Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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Last updated on

September 2, 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 10. Binomial Option Pricing: Basic Concepts

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Binomial option pricing

The
binomial option pricing model
or
Cox-Ross-Rubinstein pricing model
assumes that

the price of the underlying asset follows a binomial distribution,

that is,

the asset price in each period can
move only up or down by a specified amount.

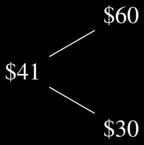
The binomial option pricing model enables us to

determine the price of an option,

given the characteristics of the stock or other underlying asset.

Example 10.1-1 Consider an European call option on the stock of XYZ, with a \$40 strike price and one year expiration. XYZ does not pay dividends and its current price is \$41.

Assume that, in a year, the price can be either \$60 or \$30.



Can one determine the call premium?

(Let the continuously compounded risk free interest rate be 8%.)

Law of one price

Positions that have the same payoff should have the same cost!

Two portfolios (positions)

- Portfolio A: Buy one 40-strike call option.
- Portfolio B: Buy $\Delta \in (0, 1)$ share of stock and borrow B at the risk-free rate.

These two positions should have the same cost.

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These two positions should have the same cost.

Solution. The cost for Portfolio B at day zero is

$$\Delta \times S_0 - B.$$

and its payoff at expiration is

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} & \text{if the stock price is 30} \\ \Delta \times 60 - B \times e^{0.08} & \text{if the stock price is 60} \end{cases}$$

On the other hand, the payoff for Portfolio A should be

$$\begin{cases} 0 & \text{if the stock price is 30} \\ (60 - 40) & \text{if the stock price is 60} \end{cases}$$

By equating the two payoffs, one obtains that

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} = 0 \\ \Delta \times 60 - B \times e^{0.08} = 60 - 40 \end{cases}$$

Solution. Hence,

$$B = 20 \times e^{-0.08} \quad \text{and} \quad \Delta = 2/3.$$

Finally, since the cost of Portfolio A has to be equal to that of Portfolio B, we find the cost of Portfolio A:

$$\Delta \times S_0 - B = \frac{2}{3} S_0 - 20 \times e^{-0.08}.$$

If we plug in $S_0 = \$41$, we have

$$B = \$18.462 \quad \text{and the cost is } \$\ 8.871.$$

$\square$

More generally, suppose the stock change its value over a period of time h as

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & u \times S \\
S \arrow[ur, dash] \arrow[dr, dash] & \\
 & d \times S
\end{tikzcd}
$$

Portfolio A

Payoff	$d \times S$	$u \times S$
Option	0	$u \times S - K$
Total	$C_d = 0$	$C_u = u \cdot S - K$

Portfolio B

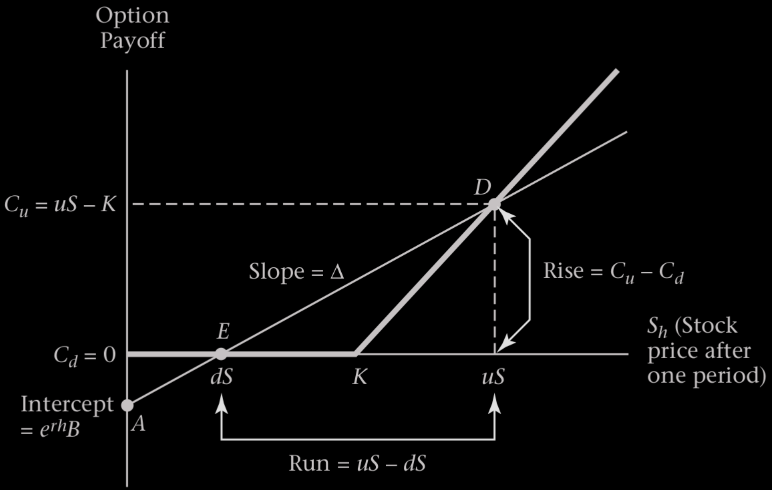
Payoff	$d \times S$	$u \times S$
Δ share	$\Delta \cdot d \cdot S \cdot e^{\delta h}$	$\Delta \cdot u \cdot S \cdot e^{\delta h}$
B bond	Be^{rh}	Be^{rh}
Total	$\Delta \cdot d \cdot S \cdot e^{\delta h} + Be^{rh}$	$\Delta \cdot u \cdot S \cdot e^{\delta h} + Be^{rh}$

For two unknowns: Δ and B, solve:

$$\begin{cases} \Delta dSe^{\delta h} + Be^{rh} = C_d \\ \Delta uSe^{\delta h} + Be^{rh} = C_u \end{cases}$$

Set S_h be either dS or uS and
C_h be either C_u or C_d.
Plot S_h (x-axis) versus C_h (y-axis).

$$\Delta S_h e^{\delta h} + Be^{rh} = C_h$$


Option
Payoff
$C_u = uS - K$
D
Slope = Δ
Rise = $C_u - C_d$
E
$C_d = 0$
S_h (Stock
price after
one period)
dS
K
uS
Intercept
= $e^{rh}B$
A
Run = $uS - dS$

$$\Delta = e^{-\delta h} \frac{C_h - C_d}{S(u-d)} \qquad \text{and} \qquad B = e^{-rh} \frac{uC_d - dC_u}{u-d}$$

$$\Delta S + B = e^{-rh} \left(C_u \underbrace{\frac{e^{(r-\delta)h} - d}{u-d}}_{:=p^*} + C_d \underbrace{\frac{u - e^{(r-\delta)h}}{u-d}}_{:=1-p^*} \right)$$

p^* the **risk-neutral probability** of an increase in the stock price.

Arbitraging a mispriced option

Example 10.1-2 Find arbitrage opportunities in Example 10.1-1 with

- the option price being overpriced with \$9.00;
- the option price being underpriced with \$8.25,

instead of the risk-neutral pricing \$8.871.

Solution. One can buy the synthetic option which cost \$8.25 and sell the real one by earning \$9.00. Hence, the present value of the profit is

$$\$9 - \$8.871 = \$0.129.$$

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