Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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Last updated on

September 28, 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 11. Binomial Option Pricing: Selected Topics

Risk-Neutral Probability

Recall the binomial option pricing formula:

$$C = \Delta S + B = e^{-rh}\left[p^* C_u + (1 - p^*) C_d\right]$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$ $\sim$ risk-neutral probability that the stock will go up

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$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \iff p^* u S e^{\delta h} + (1 - p^*) d S e^{\delta h} = e^{rh} S$$

Two offers:

(a) \$1000 cash

(b) \$2000 or \$0 cash with probability 1/2 for each

Both offers have the same expected return, while (b) bears risk and (a) does not.

A risk-averse investor prefers (a).

A risk-neutral investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing. Hence, he/she prefers equally to (a) and (b).

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The option pricing formula can be said to price options
as if investors are risk-neutral

Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return.

Pricing an option using real probability

- Suppose that the continuously compounded expected return on the stock is α and that the stock does not pay dividends.
- If p is the true probability of the stock going up, p must be consistent with u, d and α

$$puS + (1 - p)dS = e^{\alpha h}S$$

- Solving for p gives us

$$p = \frac{e^{\alpha h} - d}{u - d}$$

- For p to be a probability, we have to have $u \geq e^{\alpha h} \geq d$.
- Using this p, the actual expected payoff to the option one period is

$$pC_u + (1 - p)C_d = \frac{e^{\alpha h} - d}{u - d}C_u + \frac{u - e^{\alpha h}}{u - d}C_d.$$

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At what rate do we discount this expected payoff?

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It is not correct to discount the option at the expected return on the stock, α, because the option is equivalent to a leveraged investment in the stock and hence is riskier than the stock

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- Denote the appropriate per-period discount rate for the option as γ
- Since an option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta+B}e^{\alpha h} + \frac{B}{S\Delta+B}e^{rh}$$

- Hence, the discounted at this appropriate discount rate, the price for the option should be

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h}-d}{u-d}C_u + \frac{u-e^{\alpha h}}{u-d}C_d\right]$$

- By setting $\alpha = r$, one obtains the simplest pricing procedure.
- This gives an alternative way to compute the option price, instead of $\Delta S + B$.

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One can use either

$$C = \Delta S + B$$

or

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d\right]$$

to compute the option price

- **First equation** is more efficient
- For the **second one**, in order to compute γ, one needs to computer Δ and B first and then obtains γ via

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Given the continuously compounded expected return of the stock α

1. Compute the probability that stock goes up

$$p = \frac{e^{\alpha h} - d}{u - d}$$

2. Compute the actual expected payoff (to be discounted)

$$X := pC_u + (1 - p)C_d$$

3. Using r and δ to compute Δ and B:

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S(u - d)} \qquad \text{and} \qquad B = e^{-rh}\frac{uC_d - dC_u}{u - d}.$$

4. Compute the discounted rate γ:

$$\gamma = \frac{1}{h}\log\left(\frac{S\Delta}{S\Delta + B}e^{\alpha h} + \frac{B}{S\Delta + B}e^{rh}\right)$$

5. Finally, the option price should be the discounted value:

$$Xe^{-\gamma h}.$$

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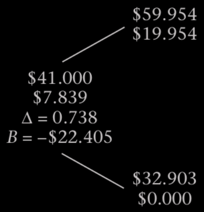
An one-period example

FIGURE 11.3

Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 1.000$. This is the same as Figure 10.3.



A multi-period example

FIGURE 11.4

Binomial tree for pricing an American call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. The continuously compounded true expected return on the stock, α, is 15%. At each node the stock price, option price, and continuously compounded true discount rate for the option, γ, are given. Option price in ***bold italic*** signify that exercise is optimal at that node.