Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 1. Introduction to Derivatives

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Definition 1.1-1 A derivative is a financial instrument that has a value determined by the price of something else.

Example 1.1-1 An agreement where

you pay \$1 if the price of corn is greater than \$3
and
you receive \$1 if the price of corn is less that \$1

is a derivative.

This contract can be used to

speculate on the price of corn
or
it can be used to reduce risk.

Hence, it is not the contract itself, but how it is used, and who uses it, that determines whether or not it is risk-reducing. It all depends on context.

The trading of a financial asset involves at least four discrete steps:

1. A buyer and a seller must locate one another and agree on a price
2. The trade must be cleared
   (the obligations of each party are specified)
3. The trade must be settled
   (the buyer and the seller must deliver the cash or securities necessary to satisfy their obligations in the required period of time)
4. Ownership records are updated.

$\boxed{\text{Striking a price}}$

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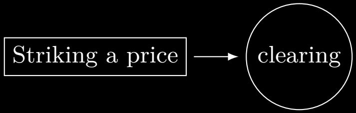

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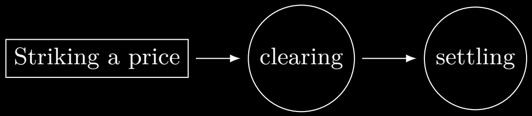

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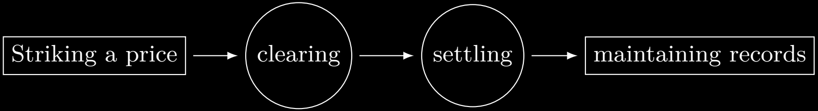

There are at least four different measures of a market and its activity:

1. Trading volume: the number of financial claims that change hands daily or annually.

2. Market value or market cap: the sum of the market value of the claims that could be traded, without regard to whether they have traded.

3. Notional value: Notional value measure the scale of a position, usually with reference to some underlying asset.

4. Open Interest. Open interest measures the total number of contracts for which counter parties have a future obligation to perform. It is an important statistic in derivatives market.

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Companies typically raise funds by

stock markets	bound markets
Selling ownership claims Securities exchanges (NYSE, NASDAQ)	Obtaining a bank loan or issuing a bond Through dealers less frequent

The introduction of derivatives in a market often coincides with an increase in price risk in that market. For example,

1. Currencies were permitted to float in 1971 when the gold standard was officially abandoned. The modern market in financial derivatives began in 1972, when the Chicago Mercantile Exchange started trading futures contracts on seven currencies.
2. OPEC's 1973 reduction in the supply of oil was followed by high and variable oil prices.
3. U.S. interest rates became more volatile following inflation and recessions in the 1970s.
4. The market for natural gas has been deregulated gradually since 1978, resulting in a volatile market in recent years.
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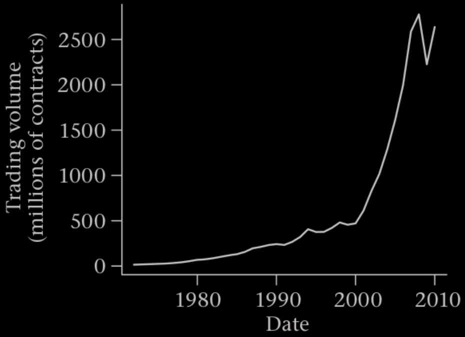
History of the crude oil prices

[2]

[2] Image from `https://www.macrotrends.net/`

Price variability leads to the development of derivatives markets to efficiently share risk.



Millions of future contracts traded annually at the Chicago Board of Trade (CBT), Chicago Mercantile Exchange (CME), and the New York Mercantile Exchange (NYMEX), 1970-2011.

Examples of underlying assets on which futures contracts are traded:

Category	Description
Stock index	S&P 500 index, Euro Stoxx 50 index, Nikkei 225, Dow-Jones Industrials, Dax, NASDAQ, Russell 2000, S&P Sectors (healthcare, utilities, technology, etc.)
Interest rate	30-year U.S. Treasury bond, 10-year U.S. Treasury notes, Fed funds rate, Euro-Bund, Euro-Bobl, LIBOR, Euribor
Foreign exchange	Euro, Japanese yen, British pound, Swiss franc, Australian dollar, Canadian dollar, Korean won
Commodity	Oil, natural gas, gold, copper, aluminum, corn, wheat, lumber, hogs, cattle, milk
Other	Heating and cooling degree-days, credit, real estate

The role of financial markets

Insurance companies and individual communities/families have traditionally helped each other to share risks.

Markets make RISK-SHARING more efficient

Diversifiable risks	vanishes	lightening strike
Non-diversifiable risks	are reallocated to those most willing to hold it	Stock market crash

The existence of risk-sharing mechanisms benefits everyone!

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§ 1.1 What is a derivative?

§ 1.2 An overview of financial markets

§ 1.3 The use of derivatives

§ 1.4 Buying and short-selling financial assets

§ 1.5 Problems

Chapter 1. Introduction to Derivatives

Common reasons to use derivatives

1. Risk management. Derivatives are a tool for companies and other users to reduce risks ($\sim$ hedging). Every form of insurance is a derivative.

2. Speculation. Derivatives can serve as investment vehicles ($\sim$ betting).

3. Reduce transaction costs. Sometimes derivatives provide a lower cost way to undertake a particular financial transaction.

4. Regulatory arbitrage. It is sometimes possible to circumvent regulatory restrictions, taxes, and accounting rules by trading derivatives.

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Three perspectives on derivatives

End users	Intermediaries	Economic Observers
Corporations Investment managers investors	Market-makers Traders	Regulators Researchers
How to use a derivative to meet the goal	Mathematical details of pricing and hedging	Make sense of the market

New securities can be designed by using existing securities

Financial engineering is the construction of a financial product from other products.

Principles for financial engineering (or security design):

1. Facilitate hedging of existing positions
2. Allow for creation of customized products
3. Enable understanding of complex positions
4. Render regulation less effective

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Transaction costs and the bid-ask spread

Definition 1.4-1 The price at which one can buy is called the offer price or ask price, and the price at which one can sell is called the bid price. The difference between the ask price and the bid price is called the bid-ask spread.

Terminology is in the perspective
of market-maker

	Ask (offer) price	Bid price
End users	Buy	Sell
Market makers	Sell	Buy

Transaction costs

Commission	bid-ask spread
Brokers Electronic trading system	Market-makers
Fixed amount per transaction or percentage of purchase price	Based on per share

Example 1.4-1 Buy and sell 100 shares of XYZ with

$$\text{bid} = \$49.75, \quad \text{offer} = \$50, \quad \text{commission} = \$15.$$

What is the transaction cost?

Solution.

1. Buy:

$$(100 \times \$50) + \$15 = \$5,015.$$

2. Sell:

$$(100 \times \$49.75) - \$15 = \$4,960.$$

3. Transaction cost:

$$\$5,015 - \$4,960 = \$55.$$

(Note that We have payed twice the commission.) □

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Ways to buy or sell

	Market order	Limited order
Pros	Filled immediately	Might not be filled
Cons	Price could be better	At a better price

1. Market order: an instruction to trade a specific quantity of the asset immediately, at the best price that is currently available.

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3. Others such as stop-loss order.

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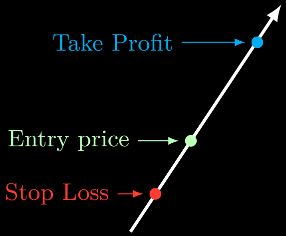
Long vs short positions

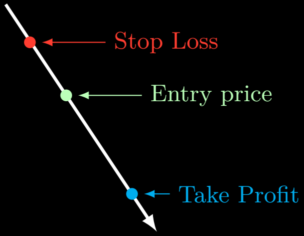
Buy when you expect
the price will go up

Long=Buy

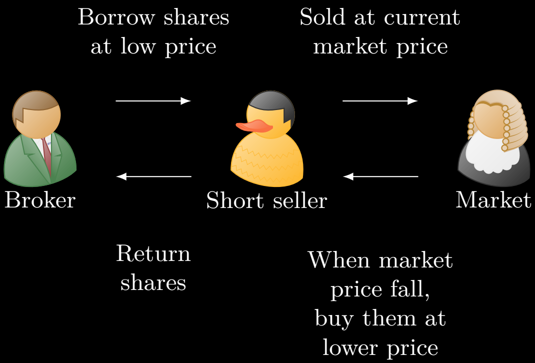


Sell when you expect
the price will go down

Short=Sell



Short-selling



Example 1.4-2 Short-sell IBM stock for 90 days.

	Day zero	Dividend Ex-Day	Day 90	Profit
Action	Borrow shares		Return shares	
Security	Shell shares		Purchase shares	
Cash flow	$+S_0$	$-D$	$-S_{90}$	$S_0 - D - S_{90}$

Three reasons to short-sell

1. Speculation
2. Financing
3. Hedging

Credit risk in short-selling

- The lender holding the money with an extra called Haircut.

Interest received from lender

- Scarcity decreases the interest rate.
- Repo rate in bound markets.
- Short rebate in the stock market.

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