Introduction to stochastic partial differential equations

Le Chen
Auburn University

Graduate Seminar

Department of Mathematics & Statistics
Auburn University

Sept. 27th 2021

Intermittency

Introduction to stochastic partial differential equations

Random Matrices

Additive *v.s.* Multiplicative

Example

Let X_n be i.i.d. Bernoulli r.v. with $P(X_n = 4) = P(X_n = 0) = 1/2$. Denote

$$\Sigma_n = \sum_{i=1}^{n} X_n.$$

$$\frac{\Sigma_n - \mathbb{E}(\Sigma_n)}{Var(\Sigma_n)} \to N(0,1), \quad \text{as } n \to \infty$$

Example

Let X_n be i.i.d. Bernoulli r.v. with $P(X_n = 4) = P(X_n = 0) = 1/2$. Denote

$$S_n = \prod_{i=1}^{n} X_n.$$

$$P\left(S_n = 2^{2n}\right) = \frac{1}{2^n} \qquad P\left(S_n = 0\right) = 1 - \frac{1}{2^n}.$$

$$\mathbb{E}\left[S_n\right] = 2^{2n} \times \frac{1}{2^n} + 0 \times \left(1 - \frac{1}{2^n}\right) = 2^n.$$

Additive *v.s.* Multiplicative

Example

Let X_n be i.i.d. Bernoulli r.v. with $P(X_n = 4) = P(X_n = 0) = 1/2$. Denote

$$\Sigma_n = \sum_{i=1}^{n} X_n.$$

$$\frac{\Sigma_n - \mathbb{E}(\Sigma_n)}{Var(\Sigma_n)} \to N(0,1), \quad \text{as } n \to \infty$$

Example

Let X_n be i.i.d. Bernoulli r.v. with $P(X_n = 4) = P(X_n = 0) = 1/2$. Denote

$$S_n = \prod_{i=1}^{n} X_n.$$

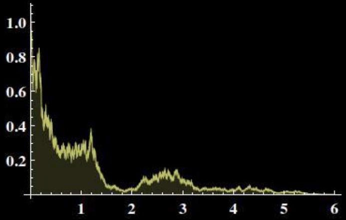
$$P\left(S_n = 2^{2^n}\right) = \frac{1}{2^n} \qquad P\left(S_n = 0\right) = 1 - \frac{1}{2^n}.$$

$$\mathbb{E}\left[S_n\right] = 2^{2n} \times \frac{1}{2^n} + 0 \times \left(1 - \frac{1}{2^n}\right) = 2^n.$$

Example

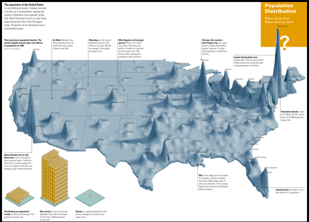
$dX_t = X_t dB_t$, $X_0 \equiv 1$. Then $X_t = \exp(B_t - t/2)$.

$X_t \to 0$ almost surely as $t \to \infty$ but $\mathbb{E}(X_t) \equiv 1$.



Intermittency

(Zeldovich *et al.* The almighty chance. 1990)



Matthew Effect

The rich get richer and the poor get poorer.

> For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away.
>
> — *Matthew* 25:29, RSV.

Intermittency

> Intermittency is a very universal phenomenon which occurs practically irrespective of detailed properties of the background instability in a random medium provided only that the random field is of multiplicative type, ...

Zeldovich *et al.* **The almighty chance**. 1990

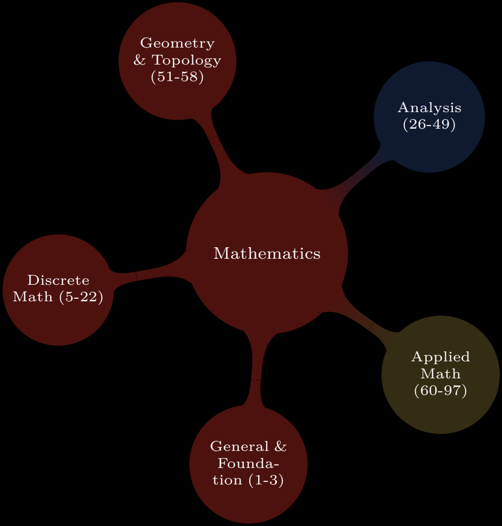
High peaks + highly concentrated on small islands | multiplicative noise

Intermittency

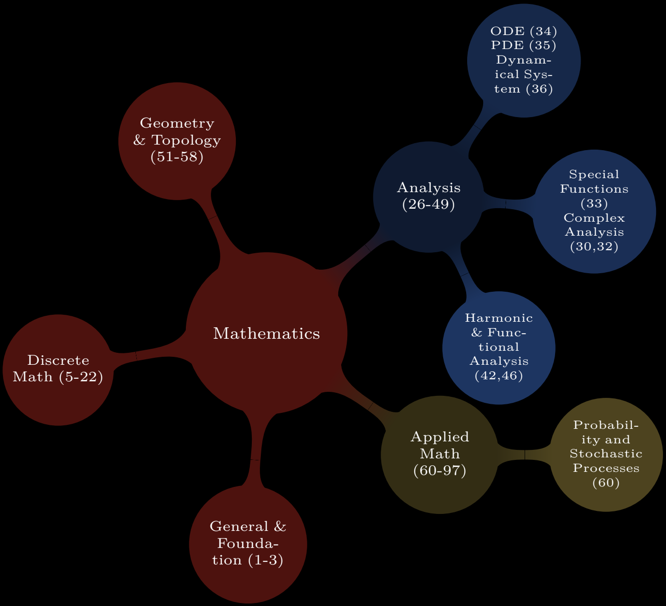
Introduction to stochastic partial differential equations

Random Matrices

SPDE as a branch of mathematics



SPDE as a branch of mathematics



SPDE as a branch of mathematics

Stochastic
Diff. Eq.

(SDE)

$u(t)$

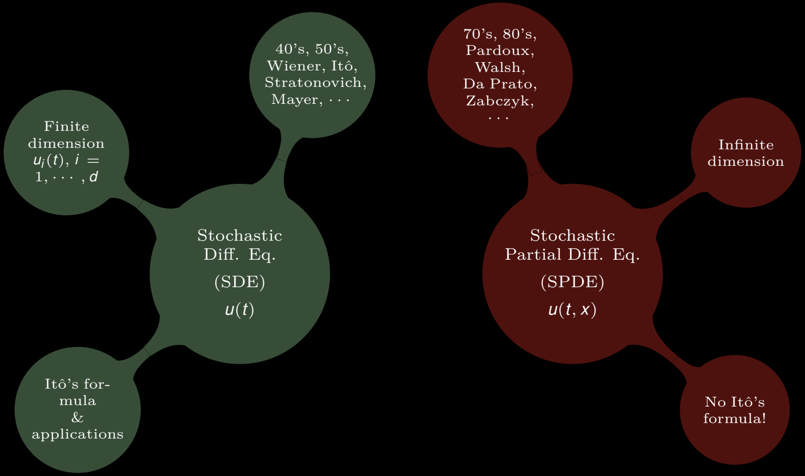
Stochastic
Partial Diff. Eq.

(SPDE)

$u(t, x)$

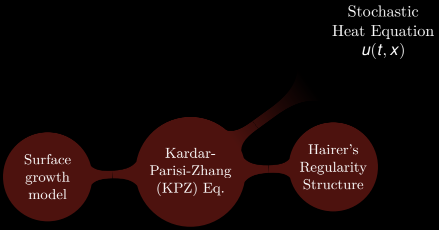

40's, 50's,
Wiener, Itô,
Stratonovich,
Mayer, ...
Stochastic
Diff. Eq.
(SDE)
$u(t)$
70's, 80's,
Pardoux,
Walsh,
Da Prato,
Zabczyk,
...
Stochastic
Partial Diff. Eq.
(SPDE)
$u(t, x)$

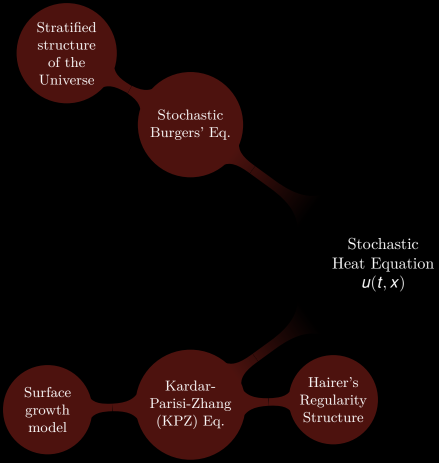

40's, 50's, Wiener, Itô, Stratonovich, Mayer, $\cdots$
Finite dimension $u_i(t)$, $i = 1, \cdots, d$
Stochastic Diff. Eq.
(SDE)
$u(t)$
70's, 80's, Pardoux, Walsh, Da Prato, Zabczyk, $\cdots$
Infinite dimension
Stochastic Partial Diff. Eq.
(SPDE)
$u(t, x)$

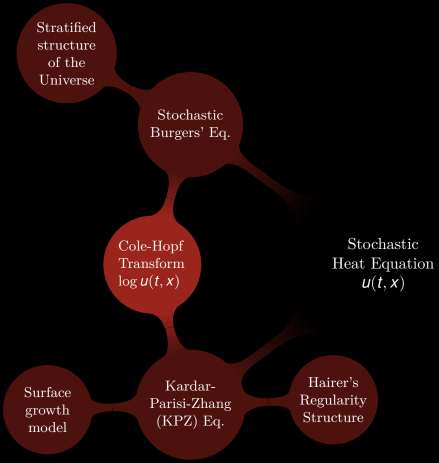

Stochastic Diff. Eq.
(SDE)
$u(t)$
40's, 50's, Wiener, Itô, Stratonovich, Mayer, $\cdots$
Finite dimension $u_i(t)$, $i = 1, \cdots, d$
Itô's formula & applications
Stochastic Partial Diff. Eq.
(SPDE)
$u(t, x)$
70's, 80's, Pardoux, Walsh, Da Prato, Zabczyk, $\cdots$
Infinite dimension
No Itô's formula!

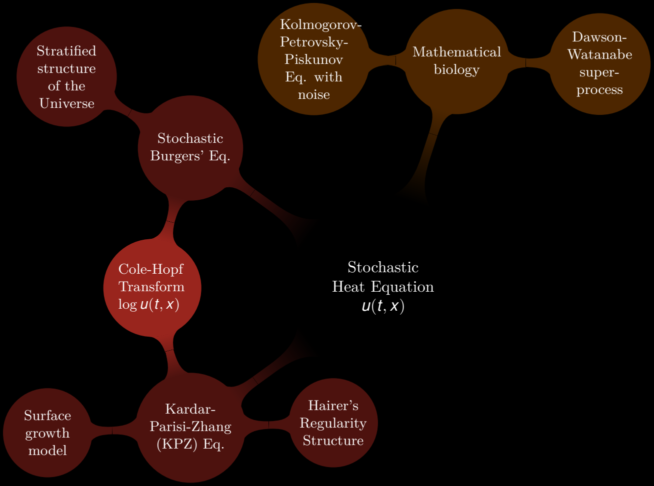

Finite dimension $u_i(t)$, $i = 1, \cdots, d$
40's, 50's, Wiener, Itô, Stratonovich, Mayer, ...
Stochastic Diff. Eq. (SDE) $u(t)$
Itô's formula & applications
Mature
70's, 80's, Pardoux, Walsh, Da Prato, Zabczyk, ...
Infinite dimension
Stochastic Partial Diff. Eq. (SPDE) $u(t, x)$
Vigorously developing ...
No Itô's formula!

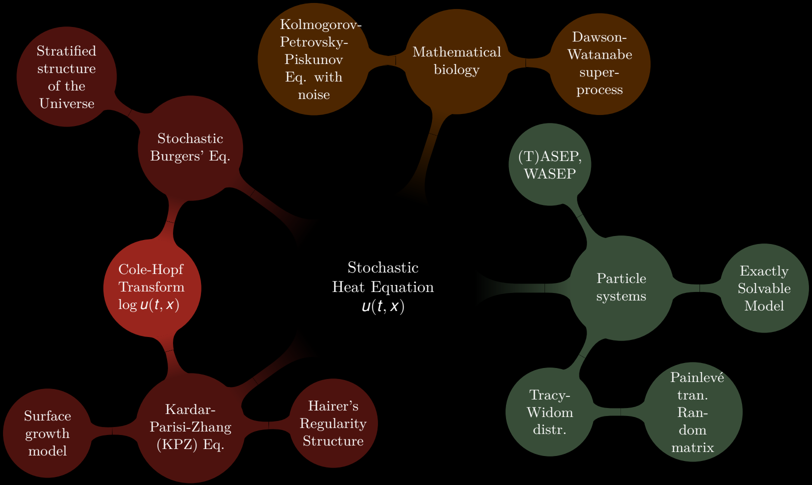
Stochastic
Heat Equation
$u(t, x)$

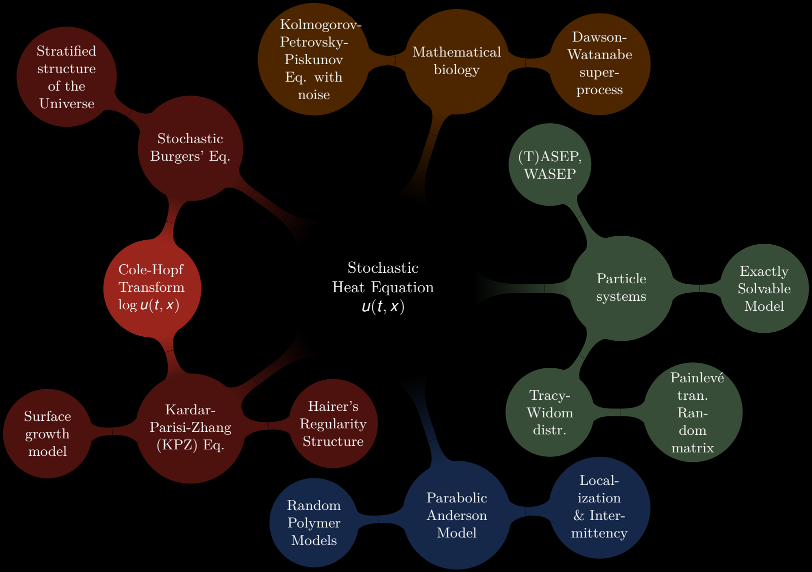

Stochastic
Heat Equation
$u(t, x)$
Surface
growth
model
Kardar-
Parisi-Zhang
(KPZ) Eq.
Hairer's
Regularity
Structure


Stratified structure of the Universe
Stochastic Burgers' Eq.
Stochastic Heat Equation $u(t, x)$
Surface growth model
Kardar-Parisi-Zhang (KPZ) Eq.
Hairer's Regularity Structure

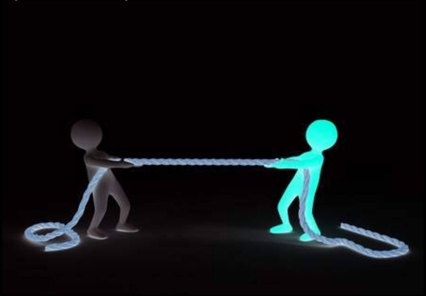

Stratified structure of the Universe
Stochastic Burgers' Eq.
Cole-Hopf Transform $\log u(t,x)$
Stochastic Heat Equation $u(t,x)$
Kardar-Parisi-Zhang (KPZ) Eq.
Surface growth model
Hairer's Regularity Structure


Stratified structure of the Universe
Stochastic Burgers' Eq.
Kolmogorov-Petrovsky-Piskunov Eq. with noise
Mathematical biology
Dawson-Watanabe super-process
Cole-Hopf Transform $\log u(t,x)$
Stochastic Heat Equation $u(t,x)$
Surface growth model
Kardar-Parisi-Zhang (KPZ) Eq.
Hairer's Regularity Structure


Stochastic Heat Equation $u(t, x)$
Mathematical biology
Kolmogorov-Petrovsky-Piskunov Eq. with noise
Dawson-Watanabe super-process
Stochastic Burgers' Eq.
Stratified structure of the Universe
Cole-Hopf Transform $\log u(t, x)$
Kardar-Parisi-Zhang (KPZ) Eq.
Surface growth model
Hairer's Regularity Structure
Particle systems
(T)ASEP, WASEP
Exactly Solvable Model
Tracy-Widom distr.
Painlevé tran. Ran-dom matrix

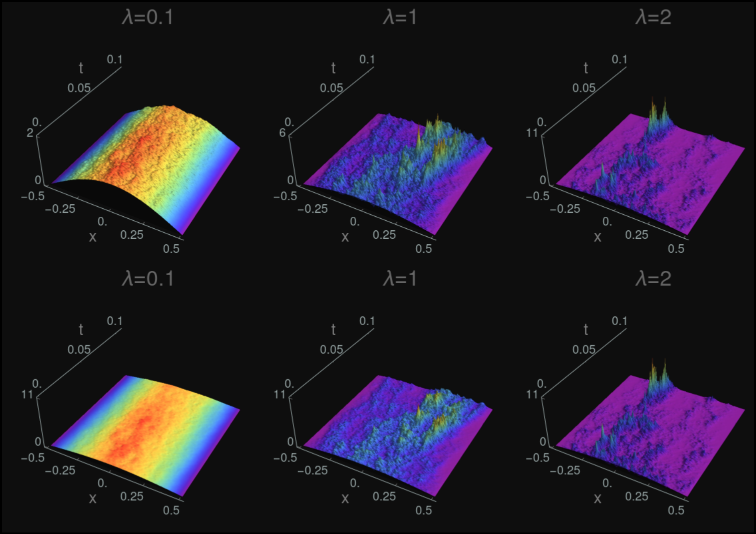

Stochastic Heat Equation $u(t,x)$
Cole-Hopf Transform $\log u(t,x)$
Stochastic Burgers' Eq.
Stratified structure of the Universe
Kardar-Parisi-Zhang (KPZ) Eq.
Surface growth model
Hairer's Regularity Structure
Mathematical biology
Kolmogorov-Petrovsky-Piskunov Eq. with noise
Dawson-Watanabe super-process
Particle systems
(T)ASEP, WASEP
Exactly Solvable Model
Tracy-Widom distr.
Painlevé tran. Ran-dom matrix
Parabolic Anderson Model
Random Polymer Models
Local-ization & Inter-mittency

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u(t,x) = \rho(u(t,x))\,\dot{W}(t,x)$$

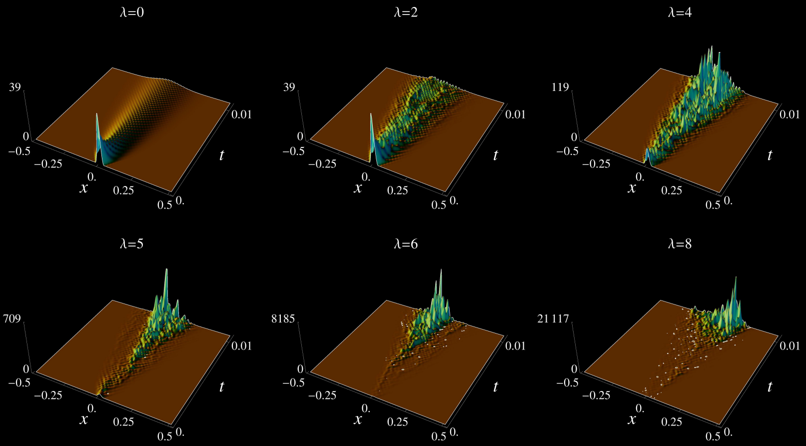


Δ	$\rho(u)\dot{W}$
Smoothing	Roughening

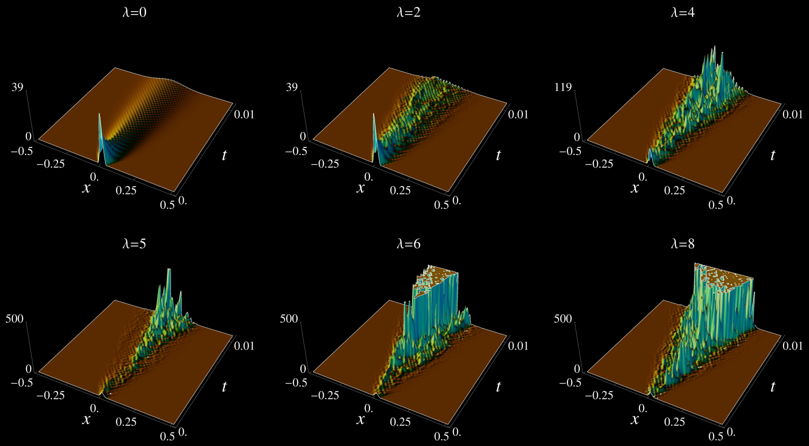
$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x),\ x \in [-1/2, 1/2] \text{ with } u(0,x) = \cos(\pi x)$$



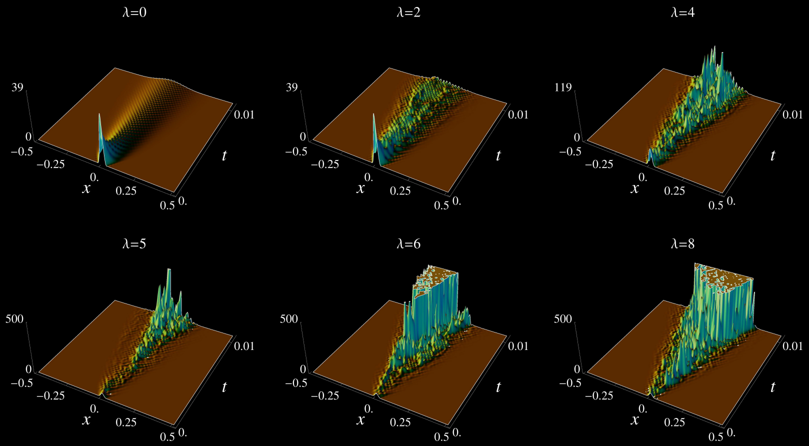
$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x) \dot{W}(t,x),\ x \in \mathbb{R}$$



$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x) \dot{W}(t,x),\ x \in \mathbb{R}$$



$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u(t,x) = \lambda u(t,x)\dot{W}(t,x),\ x \in \mathbb{R}$$



The rate of the propagation of the tall peaks $\asymp \lambda^2$

L.C. & Dalang, 15.

Various approaches for SPDE's

- – Semigroup approach (Da Prato, et al). $u(t) \in L^p(\Omega, \mathcal{H})$

  – Variational approach (Röckner, et al). Gel'fand triple

- Random field approach (Walsh, Dalang, et al.) $u(t, x) \in L^p(\Omega)$

- – Regularity structure (Hairer): Singular SPDEs

  – Paracontrolled distributions (Gubinelli, et al)

Various approaches for SPDE's

- – Semigroup approach (Da Prato, et al). $u(t) \in L^p(\Omega, \mathcal{H})$

  – Variational approach (Röckner, et al). Gel'fand triple

- Random field approach (Walsh, Dalang, et al.) $u(t,x) \in L^p(\Omega)$

  The approach that we are using!

- – Regularity structure (Hairer): Singular SPDEs

  – Paracontrolled distributions (Gubinelli, et al)

Stochastic Heat Equation on $\mathbb{R}^d$

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{1}{2}\Delta\right) u(t,x) = \rho(u(t,x))\dot{W}(t,x), \quad t > 0\ ,\ x \in \mathbb{R}^d \\ u(0,\cdot) = \mu(\cdot) \end{cases} \tag{SHE}$$

1. ρ is globally Lipschitz continuous s.t. $\rho(0) = 0$.
2. μ is *rough* (measure+integrability condition).
3. $\dot{W}$ is white in time and colored in space (f nonneg. & nonneg. definite):

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)f(x-y).$$

4. $\rho(u) = \lambda u$: *Parabolic Anderson Model*.

Mild solution

$$u(t,x) = J_0(t,x) + \underbrace{\int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)\rho(u(s,y))W(\mathrm{d}s\mathrm{d}y)}_{=: I(t,x)}$$

† $G(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$

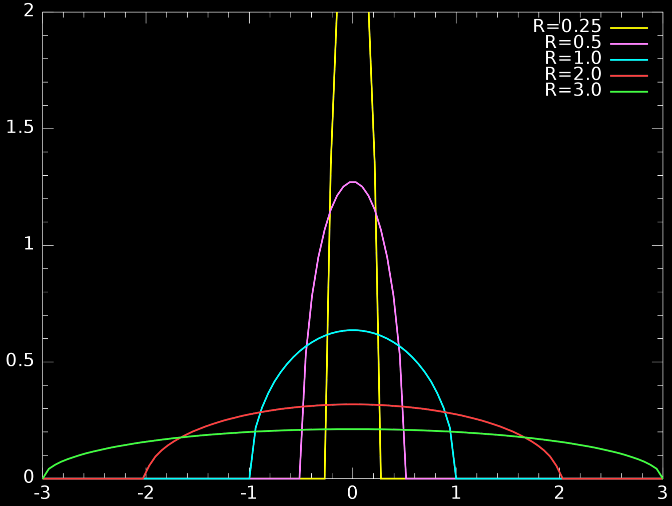
† $J_0(t,x) = \int_{\mathbb{R}^d} G(t, x-y)\mu(\mathrm{d}y)$

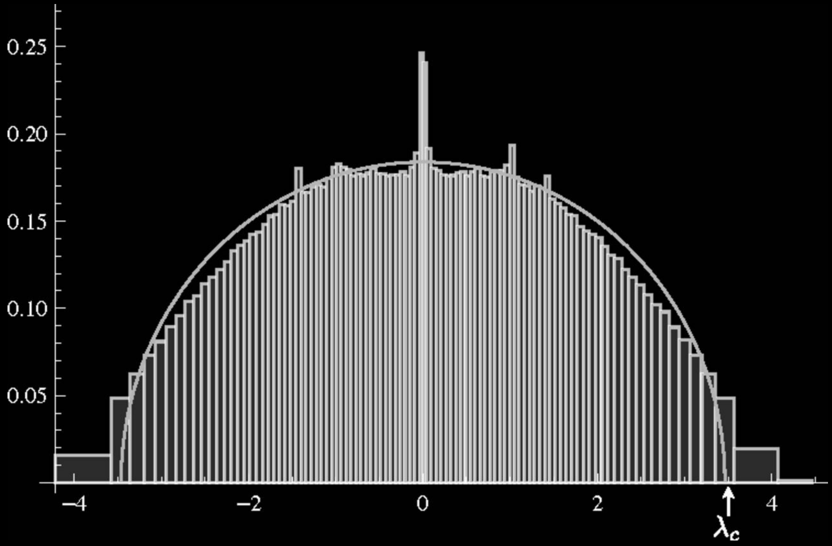
Intermittency

Introduction to stochastic partial differential equations

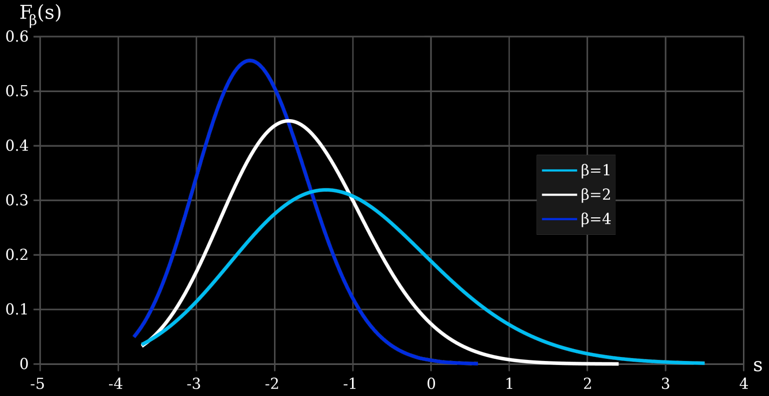
Random Matrices

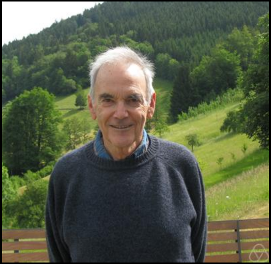
Wigner's Semi-circle law




0.25
0.20
0.15
0.10
0.05
−4
−2
0
2
4
λ_c

Tracy-Widom Distribution





Universality
— Tracy-Widom Distribution

- Distribution of the largest eigenvalue of a random matrix.
- The distribution of the length of the longest increasing subsequence of random permutations.
- The distribution of the fluctuations of the asymmetric simple exclusion process (ASEP) with step initial condition.
- SPDE...

Universality — Tracy-Widom Distribution

- Distribution of the largest eigenvalue of a random matrix.
- The distribution of the length of the longest increasing subsequence of random permutations.
- The distribution of the fluctuations of the asymmetric simple exclusion process (ASEP) with step initial condition.
- SPDE...

Universality
— Tracy-Widom Distribution

- Distribution of the largest eigenvalue of a random matrix.
- The distribution of the length of the longest increasing subsequence of random permutations.
- The distribution of the fluctuations of the asymmetric simple exclusion process (ASEP) with step initial condition.
- SPDE...

Universality — Tracy-Widom Distribution

- Distribution of the largest eigenvalue of a random matrix.
- The distribution of the length of the longest increasing subsequence of random permutations.
- The distribution of the fluctuations of the asymmetric simple exclusion process (ASEP) with step initial condition.
- SPDE...

Amir, Corwin, Quastel proved in 2011 that

$$\mathbb{P}\left(\frac{\log u(t, t^{3/2}x) + \frac{t}{4!}}{t^{1/3}}\right) \to F_{\mathrm{GUE}}\left(2^{1/3}s\right), \quad \text{as } t \to \infty,$$

Open Question

Can one give an intrinsic proof of this limit?

Thank you for listening!

Le Chen (le.chen@auburn.edu)