# Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class


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Auburn University



Acknwolegement

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Math 7820/30: Applied Stochastic Processes (2023/24):

Mauricio Montes and Ian Ruau

# Plan

Introduction to growth model and SPDE

Tetromino Pieces

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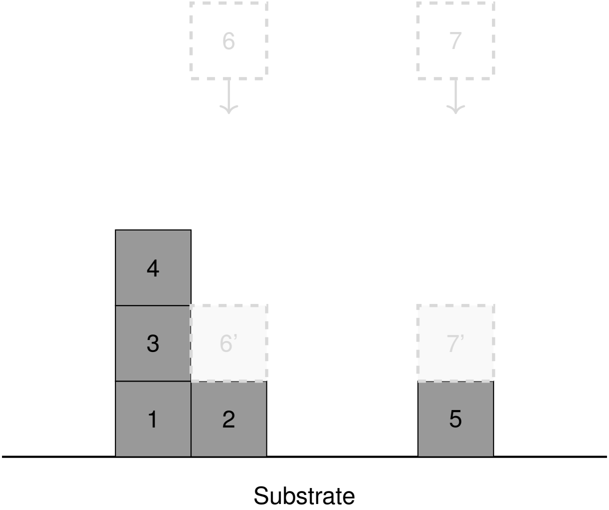
Tetromino Pieces

# Random deposition



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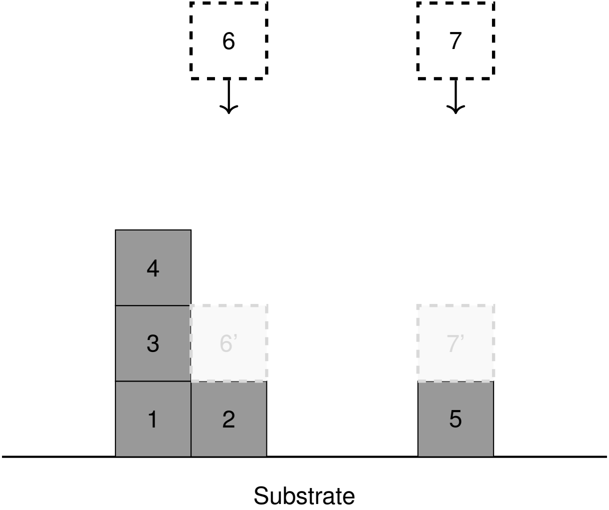

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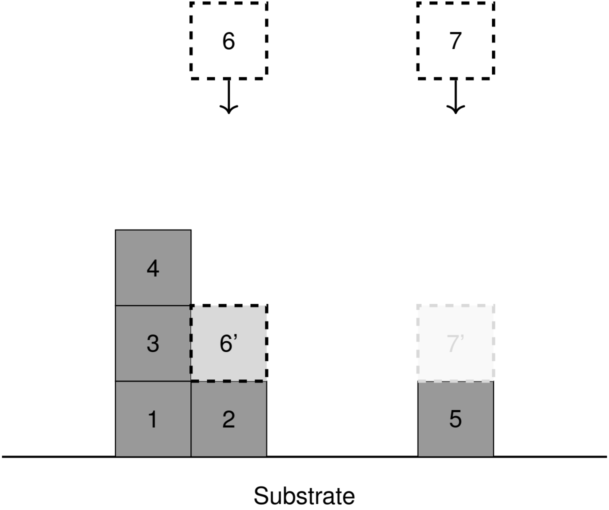

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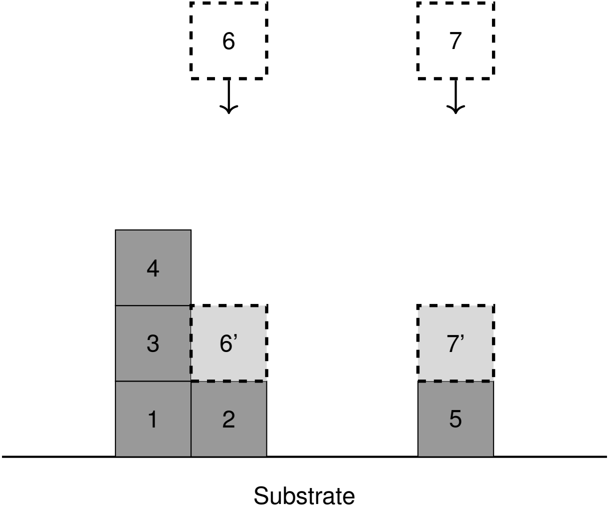
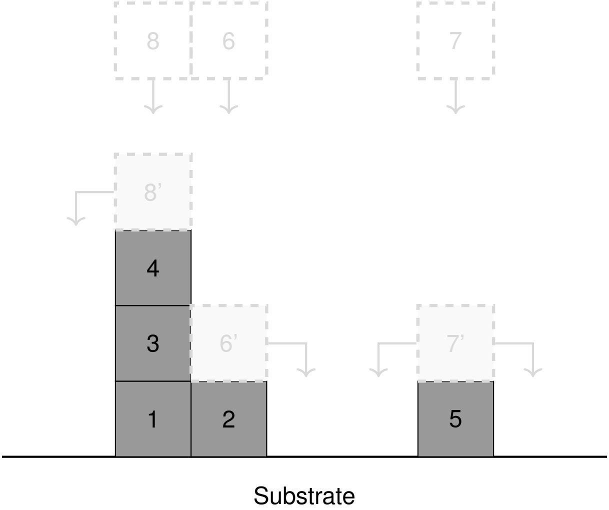


# Random deposition with surface relaxation



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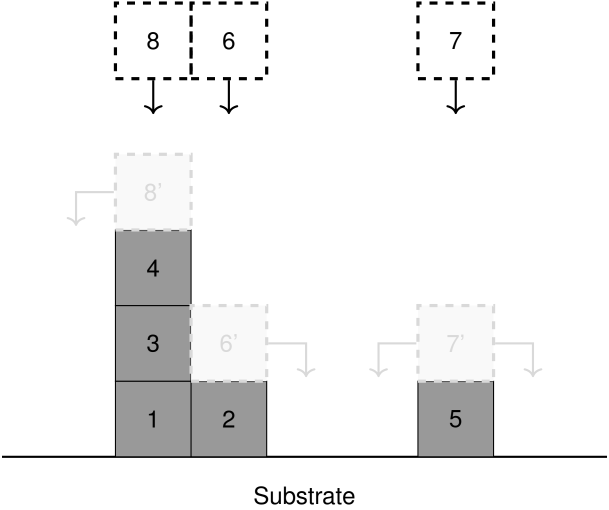

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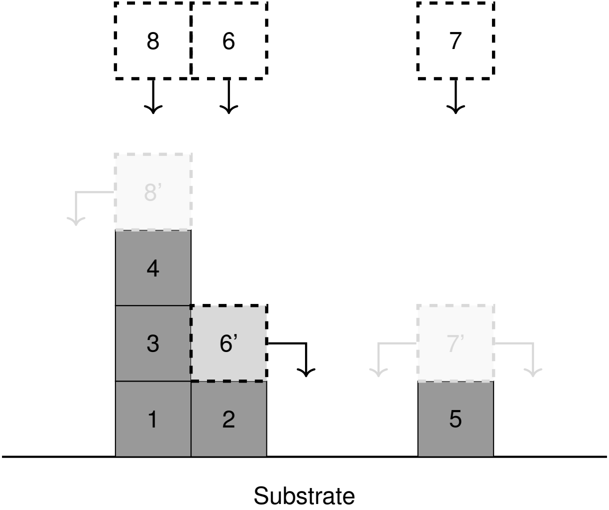

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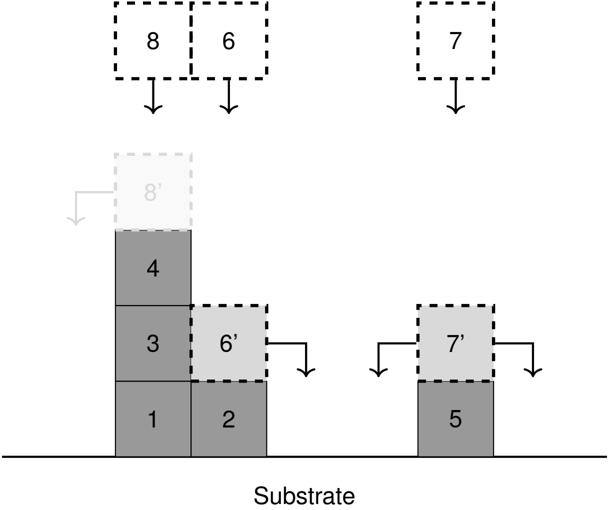

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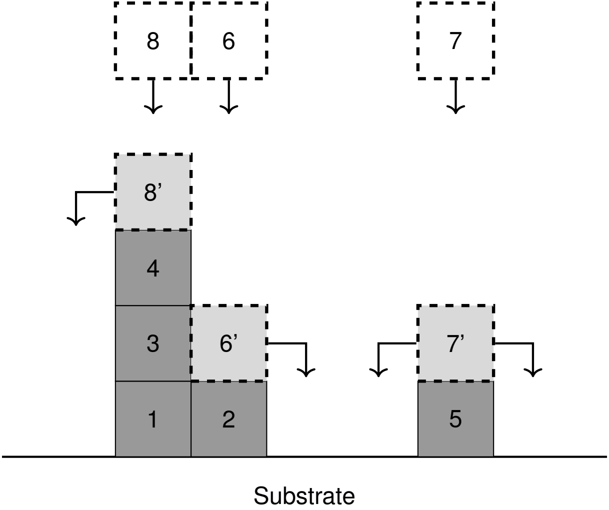

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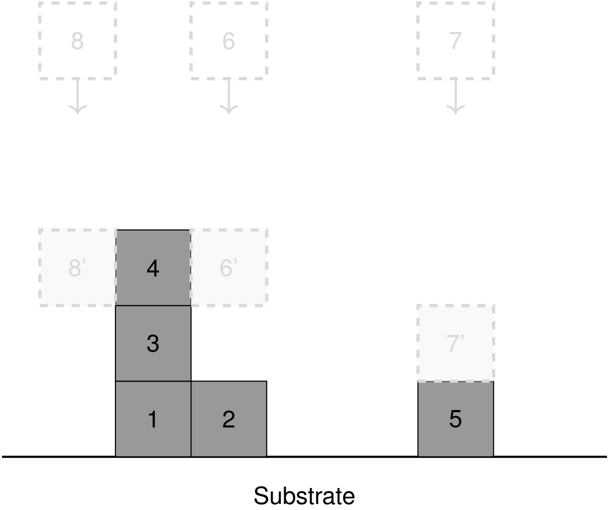

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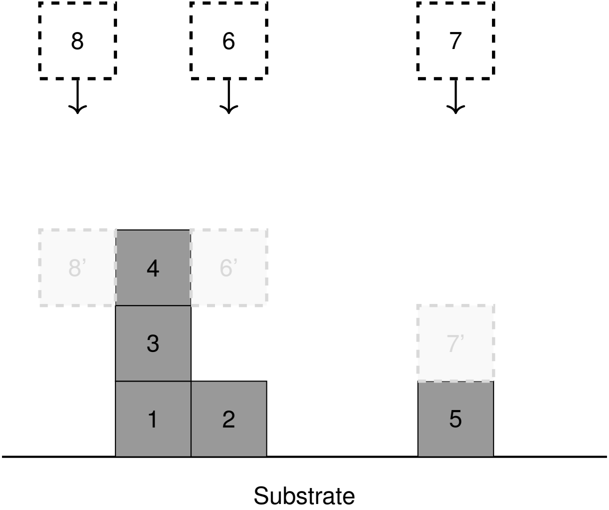

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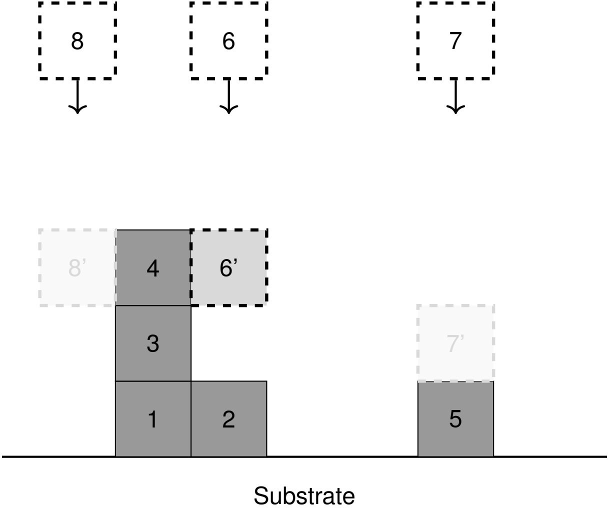

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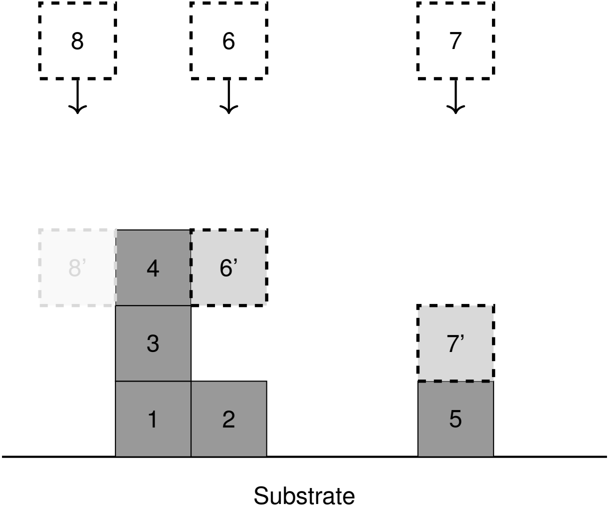

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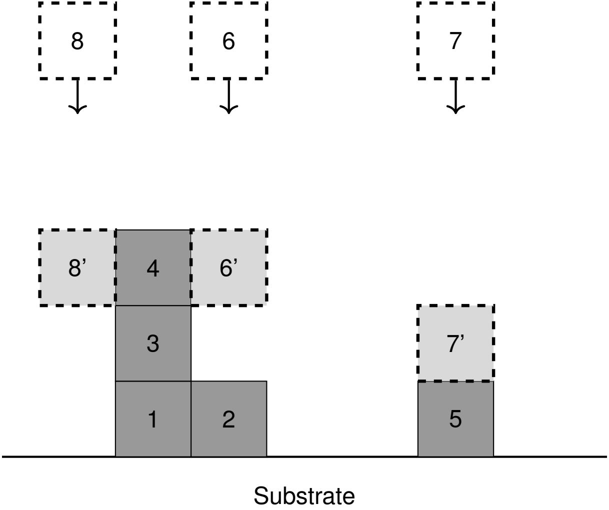
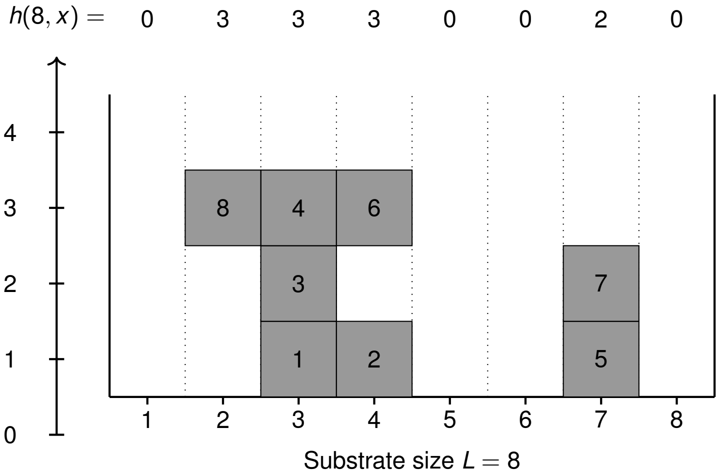


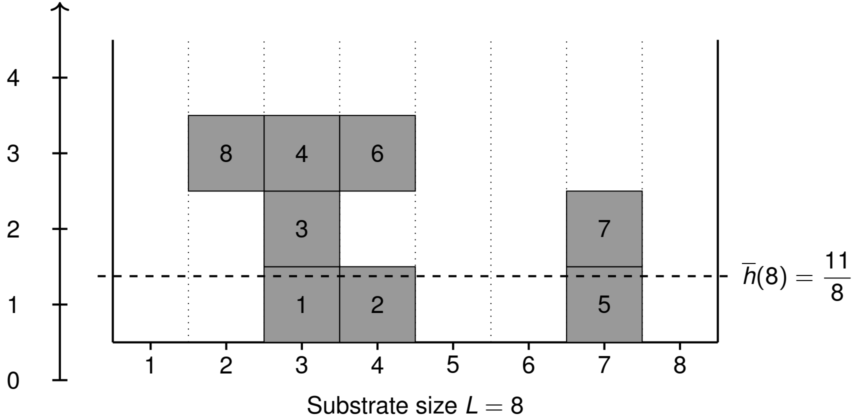
# Average height and fluctuation



# Average height and fluctuation

$$\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t,x) \qquad \text{Fluctuation } W(L,t) = \sqrt{\frac{1}{L}\sum_{x=1}^{L}\left[h(t,x) - \overline{h}(t)\right]^2}$$

$$h(8,x) = \quad 0 \quad 3 \quad 3 \quad 3 \quad 0 \quad 0 \quad 2 \quad 0$$



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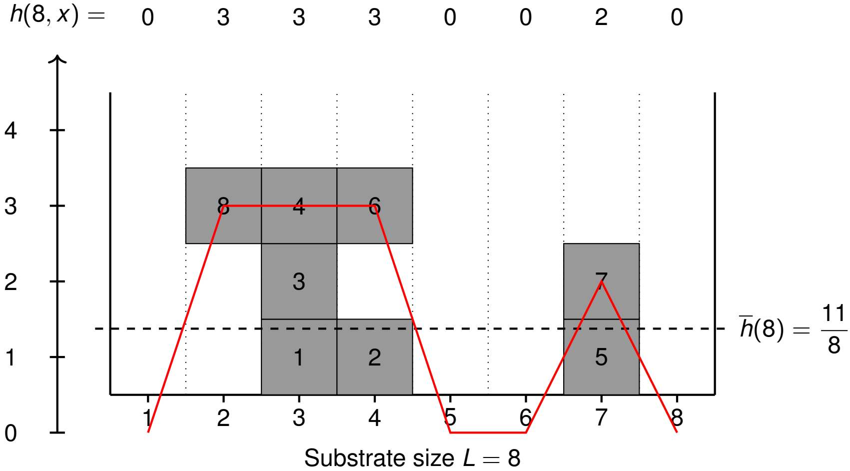

# Random Deposition (independent columns, nonsticky)

**Model.** $L$ independent columns. At each integer time $t = 1, 2, \dots$, drop *one* particle on a uniformly random column. Heights $h(t, x)$, mean $\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t, x) = \frac{t}{L}$, width

$$W^2(L, t) = \frac{1}{L}\sum_{x=1}^{L}\left(h(t, x) - \overline{h}(t)\right)^2.$$

**Single–column law:** After $t$ drops total,

$$h(t, x) \sim \mathrm{Binomial}\left(t, \frac{1}{L}\right), \qquad \mathbb{E}[h(t, x)] = \frac{t}{L}, \quad \mathrm{Var}(h(t, x)) = t\,\frac{1}{L}\left(1 - \frac{1}{L}\right).$$

**Fluctuation:** By i.i.d. columns,

$$\mathbb{E}\left[W^2(L, t)\right] = \frac{1}{L}\sum_{x=1}^{L}\mathbb{E}\left[h(t, x)^2\right] - \mathbb{E}\left[\overline{h}^2(t)\right] = \mathbb{E}\left[h(t, 1)^2\right] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right)\mathrm{Var}\big(h(t, 1)\big).$$

Hence

$$\boxed{\mathbb{E}\left[W^2(L, t)\right] = \left(1 - \frac{1}{L}\right) t\,\frac{1}{L}\left(1 - \frac{1}{L}\right) = \frac{t}{L}\left(1 - \frac{1}{L}\right)^2}$$

and

$$\boxed{W(L, t) \simeq \left(1 - \frac{1}{L}\right)\left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \frac{1}{2}$.

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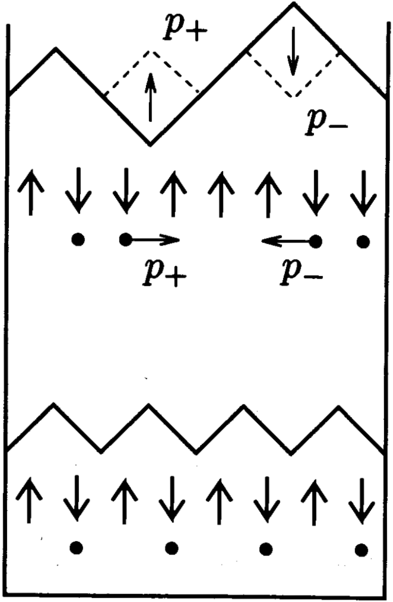
For ballistic deposition (sticky), lateral interactions appear, can we view it as a perturbation of random deposition (nonsticky)?

Simulations on

Random deposition vs. Ballistic decomposition

# More models? Even more simpler?

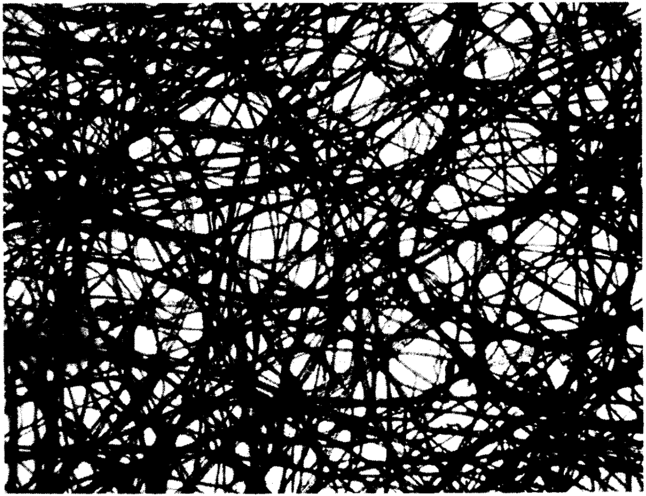



Solid-on-Solid

Ising

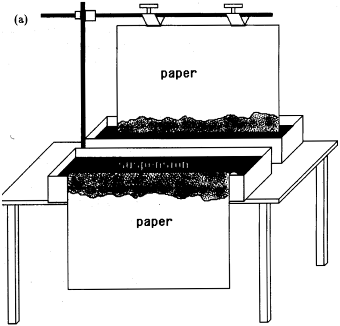
Lattice Gas

# Paper – a random environment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

# Paper wetting experiment




Barabási, A.-L., Stanley, H. E., 1995

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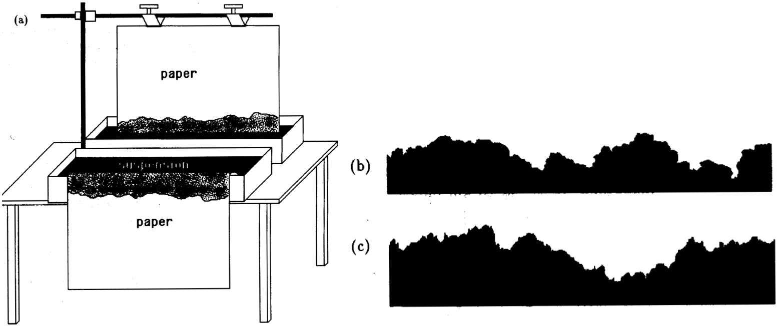


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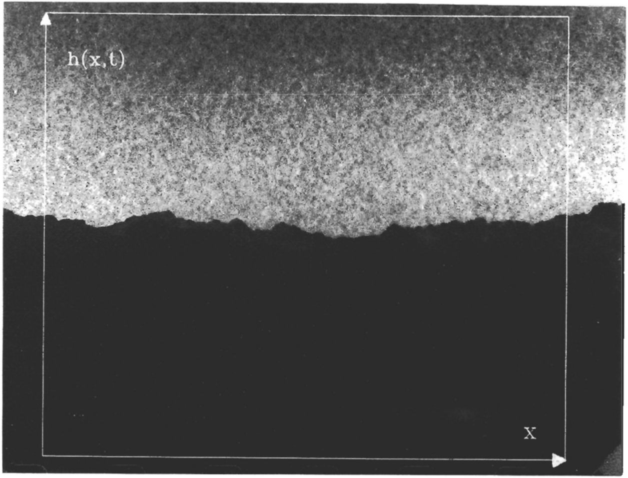
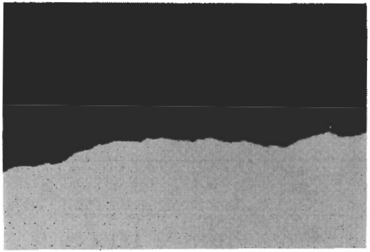
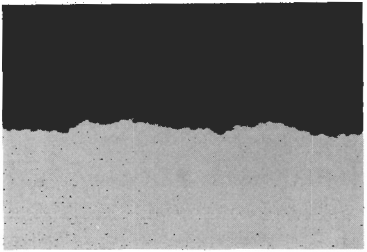
# Paper burning experiment




Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

# Paper rupture experiment

Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

# Study of growing interfaces in a thin film

## — Convection of nematic liquid crystal*

Show movies !

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

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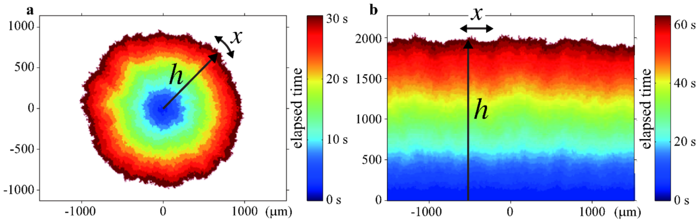


Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

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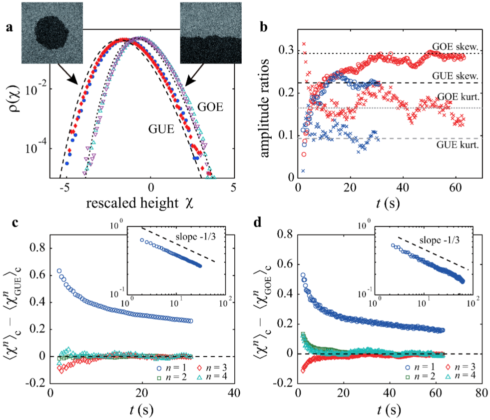



Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

# KPZ Equation '86

$$\frac{\partial}{\partial t} h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{\lambda}{2}(\nabla h)^2 + \dot{W}(t,x) \tag{KPZ}$$


Mehran Kardar (1957 –)


Giorgio Parisi (1948 –)


Yicheng Zhang

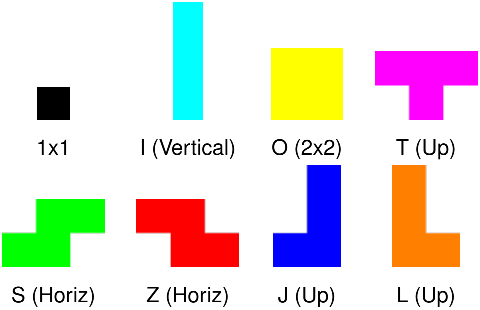
Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

# Plan

Introduction to growth model and SPDE

Tetromino Pieces

# Tetrominoes

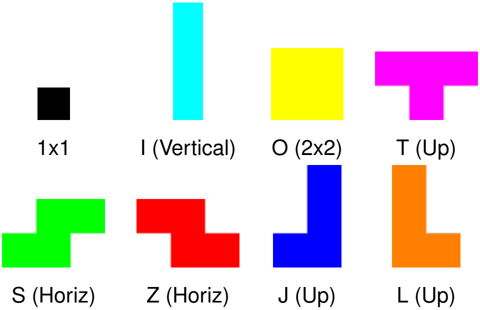



- "1x1": Single (extra single-site particle)
- "I": Horizontal, Vertical
- "J, L, T": Up, Right, Down, Left
- "S, Z": Horizontal, Vertical
- "O": Single (2x2 square)

- Sticky
- Nonstikcy

$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40$ types of pieces

# Tetrominoes

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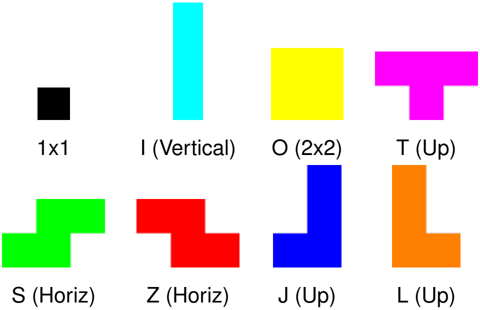


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- Sticky
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$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

# Configure files

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [20, 0]
Piece-01: [20, 0]
Piece-02: [20, 0]
Piece-03: [20, 0]
Piece-04: [20, 0]
Piece-05: [20, 0]
Piece-06: [20, 0]
Piece-07: [20, 0]
Piece-08: [20, 0]
Piece-09: [20, 0]
Piece-10: [20, 0]
Piece-11: [20, 0]
Piece-12: [20, 0]
Piece-13: [20, 0]
Piece-14: [20, 0]
Piece-15: [20, 0]
Piece-16: [20, 0]
Piece-17: [20, 0]
Piece-18: [20, 0]
Piece-19: [20, 0]
```

All nonsticky pieces
with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 20]
Piece-01: [0, 20]
Piece-02: [0, 20]
Piece-03: [0, 20]
Piece-04: [0, 20]
Piece-05: [0, 20]
Piece-06: [0, 20]
Piece-07: [0, 20]
Piece-08: [0, 20]
Piece-09: [0, 20]
Piece-10: [0, 20]
Piece-11: [0, 20]
Piece-12: [0, 20]
Piece-13: [0, 20]
Piece-14: [0, 20]
Piece-15: [0, 20]
Piece-16: [0, 20]
Piece-17: [0, 20]
Piece-18: [0, 20]
Piece-19: [0, 20]
```

All sticky pieces
with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 0]
Piece-01: [0, 0]
Piece-02: [0, 0]
Piece-03: [0, 0]
Piece-04: [0, 0]
Piece-05: [0, 0]
Piece-06: [0, 0]
Piece-07: [0, 0]
Piece-08: [0, 0]
Piece-09: [0, 0]
Piece-10: [0, 0]
Piece-11: [0, 0]
Piece-12: [0, 0]
Piece-13: [0, 0]
Piece-14: [0, 0]
Piece-15: [0, 0]
Piece-16: [0, 0]
Piece-17: [0, 0]
Piece-18: [0, 0]
Piece-19: [20, 80]
```

20% nonsticky
+ 80% sticky
of 1x1 piece

# Main References:


Barabási, A.-L., & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press, Cambridge.

Family, F., & Vicsek, T. (1985).Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model. *Journal of Physics A: Mathematical and General*, *18*(2), L75.

Kardar, M., Parisi, G., & Zhang, Y.-C. (1986).Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, *56*(9), 889.

Kertész, J., Horváth, V. k., & Weber, F. (1993).Self-affine rupture lines in paper sheets. *Fractals*, *01*(01), 67–74.

Takeuchi, K. A., Sano, M., Sasamoto, T., & Spohn, H. (2011).Growing interfaces uncover universal fluctuations behind scale invariance. *Sci. Rep.*, *1*(1), 1–5.

Zhang, J., Zhang, Y.-C., Alstrøm, P., & Levinsen, M. (1992).Modeling forest fire by a paper-burning experiment, a realization of the interface growth mechanism. *Phys. A: Stat. Mech. Appl.*, *189*(3), 383–389.

Thank you!

Questions?