Financial Mathematics

MATH 5870/6870[1]
Fall 2021

Le Chen

lzc0090@auburn.edu

Last updated on

August 8, 2021

Auburn University
Auburn AL

[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 5. Financial Forwards and Futures

Chapter 5. Financial Forwards and Futures

§ 5.1 Alternative ways to buy a stock

§ 5.2 Prepaid forward contracts on stock

§ 5.3 Forward contracts on stock

§ 5.4 Futures contracts

§ 5.5 Problems

Four different payment and receipt timing combinations

1. Outright purchase: ordinary transaction
2. Fully leveraged purchase: investor borrows the full amount
3. Prepaid forward contract: pay today, receive the share later
4. Forward contract: agree on price now, pay/receive later

	Day 0	Day T	Payment
Outright purchase	pay+receive	—	S_0
Fully leveraged purchase	receive	pay	$S_0 e^{rT}$
Prepaid forward contract	pay	receive	?
Forward contract	—	pay+receive	$? \times e^{rT}$

Three ways to determine the payment for the prepaid forward contracts (no dividend case)

- Pricing the prepaid forward by analogy
- Pricing the prepaid forward by discounted present value
- Pricing the prepaid forward by arbitrage

Pricing the prepaid forward by analogy

In the absence of dividends, whether you receive physical possession today or at time T is irrelevant: In either case you own the stock, and at time T it will be exactly as if you had owned the stock the whole time. Hence,

$$F^{p}_{0,T} = S_0$$

Pricing the prepaid forward by discounted present value

Let α be the expected return on the stock.

Let $\mathbb{E}_0(S_T)$ be the expected stock price at time T.

Hence,

$$F^p_{0,T} = \underbrace{\mathbb{E}_0(S_T)}_{=S_0 \times e^{\alpha T}} \times e^{-\alpha T} = S_0$$

Pricing the prepaid forward by arbitrage

Arbitrage = Free money

The price of a derivative should be such that

no arbitrage is possible.

---

1. If $F^{p}_{0,T} > S_0$: find the arbitrage.

2. If $F^{p}_{0,T} < S_0$: find the arbitrage.

   Hence, $F^{p}_{0,T} = S_0$.

Pricing prepaid forwards with dividends – Discrete dividends

Suppose a stock is expected to make dividend payments of D_{t_i} at time t_i, $i = 1, \cdots, n$. Then

$$F^P_{0,T} = S_0 - \sum_{i=1}^{n} \mathrm{PV}_{0,t_i}\left(D_{t_i}\right),$$

where $\mathrm{PV}_{0,t}(\cdot)$ is the present value at time zero of a time t_j payment.

Example 5.2-1 Suppose XYZ stock costs \$100 today and is expected to pay a \$1.25 quarterly dividend, with the first coming 3 months from today and the last just prior to the delivery of the stock. Suppose the annual continuously compounded risk-free rate is 10%. The quarterly continuously compounded rate is therefore 2.5%. Find a 1-year prepaid forward contract for the stock would cost.

Solution.

$$F_{0,1}^{T} = \$100 - \sum_{i=1}^{4} \$1.25 \times e^{-0.025i} = \$93.30.$$

$\square$

Pricing prepaid forwards with dividends – Continuous dividends

Let δ be the compounded dividend yield. Then

$$F^P_{0,T} = S_0 e^{-\delta T}$$

Example 5.2-2 Suppose that the index is $\$125$ and the annualized daily compounded dividend yield is 3%. Find the prepaid forward price at one year.

Solution.

$$F_{0,1}^{p} = \$125e^{-0.03\times 1} = \$121.306.$$

$\square$

Forward price is the future value of the prepaid forward price:

- No dividends

$$F_{0,T} = \mathrm{FV}\left(F_{0,T}^{p}\right)$$

- Continuous dividends

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

$$\text{Forward premium} = \frac{F_{0,T}}{S_0}$$

$$\text{Annualized forward premium} = \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right)$$

Does the forward price predict the future spot price?

Buying a stock

Compensation for	Earn	Buying a stock
time value of the money	interest	✓
the risk of the stock	risk premium	✓

Entering a forward contract

Compensation for	Earn	Entering a forward contract
time value of the money	interest	✗
the risk of the stock	risk premium	✓

The forward price is the expected future spot price, discounted at the risk premium.

$$F_{0,T} = e^{rT} \times \underbrace{F^{p}_{0,T}}_{=\mathbb{E}_0(S_T)e^{-\alpha T}} = \mathbb{E}_0(S_T)e^{-(\alpha - r)T}$$

Creating a synthetic forward contract

Assuming that the dividends are continuous and paid at the rate δ.

Recall that

Payoff of a long forward position at expiration

$$||$$

$$S_T - F_{0,T}$$

$$||$$

$$S_T - S_0 e^{(r-\delta)T}$$

$$\text{Forward} = \text{Stock} - \text{Zero-coupon bond}$$

	Cash Flows	
Transaction	**Time 0**	**Time T (expiration)**
Buy $e^{-\delta T}$ units of the index	$-S_0 e^{-\delta T}$	$+ S_T$
Borrow $S_0 e^{-\delta T}$	$+S_0 e^{-\delta T}$	$- S_0 e^{(r-\delta)T}$
Total	0	$S_T - S_0 e^{(r-\delta)T}$

$$\text{Stock} = \text{Forward} + \text{Zero-coupon bond}$$

Transaction	Cash Flows	
	Time 0	**Time T (expiration)**
Long one forward	0	$S_T - F_{0,T}$
Lend $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_0e^{(r-\delta)T}$
Total	$-S_0e^{-\delta T}$	S_T

Zero-coupon bond = Stock − Forward

	Cash Flows	
Transaction	**Time 0**	**Time T (expiration)**
Long one forward	0	$S_T - F_{0,T}$
Lend $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_0e^{(r-\delta)T}$
Total	$-S_0e^{-\delta T}$	S_T

Cash-and-carry is a transaction in which one buys the underlying asset and short the offsetting forward contract.

A cash-and-carry has no risk because
You have an obligation to deliver the asset
that you have already owned.

Transaction	Cash Flows: Time 0	Cash Flows: Time T (expiration)
Buy tailed position in stock, paying $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

Cash-and-carry

Transaction	Cash Flows: Time 0	Cash Flows: Time T (expiration)
Buy tailed position in stock, paying $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

Arbitrage when $F_{0,T} > S_0 e^{(r-\delta)T}$

Reverse cash-and-carry

Transaction	Cash Flows: Time 0	Cash Flows: Time T (expiration)
Short tailed position in stock, receiving $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_T$
Lend $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_0e^{(r-\delta)T}$
Long forward	0	$S_T - F_{0,T}$
Total	0	$S_0e^{(r-\delta)T} - F_{0,T}$

Arbitrage when $F_{0,T} < S_0 e^{(r-\delta)T}$

Problems: 5.2, 5.3, 5.4, 5.5, 5.8, 5.10, 5.11, 5.12, 5.16, 5.20.

Due Date: TBD