Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 12. The Black-Scholes Formula

The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

Check Python code Figure12-1.py

- Consider an European call (or put) option written on a stock
- Assume that the stock pays dividend at the continuous rate δ

Call options	Put options
$C(S, K, \sigma, r, T, \delta)$	$P(S, K, \sigma, r, T, \delta)$
$\vert\vert$	$\vert\vert$
$Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$	$Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Put-call Parity

$$P = C + Ke^{-rT} - Se^{-\delta T}$$

$$d_1 - d_2 = \sigma\sqrt{T}$$

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Put-call Parity

$$P = C + Ke^{-rT} - Se^{-\delta T}$$

$$d_1 - d_2 = \sigma\sqrt{T}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx$$

Example 12.1-1 Verify that the Black-Scholes formula for call and put

$$C := C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$
$$P := P(S, K, \sigma, r, T, \delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

with

$$d_i = \frac{\ln(S/K) + (r - \delta - (-1)^i \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad i = 1, 2$$

satisfies the call-put parity: $C - P = Se^{-\delta T} - Ke^{-rT}$.

Solution.

$\square$

Example 12.1-2 Plot the functions

$$S \to C(S, K, \sigma, r, T-t, \delta) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$S \to P(S, K, \sigma, r, T-t, \delta) = Ke^{-r(T-t)}N(-d_2) - Se^{-\delta(T-t)}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

with σ, r, δ, K fixed for various values of $T - t = 2, 1.5, 1, 0.5, 0$.

Solution. Try code

CallPut_vs_T-t.nb

□

Example 12.1-3 Let $S = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 8\%$, $T = 0.25$ (3 months), and $\delta = 0$. Compute the Black-Scholes call and put prices. Compare what you obtained with the results obtained from the binomial tree.

Check code
Example12-1.py

When is the Black-Scholes formula valid?

Assumptions about stock return distribution

- Continuously compounded returns on the stock are normally distributed and independent over time (no "jumps")
- The volatility of continuously compounded returns is known and constant
- Future dividends are known, either as dollar amount or as a fixed dividend yield

Assumptions about the economic environment

- The risk-free rate is known and constant
- There are no transaction costs or taxes
- It is possible to short-sell costlessly and to borrow at the risk-free rate

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Chapter 12. The Black-Scholes Formula

§ 12.1 Introduction to the Black-Scholes formula

§ 12.2 Applying the formula to other assets

§ 12.3 Option Greeks

§ 12.4 Problems

This section is left to motivated students to study.

What happens to the option price when one and only one input changes?

- Delta (Δ): change in option price when stock price increases by \$1
- Gamma (Γ): change in delta when option price increases by \$1
- Vega: change in option price when volatility increases by 1%
- Theta (θ): change in option price when time to maturity decreases by 1 day
- Rho (ρ): change in option price when interest rate increases by 1%
- Psi (ψ): change in the option premium due to a change in the dividend yield

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- The **Greek measure of a portfolio** is weighted average of Greeks of individual portfolio components

$$\Delta_{\text{portfolio}} = \sum_{i=1}^{N} n_i \Delta_i$$

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$$
\begin{tikzcd}[column sep=tiny]
\Gamma \arrow[r] & \Delta \arrow[d] & & & & \theta \arrow[d] & \\
& C(S, & K, & \sigma, & r, & T-t, & \delta) \\
& & & \text{Vega} \arrow[u] & \rho \arrow[u] & & \psi \arrow[u]
\end{tikzcd}
$$

Delta

Delta (Δ): change in option price when stock price increases by \$1.

$$\Delta = \begin{cases} \dfrac{\partial C(S, K, \sigma, T-t, \delta)}{\partial S} = +e^{-\delta(T-t)} N(+d_1) & \text{Call} \\ \dfrac{\partial P(S, K, \sigma, T-t, \delta)}{\partial S} = -e^{-\delta(T-t)} N(-d_1) & \text{Put} \end{cases}$$

Example 12.3-1 Demonstrate that

$$\Delta = \begin{cases} \dfrac{\partial C(S, K, \sigma, T-t, \delta)}{\partial S} = +e^{-\delta(T-t)} N(+d_1) & \text{Call} \\ \dfrac{\partial P(S, K, \sigma, T-t, \delta)}{\partial S} = -e^{-\delta(T-t)} N(-d_1) & \text{Put.} \end{cases}$$

Solution. We only show the call part. By the chain rule:

$$\begin{aligned} \frac{\partial C}{\partial S} = \quad & e^{-\delta(T-t)} N(d_1) \\ & + S e^{-\delta(T-t)} N'(d_1) \frac{\partial d_1}{\partial S} - K e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S}. \end{aligned}$$

Because $d_2 = d_1 - \sigma\sqrt{T-t}$, we see that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}.$$

It suffices to prove that

$$S e^{\delta(T-t)} N'(d_1) = K e^{-r(T-t)} N'(d_2).$$

Solution. (Continued) Notice that

$$N'(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}.$$

The above relation is equivalent to

$$\frac{Se^{(r-\delta)(T-t)}}{K} = \exp\left(\frac{d_1^2 - d_2^2}{2}\right). \tag{$\star$}$$

Now, from the definitions of d_1 and d_2, we see that

$$\begin{aligned} d_1^2 - d_2^2 &= d_1^2 - \left(d_1 - \sigma\sqrt{T-t}\right)^2 \\ &= 2d_1\sigma\sqrt{T-t} - \sigma^2(T-t) \\ &= 2\left(\ln\left(S/K\right) + (r-\delta)(T-t)\right) \\ &= 2\ln\left(\frac{Se^{(r-\delta)(T-t)}}{K}\right). \end{aligned}$$

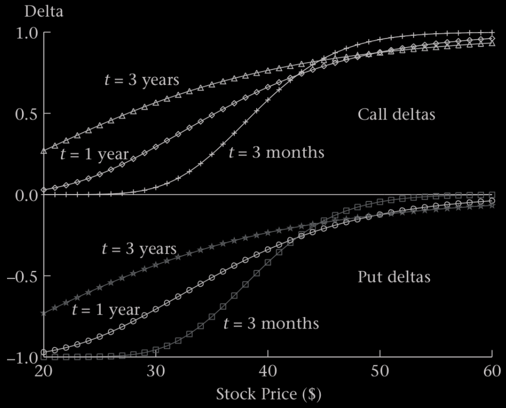
Plugging the above expression back to $(\star)$ proves the case. $\square$

In the above proof, we have showed the following relation, which will be useful in the computations of other Greeks:

$$\boxed{Se^{-\delta(T-t)}N'(d_1) = Ke^{-r(T-t)}N'(d_2)}$$

FIGURE 12.1

Call (top graph) and put (bottom graph) deltas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Gamma and Vega

Gamma (Γ): change in delta when option price increases by \$1

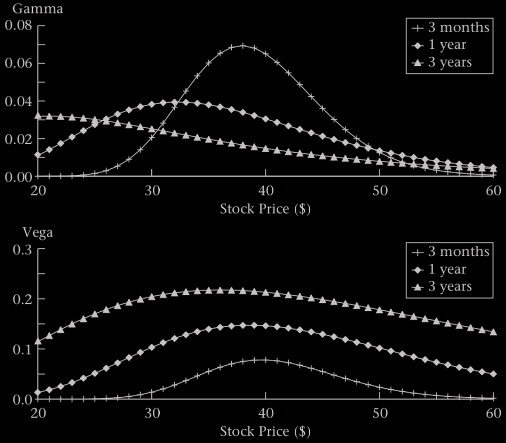
$$\Gamma = \frac{\partial^2 C(S, K, \sigma, r, T-t, \delta)}{\partial S^2} = \frac{\partial^2 P(S, K, \sigma, r, T-t, \delta)}{\partial S^2} = \frac{e^{-\delta(T-t)} N'(d_1)}{S\sigma\sqrt{T-t}}$$

Vega: change in option price when volatility increases by 1%

$$\text{Vega} = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial \sigma} = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial \sigma} = Se^{-\delta(T-t)} N'(d_1)\sqrt{T-t}$$

FIGURE 12.2

Gamma (top panel) and vega (bottom panel) for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$. Vega is the sensitivity of the option price to a 1 percentage point change in volatility. Otherwise identical calls and puts have the same gamma and vega.



Theta

Theta (θ): change in option price when time to maturity decreases by 1 day

$$\begin{aligned}
\text{Call } \theta &= \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial t} \\
&= \delta S e^{-\delta(T-t)} N(d_1) - rKe^{-r(T-t)} N(d_2) - \frac{Ke^{r(T-r)} N'(d_2)\sigma}{2\sqrt{T-t}}
\end{aligned}$$

$$\begin{aligned}
\text{Put } \theta &= \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial t} \\
&= \text{Call } \theta + rKe^{-r(T-t)} + \delta S e^{-\delta(T-t)}
\end{aligned}$$

FIGURE 12.3

Call (top panel) and put (bottom panel) prices for options with different strikes at different times to expiration. Assumes $S = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.

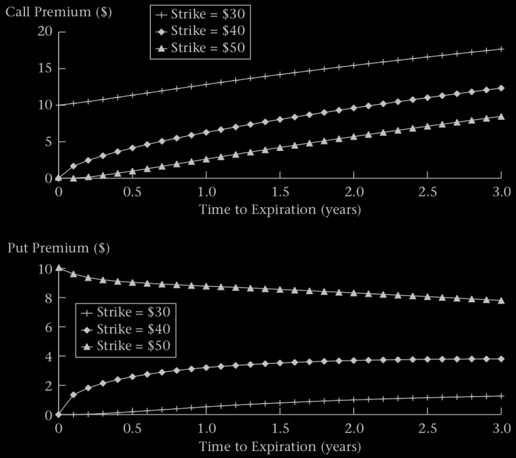
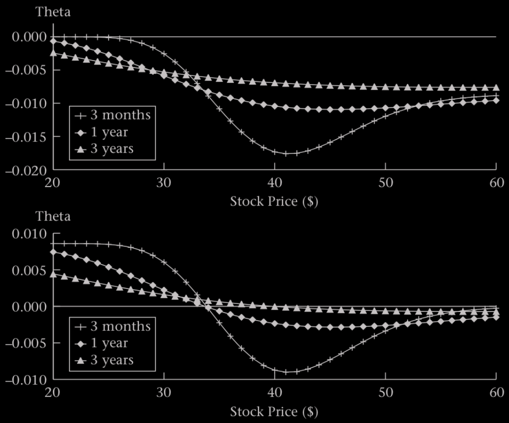


FIGURE 12.4

Theta for calls (top panel) and puts (bottom panel) with different expirations at different stock prices. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Rho and Psi

Rho (ρ): change in option price when interest rate increases by 1%

$$\text{Call } \rho = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial r} = +(T-t)Ke^{-r(T-t)}N(+d_2)$$

$$\text{Put } \rho = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial r} = -(T-t)Ke^{-r(T-t)}N(-d_2)$$

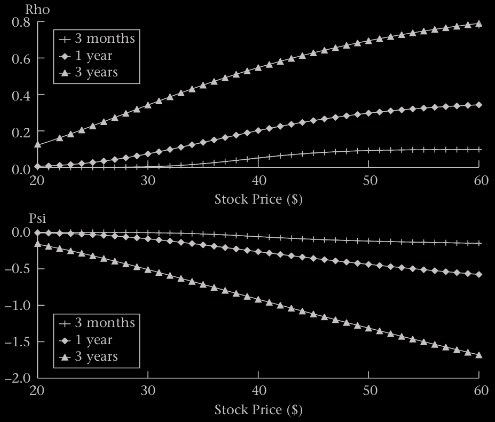
Psi (ψ): change in the option premium due to a change in the dividend yield

$$\text{Call } \psi = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial \delta} = -(T-t)Ke^{-\delta(T-t)}N(+d_1)$$

$$\text{Put } \psi = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial \delta} = +(T-t)Ke^{-\delta(T-t)}N(-d_1)$$

FIGURE 12.5

Rho (top panel) and psi (bottom panel) at different stock prices for call options with different maturities. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Do these Greeks satisfy some relation?

Theorem 12.3-1 Let $V(t, S)$ denote the option price for either European call or put. Recall that

$$V_t = \theta, \quad V_S = \Delta, \quad \text{and} \quad V_{SS} = \Gamma.$$

Then, these three Greeks have to satisfy the Black-Scholes equation:

$$\boxed{V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r-\delta)SV_S - rV = 0} \qquad 0 \le t \le T, \qquad \text{(BS)}$$

with the boundary conditions:

Condition	call	put
$V(T,S)$	$\max(S-K,0)$	$\max(K-S,0)$
$V(t,S)$	0	$Ke^{-r(T-t)}$
$\lim_{S\to\infty} V(t,S)$	S	0

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Then, these three Greeks have to satisfy the Black-Scholes equation:

$$\boxed{V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta) S V_S - rV = 0} \qquad 0 \leq t \leq T, \qquad \text{(BS)}$$

with the boundary conditions:

Condition	call	put
$V(T, S)$	$\max(S - K, 0)$	$\max(K - S, 0)$
$V(t, S)$	0	$Ke^{-r(T-t)}$
$\lim_{S\to\infty} V(t, S)$	S	0

Proof. We will only verify (BS). This can be easily done by the symbolic computations via Mathematica. Check

Greeks-BS-Equation.nb

□

Questions:

(1) How to derive this Black-Scholes equation?

(2) How to solve this equation to get the Black-Scholes formula?

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TABLE 12.2 Greeks for a bull spread where $S = \$40$, $\sigma = 0.3$, $r = 0.08$, and $T = 91$ days, with a purchased 40-strike call and a written 45-strike call. The column titled "combined" is the difference between column 1 and column 2.

	40-Strike Call	45-Strike Call	Combined
ω_i	1	−1	—
Price	2.7804	0.9710	1.8094
Delta	0.5824	0.2815	0.3009
Gamma	0.0652	0.0563	0.0088
Vega	0.0780	0.0674	0.0106
Theta	−0.0173	−0.0134	−0.0040
Rho	0.0511	0.0257	0.0255

Delta (Δ): change in option price when stock price increases by \$1

Option Elasticity (Ω): If stock price S changes by 1%, what is the percentage change in the value of the option C:

$$\Omega = \frac{\text{Percentage change in option price}}{\text{Percentage change in stock price}} = \frac{\frac{\epsilon \Delta}{C}}{\frac{\epsilon}{S}} = \frac{S\Delta}{C}.$$

Chapter 12. The Black-Scholes Formula

Problems: 12.3, 12.4, 12.6, 12.7, 12.9,

Due Date: TBA