

Nonlinear stochastic wave equation driven by rough noise

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Outline of the talk

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3. Difficulty
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5. Some ideas

1. Problem

$$\frac{\partial^2 u(t, x)}{\partial t^2} = \Delta u(t, x) + \sigma(t, x, u(t, x)) \dot{W}, \quad t > 0, x \in \mathbb{R}.$$

- $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a nice function (Lipschitz).
- initial condition $u(0, x) = u_0(x)$ and $\frac{\partial}{\partial t} u(0, x) = v(x)$ are nice.
- $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$ is centered Gaussian field with covariance

$$\mathbb{E}(\dot{W}(s, x) \dot{W}(t, y)) = \delta(s - t) |x - y|^{2H-2}.$$

Here $1/4 < H < 1/2$

- The product $\sigma(u) \dot{W}$ is taken in Skorohod sense.

Stochastic integral

For a function $\phi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, the Marchaud fractional derivative D_-^β is defined as:

$$\begin{aligned} D_-^\beta \phi(t, x) &= \lim_{\varepsilon \downarrow 0} D_{-, \varepsilon}^\beta \phi(t, x) \\ &= \lim_{\varepsilon \downarrow 0} \frac{\beta}{\Gamma(1 - \beta)} \int_\varepsilon^\infty \frac{\phi(t, x) - \phi(t, x + y)}{y^{1+\beta}} dy. \end{aligned}$$

The Riemann-Liouville fractional integral is defined by

$$I_-^\beta \phi(t, x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t, y) (y - x)^{\beta-1} dy.$$

Set

$$\mathbb{H} = \{\phi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \mid \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \text{ s.t. } \phi(t, x) = I_-^{\frac{1}{2}-H} \psi(t, x)\}.$$

Proposition

\mathbb{H} is a Hilbert space equipped with the scalar product

$$\begin{aligned} \langle \phi, \psi \rangle_{\mathbb{H}} &= c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s, \xi) \overline{\mathcal{F}\psi(s, \xi)} |\xi|^{1-2H} d\xi ds \\ &= c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_-^{\frac{1}{2}-H} \phi(t, x) D_-^{\frac{1}{2}-H} \psi(t, x) dx dt \\ &= c_{3,\beta}^2 \int_{\mathbb{R}^2} [\phi(x+y) - \phi(x)][\psi(x+y) - \psi(x)] |y|^{2H-2} dx dy, \end{aligned}$$

where

$$c_{1,H} = \frac{1}{2\pi} \Gamma(2H+1) \sin(\pi H);$$

$$c_{2,H} = \left[\Gamma\left(H + \frac{1}{2}\right) \right]^2 \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}} \right]^2 dt + \frac{1}{2H} \right)^{-1};$$

$$c_{3,\beta}^2 = \left(\frac{1}{2} - \beta\right) \beta c_{2,\frac{1}{2}-\beta}^{-1}.$$

The space $D(\mathbb{R}_+ \times \mathbb{R})$ is dense in \mathbb{H} .

Definition

An elementary process g is a process of the following form

$$g(t, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \mathbf{1}_{(a_i, b_i]}(t) \mathbf{1}_{(h_j, l_j]}(x),$$

where n and m are finite positive integers,

$-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are \mathcal{F}_{a_i} -measurable random variables for $i = 1, \dots, n$. The stochastic integral of such an elementary process with respect to W is defined as

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t, x) W(dx, dt) &= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W(\mathbf{1}_{(a_i, b_i]} \otimes \mathbf{1}_{(h_j, l_j]}) \\ &= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} [W(b_i, l_j) - W(a_i, l_j) - W(b_i, h_j) + W(a_i, h_j)]. \end{aligned}$$

Definition

Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathbb{H}$ and $\mathbb{E}[\|g\|_{\mathbb{H}}^2] < \infty$. Then, the space of elementary processes defined as above is dense in Λ_H .

For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+ \times \mathbb{R}} g(t, x) W(dx, dt)$ is defined as the $L^2(\Omega)$ -limit of stochastic integrals of the elementary processes approximating $g(t, x)$ in Λ_H , and we have the following isometry equality

$$\begin{aligned} \mathbb{E} \left(\left[\int_{\mathbb{R}_+ \times \mathbb{R}} g(t, x) W(dx, dt) \right]^2 \right) &= \mathbb{E} \left(\|g\|_{\mathbb{H}}^2 \right) \\ &+ c_{3,H}^2 \int_0^\infty \int_{\mathbb{R}^2} \mathbb{E} |g(t, x+y) - g(t, x)|^2 |y|^{2H-2} dx dy dt. \end{aligned}$$

Definition (Strong solution)

$u(t, x)$ is a **strong (mild random field) solution** if for all $t \in [0, T]$ and $x \in \mathbb{R}$ the process $\{G_{t-s}(x - y)\sigma(u(s, y))\mathbf{1}_{[0,t]}(s)\}$ is integrable with respect to W , where $G_t(x) := \frac{1}{2}I_{\{|x|<t\}}$ is heat kernel, and

$$u(t, x) = l_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y)\sigma(s, y, u(s, y))W(dy, ds) \quad (1)$$

almost surely, where

$$\begin{aligned} l_0(t, x) &:= \frac{\partial}{\partial t} G_t * u_0(x) + G_t * v_0(x) \\ &= \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy + \frac{1}{2} [u_0(x+t) + u_0(x-t)]. \end{aligned}$$

Definition (Weak solution)

We say the spde has a *weak solution* if there exists a probability space with a filtration $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\mathcal{F}}_t)$, a Gaussian noise \tilde{W} identical to W in law, and an adapted stochastic process $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ on this probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\mathcal{F}}_t)$ such that $u(t, x)$ is a strong (mild) solution with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\mathcal{F}}_t)$ and \tilde{W} .

Want to study the existence and uniqueness of the solution (strong or weak).

2. Main results

Solution space

Let $(B, \|\cdot\|_B)$ be a Banach space with the norm $\|\cdot\|_B$ (we take $\beta = \frac{1}{2} - H$). Let $\beta \in (0, 1)$ be a fixed number. For any function $f : \mathbb{R} \rightarrow B$ denote

$$\mathcal{N}_\beta^B f(x) := \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_B^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}, \quad (2)$$

if the above quantity is finite. When $B = \mathbb{R}$, we abbreviate the notation as $\mathcal{N}_\beta f$. With this notation, the norm of the homogeneous Sobolev space \dot{H}^β can be given by using $\mathcal{N}_\beta f$:
 $\|f\|_{\dot{\mathcal{H}}_\beta} = \|\mathcal{N}_\beta f\|_{L^2(\mathbb{R})}.$

We are particularly interested in the case $B = L^p(\Omega)$.

$$\mathcal{N}_{\beta,p}f(x) := \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_{L^p(\Omega)}^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}.$$

Definition of the solution space $\mathcal{Z}^p(T)$.

It consists of all continuous functions f from $[0, T] \times \mathbb{R}$ to $L^p(\Omega)$ such that the following norm is finite:

$$\begin{aligned} \|f\|_{\mathcal{Z}^p(T)} &= \|f\|_{\mathcal{Z}_1^p(T)} + \|f\|_{\mathcal{Z}_2^p(T)} \\ &:= \sup_{t \in [0, T]} \|f(t, \cdot)\|_{L_\lambda^p(\Omega \times \mathbb{R})} + \sup_{t \in [0, T]} \mathcal{N}_{\frac{1}{2}-H}^* f(t), \end{aligned} \quad (3)$$

where $\|f(t, \cdot)\|_{L_\lambda^p(\Omega \times \mathbb{R})} = [\int_{\mathbb{R}} \mathbb{E}[|f(t, x)|^p] dx]^{1/p}$ and

$$\mathcal{N}_{\frac{1}{2}-H}^* f(t) := \left[\int_{\mathbb{R}} \|f(t, \cdot + h) - f(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 |h|^{2H-2} dh \right]^{\frac{1}{2}}.$$

It is proved that $\mathcal{Z}^p(T)$ is a Banach space.

(1). $\sigma(t, x, u)$ is jointly continuous over $[0, T] \times \mathbb{R}^2$,
 $\sigma(t, x, 0) = 0$.

(2). Assume

$$\sup_{t \in [0, T], x \in \mathbb{R}, u \in \mathbb{R}} \left| \frac{\partial}{\partial u} \sigma(t, x, u) \right| \leq C;$$

$$\sup_{t \in [0, T], x \in \mathbb{R}, u \in \mathbb{R}} \left| \frac{\partial^2}{\partial x \partial u} \sigma(t, x, u) \right| \leq C;$$

$$\sup_{t \in [0, T], x \in \mathbb{R}} \left| \frac{\partial}{\partial u} \sigma(t, x, u_1) - \frac{\partial}{\partial u} \sigma(t, x, u_2) \right| \leq C |u_1 - u_2|$$

Theorem

Assume that $\sigma(t, x, u)$ satisfies the above hypothesis and that $l_0(t, x)$ is in $\mathcal{Z}^p(T)$ for some $p > \frac{2}{4H-1}$. Then the nonlinear SDE has a unique strong solution with sample paths in $\mathcal{C}([0, T] \times \mathbb{R})$ almost surely. Moreover, for any $\gamma < H - \frac{1}{p}$, the process $u(t, x)$ is almost surely Hölder continuous of exponent γ with respect to t and x on any compact sets in $[0, T] \times \mathbb{R}$.

Theorem

If the hyperbolic Anderson model has a solution in $\mathcal{Z}^p(T)$ for some $p \geq 2$ and for some $T \geq 0$, then the Hurst parameter H must satisfy $H > 1/4$.

3. Difficulty

Naive application of Picard iteration ($v = u^{n+1}$ and $u = u^n$):

$$v(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x - y) \sigma(s, y, u(s, y)) W(dy, ds)$$

Then following isometry equality

$$\begin{aligned} \mathbb{E} \left(v^2(t, x) \right) &= \xi_t^2(x) \\ &\quad + c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E} |G_{t-s}(x - y - z) \sigma(s, y + z, u(s, y + z)) \\ &\quad - G_{t-s}(x - y) \sigma(s, y, u(s, y))|^2 |z|^{2H-2} dy dz ds \\ &\leq \dots + \\ &\quad c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E} G_{t-s}^2(x - y) |u(s, y + z) - u(s, y)|^2 |z|^{2H-2} dy dz ds \end{aligned}$$

If $\sigma(t, x, u) = \sigma(u)$, then we may bound $\mathbb{E} (|v(t, x + h) - v(x)|^2)$ by

$$\sigma(v(s, y + h + z)) - \sigma(v(s, y + h)) - \sigma(v(s, y + z)) + \sigma(v(s, y))$$

If we want to consider

$\mathbb{E} (|v_1(t, x + h) - v_1(x) - v_2(t, x + h) + v_2(x)|^2)$ then we need to consider

the difference of

$$\sigma(v_1(s, y + h + z)) - \sigma(v_1(s, y + h)) - \sigma(v_1(s, y + z)) + \sigma(v_1(s, y))$$

and

$$\sigma(v_2(s, y + h + z)) - \sigma(v_2(s, y + h)) - \sigma(v_2(s, y + z)) + \sigma(v_2(s, y))$$

One difficulty is that we cannot no longer bound

$|\sigma(x_1) - \sigma(x_2) - \sigma(y_1) + \sigma(y_2)|$ by a multiple of $|x_1 - x_2 - y_1 + y_2|$ (which is possible only in the affine case).

4. Background

When $H > 1/2$ or when the noise is more regular, the equation was studied by many researchers.

R. Dalang, M. Sanz, C. Mueller, D. Nualart,

$\sigma(u) = au + b$: $H > 1/4$.

Balan, R.; Jolis, M. and Quer-Sardanyons, L.

SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4} < H < \frac{1}{2}$.

Electronic Journal of Probability 20 (2015).

Jian Song, Xiaoming Song, and Fangjun Xu

Fractional stochastic wave equation driven by a Gaussian noise rough in space.

Bernoulli 26(2020):2699-2726

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + \sigma(t, x, u(t, x)) \dot{W}, \quad t > 0, x \in \mathbb{R}.$$

General $\sigma(u)$ but with $\sigma(0) = 0$.

Hu, Yaozhong; Huang, Jingyu; Le, Khoa; Nualart, David;
Tindel, Samy

Stochastic heat equation with rough dependence in space.

Ann. Probab. 45 (2017), 4561-4616.

The condition $\sigma(t, x, 0) = 0$ is removed for the stochastic heat equation in

Hu, Y. and Wang, X.

Stochastic Heat Equation with general noise.

To appear in Ann.IHP.

Let $u_{\text{aff}}(t, x)$ be the solution to the stochastic heat equation with $\sigma(t, x, u) = 1$ and $u_0(x) = 0$:

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x) + \dot{W}, \quad t > 0, x \in \mathbb{R}.$$

Then, there are two positive constants c_H and C_H , independent of T and L , such that

$$\begin{aligned} c_H \rho(T, L) &\leq \mathbb{E} \left(\sup_{\substack{0 \leq t \leq T \\ -L \leq x \leq L}} u_{\text{aff}}(t, x) \right) \\ &\leq \mathbb{E} \left(\sup_{\substack{0 \leq t \leq T \\ -L \leq x \leq L}} |u_{\text{aff}}(t, x)| \right) \leq C_H \rho(T, L), \end{aligned}$$

where

$$\rho(T, L) = \begin{cases} T^{\frac{H}{2}} + T^{\frac{H}{2}} \sqrt{\log_2 \left[\frac{L}{\sqrt{T}} \right]} & \text{if } L^2 > T, \\ T^{\frac{H}{2}} & \text{if } L^2 \leq T. \end{cases}$$

5. Some ideas

We need to use localization argument.

$$\tau_k = \inf \left\{ t \in [0, T]; \sup_{0 \leq s \leq t, x \in \mathbb{R}} \left| \mathcal{N}_{\frac{1}{2}-H, p} u(s, x) \right| \geq k \right\}.$$

We need $\tau_k \uparrow T$ as $k \rightarrow \infty$. This imposes $\sigma(t, x, 0) = 0$.

To show this we need to bound

$$\mathbb{E} \left[\sup_{0 \leq t \leq T, x \in \mathbb{R}} |\mathcal{N}_{\frac{1}{2}-H, p} u(s, x)|^p \right].$$

Because the equation satisfied by the mild solution u , we need to bound

$$\mathbb{E} \left[\sup_{0 \leq t \leq T, x \in \mathbb{R}} \left| \int_0^s G_{s-r}(x-y) \sigma(r, y, u(r, y)) W(dr, dy) \right|^p \right].$$

But

$$\Phi(s, x) = \int_0^s G_{s-r}(x-y) \sigma(r, y, u(r, y)) W(dr, dy)$$

is not a martingale. We cannot use the Burkholder-Davis-Gundy inequality.

In the case of heat equation one can use the semigroup property of the heat kernel.

$$\Phi(t, x) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} G_{t-r}(x-z) Y(r, z) dz dr ,$$

with

$$Y(r, z) = \int_0^r \int_{\mathbb{R}} (r-s)^{-\alpha} G_{r-s}(z-y) v(s, y) W(ds, dy) ,$$

Then

$$|\Phi(t, x)| \leq C \int_0^t \left(\int_{\mathbb{R}} (t-r)^{p(\alpha-1)} G_{t-r}^p(x-z) dz \right)^{1/p} \left(\int_{\mathbb{R}} |Y(s, z)|^q dz \right)^{1/q} dr .$$

Of course we need to bound something like

$$|\Phi(t, x + h) - \Phi(t, x)|$$

The wave kernel $G_t(x) = \frac{1}{2} \mathbf{1}_{\{|x| < t\}}$ can be expressed as

$$\begin{aligned} G_{t-s}(x-y) &= \int_{\mathbb{R}} \mathcal{C}_{\beta}(t-r, x-z) \mathcal{S}_{1-\beta}(r-s, z-y) dz \\ &+ \int_{\mathbb{R}} \mathcal{S}_{\alpha}(t-r, x-z) \mathcal{C}_{1-\alpha}(r-s, z-y) dz \\ &+ \int_{\mathbb{R}} \mathcal{S}(t-r, x-z) \mathcal{E}(r-s, z-y) dz \\ &+ \int_{\mathbb{R}} \mathcal{E}(t-r, x-z) \mathcal{S}(r-s, z-y) dz, \end{aligned} \tag{4}$$

where $\alpha, \beta \in (0, 1)$, $\mathcal{S}(t, x) = \mathcal{S}_1(t, x) = G_t(x) = \frac{1}{2} \mathbf{1}_{\{|x| < t\}}$ and

$$\left\{ \begin{array}{l} \mathcal{E}(t, x) := \frac{1}{\pi} \frac{t}{t^2 + x^2}, \\ \mathcal{S}_\alpha(t, x) := \frac{\Gamma(1 - \alpha)}{2\pi} \cos \left[\frac{\alpha\pi}{2} \right] \left((t + |x|)^{\alpha-1} \right. \\ \quad \left. + \operatorname{sgn}(t - |x|) |t - |x||^{\alpha-1} \right), \\ \mathcal{C}_{1-\alpha}(t, x) := \frac{\Gamma(\alpha)}{2\pi} \left[\cos \left[\frac{\alpha\pi}{2} \right] [|t + |x||^{-\alpha} + |t - |x||^{-\alpha}] \right. \\ \quad \left. - 2 \cos \left[\alpha \tan^{-1} \left[\frac{|x|}{t} \right] \right] [t^2 + x^2]^{-\frac{\alpha}{2}} \right]. \end{array} \right. \quad (5)$$

THANKS