Introduction to stochastic partial differential equations

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Graduate Student Seminar

Department of Mathematics & Statistics
Auburn University

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Intermittency

Introduction to stochastic partial differential equations

Random Matrices

Plan

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Introduction to stochastic partial differential equations

Random Matrices

Additive v.s. Multiplicative

Let X_n be independent Bernoulli random variables with

$$\mathbb{P}(X_n = 4) = \mathbb{P}(X_n = 0) = {}^1\!/_2.$$

$\boxed{\Sigma_n = \sum_{i=1}^n X_n}$

$\boxed{S_n = \prod_{i=1}^n X_n}$

$$\mathbb{P}\left(S_n = 2^{2n}\right) = \frac{1}{2^n}$$

$$\frac{\Sigma_n - \mathbb{E}(\Sigma_n)}{\sqrt{Var(\Sigma_n)}} \to N(0,1)$$

$$\mathbb{P}\left(S_n = 0\right) = 1 - \frac{1}{2^n}$$

$$\mathbb{E}\left[S_n\right] = 2^{2n} \times \frac{1}{2^n} + 0 \times \left(1 - \frac{1}{2^n}\right) = 2^n$$

Additive v.s. Multiplicative

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Exponential Brownian Motion

$dX_t = X_t dB_t$, $X_0 \equiv 1$. Then $X_t = \exp(B_t - t/2)$.

$X_t \to 0$ almost surely as $t \to \infty$ but $\mathbb{E}(X_t) \equiv 1$.

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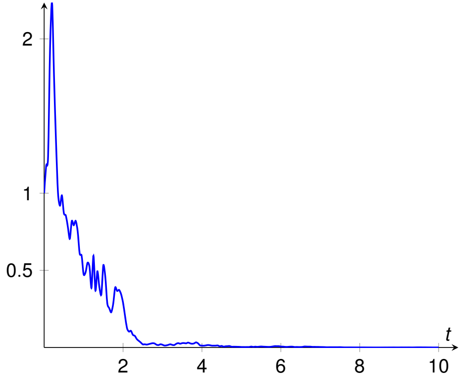

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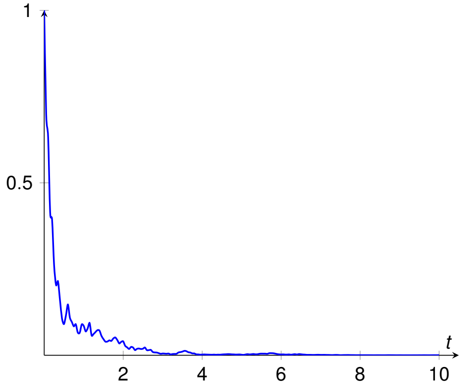

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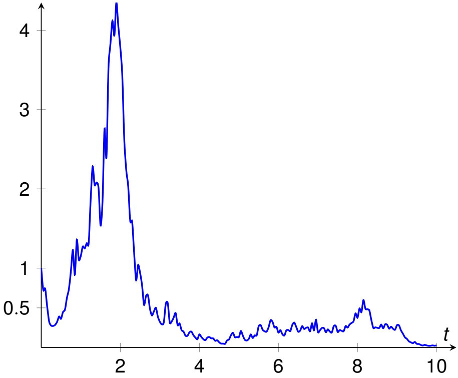

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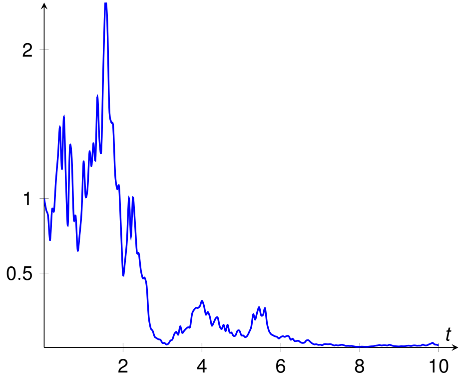

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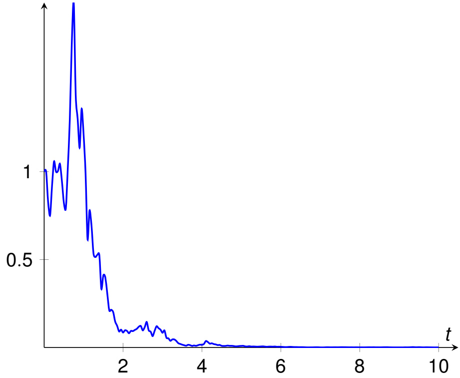

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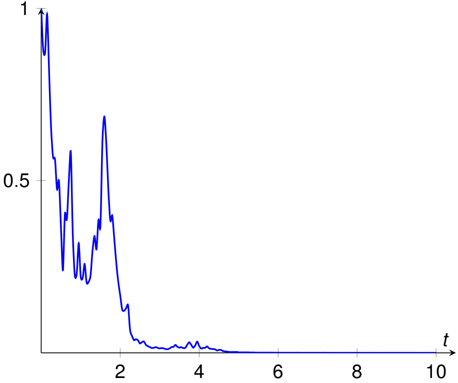

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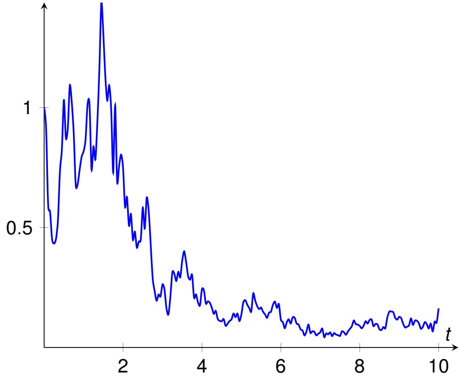

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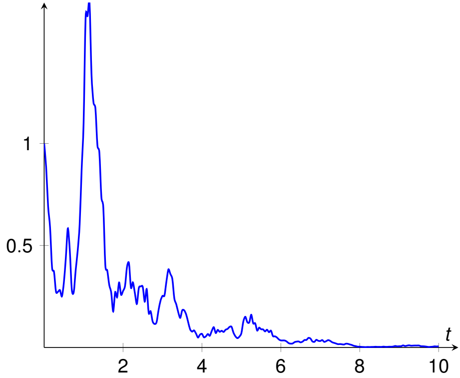

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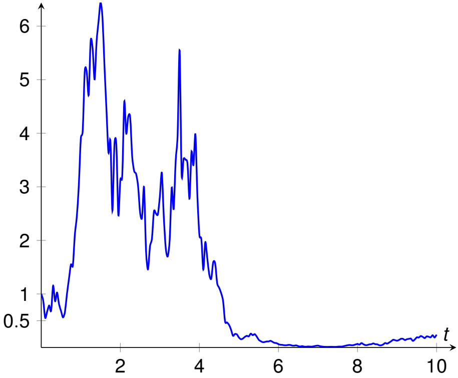

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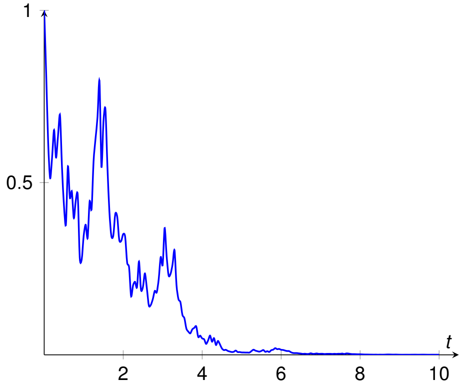

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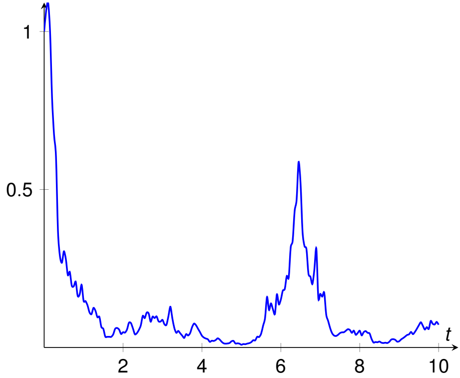

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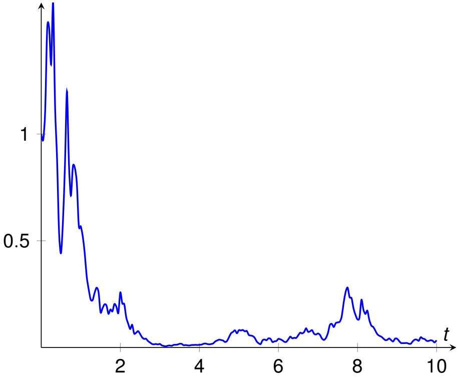

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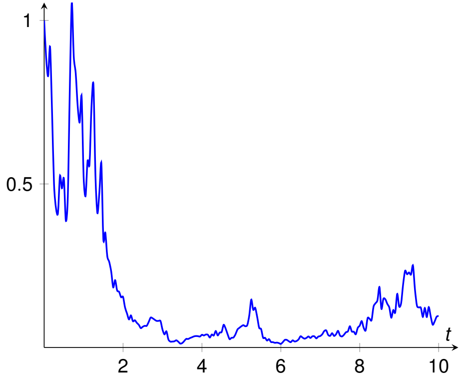

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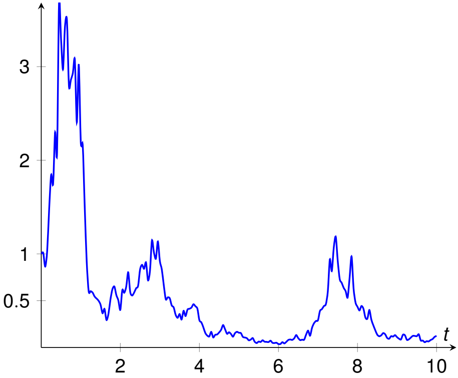

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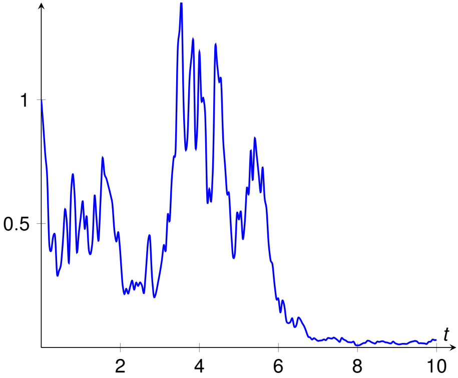

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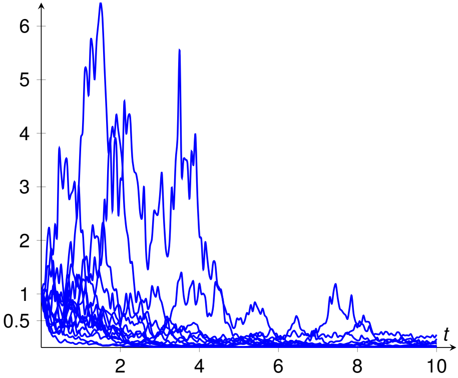
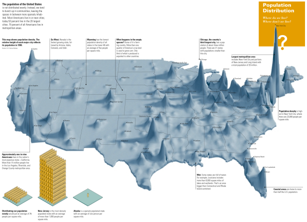


Intermittency

(Zeldovich *et al.* The almighty chance. 1990)



Matthew Effect

The rich get richer and the poor get poorer.

> *For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away.*
>
> *—Matthew 25:29, RSV.*

Intermittency

Intermittency is a very universal phenomenon which occurs practically irrespective of detailed properties of the background instability in a random medium provided only that the random field is of multiplicative type, ...

Zeldovich et al. **The almighty chance**. *1990*

High peaks + highly concentrated on small islands | multiplicative noise

Plan

Intermittency

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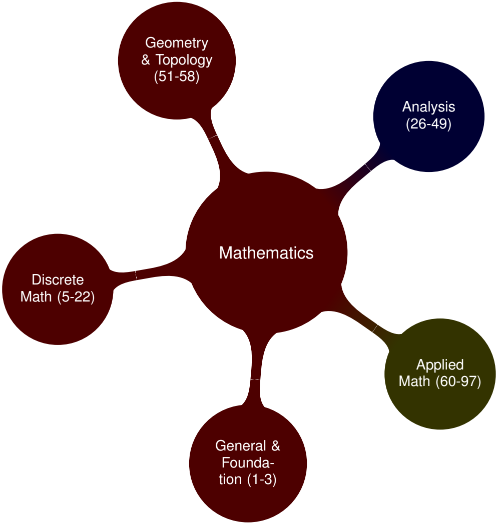
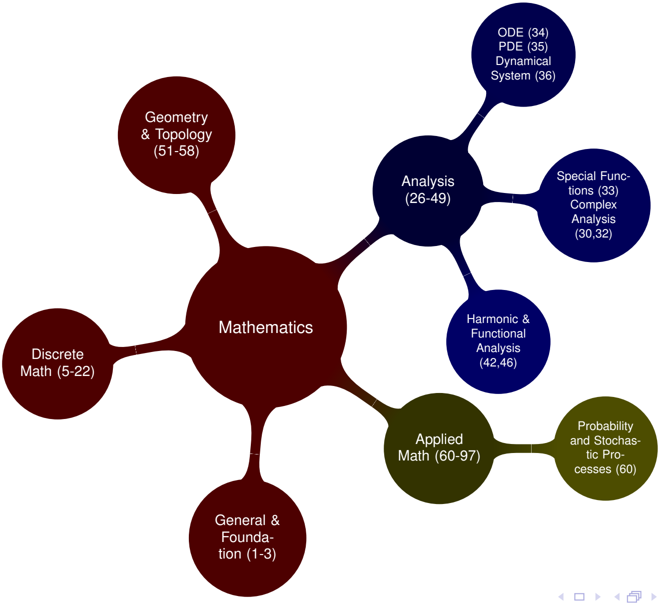
Random Matrices

SPDE as a branch of mathematics



SPDE as a branch of mathematics



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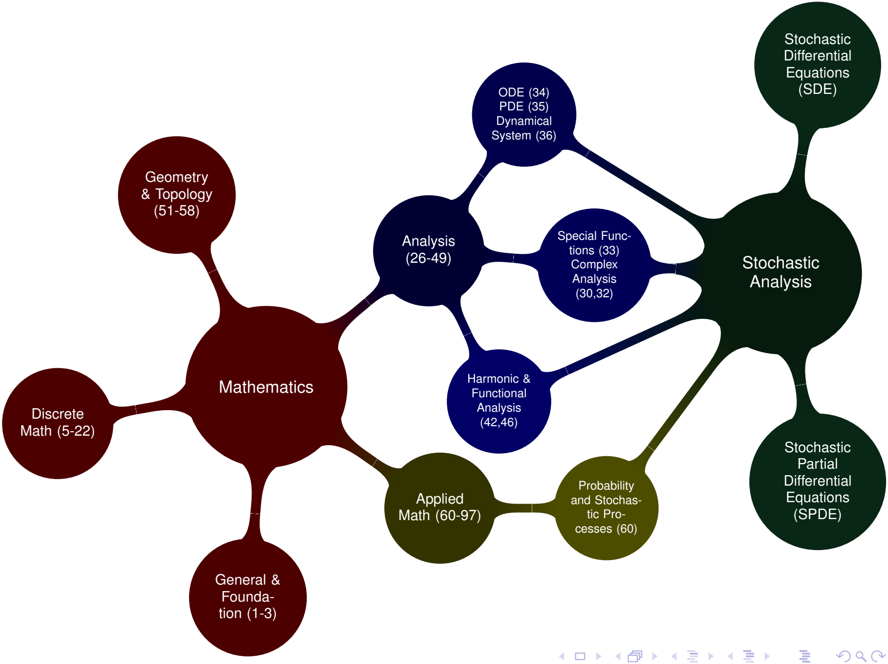


Stochastic
Diff. Eq.

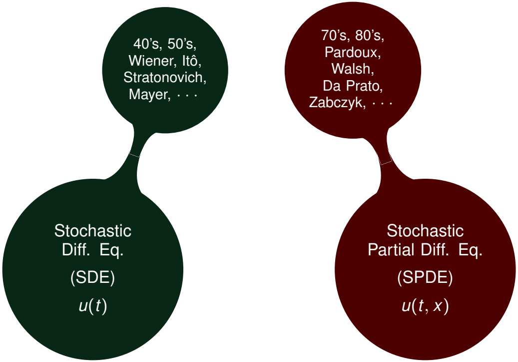
(SDE)

$u(t)$

Stochastic
Partial Diff. Eq.

(SPDE)

$u(t, x)$


40’s, 50’s, Wiener, Itô, Stratonovich, Mayer, · · ·
Stochastic Diff. Eq.
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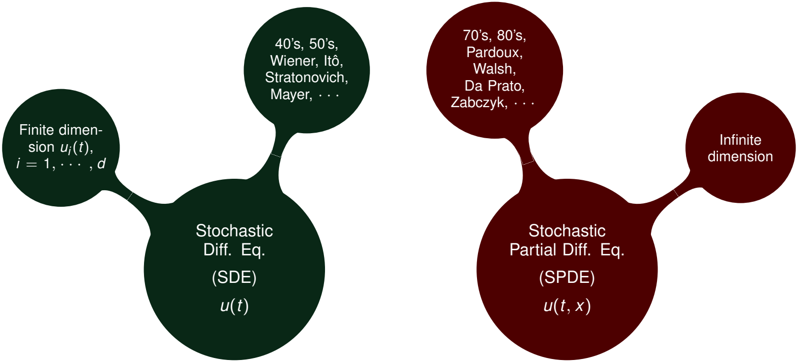
Finite dimension $u_i(t)$, $i = 1, \cdots, d$
40's, 50's, Wiener, Itô, Stratonovich, Mayer, $\cdots$
Stochastic Diff. Eq. (SDE) $u(t)$
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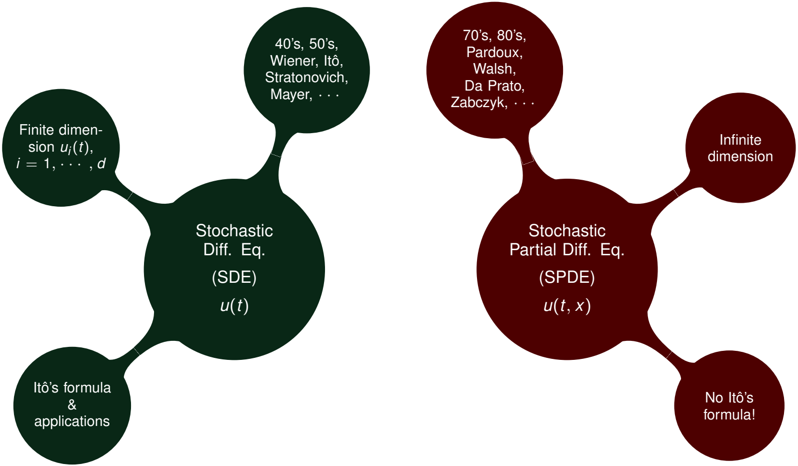
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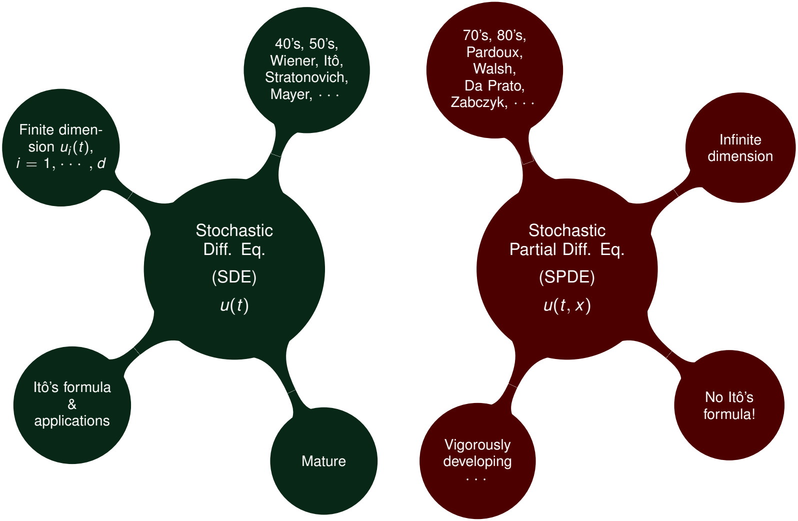
No Itô's formula!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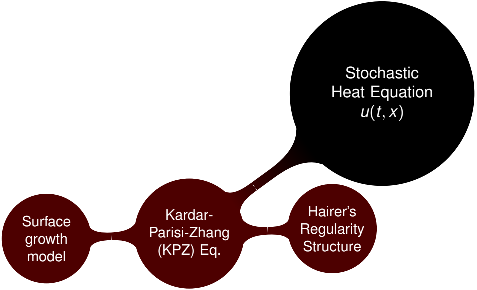
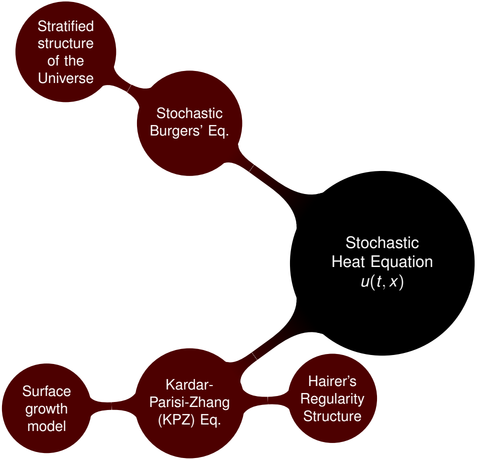
70's, 80's, Pardoux, Walsh, Da Prato, Zabczyk, $\cdots$
Infinite dimension
No Itô's formula!
Vigorously developing
$\cdots$

Stochastic
Heat Equation
$u(t, x)$


Stochastic Heat Equation $u(t, x)$
Surface growth model
Kardar-Parisi-Zhang (KPZ) Eq.
Hairer's Regularity Structure



Stratified structure of the Universe
Stochastic Burgers' Eq.
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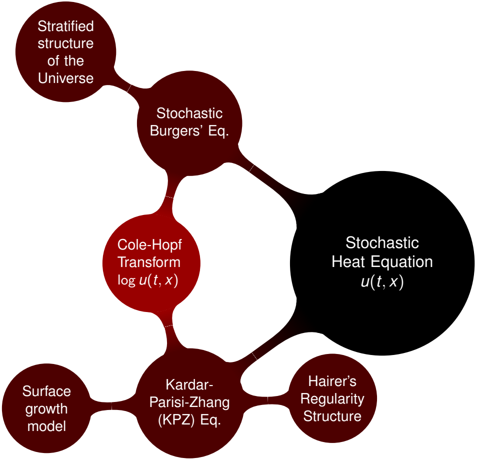
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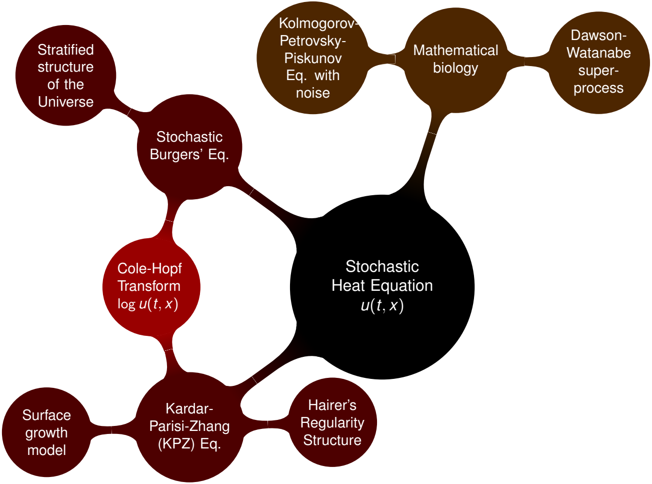
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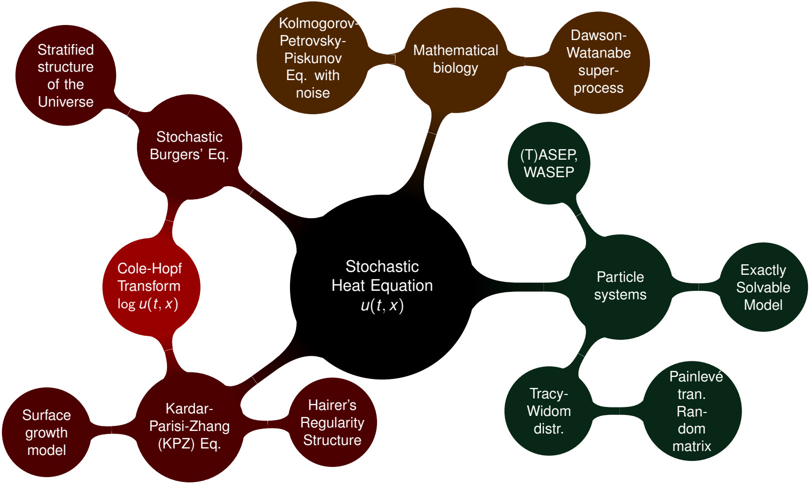
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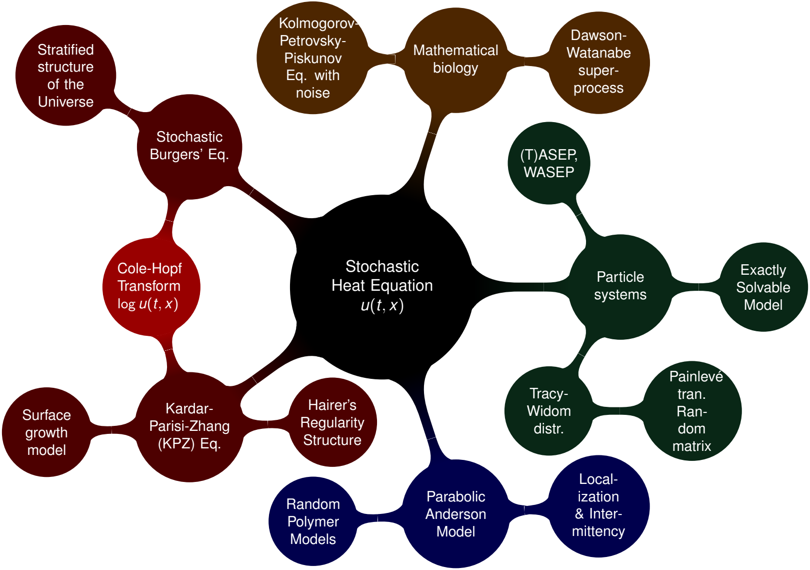
Particle systems
(T)ASEP, WASEP
Exactly Solvable Model
Tracy-Widom distr.
Painlevé tran. Ran-dom matrix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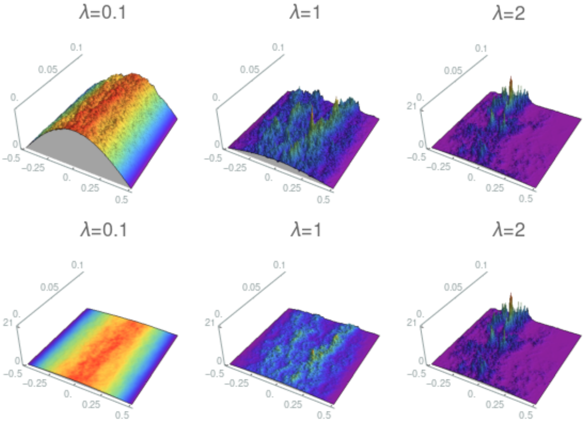
Parabolic Anderson Model
Random Polymer Models
Local-ization & Inter-mittency

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x)$$

Δ	$\rho(u)\dot{W}$
Smoothing	Roughening

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x),\ x \in [-1/2, 1/2] \text{ with } u(0,x) = \cos(\pi x)$$



Stochastic heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), \quad x \in \mathbb{R}.$$

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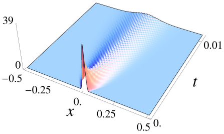
λ=0



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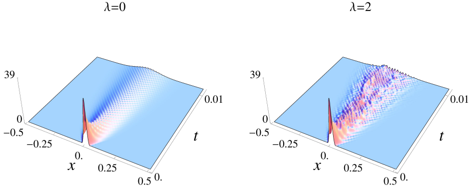

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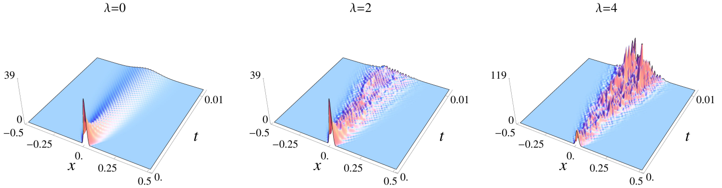

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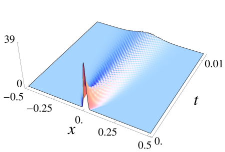
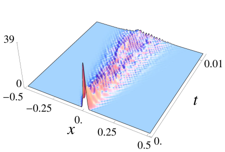
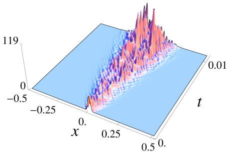


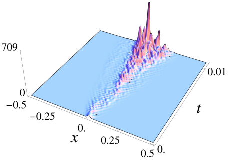
λ=2



λ=4

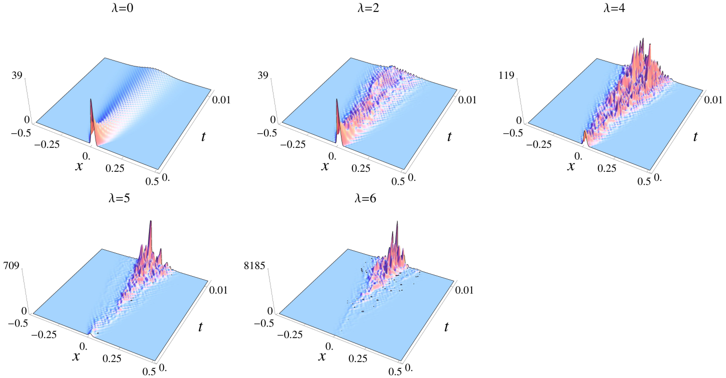


λ=5



Stochastic heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x), \quad x \in \mathbb{R}.$$



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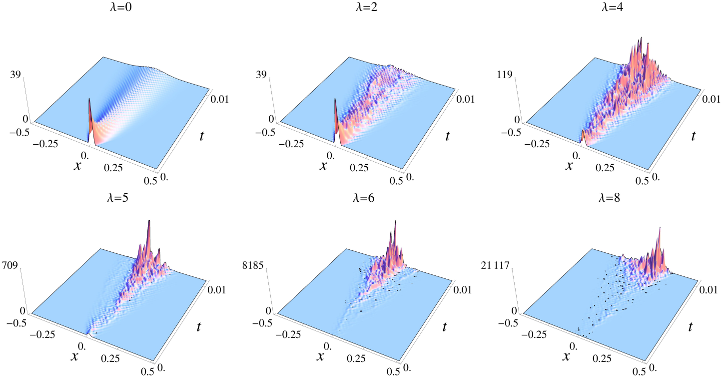

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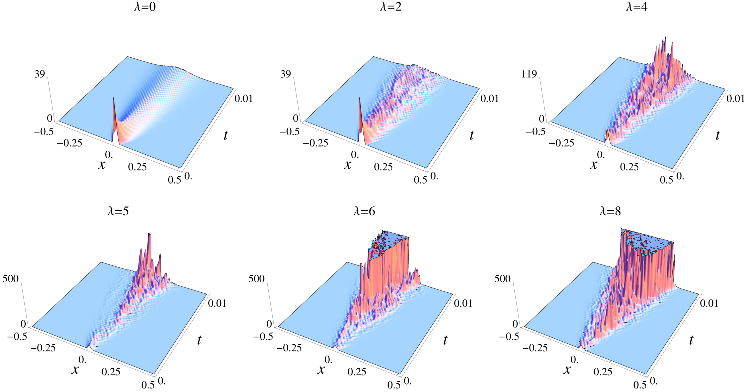

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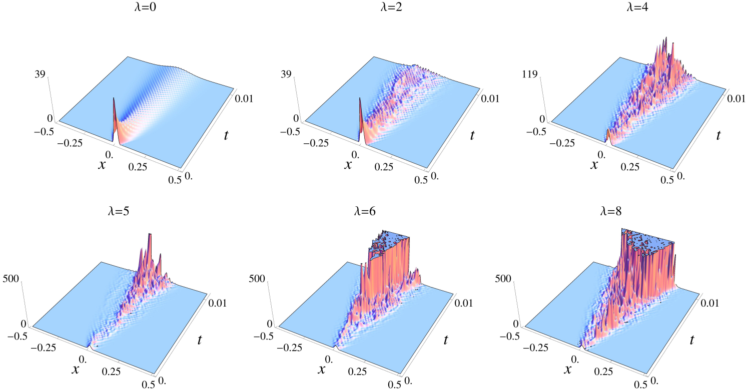

The rate of the propagation of the tall peaks $\asymp \lambda^2$

C. & Dalang, 15.

Various approaches for SPDE's

- – Semigroup approach (Da Prato, et al). $u(t) \in L^p(\Omega, \mathcal{H})$

  – Variational approach (Röckner, et al). Gel'fand triple

- Random field approach (Walsh, Dalang, et al.) $u(t, x) \in L^p(\Omega)$

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  The approach that we are using!

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Stochastic Heat Equation on $\mathbb{R}^d$

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{2}\Delta\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x), \quad t > 0\ ,\ x \in \mathbb{R}^d \\ u(0,\cdot) = \mu(\cdot) \end{cases} \tag{SHE}$$

1. ρ is globally Lipschitz continuous s.t. $\rho(0) = 0$.
2. μ is *rough* (measure+integrability condition).
3. $\dot{W}$ is white in time and colored in space (f nonneg. & nonneg. definite):

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)f(x-y).$$

4. $\rho(u) = \lambda u$: *Parabolic Anderson Model*.

Mild solution

$$u(t,x) = J_0(t,x) + \underbrace{\int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\rho(u(s,y))W(\mathrm{d}s\mathrm{d}y)}_{=:\, I(t,x)}$$

† $G(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$

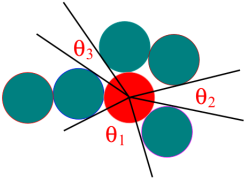
† $J_0(t,x) = \int_{\mathbb{R}^d} G(t, x-y)\mu(\mathrm{d}y)$

Surface growth

Rule of replication of **cells** [1]

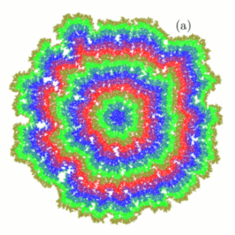
Replication probability $\propto$ Aperture angle θ_i

Microscopic world



$\Rightarrow$

Macroscopic world



$$\frac{\partial}{\partial t} h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{\lambda}{2}(\nabla h)^2 + \dot{W}(t,x) \qquad \text{(KPZ)}$$

[1] S. Santalla and S. Ferreira. *Phys. Rev. E.* 98, 2018

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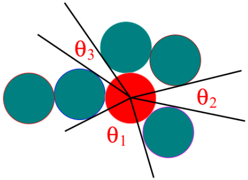
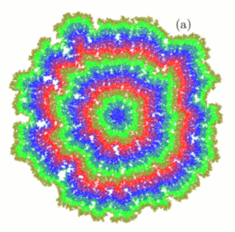


Some rules

$\Rightarrow$

Macroscopic world



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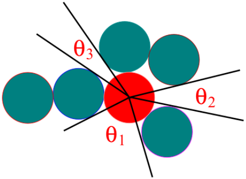

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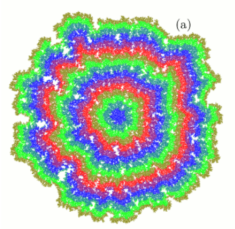

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Universal object – KPZ!

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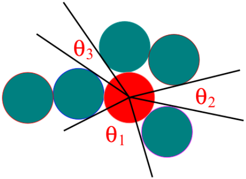

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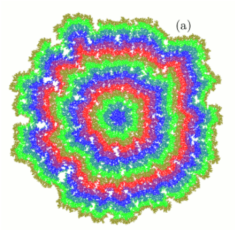

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Study of growing interfaces in a thin film

— Convection of nematic liquid crystal [2]

Show movies !

[2] K. Takeuchi, M. Sano, T. Sasamoto *et al*. *Sci. Rep. (Nature)*, 1, 34 (2011).

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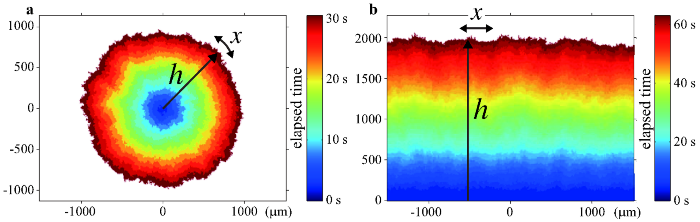

Prediction from KPZ equation:

$$h \asymp v_\infty t + (\Gamma t)^{1/3} \xi$$

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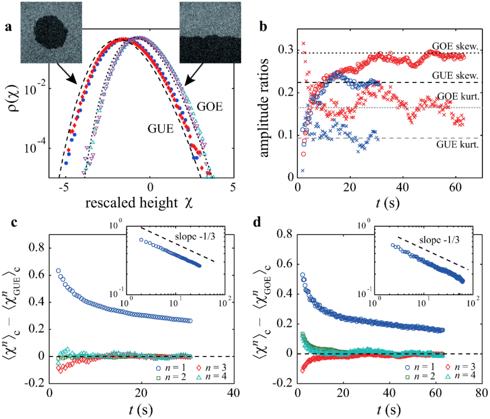

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KPZ Equation '86 [3]

Mehran Kardar (1957 –)

Giorgio Parisi (1948 –)

Yicheng Zhang

[3] "Dynamic Scaling of Growing Interfaces". Physical Review Letters. 56 (9): 889–892, 1986.

Giorgio Parisi
Facts


Ill. Niklas Elmehed © Nobel Prize Outreach

Giorgio Parisi
The Nobel Prize in Physics 2021

Born: 4 August 1948, Rome, Italy

Affiliation at the time of the award: Sapienza University of Rome, Rome, Italy

Prize motivation: "for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales."

Prize share: 1/2

https://www.nobelprize.org/prizes/physics/2021/parisi/facts/

Sir
Martin Hairer
KBE FRS

Hairer at the Royal Society admissions day in London, July 2014

Born	14 November 1975 (age 47) Geneva, Switzerland
Citizenship	Austrian British
Education	College Claparede, Geneva
Alma mater	University of Geneva
Spouse	Xue-Mei Li (m. 2003)[1][2]
Awards	Whitehead Prize (2008) Philip Leverhulme Prize (2008) Wolfson Research Merit Award (2009) Fermat Prize (2013) Fröhlich Prize (2014) Fields Medal (2014) Breakthrough Prize in Mathematics (2021) King Faisal Prize (2022)
Scientific career	
Fields	Probability theory[3] Analysis[3]
Institutions	École Polytechnique Fédérale de Lausanne Imperial College London University of Warwick New York University[1]
Thesis	*Comportement Asymptotique d'Équations à Dérivées Partielles Stochastiques* (2001)
Doctoral advisor	Jean-Pierre Eckmann[4]
Website	hairer.org

- M. Hairer. Solving the KPZ equation. *Annals of Mathematics*, vol. 178, 2013.
- M. Hairer. A theory of regularity structures. *Inventiones mathematicae*, vol. 198, 2014.

Fields Medal in 2014

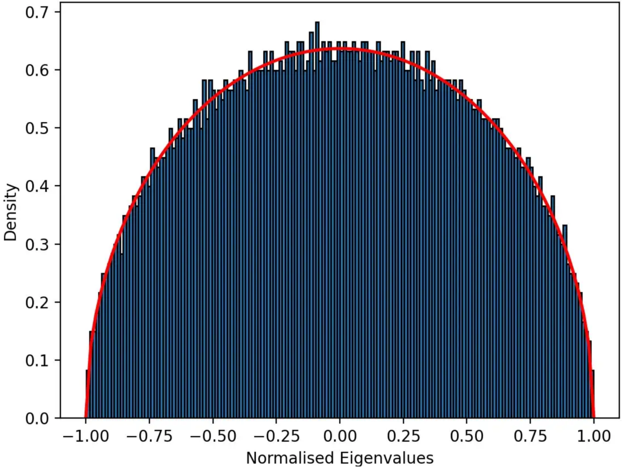
Plan

Intermittency

Introduction to stochastic partial differential equations

Random Matrices

Wigner's Semi-circle law



$$f(x) = \frac{2}{\pi}\sqrt{1 - x^2}, \quad |x| \leq 1.$$

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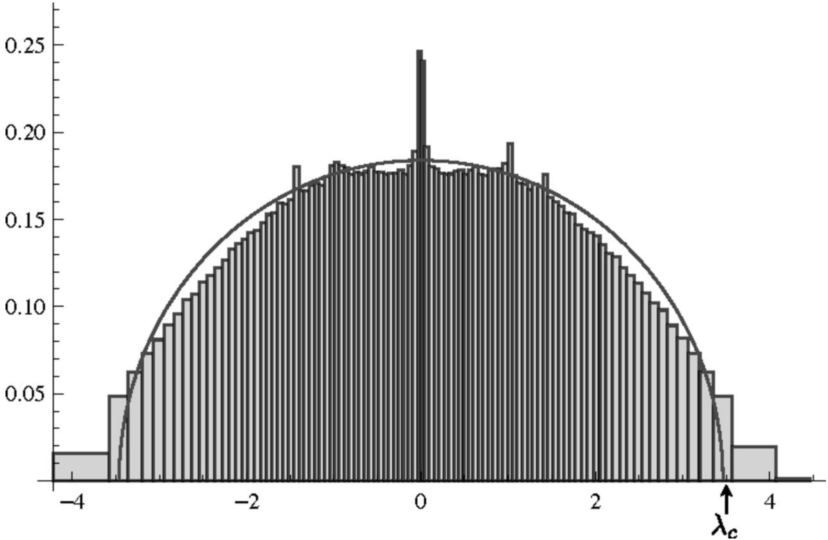
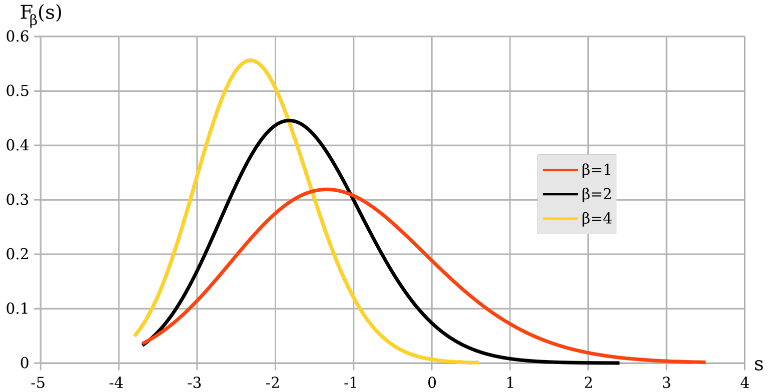
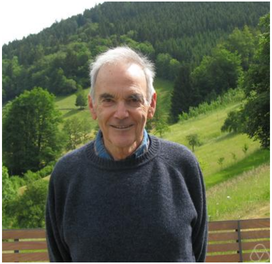
* Image is from https://www.cantorsparadise.com/elementary-results-in-random-matrix-theory-5abe7dab11ef.


0.25
0.20
0.15
0.10
0.05
−4
−2
0
2
4
λ_c

Tracy-Widom Distribution





G B C F D J B M T U W X Y Z S · · ·

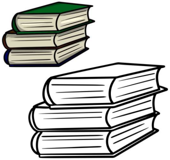
Longest increasing subsequence

G B C F D J B M T U W X Y Z S $\cdots$

$\underbrace{\text{B M T U W X Y Z}}_{\text{Longest increasing subsequence}}$

G B C F D J B M T U W X Y Z S · · ·

Longest increasing subsequence



Universality
— Tracy-Widom Distribution

- Distribution of the largest eigenvalue of a random matrix.
- The distribution of the length of the longest increasing subsequence of random permutations.
- The distribution of the fluctuations of the asymmetric simple exclusion process (ASEP) with step initial condition.
- SPDE...

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Amir, Corwin, Quastel proved in 2011 that

$$\mathbb{P}\left(\frac{\log u(t, t^{3/2}x) + \frac{t}{4!}}{t^{1/3}}\right) \to F_{\mathrm{GUE}}\left(2^{1/3}s\right), \quad \text{as } t \to \infty,$$

Open Question

Can one give an intrinsic proof of this limit?

Thank you for listening!

Le Chen (le.chen@auburn.edu)

Acknowledgment:

- Chatgpt
- Github Copilot
- Wikipeidia