# Some symbolic tools for the Fox $H$-function

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## Contents

## 1 Introduction

In this note, we explain the code for checking the conditions of the Fox $H$-function [**fox:61:g**]. Here we follow the notation from Kilbas and Saigo [**kilbas.saigo:04:h-transforms**].

Let $m, n, p, q$ be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

| $\in (\mathbb{C}, \mathbb{R}_+)$ | Front list | Rear list | |
|---|---|---|---|
| $p$ | $(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$ | $(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$ | Upper list |
| $q$ | $(b_1, \beta_1), \cdots, (b_m, \beta_m)$ | $(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$ | Lower list |

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod\limits_{i=1}^{n} \Gamma\left(1 - a_i - \alpha_i s\right)}{\prod\limits_{i=n+1}^{p} \Gamma\left(a_j + \alpha_i s\right)} \times \frac{\prod\limits_{j=1}^{m} \Gamma\left(b_j + \beta_j s\right)}{\prod\limits_{j=m+1}^{q} \Gamma\left(1 - b_j - \alpha_j s\right)}. \tag{1}$$

Then the Fox $H$-function $H_{2,3}^{2,1}\left( z \left| \begin{array}{c} \cdots \\ \cdots \end{array} \right. \right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q}\left( z \left| \begin{array}{c} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{array} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} \mathrm{d}s. \tag{2}$$

It is more convenient to use the following notation:

$$H_{m,n}^{p,q}\left( z \left| \begin{array}{c|c} (a_1, \alpha_1), \cdots, (a_n, \alpha_n) & (a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p) \\ (b_1, \beta), \cdots, (b_m, \beta_m) & (b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q) \end{array} \right. \right),$$

in order to emphasize the front and rear lists.

The basic assumption for the well-posedness of the Fox $H$-function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \cdots \right\} \bigcap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \cdots \right\} = \emptyset. \tag{3}$$

The contour $\mathcal{L}$ in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [**kilbas.saigo:04:h-transforms**]). First denote

$$a_1^* := \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i,$$

$$a_2^* := \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.$$

Similar to $a^*$, we define

$$\xi := \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_i};$$

$$\mu := \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}$$

The well-posedness of the Fox $H$-function is given by Theorems 1.1 and 1.2 of [**kilbas.saigo:04:h-transforms**], which are summarized in the following figure 2:

We will need the following two parameters (see (1.5.15)/(1.5.16) and (1.8.10)/(1.8.11), respectively, of [**kilbas.saigo:04:h-transforms**] for $\rho$ and $\rho^*$):

$$\rho := \max_{1 \le i \le n} \left[ \frac{\Re(a_i) - 1}{\alpha_i} \right] \quad \text{and} \quad \rho^* := \min_{1 \le j \le m} \left[ \frac{\Re(b_i)}{\beta_j} \right].$$
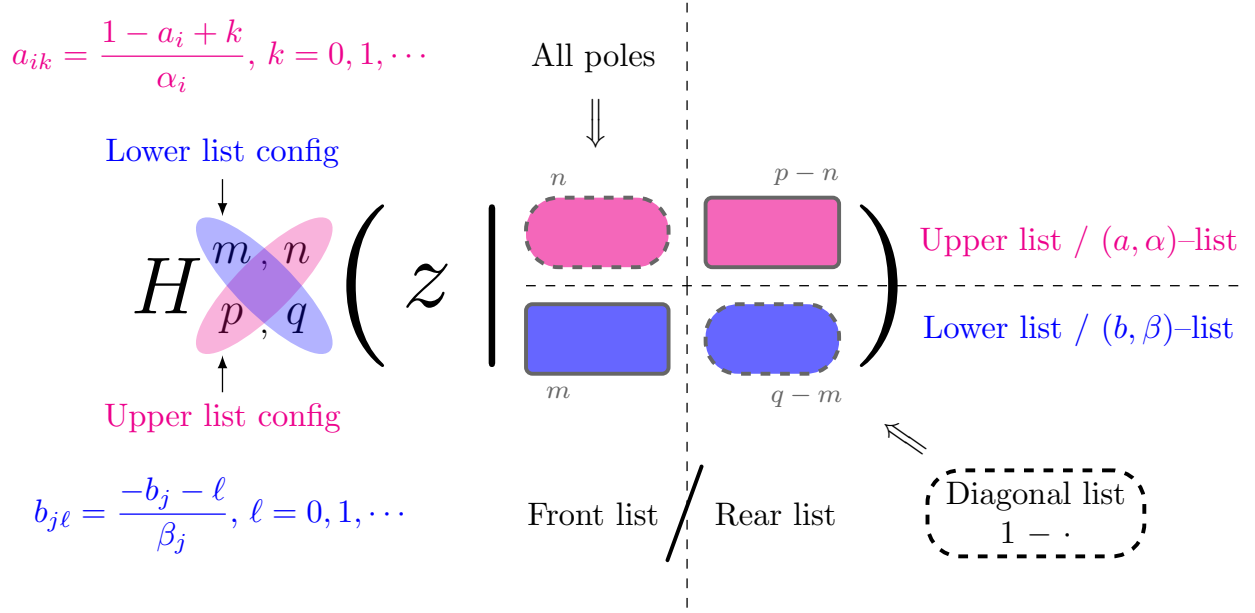
3

$$a_{ik} = \frac{1 - a_i + k}{\alpha_i}, \; k = 0, 1, \cdots$$

Lower list config

$$H^{m,\,n}_{p,\,q}\left(z \;\middle|\; \right)$$

Upper list config

$$b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \; \ell = 0, 1, \cdots$$

All poles $\Downarrow$

$n$  $p - n$

$m$  $q - m$

Upper list / $(a, \alpha)$–list

Lower list / $(b, \beta)$–list

Front list / Rear list

Diagonal list
$1 - \cdot$

Figure 1: Diagram for the parameterization of the Fox $H$-function.

No overlap between $a_{ik}$ poles and $b_{j\ell}$ poles

$\Delta < 0$

$\Delta = 0$

$\Delta > 0$

$z \neq 0$

$|z| > \delta$

$|z| = \delta$

$0 < |z| < \delta$

$z \neq 0$

$\mathrm{Re}(\mu) < -1$

$a_{ik}$ poles

$b_{j\ell}$ poles

$$H^{m,n}_{p,q}(z) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \operatorname*{Res}_{s=a_{ik}} \left[ \mathcal{H}^{m,n}_{p,q}(s) z^{-s} \right] \qquad H^{m,n}_{p,q}(z) = \sum_{j=1}^{m} \sum_{\ell=0}^{\infty} \operatorname*{Res}_{s=b_{j\ell}} \left[ \mathcal{H}^{m,n}_{p,q}(s) z^{-s} \right]$$
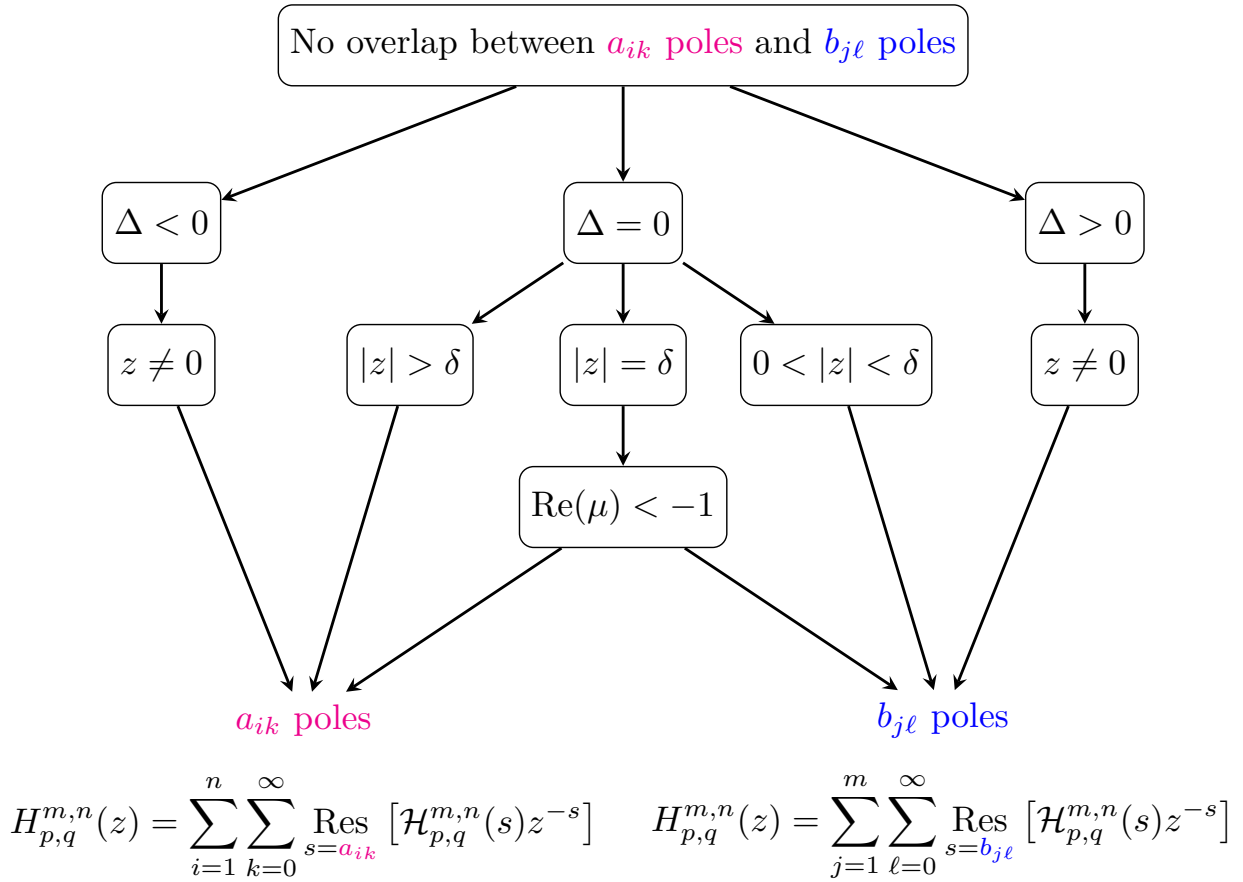
Figure 2: Well-posedness of the Fox $H$-function.

# 2 Examples

## 2.1 Example `FoxH-2_9_11.wls`

**File content**

```
(* (2.9.11) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{1,1},{1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1,1}},
    (* Lower Rear List *)  {{0,0}}
  }
}
```

**Fox H-function**

$$H_{2,2}^{1,0}\left(\cdot\left|\begin{array}{c}(1,1),(1,1)\\\\(1,1),(0,0)\end{array}\right.\right)$$

$$H_{2,2}^{1,0}\left(\cdot\left|\begin{array}{c|c}&(1,1),(1,1)\\\hline(1,1)&(0,0)\end{array}\right.\right)$$

**Summary**

$$a^* = -1$$
$$\Delta = -1$$
$$\delta = \text{Indeterminate}$$
$$\mu = -1$$
$$a_1^* = -1$$
$$a_2^* = 0$$
$$\xi = -1$$
$$c^* = -1$$

**Poles** **1. First eight poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \left(\begin{array}{cccccccc}-1 & -2 & -3 & -4 & -5 & -6 & -7 & -8\end{array}\right)$$

## 2.2 Example `FoxH-2_9_12.wls`

**File content**

```
(* (2.9.12) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1/2,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{-1/2,1}}
  }
}
```

**Fox H-function**

$$H_{2,2}^{1,2}\left(\,\cdot\,\left|\begin{array}{c} \left(\frac{1}{2},1\right),\left(\frac{1}{2},1\right) \\[2mm] (0,1),\left(-\frac{1}{2},1\right) \end{array}\right.\right)$$

$$H_{2,2}^{1,2}\left(\,\cdot\,\left|\begin{array}{c|c} \left(\frac{1}{2},1\right),\left(\frac{1}{2},1\right) & \\ \hline (0,1) & \left(-\frac{1}{2},1\right) \end{array}\right.\right)$$

**Summary**

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = 1$$
$$\mu = -\frac{3}{2}$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = \frac{3}{2}$$
$$c^* = 1$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\[2mm] \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \end{pmatrix}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

## 2.3 Example `FoxH-2_9_13.wls`

**File content**

```
(* (2.9.13) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,1}},
    (* Lower Rear List *)  {{0,1}}
  }
}
```

**Fox H-function**

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{l} \left(\frac{1}{2},1\right),(1,1) \\[2mm] \left(\frac{1}{2},1\right),(0,1) \end{array} \right.\right)$$

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{c|c} \left(\frac{1}{2},1\right),(1,1) & \\ \hline \left(\frac{1}{2},1\right) & (0,1) \end{array} \right.\right)$$

**Summary**

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = 1$$
$$\mu = -1$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = 2$$
$$c^* = 1$$

**Poles 1. First eight poles from upper front list**

$$a_{i,k} = \left( \begin{array}{cccccccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\[2mm] 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right)$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} \end{pmatrix}$$

## 2.4   Example `FoxH-2_9_4.wls`

**File content**

```
(* (2.9.4) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{b, β}},
    (* Lower Rear List *)  {}
  }
}
```

**Fox H-function**

$$H_{0,1}^{1,0}\left(\cdot \left| \begin{array}{c} \\ (b,\beta) \end{array}\right.\right)$$

$$H_{0,1}^{1,0}\left(\cdot \left| \begin{array}{c|c} \\ \hline (b,\beta) & \end{array}\right.\right)$$

**Summary**

$$a^* = \beta$$
$$\Delta = \beta$$
$$\delta = \beta^\beta$$
$$\mu = b - \frac{1}{2}$$
$$a_1^* = \beta$$
$$a_2^* = 0$$
$$\xi = b$$
$$c^* = \frac{1}{2}$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \left( -\frac{b}{\beta} \quad -\frac{b+1}{\beta} \quad -\frac{b+2}{\beta} \quad -\frac{b+3}{\beta} \quad -\frac{b+4}{\beta} \quad -\frac{b+5}{\beta} \quad -\frac{b+6}{\beta} \quad -\frac{b+7}{\beta} \right)$$

**Source**   This example is from (2.9.4) of [**kilbas.saigo:04:h-transforms**]:

$$H_{0,1}^{1,0}\left(z \;\middle|\; {(b,\beta)}\right) = \frac{1}{\beta} z^{b/\beta} \exp\left(-z^{1/\beta}\right).$$

## 2.5 Example `FoxH-2_9_5.wls`

**File content**

```
(* (2.9.5) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {}
  }
}
```

**Fox H-function**

$$H_{1,1}^{1,1}\left(\cdot\left|\begin{array}{c}(1-a,1)\\[2mm](0,1)\end{array}\right.\right)$$

$$H_{1,1}^{1,1}\left(\cdot\left|\begin{array}{c|}(1-a,1)\\\hline(0,1)\end{array}\right.\right)$$

**Summary**

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = 1$$
$$\mu = a - 1$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = 1 - a$$
$$c^* = 1$$

**Poles**  **1. First eight poles from upper front list**

$$a_{i,k} = \left(\begin{array}{cccccccc} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \end{array}\right)$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \left(\begin{array}{cccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array}\right)$$

## 2.6 Example `FoxH-2_9_6.wls`

**File content**

```
(* (2.9.6) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{α+β+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{α, 1}},
    (* Lower Rear List *)  {}
  }
}
```

**Fox H-function**

$$H_{1,1}^{1,0}\left(\cdot \left| \begin{array}{c} (\alpha+\beta+1,1) \\ \\ (\alpha,1) \end{array} \right. \right)$$

$$H_{1,1}^{1,0}\left(\cdot \left| \begin{array}{c|c} & (\alpha+\beta+1,1) \\ \hline (\alpha,1) & \end{array} \right. \right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 0$$
$$\delta = 1$$
$$\mu = -\beta - 1$$
$$a_1^* = 0$$
$$a_2^* = 0$$
$$\xi = -\beta - 1$$
$$c^* = 0$$

**Poles** **1. First eight poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \left( \begin{array}{cccccccc} -\alpha & -\alpha-1 & -\alpha-2 & -\alpha-3 & -\alpha-4 & -\alpha-5 & -\alpha-6 & -\alpha-7 \end{array} \right)$$

13

## 2.7 Example `FoxH32-21-Y.wls`

**File content**

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$H_{2,3}^{2,1}\left(\cdot\left|\begin{array}{c} (1,1),(\beta+\gamma,\beta) \\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)$$

$$H_{2,3}^{2,1}\left(\cdot\left|\begin{array}{c|c} (1,1) & (\beta+\gamma,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)$$

**Summary**

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha}\alpha^{\alpha}\beta^{-\beta}$$
$$\mu = \frac{1}{2}(-2\beta - 2\gamma + d + 1)$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{1}{2}(d - 2(\beta + \gamma - 1))$$
$$c^* = \frac{1}{2}$$

**Poles    1. First eight poles from upper front list**

$$a_{i,k} = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}\right)$$

## 2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

## 2.8   Example `FoxH32-21-Z.wls`

**File content**

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$
H_{2,3}^{2,1}\left(\cdot\ \left|\ \begin{array}{c} (1,1),(\lceil\beta\rceil,\beta) \\[2ex] \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)
$$

$$
H_{2,3}^{2,1}\left(\cdot\ \left|\ \begin{array}{c|c} (1,1) & (\lceil\beta\rceil,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)
$$

**Summary**

$$
\begin{aligned}
a^* &= 2 - \beta \\
\Delta &= \alpha - \beta \\
\delta &= 2^{-\alpha}\alpha^{\alpha}\beta^{-\beta} \\
\mu &= \frac{1}{2}(-2\lceil\beta\rceil + d + 1) \\
a_1^* &= \frac{1}{2}(\alpha - 2\beta + 2) \\
a_2^* &= 1 - \frac{\alpha}{2} \\
\xi &= \frac{1}{2}(-2\lceil\beta\rceil + d + 2) \\
c^* &= \frac{1}{2}
\end{aligned}
$$

**Poles   1. First eight poles from upper front list**

$$
a_{i,k} = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}\right)
$$

16

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

**Source** This is the fundamental solution to the fractional diffusion equation used, e.g., in [**chen.hu.ea:17:space-t** **chen.hu.ea:19:nonlinear**; **chen.eisenberg:22:interpolating**; **chen.guo.ea:22:moments**].

## 2.9 Example `FoxH32-21-Z-Star.wls`

**File content**

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$H_{2,3}^{2,1}\left(\cdot\left|\begin{array}{c}(1,1),(1,\beta)\\[2em]\left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right)\end{array}\right.\right)$$

$$H_{2,3}^{2,1}\left(\cdot\left|\begin{array}{c|c}(1,1)&(1,\beta)\\\hline\left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1)&\left(1,\frac{\alpha}{2}\right)\end{array}\right.\right)$$

**Summary**

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha}\alpha^{\alpha}\beta^{-\beta}$$
$$\mu = \frac{d-1}{2}$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{d}{2}$$
$$c^* = \frac{1}{2}$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \left(\begin{array}{cccccccc}0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\end{array}\right)$$

**2. First eight poles from lower front list**

$$
b_{j,\ell} = \begin{pmatrix}
-\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\
-1 & -2 & -3 & -4 & -5 & -6 & -7 & -8
\end{pmatrix}
$$

## 2.10 Example `FoxH-Bessel-J_2_9_18.wls`

**File content**

```
(* (2.9.18) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a+η)/2, 1}},
    (* Lower Rear List *)  {{(a-η)/1, 1}}
  }
}
```

**Fox H-function**

$$H_{0,2}^{1,0}\left(\cdot\ \middle|\ \begin{matrix} \\ \left(\frac{a+\eta}{2},1\right),(a-\eta,1) \end{matrix}\right)$$

$$H_{0,2}^{1,0}\left(\cdot\ \middle|\ \begin{array}{c|c} & \\ \hline \left(\frac{a+\eta}{2},1\right) & (a-\eta,1) \end{array}\right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = 1$$
$$\mu = \frac{1}{2}(3a - \eta - 2)$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = \frac{1}{2}(3\eta - a)$$
$$c^* = 0$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \{\}^T$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-a - \eta) \\ \frac{1}{2}(-a - \eta - 2) \\ \frac{1}{2}(-a - \eta - 4) \\ \frac{1}{2}(-a - \eta - 6) \\ \frac{1}{2}(-a - \eta - 8) \\ -\frac{a}{2} - \frac{\eta}{2} - 5 \\ -\frac{a}{2} - \frac{\eta}{2} - 6 \\ -\frac{a}{2} - \frac{\eta}{2} - 7 \end{pmatrix}^{T}$$

## 2.11   Example `FoxH-Bessel-K_2_9_19.wls`

**File content**

```
(* (2.9.19) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *)  {}
  }
}
```

**Fox H-function**

$$H_{0,2}^{2,0}\left(\cdot\left|\begin{array}{c} \\ \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right)\end{array}\right.\right)$$

$$H_{0,2}^{2,0}\left(\cdot\left|\begin{array}{c|c} & \\ \hline \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right) & \end{array}\right.\right)$$

**Summary**

$$a^* = 2$$
$$\Delta = 2$$
$$\delta = 1$$
$$\mu = a - 1$$
$$a_1^* = 2$$
$$a_2^* = 0$$
$$\xi = a$$
$$c^* = 1$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \{\}^T$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a-\eta) \\[2mm] \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a-\eta-2) \\[2mm] \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a-\eta-4) \\[2mm] \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a-\eta-6) \\[2mm] \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a-\eta-8) \\[2mm] \frac{1}{2}(-a+\eta-10) & -\frac{a}{2}-\frac{\eta}{2}-5 \\[2mm] \frac{1}{2}(-a+\eta-12) & -\frac{a}{2}-\frac{\eta}{2}-6 \\[2mm] \frac{1}{2}(-a+\eta-14) & -\frac{a}{2}-\frac{\eta}{2}-7 \end{pmatrix}^{T}$$

## 2.12 Example `FoxH-Bessel-Y_2_9_20.wls`

**File content**

```
(* (2.9.20) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{(a-η-1)/2, 1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *)  {{(a-η-1)/2, 1}}
  }
}
```

**Fox H-function**

$$H_{1,3}^{2,0} \left( \cdot \left| \begin{array}{c} \left(\frac{1}{2}(a - \eta - 1), 1\right) \\ \left(\frac{a-\eta}{2}, 1\right), \left(\frac{a+\eta}{2}, 1\right), \left(\frac{1}{2}(a - \eta - 1), 1\right) \end{array} \right. \right)$$

$$H_{1,3}^{2,0} \left( \cdot \left| \begin{array}{c|c} & \left(\frac{1}{2}(a - \eta - 1), 1\right) \\ \hline \left(\frac{a-\eta}{2}, 1\right), \left(\frac{a+\eta}{2}, 1\right) & \left(\frac{1}{2}(a - \eta - 1), 1\right) \end{array} \right. \right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = 1$$
$$\mu = a - 1$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = \eta + 1$$
$$c^* = 0$$

**Poles  1. First eight poles from upper front list**

$$a_{i,k} = \{\}^T$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a-\eta) \\[4pt] \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a-\eta-2) \\[4pt] \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a-\eta-4) \\[4pt] \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a-\eta-6) \\[4pt] \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a-\eta-8) \\[4pt] \frac{1}{2}(-a+\eta-10) & -\frac{a}{2}-\frac{\eta}{2}-5 \\[4pt] \frac{1}{2}(-a+\eta-12) & -\frac{a}{2}-\frac{\eta}{2}-6 \\[4pt] \frac{1}{2}(-a+\eta-14) & -\frac{a}{2}-\frac{\eta}{2}-7 \end{pmatrix}^{T}$$

## 2.13 Example `FoxH-Cos.wls`

**File content**

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1/2,1}}
  }
}
```

**Fox H-function**

$$H_{0,2}^{1,0} \left( \cdot \left| \begin{array}{c} \\ (0,1), \left(\frac{1}{2},1\right) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left( \cdot \left| \begin{array}{c|c} & \\ \hline (0,1) & \left(\frac{1}{2},1\right) \end{array} \right. \right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = 1$$
$$\mu = -\frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = -\frac{1}{2}$$
$$c^* = 0$$

**Poles**   **1. First eight poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \left( \begin{array}{cccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \right)$$

## 2.14 Example `FoxH-H.G_2_9_14.wls`

**File content**

```
(* (2.9.14) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{1-c,1}}
  }
}
```

**Fox H-function**

$$H_{1,2}^{1,1}\left(\cdot \left|\begin{array}{c} (1-a,1) \\ \\ (0,1),(1-c,1) \end{array}\right.\right)$$

$$H_{1,2}^{1,1}\left(\cdot \left|\begin{array}{c|c} (1-a,1) & \\ \hline (0,1) & (1-c,1) \end{array}\right.\right)$$

**Summary**

$$a^* = 1$$
$$\Delta = 1$$
$$\delta = 1$$
$$\mu = a - c - \frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = 0$$
$$\xi = c - a$$
$$c^* = \frac{1}{2}$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \left(\begin{array}{cccccccc} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \end{array}\right)$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

## 2.15   Example `FoxH-H.G_2_9_15.wls`

**File content**

```
(* (2.9.15) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1},{1-b,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{1-c,1}}
  }
}
```

**Fox H-function**

$$H_{2,2}^{1,2}\left(\cdot\left|\begin{array}{c}(1-a,1),(1-b,1)\\[6pt](0,1),(1-c,1)\end{array}\right.\right)$$

$$H_{2,2}^{1,2}\left(\cdot\left|\begin{array}{c|c}(1-a,1),(1-b,1)&\\\hline(0,1)&(1-c,1)\end{array}\right.\right)$$

**Summary**

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = 1$$
$$\mu = a + b - c - 1$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = -a - b + c + 1$$
$$c^* = 1$$

**Poles   1. First eight poles from upper front list**

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \\ b & b+1 & b+2 & b+3 & b+4 & b+5 & b+6 & b+7 \end{pmatrix}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

## 2.16  Example `FoxH-Lommel_2_9_22.wls`

**File content**

```
(* (2.9.22) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{(1+µ)/2,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(1+µ)/2,1},{η/2,1},{-η/2,1}},
    (* Lower Rear List *)  {}
  }
}
```

**Fox H-function**

$$H_{1,3}^{3,1}\left(\cdot \left| \begin{array}{c} \left(\frac{\mu+1}{2},1\right) \\ \left(\frac{\mu+1}{2},1\right),\left(\frac{\eta}{2},1\right),\left(-\frac{\eta}{2},1\right) \end{array} \right.\right)$$

$$H_{1,3}^{3,1}\left(\cdot \left| \begin{array}{c} \left(\frac{\mu+1}{2},1\right) \\ \hline \left(\frac{\mu+1}{2},1\right),\left(\frac{\eta}{2},1\right),\left(-\frac{\eta}{2},1\right) \end{array} \right| \right)$$

**Summary**

$$a^* = 4$$
$$\Delta = 2$$
$$\delta = 1$$
$$\mu = -1$$
$$a_1^* = 3$$
$$a_2^* = 1$$
$$\xi = \mu + 1$$
$$c^* = 2$$

**Poles  1. First eight poles from upper front list**

$$a_{i,k} = \begin{pmatrix} \frac{1-\mu}{2} \\[6pt] \frac{3-\mu}{2} \\[6pt] \frac{5-\mu}{2} \\[6pt] \frac{7-\mu}{2} \\[6pt] \frac{9-\mu}{2} \\[6pt] \frac{11-\mu}{2} \\[6pt] \frac{13-\mu}{2} \\[6pt] \frac{15-\mu}{2} \end{pmatrix}^{T}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-\mu-1) & -\frac{\eta}{2} & \frac{\eta}{2} \\[6pt] \frac{1}{2}(-\mu-3) & -\frac{\eta}{2}-1 & \frac{\eta-2}{2} \\[6pt] \frac{1}{2}(-\mu-5) & -\frac{\eta}{2}-2 & \frac{\eta-4}{2} \\[6pt] \frac{1}{2}(-\mu-7) & -\frac{\eta}{2}-3 & \frac{\eta-6}{2} \\[6pt] \frac{1}{2}(-\mu-9) & -\frac{\eta}{2}-4 & \frac{\eta-8}{2} \\[6pt] \frac{1}{2}(-\mu-11) & -\frac{\eta}{2}-5 & \frac{\eta}{2}-5 \\[6pt] \frac{1}{2}(-\mu-13) & -\frac{\eta}{2}-6 & \frac{\eta}{2}-6 \\[6pt] \frac{1}{2}(-\mu-15) & -\frac{\eta}{2}-7 & \frac{\eta}{2}-7 \end{pmatrix}^{T}$$

## 2.17 Example `FoxH-Mittag-Leffler.wls`

**File content**

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1 - μ, ρ}}
  }
}
```

**Fox H-function**

$$H_{1,2}^{1,1} \left( \cdot \left| \begin{array}{c} (0,1) \\ \\ (0,1) , (1-\mu,\rho) \end{array} \right. \right)$$

$$H_{1,2}^{1,1} \left( \cdot \left| \begin{array}{c|c} (0,1) & \\ \hline (0,1) & (1-\mu,\rho) \end{array} \right. \right)$$

**Summary**

$$a^* = 2 - \rho$$
$$\Delta = \rho$$
$$\delta = \rho^\rho$$
$$\mu = \frac{1}{2} - \mu$$
$$a_1^* = 1$$
$$a_2^* = 1 - \rho$$
$$\xi = \mu - 1$$
$$c^* = \frac{1}{2}$$

**Poles  1. First eight poles from upper front list**

$$a_{i,k} = \left( \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right)$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

## 2.18 Example `FoxH-Sin.wls`

**File content**

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *)  {{0, 1}}
  }
}
```

**Fox H-function**

$$H_{0,2}^{1,0}\left(\cdot\left|\begin{array}{c} \\ \left(\frac{1}{2},1\right),(0,1) \end{array}\right.\right)$$

$$H_{0,2}^{1,0}\left(\cdot\left|\begin{array}{c|c} & \\ \hline \left(\frac{1}{2},1\right) & (0,1) \end{array}\right.\right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = 1$$
$$\mu = -\frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = \frac{1}{2}$$
$$c^* = 0$$

**Poles** **1. First eight poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \left(\begin{array}{cccccccc} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} \end{array}\right)$$

**Source** ....

## 2.19 Example `FoxH-Whittaker_2_9_21.wls`

**File content**

```
(* (2.9.21) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{a-λ+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{a+µ+1/2,1},{a-µ+1/2,1}},
    (* Lower Rear List *)  {}
  }
}
```

**Fox H-function**

$$
H_{1,2}^{2,0} \left( \cdot \left| \begin{array}{c} (a - \lambda + 1, 1) \\ \\ \left(a + \mu + \frac{1}{2}, 1\right), \left(a - \mu + \frac{1}{2}, 1\right) \end{array} \right. \right)
$$

$$
H_{1,2}^{2,0} \left( \cdot \left| \frac{}{\left(a + \mu + \frac{1}{2}, 1\right), \left(a - \mu + \frac{1}{2}, 1\right)} \right| \frac{(a - \lambda + 1, 1)}{} \right)
$$

**Summary**

$$
a^* = 1
$$
$$
\Delta = 1
$$
$$
\delta = 1
$$
$$
\mu = a + \lambda - \frac{1}{2}
$$
$$
a_1^* = 1
$$
$$
a_2^* = 0
$$
$$
\xi = a + \lambda
$$
$$
c^* = \frac{1}{2}
$$

**Poles** **1. First eight poles from upper front list**

$$
a_{i,k} = \{\}^T
$$

**2. First eight poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -a-\mu-\frac{1}{2} & -a+\mu-\frac{1}{2} \\ -a-\mu-\frac{3}{2} & -a+\mu-\frac{3}{2} \\ -a-\mu-\frac{5}{2} & -a+\mu-\frac{5}{2} \\ -a-\mu-\frac{7}{2} & -a+\mu-\frac{7}{2} \\ -a-\mu-\frac{9}{2} & -a+\mu-\frac{9}{2} \\ -a-\mu-\frac{11}{2} & -a+\mu-\frac{11}{2} \\ -a-\mu-\frac{13}{2} & -a+\mu-\frac{13}{2} \\ -a-\mu-\frac{15}{2} & -a+\mu-\frac{15}{2} \end{pmatrix}^{T}$$