

Financial Mathematics

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¹Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 2. An Introduction to Forwards and Options

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§ 2.1 Forward contracts

§ 2.2 Call options

§ 2.3 Put options

§ 2.4 Summary of forward and option positions

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Definition 2.1-1 **Forward contract** is a binding agreement (obligation) to buy or sell an underlying asset in the future, at a price set today. The time at which the contract settles is called the **expiration date**. A forward contract specifies

- ▶ The features and quantity of the asset to be delivered.
- ▶ The delivery logistics, such as time, date, and place.
- ▶ The price the buyer will pay at the time of delivery.

Remark 2.1-1

1. **Futures contracts** are the same as forwards in principle except for some institutional and pricing differences. We will study future contracts in Chapter 5.
2. A forward contract requires no initial payment or **premium**.

Long = buy short = sell

Definition 2.1-2 Payoff for a contract is its value at expiration. In particular, for forward contracts,

Payoff for Long forward = Spot price at expiration – Forward price

Payoff for Short forward = Forward price – Spot price at expiration

Remark 2.1-2 Payoff and profit (net payoff) are the same for forward contracts because there is no initial payment – premium.

Example 2.1-1 S&R (special and rich) index:

Today: Spot price = \$1,000

6-month forward price = \$1,020

In six months at contract expiration: Spot price = \$1,050.

What are the payoff of long/short forward?

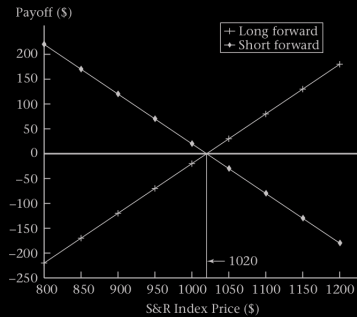
Solution.

Long position payoff = $\$1,050 - \$1,020 = \$30$,

Short position payoff = $\$1,020 - \$1,050 = (\$30)$.



Payoff diagram for a forward price = \$1,020



Forward versus outright purchase

We will see this through the following example:

Example 2.1-2 S&R 6-month forward contract with a zero-coupon bond (e.g., Treasury bills). The 6-month interest rate is 2%. Spot price today = \$1,000.

\$1,000 today is worth $\$1,000 \times 1.02 = \$1,020$ in 6 months.

Outright purchase² is equivalent to forward + bond³

because

$$\begin{aligned}\text{Payoff of forward+bond} &= \underbrace{\text{Spot price at expiration} - \$1,020}_{\text{Forward payoff}} + \underbrace{\$1,020}_{\text{Bond payoff}} \\ &= \text{Spot price at expiration} \\ &= \text{Payoff of outright purchase}\end{aligned}$$

²It is also called long physical index.

³Invest \$1,000 to bond for 6 month and enter long position of forward contract at the same time.

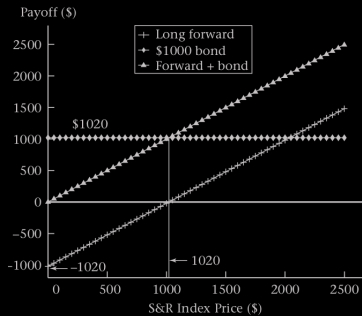
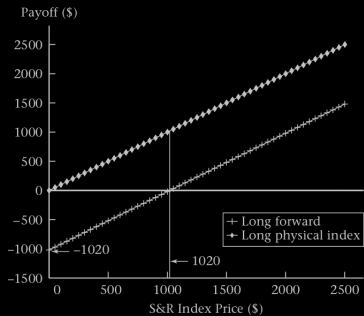
\$1,000 today is worth $\$1,000 \times 1.02 = \$1,020$ in 6 months.

Long forward is equivalent to borrow-to-buy⁴

because

$$\begin{aligned}\text{Payoff of borrow-to-buy} &= \underbrace{\text{Spot price at expiration}}_{\text{Payoff for outright buy}} - \underbrace{\$1,020}_{\text{Return borrowed money}} \\ &= \text{Payoff of long forward}.\end{aligned}$$

⁴Borrow money (\$1,000) to outright buy physical index and at expiration pay back the money (\$1,020).



Cash settlement versus physical delivery

– Type of settlement

- ▶ Cash settlement: less costly and more practical
- ▶ Physical delivery: often avoided due to significant costs

Example 2.1-3 Consider the S&R index with the forward price \$1,020.

- ▶ Suppose that the S&R index at expiration is \$1,040.
- ▶ The long position has a payoff of \$20.
- ▶ Similarly, the short position loses \$20.

With **cash settlement**, the short simply pays \$20 to the long, with **no transfer of the physical asset**, and hence **no transaction costs**. It is as if the long paid \$1,020, acquired the index worth \$1,040, and then immediately sold it with no transaction costs.

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- ▶ Suppose that the S&R index price at expiration had instead been \$960.
 - ▶ The long position would have a payoff of $-\$60$.
 - ▶ The short would have a payoff of \$60.

Cash settlement in this case entails the long paying \$60 to the short.

Credit risk

All derivatives contracts have **credit risk**, which is the possibility that the counterparty who owes money fails to make a payment.

- ▶ Major issue for **over-the-counter (OTC) contracts**

Credit check

Credit **protections** such as collateral and bank letter of credit

- ▶ Less severe for **exchange-traded contracts**

Exchange guarantees transactions, requires collateral

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Can one modify the forward contract so that the buyer can walk away from the deal at expiration?

Definition 2.2-1 A **call option** is a contract where the buyer has the right to buy, but not the obligation to buy.

Example 2.2-1 S&R index: Buyers' perspective

- ▶ Today: call buyer acquires the right to pay \$1,020 in six months for the index, but is not obligated to do so
 - ▶ In six months at contract expiration:
 - if the spot price is \$1,100, call buyers payoff = $\$1,100 - \$1,020 = \$80$
 - if the spot price is \$900, call buyer walks away, buyers payoff = \$0.
-

Example 2.2-2 S&R index: Sellers' perspective

- ▶ Today: call seller is obligated to sell the index for \$1,020 in six months, if asked to do so
- ▶ In six months at contract expiration:
 - if the spot price is \$1,100, call sellers payoff = $\$1,020 - \$1,100 = -\$80$
 - if the spot price is \$900, call buyer walks away, sellers payoff = \$0.

Buyer preserves the upside potential, while at the same time eliminates the unpleasant downside.

However

Seller has to be compensated by a initial premium for being at a disadvantage at expiration.

- ▶ **Strike (or exercise) price:** the amount paid by the option buyer for the asset if he/she decides to exercise.
- ▶ **Exercise:** the act of paying the strike price to buy the asset.
- ▶ **Expiration:** the date by which the option must be exercised or become worthless.
- ▶ **Exercise style:** specifies when the option can be exercised.

Style	can be exercised
European	only at expiration date
American	at any time before expiration
Bermudan	during specified periods

Payoff of purchased call = $\max(0, \text{spot price at expiration} - \text{strike price})$

Profit of purchased call = **payoff** of purchased call
– future value of option premium

Payoff of written call = $-\max(0, \text{spot price at expiration} - \text{strike price})$

Profit of written call = **payoff** of written call
+ future value of option premium

Example 2.2-3 S&R Index 6-month European call option

Strike price = \$1,000,

Premium = \$93.81,

6-month risk-free rate = 2%.

Compute both payoff and profit of the **purchased** call option if the index value in six months \$1,100 (resp. \$900).

Solution.

If index value in six months = \$1,100,

$$\begin{aligned}\text{Payoff} &= \max(0, \$1,100 - \$1,000) \\ &= \$100\end{aligned}$$

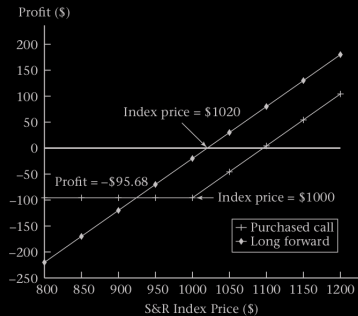
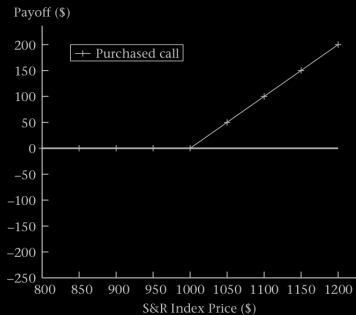
$$\begin{aligned}\text{Profit} &= \$100 - \$93.81 \times 1.02 \\ &= \$4.32.\end{aligned}$$

If index value in six months = \$900,

$$\begin{aligned}\text{Payoff} &= \max(0, \$900 - \$1,000) \\ &= \$0\end{aligned}$$

$$\begin{aligned}\text{Profit} &= \$0 - \$93.81 \times 1.02 \\ &= -\$95.68.\end{aligned}$$

□



Example 2.2-4 S&R Index 6-month European call option

Strike price = \$1,000,

Premium = \$93.81,

6-month risk-free rate = 2%.

Compute both payoff and profit of the **written** call option if the index value in six months \$1,100 (resp. \$900).

Solution.

If index value in six months = \$1,100,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$1,100 - \$1,000) \\ &= -\$100\end{aligned}$$

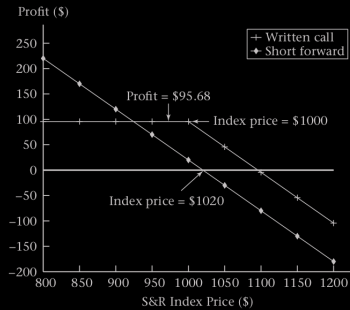
$$\begin{aligned}\text{Profit} &= -\$100 + \$93.81 \times 1.02 \\ &= -\$4.32.\end{aligned}$$

If index value in six months = \$900,

$$\begin{aligned}\text{Payoff} &= -\max(0, \$900 - \$1,000) \\ &= \$0\end{aligned}$$

$$\begin{aligned}\text{Profit} &= \$0 + \$93.81 \times 1.02 \\ &= \$95.68.\end{aligned}$$

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