A class of SPDEs in fluid dynamics: Large and moderate deviation asymptotics.

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Motivation

Consider the two-dimensional Navier-Stokes equation:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

- $\mathbf{u} = 0$ on ∂G; Initial condition: $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ given.
- $t \geq 0$ and $x \in G \subset \mathbb{R}^2$
- $\mathbf{u} = \binom{u_1}{u_2}$ is a vector function of t and x
- The unknowns are $\mathbf{u}$ and p
- $\mathbf{f}$ is an external body force.

The Analytic Form

The functional analytic setup (see Temam)

$$\begin{cases} \frac{\partial \mathbf{u}(t)}{\partial t} + \mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t)) = \mathbf{f}(t) \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) \end{cases} \tag{1}$$

- **Au** is the dissipative term. **A** = Stokes operator.
- $\mathbf{Au} \doteq -\mathrm{P}(\triangle \mathbf{u})$ where P is the Leray projection to the space of divergence free vector fields H
- The nonlinear term **B** is denoted by $\mathbf{B}(\mathbf{u}, \mathbf{v}) \doteq \mathrm{P}((\mathbf{u} \cdot \nabla)\mathbf{v})$
- We abbreviate $\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u}, \mathbf{u})$

Function Spaces

Let $\mathcal{V} = C_0^\infty(G)$ functions which are divergence free.

Define
$H =$ the completion of $\mathcal{V}$ in $L^2(G)$

$V =$ the completion of $\mathcal{V}$ in $H^1(G)$ $(= W^{1,2}(G))$.

Let V' be the dual of V. We have the dense, continuous embedding:

$$V \subset_{\rightarrow} H = H' \subset_{\rightarrow} V'.$$

The operator $\mathbf{A} : V \rightarrow V'$ and $\mathbf{B} : V \times V \rightarrow V'$.

$\mathbf{f}(t)$ is assumed to be V'-valued for all t.

Notation: Throughout, $||\cdot|| = V$-norm whereas $|\cdot|$ will denote the H-norm.

Basic observations

- $\langle \mathbf{Au}, \mathbf{u} \rangle = ||\mathbf{u}||^2$.
- Define $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_G (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) dx$.
- $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v})$. Therefore, $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.
- By Hölder inequality, $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq |\mathbf{u}|_{L^4} \, ||\mathbf{v}|| |\mathbf{w}|_{L^4}$.

  By Sobolev embedding, $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq ||\mathbf{u}||_{H^{1/2}} \, ||\mathbf{v}|| \, ||\mathbf{w}||_{H^{1/2}}$.

  Interpolation yields, $|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq ||\mathbf{u}||^{1/2} \, |\mathbf{u}|^{1/2} \, ||\mathbf{v}|| \, ||\mathbf{w}||^{1/2} \, |\mathbf{w}|^{1/2}$.
- In particular, $||\mathbf{B(u)}||_{V'} \leq ||\mathbf{u}|| \, |\mathbf{u}|$.

Example 2

Shell Model of Turbulence (GOY):

$$\frac{du_n}{dt} + \nu k_n^2 u_n + i(ak_{n+1} u_{n+1}^* u_{n+2}^* + bk_n u_{n-1}^* u_{n+1}^* + ck_{n-1} u_{n-1}^* u_{n-2}^*) = f_n \qquad (2)$$

for $n = 1, 2, \cdots$, and $u_n \in \mathbb{C}$. The boundary conditions are $u_{-1} = u_0 = 0$.

$$H := \left\{ u = (u_1, u_2, \ldots) \in \mathbb{C}^\infty : \sum_{n=1}^{\infty} |u_n|^2 < \infty \right\},$$

$$V := \left\{ u \in H : \sum_{n=1}^{\infty} k_n^2 |u_n|^2 < \infty \right\}$$

$$Au = ((Au)_1, (Au)_2, \ldots), \text{ where } (Au)_n = k_n^2 u_n,$$

$$B(u, v) = (B_1(u, v), B_2(u, v), \ldots), \text{where}$$

$$B_n(u, v) = ik_n \left(\frac{1}{4} u_{n+1}^* v_{n-1}^* - \frac{1}{2} (u_{n+1}^* v_{n+2}^* + u_{n+2}^* v_{n+1}^*) + \frac{1}{8} u_{n-1}^* v_{n-2}^* \right).$$

Example 3

MHD equation:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \frac{1}{R_e}\Delta v - S(B \cdot \nabla)B + \nabla(p + \frac{S|B|^2}{2}) = 0, \tag{3}$$

where v denotes the velocity, B, the magnetic field, and p, the pressure field. The Maxwell equation is

$$\frac{\partial B}{\partial t} + (v \cdot \nabla)B + \frac{1}{R_m}\mathrm{curl}(\mathrm{curl}\ B) - (B \cdot \nabla)v = 0, \tag{4}$$

where $\nabla \cdot v = 0$ and $\nabla \cdot B = 0$.

Boundary conditions:
$v = 0$ on ∂G ; $B \cdot \mathbf{n} = 0$ and $\mathrm{curl}\ B = 0$ on ∂G, where $\mathbf{n}$ is the unit outer normal on ∂G.

Initial conditions: $v(x, 0) = v_0(x)$ and $B(x, 0) = B_0(x)$ for $x \in G$.

General Class of PDEs

Let H and V be Hilbert spaces such that $V \subset\to H$ is dense and continuous.

Let $\mathbf{A}$ be an unbounded self-adjoint operator in H with compact resolvent.
Let $\mathbf{B} : V \times V \to V'$ be a bilinear operator.

Hypotheses H:

H.1 The map $\mathbf{A} : V \to V'$ is a linear operator such that for all $u, v \in V$, $\langle \mathbf{A}u, v\rangle = ((u, v))$.

H.2 The map $\mathbf{B} : V \times V \to V'$ is a bilinear operator, and there exists a constant $C > 0$ such that

$$|\mathbf{B}(u, v)|_{V'} \le C\|u\|\|v\| \text{ for all } u, v \in V.$$

H.3 The operator $\mathbf{B}$ satisfies the condition

$$\langle \mathbf{B}(u, v), w\rangle = -\langle \mathbf{B}(u, w), v\rangle \text{ for all } u, v, w \in V.$$

H.4 For every $r > 0$, there exists a constant L_r such that

$$|\mathbf{B}(u) - \mathbf{B}(v)|_{V'} \le L_r\|u - v\| \text{ for all } u, v \in V, \text{ and } \|u\|, \|v\| \le r.$$

Class of SPDEs

Under hypotheses **H**, consider

$$\mathbf{u}(t) + \int_0^t [\mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s))]ds = \mathbf{u}(0) + \int_0^t f(s)ds.$$

Random perturbations: White noise, jump noise, or both.

Integral form of noise: Brownian motion, Poisson random measure (Prm), or Lèvy process.

Multiplicative noise: stochastic integral with integrand as a function of **u**.

We will consider SPDEs with stochastic integral wrt Prm.

Poisson Random Measures

We will consider the SPDE with small noise:

$$\begin{aligned}\mathbf{u}(t) + \int_0^t [\mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s))]ds =& \mathbf{u}(0) + \int_0^t f(s)ds \\ &+ \epsilon \int_0^t \int_Z g(s, \mathbf{u}(s-), z)\tilde{N}^{\epsilon^{-1}}(ds, dz). \quad (5)\end{aligned}$$

$\tilde{N}^{\epsilon^{-1}}$ is a compensated Poisson random measure with compensator $\epsilon^{-1} d\nu ds$.

$g : [0, T] \times H \times Z \to H$ satisfies a linear growth and Lipschitz condition.

H.5: Let ν be a σ-finite measure on a space $(Z, \mathcal{Z})$. There exists an $L^2(\nu)$ function $M_g(z) > 0$ such that

$$|g(t, u, z)| \leq M_g(z)(1 + |u|) \text{ for all } t \in [0, T]\ u \in H, \text{ and } z \in Z.$$

H.6: For all $u_1, u_2 \in H$, and $t \in [0, T]$, there exists an $L^2(\nu)$ function $L_g(z) > 0$ such that

$$|g(t, u_1, z) - g(t, u_2, z)| \leq L_g(z)|u_1 - u_2| \text{ for all } z \in Z.$$

Problem

- Existence and uniqueness of solutions.
- LDP for the class of SPDEs as $\epsilon \to 0$.
- MDP for the class of SPDEs as $\epsilon \to 0$.

(Sritharan, S. (1999, 2006): MHD and LDP for SNSE;

Manna, Sritharan and S. (2009): LDP for Shell Model;

S. (2010): LDP for MHD system.

Hsu and S. (2021: preprint): 3-D SNSEs;

Wang, Zhai and Zhang (2014): MDP for SNSE;

Dong, Xiong, Zhai and Zhang (2017): MDP for SNSE) - a sampler!

Existence and Uniqueness

- Form Galerkin approximations $\mathbf{u}_n$.
- Using energy equality, obtain a priori estimates uniformly in n.
- Show convergence of martingale problems posed by $\mathbf{u}_n$ to a limit.
- Identify the limit mg. problem with that posed by the SPDE.
- Prove pathwise uniqueness.

Existence and Uniqueness

Theorem

Assume hypotheses **H**. *Let* $E|\mathbf{u}(0)|^4 < \infty$, *and* $\mathbf{f}$ *be in* $L^4([0,T]:V')$. *Then, for any* $\epsilon > 0$, *there exists a strong solution of (5) which is a cadlag adapted process taking values in* $L^2(\Omega \times (0,T);V) \cap L^\infty(\Omega \times (0,T);H)$. *It is pathwise unique.*

We can write $\mathbf{u}$ as $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$. If ϵ changes, we will call $\mathbf{u}$ as $\mathbf{u}^\epsilon$.

Existence and Uniqueness

For a class of good φ, there is an equivalent change of probability measure, Q^ϵ under which $\epsilon N^{\epsilon^{-1}\varphi}$ has the same law as $\epsilon N^{\epsilon^{-1}}$ under P.

Hence, we have also solved the equation

$$\begin{aligned}
&\mathbf{v}(t) + \int_0^t [\mathbf{A}\mathbf{v}(s) + \mathbf{B}(\mathbf{v}(s))]ds \\
&= \mathbf{v}(0) + \int_0^t f(s)ds + \epsilon \int_0^t \int_Z g(s, \mathbf{v}(s-), z)\tilde{N}^{\epsilon^{-1}\varphi}(ds, dz) \\
&+ \int_0^t \int_Z g(s, \mathbf{v}(s), z)(\varphi(s, z) - 1)\nu(dz)ds
\end{aligned}$$

for a class of functions φ. We write $\mathbf{v}$ as $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi})$.

As $\epsilon \to 0$, the limit equation is

$$\begin{aligned}
&\mathbf{w}(t) + \int_0^t [\mathbf{A}\mathbf{w}(s) + \mathbf{B}(\mathbf{w}(s))]ds \\
&= \mathbf{v}(0) + \int_0^t f(s)ds + \int_0^t \int_Z g(s, \mathbf{w}(s), z)(\varphi(s, z) - 1)\nu(dz)ds
\end{aligned}$$

We will call $\mathbf{w}$ as $\mathcal{G}^0(\nu_T^\varphi)$.

LDP

Let $\{X^\epsilon\}$ be a family of random variables defined on $(\Omega, \mathcal{F}, P)$ with values in a Polish space (E, d). For several Borel sets A, $P\{X^\epsilon \in A\} \to 0$ exponentially fast. Our aim is to find the exponential rate.

Definition: $I : E \to [0, \infty]$ is called a rate function if I is lower semicontinuous. The function I is a good rate function if I has compact level sets.

$I(A) := \inf_A I(x)$.

Definition: The sequence $\{X^\epsilon\}$ is said to satisfy the LDP on E with a good rate I if:

1. For each closed subset F in E,

$$\limsup_{\epsilon \to 0} \epsilon \log P\{X^\epsilon \in F\} \leq -I(F).$$

2. For each open subset G in E,

$$\liminf_{\epsilon \to 0} \epsilon \log P\{X^\epsilon \in G\} \geq -I(G).$$

Laplace Principle

The Laplace method says that for $h \in C_b([0,1])$,

$$\lim_{n\to\infty} 1/n \log \int_0^1 e^{-nh(x)}\, dx = -\min_{[0,1]} h(x).$$

Theorem

(Varadhan) Let $\{X^\epsilon\}$ satisfy LDP on E with good rate function I. Then for all $h \in C_b(E)$,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E} \exp\{-\frac{h(X^\epsilon)}{\epsilon}\} = -\inf_{x\in E}\{I(x) + h(x)\}.$$

(Laplace Principle with rate I)

Theorem

(Converse) LP implies LDP. i.e. If I is a rate function on E and the limit

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E} \exp\{-\frac{h(X^\epsilon)}{\epsilon}\} = -\inf_{E}\{I(x) + h(x)\}$$

is valid for all $h \in C_b(E)$, then $\{X^\epsilon\}$ satisfies LDP with a good rate function I.

Sufficient conditions for LDP on $D([0,T];H) \cap L^2(0,T;V)$:

(1) Let φ_n, φ such that $\varphi_n \to \varphi$ in a certain bounded ball, as $n \to \infty$. Then, $\mathcal{G}^0(\nu_T^{\varphi_n}) \to \mathcal{G}^0(\nu_T^{\varphi})$.

(2) Let $\varphi_\epsilon, \varphi$ be predictable processes taking values in a certain bounded ball such that $\varphi_\epsilon \to \varphi$ in distribution as $\epsilon \to 0$. Then, $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}) \implies \mathcal{G}^0(\nu_T^{\varphi})$.

(Budhiraja and Dupuis (2000))

Theorem

Assume hypotheses **H**. *Let* $\mathbf{v}(0) \in H$ *, and* $\mathbf{f}$ *be in* $L^4([0,T] : V')$. *Then the family* $\{\mathbf{u}^\epsilon\}$ *taking values in* $D([0,T];V') \cap L^2(0,T;V)$ *satisfies the Laplace principle with a good rate function*

$$I(w) = \inf_{A_w}\{1/2 \int_0^T \|w(s)\|^2 ds\}.$$

$A_w := \{\varphi : dw(t) + [\mathbf{A}w + \mathbf{B}(w) - f](t)dt = \int_Z g(t, w(t), z)(\varphi(t,z) - 1)\nu(dz)dt\}$ *and* $w(0) = v(0)$.

Introduction

Let $\{X_i\}$ be an iid sequence.
Let $E(X_1) = 0$, and the MGF $M(\theta) < \infty$ for all θ.
Let $S_n := \sum_{i=1}^n X_i$.

Cramér's theorem: For any $r > 0$,

$$P(|S_n| > nr) \approx \exp\{-n \inf\{I(x) : |x| \geq r\}\} \tag{6}$$

where $I(x) = \sup_\theta\{\theta x - \log M(\theta)\}$.

Let $a_n \to \infty$ and $\dfrac{a_n}{\sqrt{n}} \to 0$. Then MDP says

$$P\left(|S_n| > a_n n^{1/2} r\right) \approx \exp\{-a_n^2 \inf\{J(x) : |x| \geq r\}\} \tag{7}$$

where $J(x) = \frac{1}{2}(x, \Sigma^{-1}x)$ where $\Sigma = \mathrm{Cov}(X_1)$.

Scaling: Between SLLN and CLT; $\{a_n\}$ can approach ∞ slowly.

Moderate Deviations

Then MDP for a family $\{Y^\epsilon\}_{\epsilon>0}$ is the same as LDP with speed $b(\epsilon)$. Hence it is equivalent to the Laplace principle (LP) with speed $b(\epsilon)$.

If the rate function is denoted by I, then, LP with speed $b(\epsilon)$ means that for all h on $C_b([0,T];H)$, we have

I.
$$\limsup_{\epsilon\to 0} b(\epsilon)\log \mathbb{E}\left\{\exp\left[-\frac{1}{b(\epsilon)}h(Y^\epsilon)\right]\right\} \le -\inf_{y\in H}\{h(y)+I(y)\}$$

II.
$$\liminf_{\epsilon\to 0} b(\epsilon)\log \mathbb{E}\left\{\exp\left[-\frac{1}{b(\epsilon)}h(Y^\epsilon)\right]\right\} \ge -\inf_{y\in H}\{h(y)+I(y)\}$$

(Budhiraja, Dupuis '00; Budhiraja, Dupuis, Ganguly '15)

Moderate Deviations

Let $a : \mathbb{R}^+ \to (0,1)$ such that $a(\epsilon) \to 0$, and $b(\epsilon) := \dfrac{\epsilon}{a^2(\epsilon)}$ also goes to 0 as $\epsilon \to 0$. For $0 \le t \le T$, and for any $\epsilon > 0$, define the process

$$Y^\epsilon(t) = \frac{\mathbf{u}^\epsilon(t) - \mathbf{u}(t)}{a(\epsilon)}.$$

where

$$d\mathbf{u}^\epsilon + \left[\mathbf{A}\mathbf{u}^\epsilon + \mathbf{B}(\mathbf{u}^\epsilon, \mathbf{u}^\epsilon)\right] dt = \mathbf{f}(t)dt + \epsilon \int_Z g(t, \mathbf{u}^\epsilon, z) \tilde{N}^{\epsilon^{-1}}(dt, dz).$$

and

$$d\mathbf{u} + \left[\mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}, \mathbf{u})\right] dt = \mathbf{f}(t)dt.$$

Aim: Prove MDP for $\{Y^\epsilon\}$, i.e., a LDP result with speed $b(\epsilon)$.

Moderate Deviations

Notation: $\mathcal{H}^\epsilon(\epsilon N^{\epsilon^{-1}}) := Y^\epsilon$.

If we choose suitable functions φ, then $\epsilon N^{\epsilon^{-1}\varphi}$ has the same law as $\epsilon N^{\epsilon^{-1}}$ under a change of measure.

Define the process $\mathbf{u}^{\epsilon,\varphi}$ as the solution of

$$d\mathbf{u}^{\epsilon,\varphi} + [\nu A\mathbf{u}^{\epsilon,\varphi}(t) + B(\mathbf{u}^{\epsilon,\varphi}(t))]\,dt = f(t)dt + \epsilon \int_Z g(\mathbf{u}^{\epsilon,\varphi}(t), z) N^{\epsilon^{-1}\varphi}(dz, dt)$$

We use $\mathbf{u}^{\epsilon,\varphi}$ in the place of $\mathbf{u}^\epsilon$ in the definition of Y^ϵ.

In fact, define

$$Y^{\epsilon,\varphi} := \frac{\mathbf{u}^{\epsilon,\varphi} - \mathbf{u}}{a(\epsilon)}.$$

Recall that $Y^\epsilon = \mathcal{H}^\epsilon(\epsilon N^{\epsilon^{-1}})$. Hence, $Y^{\epsilon,\varphi} = \mathcal{H}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi})$.

Define $\mathcal{H}^0(\varphi)$ is used to denote the solution of

$$dY^0(t) + \mathbf{A}Y^0(t)dt + \left[\mathbf{B}(Y^0(t), \mathbf{u}) + \mathbf{B}(\mathbf{u}, Y^0(t))\right] dt = \int_Z g(t, \mathbf{u}(t), z)\varphi(t, z)\nu(dz)dt.$$

Moderate Deviations

For $\mathcal{H}_0$ with values in $C([0,T];H) \cap L^2([0,T];V)$, the following two conditions hold:

I Suppose that $\varphi_\epsilon, \varphi$ are in the M-ball in $L^2(Z \times [0,T])$, and $\varphi_\epsilon \to \varphi$ weakly. Then

$$\mathcal{H}_0(\varphi_\epsilon) \to \mathcal{H}_0(\varphi).$$

II Given $\{\varphi_\epsilon\}_{\epsilon>0}$, define

$$\psi_\epsilon := \frac{\varphi_\epsilon - 1}{a(\epsilon)}.$$

Then, if for some $\beta \in [0,1]$, $\psi_\epsilon 1_{\{|\psi_\epsilon| \leq \beta/a(\epsilon)\}} \to \psi$ in law, then as $\epsilon \to 0$,

$$\mathcal{H}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}) \implies \mathcal{H}_0(\psi)$$

(cf. Budhiraja, Dupuis '00; Budhiraja, Dupuis, Ganguly '15)

MDP Result

Theorem

The sequence $\{Y^\epsilon\}_{\epsilon>0}$ *satisfies LP with speed* $b(\epsilon)$ *and rate function*

$$I(y) = \inf\{\int_0^T \|\psi\|_2^2 ds\}$$

where infimum is over ψ *such that* $y(0) = 0$ *and*

$$dy + \mathbf{A}y + \mathbf{B}(y, \mathbf{u}) + \mathbf{B}(\mathbf{u}, y)]dt$$
$$= \int_Z g(\mathbf{u}, z)\psi(z, t)\nu(dz)dt$$

Thank You!