Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

Le Chen
Auburn University

Acknwolegement

NSF 2246850, NSF 2443823, & Simons Foundation Travel Grant (2022-2027)

Emerging Synergies between Stochastic Analysis and Statistical Mechanics
Banff, Alberta, Canada
October 28, 2025

Motivation

Outreach

- Auburn Summer Science Institute (AU-SSI): 2024, 2025
  Selected talented high school students
- Destination STEM: 2023, 2024
  Junior middle school students
- Graduate Student Seminars (Mathematics), Auburn: 2022–2025

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- **Math 7820/7830: Applied Stochastic Processes**
  Course project, 2023/24

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Math 7820/30: Applied Stochastic Processes (2023/24):

Mauricio Montes and Ian Ruau

Plan

Introduction to growth model and SPDE

Tetromino Pieces

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Introduction to growth model and SPDE

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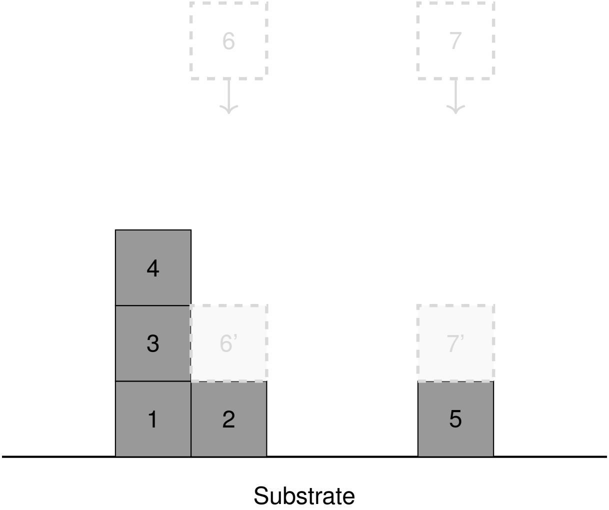
Random deposition



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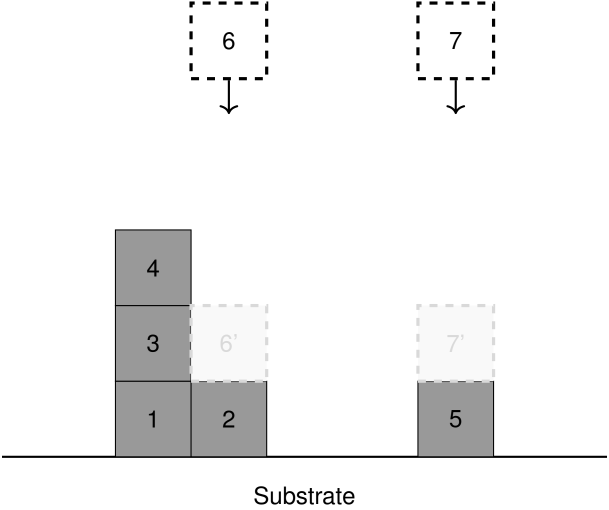

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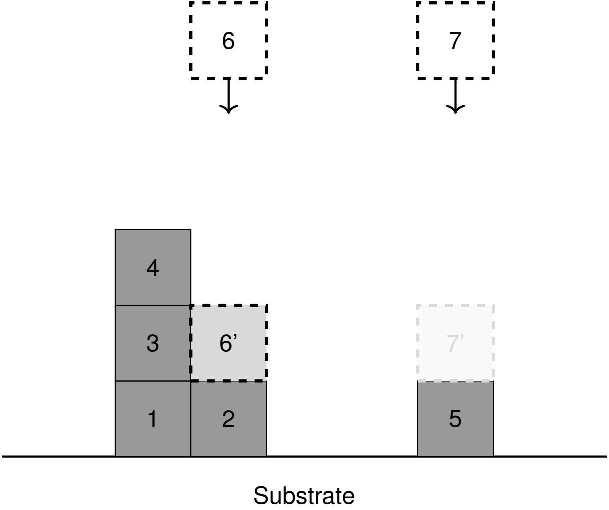

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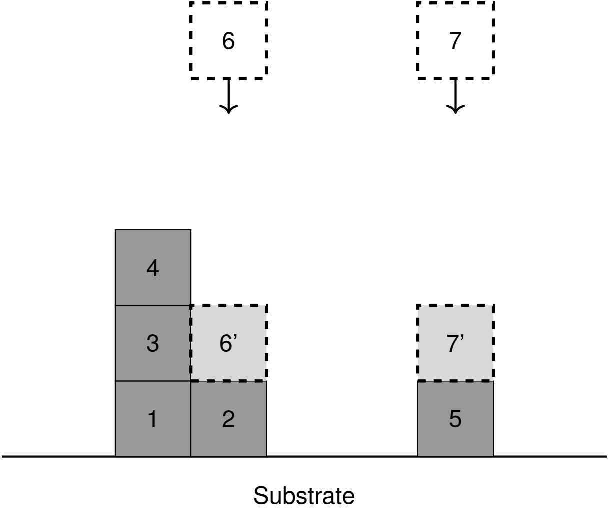
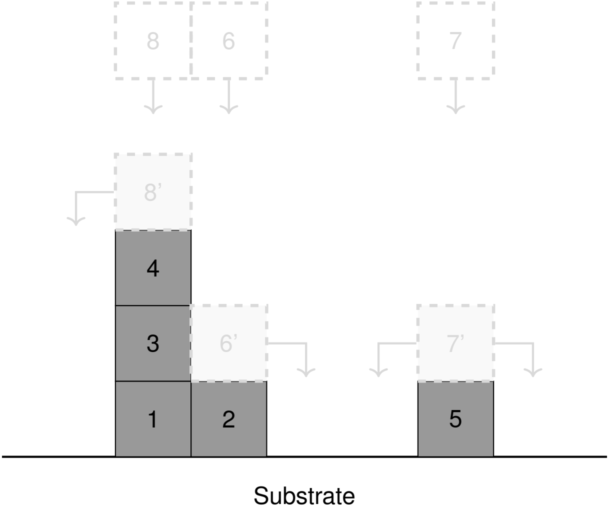


Random deposition with surface relaxation



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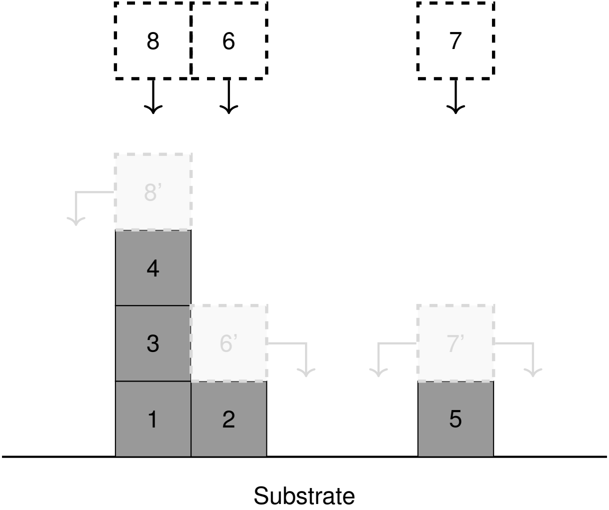

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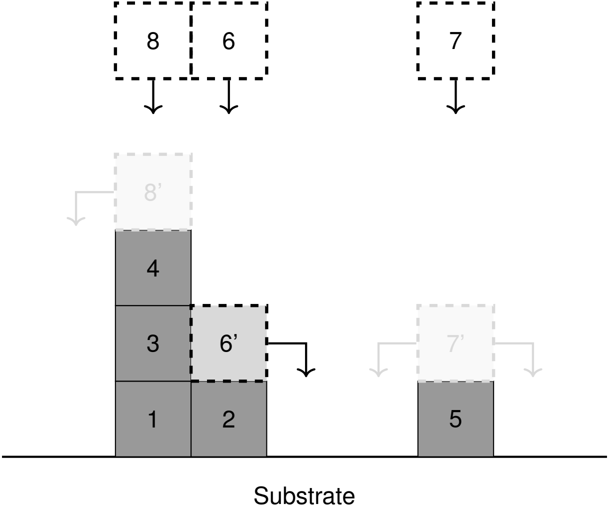

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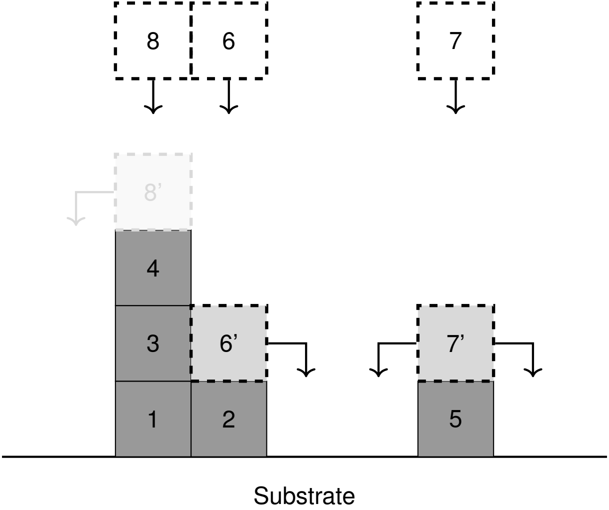

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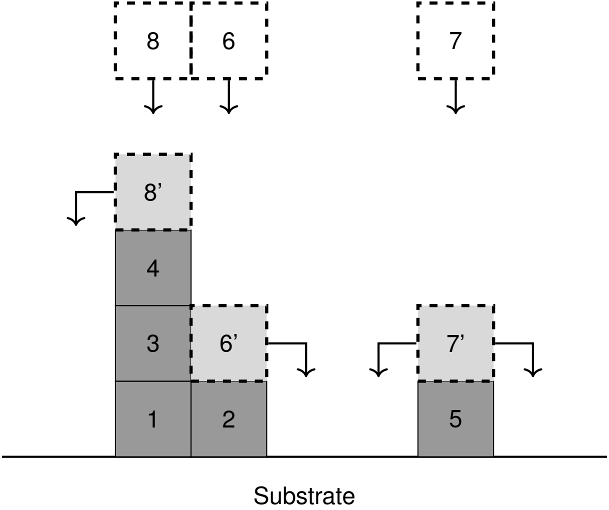
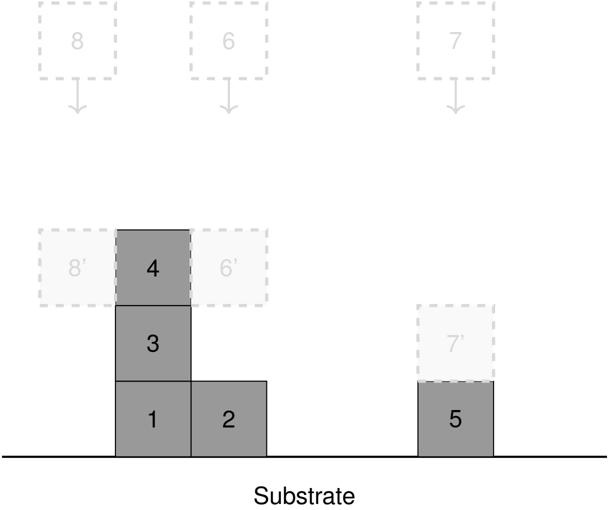


Ballistic deposition



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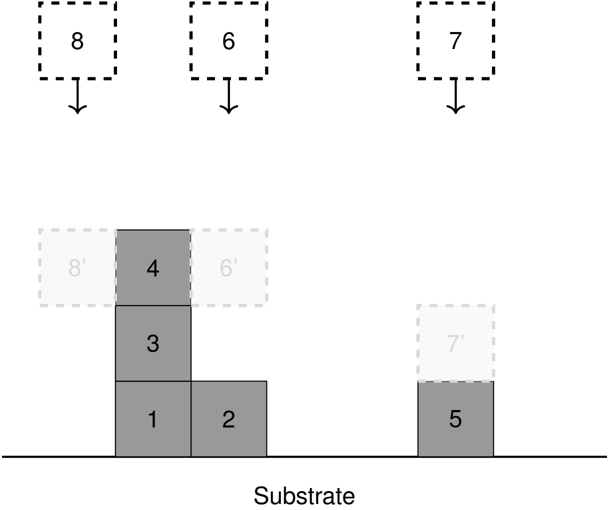

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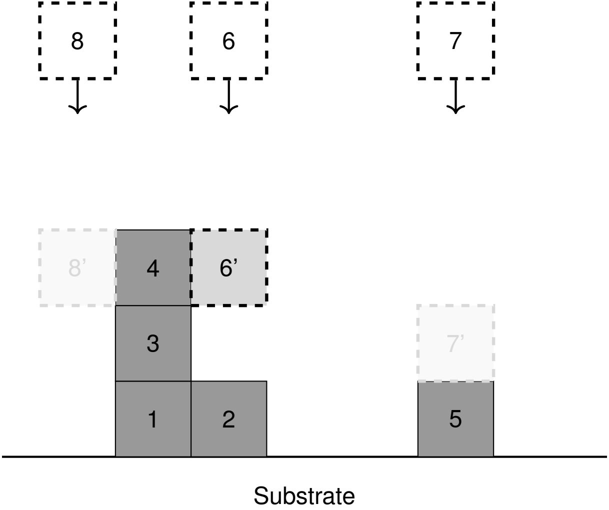

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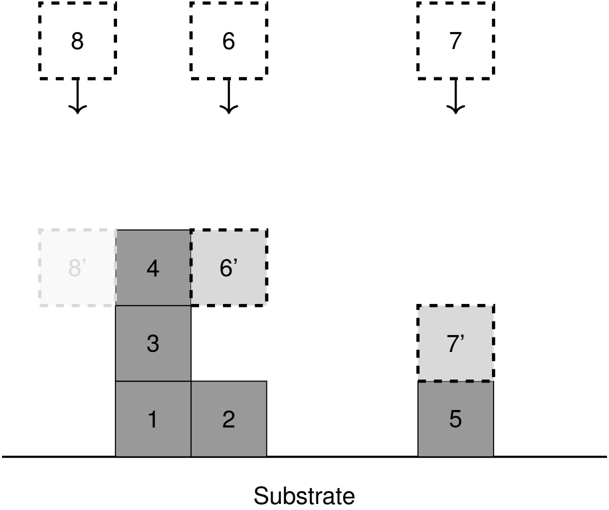

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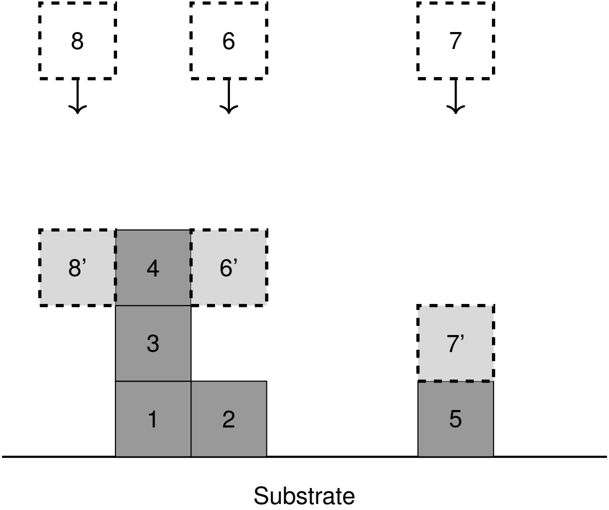
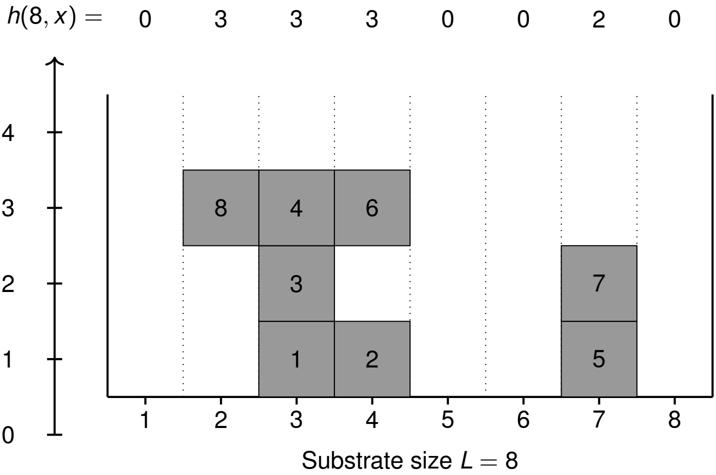


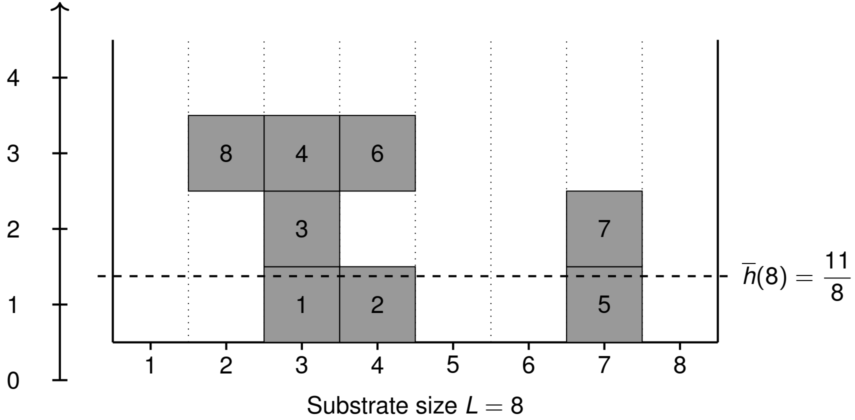
Average height and fluctuation



Average height and fluctuation

$$\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t,x) \qquad \text{Fluctuation } W(L,t) = \sqrt{\frac{1}{L}\sum_{x=1}^{L}\left[h(t,x) - \overline{h}(t)\right]^2}$$

$h(8,x) =$ 0 3 3 3 0 0 2 0



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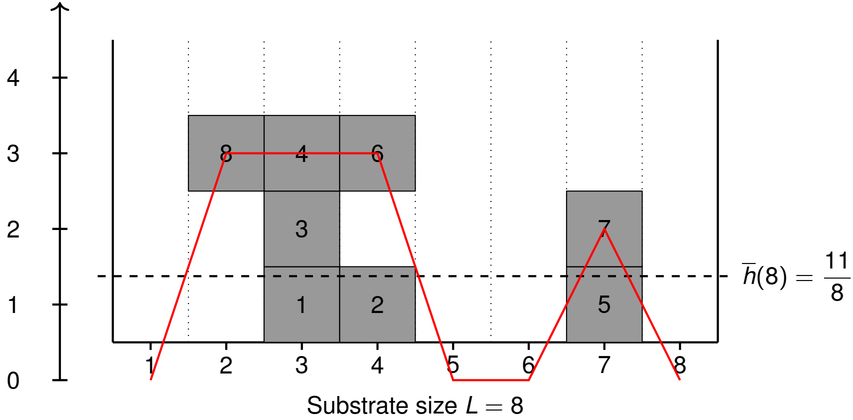

Random Deposition (independent columns, nonsticky)

Model. L independent columns. At each integer time $t = 1, 2, \ldots$, drop *one* particle on a uniformly random column. Heights $h(t,x)$, mean $\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t,x) = \frac{t}{L}$, width

$$W^2(L,t) = \frac{1}{L}\sum_{x=1}^{L}\left(h(t,x) - \overline{h}(t)\right)^2.$$

Single–column law: After t drops total,

$$h(t,x) \sim \mathrm{Binomial}\left(t, \frac{1}{L}\right), \qquad \mathbb{E}[h(t,x)] = \frac{t}{L}, \quad \mathrm{Var}(h(t,x)) = t\,\frac{1}{L}\left(1 - \frac{1}{L}\right).$$

Fluctuation: By i.i.d. columns,

$$\mathbb{E}\left[W^2(L,t)\right] = \frac{1}{L}\sum_{x=1}^{L}\mathbb{E}\left[h(t,x)^2\right] - \mathbb{E}\left[\overline{h}^2(t)\right] = \mathbb{E}\left[h(t,1)^2\right] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right)\mathrm{Var}(h(t,1)).$$

Hence

$$\boxed{\mathbb{E}\left[W^2(L,t)\right] = \left(1 - \frac{1}{L}\right) t\,\frac{1}{L}\left(1 - \frac{1}{L}\right) = \frac{t}{L}\left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L,t) \simeq \left(1 - \frac{1}{L}\right)\left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \frac{1}{2}$.

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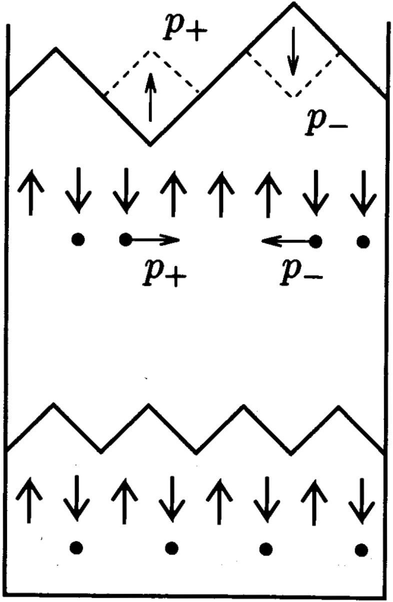
For ballistic deposition (sticky), lateral interactions appear, can we view it as a perturbation of random deposition (nonsticky)?

Simulations on

Random deposition vs. Ballistic decomposition

More models? Even more simpler?

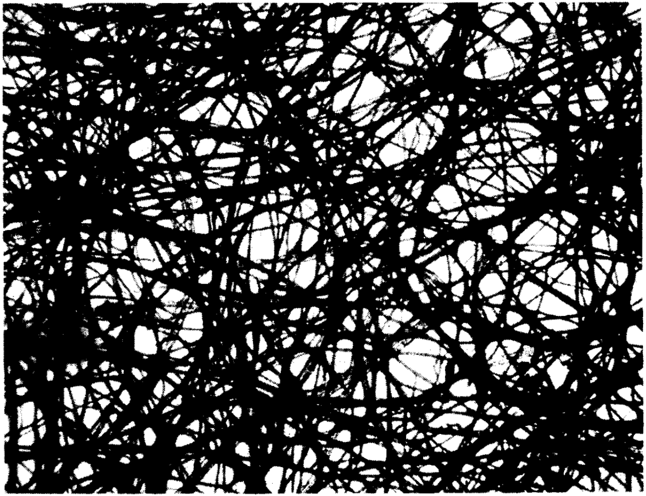


Solid-on-Solid

Ising

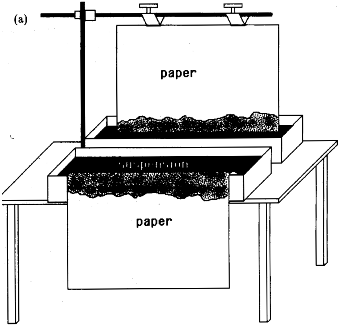
Lattice Gas

Paper – a random environment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper wetting experiment



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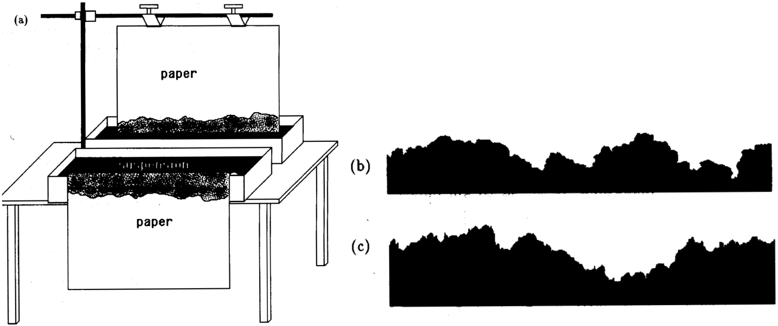

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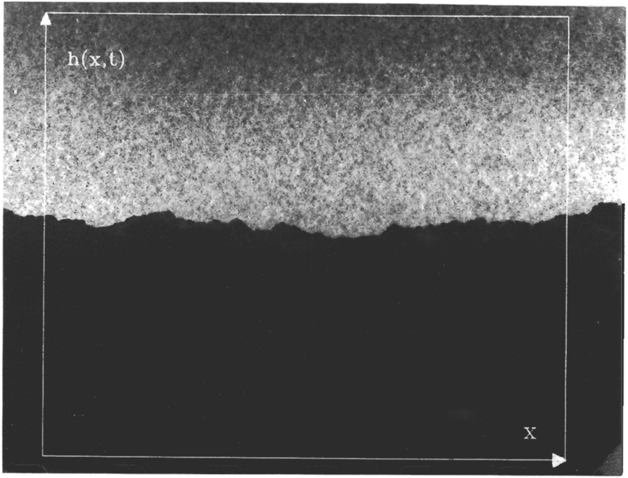
Paper burning experiment



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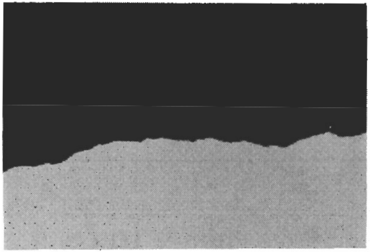
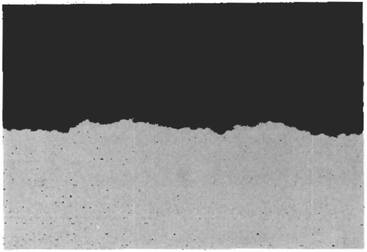
Paper rupture experiment

Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*

Show movies !

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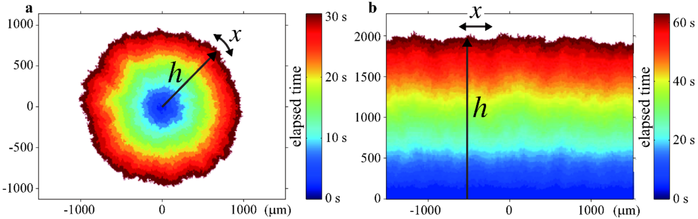
*Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*



Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

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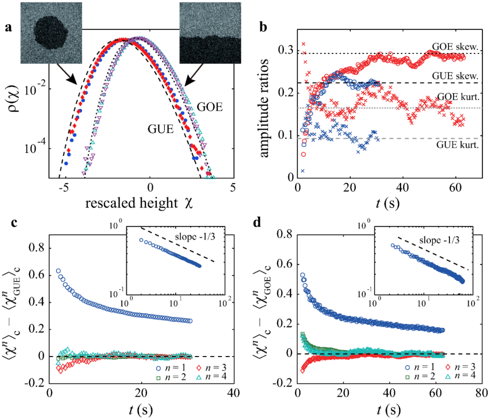

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KPZ Equation '86

$$\frac{\partial}{\partial t} h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{\lambda}{2}(\nabla h)^2 + \dot{W}(t,x) \qquad \text{(KPZ)}$$

Mehran Kardar (1957 –)

Giorgio Parisi (1948 –)

Yicheng Zhang

Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

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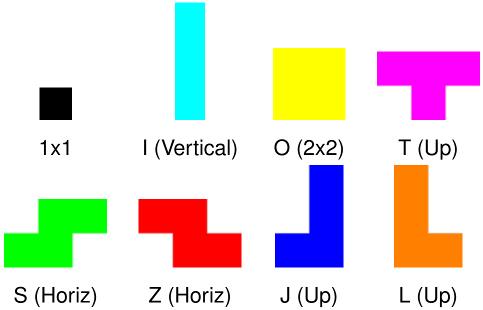
Tetromino Pieces

Tetrominoes

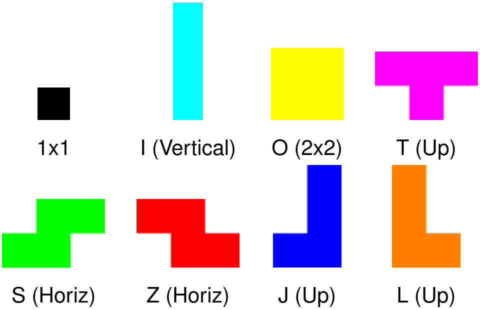


- “1x1”: Single (extra single-site particle)
- “I”: Horizontal, Vertical
- “J, L, T”: Up, Right, Down, Left
- “S, Z”: Horizontal, Vertical
- “O”: Single (2x2 square)

- Sticky
- Nonstikcy

$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40$ types of pieces

Tetrominoes



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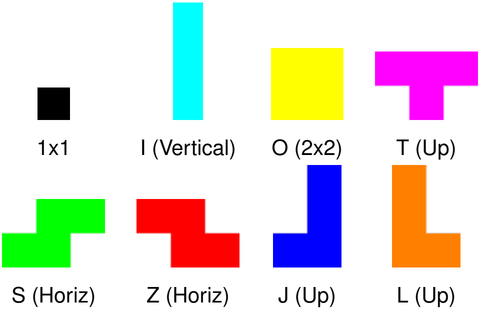

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$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

Configure files

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [20, 0]
Piece-01: [20, 0]
Piece-02: [20, 0]
Piece-03: [20, 0]
Piece-04: [20, 0]
Piece-05: [20, 0]
Piece-06: [20, 0]
Piece-07: [20, 0]
Piece-08: [20, 0]
Piece-09: [20, 0]
Piece-10: [20, 0]
Piece-11: [20, 0]
Piece-12: [20, 0]
Piece-13: [20, 0]
Piece-14: [20, 0]
Piece-15: [20, 0]
Piece-16: [20, 0]
Piece-17: [20, 0]
Piece-18: [20, 0]
Piece-19: [20, 0]
```

All nonsticky pieces
with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 20]
Piece-01: [0, 20]
Piece-02: [0, 20]
Piece-03: [0, 20]
Piece-04: [0, 20]
Piece-05: [0, 20]
Piece-06: [0, 20]
Piece-07: [0, 20]
Piece-08: [0, 20]
Piece-09: [0, 20]
Piece-10: [0, 20]
Piece-11: [0, 20]
Piece-12: [0, 20]
Piece-13: [0, 20]
Piece-14: [0, 20]
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Piece-06: [0, 0]
Piece-07: [0, 0]
Piece-08: [0, 0]
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Piece-11: [0, 0]
Piece-12: [0, 0]
Piece-13: [0, 0]
Piece-14: [0, 0]
Piece-15: [0, 0]
Piece-16: [0, 0]
Piece-17: [0, 0]
Piece-18: [0, 0]
Piece-19: [20, 80]
```

20% nonsticky
+ 80% sticky
of 1x1 piece

Main References:

Barabási, A.-L., & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press, Cambridge.

Family, F., & Vicsek, T. (1985).Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model. *Journal of Physics A: Mathematical and General*, *18*(2), L75.

Kardar, M., Parisi, G., & Zhang, Y.-C. (1986).Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, *56*(9), 889.

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Thank you!

Questions?