Financial Mathematics

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 2. An Introduction to Forwards and Options

Chapter 2. An Introduction to Forwards and Options

§ 2.1 Forward contracts

§ 2.2 Call options

§ 2.3 Put options

§ 2.4 Summary of forward and option positions

§ 2.5 Problems

Chapter 2. An Introduction to Forwards and Options

Definition 2.1-1 Forward contract is a binding agreement (obligation) to buy or sell an underlying asset in the future, at a price set today. The time at which the contract settles is called the expiration date. A forward contract specifies

- The features and quantity of the asset to be delivered.
- The delivery logistics, such as time, date, and place.
- The price the buyer will pay at the time of delivery.

Remark 2.1-1

1. Futures contracts are the same as forwards in principle except for some institutional and pricing differences. We will study future contracts in Chapter 5.
2. A forward contract requires no initial payment or premium.

Long = buy short = sell

Definition 2.1-2 Payoff for a contract is its value at expiration. In particular, for forward contracts,

Payoff for Long forward = Spot price at expiration − Forward price

Payoff for Short forward = Forward price − Spot price at expiration

Remark 2.1-2 Payoff and profit (net payoff) are the same for forward contracts because there is no initial payment – premium.

Example 2.1-1 S&R (special and rich) index:

Today: Spot price = \$1,000
6-month forward price = \$1,020
In six months at contract expiration: Spot price = \$1,050.

What are the payoff of long/short forward?

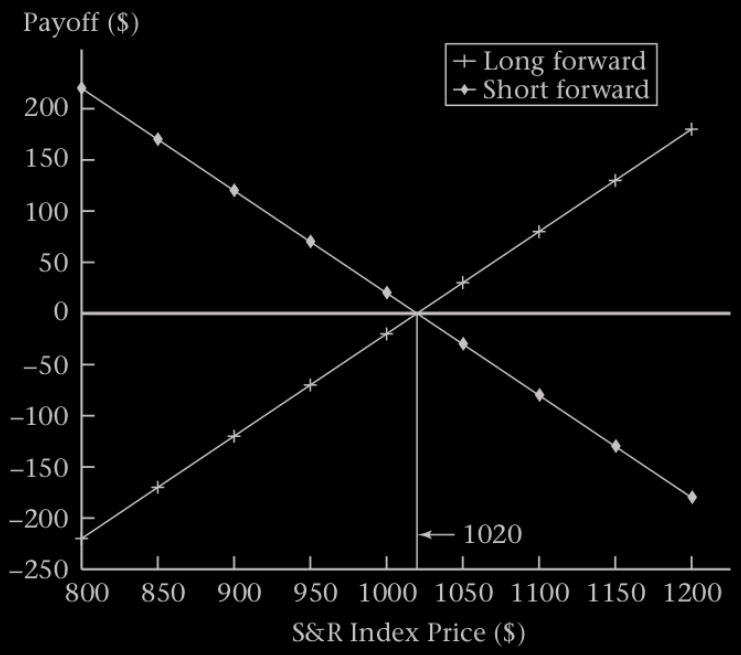
Solution.

$$\text{Long position payoff} = \$1,050 - \$1,020 = \$30,$$

$$\text{Short position payoff} = \$1,020 - \$1,050 = (\$30).$$

□

Payoff diagram for a forward price = \$1,020



Forward versus outright purchase

We will see this through the following example:

Example 2.1-2 S&R 6-month forward contract with a zero-coupon bound (e.g., Treasury bills). The 6-month interest rate is 2%. Spot price today $= \$1,000$.

$1,000 today is worth $\$1,000 \times 1.02 = \$1,020$ in 6 months.

Outright purchase[2] is equivalent to forward + bond[3]

because

$$\text{Payoff of forward+bond} = \underbrace{\text{Spot price at expiration} - \$1,020}_{\text{Forward payoff}} + \underbrace{\$1,020}_{\text{Bound payoff}}$$
$$= \text{Spot price at expiration}$$
$$= \text{Payoff of outright purchase}$$

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[2] It is also called long physical index.

[3] Invest $1,000 to bond for 6 month and enter long position of forward contract at the same time.

$1,000 today is worth $\$1,000 \times 1.02 = \$1,020$ in 6 months.

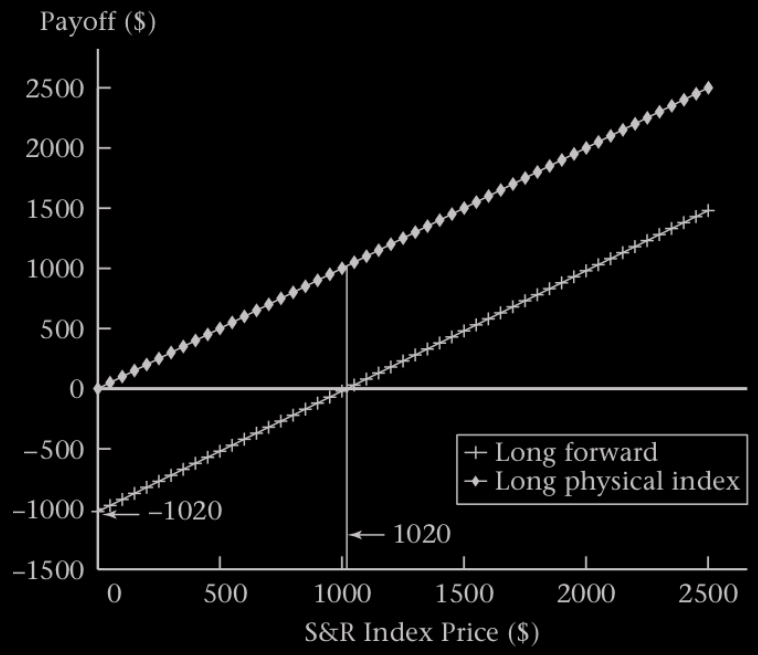
Long forward is equivalent to borrow-to-buy[4]

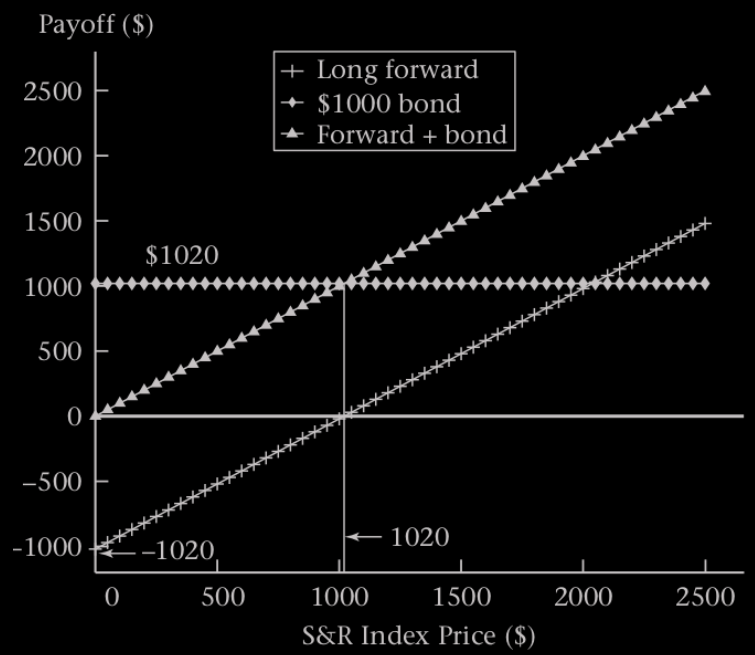
because

$$\text{Payoff of borrow-to-buy} = \underbrace{\text{Spot price at expiration}}_{\text{Payoff for outright buy}} - \underbrace{\$1,020}_{\text{Return borrowed money}}$$

$$= \text{Payoff of long forward}.$$

[4] Borrow money ($1,000) to outright buy physical index and at expiration pay back the money ($1,020).


Payoff ($)
2500
2000
1500
1000
500
0
−500
−1000
−1500
0
500
1000
1500
2000
2500
S&R Index Price ($)
Long forward
Long physical index
−1020
1020


Payoff ($)
2500
2000
1500
1000
500
0
−500
-1000
0
500
1000
1500
2000
2500
S&R Index Price ($)
Long forward
$1000 bond
Forward + bond
$1020
−1020
1020

Cash settlement versus physical delivery

– Type of settlement

- Cash settlement: less costly and more practical
- Physical delivery: often avoided due to significant costs

Example 2.1-3 Consider the S&R index with the forward price \$1,020.

- Suppose that the S&R index at expiration is \$1,040.
- The long position has a payoff of \$20.
- Similarly, the short position loses \$20.

With cash settlement, the short simply pays \$20 to the long, with no transfer of the physical asset, and hence no transaction costs. It is as if the long paid \$1,020, acquired the index worth \$1,040, and then immediately sold it with no transaction costs.

- Suppose that the S&R index price at expiration had instead been \$960.
- The long position would have a payoff of $-\$60$.
- The short would have a payoff of \$60.

Cash settlement in this case entails the long paying \$60 to the short.

Credit risk

All derivatives contracts have credit risk, which is the possibility that the counterparty who owes money fails to make a payment.

- Major issue for over-the-counter (OTC) contracts

  Credit check
  Credit protections such as collateral and bank letter of credit

- Less severe for exchange-traded contracts

  Exchange guarantees transactions, requires collateral

Chapter 2. An Introduction to Forwards and Options

Can one modify the forward contract so that
the buyer can walk away from the deal at expiration?

Definition 2.2-1 A call option is a contract where the buyer has the right to buy, but not the obligation to buy.

Example 2.2-1 S&R index: Buyers' perspective

- Today: call buyer acquires the right to pay \$1,020 in six months for the index, but is not obligated to do so
- In six months at contract expiration:
  if the spot price is \$1,100, call buyers payoff $= \$1,100 - \$1,020 = \$80$
  if the spot price is \$900, call buyer walks away, buyers payoff = \$0.

Example 2.2-2 S&R index: Sellers' perspective

- Today: call seller is obligated to sell the index for \$1,020 in six months, if asked to do so
- In six months at contract expiration:
  if the spot price is \$1,100, call sellers payoff $= \$1,020 - \$1,100 = -\$80$
  if the spot price is \$900, call buyer walks away, sellers payoff = \$0.

Buyer preserves the upside potential, while at the same time eliminates the unpleasant downside.

However

Seller has to be compensated by a initial premium for being at a disadvantage at expiration.

- Strike (or exercise) price: the amount paid by the option buyer for the asset if he/she decides to exercise.

- Exercise: the act of paying the strike price to buy the asset.

- Expiration: the date by which the option must be exercised or become worthless.

- Exercise style: specifies when the option can be exercised.

Style	can be exercised
European	only at expiration date
American	at any time before expiration
Bermudan	during specified periods

Payoff of purchased call $= \max(0, \text{spot price at expiration} - \text{strike price})$

Profit of purchased call = payoff of purchased call
$-$ future value of option premium

Payoff of written call $= -\max(0, \text{spot price at expiration} - \text{strike price})$

Profit of written call = payoff of written call
$+$ future value of option premium

Example 2.2-3 S&R Index 6-month European call option

$$\begin{aligned}\text{Strike price} &= \$1,000,\\ \text{Premium} &= \$93.81,\\ \text{6-month risk-free rate} &= 2\%.\end{aligned}$$

Compute both payoff and profit of the purchased call option if the index value in six months $\$1,100$ (resp. $\$900$).

Solution.

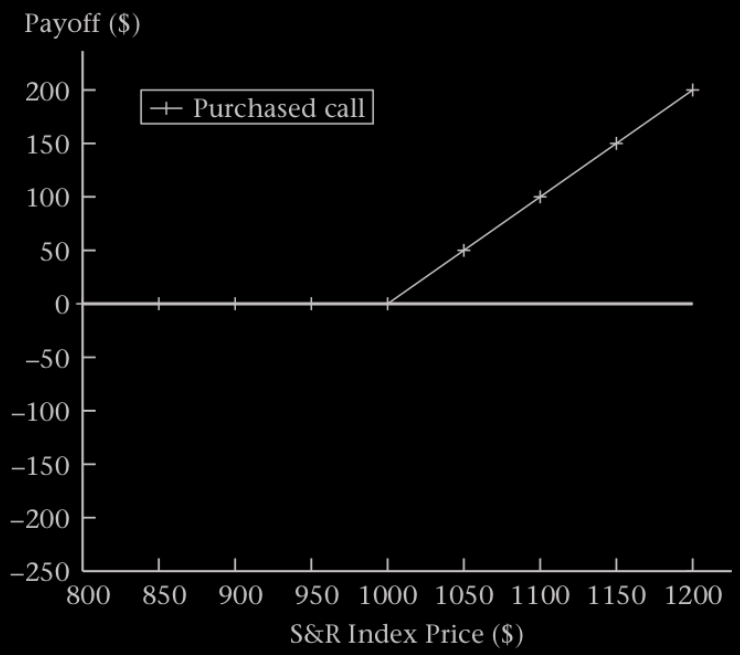
If index value in six months = $\$1{,}100$,

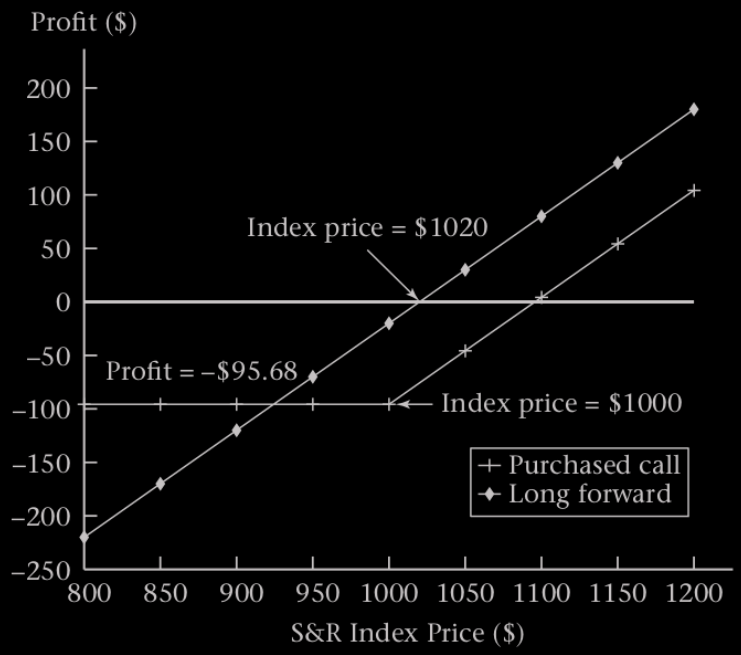
$$\begin{aligned}\text{Payoff} &= \max(0, \$1,100 - \$1,000)\\ &= \$100\\ \text{Profit} &= \$100 - \$93.81 \times 1.02\\ &= \$4.32.\end{aligned}$$

If index value in six months = $\$900$,

$$\begin{aligned}\text{Payoff} &= \max(0, \$900 - \$1,000)\\ &= \$0\\ \text{Profit} &= \$0 - \$93.81 \times 1.02\\ &= -\$95.68.\end{aligned}$$

□


Payoff ($)
Purchased call
200
150
100
50
0
–50
–100
–150
–200
–250
800 850 900 950 1000 1050 1100 1150 1200
S&R Index Price ($)


Profit ($)
Index price = $1020
Profit = –$95.68
Index price = $1000
Purchased call
Long forward
200
150
100
50
0
–50
–100
–150
–200
–250
800 850 900 950 1000 1050 1100 1150 1200
S&R Index Price ($)

Example 2.2-4 S&R Index 6-month European call option

$$\begin{aligned}\text{Strike price} &= \$1,000,\\ \text{Premium} &= \$93.81,\\ \text{6-month risk-free rate} &= 2\%.\end{aligned}$$

Compute both payoff and profit of the written call option if the index value in six months $\$1,100$ (resp. $\$900$).

Solution.

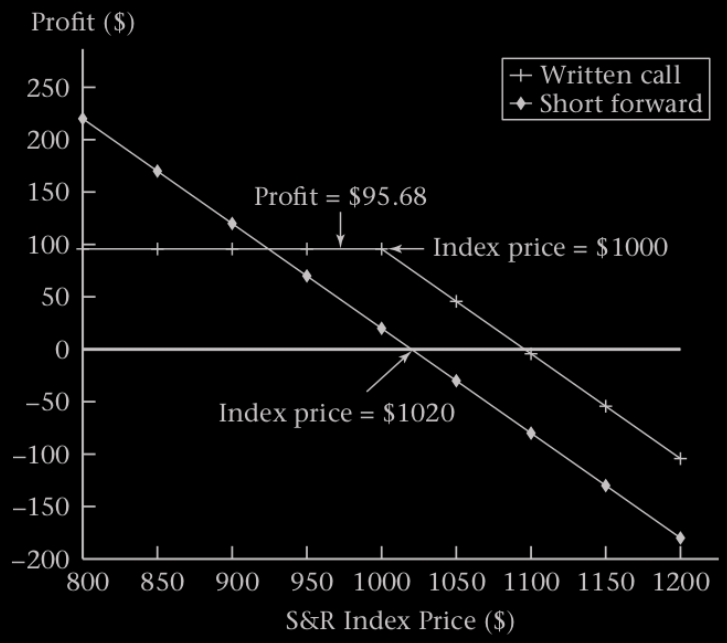
If index value in six months = $\$1{,}100$,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,100 - \$1,000)\\ &= -\$100\\ \text{Profit} &= -\$100 + \$93.81 \times 1.02\\ &= -\$4.32.\end{aligned}$$

If index value in six months = $\$900$,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$900 - \$1,000)\\ &= \$0\\ \text{Profit} &= \$0 + \$93.81 \times 1.02\\ &= \$95.68.\end{aligned}$$

□


Profit ($)
Written call
Short forward
250
200
150
100
50
0
−50
−100
−150
−200
Profit = $95.68
Index price = $1000
Index price = $1020
800
850
900
950
1000
1050
1100
1150
1200
S&R Index Price ($)

Chapter 2. An Introduction to Forwards and Options

Call option : Buyer can walk away.

???? option : Seller can walk away.

Definition 2.3-1 A put option gives the owner the right but not the obligation to sell the underlying asset at a predetermined price during a predetermined time period.

Remark 2.3-1 Similar to the call option case, a premium paid by the put buyer at the time the option is purchased is needed in order to compensate the put seller for being in a disadvantage position.

... of put option	someone needs to		premium
seller	buy	has to buy if asked	receive
buyer	sell	can walk away	pay

$$\text{Payoff of purchased put} = \max(0, \text{strike price} - \text{spot price at expiration})$$

$$\begin{aligned}\text{Profit of purchased put} = {} & \text{payoff of purchased put} \\ & - \text{future value of option premium}\end{aligned}$$

$$\text{Payoff of written put} = -\max(0, \text{strike price} - \text{spot price at expiration})$$

$$\begin{aligned}\text{Profit of written put} = {} & \text{payoff of written put} \\ & + \text{future value of option premium}\end{aligned}$$

Example 2.3-1 S&R Index 6-month European put option

$$\begin{aligned} \text{Strike price} &= \$1,000, \\ \text{Premium} &= \$74.20, \\ \text{6-month risk-free rate} &= 2\%. \end{aligned}$$

Compute both payoff and profit of the purchased put option if the index value in six months $\$1,100$ (resp. $\$900$).

Solution.

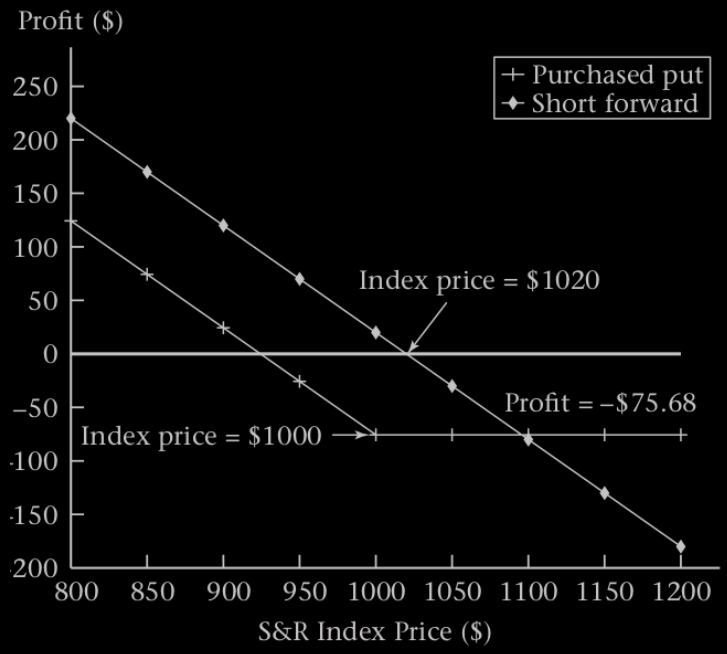
If index value in six months = $\$1{,}100$,

$$\begin{aligned} \text{Payoff} &= \max(0, \$1,000 - \$1,100) \\ &= \$0 \\ \text{Profit} &= \$0 - \$74.20 \times 1.02 \\ &= -\$75.68. \end{aligned}$$

If index value in six months = $\$900$,

$$\begin{aligned} \text{Payoff} &= \max(0, \$1,000 - \$900) \\ &= \$100 \\ \text{Profit} &= \$100 - \$74.20 \times 1.02 \\ &= \$24.32. \end{aligned}$$

□


Profit ($)
Purchased put
Short forward
250
200
150
100
50
0
−50
−100
−150
−200
Index price = $1020
Profit = −$75.68
Index price = $1000
800
850
900
950
1000
1050
1100
1150
1200
S&R Index Price ($)

Example 2.3-2 S&R Index 6-month European put option

$$\begin{aligned}\text{Strike price} &= \$1,000,\\ \text{Premium} &= \$74.20,\\ \text{6-month risk-free rate} &= 2\%.\end{aligned}$$

Compute both payoff and profit of the written put option if the index value in six months $\$1,100$ (resp. $\$900$).

Solution.

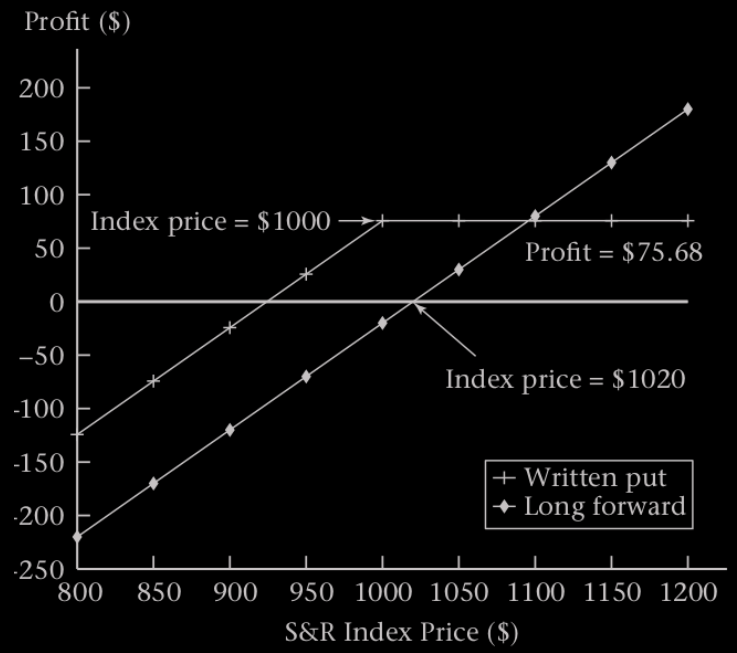
If index value in six months = $\$1{,}100$,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,000 - \$1,100)\\ &= \$0\\ \text{Profit} &= \$0 + \$74.20 \times 1.02\\ &= \$75.68.\end{aligned}$$

If index value in six months = $\$900$,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,000 - \$900)\\ &= -\$100\\ \text{Profit} &= -\$100 + \$74.20 \times 1.02\\ &= -\$24.32.\end{aligned}$$

□


Profit ($)
200
150
100
50
0
–50
-100
-150
-200
-250
Index price = $1000
Profit = $75.68
Index price = $1020
Written put
Long forward
800
850
900
950
1000
1050
1100
1150
1200
S&R Index Price ($)

A call option becomes more profitable
when the underlying asset
appreciates in value

A put option becomes more profitable
when the underlying asset
depreciates in value

Definition 2.3-2 Moneyness of an option describes whether the option payoff would be positive if the option were exercised immediately.

In particular, one has

Moneyness	payoff if exercised immediately
In-the-money option	> 0
At-the-money option	$= 0$
Out-of-the money option	< 0

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