Financial Mathematics

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 9. Parity and other option relationships

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European options

$$C(K,T) - P(K,T) = \mathrm{PV}_{0,T}\left(F_{0,T} - K\right)$$
$$= e^{-rT}\left(F_{0,T} - K\right)$$

Buying a call and selling a put
with the strike both equal to the forward price (i.e., $K = F_{0,T}$)
creates a synthetic forward contract
and hence must have a zero price.

Parity generally fails for American options!

Parity for stocks

$$C(K, T) = P(K, T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT} K$$

Example 9.1-1 Suppose that the price of a non-dividend-paying stock is \$40, the continuously compounded interest rate is 8%, and options have 3 months to expiration. If a 40-strike European call sells for \$2.78, find the price for a 40-strike European put sells.

Solution. Let the price for put be y. Then

$$\$2.78 = y + \$40 - \$40e^{-0.08 \times 0.25}$$

Hence,

$$y = \$1.99.$$

$\square$

Why is a call more expensive than a put?

When $S_0 = K$ and $\text{Div} = 0$, then

$$C(K,T) - P(K,T) = K\left(1 - e^{-rT}\right)$$

The difference of a call and put is the time value of money.

Example 9.1-2 Make the same assumptions as in Example 9.1-1, except suppose that the stock pays a \$5 dividend just before expiration. If the price of the European call is \$0.74, what would be the price of the European put?

Solution. Let the price for put be y. Then

$$\$0.74 = y + \left(\$40 - \$5e^{-0.08\times 0.25}\right) - \$40e^{-0.08\times 0.25}$$

Hence,

$$y = \$4.85.$$

$\square$

Synthetic securities

$$C(K,T) = P(K,T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

- Synthetic stock

$$S_0 = C(K,T) - P(K,T) + \mathrm{PV}_{0,T}(\mathrm{Div}) + e^{-rT}K$$

$$C(K,T) = P(K,T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

- Synthetic Treasury bill (T-bill)

$$\underbrace{S_0 - C(K,T) + P(K,T)}_{\text{a } \textbf{conversion}} = \mathrm{PV}_{0,T}\,(\mathrm{Div}) + e^{-rT}K$$

Motivation:

A hedged position that has no risk but requires investment.

T-bills are taxed differently than stocks.

Synthetic securities

$$C(K,T) = P(K,T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

- Synthetic options

$$C(K,T) = P(K,T) + (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) - e^{-rT}K$$

$$P(K,T) = C(K,T) - (S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})) + e^{-rT}K$$

Generalize the parity to apply to the case where the strike asset is not necessarily cash but could be any other asset.

We will skip this section and
leave it for motivated students.

European versus American options

$$C_{\mathrm{Amer}}(S, K, T) \geq C_{\mathrm{Eur}}(S, K, T)$$

$$P_{\mathrm{Amer}}(S, K, T) \geq P_{\mathrm{Eur}}(S, K, T)$$

Maximum and minimum option prices

$$S \geq C_{\mathrm{Amer}}(S, K, T) \geq C_{\mathrm{Eur}}(S, K, T) \geq \max\left(0, \mathrm{PV}_{0,T}(F_{0,T}) - \mathrm{PV}_{0,T}(K)\right)$$

$$K \geq P_{\mathrm{Amer}}(S, K, T) \geq P_{\mathrm{Eur}}(S, K, T) \geq \max\left(0, \mathrm{PV}(K) - \mathrm{PV}_{0,T}(F_{0,T})\right)$$

Early exercise for American options

Calls on stocks with no dividend

$$C_{\text{Amer}} \geq C_{\text{Eur}} > S_t - K$$

No early exercise!

See p. 277 for the proof of the first set of inequalities.

Calls on stock with dividends

Interest beats dividends? $K - \text{PV}_{t,T}(K) > \text{PV}_{t,T}(\text{Div})$	Early exercise?
✓	✗
✗	possibly

When dividends do make early exercise rational, one should exercise at the last moment before the ex-dividend date.

Early exercise for puts
(no dividend case)

In order to receive interest, one may exercise early
(think about the case when $S_t = 0$)

No-exercise condition:

$$P(S_t, K, T-t) > K - S_t$$
$$\Updownarrow$$
$$C(S_t, K, T-t) > K - \mathrm{PV}_{t,T}(K)$$

	calls	puts
Receive	stock	cash
Motivation for early exercise	sufficient dividends	sufficient interest

One can view interest as the dividend on cash.

Dividends are the sole reason to early-exercise an option.

Time to expiration – the K fixed

The longer the more expensive

- American call/put options
- European call option on stock with no dividend

The longer, might be cheaper

- European call option on stock with dividend
- European put option

Time to expiration
– $K_t = ke^{rt}$

Theorem 9.3-1 When $K_t = e^{rt}K$, i.e., the strike grows at the interest rate, the premiums on European calls and puts on a non-dividend-paying stock increases with time to maturity.

Proof. We only prove the case for puts and leave the calls as exercise.
Let $T > t$. In order to show that

$$P_{\text{Euro}}(S_T, K_T, T) > P_{\text{Euro}}(S_t, K_t, t),$$

it suffices to find an arbitrage when

$$P_{\text{Euro}}(S_T, K_T, T) \leq P_{\text{Euro}}(S_t, K_t, t).$$

Proof (continued).

		Payoff at Time T			
		$S_T < K_T$		$S_T > K_T$	
		Payoff at Time t			
Transaction	**Time 0**	$S_t < K_t$	$S_t > K_t$	$S_t < K_t$	$S_t > K_t$
Sell $P(t)$	$P(t)$	$S_T - K_T$	0	$S_T - K_T$	0
Buy $P(T)$	$-P(T)$	$K_T - S_T$	$K_T - S_T$	0	0
Total	$P(t) - P(T)$	0	$K_T - S_T$	$S_T - K_T$	0

$\square$

Different strike prices

$$K_1 \le K_2 \le K_3$$

Relation	Ideas in proof, arbitrage in
$C(K_1) \ge C(K_2)$	a call bull spread
$P(K_1) \le P(K_2)$	a put bear spread
$C(K_1) - C(K_2) \le K_2 - K_1$	a call bear spread
$P(K_2) - P(K_1) \le K_2 - K_1$	a put bull spread
$\dfrac{C(K_1) - C(K_2)}{K_2 - K_1} \le \dfrac{C(K_2) - C(K_3)}{K_3 - K_2}$	an asymmetric butterfly spread
$\dfrac{P(K_2) - P(K_1)}{K_2 - K_1} \le \dfrac{P(K_3) - P(K_2)}{K_3 - K_2}$	an asymmetric butterfly spread

Convexity revisited

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

$$\Updownarrow$$

$$C(K_2) \leq \lambda C(K_1) + (1 - \lambda) C(K_3).$$

with

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1}$$

Example 9.3-1 Suppose that

Strike	50	55
Call Premium	18	12

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check ∃. 9.4 on p. 283. □

Example 9.3-2 Suppose that

Strike	50	59	65
Call premium	14	8.9	5

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check ∃. 9.5 on p. 284. □

Example 9.3-3 Suppose that

Strike	50	55	70
Put premium	4	8	16

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check ∃. 9.6 on p. 284. □

Problems: 9.1, 9.2, 9.3, 9.4, 9.8, 9.9, 9.10, 9.11, 9.15.

Due Date: TBA