Phase transition of extremes for a family of stationary multiple-stable processes: the macroscopic level

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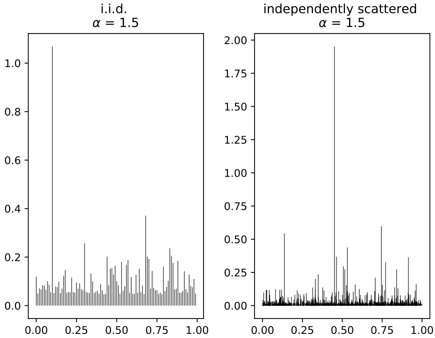
Joint work with Shuyang Bai

Extremes of i.i.d.



Assume $\mathbb{P}(X_1 > x) \sim x^{-\alpha}$. Point-process convergence recording the **magnitude and location** of each extreme point:
(Pickands 1971, Resnick 1975, 1987, **LePage, Woodroofe amd Zinn 1981**)

(i.i.d. case) PP convergence tells essentially everything regarding limit of extremes.

Extremes of i.i.d., via random sup-measures

PP convergence:

$$\sum_{i=1}^{n} \delta_{(X_i/n^{1/\alpha}, i/n)} \Rightarrow \sum_{\ell=1}^{\infty} \delta_{\left(\Gamma_\ell^{-1/\alpha}, U_\ell\right)},$$

random-sup-measure (RSM) convergence:

$$\left\{ M_n(G) \equiv \frac{1}{n^{1/\alpha}} \max_{i/n \in G} X_i \right\}_{G \in \mathcal{G}} \stackrel{f.d.d.}{\to} \left\{ \max_{\ell : U_\ell \in G} \Gamma_\ell^{-1/\alpha} \equiv \mathcal{M}_\alpha^{\mathrm{is}}(G) \right\}_{G \in \mathcal{G}}.$$

The limit RSM is

- essentially a set-indexed stochastic process $\{\mathcal{M}_\alpha^{\mathrm{is}}(G)\}_{G \in \mathcal{G}}$,
- **independently scattered**,
- α-**Fréchet**,
- not characterizing local clustering of extremes.

The same RSM convergence holds (up to a multiplicative constant) for stationary processes with regularly-varying tails with **short-range dependence**.

O'Brien, Torfs and Vervaat (1990) advocated that the RSM convergence is the right framework for macroscopic limit of extremes with non-trivial dependence. **Very few limit theorems** with limit RSM not independently scattered.

A non-technical overview of phase transition

For stationary multiple-stable stable-regenerative model (Shuyang's talk earlier), with memory parameter $\beta \in (0,1)$ and multiplicity parameter $p \in \mathbb{N}$,

$$\beta_p := p\beta - p + 1 \in (-\infty, 1),$$

we showed the following phase transition regarding extremes.

regime	tail process (microscopic)	limit RSM (macroscopic)	extremal index
super-crit., $\beta_p > 0$	'infinite'	with long-range clustering	$\theta = 0$
crit., $\beta_p = 0$	'infinite'	independently scattered	$\theta = 0$
sub-crit., $\beta_p < 0$	'geometric'	independently scattered	$\theta \in (0,1)$

Remark

- $\beta_p \leq 0$: delicate differences revealed by tail processes (Shuyang's talk). Much harder to prove! By a refined analysis of two-moment method for Poisson convergence **Arratia, Goldstein and Gordon (1989)**.
- $\beta_p > 0$: the limit RSM **does not even have standard extreme-value distributions, due to aggregations**, except for a small class.
- Inspiration/motivation: Lacaux and Samorodnitsky (2016) and Samorodnitsky and W (2019), limit RSM for $p = 1$ (having only the supercritical regime).

Theorem (Bai and W 2021) We have

$$\frac{1}{c_n}\left\{M_n(G)\right\}_{G\in\mathcal{G}} \stackrel{f.d.d.}{\to} \begin{cases} C_1\left\{\mathcal{M}_{\alpha,\beta,p}(G)\right\}_{G\in\mathcal{G}}, & \text{if } \beta_p > 0, \\ C_2\left\{\mathcal{M}_{\alpha}^{\text{is}}(G)\right\}_{G\in\mathcal{G}}, & \text{if } \beta_p = 0, \\ C_3\left\{\mathcal{M}_{\alpha}^{\text{is}}(G)\right\}_{G\in\mathcal{G}}, & \text{if } \beta_p < 0, \end{cases}$$

with

$$c_n = \begin{cases} n^{(1-\beta_p)/\alpha}, & \text{if } \beta_p > 0, \\ \left(\frac{n(\log\log n)^{p-1}}{\log n}\right)^{1/\alpha}, & \text{if } \beta_p = 0, \\ (n\log^{p-1} n)^{1/\alpha}, & \text{if } \beta_p < 0. \end{cases}$$

Recall $\beta_p = p\beta - p + 1$.

$\mathcal{M}_{\alpha,\beta,p}$: a new family of RSMs with **long-range clustering**.

Stable-regenerative model (Rosinski and Samorodnitsky 1996)

A fundamental stationary stable process with long-range dependence (mixing but non-standard limit theorems):

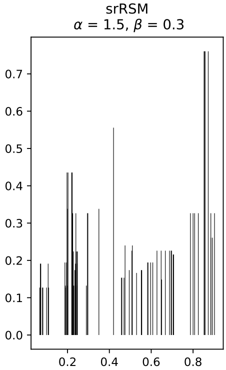
$$\{X_k\}_{k=1,\dots,n} \stackrel{d}{=} \left\{ w_n^{1/\alpha} \sum_{\ell=1}^{\infty} \frac{\varepsilon_\ell}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{k/n \in R_{n,\ell}\}} \right\}_{k=1,\dots,n}$$

with $\{R_{n,\ell}\}_{\ell\in\mathbb{N}}$ i.i.d. random closed sets. Commonly represented in terms of stochastic integrals in the literature.

Remark

- $R_{n,1} \Rightarrow \widetilde{\mathcal{R}}_\beta$, a randomly shifted β-stable regenerative set, with $\dim_{\mathrm{H}}(\mathcal{R}_\beta) = \beta$.

Long-range clustering of extremes



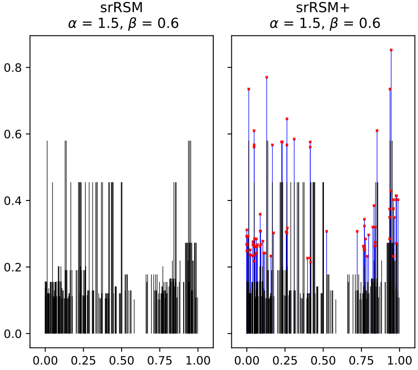
$p = 1, \beta \leq 1/2$: Lacaux and Samorodnitsky (2016)

Limit RSM:

$$\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}(G) = \sup_{\ell \in \mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{\widetilde{\mathcal{R}}_{\beta,\ell} \cap G \neq \emptyset\}}.$$

$\mathcal{M}^{\mathrm{sr}}_{\alpha,\beta}$ is α-Fréchet, but not independently scattered. $\widetilde{\mathcal{R}}_{\beta,1} \cap \widetilde{\mathcal{R}}_{\beta,2} = \emptyset$ a.s.

Aggregation of LRC of extremes



$p = 1, \beta \in (1/2, 1)$: Samorodnitsky and W (2019)

Limit RSM:

$$\mathcal{M}^{\mathrm{sr}+}_{\alpha,\beta}(G) = \sup_{t \in G} \left(\sum_{\ell \in \mathbb{N}} \frac{1}{\Gamma_\ell^{1/\alpha}} \mathbf{1}_{\{t \in \widetilde{\mathcal{R}}_{\beta,\ell}\}} \right).$$

$\mathcal{M}^{\mathrm{sr}+}_{\alpha,\beta}$ **not α-Fréchet due to aggregations** caused by **intersections of random fractals**.

Multiple-stable model: $p \geq 2$

$p = 2$:

$$\{X_k\}_{k=1,\dots,n} \stackrel{d}{=} \left\{ w_n^{2/\alpha} \sum_{1 \le i_1 < i_2} \frac{\varepsilon_{i_1}\varepsilon_{i_2}}{(\Gamma_{i_1}\Gamma_{i_2})^{1/\alpha}} \mathbf{1}_{\{k/n \in R_{n,i_1} \cap R_{n,i_2}\}} \right\}_{k=1,\dots,n}.$$

Supercritical regime ($\beta > 1/2$) (Bai and W 2021): limit RSM with **aggregations of long-range clustering**, can be constructed from $\{\varepsilon_\ell, \Gamma_\ell^{-1/\alpha}, \widetilde{\mathcal{R}}_{\beta,\ell}\}_{\ell \in \mathbb{N}}$ (intersections involved). Much more delicate than $p = 1$, needed a new representation.

A detour through FCLT

- Owada and Samorodnitsky (2015) ($p = 1$), Bai, Owada and W (2020): supercritical regime, an sssi limit process with **multiple-stochastic-integral representation**.
- Bai (2020) a new stochastic-integral representation of **Hermite processes** in terms of local times of stable-regenerative sets and their intersections.

Summary

Two recent preprints (Bai and W 2021):

- Phase transition for extremes of a family of stationary multiple-stable processes.
  arxiv 2110.07497
- Tail processes for stable-regenerative model.
  arxiv 2110.07499

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