Numerical approximations for stochastic rough integrations

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A joint work with Z. Selk and S. Tindel.

Outline

Presentation Outline

Signatures

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$$\mathbf{x}^1_{st} := \int_s^t dx_u = x_t - x_s \qquad \mathbf{x}^2_{st} := \int_s^t \int_s^u dx_v \otimes dx_u$$

$$\mathbf{x}^3_{st} := \int_s^t \int_s^u \int_s^v dx_w \otimes dx_v \otimes dx_u$$

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and similar definition for m-th order multiple integral $\mathbf{x}^m$.

- The n-th order signature of x:

$$S_n(x)_{st} = (\mathbf{x}^1_{st}, \mathbf{x}^2_{st}, \dots, \mathbf{x}^n_{st}).$$

It is known that for fixed s and t, the $S_n(x)_{st}$ contains all information of x for $n = \infty$.

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- Suppose that there exists a smooth path x^ϵ such that for $n = \lfloor p \rfloor$ the nth order signature $S_n(x^\epsilon)$ of x^ϵ converges under the p-variation norm.
- Denote the limit by $S_n(x) := \mathbf{x}$. $\mathbf{x}$ is called a p-rough path.

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- We extend the Itô map I to p-variation rough paths. We define $\mathbf{y} = I(\mathbf{x})$ as the solution of the differential equation

$$d\mathbf{y}_t = V(y_t)d\mathbf{x}_t.$$

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- The rough paths framework provides the stability of the Itô map.
- The signature has been applied to model complex data streams.

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$$\int_{t_k}^{t_{k+1}} V(x_t)d\mathbf{x}_t \approx \int_{t_k}^{t_{k+1}} V(x_{t_k})dx_t = V(x_{t_k})\delta x_{t_k t_{k+1}}.$$

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- Note that if $p \geq 2$ the Riemann sum $\sum_k V(x_{t_k}) \delta x_{t_k t_{k+1}}$ diverges.

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- The compensated Riemann sum converges to the rough integral $\int_0^T V(x) d\mathbf{x}$ when $n > p - 1$.
- Note that the compensated Riemann sum requires the computations of signatures of $\mathbf{x}$.

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- (y, y', y'') is called a controlled paths of $\mathbf{x}$ of order 2.
- Such path contains most of the interesting examples. e.g. $y = V(x)$ or y is the solution of a RDE.

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$$\|R\|_{\rho\text{-var}} = \sup_{(t_i),(t'_j)} \left(\sum_{i,j} \left| R^{t'_j t'_{j+1}}_{t_i t_{i+1}} \right|^{\rho} \right)^{1/\rho},$$

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where (t_i) and (t'_j) are partitions on $[0,T]$.

- If R has finite ρ-variation for $\rho \in [1,2)$, then X gives raise to a p-rough path, provided $p > 2\rho$. (Friz-Victoir '11).

Define the trapezoid rule:

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Theorem (Liu-Selk-Tindel '21)

- Suppose that $\|R\|_{\rho\text{-var}} < \infty$. Then as the mesh size of the partition (t_k) goes to 0 we have

$$\text{tr-}\mathcal{J}_0^T(y, X) \to \int_0^T y_t \, d\mathbf{X}_t \qquad \text{in probability.}$$

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- Suppose that there exists a constant $C > 0$ such that $\|R\|_{\rho\text{-var};[s,t]\times[0,T]} \leq C|t-s|$ for all $[s,t] \subset [0,T]$. Then we have

$$\text{tr-}\mathcal{J}_0^T(y, X) \to \int_0^T y_t d\mathbf{X}_t \qquad \text{almost surely.}$$

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I_2 &= \frac{1}{2} y'_{t_k} \mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - y'_{t_k} \cdot \mathbf{X}^2_{t_k t_{k+1}} \\
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Ingredients of proof.

With a careful rearrangement of the trapezoid rule we can recast it as

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- The convergences of I_i, $i = 2, 3, 4$ can be shown based on a transfer principle combined with some 2d young-type estimates.

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- For example, in order to estimate $I_2 = \frac{1}{2}\sum_k y'_{t_k}(\mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - \cdot \mathbf{X}^2_{t_k t_{k+1}})$ it suffices to bound

$$\sum_k (\mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - \mathbf{X}^2_{t_k t_{k+1}}) \quad \text{and} \quad \sum_k \mathbf{X}^1_{t_k}(\mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - \mathbf{X}^2_{t_k t_{k+1}})$$

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- Such transfer principle for limit theorems of weighted sums are obtained and has been applied to very general weighted sum.

Let f be a smooth function on $\mathbb{R}^m$. Define the midpoint rule:

$$\text{m-}\mathcal{J}_0^T(f(X), X) = \sum_{k=0}^{n-1} f\Big(\frac{X_{t_k} + X_{t_{k+1}}}{2}\Big) \cdot \delta X_{t_k t_{k+1}}.$$

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Corollary

- Suppose that $\|R\|_{\rho\text{-var}} < \infty$. Then as the mesh size of the partition (t_k) goes to 0 we have

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- If we assume further that $\|R\|_{\rho\text{-var};[s,t]\times[0,T]} \leq C|t-s|$. Then the convergence holds almost surely.

Ingredients of proof:

For $a, b \in \mathbb{R}^d$ we consider the following mean value identity

$$\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) = \frac{1}{2}\partial^2 f(c)\left(\frac{b-a}{2}\right)^{\otimes 2},$$

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- We conclude that the two numerical integral methods converge to the same limit.

Thank you very much for your attention!