Intermittency properties for a large class of stochastic PDEs driven by fractional space-time noises

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Joint work with Yaozhong Hu

AMS Fall Southeastern Virtual Sectional Meeting
Special Sessions on Stochastic Analysis

2021-11-20

Overview

We consider the following stochastic partial differential equation (Anderson models) in the whole d-dimensional Eulcidean space $\mathbb{R}^d$:

$$\mathscr{L} u(t,x) = u(t,x)\dot{W}(t,x)\,, \qquad t>0,\ x\in\mathbb{R}^d \tag{1}$$

with some given initial condition(s). Here $\mathscr{L}$ denotes a general (including fractional order) partial differential operator.

The examples include $\mathscr{L} = \partial_t - \frac{1}{2}\Delta$ (heat operator), $\mathscr{L} = \partial_t^2 - \Delta$ (wave operator), $\mathscr{L} = \partial_t - (-\nabla(A(x)\nabla))^{\alpha/2}$ (α-heat operator), and $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$ (fractional diffusion operator), etc.

Intro to SPDEs

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Mild solution

By Duhamel's principle, the solution to (1) is given explicitly by the mild solution form

$$u(t,x) = \underbrace{I_0(t,x)}_{=1} + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x,y)u(s,y)W(ds,dy)\,, \tag{2}$$

where the term $I_0(t,x)$ depends on the initial data and $G_t(x,y)$ is the Green's function associated with $\mathscr{L}$.

The Green's function and Fourier transform in spatial variable of heat operator are respectively:

$$G_t^{\mathrm{h}}(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right) \text{ and } \mathcal{F}[G_t^{\mathrm{h}}(\cdot)](\xi) = \exp\left(-\frac{t|\xi|^2}{2}\right). \quad (3)$$

And for the wave kernel

$$\begin{cases} G_t^{\mathrm{w}}(x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}, & d=1, \\ G_t^{\mathrm{w}}(x) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}\mathbf{1}_{\{|x|<t\}}, & d=2, \\ G_t^{\mathrm{w}}(dx) = \frac{1}{4\pi}\frac{\sigma_t(dx)}{t}, & d=3, \end{cases} \text{ and } \mathcal{F}[G_t^{\mathrm{w}}(\cdot)](\xi) = \frac{\sin(t|\xi|)}{|\xi|}. \quad (4)$$

For the Green kernels of α-heat operator $\mathscr{L} = \partial_t - (-\nabla(A(x)\nabla))^{\alpha/2}$, and fractional diffusion operator $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$, we refer [9, Section 7] and reference therein.

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Some known results

There there is a vast literature on the topic of intermittency properties or upper/lower p-th moments. We mainly mention these driven by fractional Gaussian noises with temporal and spatial Hurst parameters are greater than or equal to one half.

1 Under the condition $2\gamma + \lambda < 2$, it is possible (see [5]) to compute the exact Lyapunov exponent for the solution of PAM with stochastic integral is interpreted in the Stratonivich sense:

$$\lim_{t\to\infty} t^{-\frac{4-\lambda-2\gamma}{2-\lambda}} \log \mathbb{E}[|u(t,x)|^p] = C \cdot p^{\frac{4-\lambda}{2-\lambda}} .$$

The same result and its extension to α-heat operator for Skorohod integral is proved in [6] by X. Chen, Y. Hu, J. Song and X. Song. Their approach highly depends on Feynman-Kac formula which is no longer applicable in general.

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2 Under Dalang's condition, R. Balan and D. Conus in [2] obtained the upper p-th moment bounds of PAM and HAM with stochastic integral is interpreted in the Skorohod sense:

$$\begin{cases} \mathbb{E}[|u^h(t,x)|^p] \leq C_1 \exp\left(C_2 \cdot t^{\frac{4-\lambda-2\gamma}{2-\lambda}} p^{\frac{4-\lambda}{2-\lambda}}\right) , \\ \mathbb{E}[|u^w(t,x)|^p] \leq C_1 \exp\left(C_2 \cdot t^{\frac{4-\lambda-\gamma}{3-\lambda}} p^{\frac{4-\lambda}{3-\lambda}}\right) . \end{cases}$$

3 For the fractional diffusion $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$, under a modified Dalang's condition, L. Chen, G. Hu, Y. Hu and J. Huang in [4] obtained the upper p-th moment bound:

$$\mathbb{E}[|u(t,x)|^p] \leq C_1 \exp\left(C_2 \cdot t^{1+\frac{\alpha(1-\gamma)}{2\alpha\beta-\alpha-\beta\lambda}} \cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}\right) .$$

4 For the time independent noise, there are some very impressive recent work [1] by R. Balan, L. Chen and X. Chen, and [3] by L. Chen and N. Eisenberg, which can be compared with the case $\gamma = 2 - 2H_0 = 0$.

5 For the (mixed) Dobric-Ojeda noise, D. Conus, R. Qiu and M. Wildman obtained the intermittency properties of SPDEs including the stochastic heat and wave equations. For the Lévy noises, Q.Berger, C. Chong and H. Lacoin obtained some results on intermittency.

6 There are still many excellent papers $\cdots$

The noises

We specially focus on what so called Gaussian noises

$$\dot{W}(t,x) = \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} W(t,x)$$

is mean zero Gaussian with the following covariance structure:

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \gamma(t-s)\Lambda(x-y)\,. \tag{5}$$

Assumptions on the noises

For $\gamma(\cdot)$ we assume

(H1) There is a $\gamma \in (0,1)$ such that

$$c|t|^{-\gamma} \le \gamma(t) \le C|t|^{-\gamma}, \quad \forall\ t \in \mathbb{R}_+$$

for some positive constants c, C. For convenience, when $\gamma = 1$ we mean $\gamma(t) = \delta(t)$.

For $\Lambda(\cdot)$ we assume that it satisfies one of the following three conditions:

(H2) There is $\lambda \in (0, d)$ such that

$$c|x|^{-\lambda} \leq \Lambda(x) \leq C|x|^{-\lambda}, \quad \forall\, x \in \mathbb{R}^d.$$

(H3) There are constants $\lambda_j \in (0,1), j = 1, \cdots, d$ such that

$$c\prod_{j=1}^{d} |x_j|^{-\lambda_j} \leq \Lambda(x) \leq C\prod_{j=1}^{d} |x_j|^{-\lambda_j}, \quad \forall\, x \in \mathbb{R}^d.$$

In this case we denote $\lambda = \sum_{i=1}^{d} \lambda_i$.

(H4) When $d = 1$ and $\gamma = 1$, we assume $\Lambda(x) = \delta(x)$.

Assumptions on the Green kernel

(G1) **[Positivity]**: $G_t(\cdot,\cdot)$ is a positive function, measure, or generalized function.

(G2) **[Small ball non-degeneracy]**: $G_t(\cdot,\cdot)$ satisfies the *small ball non-degeneracy (B(*a, b*))*. This is, there exist real numbers a and b (depending on the Green's function) satisfying

$$\mathrm{a} > -1\,, \quad \mathrm{b} > 0\,, \quad \text{and} \quad \mathrm{b}(2\mathrm{a}+1) - \lambda > 0\,, \tag{6}$$

and there is a constant $C > 0$ such that

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G_t(y,z)dz \geq C \cdot t^{\mathrm{a}}\,, \tag{7}$$

for all $0 < t \leq \varepsilon^{\mathrm{b}} (\leq 1)$ and $x \in \mathbb{R}^d$, where $B_\varepsilon(x)$ is the ball of center x with radius ε.

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To obtain the upper bound for moments, we need is the following hypothesis for the Green's function.

(G3) [**HLS-type mass property**]: $G_t(\cdot,\cdot)$ satisfies what we shall call the *Hardy-Littelewood-Sobolev type mass property* $M(\hbar)$. That is, there exist a real number $\hbar$ and a constant $C > 0$ satisfying

$$\hbar > -1\,, \tag{8}$$

and

$$\sup_{x,x'\in\mathbb{R}^d} \int_{\mathbb{R}^{2d}} G_t(x,y)\Lambda(y-y')G_t(x',y')dydy' \leq C\cdot t^{\hbar}\,. \tag{9}$$

Recall the heat kernel $G^{\mathrm{h}}_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right)$.

(G1) It is obvious heat kernel is positive.

(G2) There exist some strict positive constants C_1 and C_2 independent of t, x and ε such that

$$\inf_{y \in B_\varepsilon(x)} \int_{B_\varepsilon(x)} G^{\mathrm{h}}_t(y-z)dz \geq C_1 \exp\left(-C_2 \frac{t}{\varepsilon^2}\right).$$

This implies Small ball non-degeneracy $B(0,2)$.

(G3) We have

$$\begin{cases} \int_{\mathbb{R}} G^{\mathrm{h}}_t(x)dx = 1\,, \\ \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} G^{\mathrm{h}}_t(x-y)\Lambda(x)dx \leq C \cdot t^{-\frac{\lambda}{2}}\,. \end{cases} \quad \Rightarrow \text{HLS mass } M(-\lambda/2)\,.$$

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Our results

Theorem 1 (Theorem 3.7 in [9], Hu-Wang 2021)

Under the assumptions (H) on the noises and (G) on the Green kernels, then the mild solution $u(t,x)$ satisfies

$$c_1 \exp\left(c_2 \cdot t^{1+\frac{b\cdot(1-\gamma)}{b(2a+1)-\lambda}} \cdot p^{1+\frac{b}{b(2a+1)-\lambda}}\right) \leq \mathbb{E}\left[|u(t,x)|^p\right] \leq C_1 \exp\left(C_2 \cdot t^{1+\frac{b\cdot(1-\gamma)}{b(2a+1)-\lambda}} \cdot p^{1+\frac{b}{b(2a+1)-\lambda}}\right). \tag{10}$$

These results of $\mathscr{L} = \partial_t - \frac{1}{2}\Delta$ (heat operator), $\mathscr{L} = \partial_t^2 - \Delta$ (wave operator), $\mathscr{L} = \partial_t - (-\nabla(A(x)\nabla))^{\alpha/2}$ (α-heat operator), and $\mathscr{L} = \partial_t^\beta - \frac{1}{2}(-\Delta)^{\alpha/2}$ (fractional diffusion operator) are summarized in Table 1 including the exponent parts of (10).

Table: Matching lower and upper moments

SPDEs	(a,b)	$\hbar$	Moment	When $\gamma = 2 - 2H_0$
SHE	*(0,2)*	$-\frac{\lambda}{2}$	$t^{1+\frac{2\cdot(1-\gamma)}{2-\lambda}} \cdot p^{\frac{4-\lambda}{2-\lambda}}$	$t^{\frac{4H_0-\lambda}{2-\lambda}} \cdot p^{\frac{4-\lambda}{2-\lambda}}$
α-SHE	$(0,\alpha)$	$-\frac{\lambda}{\alpha}$	$t^{1+\frac{\alpha\cdot(1-\gamma)}{\alpha-\lambda}} \cdot p^{\frac{2\alpha-\lambda}{\alpha-\lambda}}$	$t^{\frac{2H_0\alpha-\lambda}{\alpha-\lambda}} \cdot p^{\frac{2\alpha-\lambda}{\alpha-\lambda}}$
SWE	*(1,1)*	$2-\lambda$	$t^{1+\frac{1-\gamma}{3-\lambda}} \cdot p^{\frac{4-\lambda}{3-\lambda}}$	$t^{\frac{2H_0+2-\lambda}{3-\lambda}} \cdot p^{\frac{4-\lambda}{3-\lambda}}$
SFD	$(\beta-1, \frac{\alpha}{\beta})$	$2(\beta-1) - \frac{\lambda\beta}{\alpha}$	$t^{1+\frac{\alpha(1-\gamma)}{2\alpha\beta-\alpha-\beta\lambda}} \cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}$	$t^{\frac{\alpha(2\beta+2H_0-2)-\beta\lambda}{2\alpha\beta-\alpha-\beta\lambda}} \cdot p^{\frac{\beta(2\alpha-\lambda)}{2\alpha\beta-\alpha-\beta\lambda}}$

Feynman diagram

Iterating mild form (2) with $\sigma(t,x,u(t,x)) = u(t,x)$, we have the **Wiener chaos expansion**:

$$
\begin{aligned}
u(t,x) =& 1 + \sum_{n=1}^{\infty} I_n(f_n(\cdot,t,x)) \\
=& 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x,y) W(ds,dy) \\
& + \int_0^t \int_0^r \int_{\mathbb{R}^{2d}} G_{t-s}(x,y) G_{s-r}(y,z) W(dr,dz) W(ds,dy) \\
& + \cdots .
\end{aligned} \tag{11}
$$

Here

$$
\begin{aligned}
& f_n(\cdot, t, x) := f_n(t_1, x_1, \cdots, t_n, x_n, t, x) \\
=& \frac{1}{n!} \sum_{\sigma \in S_n} G_{t-t_{\sigma(n)}}(x, x_{\sigma(n)}) G_{t_{\sigma(n)}-t_{\sigma(n-1)}}(x_{\sigma(n)}, x_{\sigma(n-1)}) \cdots \\
& \times G_{t_{\sigma(2)}-t_{\sigma(1)}}(x_{\sigma(2)}, x_{\sigma(1)}) \mathbf{1}_{\{0<t_{\sigma(1)}<\cdots<t_{\sigma(n)}<t\}}
\end{aligned} \tag{12}
$$

is the symmetrization of

$$
\begin{aligned}
& G_{t-t_n}(x, x_n) G_{t_n-t_{n-1}}(x_n, x_{n-1}) \cdots \\
& \times G_{t_2-t_1}(x_2, x_1) \mathbf{1}_{\{0<t_1<\cdots<t_n<t\}} ,
\end{aligned} \tag{13}
$$

where S_n denotes the set of all permutations of $\{1, \ldots, n\}$.

Feynman diagram

A **Feynman diagram** D is a set of some vertices and some edges connecting them so that the vertices are arranged into some finite rows and each row contains some finite many vertices. The set of vertices of the diagram D can then be represented by

$$\mathscr{V}(\mathcal{D}) = \{(k, r) : 1 \le k \le m, 1 \le r \le n_m\}.$$

We use $\mathscr{E}(\mathcal{D}) = \{[(\overline{k}, \overline{r}), (\underline{k}, \underline{r})] : \overline{k} < \underline{k}\}$ denote the set of all edges of a diagram $\mathcal{D}$.

Definition 2

A diagram $\mathcal{D} = (\mathscr{V}(\mathcal{D}), \mathscr{E}(\mathcal{D}))$ *is called* **admissible** *if every vertice is associated with one and only one edge. The set of all* **admissible diagrams** *associated with the vertices* $\{(k,r), 1 \le k \le m, 1 \le r \le n_k\}$ *is denoted by* $\mathbb{D}(n_1, \cdots, n_m)$*. We use* $\mathbb{D}(f_1, \cdots, f_m)$ *to denote the set of all admissible Feynman diagrams associated with* $f_1, \cdots, f_m$*.*

It is clear that if a diagram $\mathcal{D}$ is admissible then $n_1 + \cdots + n_m = 2|\mathscr{E}(\mathcal{D})|$, in particular, $n_1 + \cdots + n_m$ is an even integer.

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Our main tool is the following Feynman diagram formula:

Theorem 3 (Theorem 5.4 in [9], Hu-Wang 2021)

Let $f_n(\cdot, t, x)$ be defined by (12) and let $I_n(f_n(\cdot, t, x))$ be the associated multiple Wiener-Itô integral. Then

$$
\begin{aligned}
&\mathbb{E}\Big[I_{n_1}(f_{n_1}(\cdot, t, x)) \cdots I_{n_m}(f_{n_m}(\cdot, t, x))\Big] \\
&\quad = \sum_{\mathcal{D} \in \mathbb{D}(f_{n_1}, \cdots, f_{n_m})} F_{\mathcal{D}}\left(f_{n_1}, \cdots, f_{n_m}\right) \\
&\quad = \sum_{\mathcal{D} \in \mathbb{D}(f_{n_1}, \cdots, f_{n_m})} \int \prod_{j=1}^{m} \prod_{r=1}^{n_j} G_{t_{(j,r+1)} - t_{(j,r)}}\left(x_{(j,r+1)} - x_{(j,r)}\right) \mathbf{1}_{\{0 < t_{(j,1)} < \cdots < t_{(j,n_j)}\}} \\
&\qquad\qquad \times \gamma\left(t_{\overline{\mathcal{V}}(\mathcal{D})} - t_{\underline{\mathcal{V}}(\mathcal{D})}\right) \Lambda\left(x_{\overline{\mathcal{V}}(\mathcal{D})} - x_{\underline{\mathcal{V}}(\mathcal{D})}\right) dt_{\mathcal{D}}\, dx_{\mathcal{D}}\,,
\end{aligned}
\tag{14}
$$

where we use the notations $t_{(j,n_j+1)} = t$ and $x_{(j,n_j+1)} = x$.

An example of admissible diagram $\mathcal{D} \in \mathbb{D}(4,4)$ can be illustrated in the Figure 1. In this diagram, we have
$T_{\overline{\mathscr{V}}(\mathcal{D})} := (t_{\overline{\mathscr{V}}(\mathcal{D})}, x_{\overline{\mathscr{V}}(\mathcal{D})}) = \{T_j^{(2)} : 1 \le j \le 4\}$ colored in red,
$T_{\underline{\mathscr{V}}(\mathcal{D})} := (t_{\underline{\mathscr{V}}(\mathcal{D})}, x_{\underline{\mathscr{V}}(\mathcal{D})}) = \{T_j^{1} : 1 \le j \le 4\}$ colored in blue. Moreover,

$$\gamma\left(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})}\right) := \gamma\left(t_1^{(2)} - t_3^1\right)\gamma\left(t_2^{(2)} - t_1^1\right)\gamma\left(t_3^{(2)} - t_4^1\right)\gamma\left(t_4^{(2)} - t_2^1\right),$$
$$\Lambda\left(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\right) := \Lambda\left(x_1^{(2)} - x_3^1\right)\Lambda\left(x_2^{(2)} - x_1^1\right)\Lambda\left(x_3^{(2)} - x_4^1\right)\Lambda\left(x_4^{(2)} - x_2^1\right).$$

and $\Lambda\left(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\right)$ is also expressed in the same way. Obviously, there are 4! such diagram.

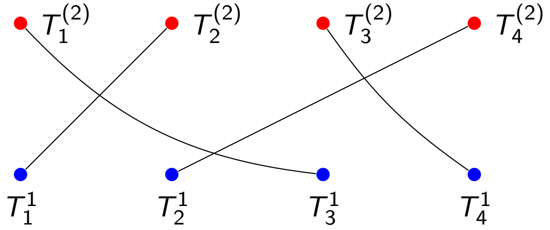

Figure: An admissible diagram $\mathcal{D} \in \mathbb{D}(4,4)$ with $T_l^{(j)} = (t_l^{(j)}, x_l^{(j)})$

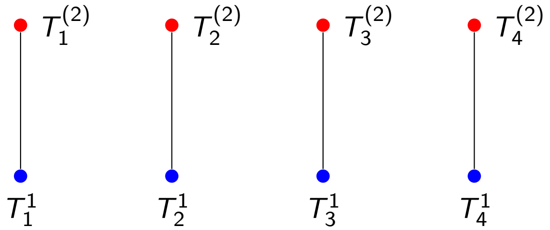

Figure: The ‘trivial’ admissible diagram $\mathcal{D} \in \mathbb{D}(4,4)$

By (14) in Theorem 3, one finds

$$\mathbb{E}\left[\left|I_n(f_n)\right|^2\right]$$
$$= \sum_{\mathcal{D}\in\mathbb{D}(n,n)} \int \prod_{j=1}^{2} \prod_{r=1}^{n} G_{t_{r+1}^{(j)}-t_r^{(j)}} \left(x_{r+1}^{(j)} - x_r^{(j)}\right) \mathbf{1}_{\{0<t_1^{(j)}<\cdots<t_n^{(j)}<t\}} \tag{15}$$
$$\times \gamma\left(t_{\overline{\mathscr{V}}(\mathcal{D})} - t_{\underline{\mathscr{V}}(\mathcal{D})}\right)\Lambda\left(x_{\overline{\mathscr{V}}(\mathcal{D})} - x_{\underline{\mathscr{V}}(\mathcal{D})}\right) dt_{\mathcal{D}} dx_{\mathcal{D}}\,.$$

If $\gamma(\cdot) = \delta(\cdot)$, then (15) reduced to

$$\mathbb{E}\left[\left|I_n^W(f_n)\right|^2\right] = \int \prod_{r=1}^{n} G_{t_{r+1}-t_r}\left(x_{r+1} - x_r\right) \Lambda\left(x_r - y_r\right) \tag{16}$$
$$\times\, G_{t_{r+1}-t_r}\left(y_{r+1} - y_r\right) \cdot \mathbf{1}_{\{0<t_1<\cdots<t_n<t\}} dt_{\mathcal{D}} dxdy\,.$$

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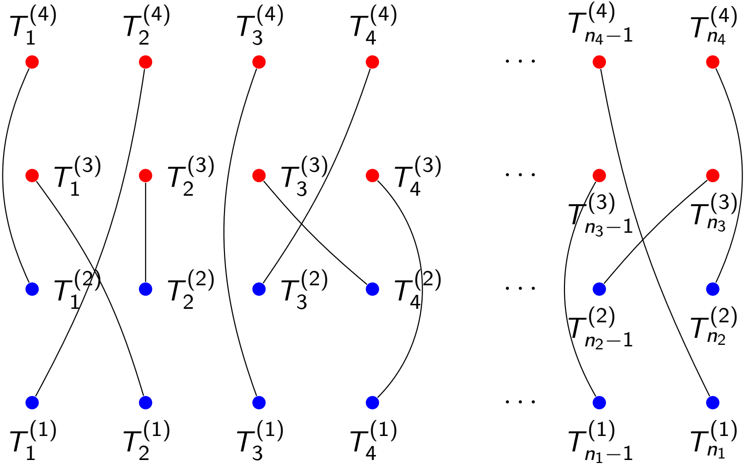

Figure: A particular scheme when $p = 4$

Sharp lower p-th moment bounds

Idea:We apply Feynman diagram formula for the moments of the solution and then select the diagrams in the principle of Small ball non-degeneracy introduced. In this way, we luckily obtain the sharp lower bounds for a large class of SPDEs.

We obtain that by choosing carefully m_p, $\mathcal{D}$, $m_0 = m_0(\varepsilon)$ and ε in Small ball non-degeneracy:

$$\mathbb{E}\Big[\prod_{j=1}^{p} u(t,x_j)\Big] = \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_p=2m} \sum_{\mathcal{D}\in\mathbb{D}(f_{m_1},\cdots,f_{m_p})} F_{\mathcal{D}}(f_{m_1},\cdots,f_{m_p})$$

$$\gtrsim \sum_{p\cdot m_p=2m_0} \sum_{\mathcal{D}\in\mathbb{D}(m_p)} F_{\mathcal{D}}(f_{m_p},\cdots,f_{m_p})$$

$$\gtrsim \exp\left(C \cdot t^{1+\frac{\mathrm{b}\cdot(1-\gamma)}{\mathrm{b}(2\mathrm{a}+1)-\lambda}} \cdot p^{1+\frac{\mathrm{b}}{\mathrm{b}(2\mathrm{a}+1)-\lambda}}\right).$$

The choice of m_p is inspired by the fundamental work [7].

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Sharp upper p-th moment bounds

It follows from the Itô isometry for the multiple Wiener-Itô integral (e.g. [8]) and HLS-type mass property that

$$\|u_n(t,x)\|_{L^2}^2 = \mathbb{E}|I_n(f_n(\cdot;t,x))|^2 \le C_H^n \frac{t^{n(\hbar+2H)}}{(n!)^{\hbar+1}}\,.$$

For the upper bound, it is now easy to bound the p-th moment from the above second moment bound by using the hypercontractivity inequality (e.g. [8, p.54, Theorem 3.20])

$$\begin{aligned}\|u_n(t,x)\|_{L^p} \le& (p-1)^{n/2}\|u_n(t,x))\|_{L^2}\\ \le& C_H^n (p-1)^{n/2}\left[\frac{t^{n(\hbar+2H)}}{(n!)^{\hbar+1}}\right]^{1/2}\,.\end{aligned}$$

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Thus we get

$$\begin{aligned}
\|u(t,x)\|_p \leq & C + \sum_{n=1}^{\infty} \|u_n(t,x))\|_{L^p} \\
\leq & C + \sum_{n=1}^{\infty} C_H^n (p-1)^{n/2} \left[\frac{t^{n(\hbar+2H)}}{(n!)^{\hbar+1}} \right]^{1/2} \\
\leq & C \exp\left(C \cdot t^{\frac{\hbar+2H}{\hbar+1}} (p-1)^{\frac{1}{\hbar+1}} \right) .
\end{aligned}$$

This means $\mathbb{E}[|u(t,x)|^p] \leq C_1 \exp\left(C_2 \cdot t^{1+\frac{1-\gamma}{\hbar+1}} p^{1+\frac{1}{\hbar+1}} \right)$ for some positive constants C_1 and C_2

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Future problems

The intermittency islands (e.g. see [10, Chapter 9]) are introduced to describe the size of 'high peaks' arose in PAM, and the length of the intermittency islands (e.g. see [10, Chapter 10]). They are also expected for other cases, such as hyperbolic Anderson model.

The exact Lyapunov exponent for time dependent case.

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Thanks for your attention!