Financial Mathematics

MATH 5870/6870¹ Fall 2021

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¹Based on Robert L. McDonald's *Derivatives Markets*. 3rd Ed. Pearson. 2013.

- § 20.1 The Black-Scholes assumption about stock prices
- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- § 20.7 Problems

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0.$$
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- ▶ S(t) is the stock price. dS(t) is the instantaneous change in the stock price. S_0 is the initial asset value.
- \triangleright α is the continuously compound expected return on the stock:
- \triangleright σ is the volatility, i.e., the standard deviation of the instantaneous return;
- \triangleright Z(t) is the standard Brownian motion
- ightharpoonup dZ(t) requires rigorous justification.

- Equation of this type is called stochastic differential equation
- ▶ Solution to this specific equation is the geometric Brownian motion

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- ► Solution to this specific equation is the geometric Brownian motion.

Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(\mathcal{S}(t)) \sim \mathcal{N}\left(\ln(\mathcal{S}_0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t, \ \sigma^2 \ t\right), \quad ext{for all } t>0.$$

Remark 20.1-2 Note that Remark 20.1-1 is valid for all t > 0. It works for the terminal time t = T. It can also help us solve path-dependent options.

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1. It starts at 0:

$$Z(0)=0.$$

2. For $0 \le s < t$, the increment Z(t) - Z(s) is normally distributed with mean zero and variance t - s:

$$Z(t) - Z(s) \sim N(0, t - s)$$

3. Its increments are independent: if

$$0 \leq t_0 \leq t_1 \leq \cdots \leq t_k$$

then

$$\mathbb{P}(Z(t_i) - Z(t_{i-1}) \in H_i, \ 1 \le i \le k) = \prod_{i=1}^k \mathbb{P}(Z(t_i) - Z(t_{i-1}) \in H_i)$$

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1. Z(t) is nowhere differentiable.

(Hence, dZ(t) requires some special treatment.)

2. Z(t) satisfies the scaling property.

$$\widetilde{Z}(t) := \frac{1}{\sqrt{c}} Z(ct)$$
 is also a B.M. for all $c>0$

3. Z(t) is a martingale, namely,

$$\mathbb{E}\left(Z(t+s)|Z(t)\right)=Z(t).$$

4. For any t > 0, $Z(t) \sim N(0, t)$ and

$$\mathbb{E}(Z(t)Z(s)) = \min(t, s)$$
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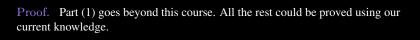
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Arithmetic Brownian motion

Definition 20.2-2 Let Z(t) be a B.M. Then the process X(t) given by

$$dX(t) = \alpha dt + \sigma dZ(t)$$

is called an arithmetic Brownian motion. Equivalently, X(t) can be written in the following integral representation:

$$X(t) = X(0) + \int_0^t \alpha ds + \int_0^t \sigma dZ(s).$$

Remark 20.2-2

1. X(t) is normally distributed:

$$X(t) = \sigma t + \sigma Z(t) \sim N(\sigma t, \sigma^2 t).$$

- 2. X(t) takes both positive and negative values almost surely.
- 3. αt is a drift term

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The Ornstein-Uhlenbeck process

Definition 20.2-3 Let Z(t) be a B.M. Then the process X(t) given by

$$dX(t) = \lambda \left(\alpha - X(t)\right) dt + \sigma dZ(t)$$

is called the Ornstein-Uhlenbeck process.

Remark 20.2-3 Equivalently, X(t) can be written in the following integral representation:

$$X(t) = X(0) + \lambda \int_0^t (\alpha - X(s)) ds + \int_0^t \sigma dZ(s),$$

which is an integral equation (unknown X appears on both sides).

Remark 20.2-4 We have introduced mean reversion in the drift term.

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Problems: 20.1, 20.2, 20.3, 20.4, 20.5, 20.6, 20.7, 20.8, $\overline{20.9}$, 20.10, 20.11, 20.12.

Due Date: TBA