

Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

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Auburn University

Acknwolegement

NSF 2246850, NSF 2443823, & Simons Foundation Travel Grant (2022-2027)

Talk available at: github.com/chenle02

Emerging Synergies between Stochastic Analysis and Statistical Mechanics
Banff, Alberta, Canada
October 28, 2025

Integration of Research, Education, and Outreach

Outreach

- ▶ Auburn Summer Science Institute (AU-SSI): **2024, 2025**
Selected talented high school students
- ▶ Destination STEM: **2023, 2024**
Junior middle school students

Education

- ▶ Graduate Student Seminars (Mathematics), Auburn: **2022–2025**
- ▶ Math 7820/7830: Applied Stochastic Processes Course project, **2023/24**

Research

- ▶ Simulation and modeling packages (open source)
- ▶ Forming conjectures and validating results

Most materials are available at

github.com/chenle02

Math 7820/30: Applied Stochastic Processes (2023/24):



Mauricio Montes and Ian Ruau

Simulation package:

https://github.com/chenle02/Simulations_on_Some_Surface_Growth_Models

```
pip install tetris-ballistic
```



Image is generated by OpenAI's *DALL-E*

Random deposition



Ballistic deposition

Image is generated by OpenAI's *DALL-E*

Plan

Introduction to growth model and SPDE

Family-Vicsek scaling and experiments

More examples

Tetromino Pieces

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Introduction to growth model and SPDE

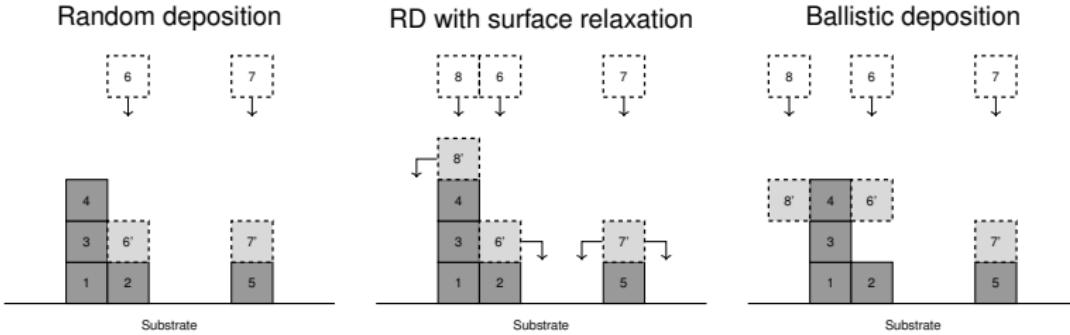
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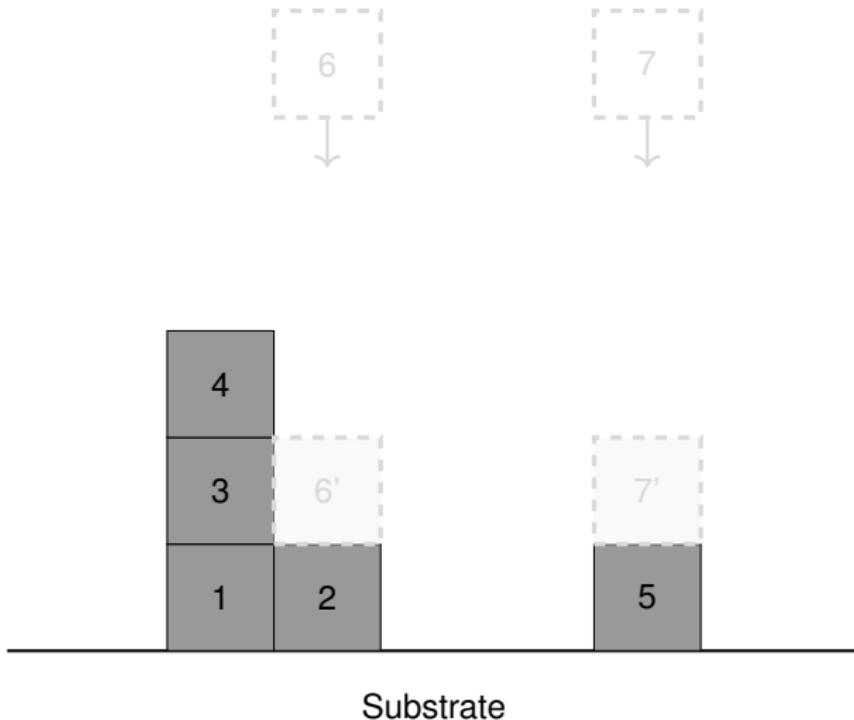
Tetromino Pieces

How do surfaces grow?

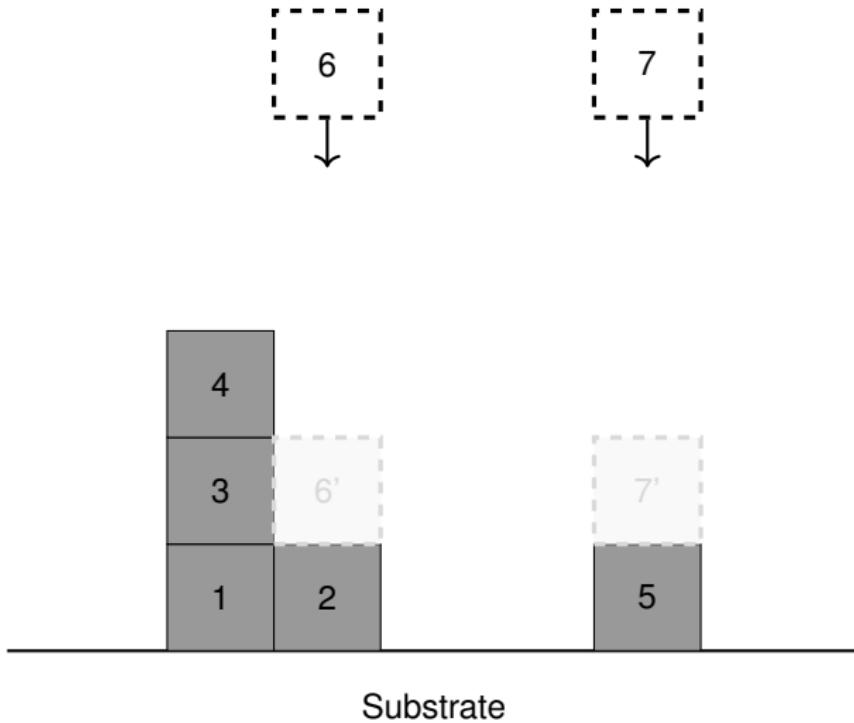
Three models



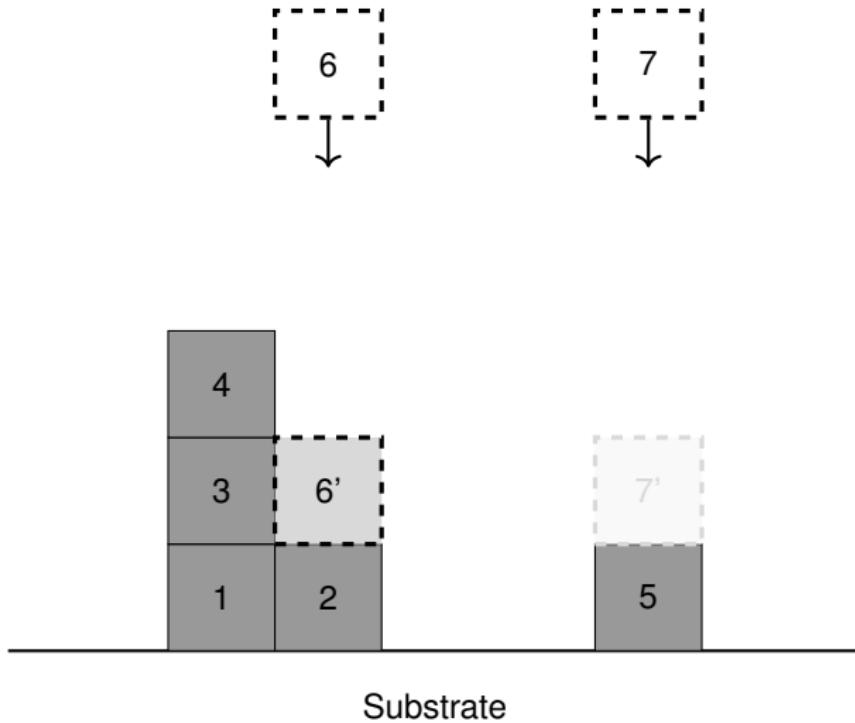
Random deposition



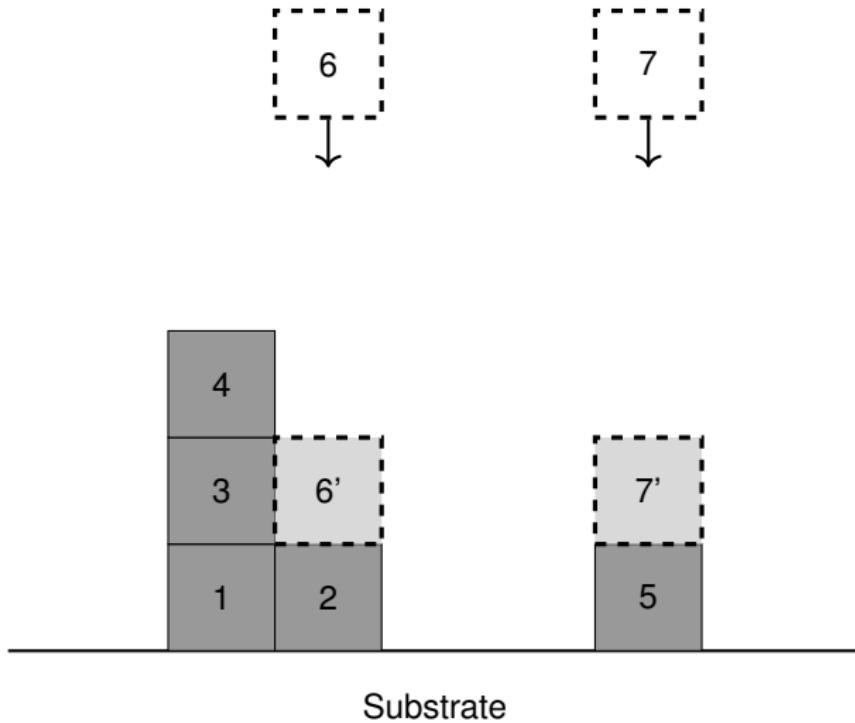
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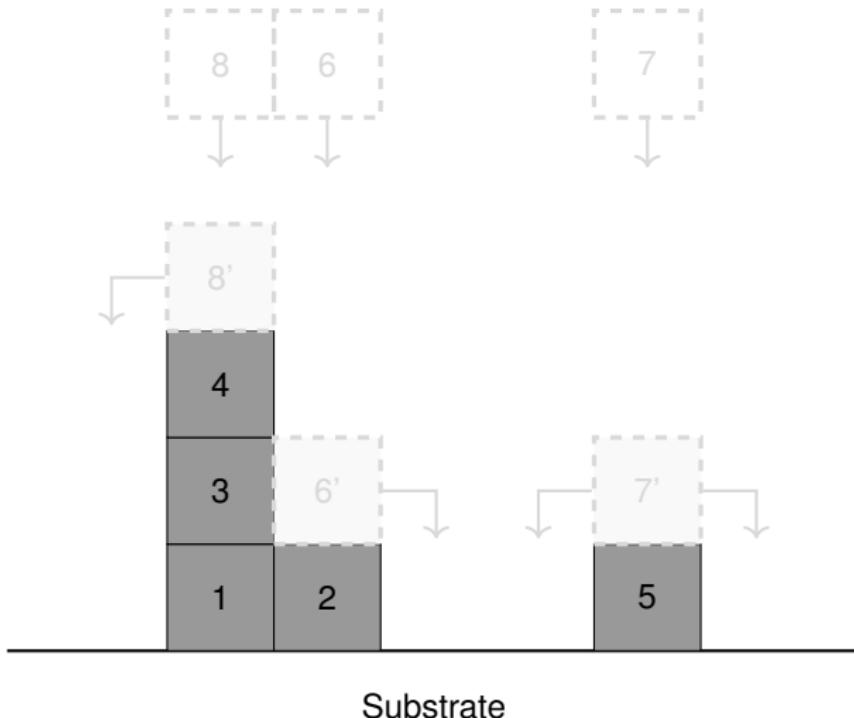
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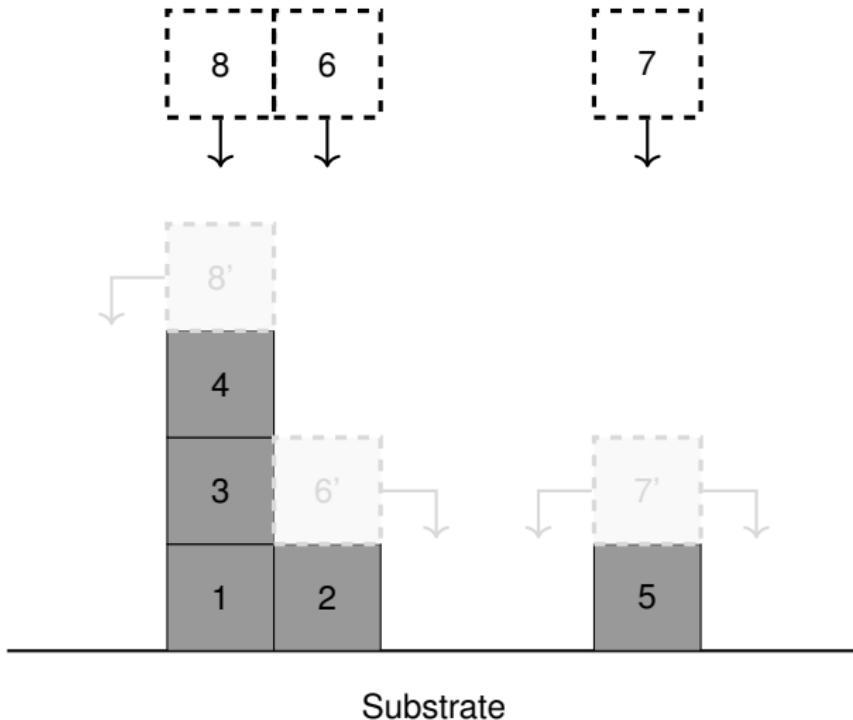
Random deposition



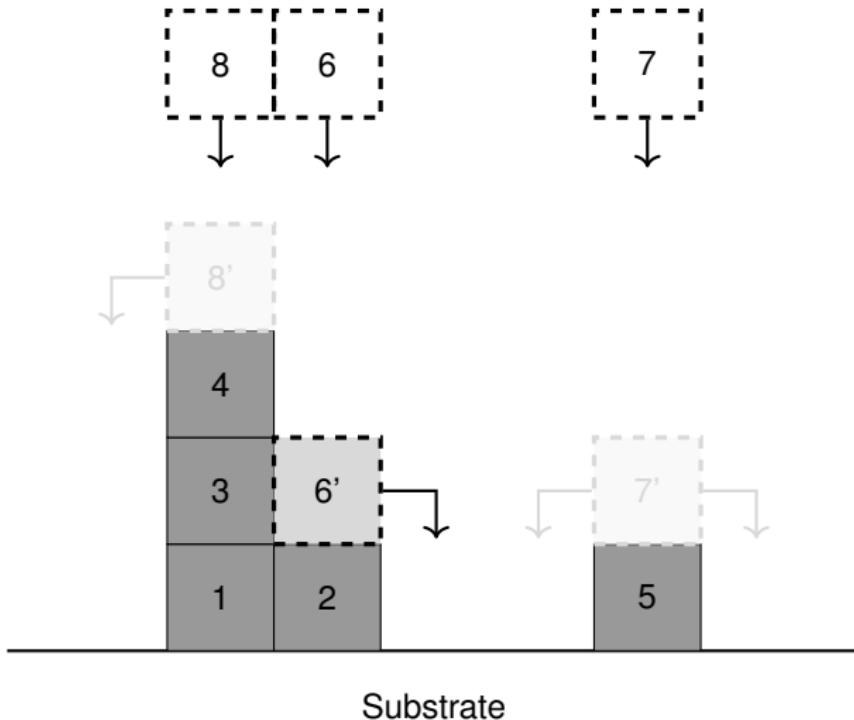
Random deposition with surface relaxation



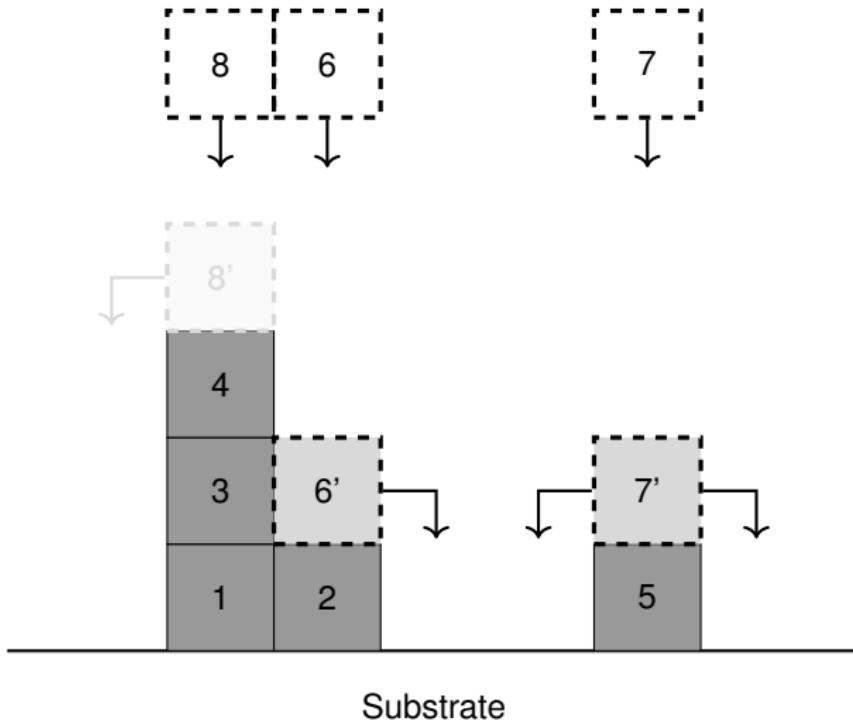
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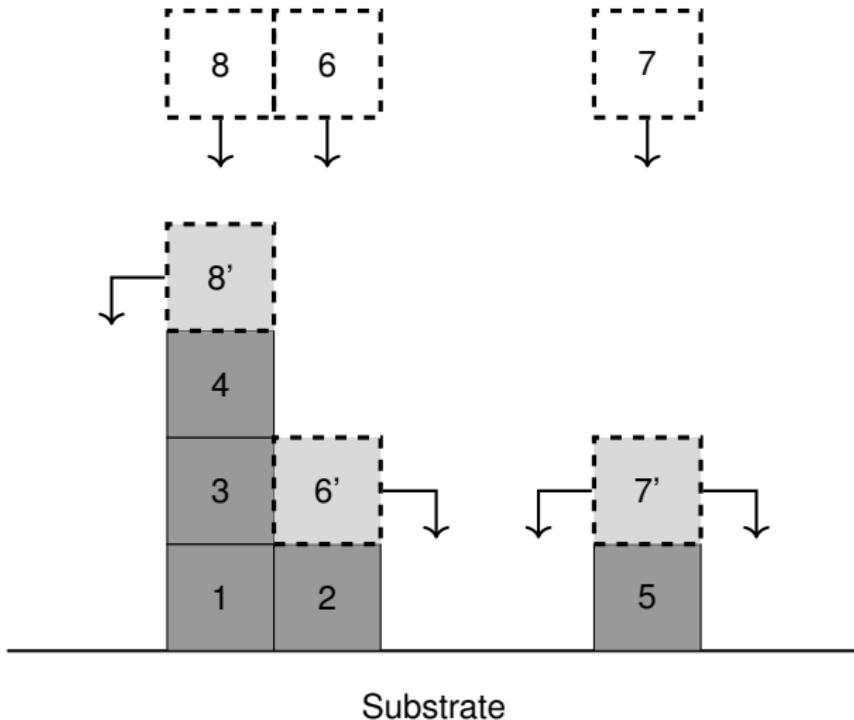
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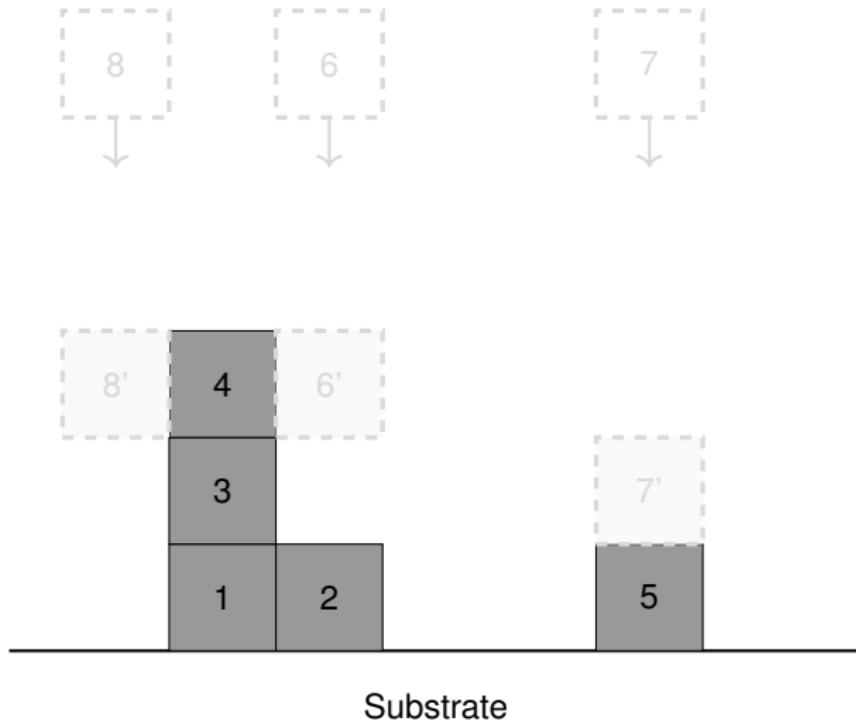
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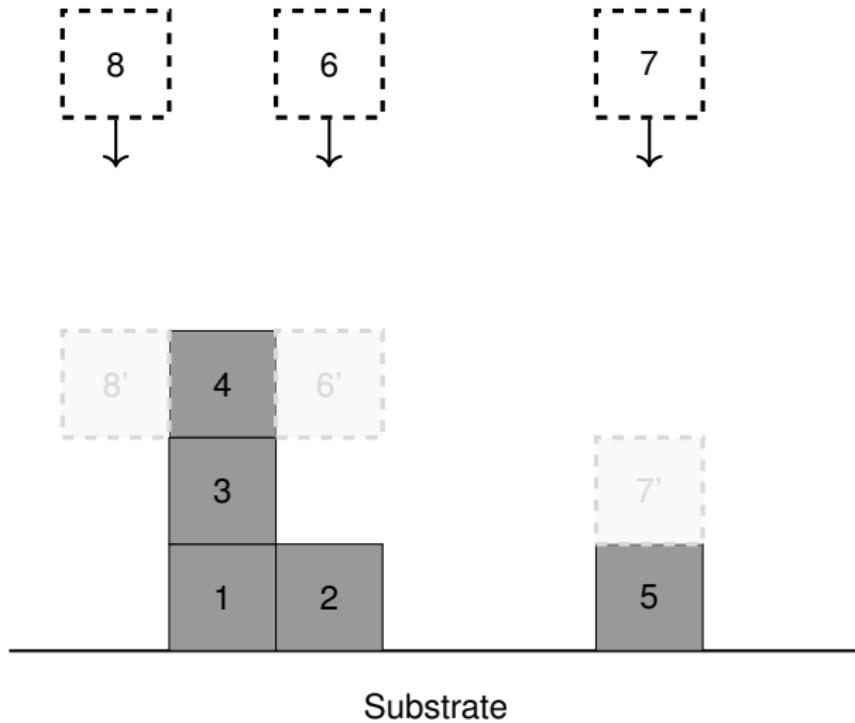
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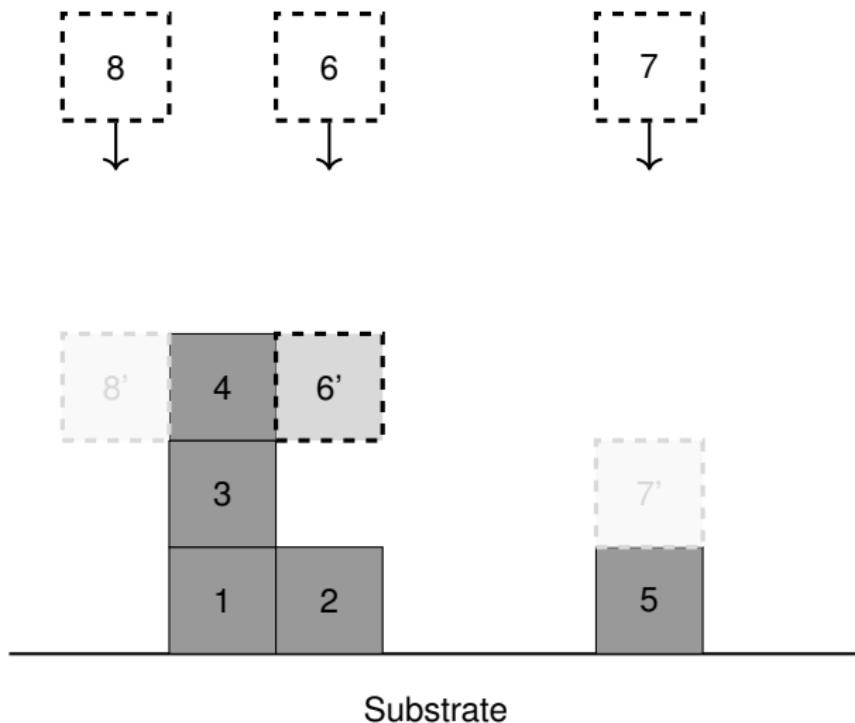
Ballistic deposition



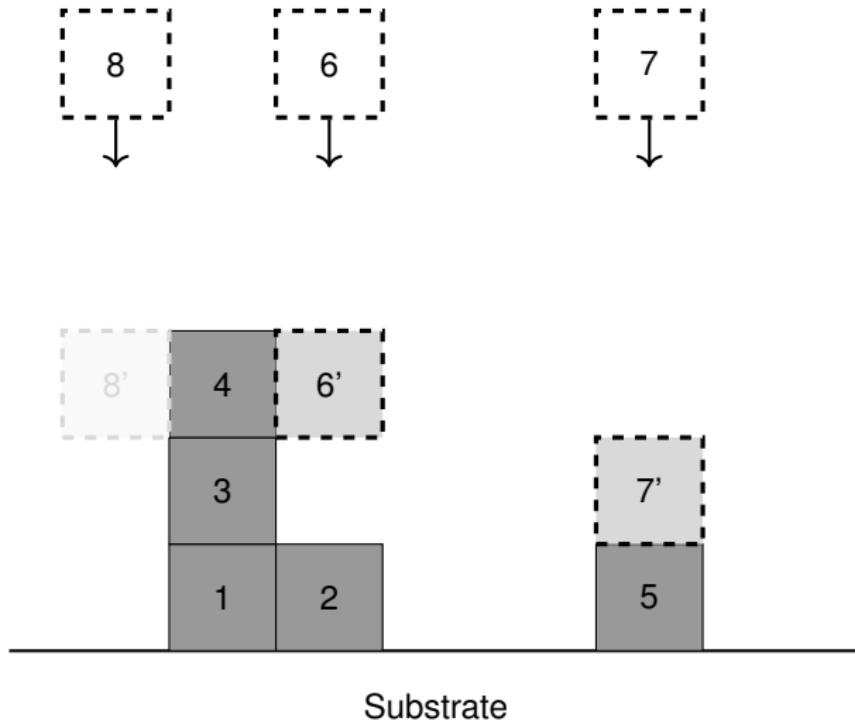
Ballistic deposition



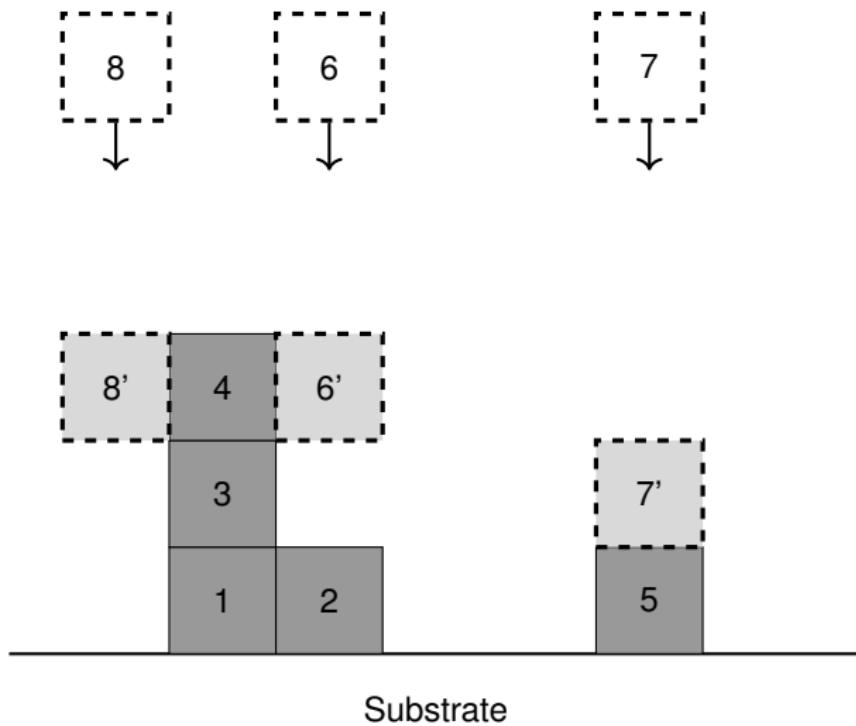
Ballistic deposition



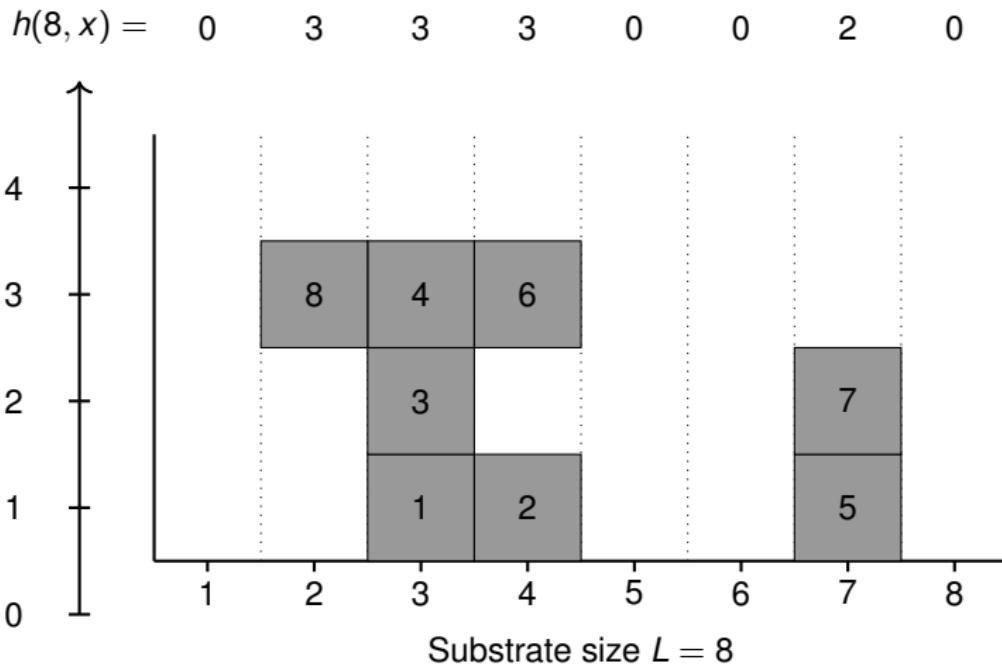
Ballistic deposition



Ballistic deposition



Average height and fluctuation

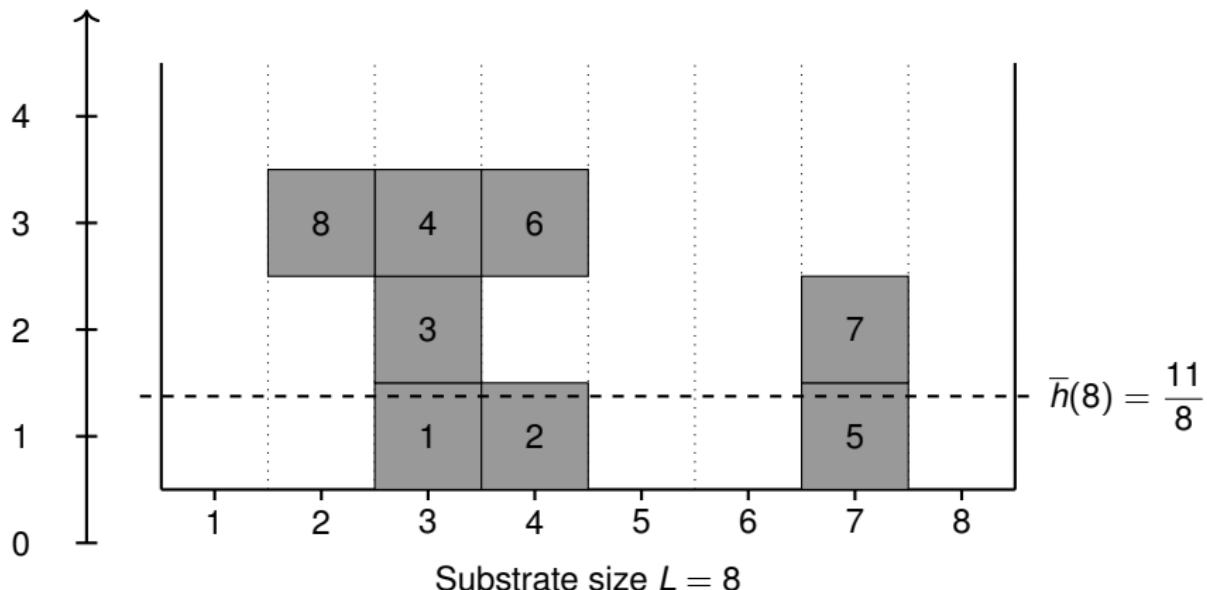


Average height and fluctuation

$$\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x)$$

$$h(8, x) = \begin{array}{ccccc} 0 & 3 & 3 & 3 & 0 \end{array}$$

$$\text{Fluctuation } W(L, t) = \sqrt{\frac{1}{L} \sum_{x=1}^L [h(t, x) - \bar{h}(t)]^2}$$

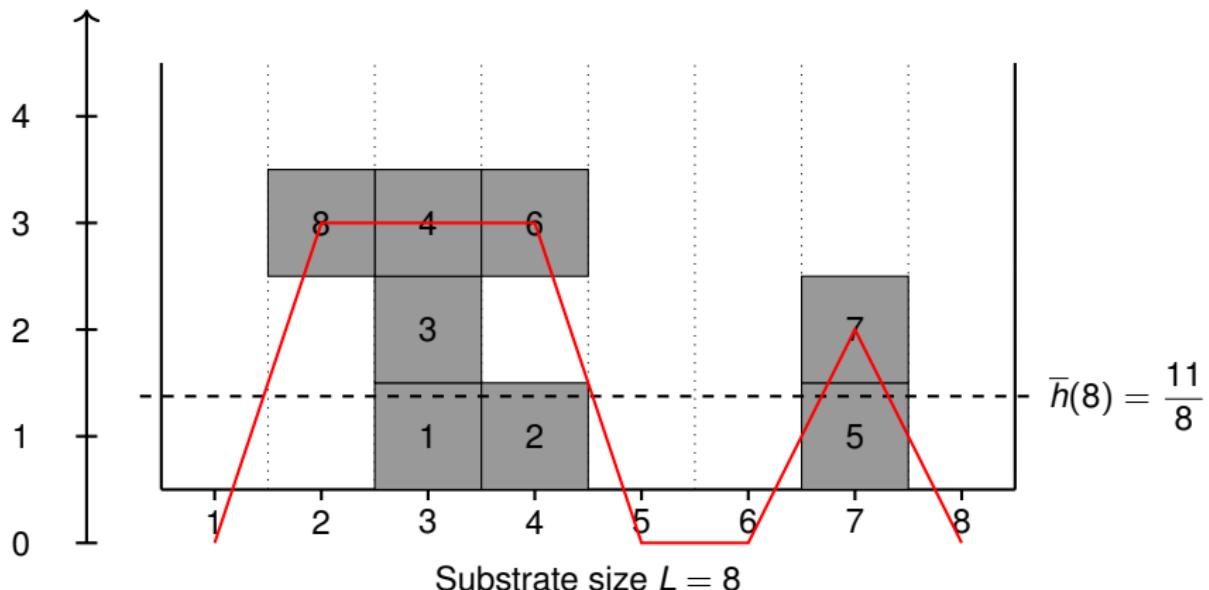


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Random Deposition (independent columns, nonsticky)

Model. L independent columns. At each integer time $t = 1, 2, \dots$, drop *one* particle on a uniformly random column. Heights $h(t, x)$, mean $\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(t, x) = \frac{t}{L}$, width

$$W^2(L, t) = \frac{1}{L} \sum_{x=1}^L (h(t, x) - \bar{h}(t))^2.$$

Single-column law: After t drops total,

$$h(t, x) \sim \text{Binomial}\left(t, \frac{1}{L}\right), \quad \mathbb{E}[h(t, x)] = \frac{t}{L}, \quad \text{Var}(h(t, x)) = t \frac{1}{L} \left(1 - \frac{1}{L}\right).$$

Fluctuation: By i.i.d. columns,

$$\mathbb{E}[W^2(L, t)] = \frac{1}{L} \sum_{x=1}^L \mathbb{E}[h(t, x)^2] - \mathbb{E}[\bar{h}^2(t)] = \mathbb{E}[h(t, 1)^2] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right) \text{Var}(h(t, 1)).$$

Hence

$$\boxed{\mathbb{E}[W^2(L, t)] = \left(1 - \frac{1}{L}\right) t \frac{1}{L} \left(1 - \frac{1}{L}\right) = \frac{t}{L} \left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L, t) \simeq \left(1 - \frac{1}{L}\right) \left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \frac{1}{2}$.

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Hence

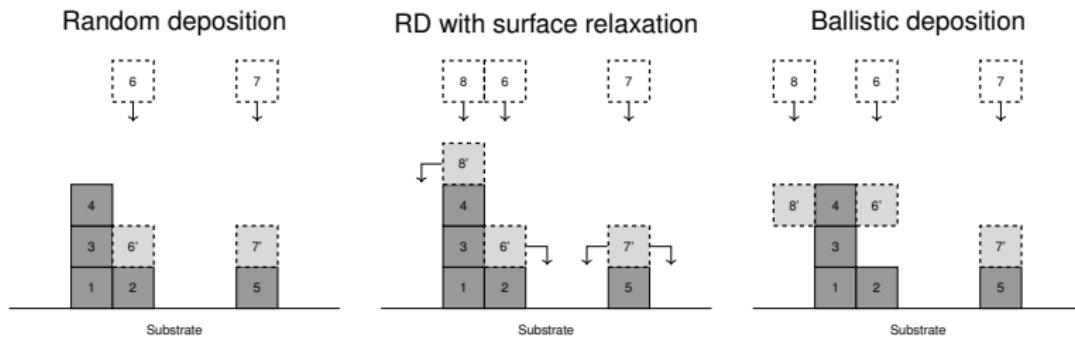
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Questions

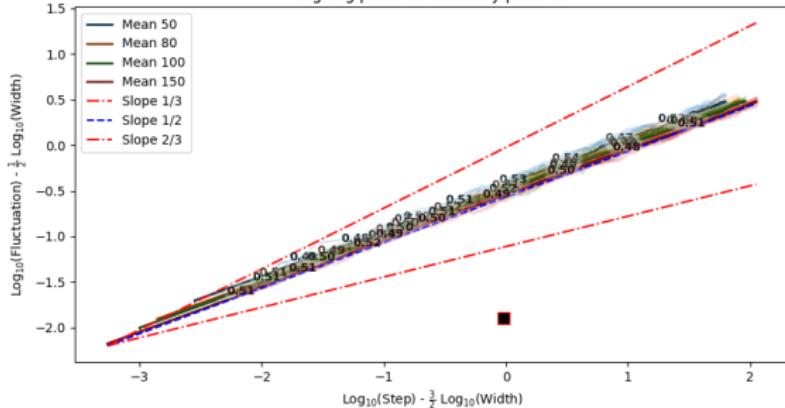


$$W(t, x) \sim t^{1/2}$$

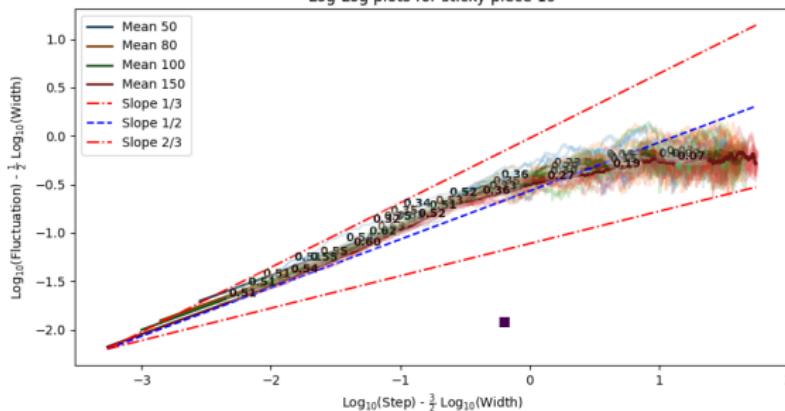
$$W(t, x) \sim t^{??}$$

$$W(t, x) \sim t^{??}$$

Log-Log plots for nonsticky piece 19



Log-Log plots for sticky piece 19



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Family–Vicsek scaling theory

Recall height function $h(x, t)$, $x = 1, \dots, L$. Mean $\bar{h}(t) = \frac{1}{L} \sum_{x=1}^L h(x, t)$.

$$w(L, t) := \sqrt{\frac{1}{L} \sum_{x=1}^L (h(x, t) - \bar{h}(t))^2} \quad (\text{Fluctuation}).$$

Empirical scaling (log–log)

Early time $w(L, t) \sim t^\beta$

β : growth exponent

Late time $w(L, t) \rightarrow w_{\text{sat}}(L) \sim L^\alpha$

α : roughness exponent

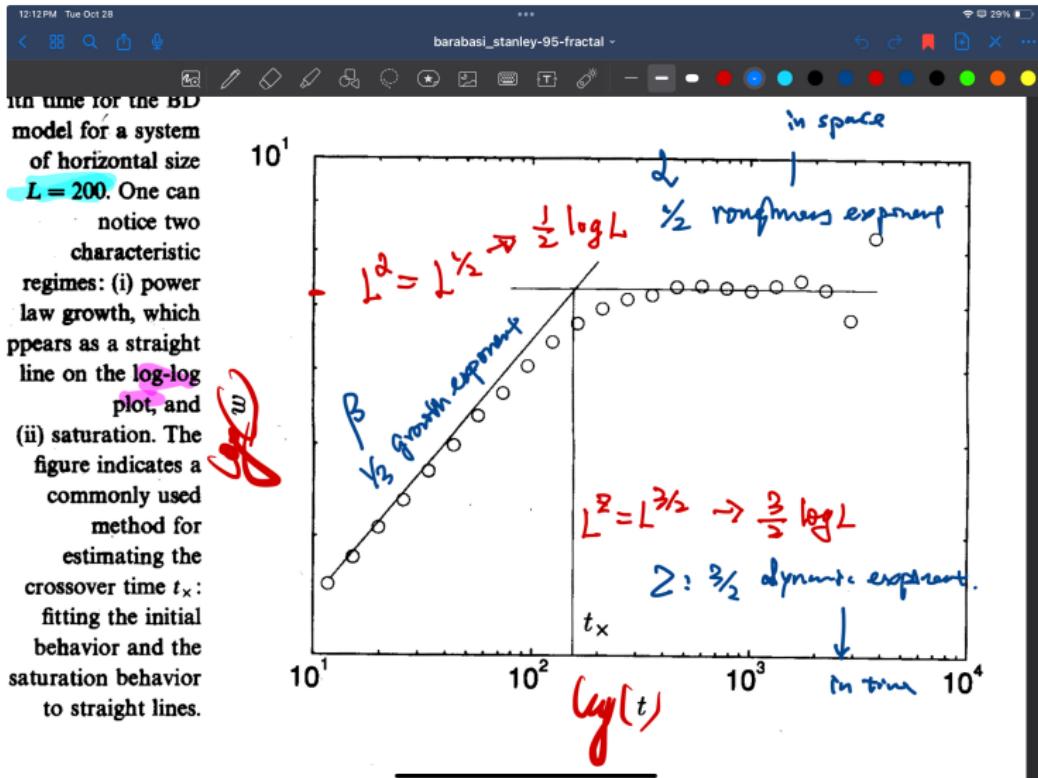
Crossover time $t_x(L) \sim L^z$

z : dynamic exponent

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

Family-Vicsek scaling theory

α : Roughness exp.; β : Growth exp.; z : Dynamic exp.



(Image from Barabási-Stanley's book 95)

Family–Vicsek scaling theory

α : Roughness exp.; β : Growth exp.; z : Dynamic exp.

$$w(L, t) \sim L^\alpha f\left(\frac{t}{L^z}\right)$$

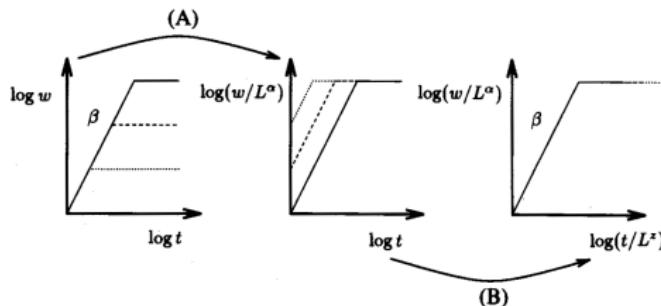
with

$$f(u) \sim \begin{cases} u^\beta, & u \ll 1, \\ \text{const}, & u \gg 1. \end{cases} \Rightarrow \beta = \frac{\alpha}{z}$$

Immediate consequences:

$$w_{\text{sat}}(L) \sim L^\alpha, \quad t_x(L) \sim L^z.$$

Interpretation: dynamic renormalization $x \rightarrow bx$, $t \rightarrow b^z t$, $h \rightarrow b^\alpha h$ leaves $w(L, t)/L^\alpha$ invariant as a function of t/L^z .



(Image from Barabási-Stanley's book 95)

Family–Vicsek (1985): Growth Exponent β

Reported from simulations

$$\beta = 0.30 \pm 0.02 \approx \frac{1}{3}$$

Practical difficulty

Robustly estimating the log–log slope β requires careful choices of fitting window and handling of finite-size effects.

Family, F., Vicsek, T., *Journal of Physics A: Mathematical and General*, 1985

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Simulations on
Random deposition vs. Ballistic decomposition

Study of growing interfaces in a thin film

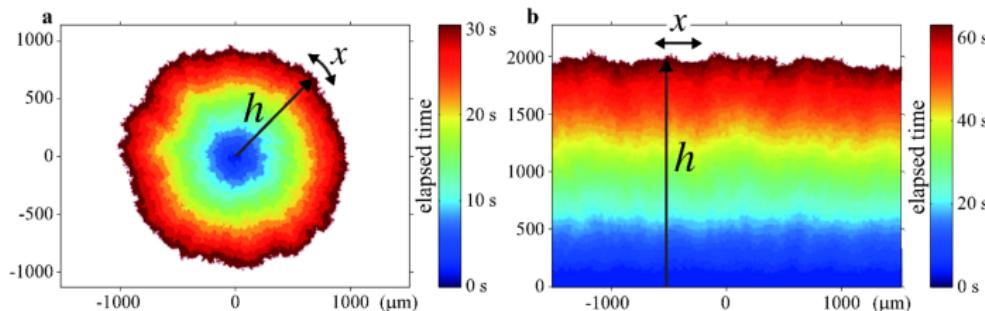
— Convection of nematic liquid crystal*

Show movies !

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*



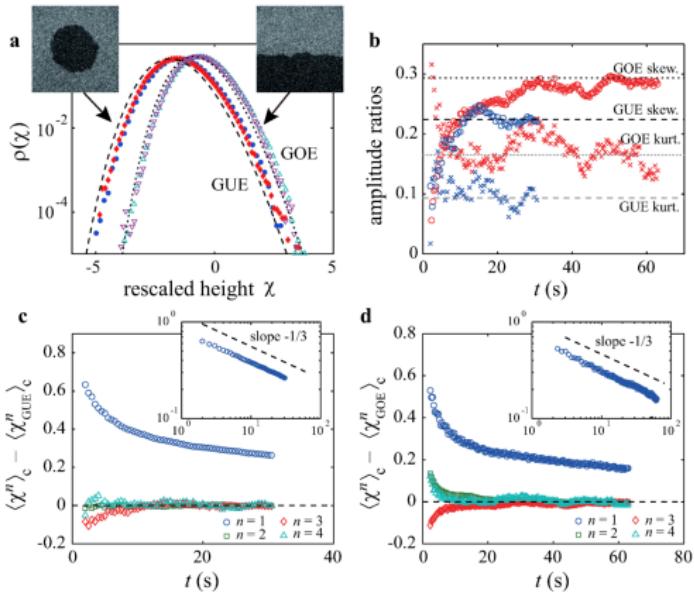
Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$



KPZ Equation '86

$$\frac{\partial}{\partial t} h(t, x) = \frac{1}{2} \Delta h(t, x) + \frac{\lambda}{2} (\nabla h)^2 + \dot{W}(t, x) \quad (\text{KPZ})$$



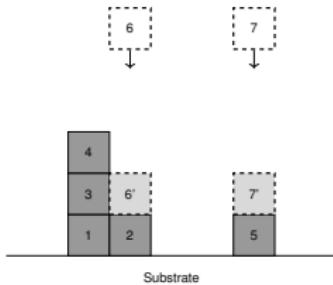
Mehran Kardar (1957 –) Giorgio Parisi (1948 –)



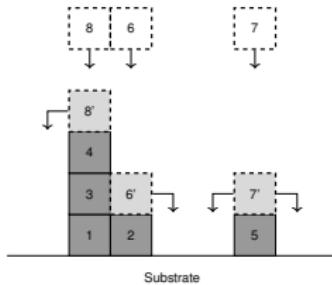
Yicheng Zhang

Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

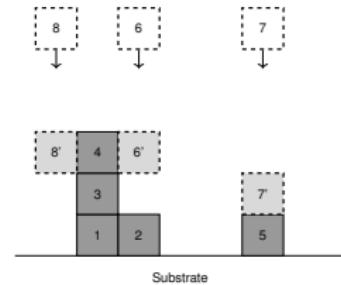
Random deposition



RD with surface relaxation



Ballistic deposition



$$W(t, x) \sim t^{1/2}$$

$$W(t, x) \sim t^{1/4}$$

$$W(t, x) \sim t^{1/3}$$

Edwards–Wilkinson

KPZ/BD

Two universality classes: EW and KPZ

Edwards–Wilkinson (EW)

Linear diffusion + noise

$$\partial_t h = \nu \nabla^2 h + \eta, \quad \langle \eta \eta \rangle \propto \delta(x) \delta(t).$$

Scale invariance (1D): $z = 2, \alpha = \frac{1}{2}$
 $\Rightarrow \beta = \frac{1}{4}$.

$$\boxed{\alpha = \frac{1}{2}, z = 2, \beta = \frac{1}{4}}$$

Kardar–Parisi–Zhang (KPZ)

Nonlinear growth

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta.$$

In 1D (exact):

$$\boxed{\alpha = \frac{1}{2}, z = \frac{3}{2}, \beta = \frac{1}{3}}$$

BD, SOS, Eden \Rightarrow KPZ universality in 1+1D.

Plan

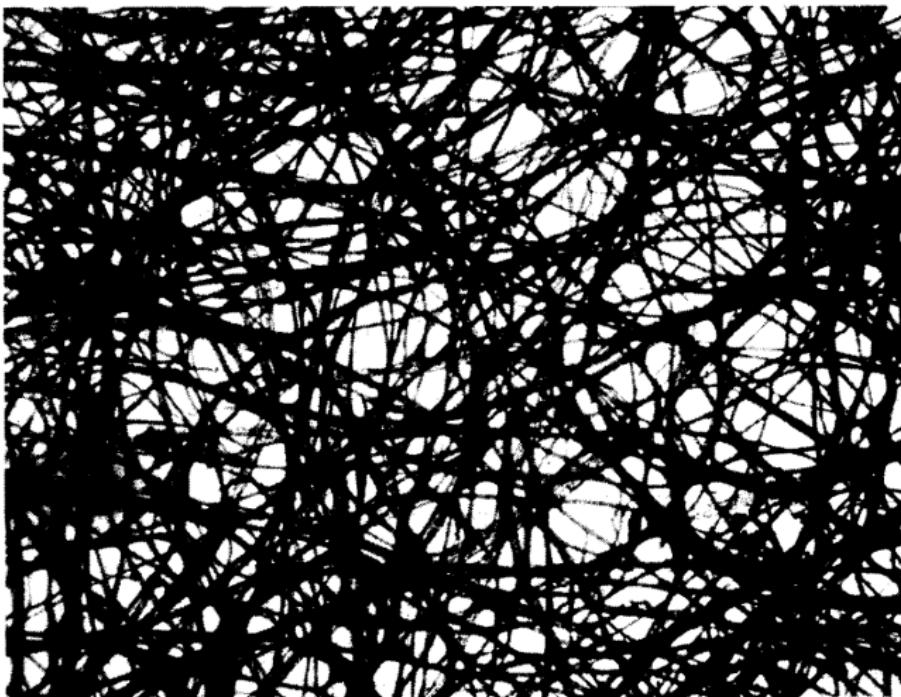
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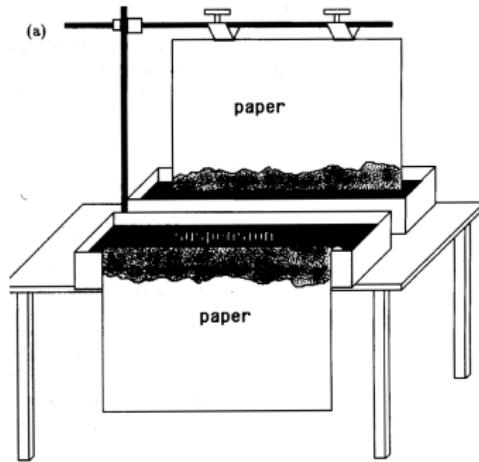
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Paper – a random environment



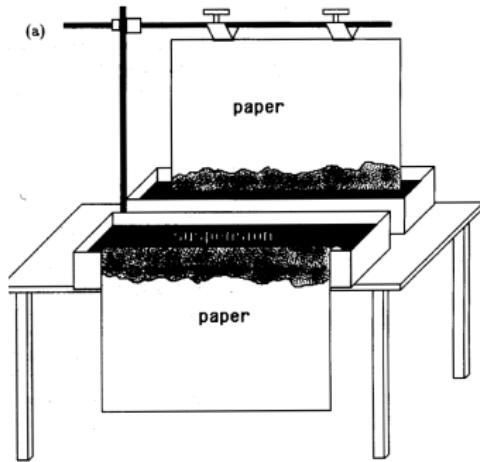
Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper wetting experiment



Barabási, A.-L., Stanley, H. E., 1995

Paper wetting experiment



(b)

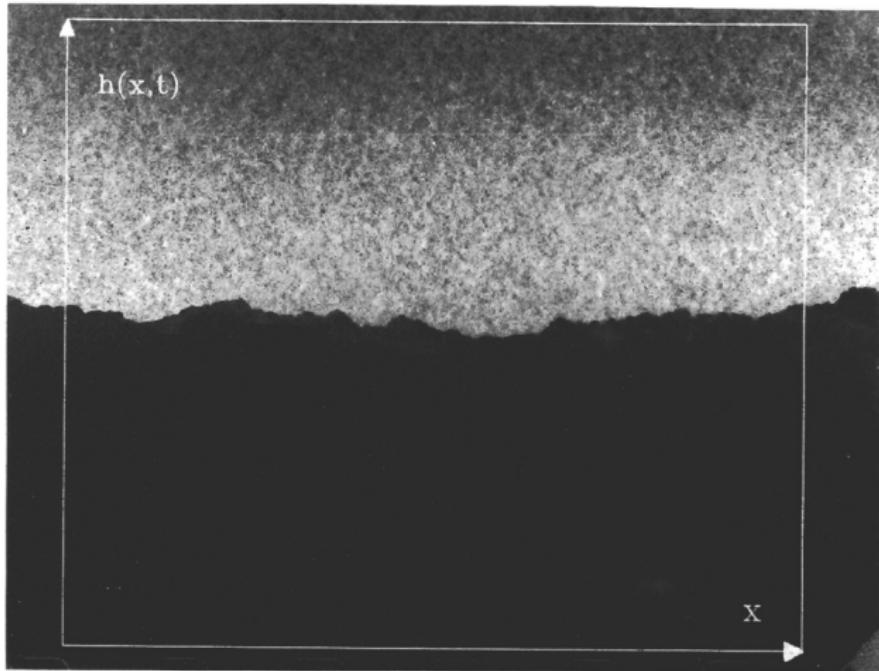


(c)



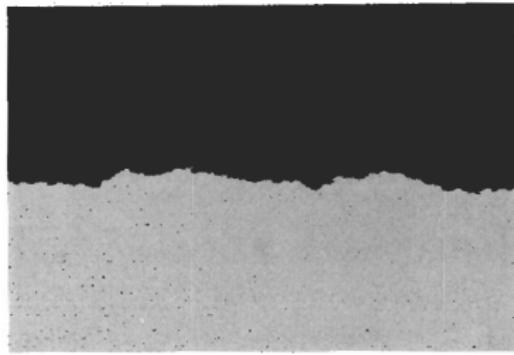
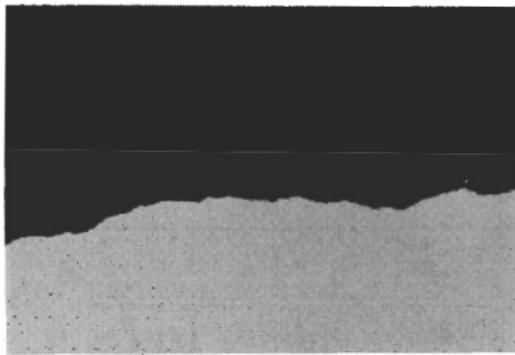
Barabási, A.-L., Stanley, H. E., 1995

Paper burning experiment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper rupture experiment



Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

Plan

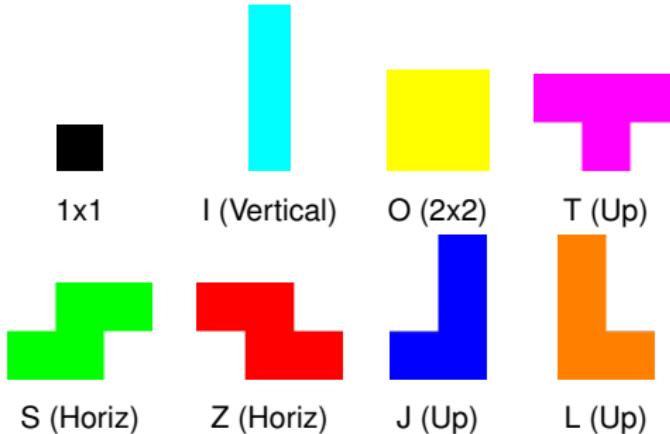
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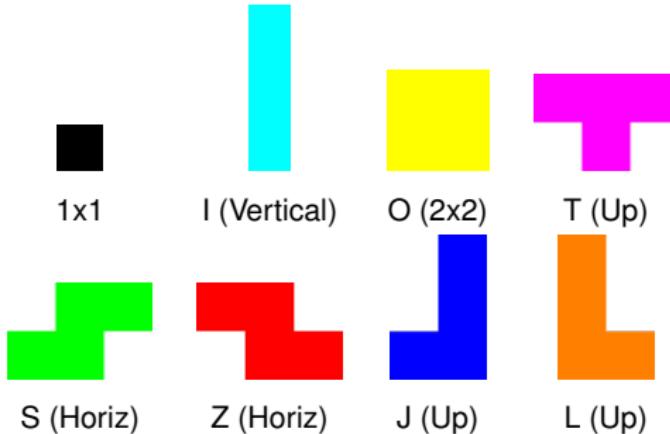
Tetrominoes



- ▶ “1x1”: Single (extra single-site particle)
- ▶ “I”: Horizontal, Vertical
- ▶ “J, L, T”: Up, Right, Down, Left
- ▶ “S, Z”: Horizontal, Vertical
- ▶ “O”: Single (2x2 square)
- ▶ Sticky
- ▶ Nonsticky

$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

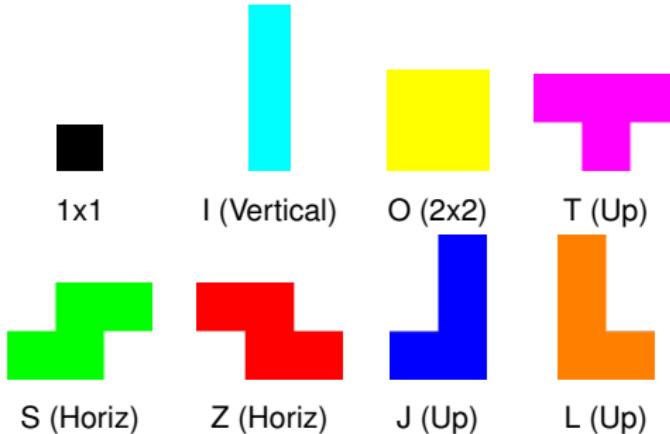
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Configure files

```
steps: 12000  
width: 100  
height: 300  
seed: 12  
Piece-00: [20, 0]  
Piece-01: [20, 0]  
Piece-02: [20, 0]  
Piece-03: [20, 0]  
Piece-04: [20, 0]  
Piece-05: [20, 0]  
Piece-06: [20, 0]  
Piece-07: [20, 0]  
Piece-08: [20, 0]  
Piece-09: [20, 0]  
Piece-10: [20, 0]  
Piece-11: [20, 0]  
Piece-12: [20, 0]  
Piece-13: [20, 0]  
Piece-14: [20, 0]  
Piece-15: [20, 0]  
Piece-16: [20, 0]  
Piece-17: [20, 0]  
Piece-18: [20, 0]  
Piece-19: [20, 0]
```

All nonsticky pieces
with equal prob.

```
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Piece-02: [0, 0]  
Piece-03: [0, 0]  
Piece-04: [0, 0]  
Piece-05: [0, 0]  
Piece-06: [0, 0]  
Piece-07: [0, 0]  
Piece-08: [0, 0]  
Piece-09: [0, 0]  
Piece-10: [0, 0]  
Piece-11: [0, 0]  
Piece-12: [0, 0]  
Piece-13: [0, 0]  
Piece-14: [0, 0]  
Piece-15: [0, 0]  
Piece-16: [0, 0]  
Piece-17: [0, 0]  
Piece-18: [0, 0]  
Piece-19: [20, 80]
```

20% nonsticky
+ 80% sticky
of 1x1 piece

Question

For various configurations of Tetromino pieces, do the resulting surface robustly exhibit Family–Vicsek scaling?

Will the scaling exponent β be always close to $\frac{1}{3}$?

Practical difficulty

Robustly estimating the log–log slope β requires careful choices of fitting window and handling of finite-size effects.

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Setup

Consider a mixture of $\ell\%$ ($\ell \in [0, 100]$) nonsticky pieces and $(100-\alpha)\%$ sticky pieces, where only the 1×1 piece is sticky.

Question

How does α influence the scaling exponents, in particular, $\beta \stackrel{?}{\approx} \frac{1}{3}$?

Simulations and log-log plots:

https://chenle02.github.io/2025-10-28_Emerging_Synergies_Banff_Le/exp13/videos_and_images_display.html

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Outreach Highlights



AU-SSI 2023



AU-SSI 2024



Destination STEM 2023

Thank you!

Questions?