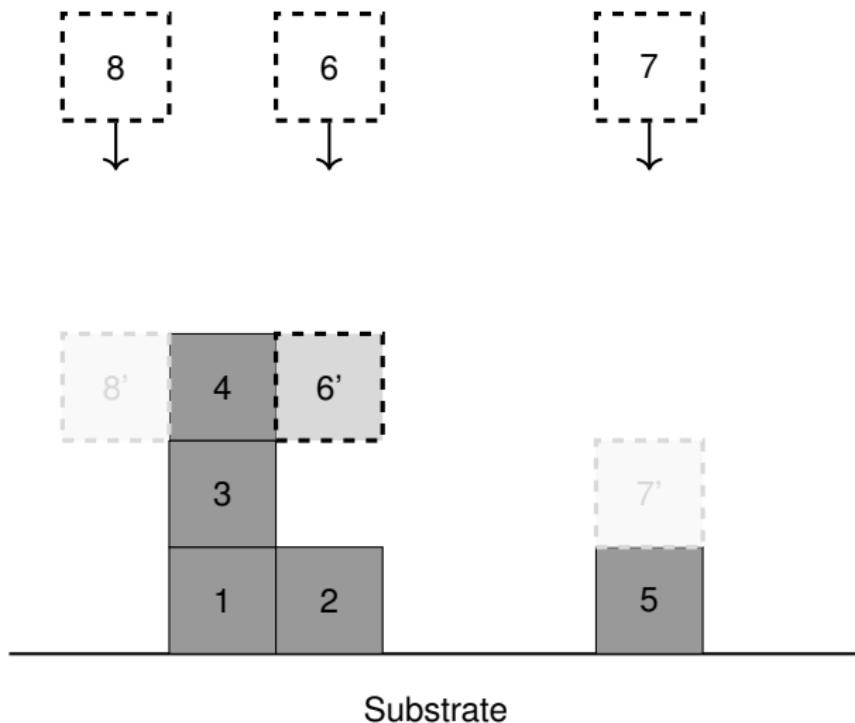
Ballistic deposition



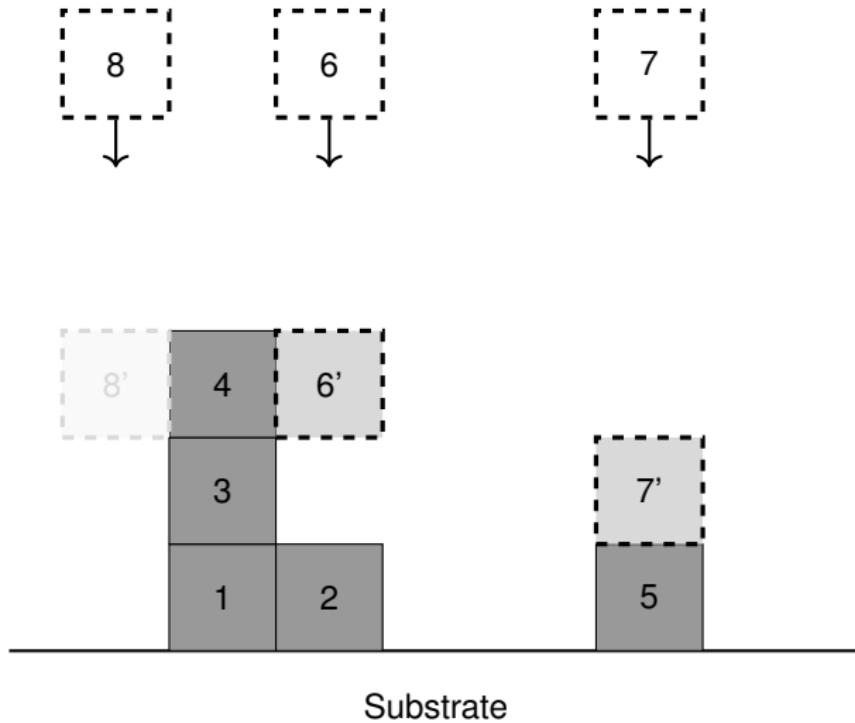
Ballistic deposition



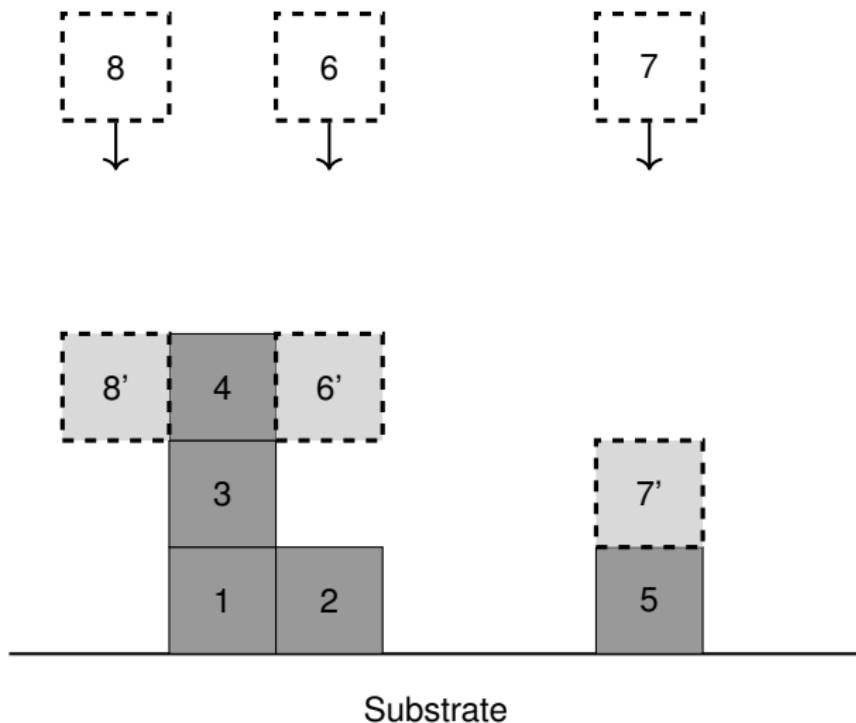
Ballistic deposition



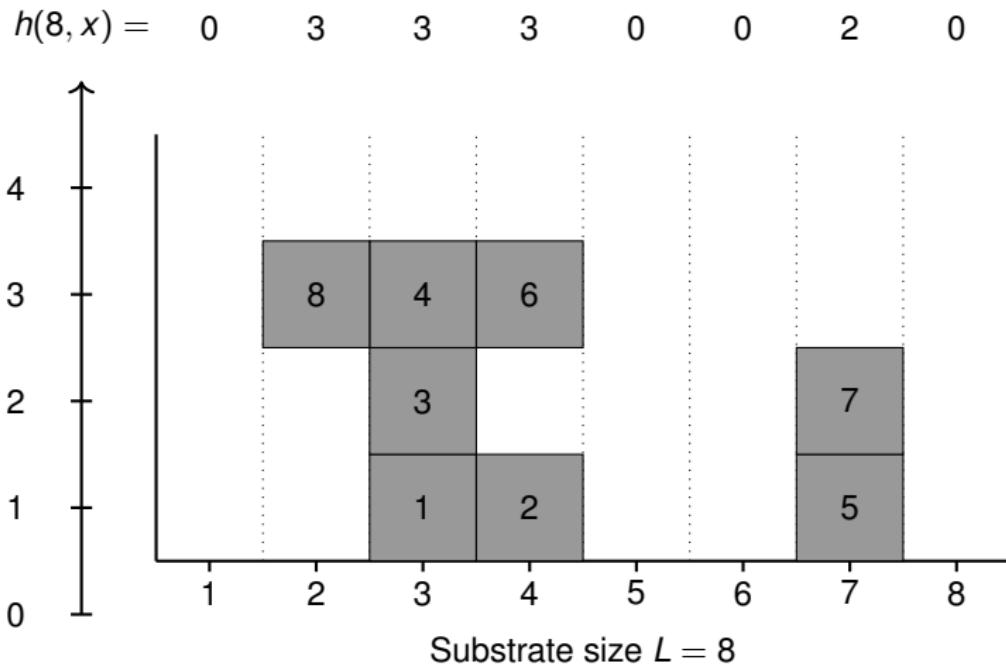
Ballistic deposition



Ballistic deposition



Average height and fluctuation

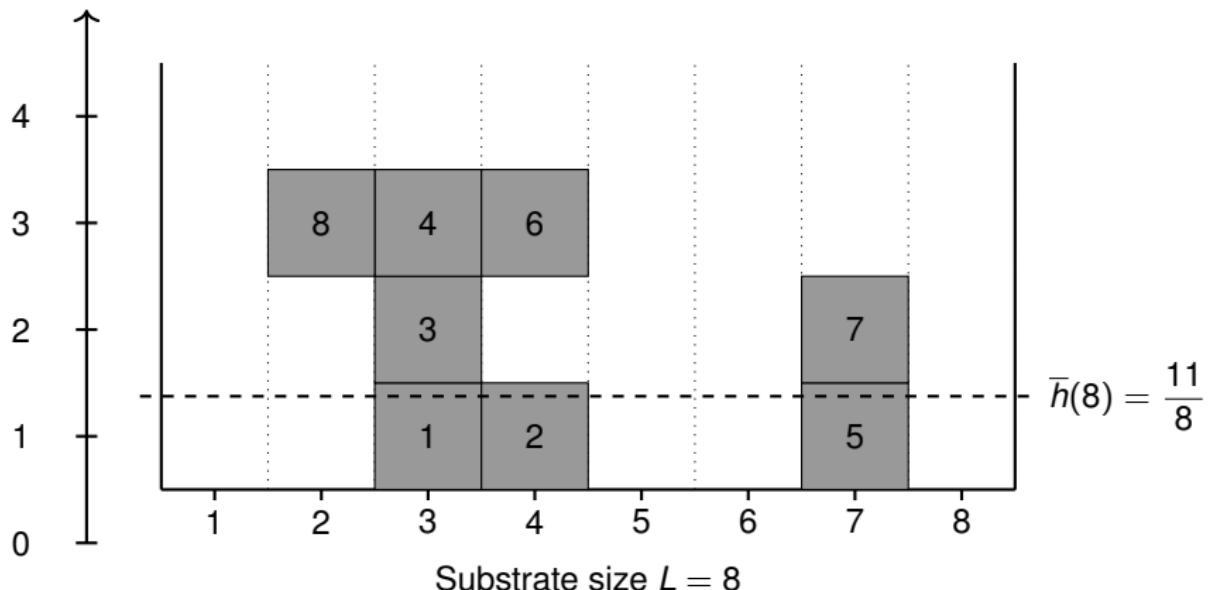


Average height and fluctuation

$$\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x)$$

$$h(8, x) = \begin{array}{ccccc} 0 & 3 & 3 & 3 & 0 \end{array}$$

$$\text{Fluctuation } W(L, t) = \sqrt{\frac{1}{L} \sum_{x=1}^L [h(t, x) - \bar{h}(t)]^2}$$

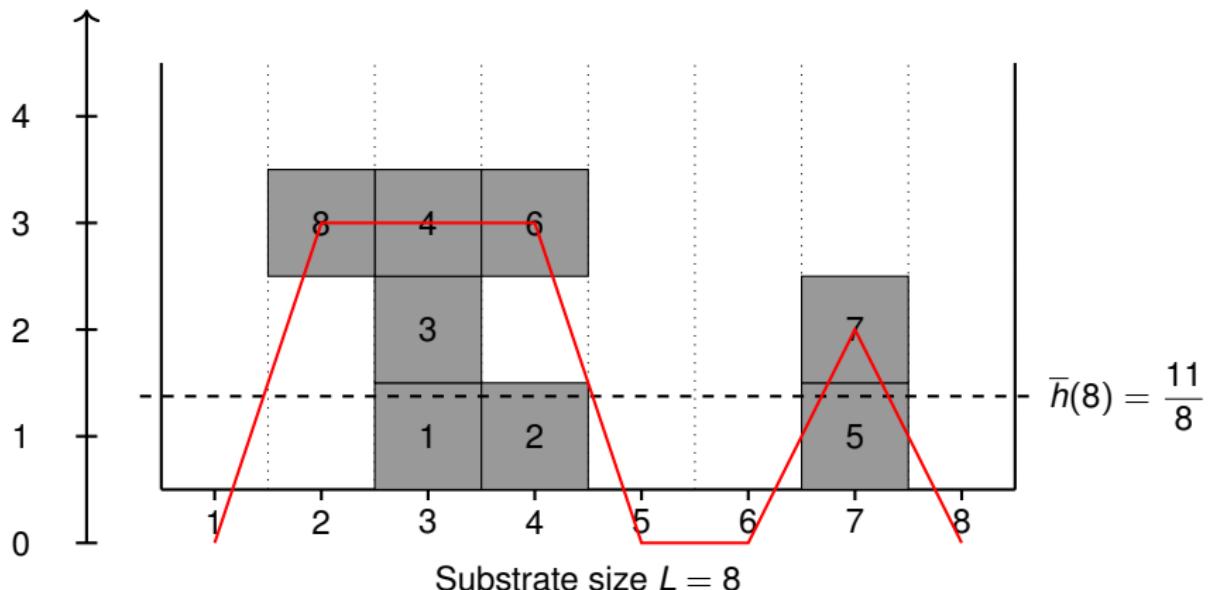


Average height and fluctuation

$$\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x)$$

$$h(8, x) = \begin{array}{ccccccc} & 0 & 3 & 3 & 3 & 0 & 2 & 0 \\ & \hline \end{array}$$

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Random Deposition (independent columns, nonsticky)

Model. L independent columns. At each integer time $t = 1, 2, \dots$, drop *one* particle on a uniformly random column. Heights $h(t, x)$, mean $\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x) = \frac{t}{L}$, width

$$W^2(L, t) = \frac{1}{L} \sum_{x=1}^L (h(t, x) - \bar{h}(t))^2.$$

Single-column law: After t drops total,

$$h(t, x) \sim \text{Binomial}\left(t, \frac{1}{L}\right), \quad \mathbb{E}[h(t, x)] = \frac{t}{L}, \quad \text{Var}(h(t, x)) = t \frac{1}{L} \left(1 - \frac{1}{L}\right).$$

Fluctuation: By i.i.d. columns,

$$\mathbb{E}[W^2(L, t)] = \frac{1}{L} \sum_{x=1}^L \mathbb{E}[h(t, x)^2] - \mathbb{E}[\bar{h}^2(t)] = \mathbb{E}[h(t, 1)^2] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right) \text{Var}(h(t, 1)).$$

Hence

$$\boxed{\mathbb{E}[W^2(L, t)] = \left(1 - \frac{1}{L}\right) t \frac{1}{L} \left(1 - \frac{1}{L}\right) = \frac{t}{L} \left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L, t) \simeq \left(1 - \frac{1}{L}\right) \left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \frac{1}{2}$.

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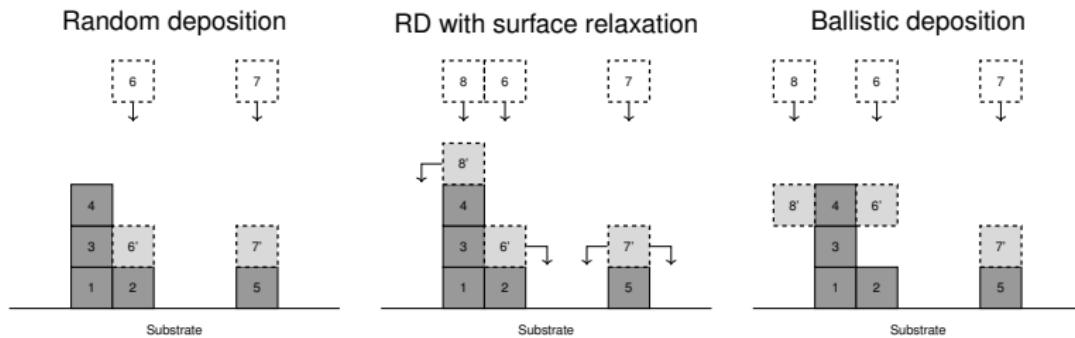
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Questions

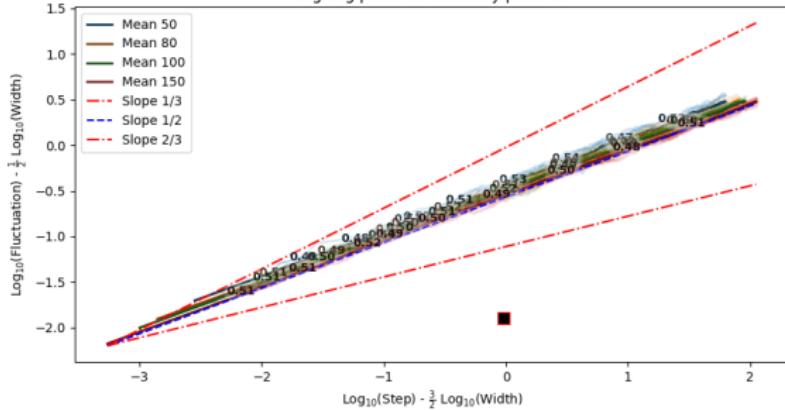


$$W(t, x) \sim t^{1/2}$$

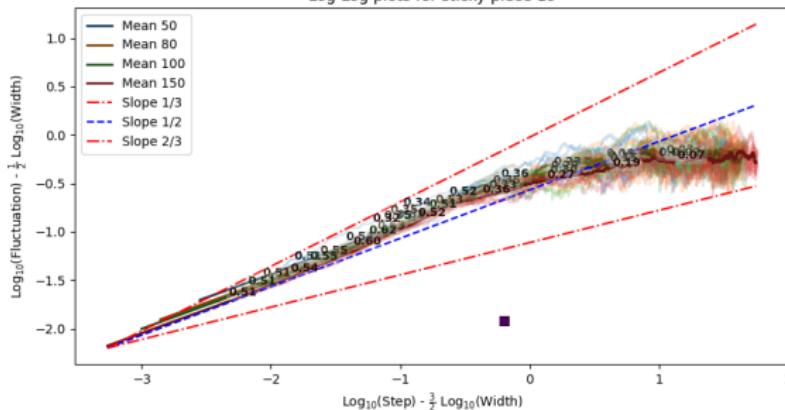
$$W(t, x) \sim t^{1/4}??$$

$$W(t, x) \sim t^{1/3}??$$

Log-Log plots for nonsticky piece 19



Log-Log plots for sticky piece 19



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Family–Vicsek scaling theory

Discrete interface with heights $h(x, t)$, $x = 1, \dots, L$. Mean $\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(x, t)$.

$$w(L, t) := \sqrt{\frac{1}{L} \sum_{x=1}^L (h(x, t) - \bar{h}(t))^2} \quad (\text{interface width / roughness}).$$

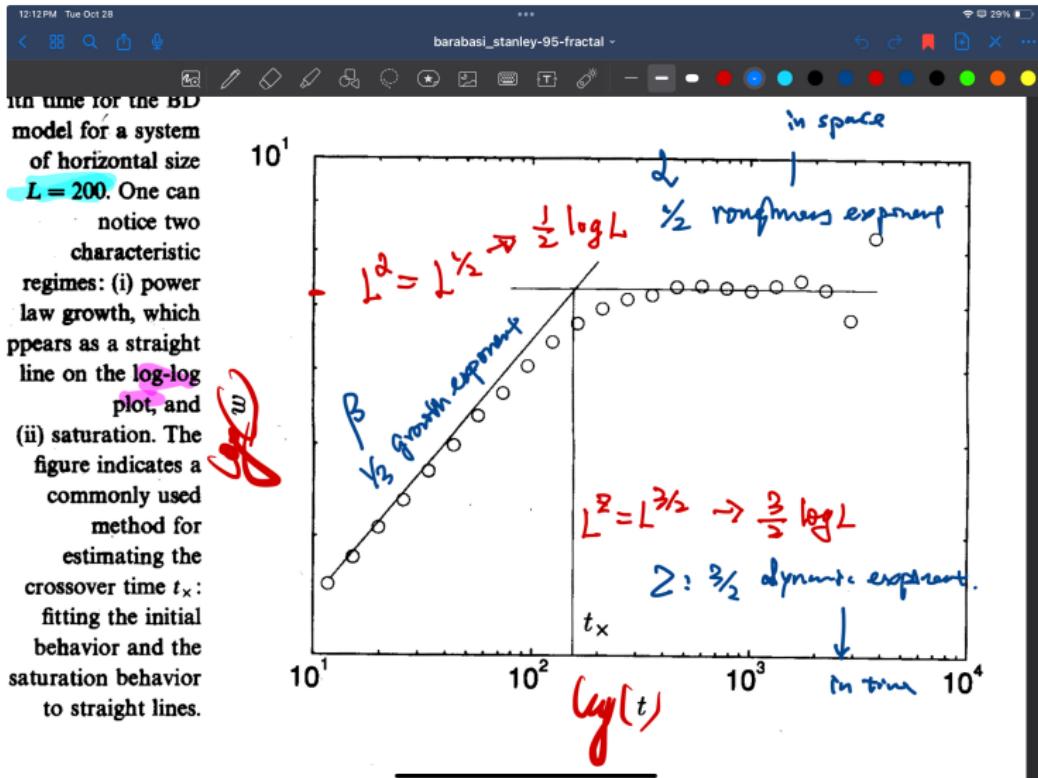
Empirically (log–log):

- ▶ Early time: $w(L, t) \sim t^\beta$ (growth exponent β).
- ▶ Late time: $w(L, t) \rightarrow w_{\text{sat}}(L) \sim L^\alpha$ (roughness exponent α).
- ▶ Crossover time: $t_x(L) \sim L^z$ (dynamic exponent z).

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

Family-Vicsek scaling theory

α : Roughness exp.; β : Growth exp.; z : Dynamic exp.



(Image from Barabási-Stanley's book 95)

Family–Vicsek scaling theory

α : Roughness exp.; β : Growth exp.; z : Dynamic exp.

$$w(L, t) \sim L^\alpha f\left(\frac{t}{L^z}\right)$$

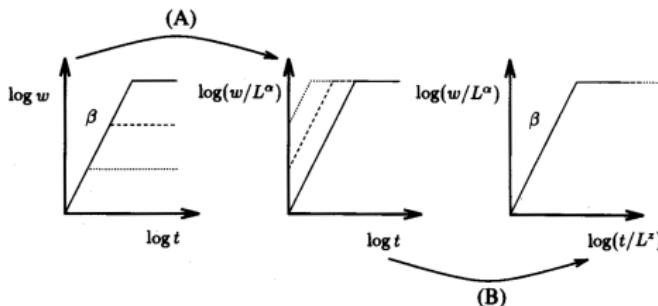
with

$$f(u) \sim \begin{cases} u^\beta, & u \ll 1, \\ \text{const}, & u \gg 1. \end{cases} \Rightarrow \beta = \frac{\alpha}{z}$$

Immediate consequences:

$$w_{\text{sat}}(L) \sim L^\alpha, \quad t_x(L) \sim L^z.$$

Interpretation: dynamic renormalization $x \rightarrow bx$, $t \rightarrow b^z t$, $h \rightarrow b^\alpha h$ leaves $w(L, t)/L^\alpha$ invariant as a function of t/L^z .



(Image from Barabási-Stanley's book 95)

Family and Vicsek reported from their simulation that

$$\beta = 0.30 \pm 0.02 \approx \frac{1}{3}.$$

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

Continuum viewpoint: EW and KPZ

EW (linear diffusion + noise):

$$\partial_t h = \nu \nabla^2 h + \eta, \quad \langle \eta \eta \rangle \propto \delta(x) \delta(t).$$

Scale invariance gives $z = 2$, $\alpha = \frac{1}{2}$ in 1D $\Rightarrow \beta = \frac{1}{4}$.

KPZ (nonlinear growth):

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta.$$

Galilean/tilt invariance $\Rightarrow \alpha + z = 2$. In 1D (exact):

$$\boxed{\alpha = \frac{1}{2}, \quad z = \frac{3}{2}, \quad \beta = \frac{1}{3}.}$$

Many lattice models (BD, RSOS, Eden) fall into KPZ in 1+1D.

Reading plots (as in Barabási–Stanley)

- ▶ On w vs. t (log–log) for several L : early slopes $\approx \beta$; plateaus $\propto L^\alpha$; horizontal shift of crossovers $\propto L^z$.
- ▶ After rescaling: plot w/L^α vs. t/L^z ; all curves collapse to $f(u)$ with $f(u) \sim u^\beta$ ($u \ll 1$) and $f(u) \rightarrow \text{const}$ ($u \gg 1$).

Rule of thumb (1D):

$$\text{EW: } (\alpha, \beta, z) = \left(\frac{1}{2}, \frac{1}{4}, 2\right), \quad \text{KPZ/BD: } (\alpha, \beta, z) = \left(\frac{1}{2}, \frac{1}{3}, \frac{3}{2}\right).$$

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Exactly solvable baseline: Random Deposition (RD)

One particle per step, random column; after t steps heights are multinomial:

$$\mathbb{E}[w^2(L, t)] = \frac{t}{L} \left(1 - \frac{1}{L}\right) \sim \frac{t}{L}.$$

Hence $w \sim (t/L)^{1/2}$ ($\beta = \frac{1}{2}$), *no lateral correlations*, no finite z and no true saturation. RD serves as a growth-only sanity check; FV scaling requires correlated dynamics (EW/KPZ).

Structure factor (optional but handy)

Fourier modes $h_k(t)$, structure factor $S(k, t) = \langle |h_k(t)|^2 \rangle$ obey

$$S(k, t) \sim k^{-(2\alpha+1)} g(k^z t), \quad g(y) \sim \begin{cases} y^{2\beta+1} & y \ll 1, \\ \text{const} & y \gg 1. \end{cases}$$

Useful when $w(L, t)$ is noisy; fit (α, z) from spectral slopes and collapse $S(k, t)$.

Cheat sheet / takeaways

$$w(L, t) \sim L^\alpha f\left(\frac{t}{L^z}\right), \quad f(u) \sim u^\beta \text{ (} u \ll 1 \text{), } f(u) \rightarrow \text{const (} u \gg 1 \text{), } \beta = \alpha/z$$

1D benchmarks: EW (α, β, z) = ($\frac{1}{2}, \frac{1}{4}, 2$); KPZ/BD ($\frac{1}{2}, \frac{1}{3}, \frac{3}{2}$).

Workflow: measure $w(L, t)$ \Rightarrow fit α and β \Rightarrow infer/fit z \Rightarrow collapse w/L^α vs. t/L^z .

Simulations on
Random deposition vs. Ballistic decomposition

Study of growing interfaces in a thin film

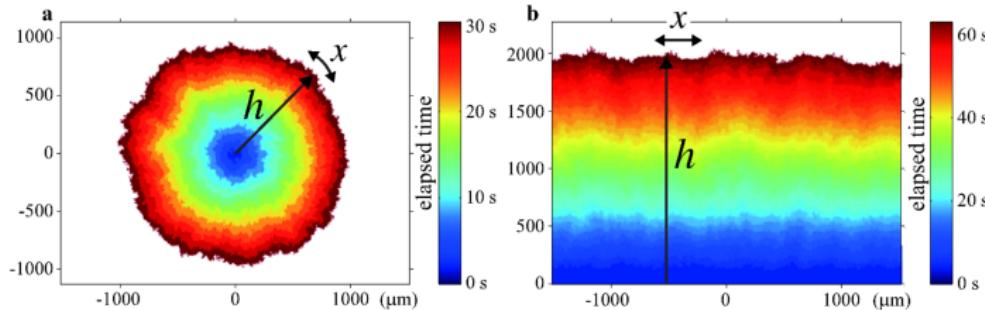
— Convection of nematic liquid crystal*

Show movies !

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

Study of growing interfaces in a thin film

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Prediction from KPZ equation:

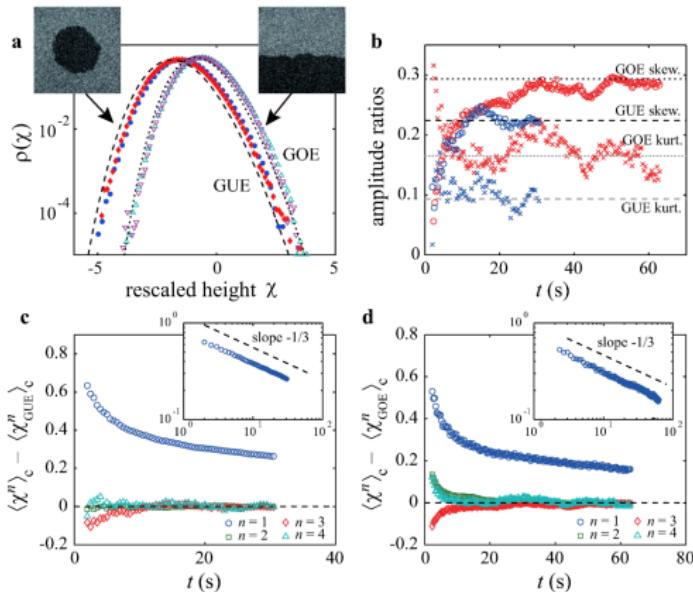
$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

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KPZ Equation '86

$$\frac{\partial}{\partial t} h(t, x) = \frac{1}{2} \Delta h(t, x) + \frac{\lambda}{2} (\nabla h)^2 + \dot{W}(t, x) \quad (\text{KPZ})$$



Mehran Kardar (1957 –) Giorgio Parisi (1948 –)



Yicheng Zhang

Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

Plan

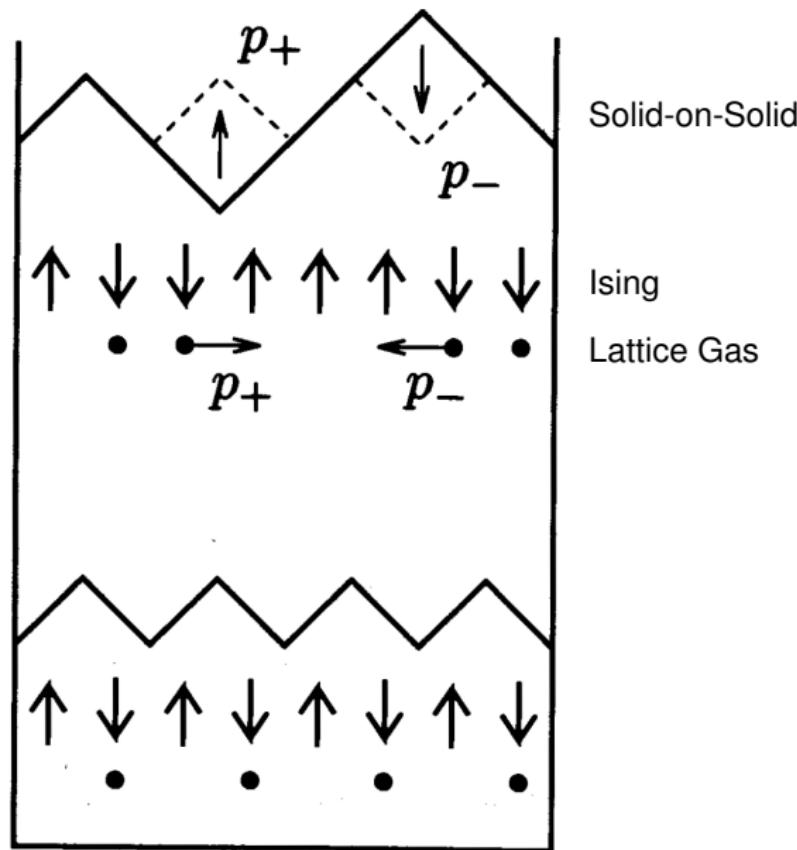
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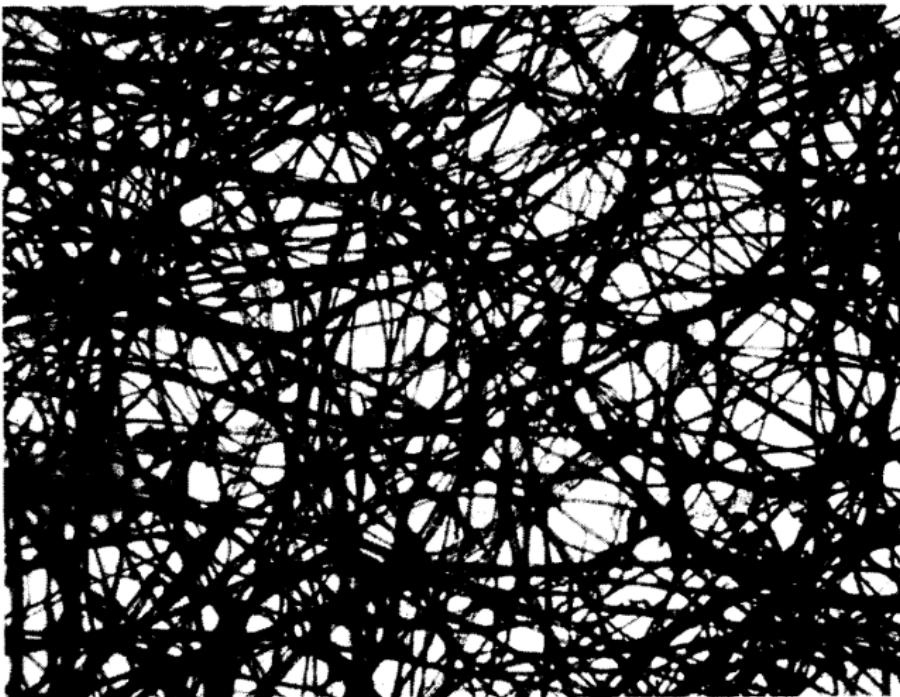
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More models? Even more simpler?

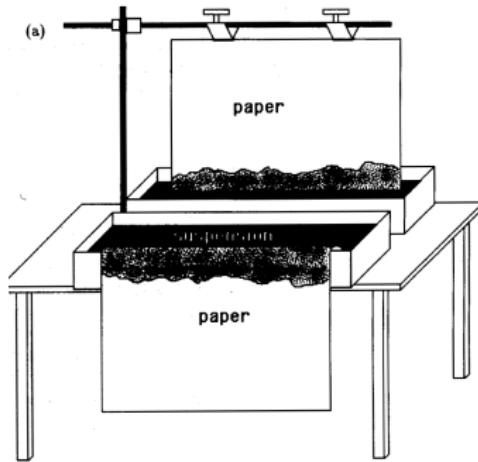


Paper – a random environment



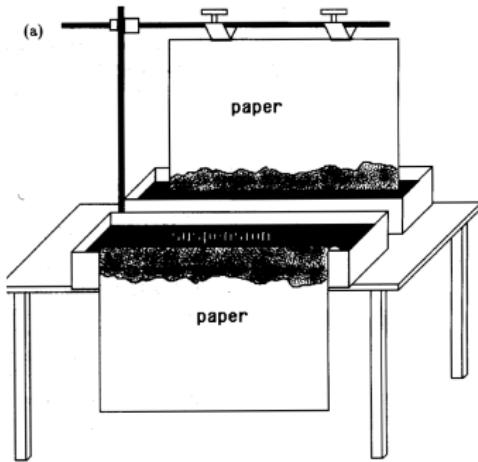
Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper wetting experiment



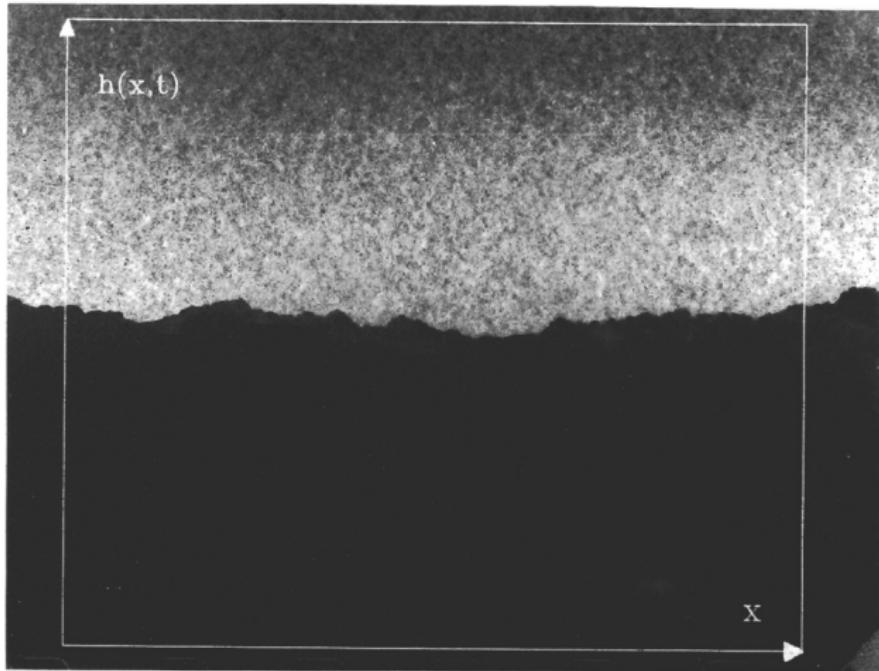
Barabási, A.-L., Stanley, H. E., 1995

Paper wetting experiment



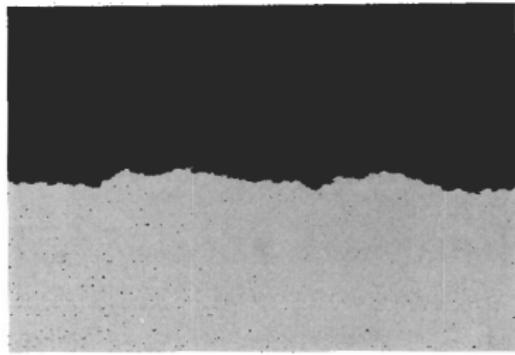
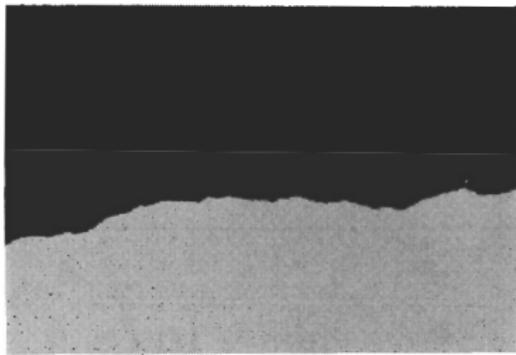
Barabási, A.-L., Stanley, H. E., 1995

Paper burning experiment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper rupture experiment



Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

Plan

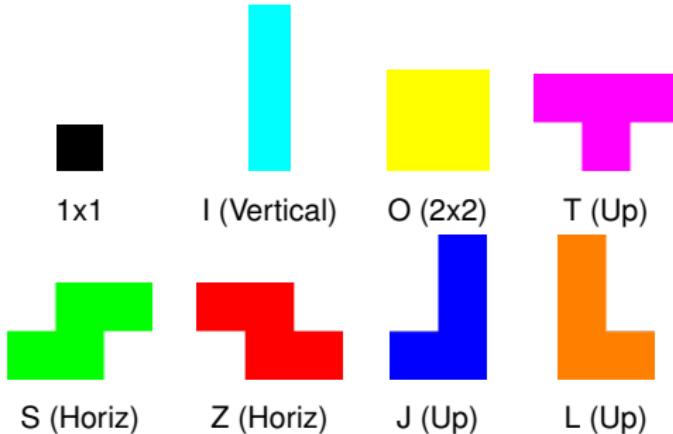
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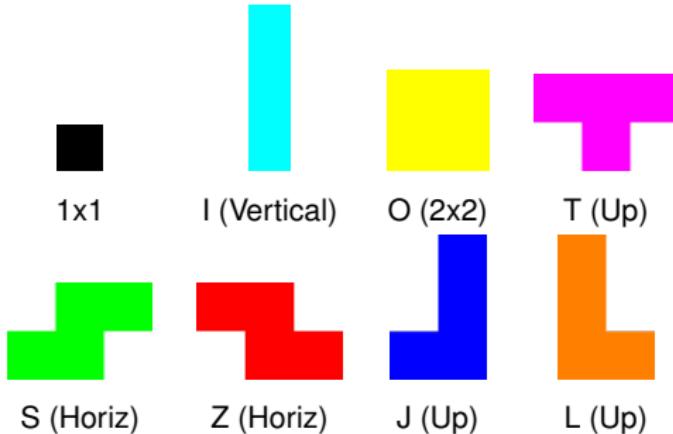
Tetrominoes



- ▶ “1x1”: Single (extra single-site particle)
- ▶ “I”: Horizontal, Vertical
- ▶ “J, L, T”: Up, Right, Down, Left
- ▶ “S, Z”: Horizontal, Vertical
- ▶ “O”: Single (2x2 square)
- ▶ Sticky
- ▶ Nonsticky

$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

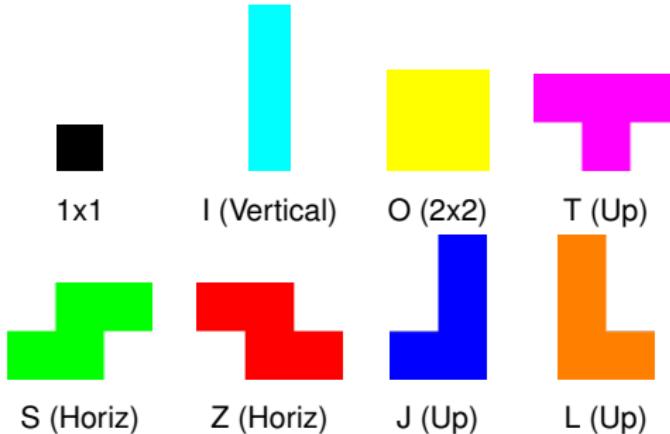
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Configure files

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [20, 0]  
Piece-01: [20, 0]  
Piece-02: [20, 0]  
Piece-03: [20, 0]  
Piece-04: [20, 0]  
Piece-05: [20, 0]  
Piece-06: [20, 0]  
Piece-07: [20, 0]  
Piece-08: [20, 0]  
Piece-09: [20, 0]  
Piece-10: [20, 0]  
Piece-11: [20, 0]  
Piece-12: [20, 0]  
Piece-13: [20, 0]  
Piece-14: [20, 0]  
Piece-15: [20, 0]  
Piece-16: [20, 0]  
Piece-17: [20, 0]  
Piece-18: [20, 0]  
Piece-19: [20, 0]
```

All nonsticky pieces
with equal prob.

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [0, 20]  
Piece-01: [0, 20]  
Piece-02: [0, 20]  
Piece-03: [0, 20]  
Piece-04: [0, 20]  
Piece-05: [0, 20]  
Piece-06: [0, 20]  
Piece-07: [0, 20]  
Piece-08: [0, 20]  
Piece-09: [0, 20]  
Piece-10: [0, 20]  
Piece-11: [0, 20]  
Piece-12: [0, 20]  
Piece-13: [0, 20]  
Piece-14: [0, 20]  
Piece-15: [0, 20]  
Piece-16: [0, 20]  
Piece-17: [0, 20]  
Piece-18: [0, 20]  
Piece-19: [0, 20]
```

All sticky pieces
with equal prob.

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [0, 0]  
Piece-01: [0, 0]  
Piece-02: [0, 0]  
Piece-03: [0, 0]  
Piece-04: [0, 0]  
Piece-05: [0, 0]  
Piece-06: [0, 0]  
Piece-07: [0, 0]  
Piece-08: [0, 0]  
Piece-09: [0, 0]  
Piece-10: [0, 0]  
Piece-11: [0, 0]  
Piece-12: [0, 0]  
Piece-13: [0, 0]  
Piece-14: [0, 0]  
Piece-15: [0, 0]  
Piece-16: [0, 0]  
Piece-17: [0, 0]  
Piece-18: [0, 0]  
Piece-19: [20, 80]
```

20% nonsticky
+ 80% sticky
of 1x1 piece

Outreach Highlights



AU-SSI 2023



AU-SSI 2024



Destination STEM 2023

Main References:

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Thank you!

Questions?