The Interpolated Stochastic Heat and Wave Equation: Solvability and Exact Moment Asymptotics

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(Chen-E. - arXiv:2108.11473)

The Stochastic Heat and Wave Equations

$$
\begin{cases}
\left(\partial_t^b - \Delta\right)\ u(t,x) = u(t,x)\ \dot{W}(x) & x \in \mathbb{R}^d,\ t > 0 \\
u(0,\cdot) = 1 & b = 1 \quad \text{(SHE)} \\
u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b = 2 \quad \text{(SWE)}
\end{cases}
$$

- $W = \{W(\phi) : \phi \in \mathcal{D}(\mathbb{R}^d)\}$ is a centered and time-independent Gaussian noise
- The choice of initial condition is such that the solution to the homogeneous equation is constant one.
- The solution is understood in the *Skorohod* sense.

For the SHE, i.e., $b = 1$ ($a = 2$, $\nu = 2$)

[X. Chen '17]

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{2}{2-\alpha}}(2-\alpha)\left(\frac{2\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2-\alpha}}$$

For the SWE, i.e., $b = 2$ ($a = 2$, $\nu = 2$)

[Balan, L. Chen, and X. Chen '21]

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{1}{3-\alpha}}\left(\frac{1}{2}\right)^{\frac{\alpha}{2(3-\alpha)}} \frac{3-\alpha}{2}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

For the SHE, i.e., $b = 1$ ($a = 2$, $\nu = 2$)
[X. Chen '17]

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{2}{2-\alpha}}(2-\alpha)\left(\frac{2\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2-\alpha}}$$

$b \in (0,2)$ and $a \in (0,2]$??

For the SWE, i.e., $b = 2$ ($a = 2$, $\nu = 2$)
[Balan, L. Chen, and X. Chen '21]

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}|u(t,x)|^p = p(p-1)^{\frac{1}{3-\alpha}}\left(\frac{1}{2}\right)^{\frac{\alpha}{2(3-\alpha)}}\frac{3-\alpha}{2}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

The Interpolated Stochastic Heat and Wave Equation

$$
\begin{cases}
\left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r \left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right] & x \in \mathbb{R}^d,\ t > 0 \\
u(0,\cdot) = 1 & b \in (0,1] \\
u(0,\cdot) = 1, \quad \partial_t u(0,\cdot) = 0 & b \in (1,2)
\end{cases}
\tag{ISHWE}
$$

- ∂_t^b is the *Caputo* fractional derivative
- $(-\Delta)^{a/2}$ is the fractional Laplacian of order $a \in (0,2]$
- I_t^r is the Riemann-Liouville fractional integral of order $r \geq 0$

The Noise, $\dot{W}$.

The time independent noise informally satisfies

$$\mathbb{E}(\dot{W}(x)\dot{W}(y)) = \gamma(x-y).$$

The spatial correlation and spectral density, γ and φ, for the noise $\dot{W}$ can be assumed to be any of the following:

$$\gamma(x) = |x|^{-\alpha}, \qquad \varphi(\xi) = C_1|\xi|^{d-\alpha} \qquad \alpha \in (0,d)$$

$$\gamma(x) = \prod_{i=1}^{d} |x_i|^{-\alpha_i}, \qquad \varphi(\xi) = C_2 \prod_{i=1}^{d} |\xi_i|^{1-\alpha_i} \qquad \alpha_i \in (0,1),\ \alpha = \sum_i \alpha_i$$

Mild Solution

Definition

For $T > 0$, a random field $u = \{u(t,x) : t \in (0,T), x \in \mathbb{R}^d\}$ is called a *mild solution* if $G(t-s, x-\cdot)u(s,\cdot)1_{\{s<t\}}$ is Skorohod integrable and the following holds almost surely:

$$u(t,x) = 1 + \sqrt{\theta}\int_0^t \left(\int_{\mathbb{R}^d} G(t-s,x-y)u(s,y)W(\delta y)\right)\mathrm{d}s$$

where G is defined through the Fox-H function.

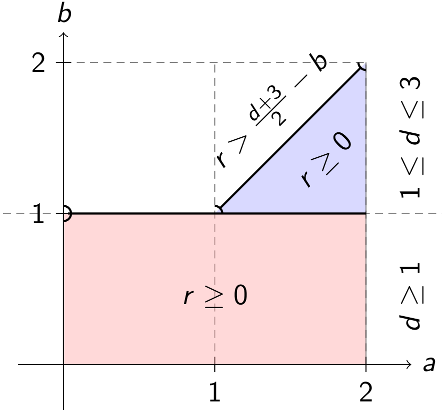
An important characteristic of G is that

$$\mathcal{F}\mathcal{L}G(s,\xi) = \frac{s^{-r}}{s^b + \frac{\nu}{2}|\xi|^a}$$

Nonnegativity assumption on G

Under any of the following cases, G is nonnegative [Chen, Hu, Nualart '19]:

- $d \geq 1$, $b \in (0,1]$, $a \in (0,2]$, $r \geq 0$;
- $1 \leq d \leq 3$, $1 < b < a \leq 2$, $r > 0$;
- $1 \leq d \leq 3$, $1 < b = a < 2$, $r > \dfrac{d+3}{2} - b$.



Global Solution

$\left(\partial_t^b + \frac{\nu}{2}(-\Delta)^{a/2}\right) u(t,x) = I_t^r \left[\sqrt{\theta}\, u(t,x)\, \dot{W}(x)\right]$

Definition

$u(t,x)$ is a *global solution* to the (ISHWE) if $\|u(t,x)\|_p < \infty$ for any $t > 0$ and $x \in \mathbb{R}^d$.

Theorem (Chen-E. '21+)

A global solution exists provided G is nonnegative and

$$0 < \alpha < \min\left(\frac{a}{b}[2(b+r)-1], 2a, d\right)$$

Local Solution

Definition

$u(t,x)$ is a *local solution* to the (ISHWE) if there exists $0 < T_a \leq T_b < \infty$ such that $\|u(t,x)\|_2 < \infty$ when $0 < t < T_a$ and $\|u(t,x)\|_2$ D.N.E. (does not exist) for $t > T_b$.

Theorem (Chen-E. '21+)

A local solution exists provided G is nonnegative and if

$$r \in [0, 1/2] \qquad \textit{and} \qquad 0 < \alpha = \frac{a}{b}[2(b+r)-1] \leq d.$$

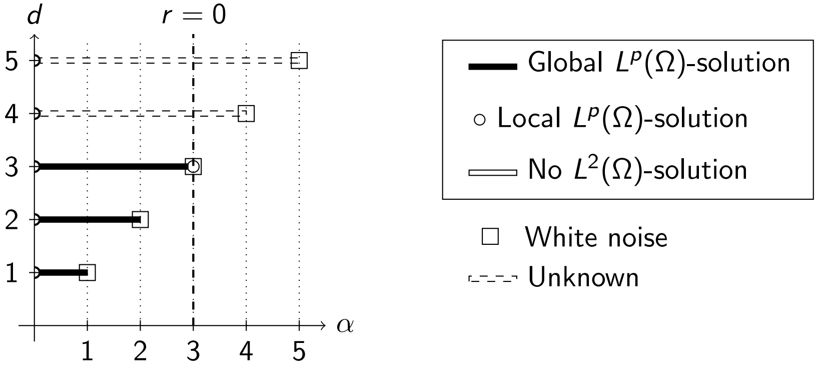
In this case, a unique $L^p(\Omega)$ solution exists for $t \in (0, T_p)$ where

$$T_p := \frac{\nu^{\alpha/a}}{2\theta(p-1)\mathcal{M}_a^{(2a-\alpha)/a}}$$

and the solution does not exist for $t > T_2$.

Example: Solvability for the SWE ($a = b = 2$)

A local solution only exists when $\alpha = 3 + 2r \leq d \leq 3$.

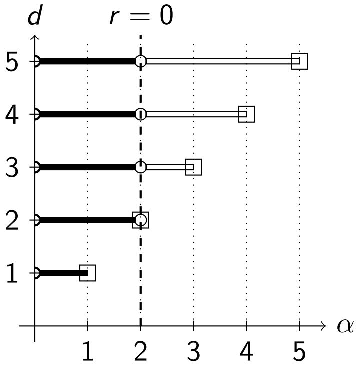


By replacing $\nu = 2$ in the following, we recover (1.12) [Balan, L. Chen & X. Chen '21]

$$T_p = \frac{\nu^{3/2}}{2\theta(p-1)\sqrt{\mathcal{M}_{2,3}(\delta_0)}}, \quad p \geq 2$$

Example: Solvability for the SHE ($a = 2$ and $b = 1$)

By setting $a = 2$, $b = 1$ and $r = 0$, we obtain the following condition for existence of a local solution: $\alpha = 2 \le d$.



- When $\alpha = d = 2$, the critical time becomes $T_p = \dfrac{\nu}{2\theta(p-1)\mathcal{M}_{2,2}(\delta_0)}$.
- Theorem 4.1 [Y. Hu, '01] proves that an $L^2(\Omega)$ solution exists for $t < 2$ but not for $t > 2\pi$.
- T_2 being precise implies $2 \le T_2 = \frac{1}{2\mathcal{M}_{2,2}(\delta_0)} \le 2\pi$

Wiener Chaos Expansion

Through a standard procedure, we may define

$$f_n(x_1, \cdots, x_n; x, t) = \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \prod_{k=1}^{n} G(t_{k+1} - t_k, x_{k+1} - x_k) \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

where $t = t_{n+1}$ and $x = x_{n+1}$, and say that

1. $u(t,x) = 1 + \sum_{k=1}^{\infty} \theta^{k/2} I_k(f_k(\cdot, x, t)), \quad (t,x) \in (0,T) \times \mathbb{R}^d$
2. $\mathbb{E}(u(t,x)^2) = \sum_{k=0}^{\infty} \theta^k \left\| f_k(\cdot, x, t) \right\|_{\mathcal{H}^{\otimes n}}^2, \quad (t,x) \in (0,T) \times \mathbb{R}^d.$

Through a change of variable, one can easily show that

1. $\mathcal{F}G(t,\cdot)(c\xi) = c^{-\frac{a}{b}(b+r-1)} \mathcal{F}G\left(c^{\frac{a}{b}} t, \cdot\right)(\xi)$
2. $\left\| \widetilde{f}_n(\cdot, 0, t) \right\|_{\mathcal{H}^{\otimes n}}^2 = t^{[2(b+r) - b\alpha/a]n} \left\| \widetilde{f}_n(\cdot, 0, 1) \right\|_{\mathcal{H}^{\otimes n}}^2$

How to Find the Critical α

For simplicity we consider the SWE ($a = b = 2$ and $r = 0$).

$$\|u(t,x)\|_2^2 = \sum_{n\geq 0} \theta^n n! \overbrace{t^{[4-\alpha]n} \left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2}^{\left\|\widetilde{f}_n(\cdot,0,t)\right\|_{\mathcal{H}^{\otimes n}}^2}$$

$$= \sum_{n\geq 0} \frac{\theta^n t^{(4-\alpha)n}}{(n!)^{3-\alpha}} R_n, \quad R_n = (n!)^{4-\alpha} \left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2$$

When $\alpha = 3$, then the above reduces down to

$$\|u(t,x)\|_2^2 = \sum_{n\geq 0} (\theta t)^n n! \left\|\widetilde{f}_n(\cdot,0,1)\right\|_{\mathcal{H}^{\otimes n}}^2 .$$

and we lose the $n!$ term in the demonstrator.

ρ

We define

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x)$$

Theorem (X. Chen '07)

$$\lim_{n\to\infty} \frac{1}{n} \log \left[\frac{1}{(n!)^2} \int_{(\mathbb{R}^d)^n} \left(\sum_{\sigma\in\Sigma_n} \prod_{k=1}^{n} \frac{1}{1+\frac{\nu}{2}\left|\sum_{j=k}^{n} \xi_{\sigma(j)}\right|^a} \right)^2 \mu(\mathrm{d}\vec{\xi}) \right]$$
$$= \log\left(\rho_{\nu,a}(\gamma)\right).$$

where γ is the spatial correlation function.

Connection of the SPDE with ρ

Recall from above that

$$\mathcal{F}\mathcal{L}G(1,\xi) = \frac{1}{1+\frac{\nu}{2}|\xi|^a}.$$

By using this relation, it can be shown that

$$\mathcal{F}\mathcal{L}(\tilde{f}_n)(1,\xi) = \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\prod_{k=1}^{n}\frac{1}{1+\frac{\nu}{2}|\sum_{j=k}^{n}\xi_{\sigma(j)}|^a}.$$

Applying this to the limit representation of ρ above yields that

$$\lim_{n\to\infty}\frac{1}{n}\log\left[\int_{(\mathbb{R}^d)^n}\left|\mathcal{F}\mathcal{L}(\tilde{f}_n)(1,\xi)\right|^2\mu(\mathrm{d}\vec{\xi})\right] = \log\left(\rho_{\nu,a}\left(\gamma\right)\right).$$

Recall the scaling property

$$\left\|\tilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}} = t^{(4-\alpha)n}\left\|\tilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}.$$

Using this scaling property we see that

$$\int_0^\infty e^{-t}\left\|\tilde{f}_n(\cdot,0,t)\right\|^2_{\mathcal{H}^{\otimes n}}\,\mathrm{d}t = \Gamma((4-\alpha)n+1)\left\|\tilde{f}_n(\cdot,0,1)\right\|^2_{\mathcal{H}^{\otimes n}}.$$

Lemma (R. Balan, L. Chen & X. Chen '21)

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\Gamma((4-\alpha)n+1)\left\|\tilde{f}_n(\cdot,0;1)\right\|^2_{\mathcal{H}^{\otimes n}}\right) = \log\left[2^{4-\alpha}\rho\right]$$

$$\lim_{n\to\infty}\frac{1}{n}\log\left((n!)^{4-\alpha}\left\|\tilde{f}_n(\cdot,0;1)\right\|^2_{\mathcal{H}^{\otimes n}}\right) = \log\left[\left(\frac{2}{4-\alpha}\right)^{4-\alpha}\rho\right]$$

How to Find the Blowup Time (SWE)

Recall from above that the critical case corresonds to $\alpha = 3 = d$.

$$
\begin{aligned}
\|u(t,x)\|_2^2 &= \sum_{n\geq 0} \theta^n n! \left\| \tilde{f}(\cdot, 0, t) \right\|_{H^{\otimes n}}^2 \\
&= \sum_{n\geq 0} (\theta t)^n n! \left\| \tilde{f}(\cdot, 0, 1) \right\|_{H^{\otimes n}}^2 \\
&=: \sum_{n\geq 0} (\theta t)^n R_n, \quad R_n = n! \left\| \tilde{f}(\cdot, 0, 1) \right\|_{H^{\otimes n}}^2 .
\end{aligned}
$$

The above lemma says that $\frac{1}{n} \log R_n \to \log(2\rho_c)$. Now the Cauchy-Hadamard theorem can be directly applied to see that the radius of convergence is precisely $(2\theta\rho_c)^{-1} = T_2$.

$\mathcal{M}$

We define

$$\mathcal{M}_{a,d}(\gamma,\theta) := \sup_{g\in\mathcal{F}_a}\left\{\left(\iint_{\mathbb{R}^{2d}} g^2(x)g^2(y)\gamma(x+y)\mathrm{d}x\mathrm{d}y\right)^{1/2} - \frac{\theta}{2}\,\mathcal{E}_a(g,g)\right\}$$
$$= \sup_{g\in\mathcal{F}_a}\left\{\langle g^2 * g^2, \gamma\rangle_{L^2(\mathbb{R}^d)}^{1/2} - \frac{\theta}{2}\,\mathcal{E}_a(g,g)\right\},$$

where

$$\mathcal{E}_a(g,g) := (2\pi)^{-d}\int_{\mathbb{R}^d} |\xi|^a |\mathcal{F}g(\xi)|^2 \mathrm{d}\xi \quad \text{and}$$
$$\mathcal{F}_a := \left\{f \in L^2(\mathbb{R}^d):\ \|f\|_{L^2(\mathbb{R}^d)} = 1,\ \mathcal{E}_a(f,f) < \infty\right\}.$$

The Connection between ρ and $\mathcal{M}$

Recall that

$$\rho_{\nu,a}(\gamma) = \sup_{\|f\|_{L^2(\mathbb{R}^d)}=1} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{f(x+y)f(y)}{\sqrt{1+\frac{\nu}{2}|x+y|^a}\sqrt{1+\frac{\nu}{2}|y|^a}} \mathrm{d}y \right]^2 \mu(\mathrm{d}x).$$

Theorem (Bass, X. Chen and Rosen '09)

$$\rho_{\nu,a}(\gamma) = \nu^{-\alpha/a} \mathcal{M}_a^{2-(\alpha/a)}(\gamma) < \infty$$

The Moment Asymptotic

Theorem (Chen-E. '21+)

Suppose a global solution to the SPDE in question exists and $\dot{W}$ is given through the generalized Riesz kernel defined above. Then,

$$
\begin{aligned}
&\lim_{t_p \to \infty} t_p^{-\beta} \log \|u(t,x)\|_p \\
&\quad = \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r) - b\alpha}\right)^{\beta} \\
&\qquad \times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right),
\end{aligned}
$$

where

$$
\beta := \frac{2(b+r) - \frac{b\alpha}{a}}{2(b+r) - \frac{b\alpha}{a} - 1} \qquad \textit{and} \qquad t_p := (p-1)^{1-1/\beta}\, t.
$$

Exact Moment Lyapunov Exponent

(Recall $\beta := \frac{2(b+r)-\frac{b\alpha}{a}}{2(b+r)-\frac{b\alpha}{a}-1}$)

Corollary (Chen-E. '21+)

For $p \geq 2$ fixed, we have that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right) \\
&\quad = p(p-1)^{\frac{1}{2(b+r)-\frac{b\alpha}{a}-1}} \left(\frac{1}{2}\right) \left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta} \\
&\qquad \times \left(\theta \nu^{-\alpha/a} \mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}} \left(2(b+r) - \frac{b\alpha}{a} - 1\right).
\end{aligned}$$

From this we can deduce that $t \mapsto \mathbb{E}(|u(t,x)|^p)$ will grow like $\exp\left(K_1 t^{\beta}\right)$, in other words, the moments will grow exponentially in time, and we obtain the exact expression for the constant K_1.

Exact Large Moment Asymptotics

(Recall $\beta := \frac{2(b+r)-\frac{b\alpha}{a}}{2(b+r)-\frac{b\alpha}{a}-1}$)

Corollary (Chen-E. '21+)

For $t > 0$ fixed , we have that

$$\lim_{p\to\infty} p^{-\beta} \log \mathbb{E}\left(|u(t,x)|^p\right)$$
$$= t^{\beta}\left(\frac{1}{2}\right)\left(\frac{2a}{2a(b+r)-b\alpha}\right)^{\beta}$$
$$\times \left(\theta\nu^{-\alpha/a}\mathcal{M}_a^{\frac{2a-\alpha}{a}}\right)^{\frac{a}{2a(b+r)-b\alpha-a}}\left(2(b+r)-\frac{b\alpha}{a}-1\right).$$

From this we can deduce that $p \mapsto \mathbb{E}(|u(t,x)|^p)$ will grow like $\exp\left(K_2 p^{\beta}\right)$, in other words, the moments will grow exponentially in p as well.

End

Thank you for listening!

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Anticipated Question: Finiteness of ρ

Recall $\varphi(x) = \prod_{i=1}^{n} |x_{(i)}|^{d_i - \alpha_i}$. By examining Lemma 1.6 (X. Chen 2007), to show ρ is finite for the generalized Riesz kernel, then we need to prove the following:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(y) G(x) \varphi(x-y) \mathrm{d}y \mathrm{d}x \leq C \|F\|_{2d/(d+\alpha)} \|G\|_{2d/(d+\alpha)}$$

This can be done by using weak Young's inequality

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x) b(x-y) c(y) \mathrm{d}x \mathrm{d}y \right| \leq K_{p,q,r,d} \|a\|_p \|b\|_{q,w} \|c\|_r$$

where $q^{-1} + q'^{-1} = 1$ and $p^{-1} + q^{-1} + r^{-1} = 2$ and

$$\|b\|_{q,w} = \sup_A |A|^{-1/q'} \int_A |b(x)| \mathrm{d}x, \quad |A| < \infty.$$

We apply this with

$$a = F, \quad b = \varphi, \quad c = G$$

and we show that $\|\varphi\|_{q,w} < \infty$ with $q = d/(d-\alpha)$.

Anticipated Question: Why does ρ appear in the Lemma.

For simplicity we will consider the second moments of stochastic wave equation ($a = b = 2$ and $r = 0$). We will also only consider the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\Gamma((4-\alpha)n+1) \left\| \tilde{f}_n(\cdot, 0; 1) \right\|^2_{\mathcal{H}^{\otimes n}} \right) = \log \left[2^{4-\alpha} \rho \right]$$

Recall that

$$\int_0^\infty e^{-t} \left\| \tilde{f}_n(\cdot, 0, t) \right\|^2_{\mathcal{H}^{\otimes n}} \mathrm{d}t = \Gamma((4-\alpha)n+1) \left\| \tilde{f}_n(\cdot, 0, 1) \right\|^2_{\mathcal{H}^{\otimes n}}.$$

Define $T_n := \int_{(\mathbb{R}^d)^n} \left| \mathcal{F}\mathcal{L}\tilde{f}_n(1,\xi) \right|^2 \mu(\mathrm{d}\vec{\xi})$. Through direct calculation and Sterlings formula:

$$\frac{c_\alpha}{C_n} 2^{(4-\alpha)n} T_n \le \int_0^\infty e^{-t} \left\| \tilde{f}(\cdot, 0, t) \right\|^2_{H^{\otimes n}} \mathrm{d}t \le 2^{(4-\alpha)n} T_n$$

Where $\log(C_n)/n \to 0$ and $c_\alpha > 0$.

Anticipated Question: Finding the moment asymptotic

$$\mathbb{E}(|u(t,x)|^2) = \sum_{n\geq 0} z_n R_n t^{(2(b+r)-b\alpha/a)n}$$

$$R_n = (n!)^{2(b+r)-b\alpha/a} \left\| \widetilde{f}_n(\cdot,0,1) \right\|^2_{\mathcal{H}^{\otimes n}} \quad \text{and} \quad z_n = \frac{\theta^n}{(n!)^{2(b+r)-(b\alpha/a)-1}}.$$

$$\frac{1}{n}\log(R_n) \to \log\left(\left(\frac{2}{2(b+r)-\frac{b\alpha}{a}} \right)^{2(b+r)-\frac{b\alpha}{a}} \rho \right) \quad \text{as } n \to \infty.$$

If we find a β and A so that

$$\lim_{t\to\infty} \frac{1}{t^\beta} \log \sum_{n\geq 0} z_n R^n \left(t^{(2(b+r)-b\alpha/a)} \right)^n = A.$$

then this is implies

$$\lim_{t\to\infty} \frac{1}{t^\beta} \log \sum_{n\geq 0} z_n R_n \left(t^{(2(b+r)-b\alpha/a)} \right)^n = A.$$