Convergence of densities for the stochastic heat equation

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Stochastic heat equation

- Consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\frac{\partial^2 W}{\partial t \partial x}, \quad x \in \mathbb{R},\ t \geq 0,$$

  with initial condition $u_0(x) = 1$.
- $\frac{\partial^2 W}{\partial t \partial x}$ is a space-time white noise.
- σ is a Lipschitz function such that $\sigma(1) \neq 0$.

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Mild solution

Theorem (Walsh '86)

There is a unique mild solution, which is an adapted random field u such that for all $p \geq 2$,

$$\sup_{x \in \mathbb{R}} \sup_{0 \leq t \leq T} \mathbb{E}[|u(t,x)|^p] < \infty,$$

and u satisfies the integral equation:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy),$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t)$.

Space averages

Remark: Fix $t > 0$. The process $x \mapsto u(t, x)$ is stationary.

- We are interested in the asymptotic behavior as $R \to \infty$ of the random variable

$$\boxed{\int_{-R}^{R} u(t,x)dx}$$

- The mean is given by

$$\mathbb{E}\left(\int_{-R}^{R} u(t,x)dx\right) = 2R.$$

- We put

$$F_{R,t} := \int_{-R}^{R} u(t,x)dx - 2R$$

$$= \int_0^t \int_{\mathbb{R}} \left(\int_{-R}^{R} p_{t-s}(x-y)dx\right) \sigma(u(s,y))W(ds,dy).$$

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Computation of the variance of $F_{R,t}$

- Using the isometry property of the stochastic integral we obtain:

$$\begin{aligned}
\mathrm{Var}(F_{R,t}) &:= \int_0^t \int_{\mathbb{R}} \left(\int_{-R}^{R} p_{t-s}(x-y)dx \right)^2 \mathbb{E}[\sigma(u(s,y))^2] dyds \\
&= \int_0^t \xi(s) \int_{\mathbb{R}} \left(\int_{-R}^{R} p_{t-s}(x-y)dx \right)^2 dyds \\
&= \int_0^t \xi(s) \int_{[-R,R]^2} p_{2(t-s)}(x-x')dxdx'ds \\
&\approx 2R \int_0^t \xi(s)ds,
\end{aligned}$$

as $R \to \infty$, with $\xi(s) = \mathbb{E}[\sigma(u(s,y))^2]$.

Functional CLT

Theorem (Huang-N.-Viitasaari '20)

Set $\xi(s) = \mathbb{E}[\sigma(u(s,y))^2]$ for any $s \geq 0$. Then

$$\left(\frac{1}{\sqrt{R}}\left(\int_{-R}^{R} u(t,x)dx - 2R\right)\right)_{t\in[0,T]} \xrightarrow{\mathcal{L}} \left(\int_0^t \sqrt{2\xi(s)}dB_s\right)_{t\in[0,T]},$$

as R tends to infinity, where B is a Brownian motion and the convergence is in the space of continuous functions $C([0,T])$.

Quantitative CLT

- The total variation distance between two random variables F and G is defined by

$$d_{TV}(F,G) = \sup_{B\in\mathcal{B}(\mathbb{R})} |P(F\in B) - P(G\in B)|.$$

Theorem (Huang-N.-Viitasaari '20)

Let $Z \sim N(0,1)$ and fix $t > 0$. Then there exists a constant $c(t)$, depending on t, such that

$$d_{TV}\left(\frac{F_{R,t}}{\sqrt{\mathrm{Var}(F_{R,t})}}, Z\right) \leq \frac{c(t)}{\sqrt{R}},$$

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Malliavin Calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $h_i \in \mathfrak{H} = L^2(\mathbb{R}_+ \times \mathbb{R})$ and $f \in C_b^\infty(\mathbb{R}^n)$.

- If $F \in \mathcal{S}$ we define its *derivative* by

$$D_{s,y}F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i(s,y).$$

DF is a random variable with values in $\mathfrak{H}$.

- Sobolev spaces: For $p \geq 1$, $\mathbb{D}^{k,p} \subset L^p(\Omega; \mathfrak{H})$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|DF\|_{k,p} = \sum_{j=0}^{k} \left(\mathbb{E}(\|D^j F\|^p_{\mathfrak{H}^{\otimes j}}) \right)^{1/p}.$$

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- The adjoint of D is the *divergence* operator δ defined by the duality relationship
$$\mathbb{E}(\langle DF, v\rangle_{\mathfrak{H}}) = \mathbb{E}(F\delta(v))$$
for any $F \in \mathbb{D}^{1,2}$ and $v \in \mathrm{Dom}\delta \subset L^2(\Omega; \mathfrak{H})$.
- If $v \in L^2(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ is a square integrable adapted random field, then v belongs to the domain of δ and $\delta(v)$ coincides with the Itô-Walsh integral of v:
$$\delta(v) = \int_{\mathbb{R}_+ \times \mathbb{R}} v(s, y) W(ds, dy).$$

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Malliavin-Stein method

Let $W = \{W(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process defined on $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is generated by W.

Theorem (Nourdin-Peccati '08)

Suppose that $F \in \mathbb{D}^{1,2}$ satisfies $\mathbb{E}[F^2] = 1$ and $F = \delta(v)$, where v belongs to $\mathrm{Dom}\delta$. Then,

$$d_{TV}(F, Z) \leq 2\sqrt{\mathrm{Var}(D_v F)},$$

where Z is $N(0,1)$ and we use the notation $D_v F = \langle DF, v\rangle_{\mathfrak{H}}$.

Remark: Because

$$\mathbb{E}[D_v F] = \mathbb{E}[F\delta(v)] = \mathbb{E}[F^2] = 1$$

we have

$$\mathrm{Var}(D_v F) = \mathbb{E}[|1 - D_v F|^2].$$

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Proof of the estimate $d_{TV}\left(\frac{F_{R,t}}{\sqrt{\mathrm{Var}(F_{R,t})}}, Z\right) \le c(t)R^{-1/2}$:

- We have

$$F_{R,t} = \int_{-R}^{R} [u(t,x) - 1]dx$$
$$= \int_0^t \int_{\mathbb{R}} \sigma(u(s,y)) \left(\int_{-R}^{R} p_{t-s}(x-y)dx \right) W(ds,dy).$$

- Thus,

$$F_{R,t} = \delta(v_{R,t}),$$

where, for $s \le t$

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Two basic ingredients:

1. Poincaré inequality: For all $F, G \in \mathbb{D}^{1,2}$,

$$|\mathrm{Cov}(F, G)| \leq \int_0^\infty \int_{\mathbb{R}} \|D_{s,y}F\|_2 \|D_{s,y}G\|_2 dyds.$$

2. Estimate on the p-norm of derivative of the solution:

$$\|D_{s,y}u(t,x)\|_p \leq Cp_{t-s}(x-y),$$

where C depends on p, t and σ.

- This inequality follows from:

$$D_{s,y}u(t,x) = p_{t-s}(x-y)\sigma(u(s,y)) + \int_s^t \int_{\mathbb{R}} p_{t-r}(x-z)\sigma'(u(r,z))D_{s,y}u(r,z)W(dr,dz).$$

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Convergence of densities

- The total variation distance is equivalent to L^1 norm of the densities:

$$d_{TV}(F,Z) = \int_{\mathbb{R}} |p_F(x) - \phi(x)| dx,$$

where ϕ is the density of the law $N(0,1)$.
Uniform convergence, however, requires stronger hypotheses.

Theorem (N.-Kuzgun '21)

Let $v \in \mathbb{D}^{1,6}(\mathfrak{H})$ and $F = \delta(v) \in \mathbb{D}^{2,6}$ with $\mathbb{E}[F^2] = 1$ and $(D_v F)^{-1} \in L^4(\Omega)$. Then, F admits a density $p_F(x)$ that satisfies

$$\sup_{x \in \mathbb{R}} |p_F(x) - \phi(x)| \leq \left(\|F\|_4 \|(D_v F)^{-1}\|_4 + 2 \right) \sqrt{\mathrm{Var}(D_v F)} + \|(D_v F)^{-1}\|_4^2 \|D_v(D_v F)\|_2.$$

- A different version of this inequality was obtained in Hu-Lu-N. '13.

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Sketch of the proof:

(i) Density formula:

$$p_F(x) = \mathbb{E}\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{v}{D_v F}\right)\right].$$

(ii) We have

$$\delta\left(\frac{v}{D_v F}\right) = \frac{F}{D_v F} - D_v((D_v F)^{-1}).$$

Therefore,

$$\begin{aligned} p_F(x) &= \mathbb{E}\left[\mathbf{1}_{\{F>x\}}\frac{F}{D_v F}\right] - \mathbb{E}[\mathbf{1}_{\{F>x\}}D_v((D_v F)^{-1})] \\ &= \mathbb{E}[\mathbf{1}_{\{F>x\}}F] + \mathbb{E}[\mathbf{1}_{\{F>x\}}F((D_v F)^{-1} - 1)] \\ &\quad - \mathbb{E}[\mathbf{1}_{\{F>x\}}D_v((D_v F)^{-1})]. \end{aligned}$$

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(iii) We easily have the estimates

$$\mathbb{E}[|F((D_vF)^{-1}-1)|] \leq \|F\|_4 \|(D_vF)^{-1}\|_4 \sqrt{\mathrm{Var}(D_vF)}$$

and

$$\mathbb{E}[|D_v((D_vF)^{-1})|] \leq \|(D_vF)^{-1}\|_4^2 \|D_v(D_vF)\|_2.$$

(iii) Taking into account that

$$\phi(x) = \mathbb{E}[\mathbf{1}_{\{Z>x\}} Z],$$

where $Z \sim N(0,1)$, it suffices to show the estimate

$$\left| \mathbb{E}[\mathbf{1}_{\{F>x\}} F] - \mathbb{E}[\mathbf{1}_{\{Z>x\}} Z] \right| \leq C \sqrt{\mathrm{Var}(D_v F)},$$

which can be done by Stein's method and Malliavin calculus.

Application to the stochastic heat equation:

Theorem (Kuzgun-N. '21)

Let $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ *be the solution to the stochastic heat equation. Assume:*

(i) $\sigma \in C^2(\mathbb{R})$, σ' *is bounded and* $|\sigma''(x)| \leq C(1+|x|^m)$ *for some* $m > 0$.

(ii) *For some* $q > 10$ *and* $t > 0$, $\mathbb{E}[|\sigma(u(t,0))|^{-q}] < \infty$.

Then,

$$\sup_{x \in \mathbb{R}} \left| p_{F_{R,t}/\sqrt{\mathrm{Var}(F_{R,t})}} - \phi(x) \right| \leq \frac{c(t)}{\sqrt{R}}.$$

- Condition (ii) is satisfied if σ is bounded below or if $\sigma(x) = x$.

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Estimate of the p-norm of the second derivative:

Proposition

Suppose $0 \le r < s < t \le T$. *Then,*

$$\|D_{r,z}D_{s,y}u(t,x)\|_p \le Cp_{t-s}(x-y)p_{s-r}(y-z) + Cp_{t-s}(x-y)\frac{p_{t-r}(y-z)+p_{t-r}(x-y)+\mathbf{1}_{\{|x-y|\ge|y-z|\}}}{(r-s)^{1/4}}.$$

- In the case $\sigma(x) = x$, the estimate

$$\|D_{r,z}D_{s,y}u(t,x)\|_p \le Cp_{t-s}(x-y)p_{s-r}(y-z)$$

  was obtained by Chen-Khoshnevisan-N.-Pu '21.

Delta initial condition

Suppose $u(0,x) = \delta_0(x)$ and $\sigma(u) = u$. The mild solution is

$$u(t,x) = p_t(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)u(s,y)W(ds,dy).$$

- For any fixed $t > 0$, the process $\left\{\frac{u(t,x)}{p_t(x)}, x \in \mathbb{R}\right\}$ is stationary (Amir-Corwin-Quastel '11).
- We set

$$G_{R,t} = \int_{-R}^{R} \frac{u(t,x)}{p_t(x)} dx - 2R.$$

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1. (Chen-Khoshnevisan-N.-Pu '21) As $R \to \infty$,

$$\left\{(R \log R)^{-1/2} G_{R,t}\right\}_{t \in [0,T]} \xrightarrow{\mathcal{L}} 2B,$$

where B is a Brownian motion.

2. (Chen-Khoshnevisan-N.-Pu '21) There exists a constant $c(t)$, such that

$$d_{TV}\left(\frac{G_{R,t}}{\sqrt{\mathrm{Var}(F_{R,t})}}, Z\right) \leq c(t)\sqrt{\frac{\log R}{R}},$$

where Z has law $N(0,1)$.

3. (Kuzgun-N. '21) Fix $\gamma > \frac{19}{2}$. Then

$$\sup_{x \in \mathbb{R}} |p_{G_{R,t}/\sqrt{\mathrm{Var}(G_{R,t})}} - \phi(x)| \leq \frac{c(t)(\log R)^{\gamma}}{\sqrt{R}}.$$

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