

Financial Mathematics

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¹Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 2. An Introduction to Forwards and Options

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§ 2.1 Forward contracts

§ 2.2 Call options

§ 2.3 Put options

§ 2.4 Options are insurance

§ 2.5 Summary of forward and option positions

§ 2.6 Problems

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Definition 2.1-1 **Forward contract** is a binding agreement (obligation) to buy or sell an underlying asset in the future, at a price set today. The time at which the contract settles is called the **expiration date**. A forward contract specifies

- ▶ The features and quantity of the asset to be delivered.
- ▶ The delivery logistics, such as time, date, and place.
- ▶ The price the buyer will pay at the time of delivery.

Remark 2.1-1

1. **Futures contracts** are the same as forwards in principle except for some institutional and pricing differences. We will study future contracts in Chapter 5.
2. A forward contract requires no initial payment or **premium**.

Long = buy short = sell

Definition 2.1-2 Payoff for a contract is its value at expiration. In particular, for forward contracts,

Payoff for Long forward = Spot price at expiration – Forward price

Payoff for Short forward = Forward price – Spot price at expiration

Remark 2.1-2 Payoff and profit (net payoff) are the same for forward contracts because there is no initial payment – premium.

Example 2.1-1 S&R (special and rich) index:

Today: Spot price = \$1,000

6-month forward price = \$1,020

In six months at contract expiration: Spot price = \$1,050.

What are the payoff of long/short forward?

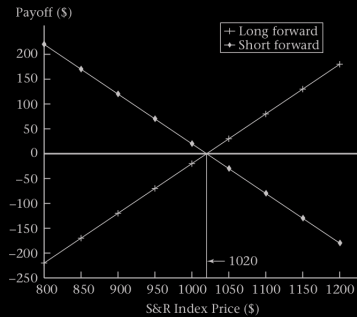
Solution.

Long position payoff = $\$1,050 - \$1,020 = \$30$,

Short position payoff = $\$1,020 - \$1,050 = (\$30)$.



Payoff diagram for a forward price = \$1,020



Forward versus outright purchase

We will see this through the following example:

Example 2.1-2 S&R 6-month forward contract with a zero-coupon bond (e.g., Treasury bills). The 6-month interest rate is 2%. Spot price today = \$1,000.

\$1,000 today is worth $\$1,000 \times 1.02 = \$1,020$ in 6 months.

Outright purchase² is equivalent to forward + bond³

because

$$\begin{aligned}\text{Payoff of forward+bond} &= \underbrace{\text{Spot price at expiration} - \$1,020}_{\text{Forward payoff}} + \underbrace{\$1,020}_{\text{Bond payoff}} \\ &= \text{Spot price at expiration} \\ &= \text{Payoff of outright purchase}\end{aligned}$$

²It is also called long physical index.

³Invest \$1,000 to bond for 6 month and enter long position of forward contract at the same time.

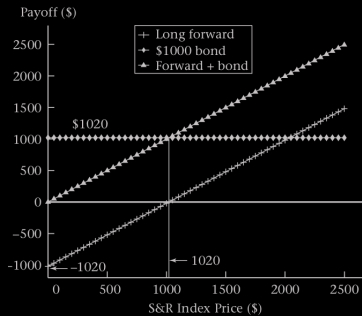
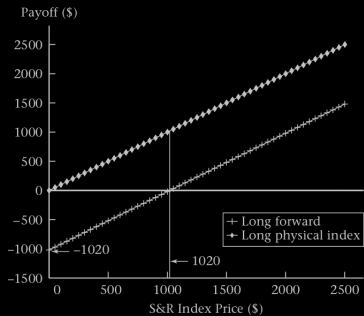
\$1,000 today is worth $\$1,000 \times 1.02 = \$1,020$ in 6 months.

Long forward is equivalent to borrow-to-buy⁴

because

$$\begin{aligned}\text{Payoff of borrow-to-buy} &= \underbrace{\text{Spot price at expiration}}_{\text{Payoff for outright buy}} - \underbrace{\$1,020}_{\text{Return borrowed money}} \\ &= \text{Payoff of long forward.}\end{aligned}$$

⁴Borrow money (\$1,000) to outright buy physical index and at expiration pay back the money (\$1,020).



Cash settlement versus physical delivery

– Type of settlement

- ▶ Cash settlement: less costly and more practical
- ▶ Physical delivery: often avoided due to significant costs

Example 2.1-3 Consider the S&R index with the forward price \$1,020.

- ▶ Suppose that the S&R index at expiration is \$1,040.
- ▶ The long position has a payoff of \$20.
- ▶ Similarly, the short position loses \$20.

With **cash settlement**, the short simply pays \$20 to the long, with **no transfer of the physical asset**, and hence **no transaction costs**. It is as if the long paid \$1,020, acquired the index worth \$1,040, and then immediately sold it with no transaction costs.

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- ▶ Suppose that the S&R index price at expiration had instead been \$960.
 - ▶ The long position would have a payoff of $-\$60$.
 - ▶ The short would have a payoff of \$60.

Cash settlement in this case entails the long paying \$60 to the short.

Credit risk

All derivatives contracts have **credit risk**, which is the possibility that the counterparty who owes money fails to make a payment.

- ▶ Major issue for **over-the-counter (OTC) contracts**

Credit check

Credit **protections** such as collateral and bank letter of credit

- ▶ Less severe for **exchange-traded contracts**

Exchange guarantees transactions, requires collateral

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Can one modify the forward contract so that the buyer can walk away from the deal at expiration?

Definition 2.2-1 A **call option** is a contract where the buyer has the right to buy, but not the obligation to buy.

Example 2.2-1 S&R index: Buyers' perspective

- ▶ Today: call buyer acquires the right to pay \$1,020 in six months for the index, but is not obligated to do so
 - ▶ In six months at contract expiration:
 - if the spot price is \$1,100, call buyers payoff = $\$1,100 - \$1,020 = \$80$
 - if the spot price is \$900, call buyer walks away, buyers payoff = \$0.
-

Example 2.2-2 S&R index: Sellers' perspective

- ▶ Today: call seller is obligated to sell the index for \$1,020 in six months, if asked to do so
- ▶ In six months at contract expiration:
 - if the spot price is \$1,100, call sellers payoff = $\$1,020 - \$1,100 = -\$80$
 - if the spot price is \$900, call buyer walks away, sellers payoff = \$0.

Buyer preserves the upside potential, while at the same time eliminates the unpleasant downside.

However

Seller has to be compensated by a initial premium for being at a disadvantage at expiration.

- ▶ **Strike (or exercise) price:** the amount paid by the option buyer for the asset if he/she decides to exercise.
- ▶ **Exercise:** the act of paying the strike price to buy the asset.
- ▶ **Expiration:** the date by which the option must be exercised or become worthless.
- ▶ **Exercise style:** specifies when the option can be exercised.

Style	can be exercised
European	only at expiration date
American	at any time before expiration
Bermudan	during specified periods

Payoff of purchased call = $\max(0, \text{spot price at expiration} - \text{strike price})$

Profit of purchased call = **payoff** of purchased call
– future value of option premium

Payoff of written call = $-\max(0, \text{spot price at expiration} - \text{strike price})$

Profit of written call = **payoff** of written call
+ future value of option premium

Example 2.2-3 S&R Index 6-month European call option

Strike price = \$1,000,

Premium = \$93.81,

6-month risk-free rate = 2%.

Compute both payoff and profit of the **purchased** call option if the index value in six months \$1,100 (resp. \$900).

Solution.

If index value in six months = \$1,100,

$$\begin{aligned}\text{Payoff} &= \max(0, \$1,100 - \$1,000) \\ &= \$100\end{aligned}$$

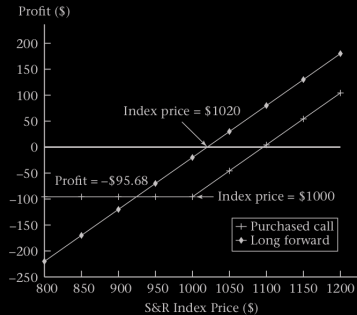
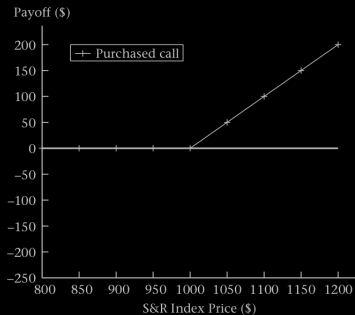
$$\begin{aligned}\text{Profit} &= \$100 - \$93.81 \times 1.02 \\ &= \$4.32.\end{aligned}$$

If index value in six months = \$900,

$$\begin{aligned}\text{Payoff} &= \max(0, \$900 - \$1,000) \\ &= \$0\end{aligned}$$

$$\begin{aligned}\text{Profit} &= \$0 - \$93.81 \times 1.02 \\ &= -\$95.68.\end{aligned}$$

□



Example 2.2-4 S&R Index 6-month European call option

Strike price = \$1,000,

Premium = \$93.81,

6-month risk-free rate = 2%.

Compute both payoff and profit of the **written** call option if the index value in six months \$1,100 (resp. \$900).

Solution.

If index value in six months = \$1,100,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,100 - \$1,000) \\ &= -\$100\end{aligned}$$

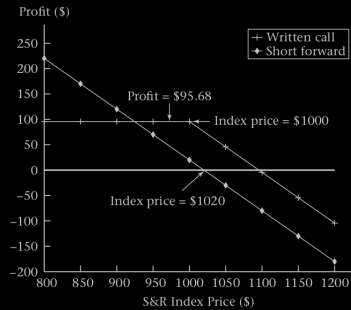
$$\begin{aligned}\text{Profit} &= -\$100 + \$93.81 \times 1.02 \\ &= -\$4.32.\end{aligned}$$

If index value in six months = \$900,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$900 - \$1,000) \\ &= \$0\end{aligned}$$

$$\begin{aligned}\text{Profit} &= \$0 + \$93.81 \times 1.02 \\ &= \$95.68.\end{aligned}$$

□



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Call option : Buyer can walk away.

???? option : Seller can walk away.

Definition 2.3-1 A **put option** gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.

Remark 2.3-1 Similar to the call option case, a premium paid by the put buyer at the time the option is purchased is needed in order to compensate the put seller for being in a disadvantage position.

... of put option	someone needs to	premium	
seller	buy	has to buy if asked	receive
buyer	sell	can walk away	pay

Payoff of purchased put = $\max(0, \text{strike price} - \text{spot price at expiration})$

Profit of purchased put = **payoff** of purchased put
– future value of option premium

Payoff of written put = $-\max(0, \text{strike price} - \text{spot price at expiration})$

Profit of written put = **payoff** of written put
+ future value of option premium

Example 2.3-1 S&R Index 6-month European put option

Strike price = \$1,000,

Premium = \$74.20,

6-month risk-free rate = 2%.

Compute both payoff and profit of the **purchased** put option if the index value in six months \$1,100 (resp. \$900).

Solution.

If index value in six months = \$1,100,

$$\begin{aligned}\text{Payoff} &= \max(0, \$1,000 - \$1,100) \\ &= \$0\end{aligned}$$

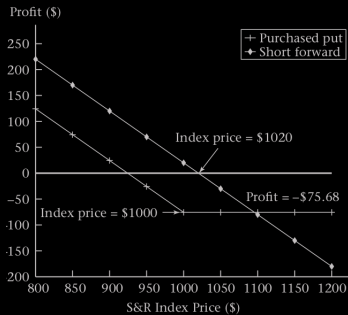
$$\begin{aligned}\text{Profit} &= \$0 - \$74.20 \times 1.02 \\ &= -\$75.68.\end{aligned}$$

If index value in six months = \$900,

$$\begin{aligned}\text{Payoff} &= \max(0, \$1,000 - \$900) \\ &= \$100\end{aligned}$$

$$\begin{aligned}\text{Profit} &= \$100 - \$74.20 \times 1.02 \\ &= \$24.32.\end{aligned}$$





Example 2.3-2 S&R Index 6-month European put option

Strike price = \$1,000,

Premium = \$74.20,

6-month risk-free rate = 2%.

Compute both payoff and profit of the **written** put option if the index value in six months \$1,100 (resp. \$900).

Solution.

If index value in six months = \$1,100,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,000 - \$1,100) \\ &= \$0\end{aligned}$$

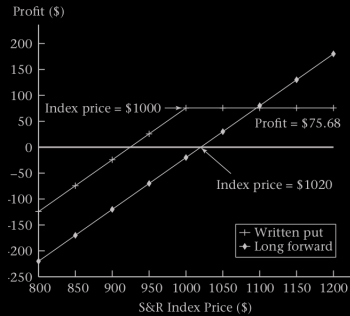
$$\begin{aligned}\text{Profit} &= \$0 + \$74.20 \times 1.02 \\ &= \$75.68.\end{aligned}$$

If index value in six months = \$900,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,000 - \$900) \\ &= -\$100\end{aligned}$$

$$\begin{aligned}\text{Profit} &= -\$100 + \$74.20 \times 1.02 \\ &= -\$24.32.\end{aligned}$$

□



A **call** option becomes more profitable
when the underlying asset
appreciates in value

A **put** option becomes more profitable
when the underlying asset
depreciates in value

Definition 2.3-2 **Moneyiness** of an option describes whether the option payoff would be positive if the option were exercised immediately.

In particular, one has

Moneyiness	payoff if exercised immediately
In-the-money option	> 0
At-the-money option	$= 0$
Out-of-the money option	< 0

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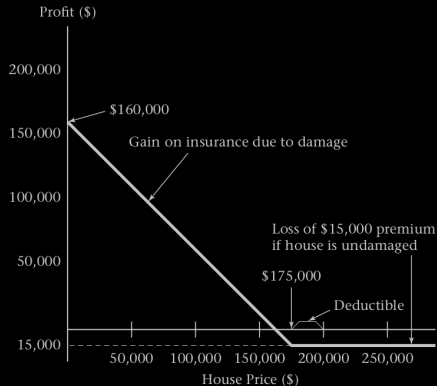
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Example 2.4-1 Homeowner's insurance is a put option:

Value of house = \$200,000

Deductible = \$25,000

Premium = \$15,000



The premium of the insurance
or
the value of the put option
depends on

- ▶ Riskiness of the underlying asset
 - ▶ The amount of deductible.
-

Difference with options

- ▶ Put option pays off no matter why the index price declines.
- ▶ Insurance pays off only if the house declines in value for specific reasons.

A put option is
an insurance

1. for an asset we already own.
 2. for a long position.
 3. against an decrease in value.
-

A call option is
an insurance

1. for an asset we plan to own in the future.
2. for a short position.
3. against an increase in price.

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$$\{\text{long, short}\} \times \{\text{forward, call, put}\}$$

||

six positions

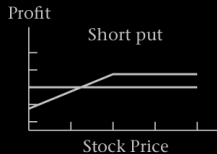
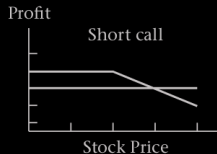
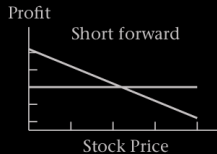
Maximum possible profit and loss at maturity for
 $\{\text{long, short}\} \times \{\text{forward, call, put}\}$

Position	Maximum Loss	Maximum Gain
Long forward	−Forward price	Unlimited
Short forward	Unlimited	Forward price
Long call	−FV(<i>premium</i>)	Unlimited
Short call	Unlimited	FV(<i>premium</i>)
Long put	−FV(<i>premium</i>)	Strike price − FV(<i>premium</i>)
Short put	FV(<i>premium</i>) − Strike price	FV(<i>premium</i>)

5

⁵ $FV(\cdot)$ denotes the function that returns the future value.

Profit diagrams for
 $\{\text{long}, \text{short}\} \times \{\text{forward}, \text{call}, \text{put}\}$



Summary of positions for
 $\{\text{long, short}\} \times \{\text{forward, call, put}\}$

Derivative Position	Position with Respect to Underlying Asset	Asset Price Contingency	Strategy
Long forward	Long (buy)	Always	Guaranteed purchase price
Short forward	Short (sell)	Always	Guaranteed sale price
Long call	Long (buy)	$> \text{Strike}$	Insures against high price
Short call	Short (sell)	$> \text{Strike}$	Sells insurance against high price
Long put	Short (sell)	$< \text{Strike}$	Insures against low price
Short put	Long (buy)	$< \text{Strike}$	Sells insurance against low price

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