Analysis of Tetris Ballistic Deposition and the Robustness of the KPZ Universality Class

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Auburn University

Acknwolegement

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Emerging Synergies between Stochastic Analysis and Statistical Mechanics
Banff, Alberta, Canada
October 28, 2025

Math 7820/30: Applied Stochastic Processes (2023/24):

Mauricio Montes and Ian Ruau

Plan

Introduction to growth model and SPDE

Tetromino Pieces

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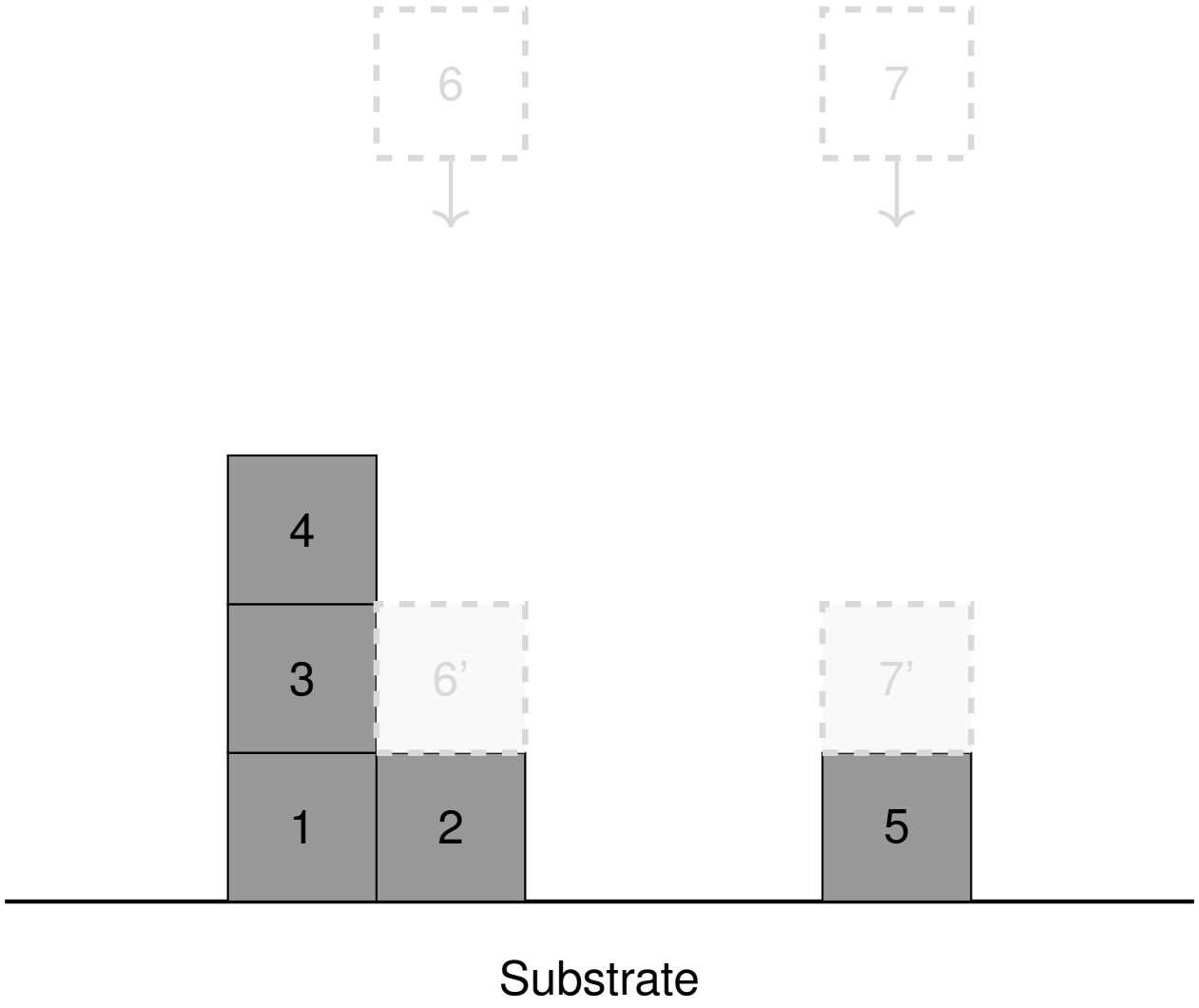
Introduction to growth model and SPDE

Tetromino Pieces

Random deposition



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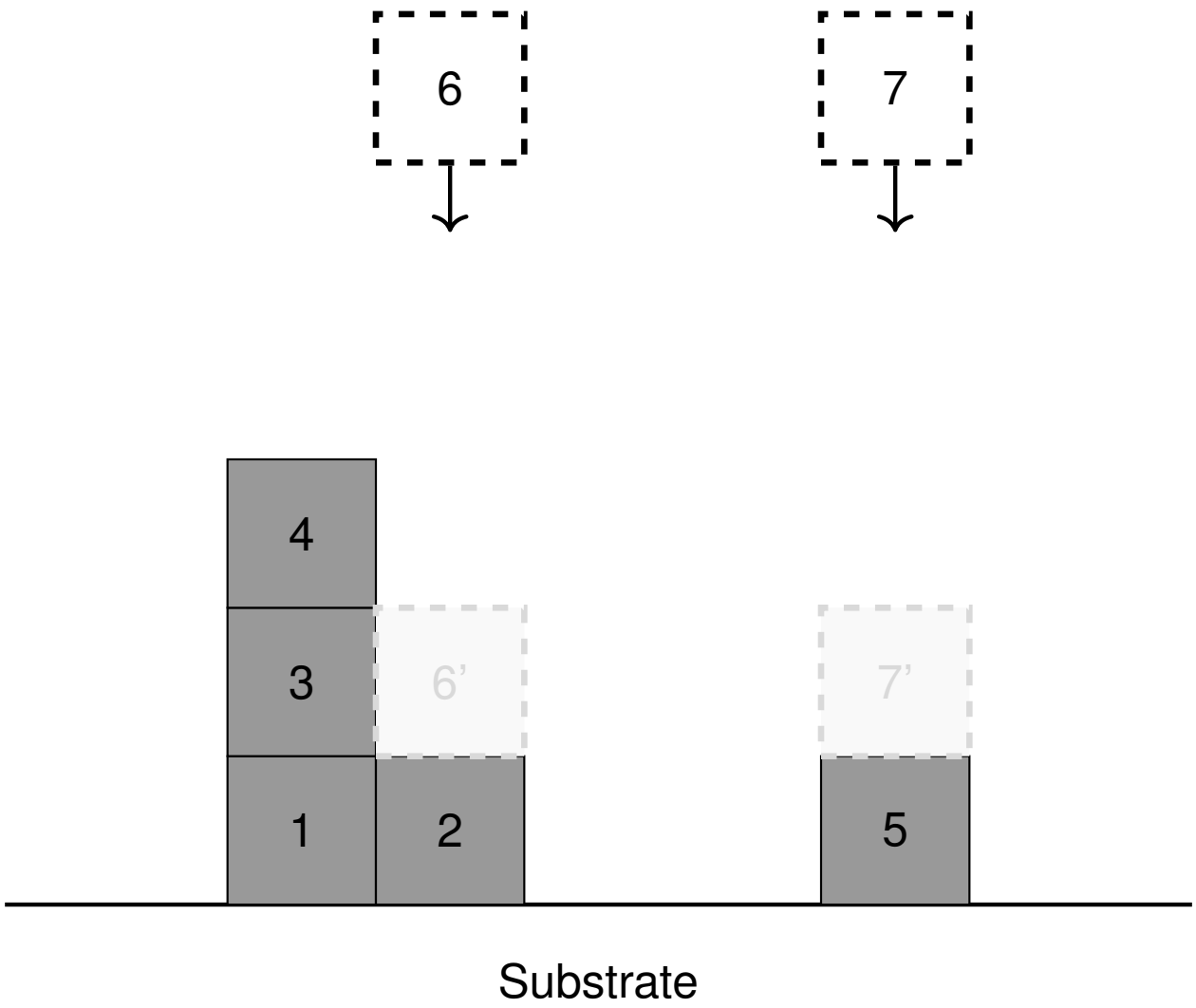

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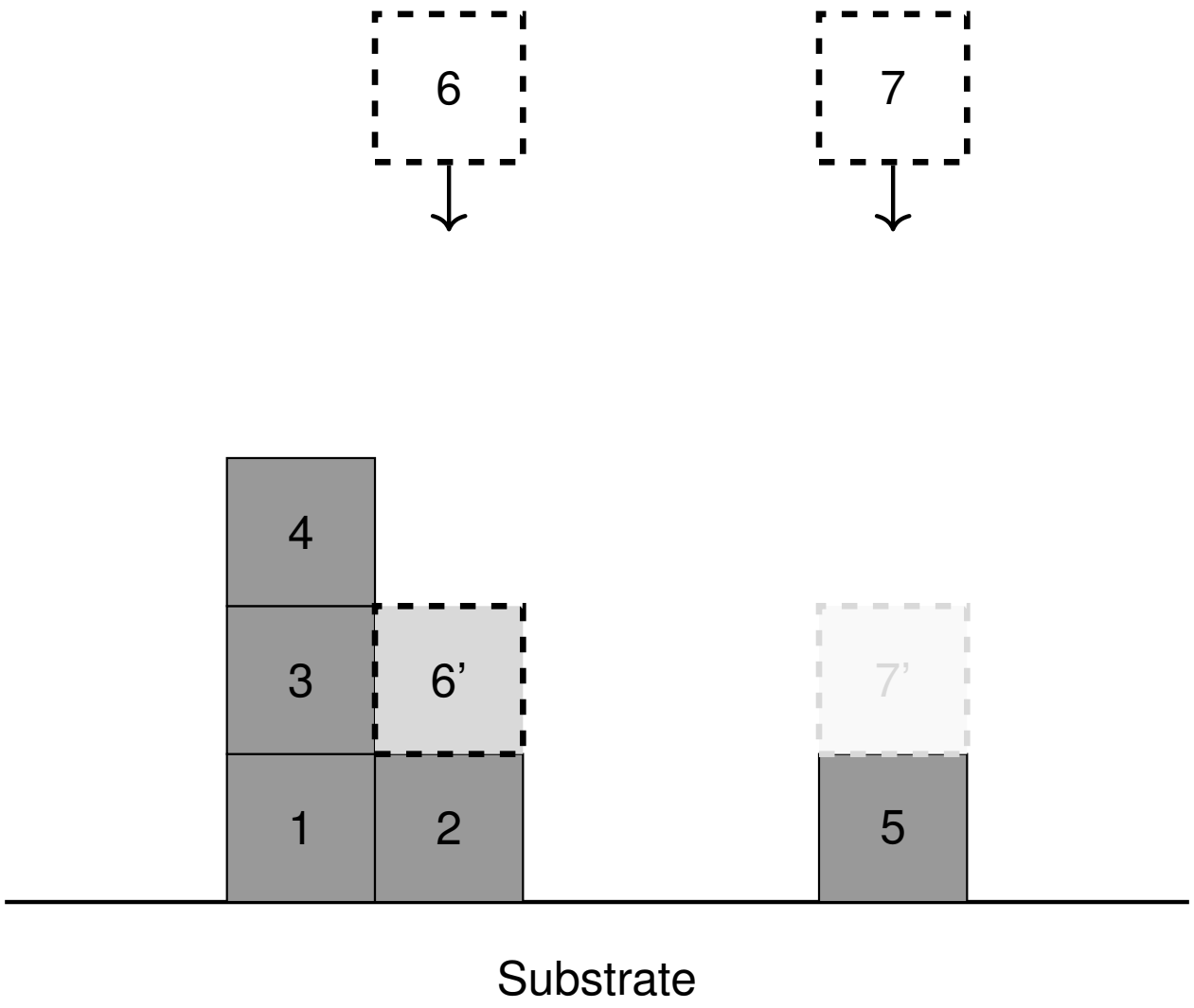

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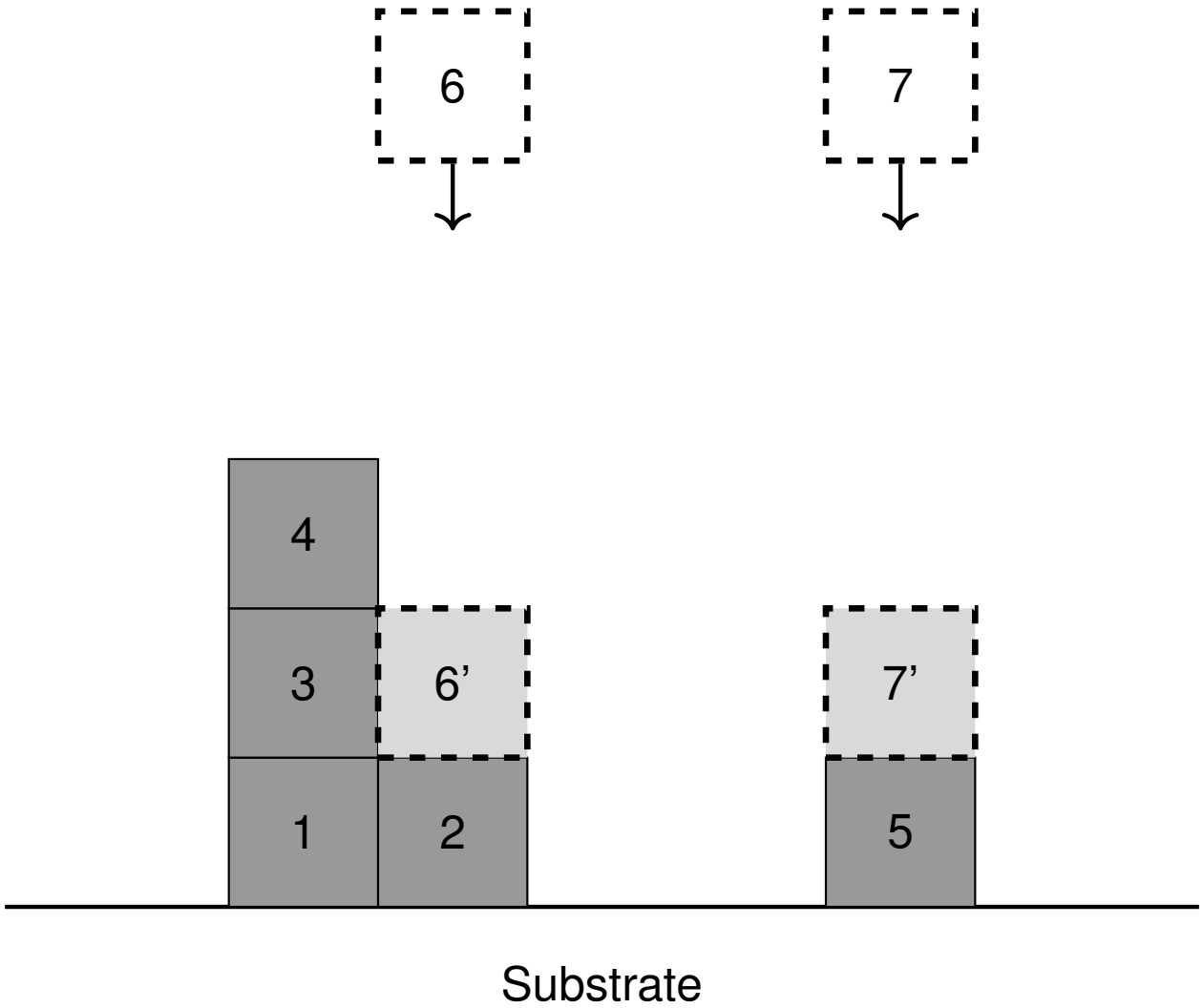
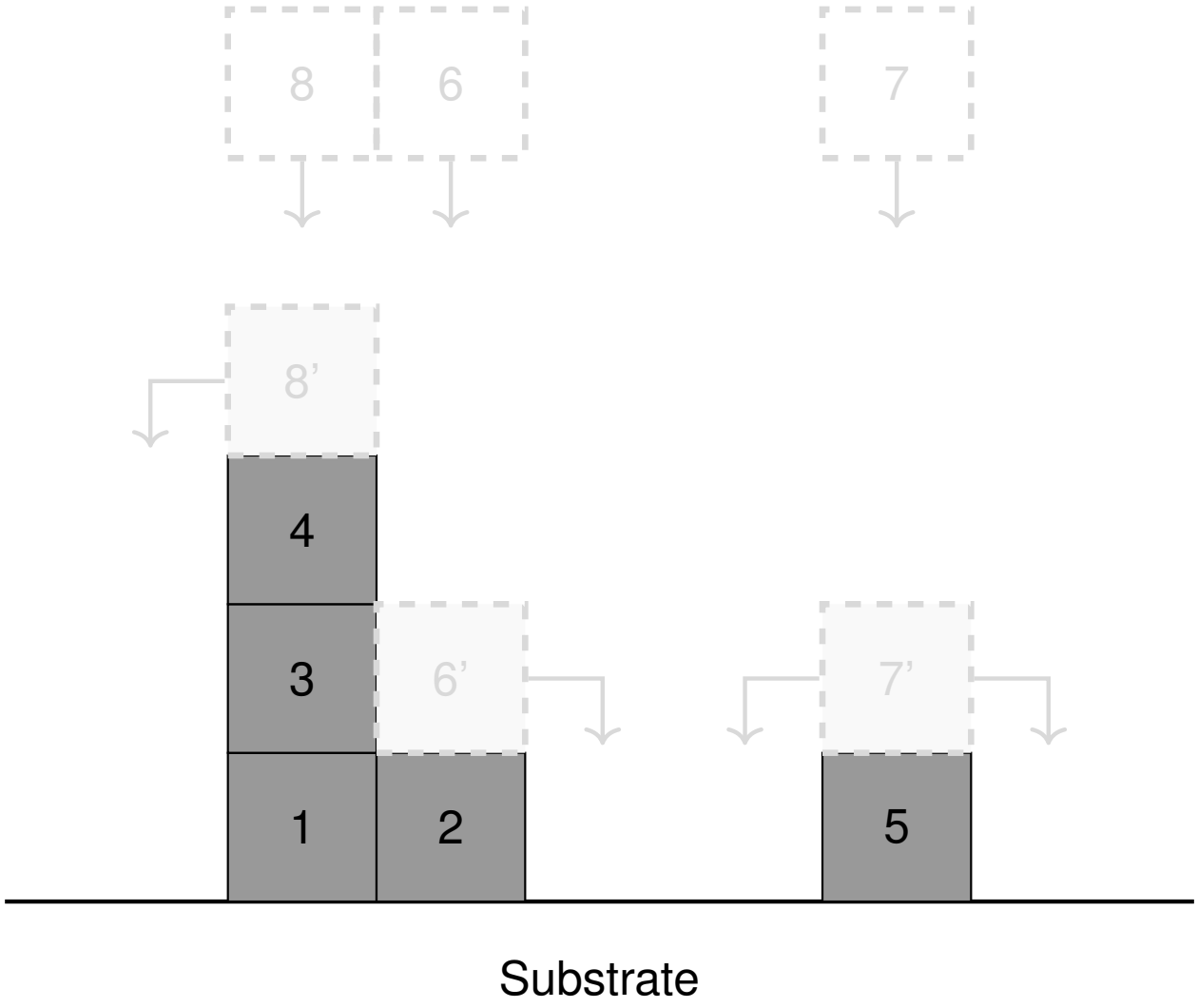


Random deposition with surface relaxation



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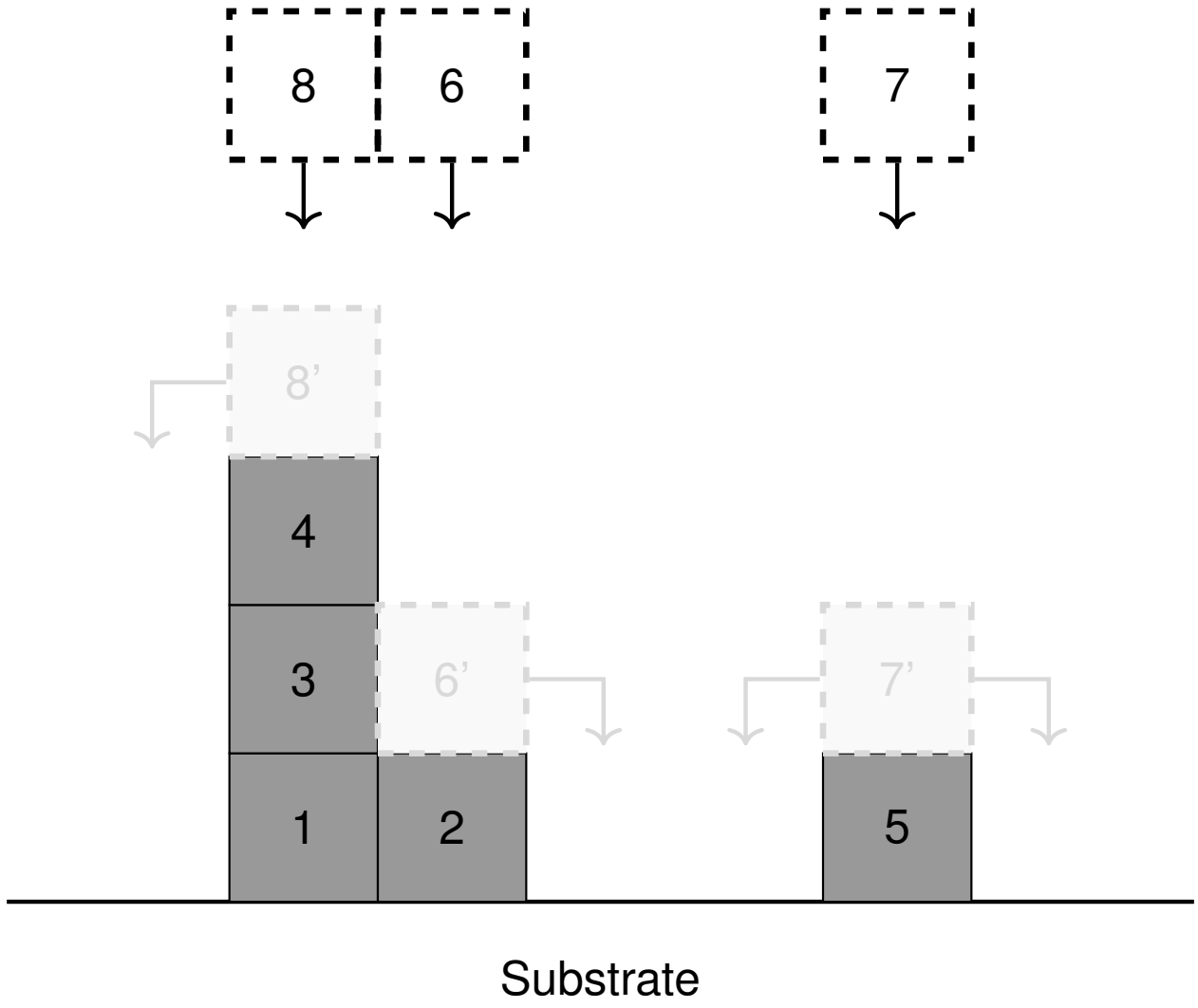

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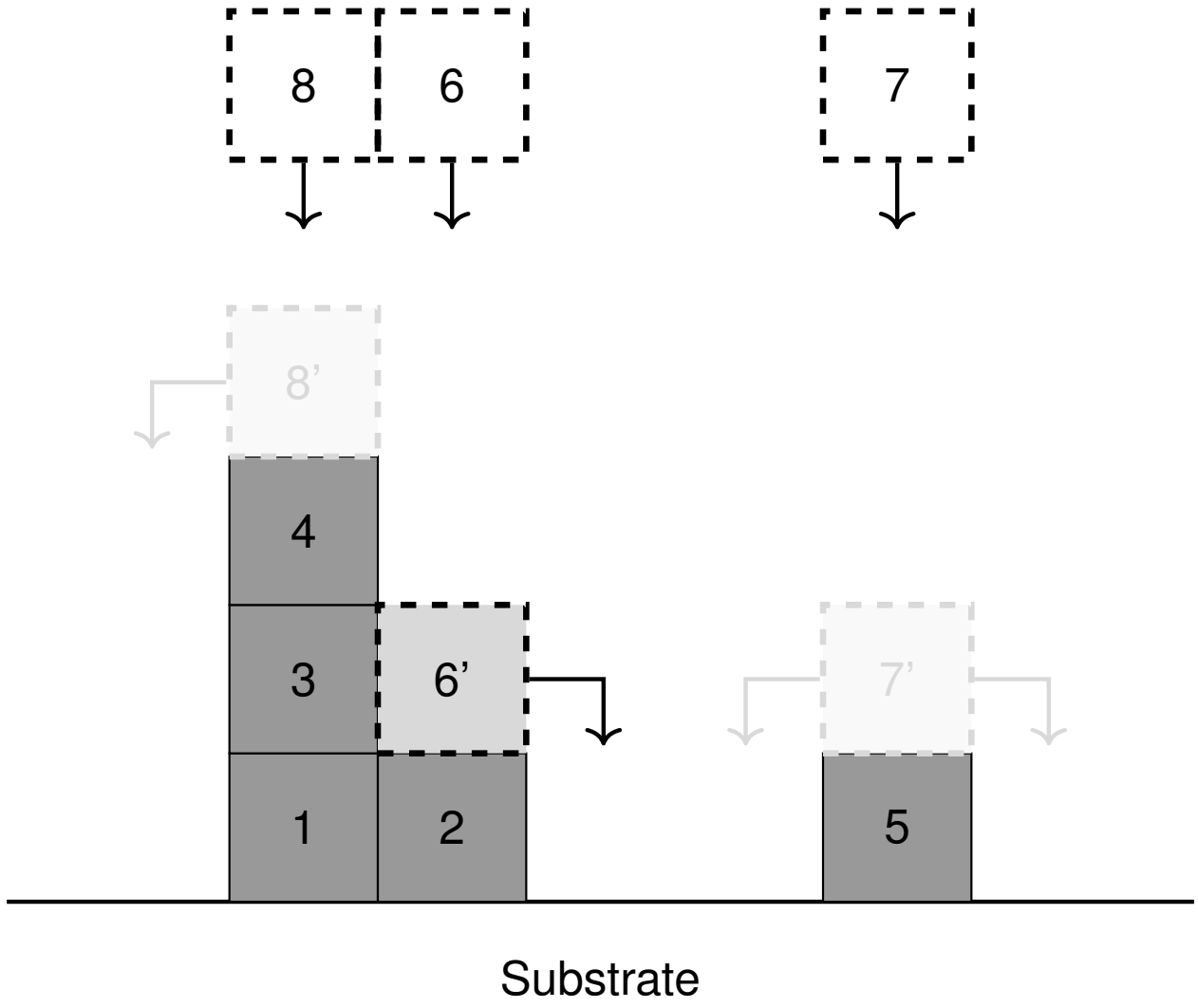

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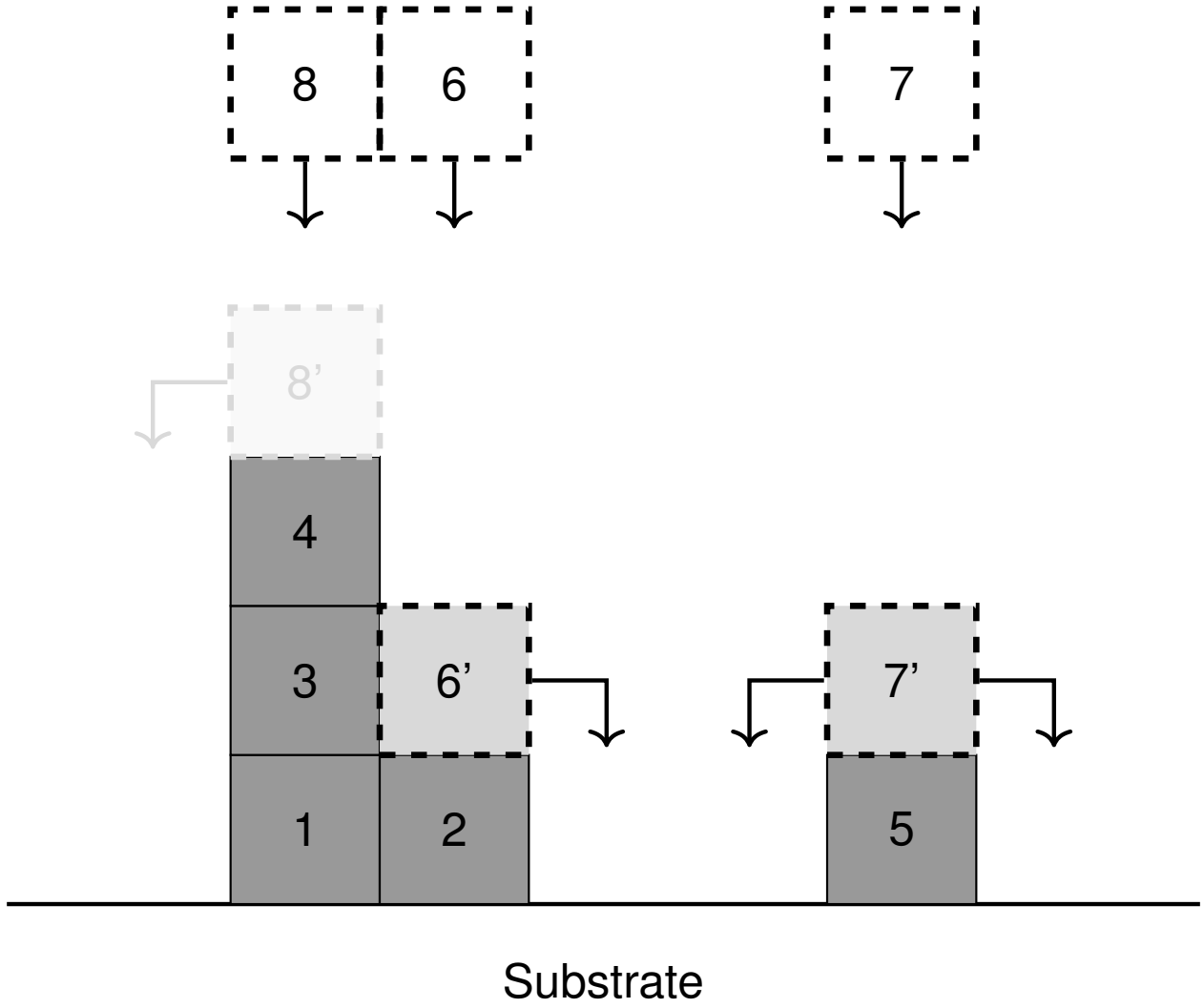

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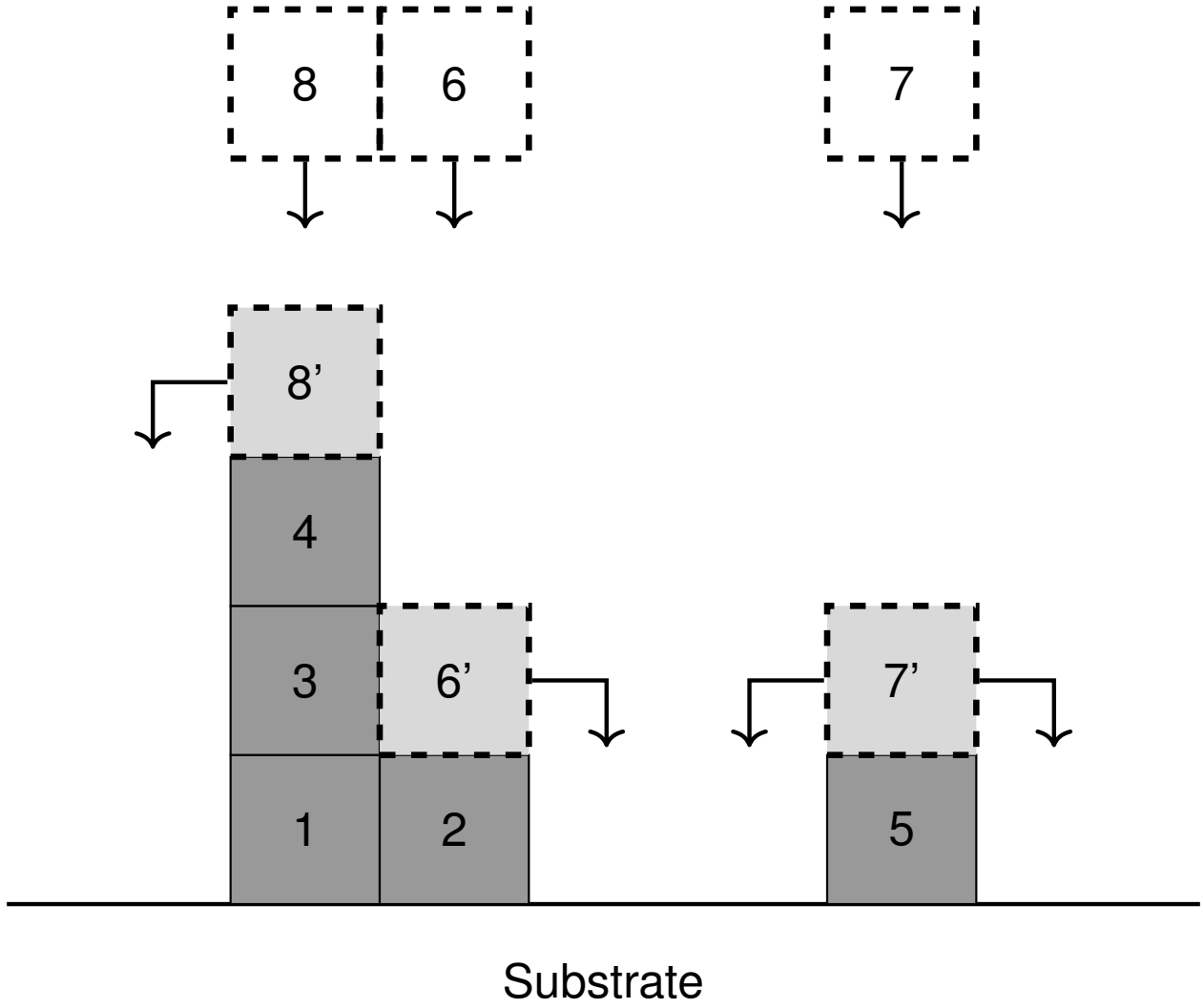
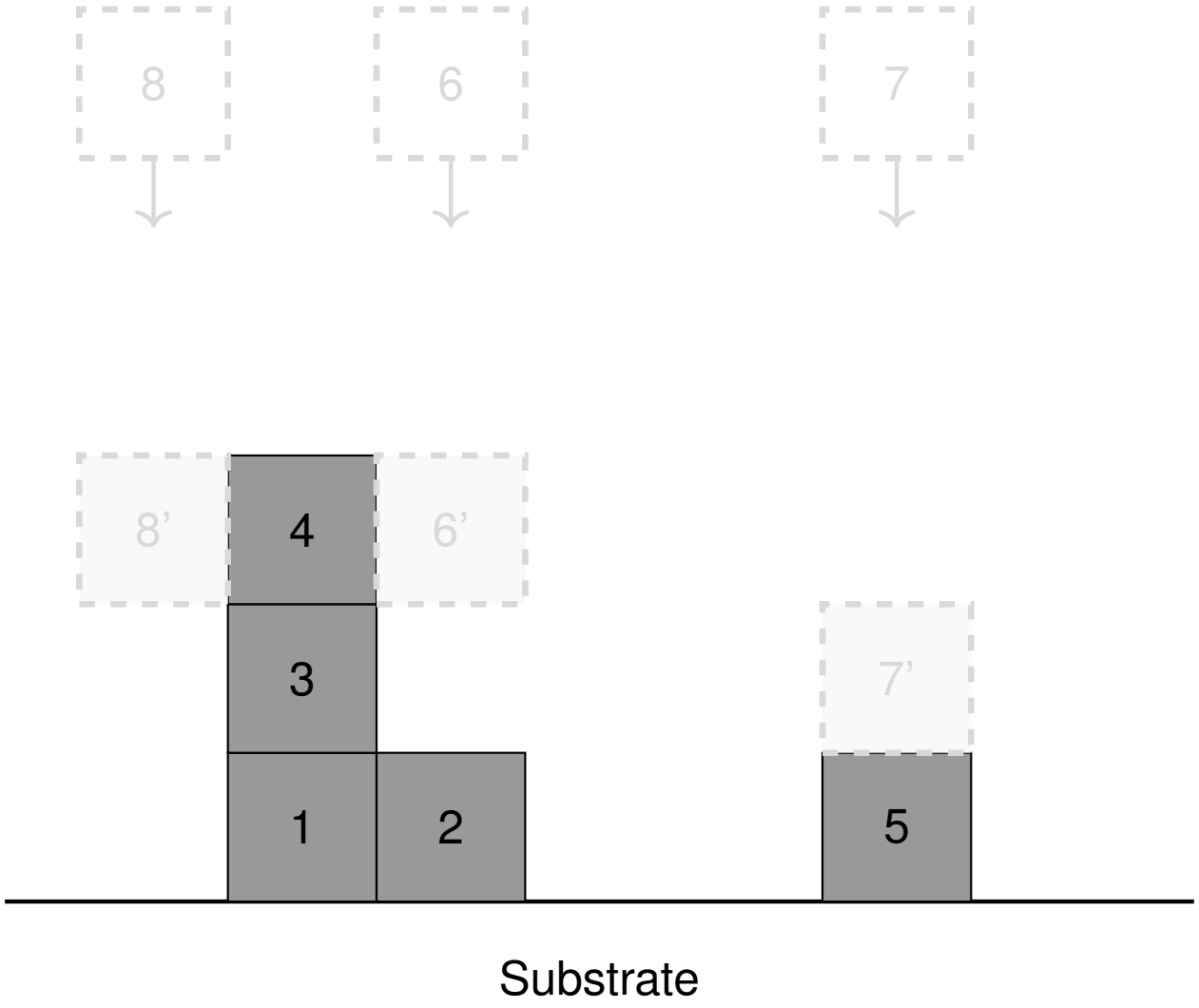


Ballistic deposition



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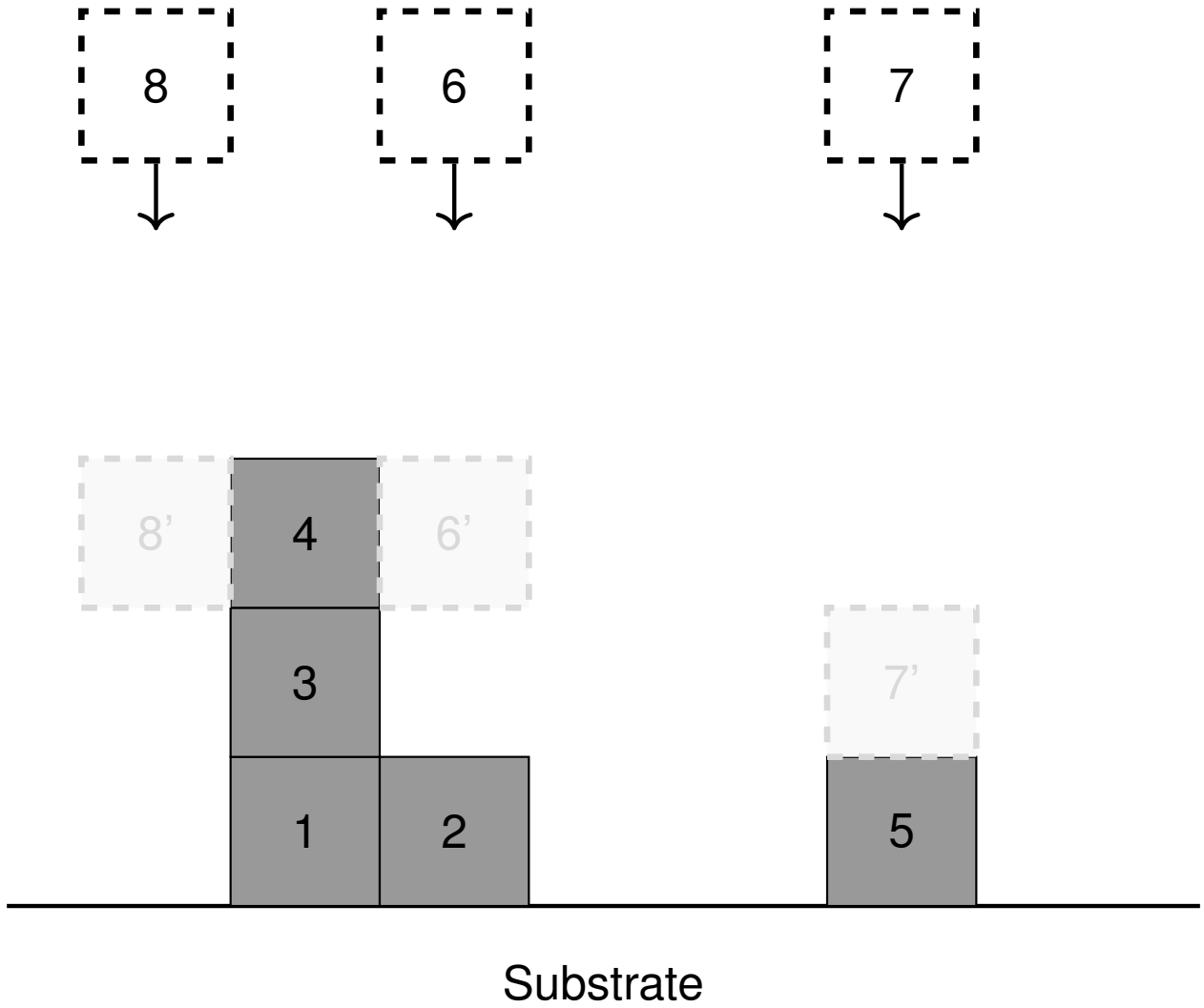

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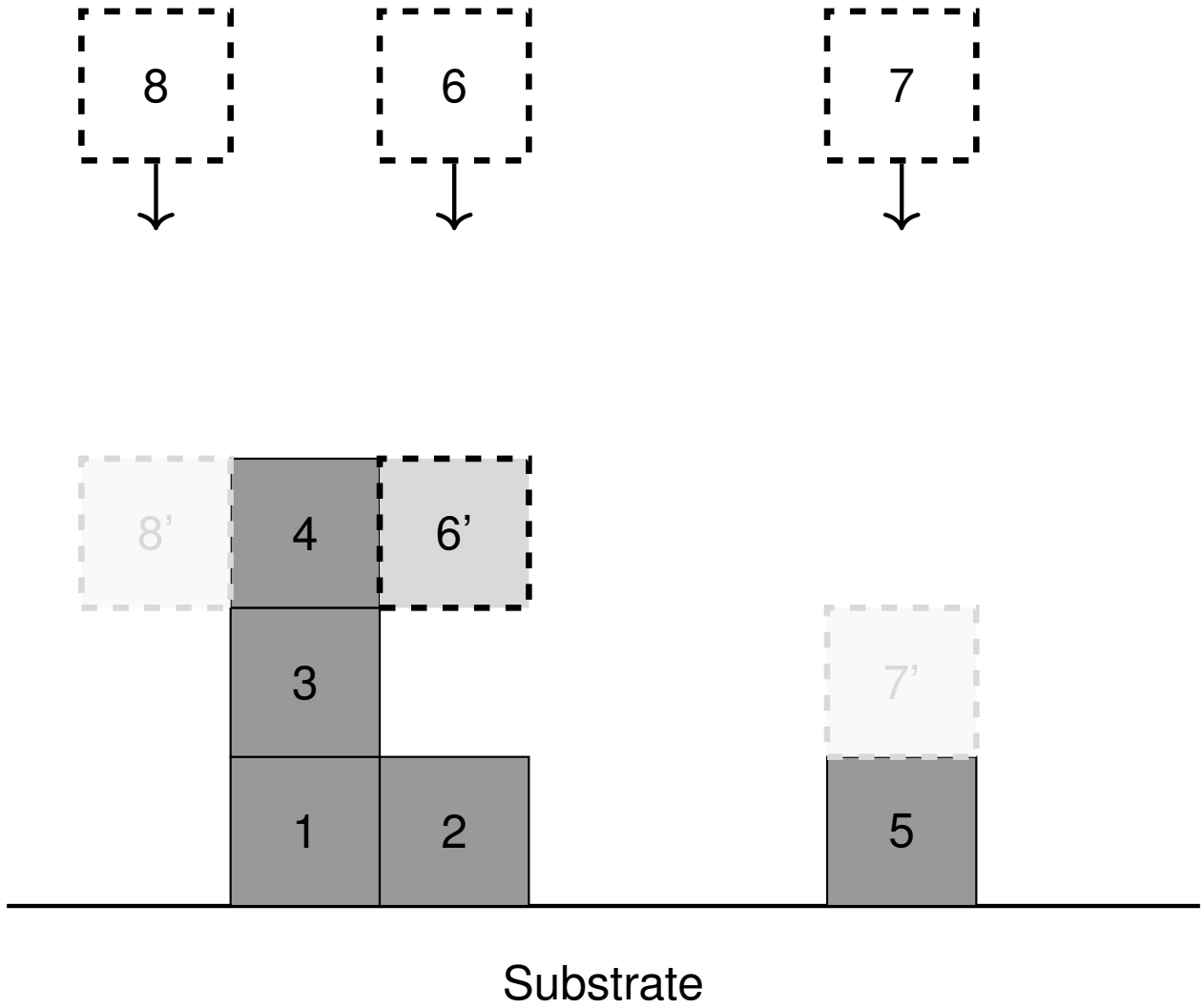

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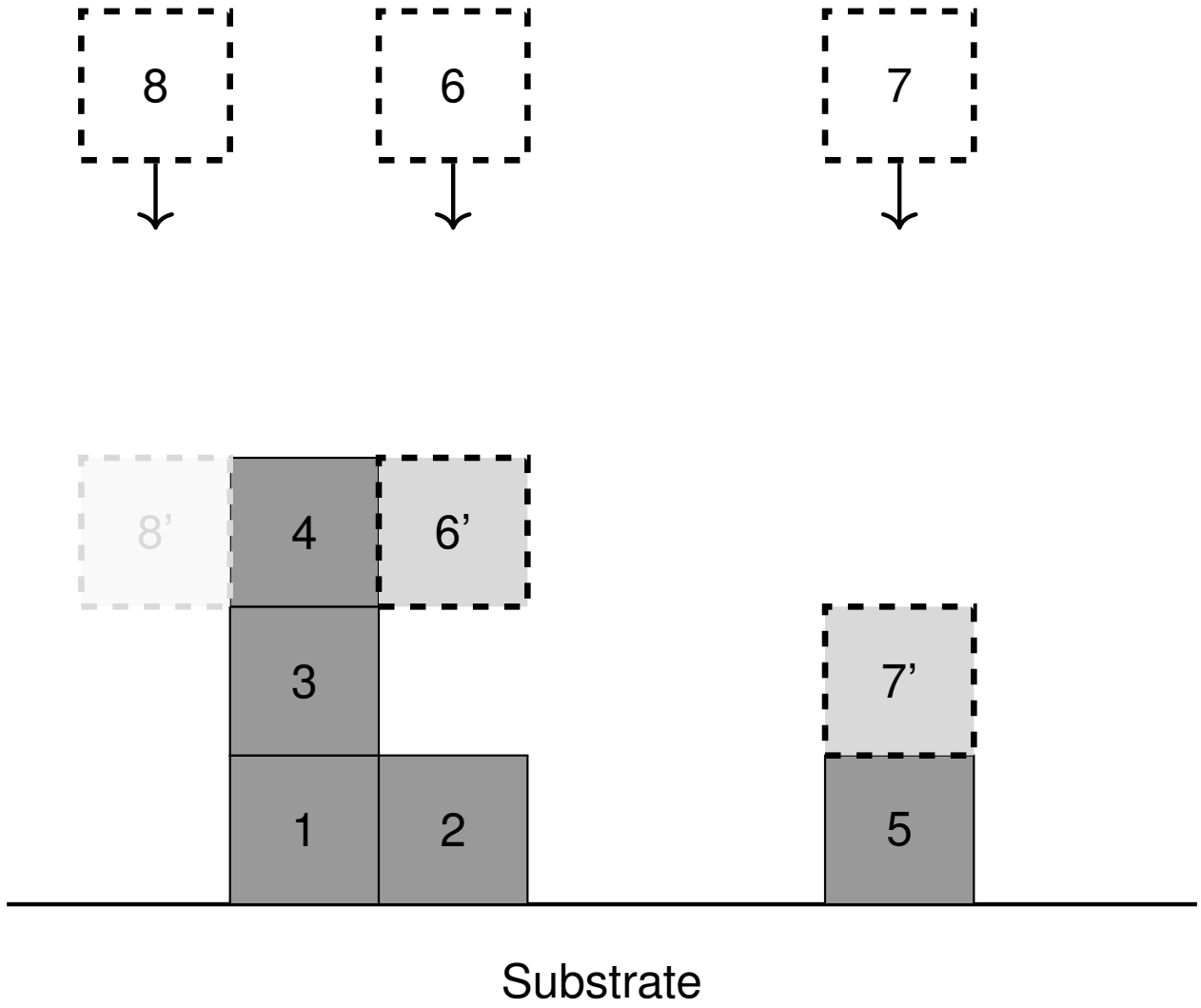

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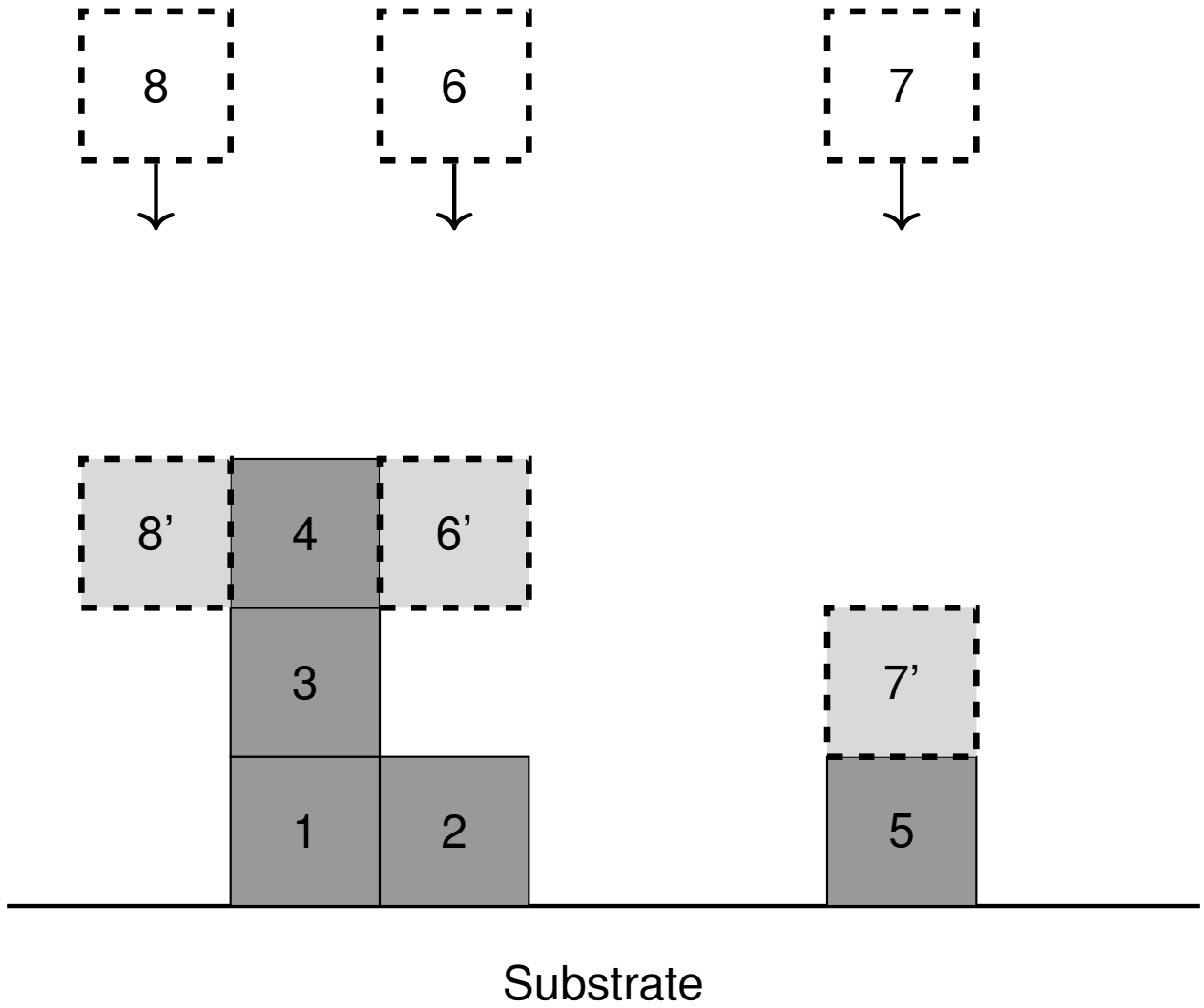
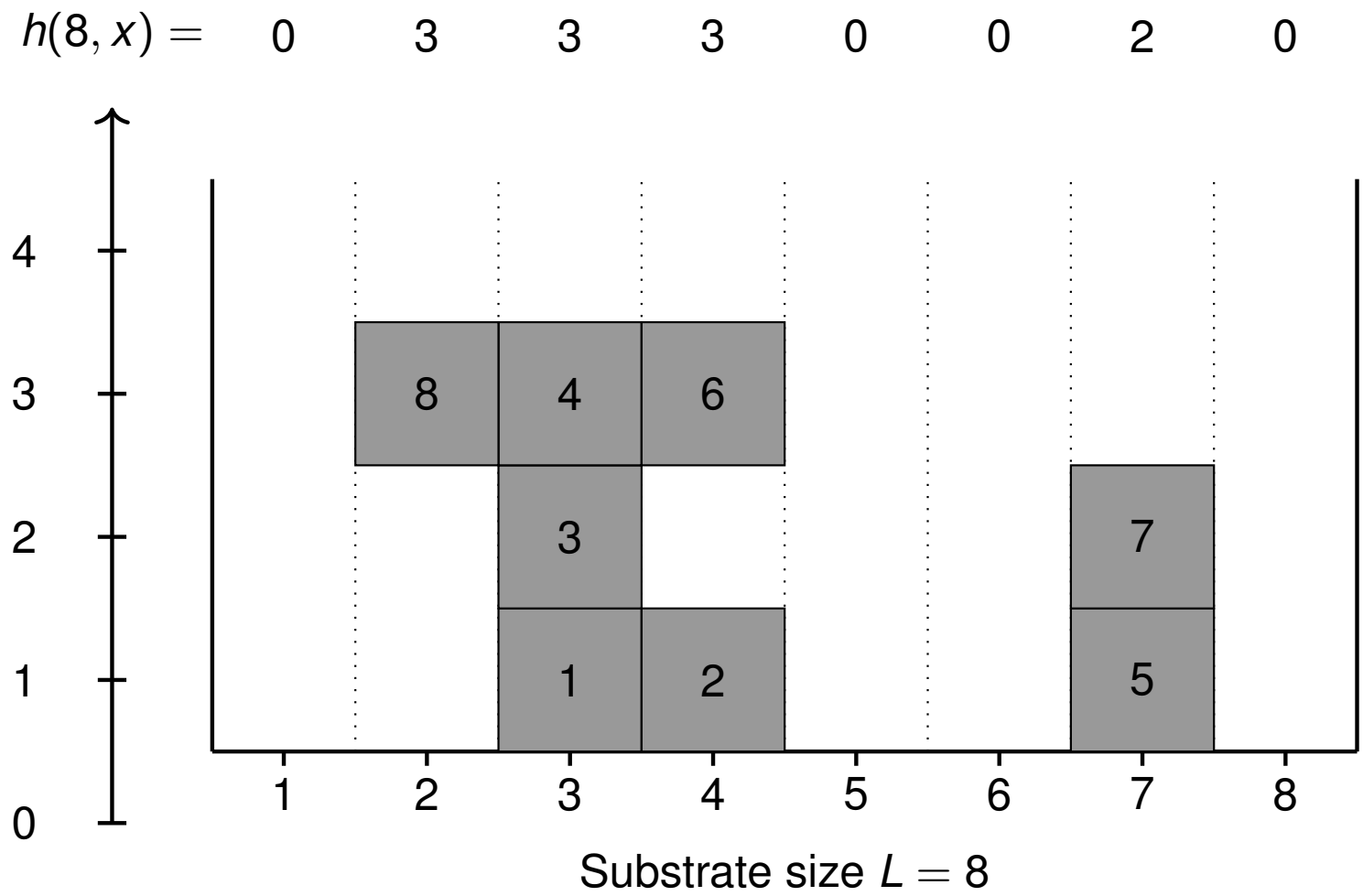
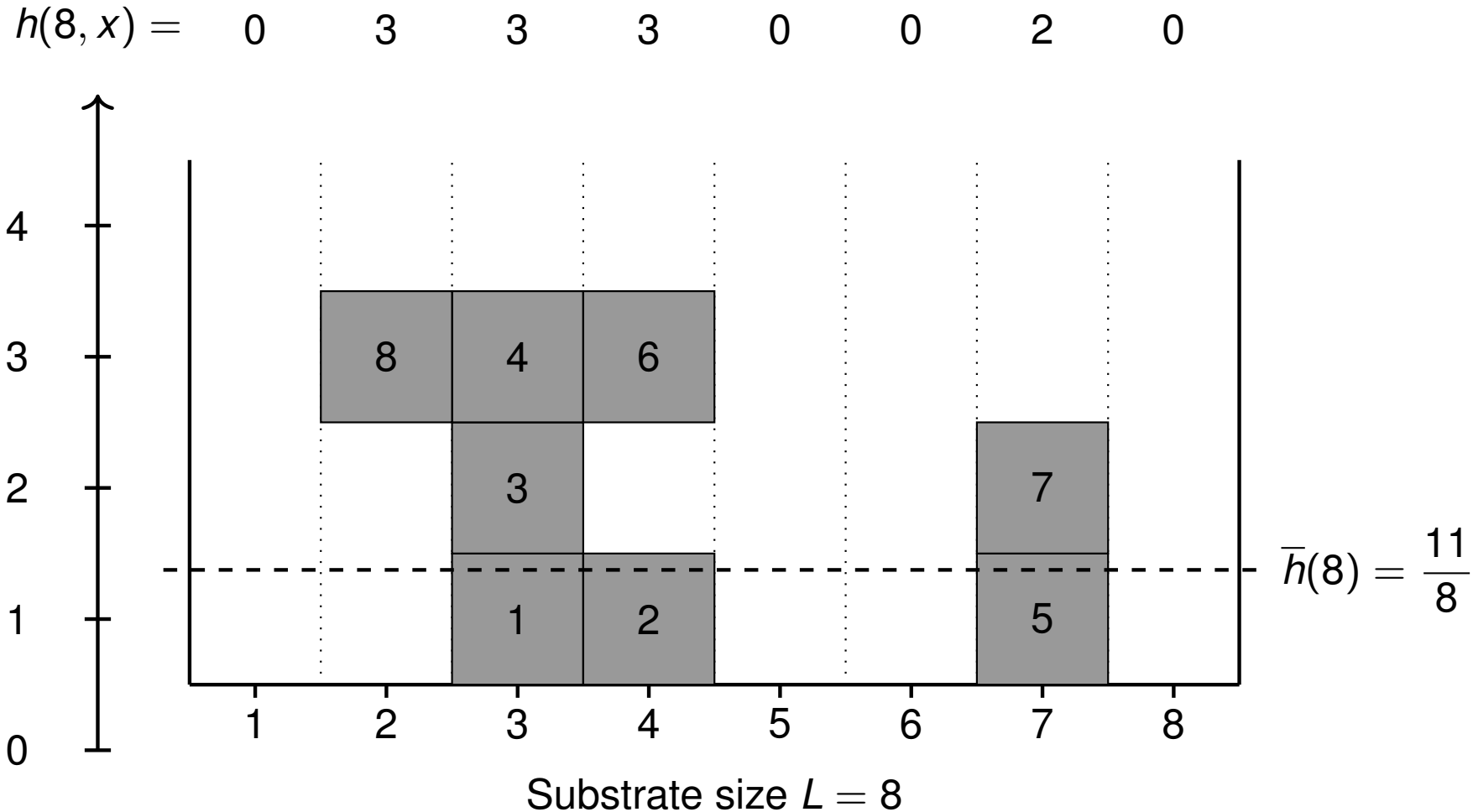


Average height and fluctuation



Average height and fluctuation

$$\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t,x) \qquad \text{Fluctuation } W(L,t) = \sqrt{\frac{1}{L}\sum_{x=1}^{L}\left[h(t,x) - \overline{h}(t)\right]^2}$$



Average height and fluctuation

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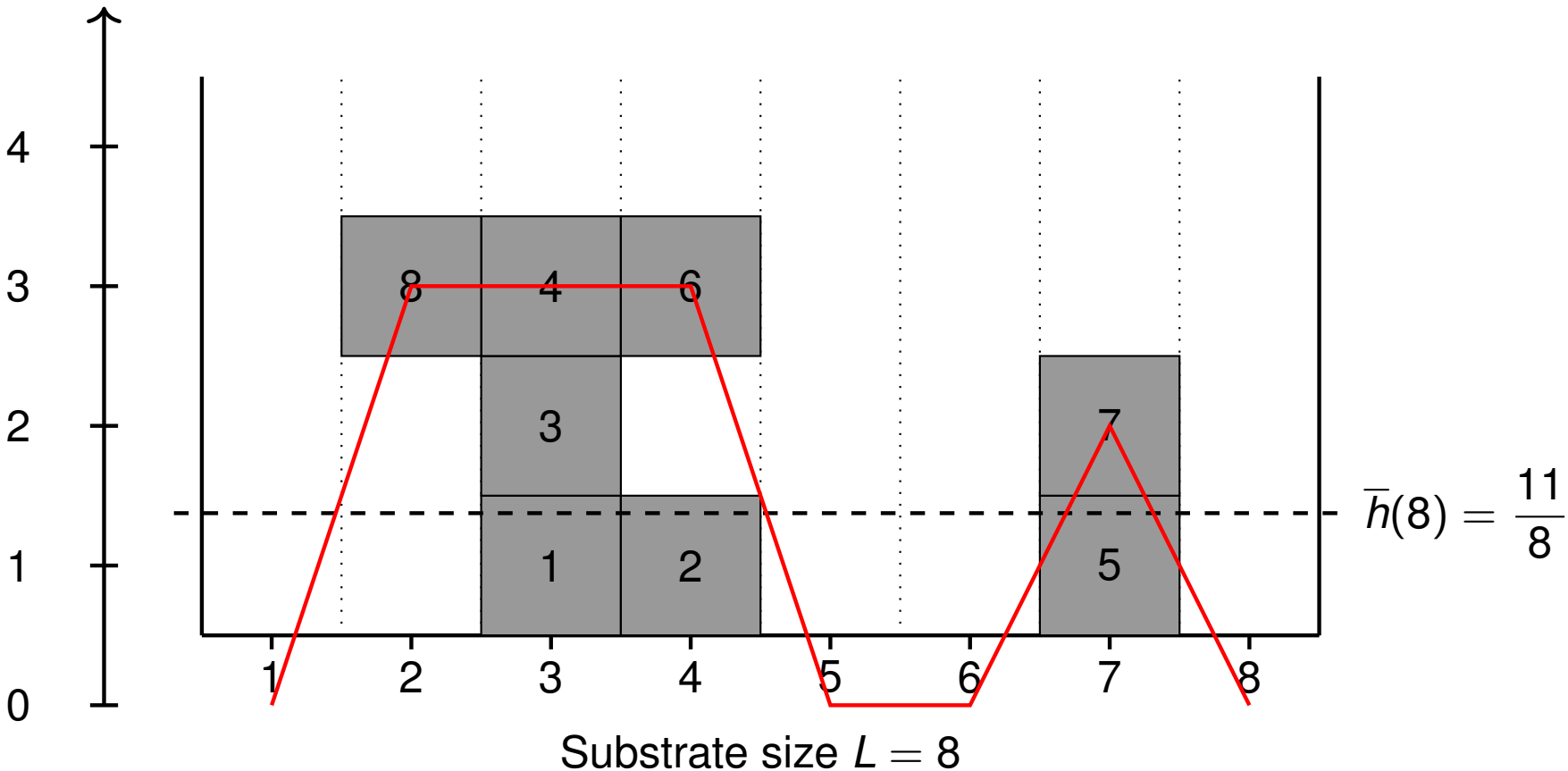
$h(8,x) =$ 0 3 3 3 0 0 2 0



Random Deposition (independent columns)

Model. L independent columns. At each integer time $t = 1, 2, \ldots$, drop *one* particle on a uniformly random column. Heights $h(t, x)$, mean $\overline{h}(t) = \frac{1}{L}\sum_{x=1}^{L} h(t, x) = \frac{t}{L}$, width

$$W^2(L, t) = \frac{1}{L}\sum_{x=1}^{L} \left(h(t, x) - \overline{h}(t)\right)^2.$$

Single–column law. After t drops total,

$$h(t, x) \sim \text{Binomial}\left(t, \frac{1}{L}\right), \qquad \mathbb{E}[h(t, x)] = \frac{t}{L}, \quad \text{Var}(h(t, x)) = t\,\frac{1}{L}\left(1 - \frac{1}{L}\right).$$

Width (variance decomposition). By i.i.d. columns,

$$\mathbb{E}\left[W^2(L, t)\right] = \frac{1}{L}\sum_{x=1}^{L} \mathbb{E}\left[h(t, x)^2\right] - \mathbb{E}\left[\overline{h}^2(t)\right] = \mathbb{E}\left[h(t, 1)^2\right] - \left(\frac{t}{L}\right)^2 = \left(1 - \frac{1}{L}\right)\text{Var}\big(h(t, 1)\big).$$

Hence

$$\boxed{\mathbb{E}\left[W^2(L, t)\right] = \left(1 - \frac{1}{L}\right) t\,\frac{1}{L}\left(1 - \frac{1}{L}\right) = \frac{t}{L}\left(1 - \frac{1}{L}\right)^2}$$

and

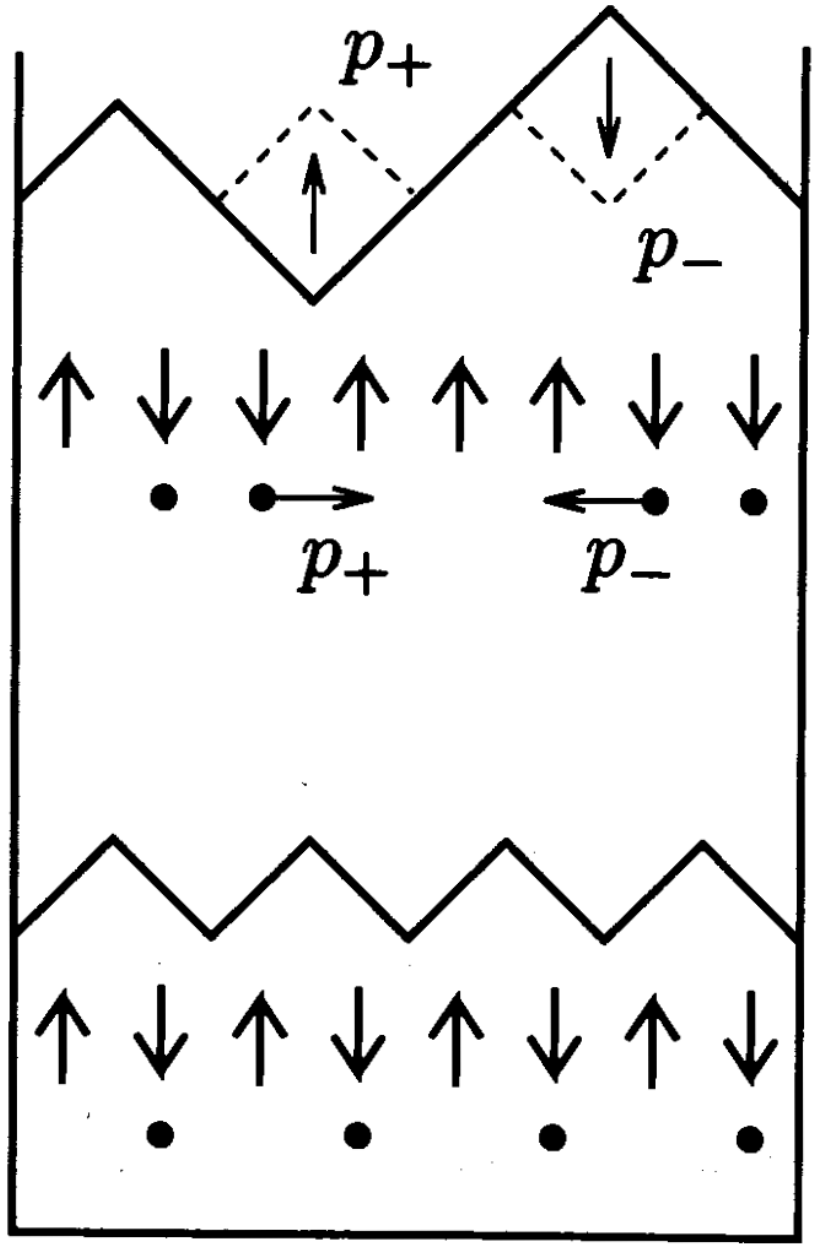
$$\boxed{W(L, t) \;\simeq\; \left(1 - \frac{1}{L}\right)\,\left(\frac{t}{L}\right)^{1/2}}$$

Scaling. Growth exponent $\beta = \dfrac{1}{2}$.

Simulations on

Random deposition vs. Ballistic decomposition

More models? Even more simpler?

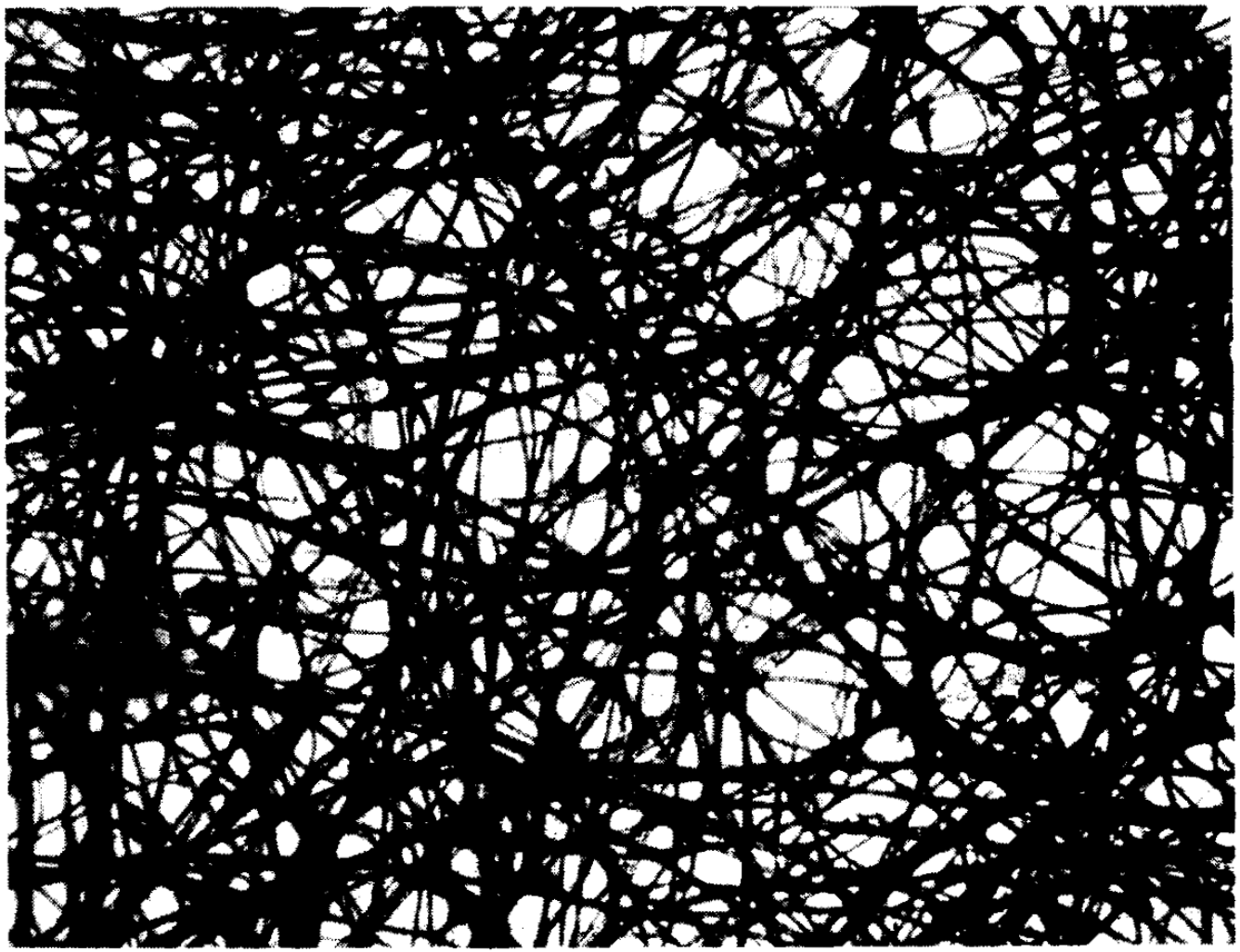


Solid-on-Solid

Ising

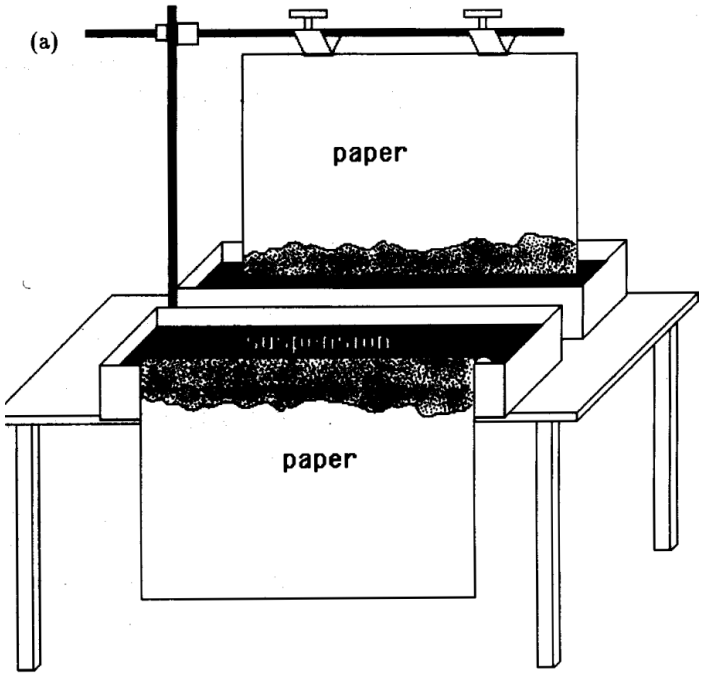
Lattice Gas

Paper – a random environment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper wetting experiment



Barabási, A.-L., Stanley, H. E., 1995

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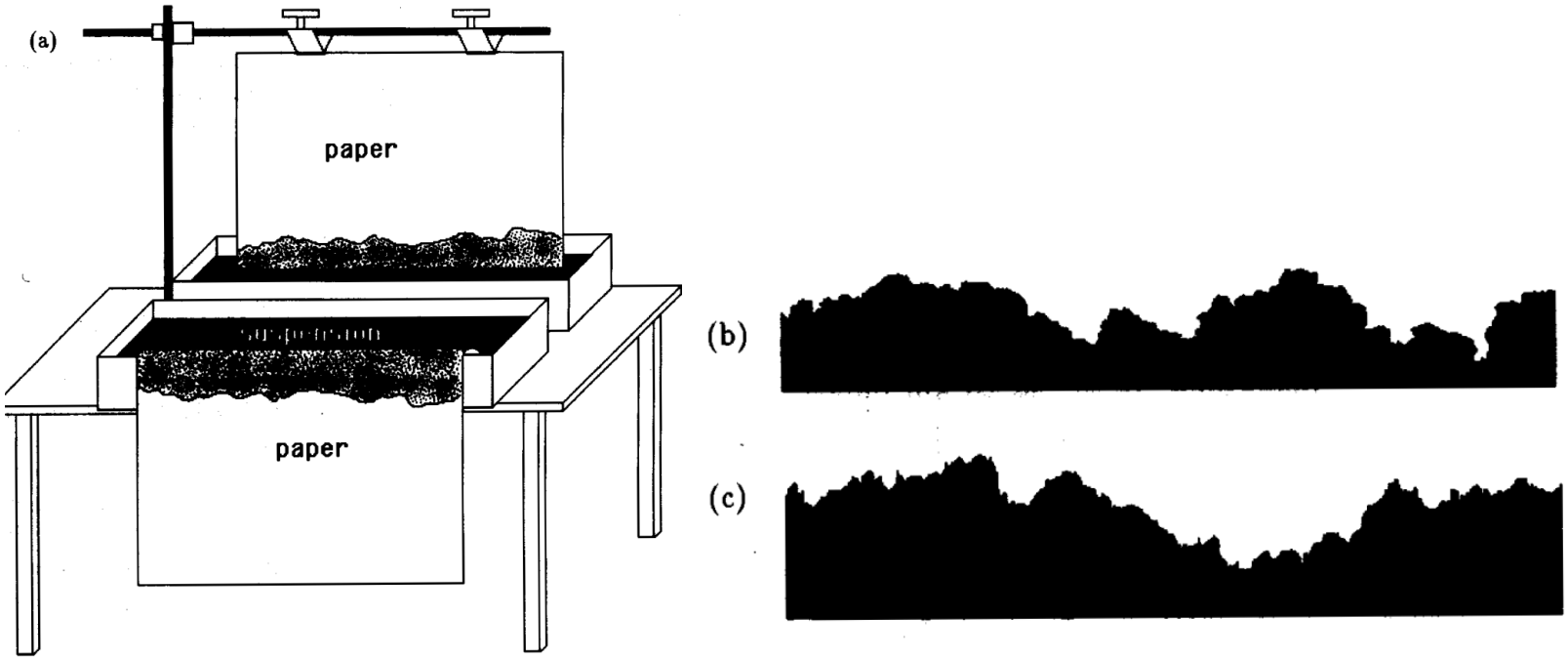

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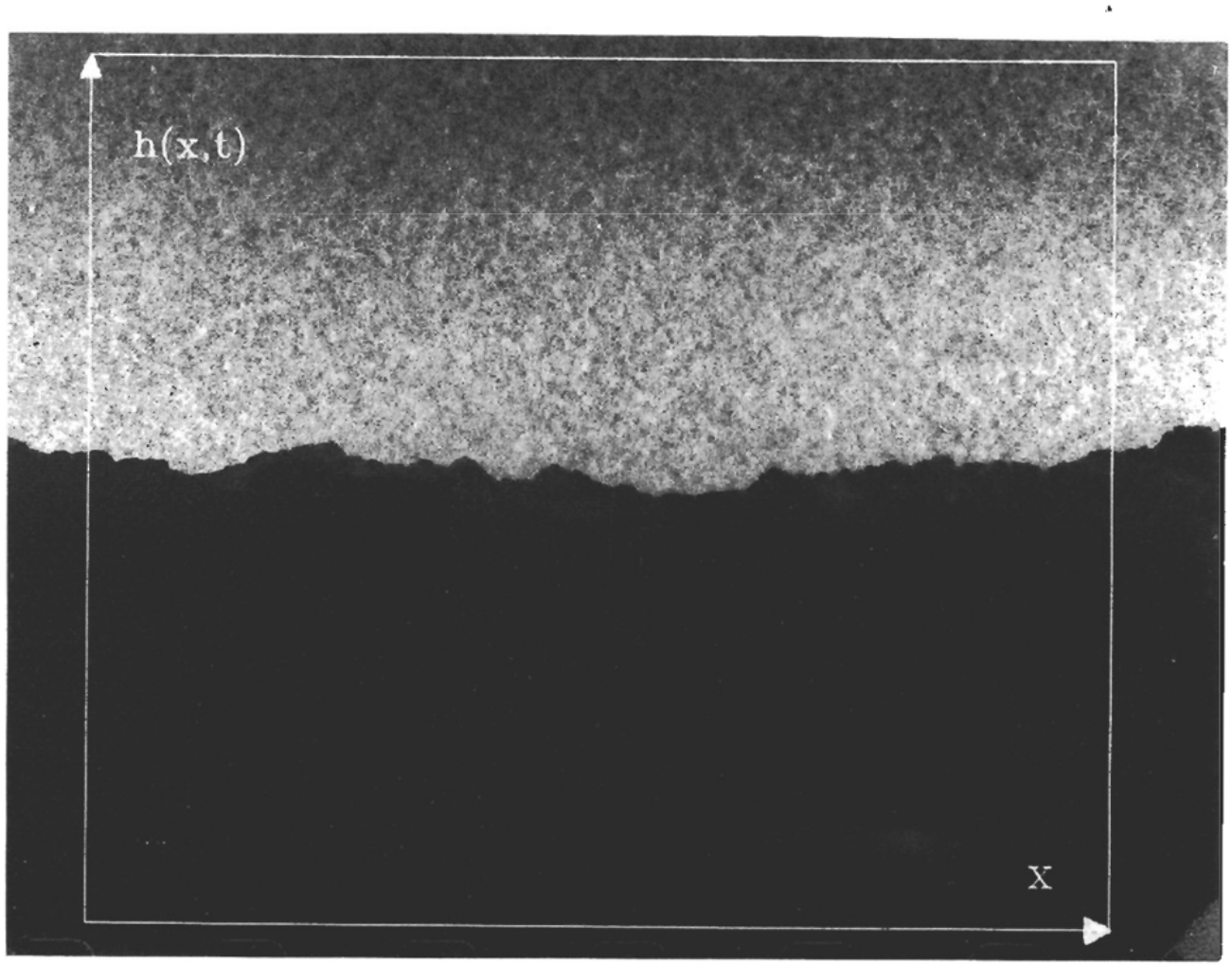
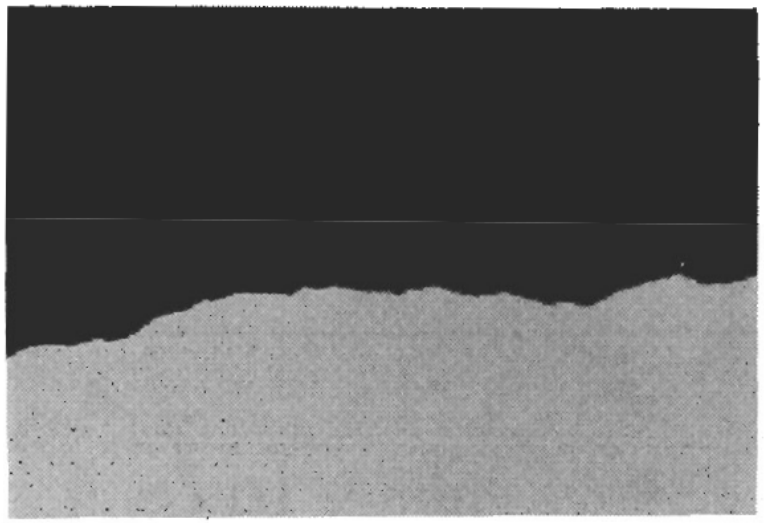
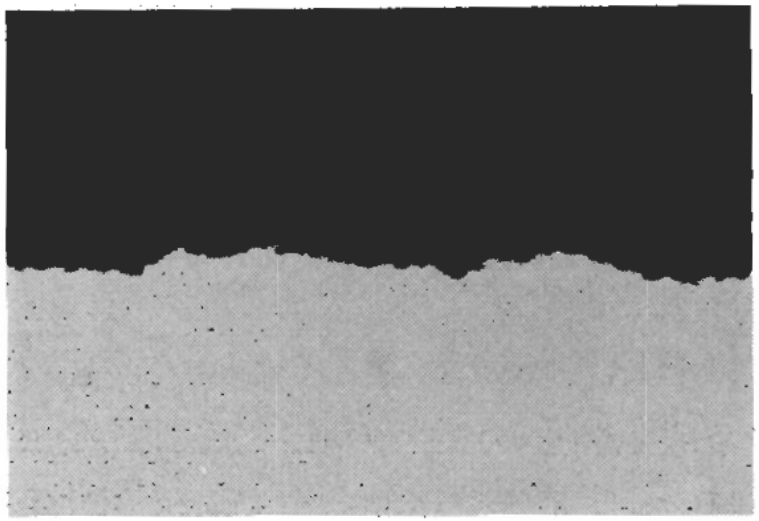
Paper burning experiment



Zhang, J., Zhang, Y.-C., Alstrøm, P., Levinsen, M., *Phys. A: Stat. Mech. Appl.*, 1992

Paper rupture experiment





Kertész, J., Horváth, V. k., Weber, F., *Fractals*, 1993

Study of growing interfaces in a thin film

— Convection of nematic liquid crystal*

Show movies !

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

Study of growing interfaces in a thin film

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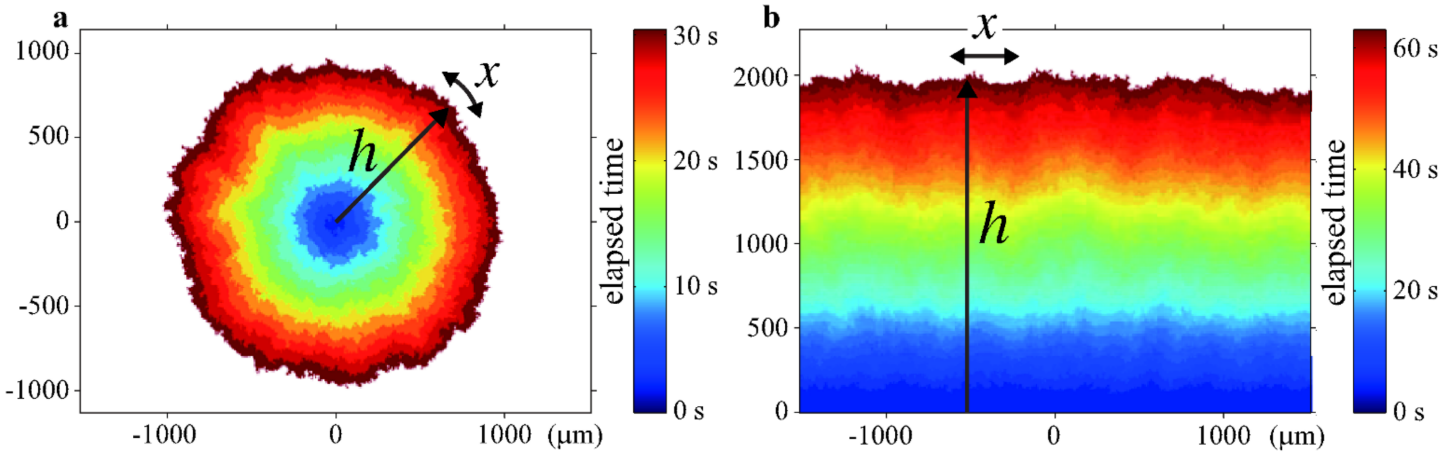

Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

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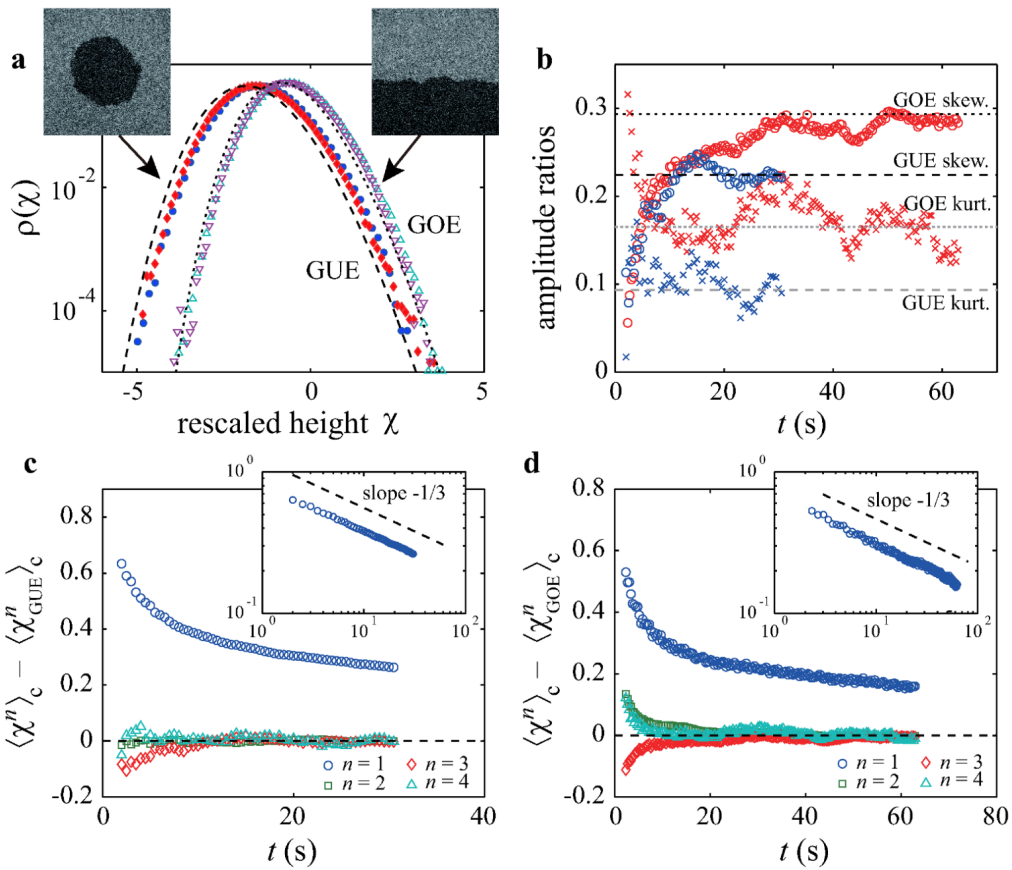


Takeuchi, K. A., Sano, M., Sasamoto, T., Spohn, H., *Sci. Rep.*, 2011

KPZ Equation '86

$$\frac{\partial}{\partial t} h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{\lambda}{2}(\nabla h)^2 + \dot{W}(t,x) \qquad \text{(KPZ)}$$

Mehran Kardar (1957 –)

Giorgio Parisi (1948 –)

Yicheng Zhang

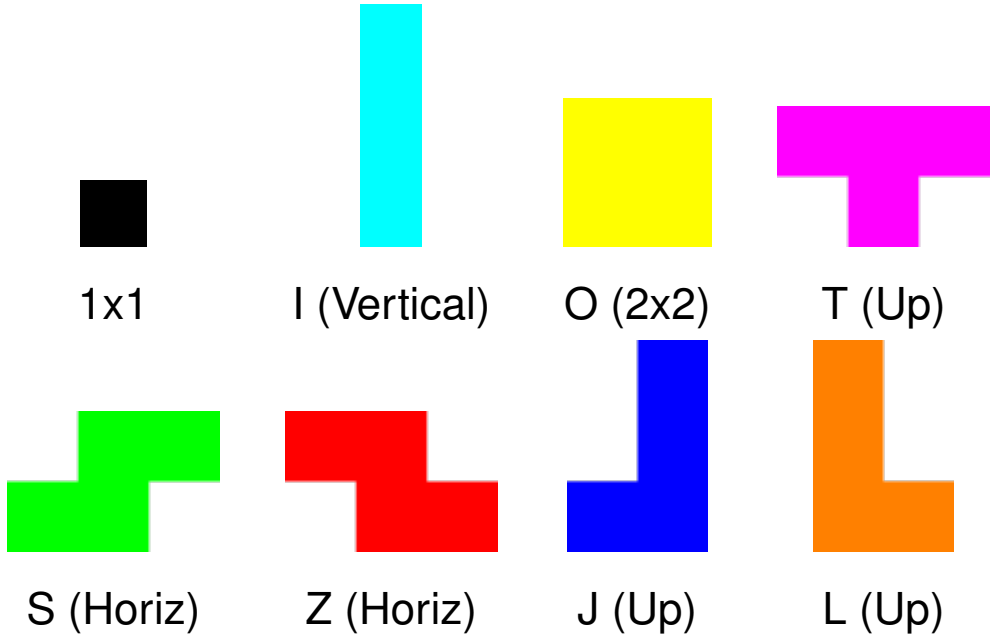
Kardar, M., Parisi, G., Zhang, Y.-C., *Phys. Rev. Lett.*, 1986

Plan

Introduction to growth model and SPDE

Tetromino Pieces

Tetrominoes

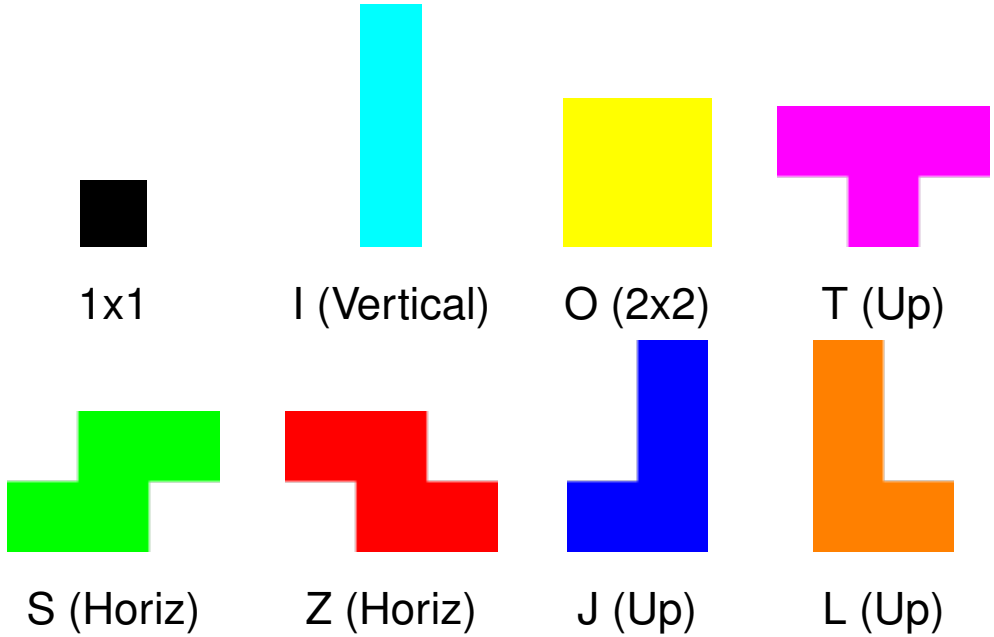


- "1x1": Single (extra single-site particle)
- "I": Horizontal, Vertical
- "J, L, T": Up, Right, Down, Left
- "S, Z": Horizontal, Vertical
- "O": Single (2x2 square)

- Sticky
- Nonstikcy

$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

Tetrominoes



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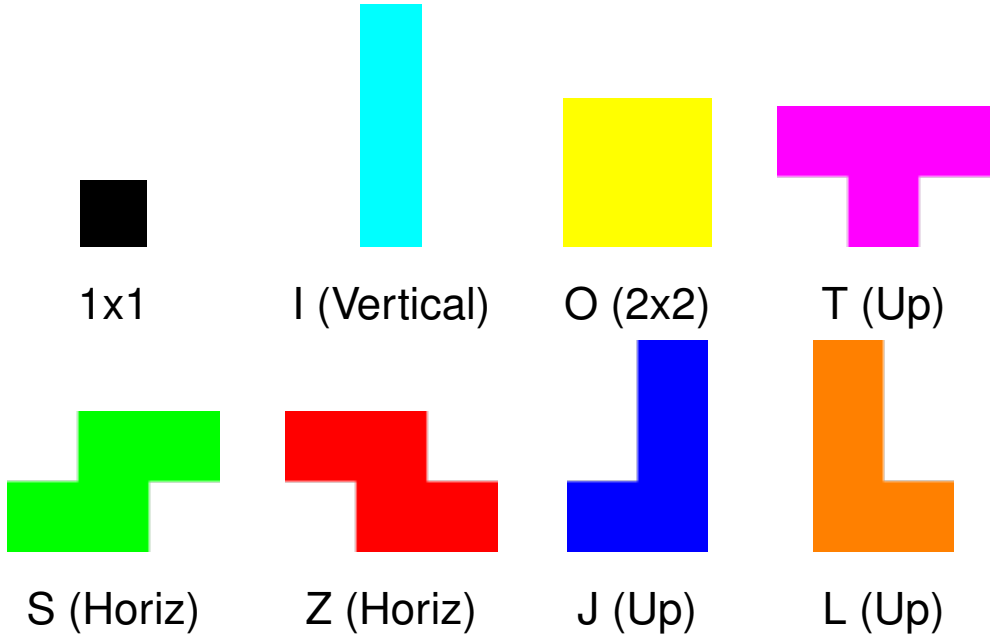

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- Sticky
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$$(1 + 1 \times 2 + 3 \times 4 + 2 \times 2 + 1) \times 2 = 20 \times 2 = 40 \text{ types of pieces}$$

Configure files

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [20, 0]
Piece-01: [20, 0]
Piece-02: [20, 0]
Piece-03: [20, 0]
Piece-04: [20, 0]
Piece-05: [20, 0]
Piece-06: [20, 0]
Piece-07: [20, 0]
Piece-08: [20, 0]
Piece-09: [20, 0]
Piece-10: [20, 0]
Piece-11: [20, 0]
Piece-12: [20, 0]
Piece-13: [20, 0]
Piece-14: [20, 0]
Piece-15: [20, 0]
Piece-16: [20, 0]
Piece-17: [20, 0]
Piece-18: [20, 0]
Piece-19: [20, 0]
```

All nonsticky pieces with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 20]
Piece-01: [0, 20]
Piece-02: [0, 20]
Piece-03: [0, 20]
Piece-04: [0, 20]
Piece-05: [0, 20]
Piece-06: [0, 20]
Piece-07: [0, 20]
Piece-08: [0, 20]
Piece-09: [0, 20]
Piece-10: [0, 20]
Piece-11: [0, 20]
Piece-12: [0, 20]
Piece-13: [0, 20]
Piece-14: [0, 20]
Piece-15: [0, 20]
Piece-16: [0, 20]
Piece-17: [0, 20]
Piece-18: [0, 20]
Piece-19: [0, 20]
```

All sticky pieces with equal prob.

```
steps: 12000
width: 100
height: 300
seed: 12
Piece-00: [0, 0]
Piece-01: [0, 0]
Piece-02: [0, 0]
Piece-03: [0, 0]
Piece-04: [0, 0]
Piece-05: [0, 0]
Piece-06: [0, 0]
Piece-07: [0, 0]
Piece-08: [0, 0]
Piece-09: [0, 0]
Piece-10: [0, 0]
Piece-11: [0, 0]
Piece-12: [0, 0]
Piece-13: [0, 0]
Piece-14: [0, 0]
Piece-15: [0, 0]
Piece-16: [0, 0]
Piece-17: [0, 0]
Piece-18: [0, 0]
Piece-19: [20, 80]
```

20% nonsticky + 80% sticky of 1x1 piece

Main References:

Barabási, A.-L., & Stanley, H. E. (1995). *Fractal concepts in surface growth*. Cambridge University Press, Cambridge.

Family, F., & Vicsek, T. (1985).Scaling of the active zone in the eden process on percolation networks and the ballistic deposition model. *Journal of Physics A: Mathematical and General*, *18*(2), L75.

Kardar, M., Parisi, G., & Zhang, Y.-C. (1986).Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, *56*(9), 889.

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Takeuchi, K. A., Sano, M., Sasamoto, T., & Spohn, H. (2011).Growing interfaces uncover universal fluctuations behind scale invariance. *Sci. Rep.*, *1*(1), 1–5.

Zhang, J., Zhang, Y.-C., Alstrøm, P., & Levinsen, M. (1992).Modeling forest fire by a paper-burning experiment, a realization of the interface growth mechanism. *Phys. A: Stat. Mech. Appl.*, *189*(3), 383–389.

Thank you!

Questions?