Stratonovich solution for the wave equation

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Outline

1. Introduction
2. Equation with mollified noise
3. Existence of Stratonovich solution

1. Introduction

Stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(x) & t>0, x \in \mathbb{R}^d \quad (d \leq 2) \\ u(0,x)=1, \quad \dfrac{\partial u}{\partial t}(0,x)=0 & x \in \mathbb{R}^d \end{cases} \tag{1}$$

Definition

A (mild) **solution** of (1) satisfies

$$u(t,x) = 1 + \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(dy) \right) ds, \tag{2}$$

where the stochastic integral $W(dy)$ has to be interpreted in some sense.

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Fundamental solution G

$$G(t,x)=\begin{cases} \frac{1}{2}1_{\{|x|<t\}} & \text{if } d=1, \\ \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|<t\}} & \text{if } d=2, \end{cases}$$

$|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$

Time-independent Gaussian noise

$W = \{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}^d)\}$ is a zero-mean Gaussian process

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\psi(y)\gamma(x-y)dxdy$$
$$= \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi) := \langle \varphi, \psi \rangle_{\mathcal{H}}$$

Covariance function γ

$\gamma : \mathbb{R}^d \to [0,\infty]$ is non-negative-definite; $\gamma = \mathcal{F}\mu$

$$\int_{\mathbb{R}^d} \varphi(x)\gamma(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi) \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d)$$

White noise case: $\gamma = \delta_0$ and $\mu(d\xi) = (2\pi)^{-d}d\xi$

$\{W(x) = W(1_{[0,x]})\}_{x\in\mathbb{R}^d}$ is a Brownian sheet

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Malliavin calculus

$\mathcal{H}$ is the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$

$W = \{W(\varphi); \varphi \in \mathcal{H}\}$ is an isonormal Gaussian process

Any $F \in L^2(\Omega)$ (measurable w.r.t. W) has the Wiener chaos expansion:

$$F = E(F) + \sum_{n \geq 1} I_n(f_n)$$

$I_0 : \mathbb{R} \to \mathbb{R}$ is the identity map ; $I_1(\varphi) = W(\varphi)$ for any $\varphi \in \mathcal{H}$

$I_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_n$ is the **multiple Wiener integral** of order n

$\mathcal{H}_n$ is the n-th Wiener chaos space corresponding to W

$$\mathbb{E}|F|^2 = \left(\mathbb{E}(F)\right)^2 + \sum_{n \geq 1} n! \|\widetilde{f}_n\|^2_{\mathcal{H}^{\otimes n}}$$

$\widetilde{f}_n$ is the symmetrization of f_n

Skorohod integral

$$\delta(u) := \int_{\mathbb{R}^d} u(x) W(\delta x) \qquad u \in \mathrm{Dom}\,\delta$$

$\delta : \mathrm{Dom}\,\delta \subset L^2(\Omega; \mathcal{H}) \to L^2(\Omega)$ is the adjoint of the Malliavin derivative $D : \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(\Omega; \mathcal{H})$

Stratonovich integral

$$\int_{\mathbb{R}^d} u(x) W^{\circ}(dx) \stackrel{P}{=} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} u(x) \dot{W}^{\varepsilon}(x) dx \quad \text{(if it exists)}$$

where $\dot{W}^{\varepsilon}(x) = W(p_{\varepsilon}(x - \cdot))$ and $p_{\varepsilon}(x) = (2\pi\varepsilon)^{-d/2} \exp(-\frac{|x|^2}{2\varepsilon})$

If $W = \{W(t)\}_{t \in [0,T]}$ is a fBm of index $H > 1/2$

$$\int_0^T u(t) W^{\circ}(dt) = \int_0^T u(t) W(\delta t) + \alpha_H \int_0^T \int_0^T D_s u(t) |t-s|^{2H-2} dt ds$$

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u is a (mild) **Skorohod solution** of stochastic wave equation (1) if

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta y) ds$$

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v is a (mild) **Stratonovich solution** of stochastic wave equation (1) if

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Fourier transform of G

$$\mathcal{F}G(t,\cdot)(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} G(t,x)dx = \frac{\sin(t|\xi|)}{|\xi|}$$

$$|\mathcal{F}G(t,\cdot)(\xi)| \leq C_t \left(\frac{1}{1+|\xi|^2}\right)^{1/2}$$

Condition (C):

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{1/2} \mu(d\xi) < \infty$$

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Dalang's condition (D):

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty,$$

If (D) holds, the **Skorohod solution exists** and has the chaos expansion

$$u(t,x) = 1 + \sum_{n \geq 1} I_n\big(f_n(\cdot, x; t)\big), \quad \text{with}$$

$$f_n(x_1, \ldots, x_n, x; t) = \int_{T_n(t)} \prod_{i=1}^{n} G(t_{i+1} - t_i, x_{i+1} - x_i) dt_1 \ldots dt_n$$

where $T_n(t) = \{0 < t_1 < \ldots < t_n < t\}$. Here $t_{n+1} = t$ and $x_{n+1} = x$.

Goal of this talk

Show that the **Stratonovich solution exists**, if (C) holds.

Note: (C) does not hold if W is white noise and $d = 1$

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2. Equation with mollified noise

Equation with mollified noise $\dot{W}^{\varepsilon}(x) = W(p_{\varepsilon}(x - \cdot))$

$$\begin{cases} \dfrac{\partial^2 v^{\varepsilon}}{\partial t^2}(t,x) = \Delta v^{\varepsilon}(t,x) + v^{\varepsilon}(t,x)\dot{W}^{\varepsilon}(x), & t > 0, x \in \mathbb{R}^d \quad (d \leq 2) \\ v^{\varepsilon}(0,x) = 1, \quad \dfrac{\partial v^{\varepsilon}}{\partial t}(0,x) = 0 & \end{cases} \tag{3}$$

Definition

v^{ε} is a **solution** of (3) if it satisfies

$$v^{\varepsilon}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) v^{\varepsilon}(s,y) \dot{W}^{\varepsilon}(y) dy ds.$$

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Series expansion

Intuitively, v^{ε} should be given by $v^{\varepsilon}(t,x) = 1 + \sum_{n\geq 1} H_n^{\varepsilon}(t,x)$ where

$$H_n^{\varepsilon}(t,x) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} G(t_{i+1}-t_i, x_{i+1}-x_i) \prod_{i=1}^{n} \dot{W}^{\varepsilon}(x_i) d\mathbf{x} d\mathbf{t}.$$

Lemma 1.

If (D) holds, for any $\varepsilon > 0$ fixed,

$$v^{\varepsilon}(t,x) := \lim_{n\to\infty} v_n^{\varepsilon}(t,x) \quad \text{exists in } L^p(\Omega)$$

uniformly in $(t,x) \in [0,T] \times \mathbb{R}^d$, for any $p \geq 1$, where

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Moreover, v^{ε} is a solution of equation (3).

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3. Existence of Stratonovich solution

Theorem 1

If (C) holds, there exists a process v such that for any $T > 0, p \geq 2$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|v^{\varepsilon}(t,x) - v(t,x)|^p \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{4}$$

Moreover, v is a Stratonovich solution of equation (1), and

$$v(t,x) = 1 + \sum_{n\geq 1} H_n(t,x),$$

where $H_n(t,x) = I_n^{\circ}\big(f_n(\cdot,x;t)\big)$ is the **multiple Stratonovich integral** of

$$f_n(x_1,\ldots,x_n,x;t) = \int_{T_n(t)} \prod_{i=1}^{n} G(t_{i+1}-t_i, x_{i+1}-x_i)d\mathbf{t}.$$

Idea for the proof of (4):

Step 1. Show that, for any n, uniformly in $t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$v_n^{\varepsilon}(t,x) \stackrel{L^p(\Omega)}{\longrightarrow} \text{some } v_n(t,x) \quad \text{as} \quad \varepsilon \downarrow 0$$

Step 2. Show that, uniformly in $t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$v_n(t,x) \stackrel{L^p(\Omega)}{\longrightarrow} \text{some } v(t,x) \quad \text{as} \quad n \to \infty$$

Step 3. Show that, uniformly in $t \in [0, T], x \in \mathbb{R}^d$ and $\varepsilon > 0$,

$$v_n^{\varepsilon}(t,x) \stackrel{L^p(\Omega)}{\longrightarrow} v^{\varepsilon}(t,x) \quad n \to \infty$$

Diagram summarizing the 3 steps:

$$\begin{array}{cccc} & v_n^{\varepsilon}(t,x) & \longrightarrow_{\varepsilon} & v_n(t,x) \quad \text{for any } n \\ \text{(uniform in } \varepsilon) & \downarrow_n & & \downarrow_n \\ & v^{\varepsilon}(t,x) & - - - \rightarrow_{\varepsilon} & v(t,x) \end{array}$$

A brief review: multiple Stratonovich integrals

Definition

The multiple Stratonovich integral is given by

$$I_n^{\circ}(f) := \int_{(\mathbb{R}^d)^n} f(x_1,\ldots,x_n)\prod_{i=1}^{n} W^{\circ}(dx_i)$$

$$\stackrel{P}{=} \lim_{\varepsilon\downarrow 0}\int_{(\mathbb{R}^d)^n} f(x_1,\ldots,x_n)\prod_{i=1}^{n}\dot{W}^{\varepsilon}(x_i)dx_1\ldots dx_n \quad \text{(if it exists)}$$

Basic idea for proving existence of $I_n^{\circ}(f)$

Compute

$$\prod_{i=1}^{n}\dot{W}^{\varepsilon}(x_i)=\prod_{i=1}^{n} I_1(p_{\varepsilon}(x_i-\cdot))$$

using the **product formula** from Malliavin calculus.

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Product Formula

$$I_n(f)I_1(g) = I_{n+1}(f \otimes g) + nI_{n-1}(f \otimes_1 g)$$

where $(f \otimes g)(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_n)g(x_{n+1})$ and

$$\begin{aligned}(f \otimes_1 g)(x_1, \ldots, x_{n-1}) &= \langle f(\cdot, x_1, \ldots, x_{n-1}), g\rangle_{\mathcal{H}} \\ &= \int_{(\mathbb{R}^d)^2} f(y, x_1, \ldots, x_{n-1})g(z)\gamma(y-z)dydz.\end{aligned}$$

Application of product formula

$$I_1(f)I_1(g) = I_2(f \otimes g) + \langle f, g\rangle_{\mathcal{H}}$$

$$I_1(f)I_1(g)I_1(h) = I_3(f \otimes g \otimes h) + I_1(f)\langle g, h\rangle_{\mathcal{H}} + I_1(g)\langle f, h\rangle_{\mathcal{H}} + I_1(h)\langle f, g\rangle_{\mathcal{H}}$$

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Notation

- $[n] = \{1, \ldots, n\}$
- $|J|$ is the cardinality of J, for any set $J \subset [n]$
- J^c is the complement of J in $[n]$
- $\lfloor x \rfloor = k$ if $k \in \mathbb{Z}$ and $k \leq x < k+1$, for any $x \in \mathbb{R}$

Product of n Wiener integrals

$$\prod_{j=1}^{n} I_1(f_j) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J \subset [n] \\ |J| = n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i = \{\ell_i, m_i\} \forall i=1,\ldots,k}} I_{n-2k}\Big(\bigotimes_{j \in J} f_j\Big) \prod_{i=1}^{k} \langle f_{\ell_i}, f_{m_i} \rangle_{\mathcal{H}},$$

Convention: $\otimes_{j \in \emptyset} f_j = 1$.

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Basic observation

$$\langle p_\varepsilon(x_1-\cdot), p_\varepsilon(x_2-\cdot)\rangle_{\mathcal{H}} = (p_{2\varepsilon} * \gamma)(x_1-x_2)$$

$p_{2\varepsilon} * \gamma = \mathcal{F}\mu_\varepsilon$ in the sense of distributions, where

$$\mu_\varepsilon(d\xi) = e^{-\varepsilon|\xi|^2}\mu(d\xi)$$

General formula

$$\prod_{i=1}^{n} \dot{W}^\varepsilon(x_i) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J\subset[n] \\ |J|=n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i=\{\ell_i,m_i\}\forall i=1,\ldots,k}} I_{n-2k}\Big(\bigotimes_{j\in J} p_\varepsilon(x_j-\cdot)\Big) \prod_{i=1}^{k}(p_{2\varepsilon}*\gamma)(x_{\ell_i}-x_{m_i})$$

Basic observation

$$\langle p_\varepsilon(x_1-\cdot), p_\varepsilon(x_2-\cdot)\rangle_{\mathcal{H}} = (p_{2\varepsilon}*\gamma)(x_1-x_2)$$

$p_{2\varepsilon}*\gamma = \mathcal{F}\mu_\varepsilon$ in the sense of distributions, where

$$\mu_\varepsilon(d\xi) = e^{-\varepsilon|\xi|^2}\mu(d\xi)$$

General formula

$$\prod_{i=1}^{n} \dot{W}^\varepsilon(x_i) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J\subset[n] \\ |J|=n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i=\{\ell_i,m_i\}\forall i=1,\ldots,k}} I_{n-2k}\Big(\bigotimes_{j\in J} p_\varepsilon(x_j-\cdot)\Big)$$
$$\prod_{i=1}^{k}(p_{2\varepsilon}*\gamma)(x_{\ell_i}-x_{m_i})$$

Back to our problem

Step 1: convergence as $\varepsilon \downarrow 0$

We have to show that for any n fixed, there exists some v_n such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|v_n^{\varepsilon}(t,x) - v_n(t,x)|^p \to 0 \quad \text{as} \quad \varepsilon \downarrow 0$$

where

$$v_n^{\varepsilon}(t,x) = 1 + \sum_{k=1}^{n} H_k^{\varepsilon}(t,x)$$

$$H_n^{\varepsilon}(t,x) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} G(t_{i+1}-t_i, x_{i+1}-x_i) \prod_{i=1}^{n} \dot{W}^{\varepsilon}(x_i) d\mathbf{x} d\mathbf{t}.$$

Theorem 2.

If (C) holds, then for any $n \geq 1, p \geq 2, T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|H_n^{\varepsilon}(t,x) - H_n(t,x)|^p \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

where

$$H_n(t,x) := I_n^{\circ}\big(f_n(\cdot,x;t)\big) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J\subset[n] \\ |J|=n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i=\{\ell_i,m_i\}\forall i=1,\ldots,k}}$$

$$\int_{T_n(t)} \int_{(\mathbb{R}^d)^{2k}} \left(\int_{(\mathbb{R}^d)^{n-2k}} \prod_{j=1}^{n} G(t_{j+1}-t_j, x_{j+1}-x_j) \prod_{j\in J} W(dx_j) \right)$$

$$\prod_{i=1}^{k} \gamma(x_{\ell_i} - x_{m_i}) d\big((x_j)_{j\in J^c}\big) dt_1 \ldots dt_n.$$

Structure of $H_n(t,x)$

- Choose $k = 0, 1, \ldots, \lfloor n/2 \rfloor$
- Choose a set $J \subset [n]$ with $|J| = n - 2k$; there are

$$\binom{n}{2k} \quad \text{possibilities}$$

- Choose an (unordered) partition $\{I_1, \ldots, I_k\}$ of J^c such that $|I_i| = 2$ for any $i = 1, \ldots, k$; there are

$$\frac{(2k)!}{2^k k!} \quad \text{possibilities}$$

- Build a mixed (stochastic/deterministic) integral:

$$\prod_{j \in J} W(dx_j) \prod_{j \in J^c} dx_j$$

Steps 2 and 3: convergence as $n \to \infty$

Theorem 3

If (C) holds, then for any $T > 0$ and $p \geq 2$, the limit

$$v(t,x) := \lim_{n\to\infty} v_n(t,x) \quad \text{exists in } L^p(\Omega),$$

uniformly in $[0,T] \times \mathbb{R}^d$. Moreover,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \sup_{\varepsilon>0} \mathbb{E}|v_n^{\varepsilon}(t,x) - v^{\varepsilon}(t,x)|^p \to 0 \text{ as } \quad n \to \infty.$$

Sketch of proof:

$$\sum_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|H_n(t,x)\|_p < \infty$$

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Thank you!