

# Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

Le Chen  
Auburn University

Acknwolegement

*NSF 2246850, NSF 2443823, & Simons Foundation Travel Grant (2022-2027)*

Talk available at: [github.com/chenle02](https://github.com/chenle02)

Emerging Synergies between Stochastic Analysis and Statistical Mechanics  
Banff, Alberta, Canada  
October 28, 2025

## Integration of Research, Education, and Outreach

### Outreach

- ▶ Auburn Summer Science Institute (AU-SSI): **2024, 2025**  
*Selected talented high school students*
- ▶ Destination STEM: **2023, 2024**  
*Junior middle school students*
- ▶ Graduate Student Seminars (Mathematics), Auburn: **2022–2025**

### Teaching

- ▶ **Math 7820/7830: Applied Stochastic Processes**  
Course project, **2023/24**

Most materials are available at

[github.com/chenle02](https://github.com/chenle02)

Math 7820/30: Applied Stochastic Processes (2023/24):



Mauricio Montes and Ian Ruau



Image is generated by OpenAI's *DALL-E*

Random deposition



Ballistic deposition

Image is generated by OpenAI's *DALL-E*

# Plan

Introduction to growth model and SPDE

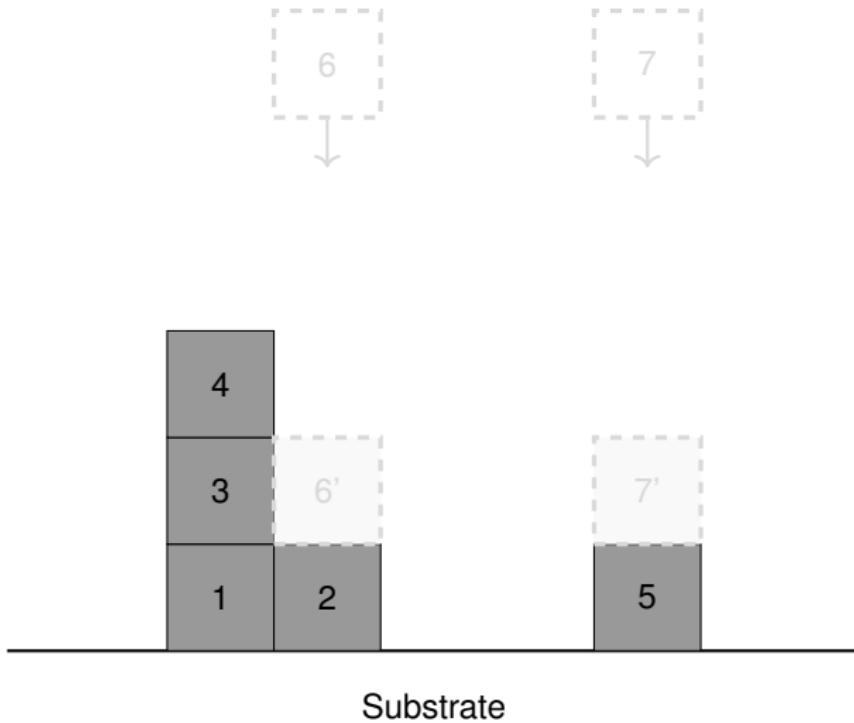
Tetromino Pieces

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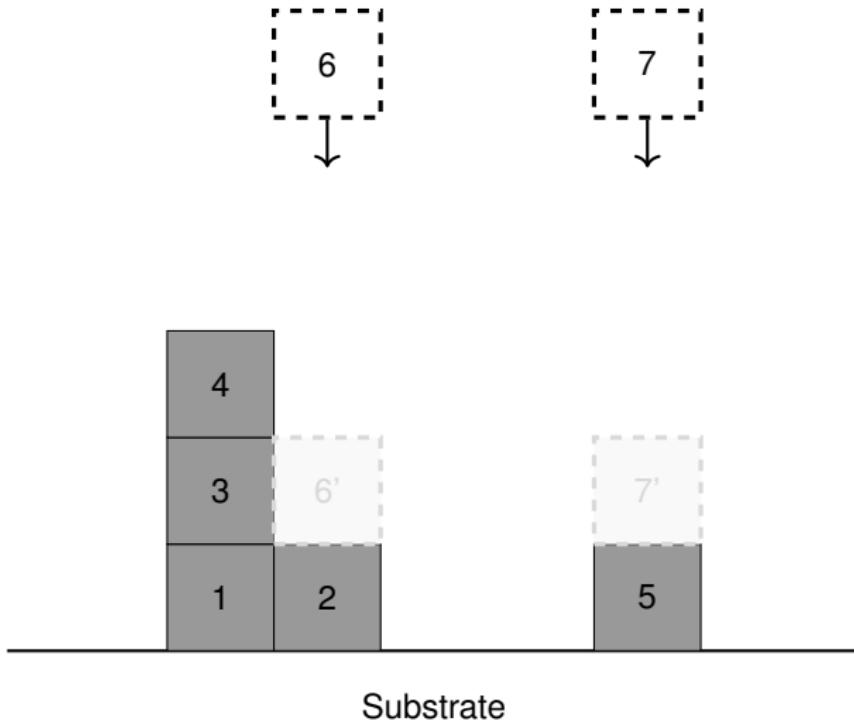
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Tetromino Pieces

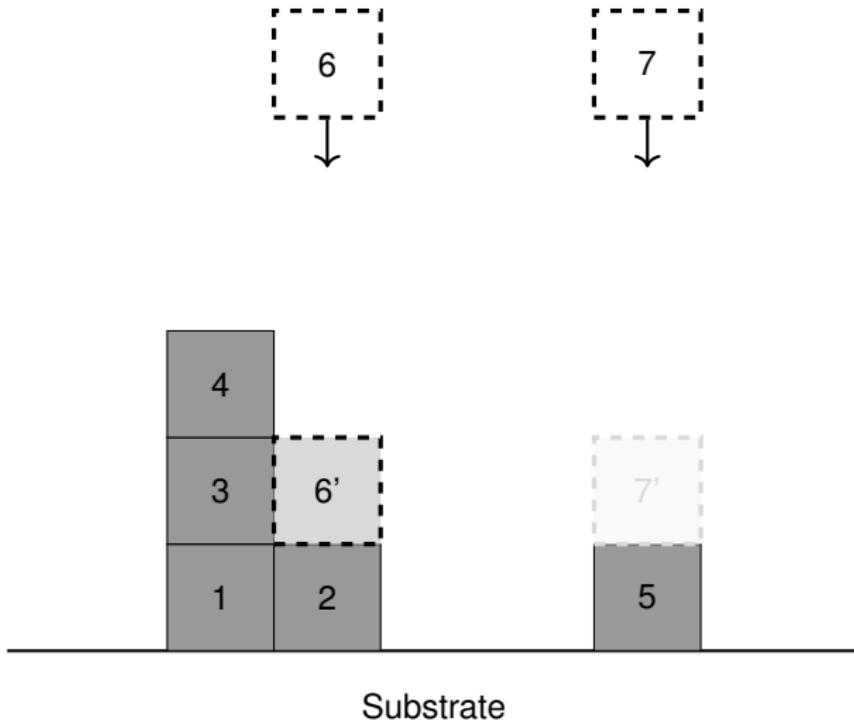
# Random deposition



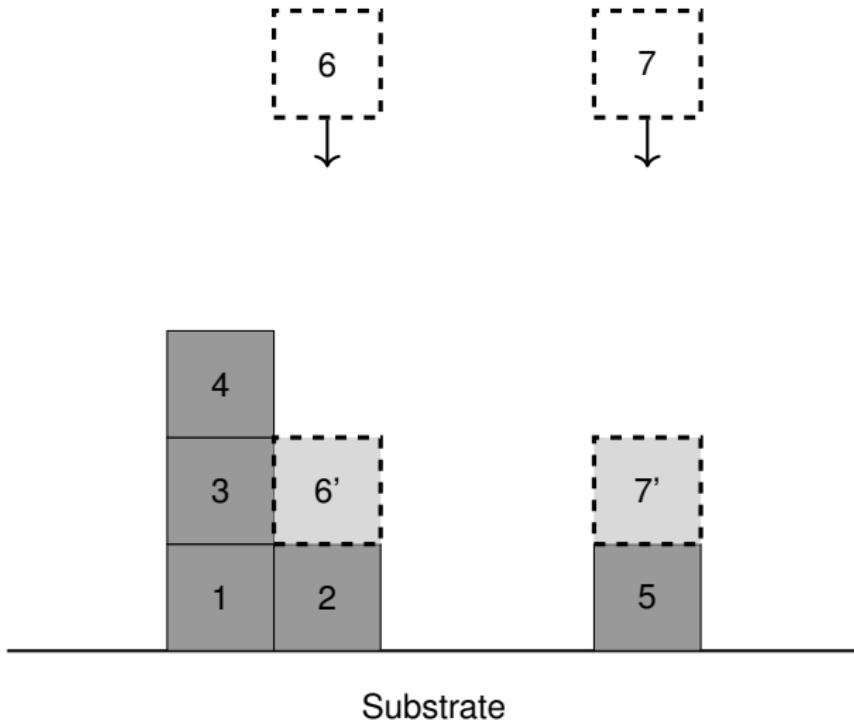
## Random deposition



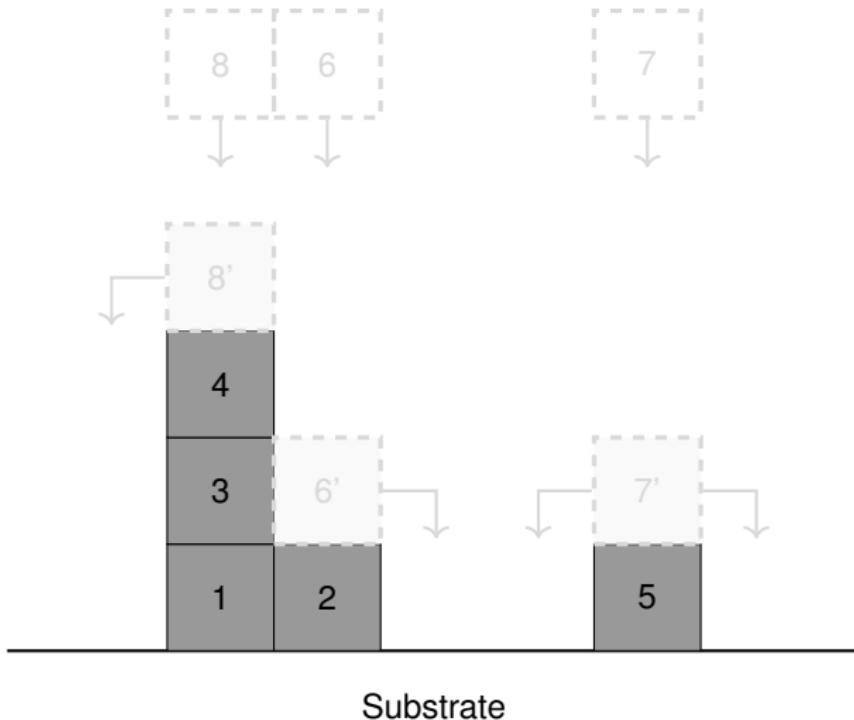
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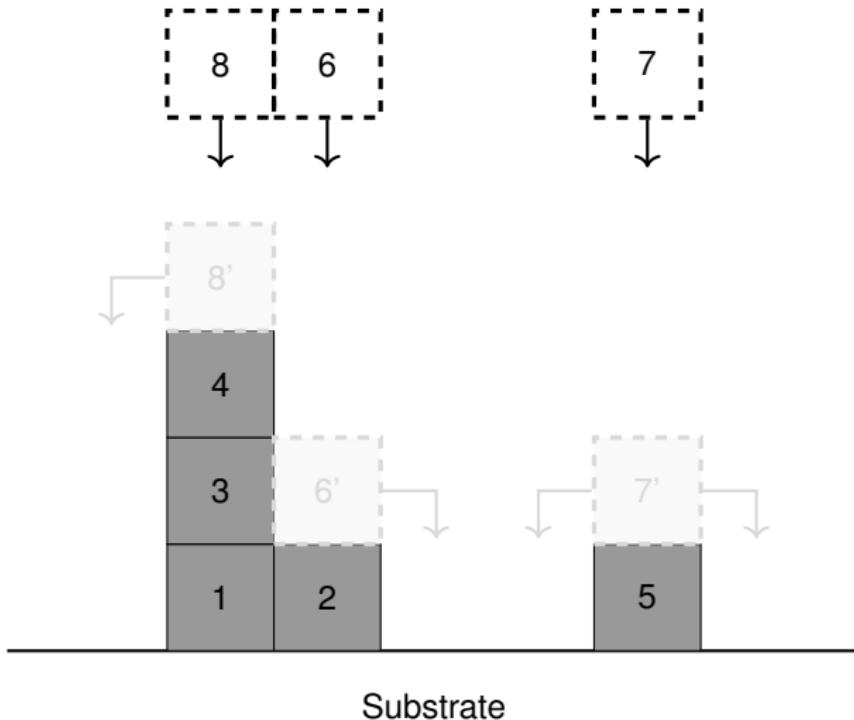
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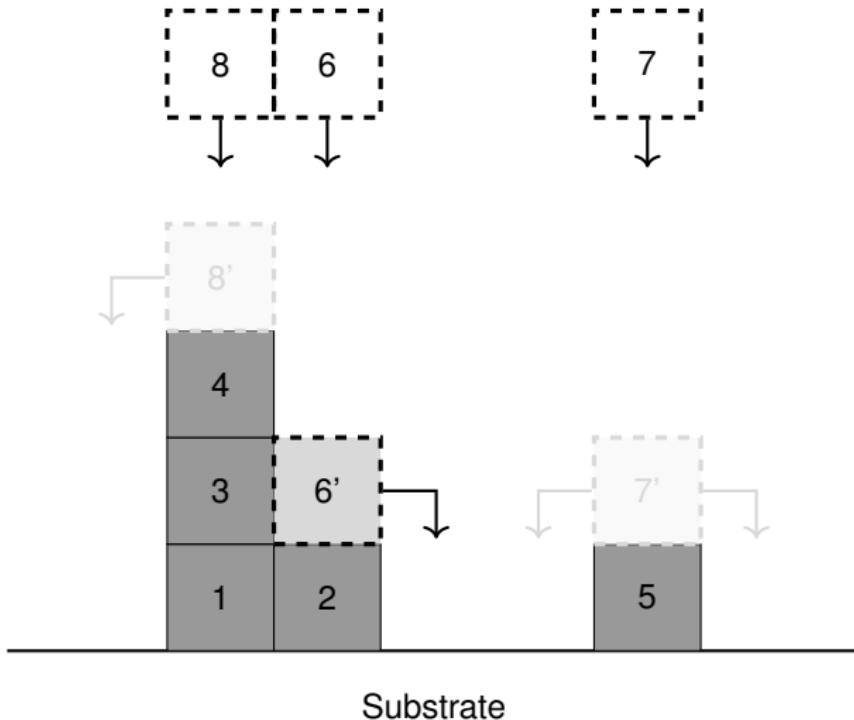
## Random deposition with surface relaxation



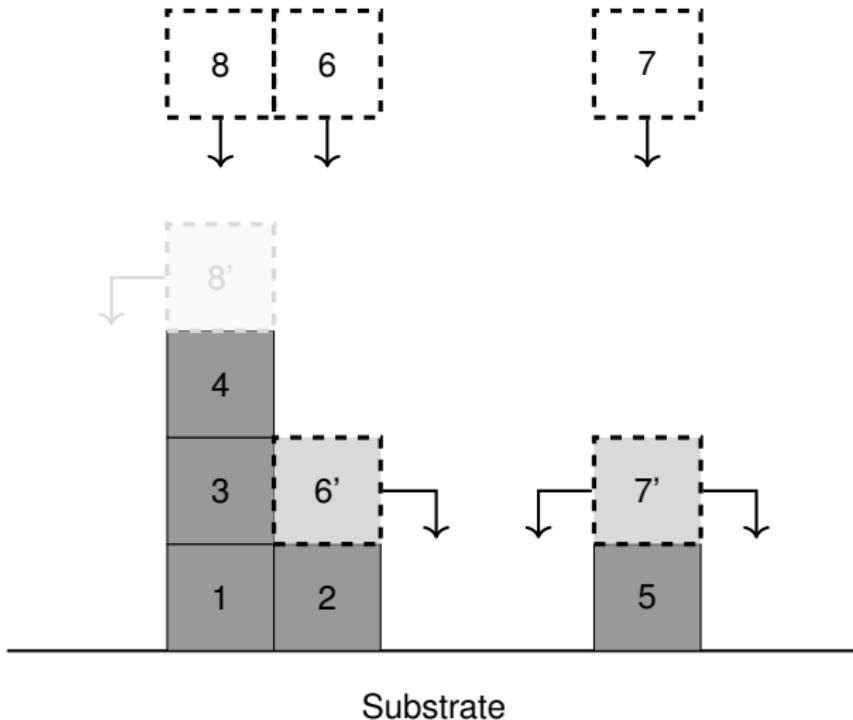
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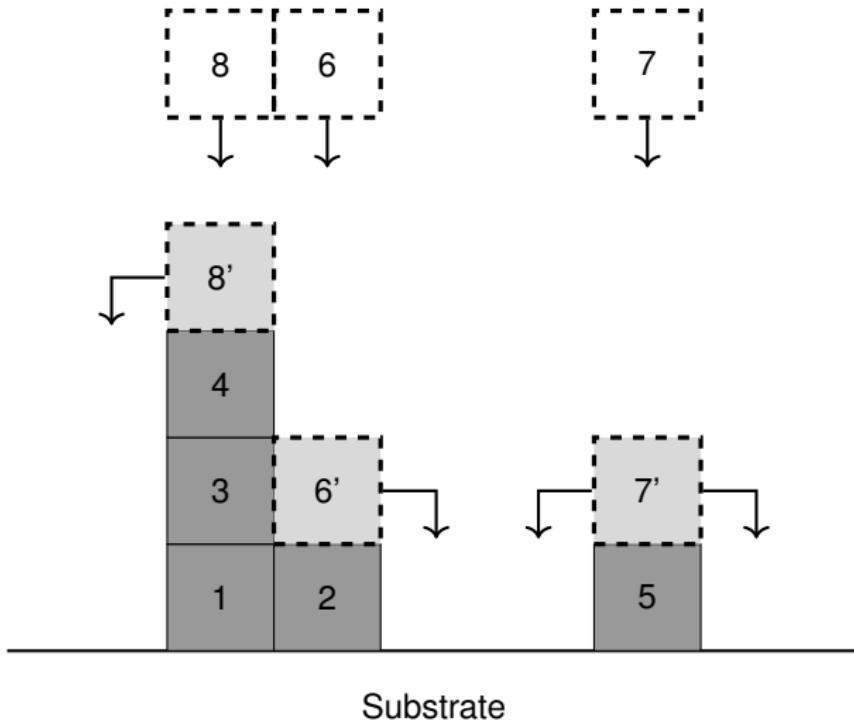
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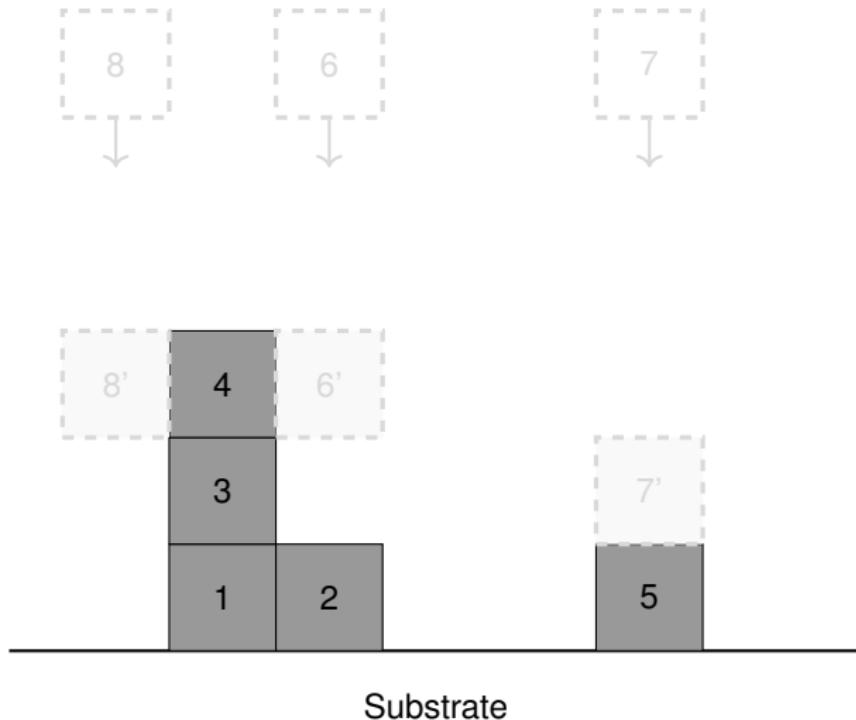
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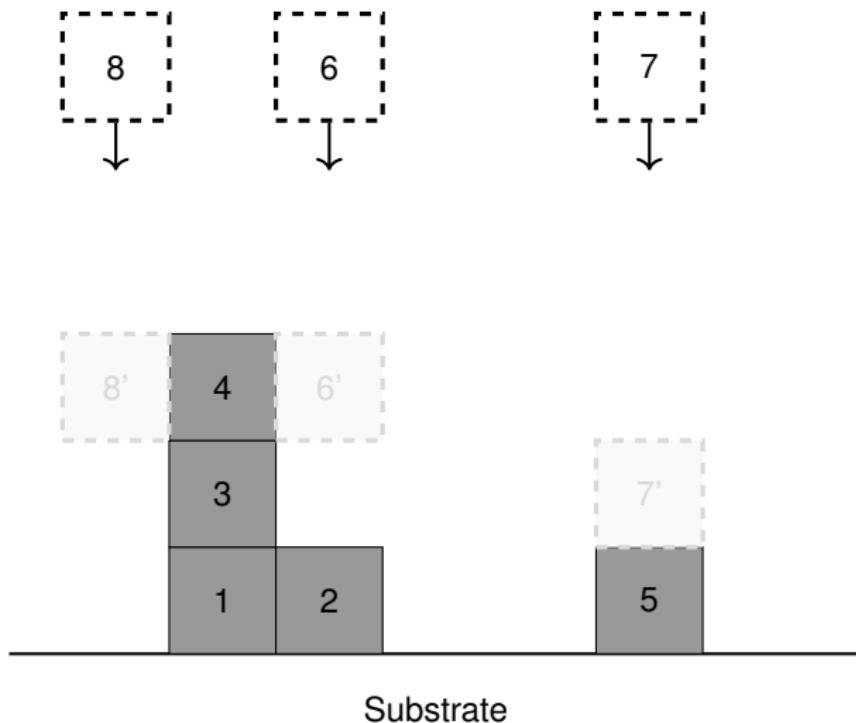
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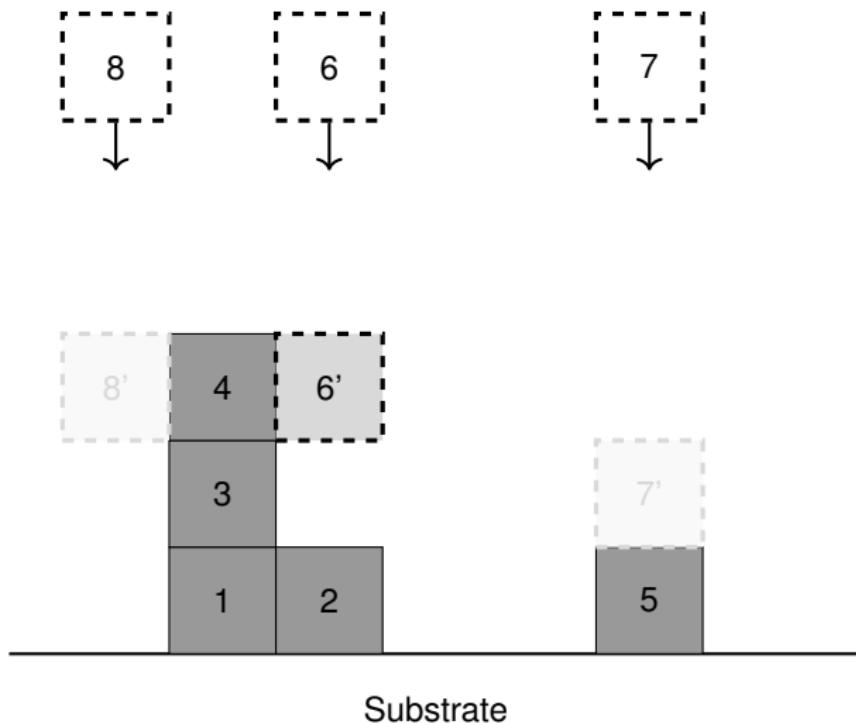
# Ballistic deposition



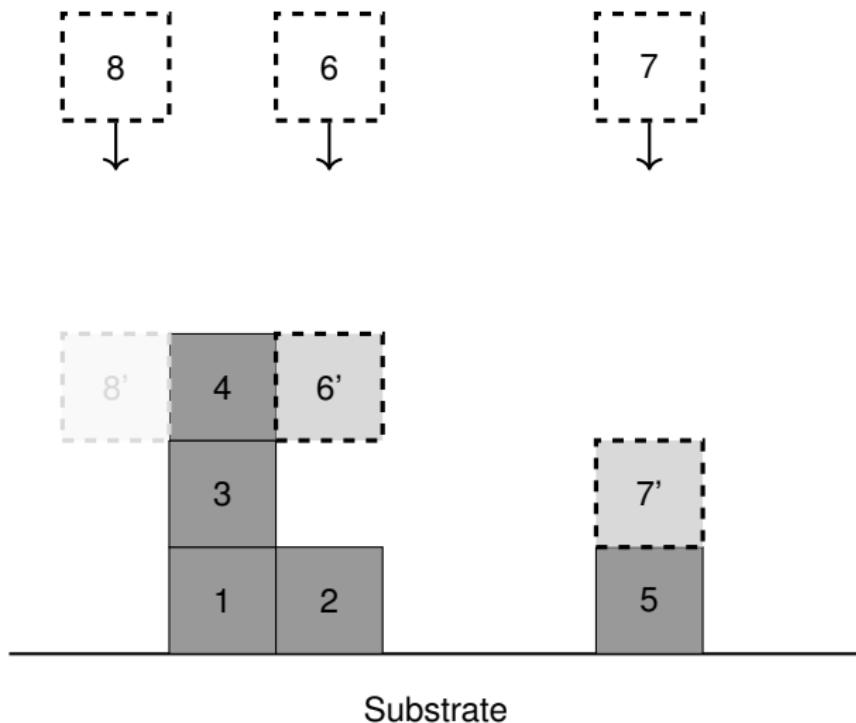
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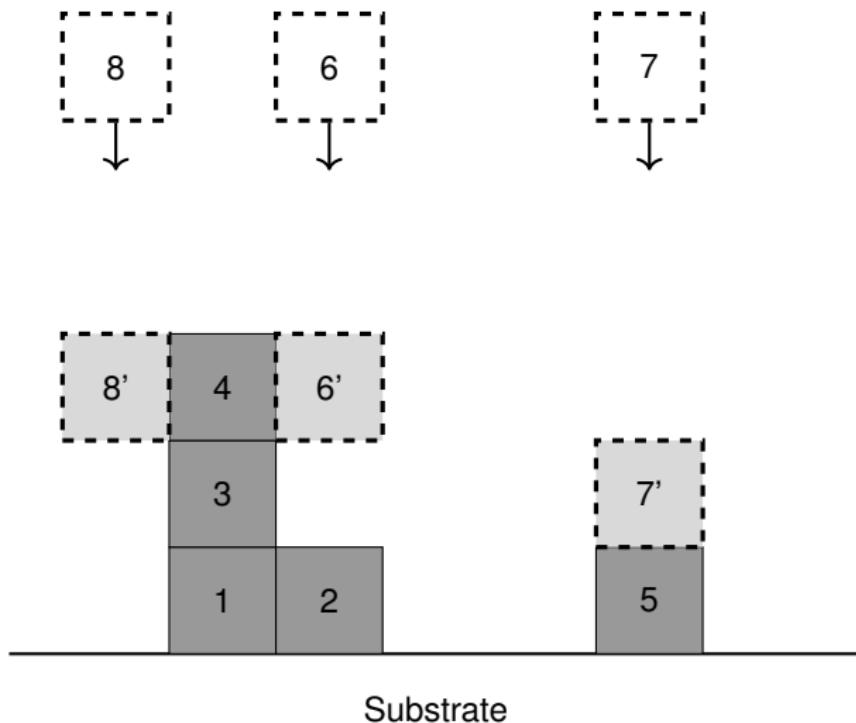
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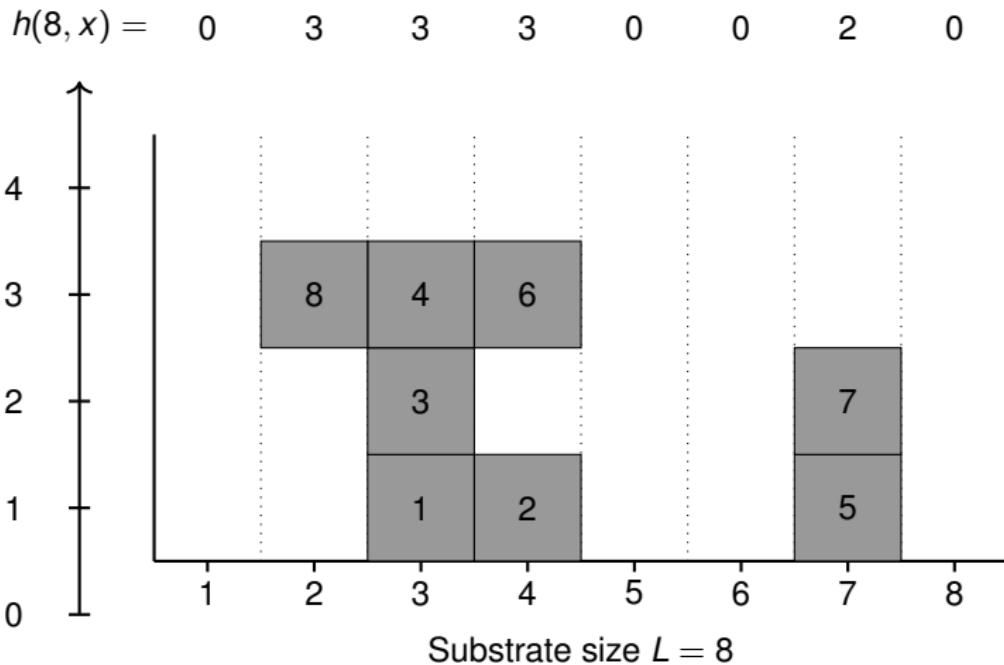
## Ballistic deposition



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## Average height and fluctuation

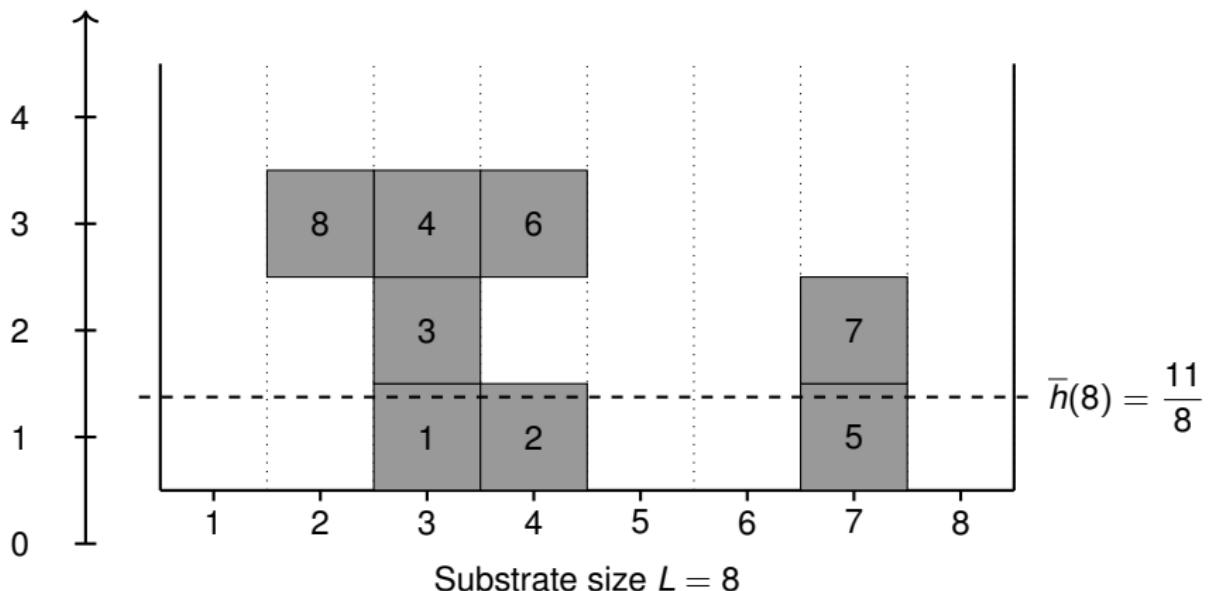


## Average height and fluctuation

$$\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x)$$

$$\text{Fluctuation } W(L, t) = \sqrt{\frac{1}{L} \sum_{x=1}^L [h(t, x) - \bar{h}(t)]^2}$$

$$h(8, x) = \begin{array}{cccccccc} 0 & 3 & 3 & 3 & 0 & 0 & 2 & 0 \end{array}$$

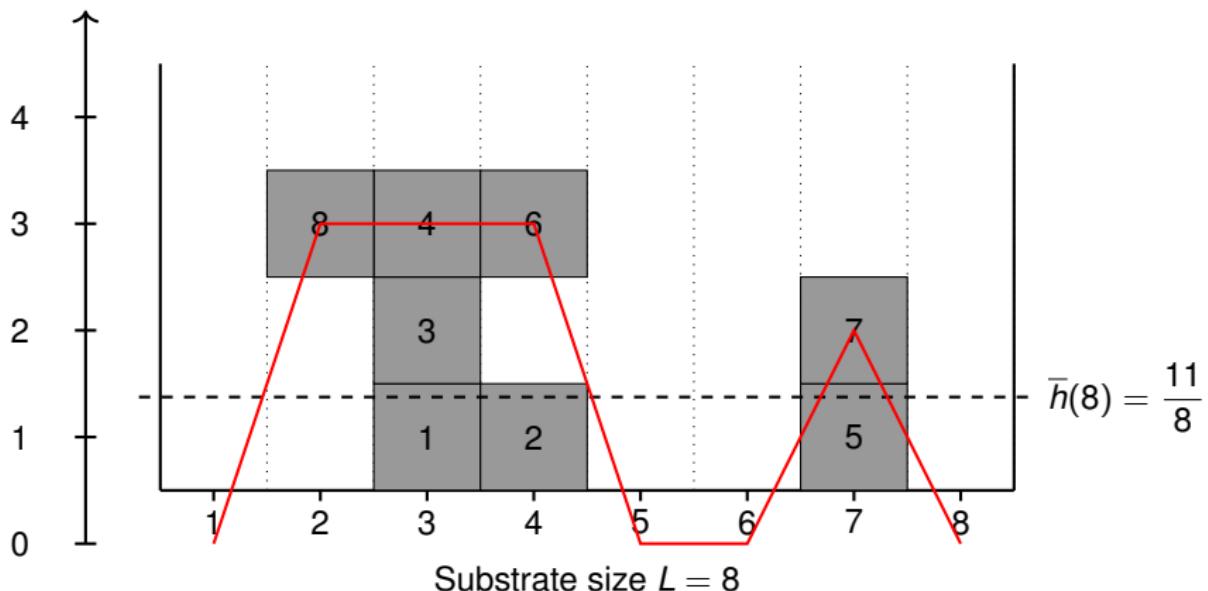


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## Random Deposition (independent columns, nonsticky)

**Model.**  $L$  independent columns. At each integer time  $t = 1, 2, \dots$ , drop *one* particle on a uniformly random column. Heights  $h(t, x)$ , mean  $\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x) = \frac{t}{L}$ , width

$$W^2(L, t) = \frac{1}{L} \sum_{x=1}^L (h(t, x) - \bar{h}(t))^2.$$

**Single-column law:** After  $t$  drops total,

$$h(t, x) \sim \text{Binomial}\left(t, \frac{1}{L}\right), \quad \mathbb{E}[h(t, x)] = \frac{t}{L}, \quad \text{Var}(h(t, x)) = t \frac{1}{L} \left(1 - \frac{1}{L}\right).$$

**Fluctuation:** By i.i.d. columns,

$$\mathbb{E}[W^2(L, t)] = \frac{1}{L} \sum_{x=1}^L \mathbb{E}[h(t, x)^2] - \mathbb{E}[\bar{h}^2(t)] = \mathbb{E}[h(t, 1)^2] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right) \text{Var}(h(t, 1)).$$

Hence

$$\boxed{\mathbb{E}[W^2(L, t)] = \left(1 - \frac{1}{L}\right) t \frac{1}{L} \left(1 - \frac{1}{L}\right) = \frac{t}{L} \left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L, t) \simeq \left(1 - \frac{1}{L}\right) \left(\frac{t}{L}\right)^{1/2}}$$

**Scaling.** Growth exponent  $\beta = \frac{1}{2}$ .

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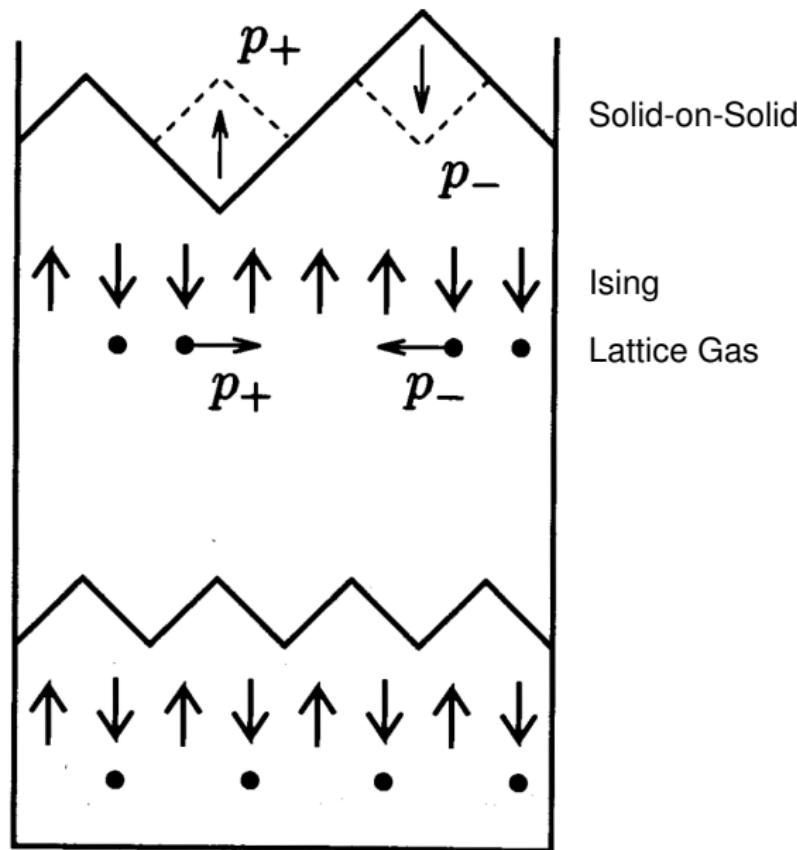
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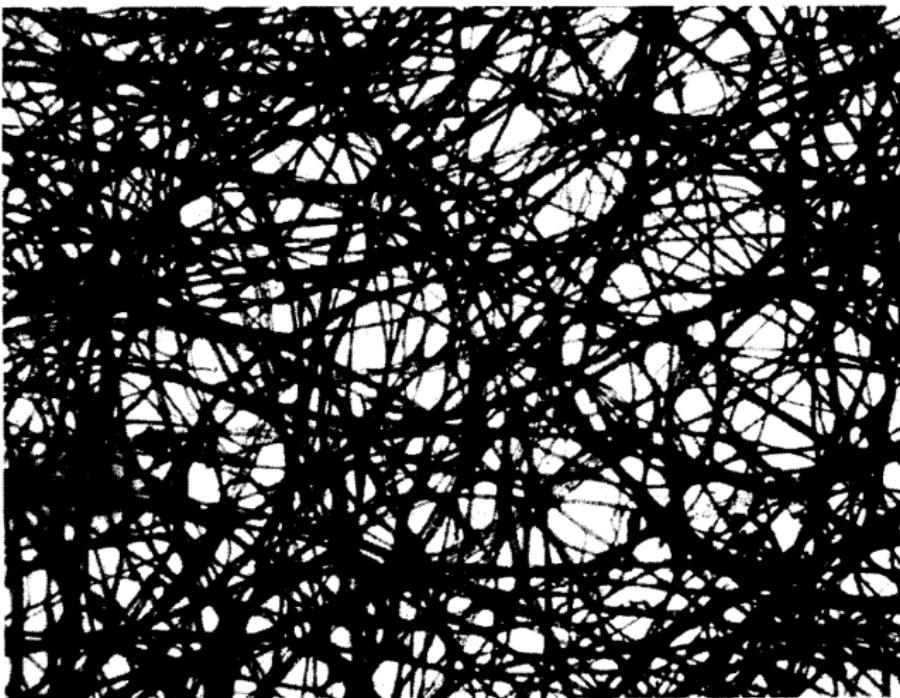
For ballistic deposition (sticky), lateral interactions appear, can we view it as a perturbation of random deposition (nonsticky)?

Simulations on  
Random deposition vs. Ballistic decomposition

## More models? Even more simpler?



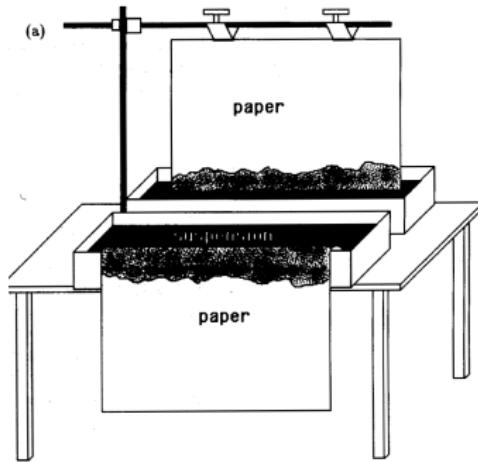
## Paper – a random environment



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Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

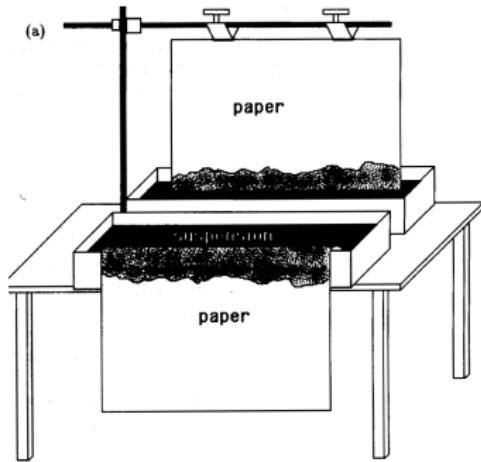
# Paper wetting experiment



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Barabási, A.-L., Stanley, H. E., 1995

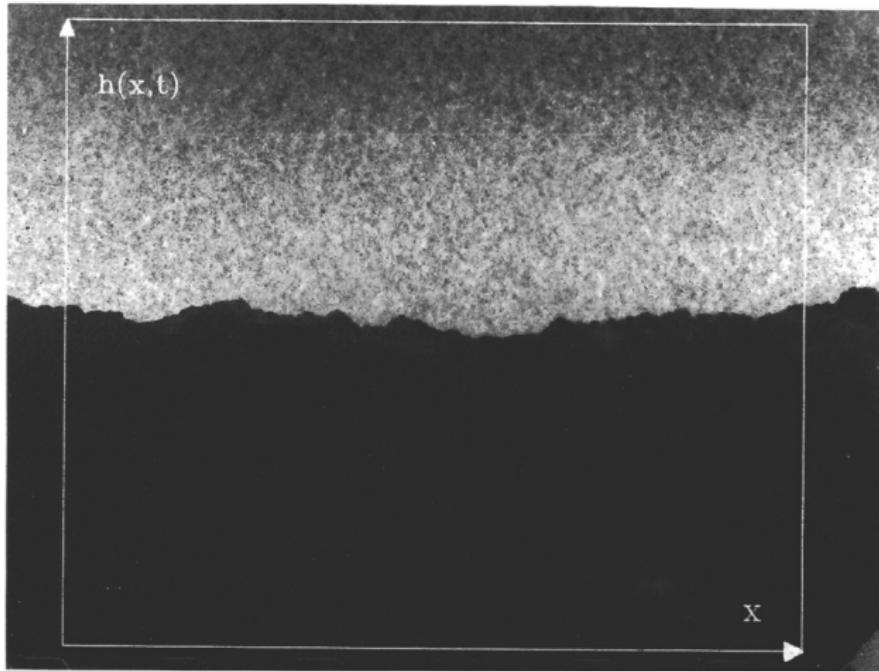
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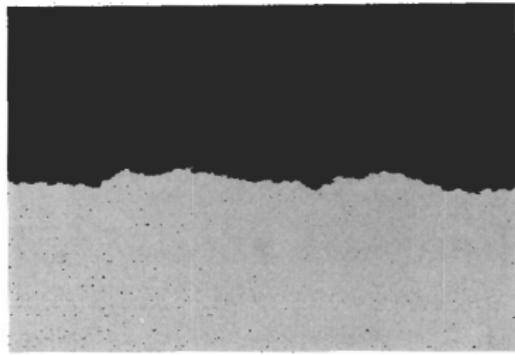
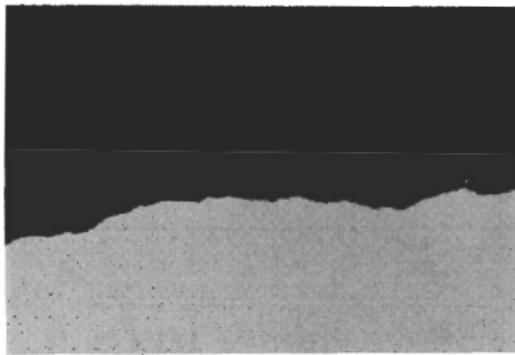
Barabási, A.-L., Stanley, H. E., 1995

# Paper burning experiment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

# Paper rupture experiment



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Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

# Study of growing interfaces in a thin film

— Convection of nematic liquid crystal\*

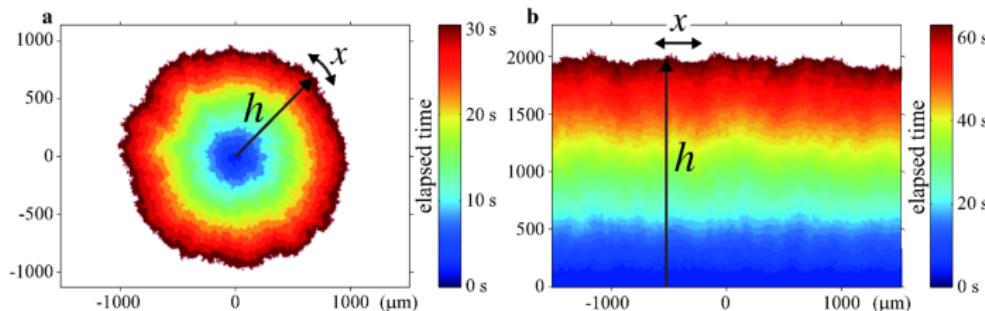
Show movies !

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Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

# Study of growing interfaces in a thin film

— Convection of nematic liquid crystal\*



Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

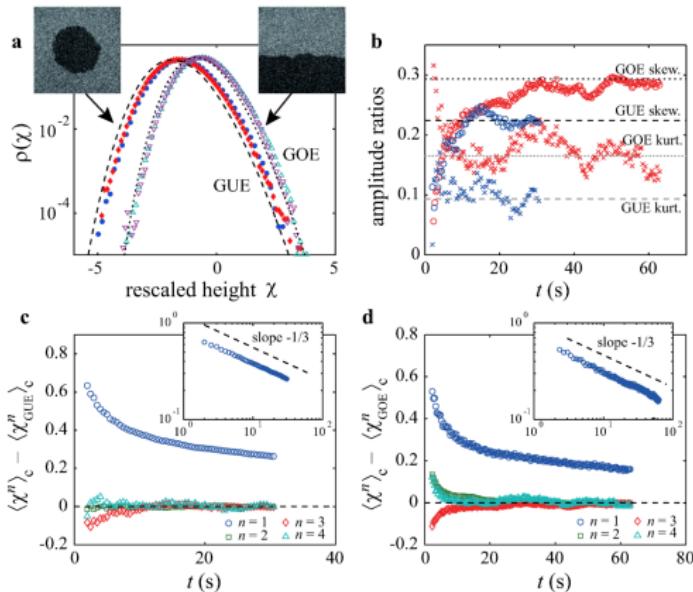
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# Study of growing interfaces in a thin film

## — Convection of nematic liquid crystal\*

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$



# KPZ Equation '86

$$\frac{\partial}{\partial t} h(t, x) = \frac{1}{2} \Delta h(t, x) + \frac{\lambda}{2} (\nabla h)^2 + \dot{W}(t, x) \quad (\text{KPZ})$$



Mehran Kardar (1957 –) Giorgio Parisi (1948 –)



Yicheng Zhang

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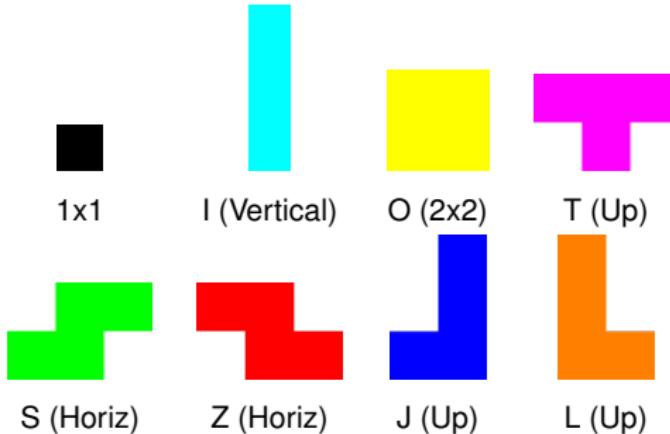
Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

# Plan

Introduction to growth model and SPDE

Tetromino Pieces

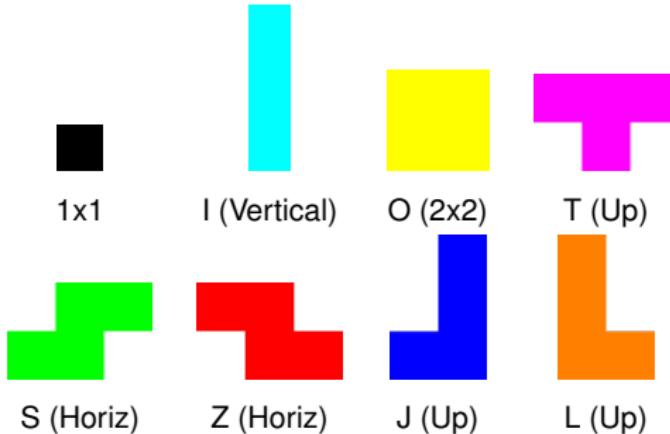
# Tetrominoes



- ▶ “1x1”: Single (extra single-site particle)
- ▶ “I”: Horizontal, Vertical
- ▶ “J, L, T”: Up, Right, Down, Left
- ▶ “S, Z”: Horizontal, Vertical
- ▶ “O”: Single (2x2 square)
- ▶ Sticky
- ▶ Nonsticky

$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

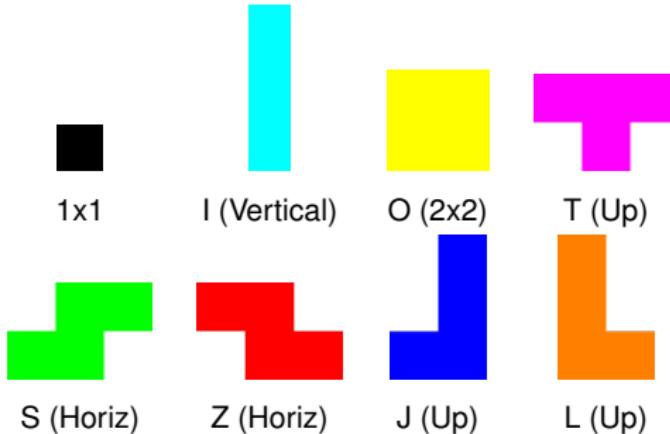
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# Configure files

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [20, 0]  
Piece-01: [20, 0]  
Piece-02: [20, 0]  
Piece-03: [20, 0]  
Piece-04: [20, 0]  
Piece-05: [20, 0]  
Piece-06: [20, 0]  
Piece-07: [20, 0]  
Piece-08: [20, 0]  
Piece-09: [20, 0]  
Piece-10: [20, 0]  
Piece-11: [20, 0]  
Piece-12: [20, 0]  
Piece-13: [20, 0]  
Piece-14: [20, 0]  
Piece-15: [20, 0]  
Piece-16: [20, 0]  
Piece-17: [20, 0]  
Piece-18: [20, 0]  
Piece-19: [20, 0]
```

All nonsticky pieces  
with equal prob.

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [0, 20]  
Piece-01: [0, 20]  
Piece-02: [0, 20]  
Piece-03: [0, 20]  
Piece-04: [0, 20]  
Piece-05: [0, 20]  
Piece-06: [0, 20]  
Piece-07: [0, 20]  
Piece-08: [0, 20]  
Piece-09: [0, 20]  
Piece-10: [0, 20]  
Piece-11: [0, 20]  
Piece-12: [0, 20]  
Piece-13: [0, 20]  
Piece-14: [0, 20]  
Piece-15: [0, 20]  
Piece-16: [0, 20]  
Piece-17: [0, 20]  
Piece-18: [0, 20]  
Piece-19: [0, 20]
```

All sticky pieces  
with equal prob.

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [0, 0]  
Piece-01: [0, 0]  
Piece-02: [0, 0]  
Piece-03: [0, 0]  
Piece-04: [0, 0]  
Piece-05: [0, 0]  
Piece-06: [0, 0]  
Piece-07: [0, 0]  
Piece-08: [0, 0]  
Piece-09: [0, 0]  
Piece-10: [0, 0]  
Piece-11: [0, 0]  
Piece-12: [0, 0]  
Piece-13: [0, 0]  
Piece-14: [0, 0]  
Piece-15: [0, 0]  
Piece-16: [0, 0]  
Piece-17: [0, 0]  
Piece-18: [0, 0]  
Piece-19: [20, 80]
```

20% nonsticky  
+ 80% sticky  
of 1x1 piece

# Outreach Highlights



AU-SSI 2023



AU-SSI 2024



Destination STEM 2023

## Main References:

- Barabási, A.-L., & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press, Cambridge.
- Family, F., & Vicsek, T. (1985). Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model. *Journal of Physics A: Mathematical and General*, 18(2), L75.
- Kardar, M., Parisi, G., & Zhang, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56(9), 889.
- Kertész, J., Horváth, V. k., & Weber, F. (1993). Self-affine rupture lines in paper sheets. *Fractals*, 01(01), 67–74.
- Takeuchi, K. A., Sano, M., Sasamoto, T., & Spohn, H. (2011). Growing interfaces uncover universal fluctuations behind scale invariance. *Sci. Rep.*, 1(1), 1–5.
- Zhang, J., Zhang, Y.-C., Alstrøm, P., & Levinsen, M. (1992). Modeling forest fire by a paper-burning experiment, a realization of the interface growth mechanism. *Phys. A: Stat. Mech. Appl.*, 189(3), 383–389.

Thank you!

Questions?