

# Financial Mathematics

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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

## Chapter 18. The Lognormal Distribution

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§ 18.1 The normal distribution

§ 18.2 The lognormal distribution

§ 18.3 A lognormal model of stock prices

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§ 18.7 Problems

# Chapter 18. The Lognormal Distribution

§ 18.1 The normal distribution

§ 18.2 The lognormal distribution

§ 18.3 A lognormal model of stock prices

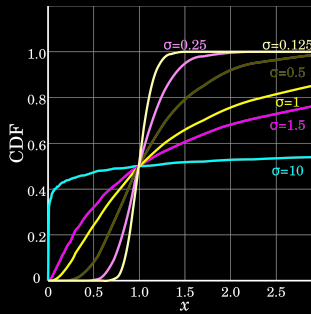
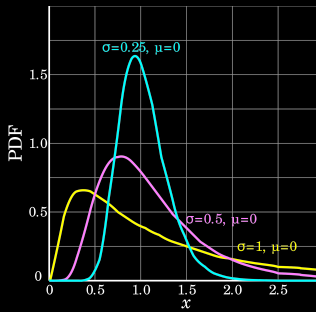
§ 18.4 Lognormal probability calculations

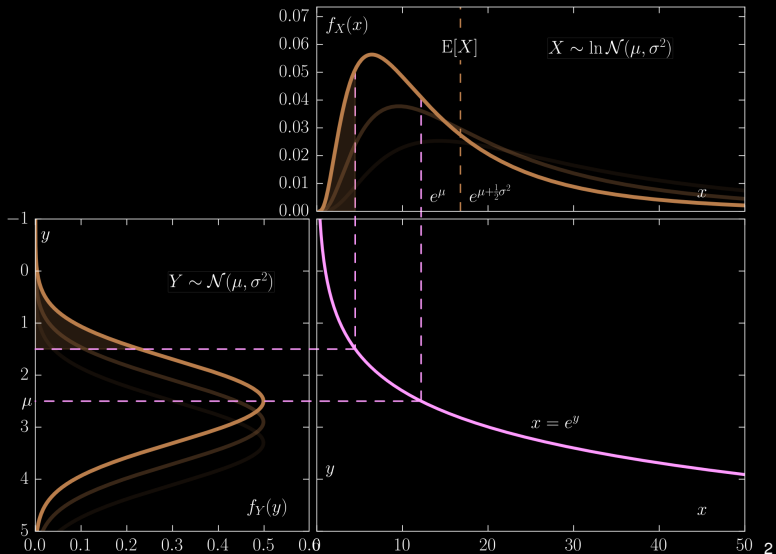
§ 18.5 A. The expectation of a lognormal variable

§ 18.6 B. Constructing a normal probability

§ 18.7 Problems

**Definition 18.2-1** A random variable  $Y$  is *lognormally distributed* with parameters  $\mu$  and  $\sigma > 0$  if  $\ln(Y) \sim N(\mu, \sigma^2)$ .





**Theorem 18.2-1** If  $Y_1$  and  $Y_2$  are lognormally distributed, so is  $Y_1 Y_2$ .

**Proof.**

Since  $Y_1$  and  $Y_2$  are lognormally distributed,  $\ln(Y_1)$  and  $\ln(Y_2)$  are normally distributed. Hence,

$$\ln(Y_1) + \ln(Y_2) = \ln(Y_1 Y_2)$$

is normally distributed too. Therefore,  $Y_1 Y_2$  is lognormally distributed.  $\square$



**Theorem 18.2-2** If  $\ln(Y) \sim N(\mu, \sigma^2)$ , then the density of  $Y$  is given by

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.$$

**Proof.** For  $y > 0$ ,

$$\begin{aligned}\mathbb{P}(Y \leq y) &= \mathbb{P}(\ln(Y) \leq \ln(y)) \\ &= \mathbb{P}\left(\frac{\ln(Y) - \mu}{\sigma} \leq \frac{\ln(y) - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right).\end{aligned}$$

Hence,

$$\begin{aligned}f_Y(y) &= \frac{d}{dy} \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2} \frac{d}{dy} \frac{\ln(y) - \mu}{\sigma} \\ &= \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.\end{aligned}$$

□

Recall that the moment generating function of the normal random variable  $X \sim N(\mu, \sigma^2)$  is

$$\mathbb{E} \left( e^{tX} \right) = e^{\mu t + \sigma^2 t^2 / 2}, \quad \text{for all } t \in \mathbb{R}. \quad (1)$$

**Remark 18.2-1** If  $Y$  is lognormally distributed with parameters  $\mu$  and  $\sigma$ , then

$$\mathbb{E} \left( Y^t \right) = e^{\mu t + \sigma^2 t^2 / 2}, \quad \text{for all } t \in \mathbb{R}.$$

**Remark 18.2-2** By *Jensen's inequality*, if  $g$  is a convex function, then

$$\mathbb{E} (g(X)) \leq g(\mathbb{E}(X)).$$

Hence, for  $g(x) = e^x$ , we see that

$$\mathbb{E} \left( e^X \right) \leq e^{\mathbb{E}(X)}.$$

This is consistent with our computations above because

$$LHS = e^{\mu + \sigma^2 / 2} \geq e^{\mu} = RHS.$$

**Theorem 18.2-3** If  $Y$  is lognormally distributed such that  $\ln(Y) \sim N(\mu, \sigma^2)$ , then

$$\mathbb{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2} \quad \text{and} \quad \text{Var}(Y) = e^{2\mu + 2\sigma^2} (e^{\sigma^2} - 1).$$

**Proof.** Let  $X = \ln(Y)$ . By (1),

$$\mathbb{E}(Y) = \mathbb{E}(e^X) = e^{\mu + \frac{1}{2}\sigma^2}$$

and

$$\mathbb{E}(Y^2) = \mathbb{E}(e^{2X}) = e^{2\mu + 2\sigma^2}.$$

Therefore,

$$\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = e^{2\mu + 2\sigma^2} - e^{2(\mu + \frac{1}{2}\sigma^2)} = e^{2\mu + 2\sigma^2} (e^{\sigma^2} - 1).$$

□