Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

Le Chen
Auburn University

Acknwolegement

NSF 2246850, NSF 2443823, & Simons Foundation Travel Grant (2022-2027)

`Talk available at: github.com/chenle02`

Emerging Synergies between Stochastic Analysis and Statistical Mechanics
Banff, Alberta, Canada
October 28, 2025

Integration of Research, Education, and Outreach

Outreach

- Auburn Summer Science Institute (AU-SSI): **2024, 2025**
  Selected talented high school students
- Destination STEM: **2023, 2024**
  Junior middle school students

Education

- Graduate Student Seminars (Mathematics), Auburn: **2022–2025**
- Math 7820/7830: Applied Stochastic Processes
  Course project, **2023/24**

Research

- Simulation and modeling packages (open source)
- Forming conjectures and validating results

Most materials are available at

`github.com/chenle02`

Math 7820/30: Applied Stochastic Processes (2023/24):


Mauricio Montes and Ian Ruau

Simulation package:

https://github.com/chenle02/Simulations_on_Some_Surface_Growth_Models

```
pip install tetris-ballistic
```

Image is generated by OpenAI's *DALL-E*

Random deposition

Ballistic deposition

Image is generated by OpenAI's *DALL-E*

Plan

Introduction to growth model and SPDE

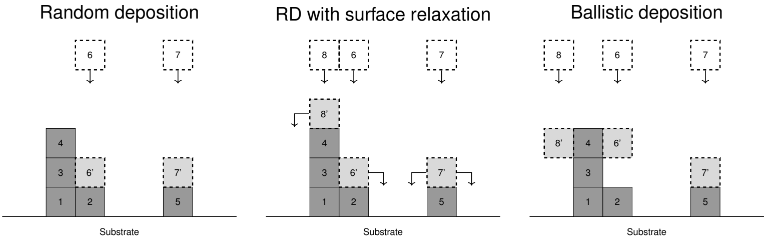
Family-Vicsek scaling and experiments

More examples

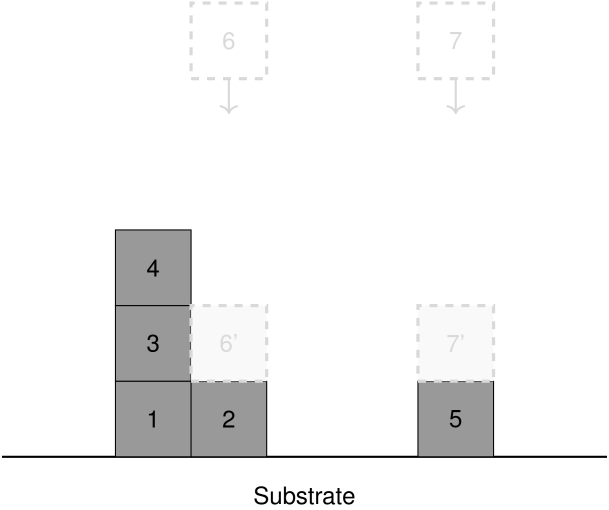
Tetromino Pieces

Plan

How do surfaces grow?
Three models



Random deposition



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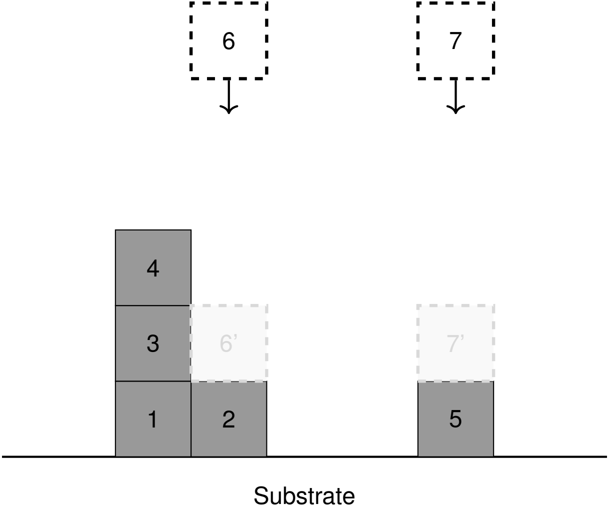

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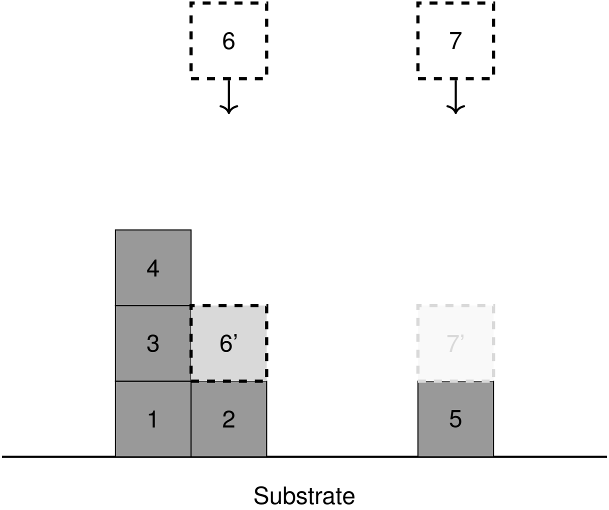

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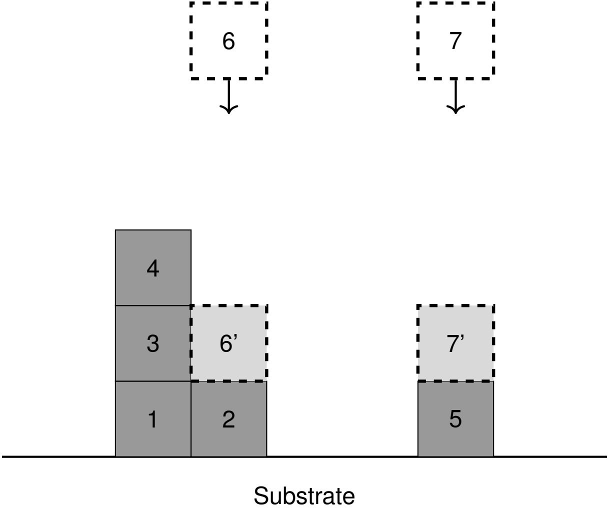
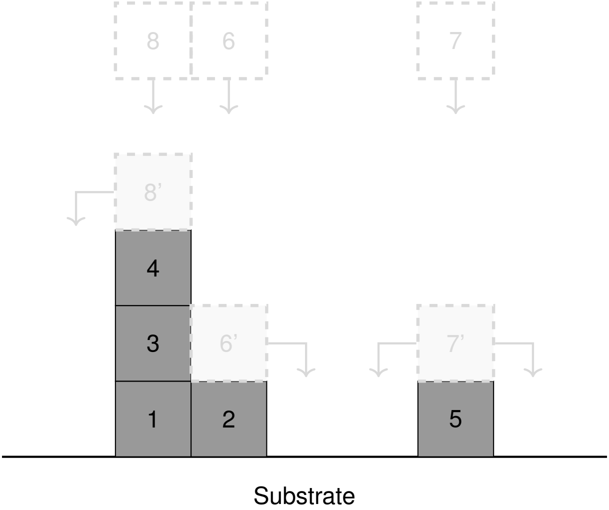


Random deposition with surface relaxation



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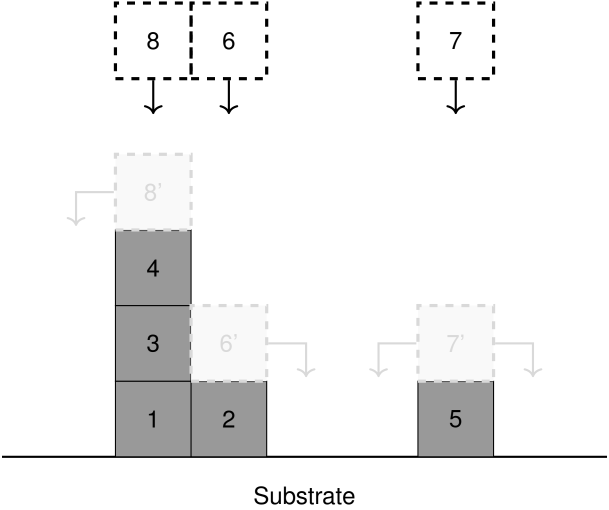

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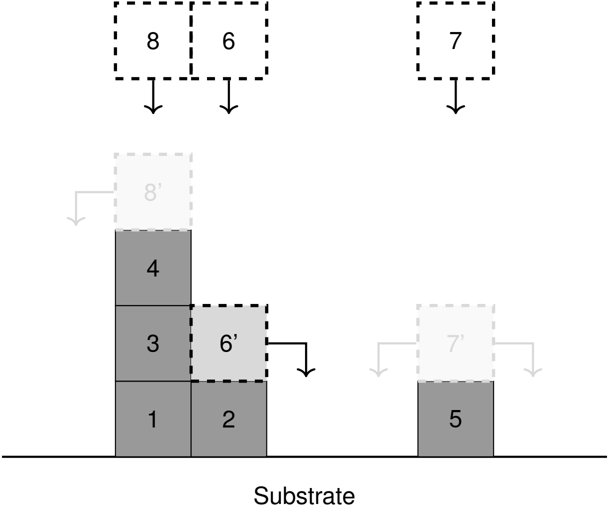
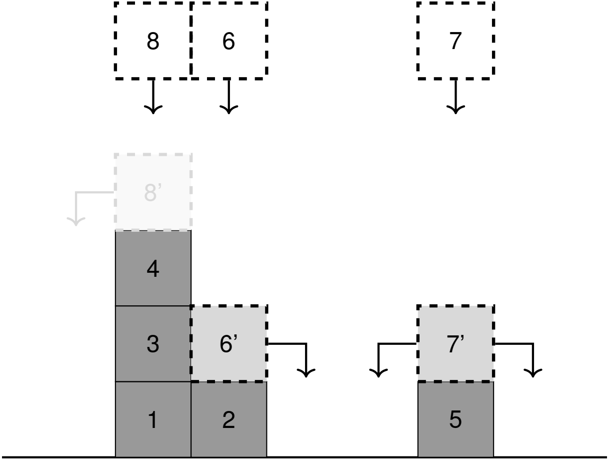
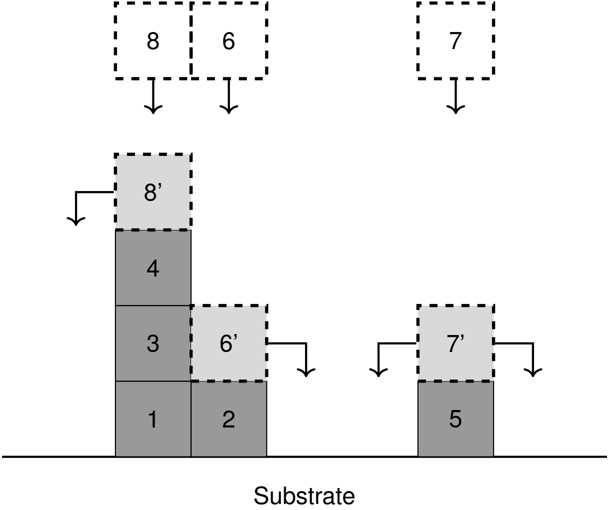


Random deposition with surface relaxation

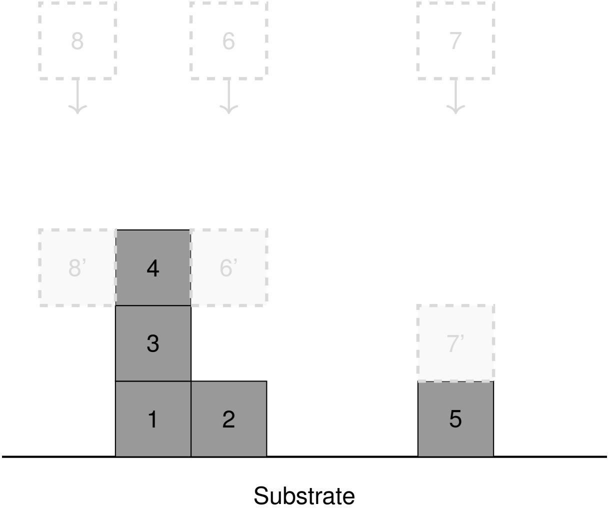


Substrate

Random deposition with surface relaxation



Ballistic deposition



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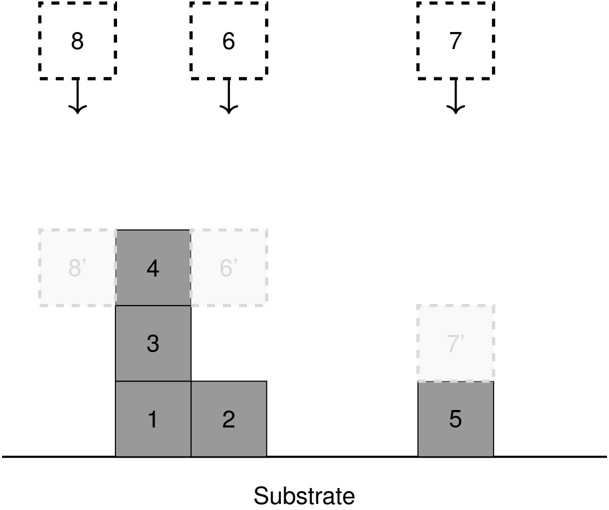

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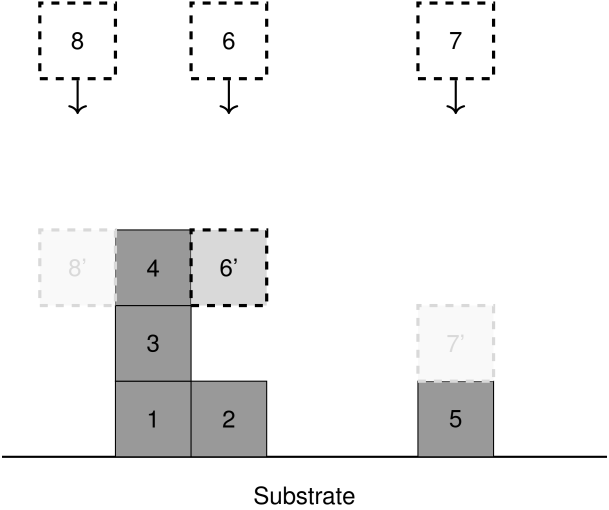

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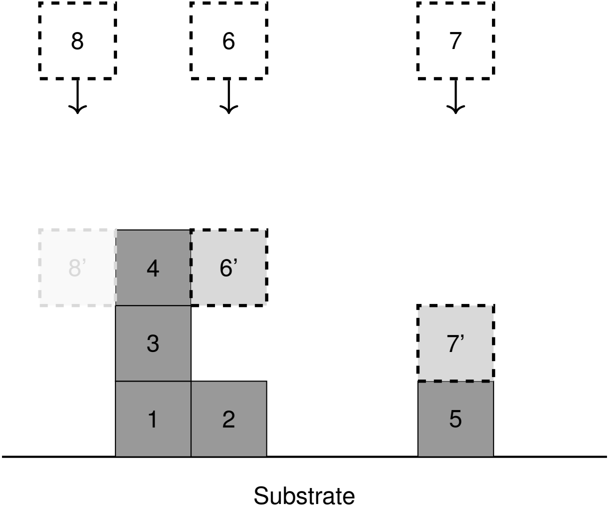

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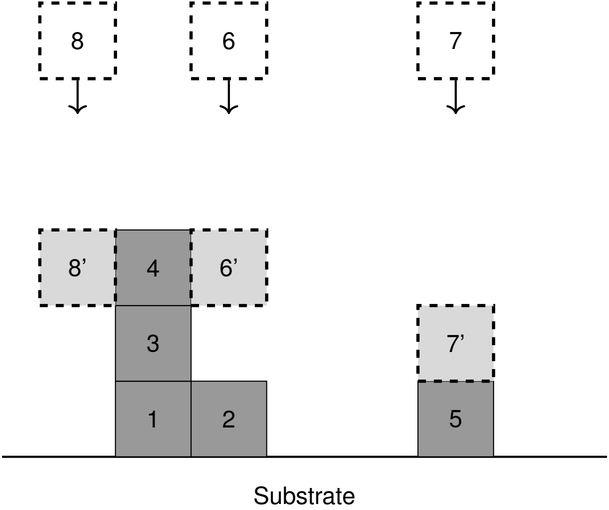
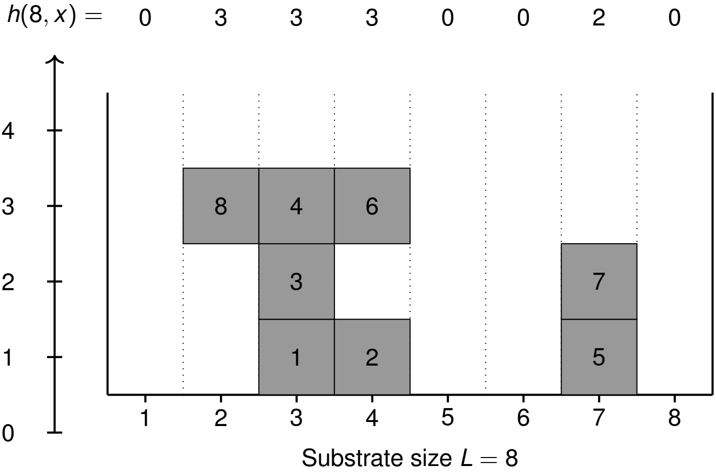


Average height and fluctuation

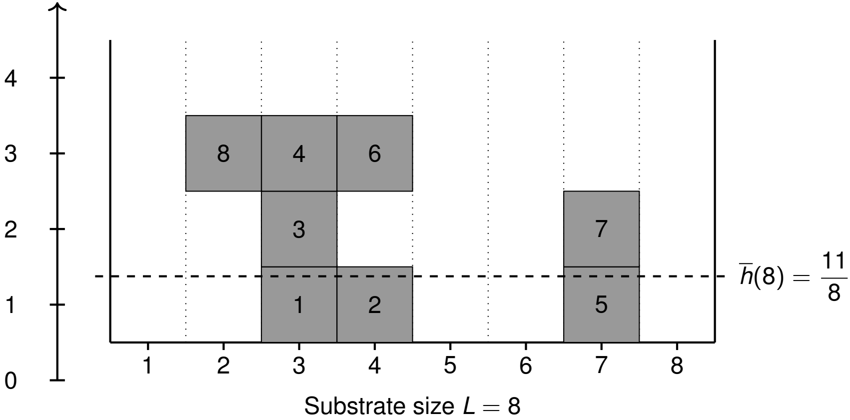


Average height and fluctuation

$$\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t,x)$$

Fluctuation $W(L,t) = \sqrt{\frac{1}{L}\sum_{x=1}^{L}\left[h(t,x) - \overline{h}(t)\right]^2}$

$h(8,x) =$ 0 3 3 3 0 0 2 0



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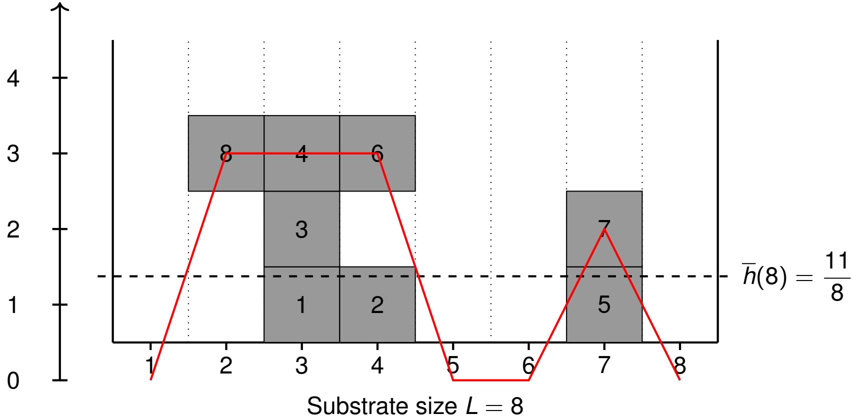

Random Deposition (independent columns, nonsticky)

Model. L independent columns. At each integer time $t = 1, 2, \ldots$, drop *one* particle on a uniformly random column. Heights $h(t,x)$, mean $\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t,x) = \frac{t}{L}$, width

$$W^2(L,t) = \frac{1}{L}\sum_{x=1}^{L}\left(h(t,x) - \overline{h}(t)\right)^2.$$

Single–column law: After t drops total,

$$h(t,x) \sim \mathrm{Binomial}\left(t, \frac{1}{L}\right), \qquad \mathbb{E}[h(t,x)] = \frac{t}{L}, \quad \mathrm{Var}(h(t,x)) = t\,\frac{1}{L}\left(1 - \frac{1}{L}\right).$$

Fluctuation: By i.i.d. columns,

$$\mathbb{E}\left[W^2(L,t)\right] = \frac{1}{L}\sum_{x=1}^{L}\mathbb{E}\left[h(t,x)^2\right] - \mathbb{E}\left[\overline{h}^2(t)\right] = \mathbb{E}\left[h(t,1)^2\right] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right)\mathrm{Var}(h(t,1)).$$

Hence

$$\boxed{\mathbb{E}\left[W^2(L,t)\right] = \left(1 - \frac{1}{L}\right) t\,\frac{1}{L}\left(1 - \frac{1}{L}\right) = \frac{t}{L}\left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L,t) \simeq \left(1 - \frac{1}{L}\right)\left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \frac{1}{2}$.

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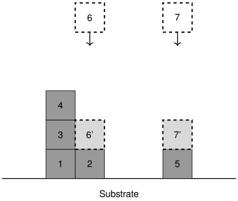
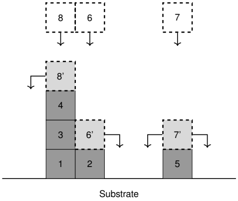
Questions

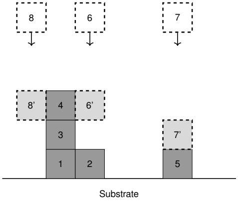
Random deposition



RD with surface relaxation



Ballistic deposition



$W(t,x) \sim t^{1/2}$

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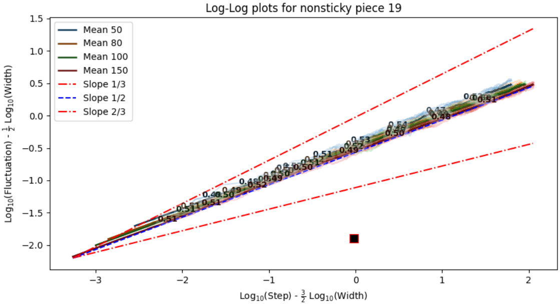
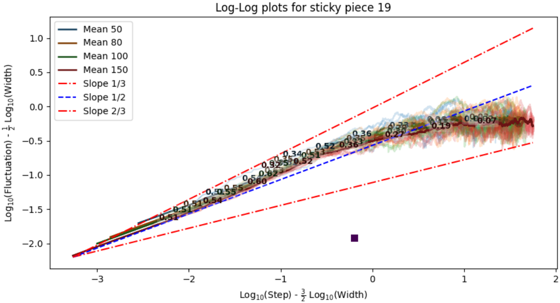

Log-Log plots for nonsticky piece 19
Mean 50
Mean 80
Mean 100
Mean 150
Slope 1/3
Slope 1/2
Slope 2/3
$\mathrm{Log}_{10}(\mathrm{Fluctuation}) - \frac{1}{2}\mathrm{Log}_{10}(\mathrm{Width})$
$\mathrm{Log}_{10}(\mathrm{Step}) - \frac{3}{2}\mathrm{Log}_{10}(\mathrm{Width})$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Plan

Introduction to growth model and SPDE

Family-Vicsek scaling and experiments

More examples

Tetromino Pieces

Family–Vicsek scaling theory

Recall height function $h(x,t)$, $x = 1, \ldots, L$. Mean $\bar{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(x,t)$.

$$w(L,t) := \sqrt{\frac{1}{L}\sum_{x=1}^{L}\left(h(x,t) - \bar{h}(t)\right)^2} \quad \text{(Fluctuation).}$$

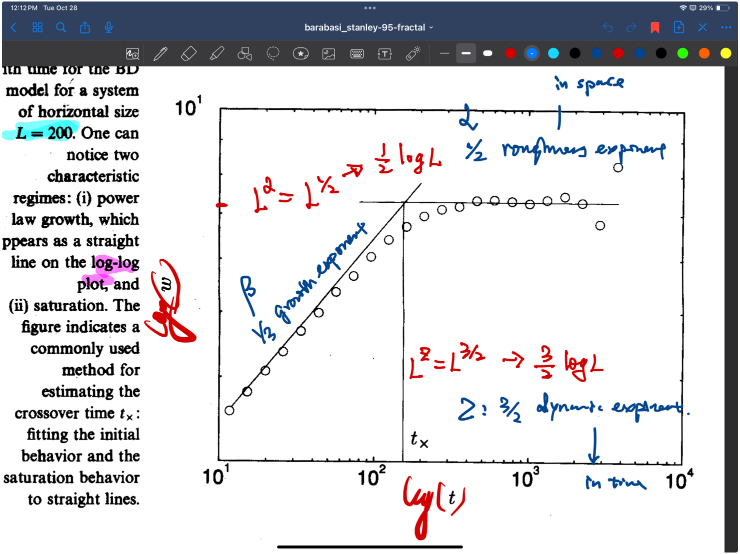
Empirical scaling (log–log)

Early time	$w(L,t) \sim t^{\beta}$	β: growth exponent
Late time	$w(L,t) \to w_{\mathrm{sat}}(L) \sim L^{\alpha}$	α: roughness exponent
Crossover time	$t_{\times}(L) \sim L^{z}$	z: dynamic exponent

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

Family–Vicsek scaling theory

α: Roughness exp.; β: Growth exp.; z: Dynamic exp.



(Image from Barabási-Stanley's book 95)

Family–Vicsek scaling theory

α: Roughness exp.; β: Growth exp.; z: Dynamic exp.

$$\boxed{w(L,t) \sim L^{\alpha} f\left(\frac{t}{L^z}\right)}$$

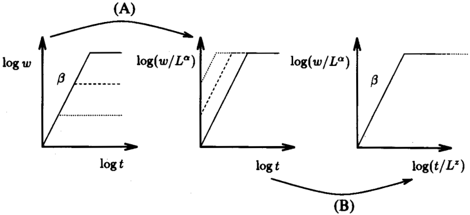
with

$$f(u) \sim \begin{cases} u^{\beta}, & u \ll 1, \\ \text{const}, & u \gg 1. \end{cases} \quad \Rightarrow \quad \boxed{\beta = \frac{\alpha}{z}}$$

Immediate consequences:

$$w_{\text{sat}}(L) \sim L^{\alpha}, \qquad t_{\times}(L) \sim L^{z}.$$

Interpretation: dynamic renormalization $x \to bx$, $t \to b^z t$, $h \to b^{\alpha} h$ leaves $w(L,t)/L^{\alpha}$ invariant as a function of t/L^z.



(Image from Barabási-Stanley's book 95)

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

Family–Vicsek (1985): Growth Exponent β

Reported from simulations

$$\boxed{\beta = 0.30 \pm 0.02 \approx \frac{1}{3}}$$

Practical difficulty

Robustly estimating the log–log slope β requires careful choices of fitting window and handling of finite-size effects.

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

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Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

Simulations on

Random deposition vs. Ballistic decomposition

Study of growing interfaces in a thin film

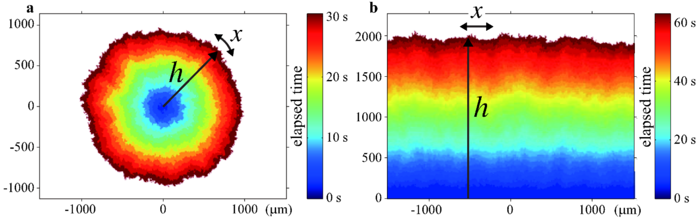
— Convection of nematic liquid crystal*

Show movies !

* Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*



Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

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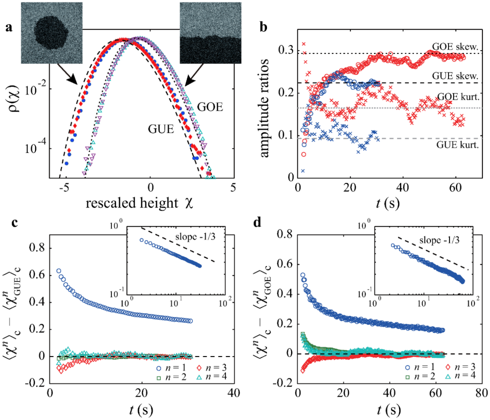


Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

KPZ Equation ’86

$$\frac{\partial}{\partial t} h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{\lambda}{2}(\nabla h)^2 + \dot{W}(t,x) \qquad \text{(KPZ)}$$

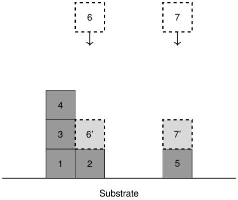
Mehran Kardar (1957 –)

Giorgio Parisi (1948 –)

Yicheng Zhang

Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

Random deposition



$$W(t,x) \sim t^{1/2}$$

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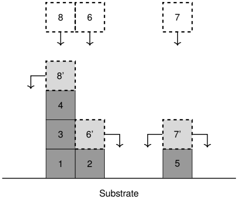
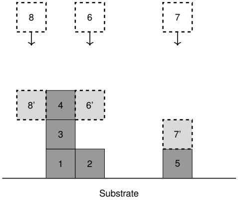


$$W(t,x) \sim t^{1/4}$$

Edwards–Wilkinson

Ballistic deposition



$$W(t,x) \sim t^{1/3}$$

KPZ/BD

Two universality classes: EW and KPZ

Edwards–Wilkinson (EW)

Linear diffusion + noise

$$\partial_t h = \nu \nabla^2 h + \eta, \qquad \langle \eta\eta \rangle \propto \delta(x)\,\delta(t).$$

Scale invariance (1D): $z = 2$, $\alpha = \frac{1}{2}$
$\Rightarrow \beta = \frac{1}{4}$.

$$\boxed{\alpha = \tfrac{1}{2},\ z = 2,\ \beta = \tfrac{1}{4}}$$

Kardar–Parisi–Zhang (KPZ)

Nonlinear growth

$$\partial_t h = \nu \nabla^2 h + \tfrac{\lambda}{2}(\nabla h)^2 + \eta.$$

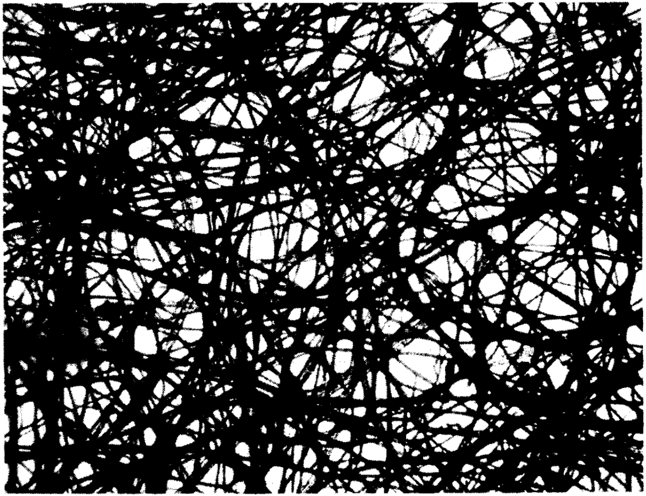
In 1D (exact):

$$\boxed{\alpha = \tfrac{1}{2},\ z = \tfrac{3}{2},\ \beta = \tfrac{1}{3}}$$

BD, SOS, Eden $\Rightarrow$ KPZ universality in 1+1D.

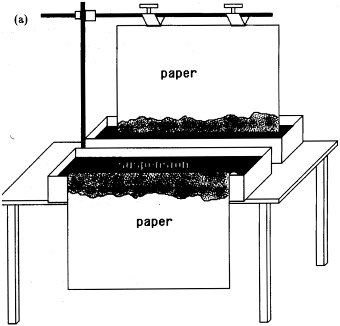
Plan

Paper – a random environment



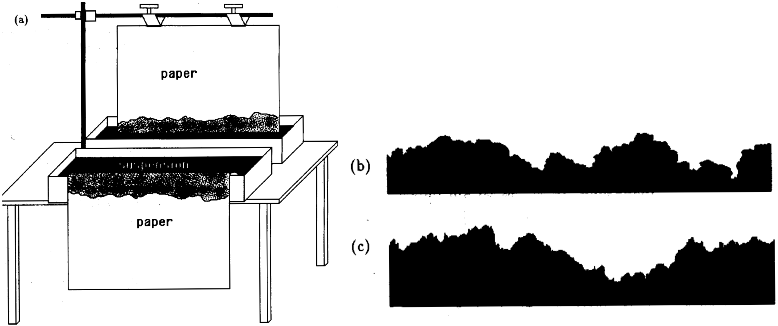
Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper wetting experiment



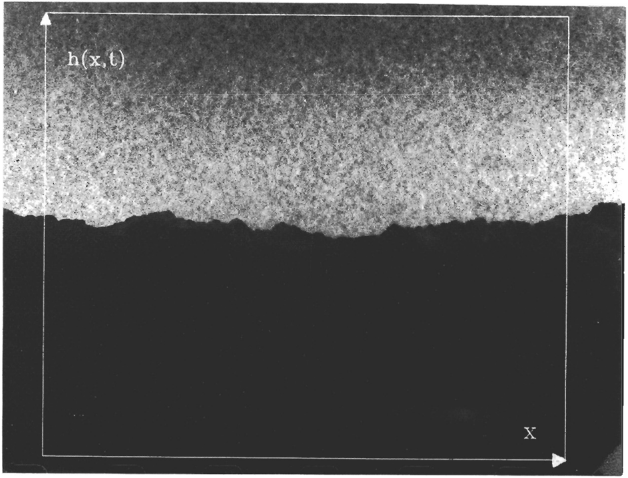
Barabási, A.-L., Stanley, H. E., 1995

Paper wetting experiment



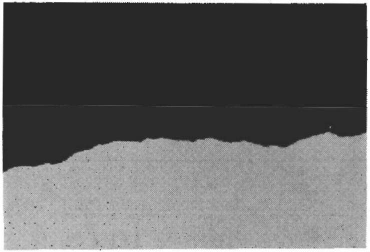
Barabási, A.-L., Stanley, H. E., 1995

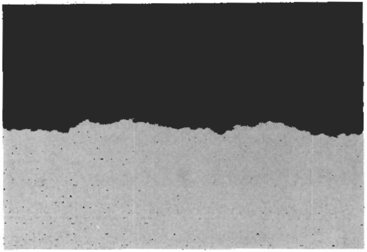
Paper burning experiment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper rupture experiment

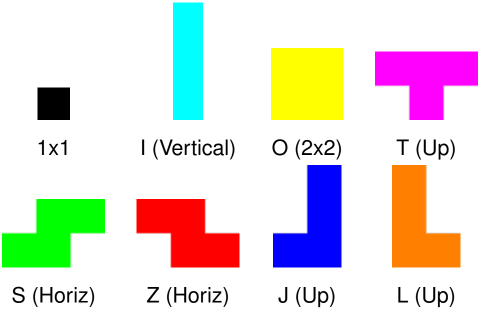




Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

Plan

Tetrominoes

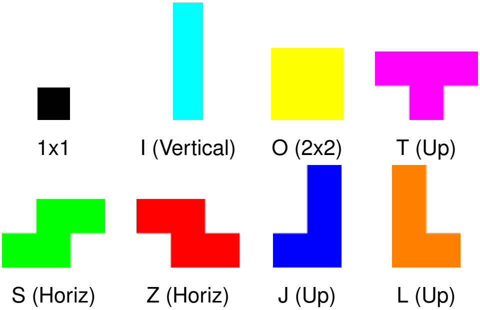


- “1x1”: Single (extra single-site particle)
- “I”: Horizontal, Vertical
- “J, L, T”: Up, Right, Down, Left
- “S, Z”: Horizontal, Vertical
- “O”: Single (2x2 square)

- Sticky
- Nonstikcy

$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40$ types of pieces

Tetrominoes



- "1x1": Single (extra single-site particle)
- "I": Horizontal, Vertical
- "J, L, T": Up, Right, Down, Left
- "S, Z": Horizontal, Vertical
- "O": Single (2x2 square)

- Sticky
- Nonstikcy

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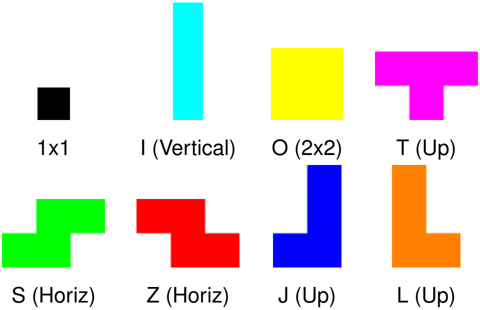

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$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

Configure files

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [20, 0]
Piece-01: [20, 0]
Piece-02: [20, 0]
Piece-03: [20, 0]
Piece-04: [20, 0]
Piece-05: [20, 0]
Piece-06: [20, 0]
Piece-07: [20, 0]
Piece-08: [20, 0]
Piece-09: [20, 0]
Piece-10: [20, 0]
Piece-11: [20, 0]
Piece-12: [20, 0]
Piece-13: [20, 0]
Piece-14: [20, 0]
Piece-15: [20, 0]
Piece-16: [20, 0]
Piece-17: [20, 0]
Piece-18: [20, 0]
Piece-19: [20, 0]
```

All nonsticky pieces with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 20]
Piece-01: [0, 20]
Piece-02: [0, 20]
Piece-03: [0, 20]
Piece-04: [0, 20]
Piece-05: [0, 20]
Piece-06: [0, 20]
Piece-07: [0, 20]
Piece-08: [0, 20]
Piece-09: [0, 20]
Piece-10: [0, 20]
Piece-11: [0, 20]
Piece-12: [0, 20]
Piece-13: [0, 20]
Piece-14: [0, 20]
Piece-15: [0, 20]
Piece-16: [0, 20]
Piece-17: [0, 20]
Piece-18: [0, 20]
Piece-19: [0, 20]
```

All sticky pieces with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 0]
Piece-01: [0, 0]
Piece-02: [0, 0]
Piece-03: [0, 0]
Piece-04: [0, 0]
Piece-05: [0, 0]
Piece-06: [0, 0]
Piece-07: [0, 0]
Piece-08: [0, 0]
Piece-09: [0, 0]
Piece-10: [0, 0]
Piece-11: [0, 0]
Piece-12: [0, 0]
Piece-13: [0, 0]
Piece-14: [0, 0]
Piece-15: [0, 0]
Piece-16: [0, 0]
Piece-17: [0, 0]
Piece-18: [0, 0]
Piece-19: [20, 80]
```

20% nonsticky + 80% sticky of 1x1 piece

Question

For various configurations of Tetromino pieces, do the resulting surface robustly exhibit Family–Vicsek scaling?

Will the scaling exponent β be always close to $\frac{1}{3}$?

Practical difficulty

Robustly estimating the log–log slope β requires careful choices of fitting window and handling of finite-size effects.

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Mixture Parameter ℓ and Scaling

Setup

Consider a mixture of ℓ% ($\ell \in [0, 100]$) nonsticky pieces and $(100-\alpha)$% sticky pieces, where only the 1×1 piece is sticky.

Question

How does α influence the scaling exponents, in particular, $\beta \overset{?}{\approx} \frac{1}{3}$?

Simulations and log–log plots:
https://chenle02.github.io/2025-10-28_Emerging_Synergies_Banff_Le/exp13/videos_and_images_display.html

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Main References:

Barabási, A.-L., & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press, Cambridge.

Family, F., & Vicsek, T. (1985).Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model. *Journal of Physics A: Mathematical and General*, *18*(2), L75.

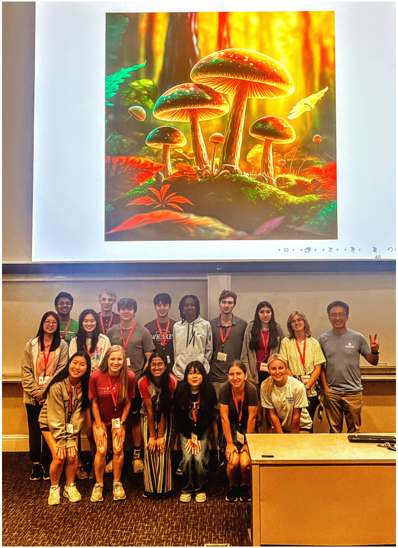
Kardar, M., Parisi, G., & Zhang, Y.-C. (1986).Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, *56*(9), 889.

Kertész, J., Horváth, V. k., & Weber, F. (1993).Self-affine rupture lines in paper sheets. *Fractals*, *01*(01), 67–74.

Takeuchi, K. A., Sano, M., Sasamoto, T., & Spohn, H. (2011).Growing interfaces uncover universal fluctuations behind scale invariance. *Sci. Rep.*, *1*(1), 1–5.

Zhang, J., Zhang, Y.-C., Alstrøm, P., & Levinsen, M. (1992).Modeling forest fire by a paper-burning experiment, a realization of the interface growth mechanism. *Phys. A: Stat. Mech. Appl.*, *189*(3), 383–389.

Outreach Highlights



AU-SSI 2023

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Destination STEM 2023

Thank you!

Questions?