

# Financial Mathematics

MATH 5870/6870<sup>1</sup>  
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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

## Chapter 9. Parity and other option relationships

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## § 9.1 Put-call parity

## § 9.2 Generalized parity and exchange options

## § 9.3 Comparing options with respect to style, maturity, and strike

## § 9.4 Problems

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§ 9.3 Comparing options with respect to style, maturity, and strike

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## European versus American options

$$C_{\text{Amer}}(S, K, T) \geq C_{\text{Eur}}(S, K, T)$$

$$P_{\text{Amer}}(S, K, T) \geq P_{\text{Eur}}(S, K, T)$$

## Maximum and minimum option prices

$$S \geq C_{\text{Amer}}(S, K, T) \geq C_{\text{Eur}}(S, K, T) \geq \max(0, \text{PV}_{0,T}(F_{0,T}) - \text{PV}_{0,T}(K))$$

$$K \geq P_{\text{Amer}}(S, K, T) \geq P_{\text{Eur}}(S, K, T) \geq \max(0, \text{PV}(K) - \text{PV}_{0,T}(F_{0,T}))$$

## Early exercise for American options

Calls on stocks **with no dividend**

$$C_{\text{Amer}} \geq C_{\text{Eur}} > S_t - K$$

No early exercise!

See p. 277 for the proof of the first set of inequalities.

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Calls on stock **with dividends**

Interest beats dividends? $K - PV_{t,T}(K) > PV_{t,T}(\text{Div})$	Early exercise?
✓	✗
✗	possibly

When dividends do make early exercise rational, one should exercise at the last moment before the ex-dividend date.



Early exercise for puts  
(no dividend case)

In order to receive interest, one may exercise early  
(think about the case when  $S_t = 0$ )

No-exercise condition:

$$P(S_t, K, T - t) > K - S_t$$



$$C(S_t, K, T - t) > K - \text{PV}_{t,T}(K)$$

	calls	puts
Receive	stock	cash
Motivation for early exercise	sufficient dividends	sufficient interest

One can view interest as the dividend on cash.

Dividends are the sole reason to early-exercise an option.

## Time to expiration – the $K$ fixed

The longer the **more expensive**

- ▶ American call/put options
  - ▶ European call option on stock with no dividend
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The longer, might be cheaper

- ▶ European call option on stock with dividend
- ▶ European put option

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## Time to expiration

$$- K_t = ke^{rt}$$

**Theorem 9.3-1** When  $K_t = e^{rt}K$ , i.e., the strike grows at the interest rate, the premiums on European calls and puts on a non-dividend-paying stock increases with time to maturity.

**Proof.** We only prove the case for puts and leave the calls as exercise.  
Let  $T > t$ . In order to show that

$$P_{\text{Euro}}(S_T, K_T, T) > P_{\text{Euro}}(S_t, K_t, t),$$

it suffices to find an arbitrage when

$$P_{\text{Euro}}(S_T, K_T, T) \leq P_{\text{Euro}}(S_t, K_t, t).$$

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Proof (continued).

Transaction		Time 0		Payoff at Time $T$			
				$S_T < K_T$		$S_T > K_T$	
				Payoff at Time $t$			
				$S_t < K_t$	$S_t > K_t$	$S_t < K_t$	$S_t > K_t$
Sell $P(t)$	$P(t)$	$S_T - K_T$	0	$S_T - K_T$	0		
Buy $P(T)$	$-P(T)$	$K_T - S_T$	$K_T - S_T$	0	0		
Total	$P(t) - P(T)$	0	$K_T - S_T$	$S_T - K_T$	0		

□

## Different strike prices

$$K_1 \leq K_2 \leq K_3$$

Relation	Ideas in proof, arbitrage in
$C(K_1) \geq C(K_2)$	a call <b>bull</b> spread
$P(K_1) \leq P(K_2)$	a put <b>bear</b> spread
$C(K_1) - C(K_2) \leq K_2 - K_1$	a call <b>bear</b> spread
$P(K_2) - P(K_1) \leq K_2 - K_1$	a put <b>bull</b> spread
$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$	an asymmetric butterfly spread
$\frac{P(K_2) - P(K_1)}{K_2 - K_1} \leq \frac{P(K_3) - P(K_2)}{K_3 - K_2}$	an asymmetric butterfly spread

## Convexity revisited

$$\frac{C(K_1) - C(K_2)}{K_2 - K_1} \leq \frac{C(K_2) - C(K_3)}{K_3 - K_2}$$

$$\Updownarrow$$

$$C(K_2) \leq \lambda C(K_1) + (1 - \lambda)C(K_3).$$

with

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1}$$

Example 9.3-1 Suppose that

Strike	50	55
Call Premium	18	12

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check  $\exists$ , 9.4 on p. 283.



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Example 9.3-2 Suppose that

Strike	50	59	65
Call premium	14	8.9	5

1. What no-arbitrage property is violated?
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Solution. Check  $\exists$ , 9.5 on p. 284.



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Example 9.3-3 Suppose that

Strike	50	55	70
Put premium	4	8	16

1. What no-arbitrage property is violated?
2. What spread position would you use to effect arbitrage?
3. Demonstrate that the spread position is an arbitrage.

Solution. Check  $\exists$ , 9.6 on p. 284.



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