Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 12. The Black-Scholes Formula

Chapter 12. The Black-Scholes Formula

The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

- Consider an European call (or put) option written on a stock
- Assume that the stock pays dividend at the continuous rate δ

Call options	Put options
$C(S, K, \sigma, r, T, \delta)$	$P(S, K, \sigma, r, T, \delta)$
$\vert\vert$	$\vert\vert$
$Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$	$Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Put-call Parity

$$P = C + Ke^{-rT} - Se^{-\delta T}$$

$$d_1 - d_2 = \sigma\sqrt{T}$$

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Example 12.1-1 Let $S = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 8\%$, $T = 0.25$ (3 months), and $\delta = 0$. Compute the Black-Scholes call and put prices.

When is the Black-Scholes formula valid?

Assumptions aboutstock return distribution

- Continuously compounded returns on the stock are normally distributed and independent over time (no "jumps")
- The volatility of continuously compounded returns is known and constant
- Future dividends are known, either as dollar amount or as a fixed dividend yield

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Assumptions about the economic environment

- The risk-free rate is known and constant
- There are no transaction costs or taxes
- It is possible to short-sell costlessly and to borrow at the risk-free rate

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Chapter 12. The Black-Scholes Formula

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This section is left to motivated students to study.

Chapter 12. The Black-Scholes Formula

What happens to the option price when one and only one input changes?

- Delta (Δ): change in option price when stock price increases by \$1
- Gamma (Γ): change in delta when option price increases by \$1
- Vega: change in option price when volatility increases by 1%
- Theta (θ): change in option price when time to maturity decreases by 1 day
- Rho (ρ): change in option price when interest rate increases by 1%
- Psi (ψ): change in the option premium due to a change in the dividend yield

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- The **Greek measure of a portfolio** is weighted average of Greeks of individual portfolio components

$$\Delta_{\text{portfolio}} = \sum_{i=1}^{N} n_i \Delta_i$$

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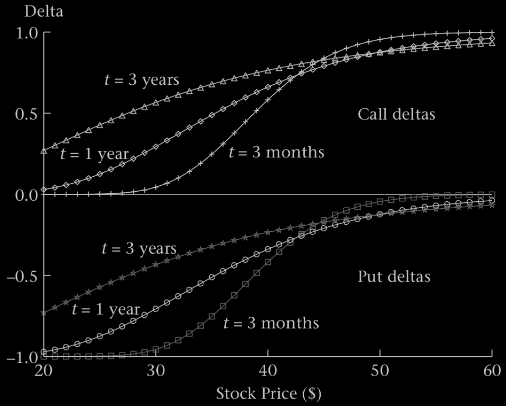
Delta

Delta (Δ): change in option price when stock price increases by \$1.

$$\Delta = \begin{cases} \dfrac{\partial C(S, K, \sigma, T-t, \delta)}{\partial S} = +e^{-\delta(T-t)} N(+d_1) & \text{Call} \\ \dfrac{\partial P(S, K, \sigma, T-t, \delta)}{\partial S} = -e^{-\delta(T-t)} N(-d_1) & \text{Put} \end{cases}$$

FIGURE 12.1

Call (top graph) and put (bottom graph) deltas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Gamma and Vega

Gamma (Γ): change in delta when option price increases by \$1

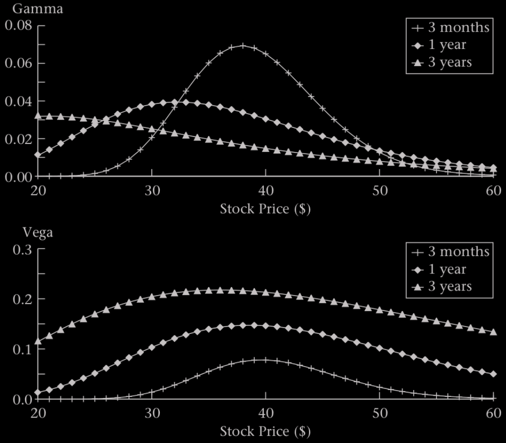
$$\Gamma = \frac{\partial^2 C(S, K, \sigma, r, T-t, \delta)}{\partial S^2} = \frac{\partial^2 P(S, K, \sigma, r, T-t, \delta)}{\partial S^2} = \frac{e^{-d(T-t)N'(d_1)}}{S\sigma\sqrt{T-t}}$$

Vega: change in option price when volatility increases by 1%

$$\text{Vega} = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial \sigma} = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial \sigma} = Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$$

FIGURE 12.2

Gamma (top panel) and vega (bottom panel) for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$. Vega is the sensitivity of the option price to a 1 percentage point change in volatility. Otherwise identical calls and puts have the same gamma and vega.



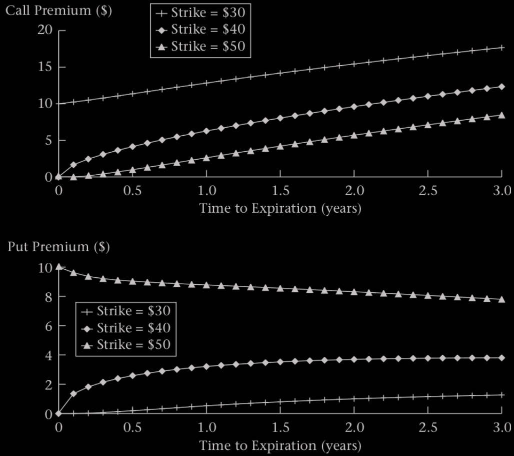
Theta

Theta (θ): change in option price when time to maturity decreases by 1 day

$$\begin{aligned}\text{Call } \theta &= \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial t} \\ &= \delta S e^{-\delta(T-t)} N(d_1) - rKe^{-r(T-t)} N(d_2) - \frac{Ke^{r(T-r)} N'(d_2)\sigma}{2\sqrt{T-t}}\end{aligned}$$

$$\begin{aligned}\text{Put } \theta &= \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial t} \\ &= \text{Call } \theta + rKe^{-r(T-t)} + \delta S e^{-\delta(T-t)}\end{aligned}$$

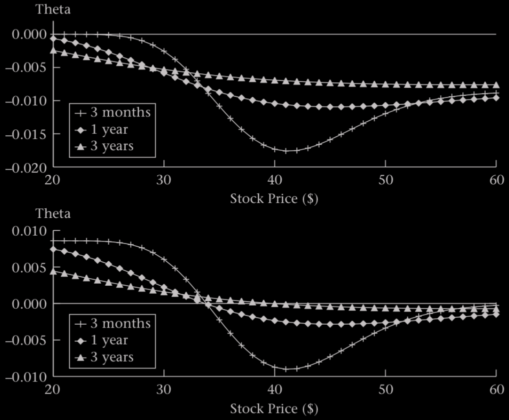
FIGURE 12.3



Call (top panel) and put (bottom panel) prices for options with different strikes at different times to expiration. Assumes $S = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.

FIGURE 12.4

Theta for calls (top panel) and puts (bottom panel) with different expirations at different stock prices. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Rho and Psi

Rho (ρ): change in option price when interest rate increases by 1%

$$\text{Call } \rho = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial r} = +(T-t)Ke^{-r(T-t)}N(+d_2)$$

$$\text{Put } \rho = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial r} = -(T-t)Ke^{-r(T-t)}N(-d_2)$$

Psi (ψ): change in the option premium due to a change in the dividend yield

$$\text{Call } \psi = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial \delta} = -(T-t)Ke^{-\delta(T-t)}N(+d_1)$$

$$\text{Put } \psi = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial \delta} = +(T-t)Ke^{-\delta(T-t)}N(-d_1)$$

FIGURE 12.5

Rho (top panel) and psi (bottom panel) at different stock prices for call options with different maturities. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.

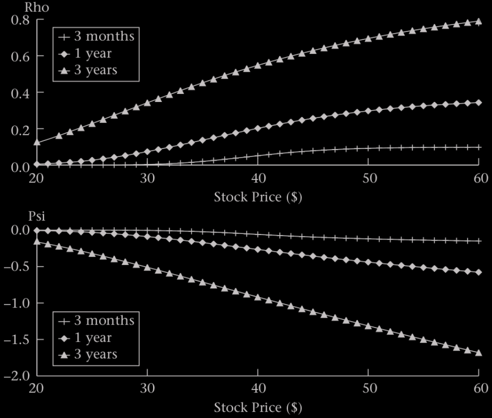


TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
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Delta (Δ): change in option price when stock price increases by \$1

Option Elasticity (Ω): If stock price S changes by 1%, what is the percentage change in the value of the option C:

$$\Omega = \frac{\text{Percentage change in option price}}{\text{Percentage change in stock price}} = \frac{\frac{\epsilon\Delta}{C}}{\frac{\epsilon}{S}} = \frac{S\Delta}{C}.$$

Chapter 12. The Black-Scholes Formula

Problems: 12.3, 12.4, 12.6, 12.7, 12.9,

Due Date: TBA