Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 18. The Lognormal Distribution

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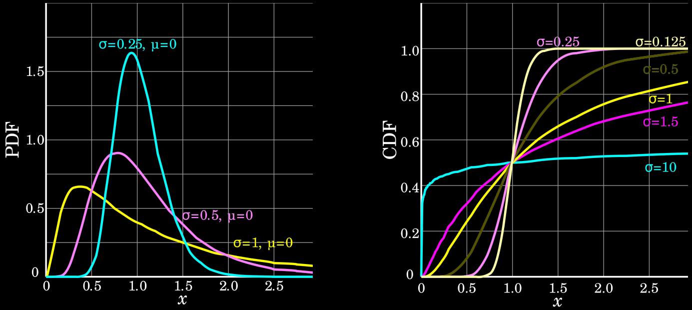
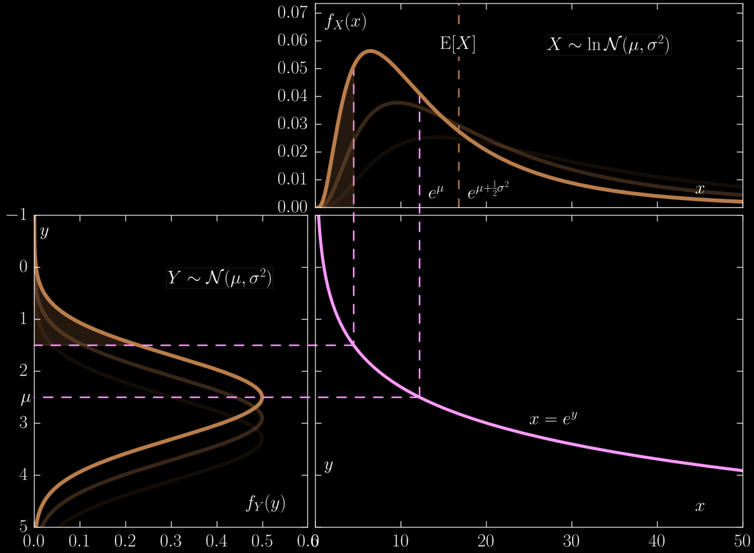
Definition 18.2-1 A random variable Y is *lognormally distributed* with parameters μ and $\sigma > 0$ if $\ln(Y) \sim N(\mu, \sigma^2)$.


σ=0.25, μ=0
σ=0.5, μ=0
σ=1, μ=0
PDF
x
0
0.5
1.0
1.5
2.0
2.5
CDF
σ=0.25
σ=0.125
σ=0.5
σ=1
σ=1.5
σ=10
0.2
0.4
0.6
0.8



[2]

[2] Image from Wikipedia.

Theorem 18.2-1 If Y_1 and Y_2 are lognormally distributed, so is $Y_1 Y_2$.

Proof.

Since Y_1 and Y_2 are lognormally distributed, $\ln(Y_1)$ and $\ln(Y_2)$ are normally distributed. Hence,

$$\ln(Y_1) + \ln(Y_2) = \ln(Y_1 Y_2)$$

is normally distributed too. Therefore, $Y_1 Y_2$ is lognormally distributed.

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is normally distributed too. Therefore, $Y_1 Y_2$ is lognormally distributed. $\square$

Theorem 18.2-2 If $\ln(Y) \sim N(\mu, \sigma^2)$, then the density of Y is given by

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.$$

Proof. For $y > 0$,

$$\begin{aligned}
\mathbb{P}(Y \leq y) &= \mathbb{P}\left(\ln(Y) \leq \ln(y)\right) \\
&= \mathbb{P}\left(\frac{\ln(Y) - \mu}{\sigma} \leq \frac{\ln(y) - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy}\Phi\left(\frac{\ln(y) - \mu}{\sigma}\right) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2} \frac{d}{dy}\frac{\ln(y) - \mu}{\sigma} \\
&= \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.
\end{aligned}$$

□

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\end{aligned}$$

$\square$

Recall that the moment generating function of the normal random variable $X \sim N(\mu, \sigma^2)$ is

$$\mathbb{E}\left(e^{tX}\right) = e^{\mu t + \sigma^2 t^2/2}, \qquad \text{for all } t \in \mathbb{R}. \tag{1}$$

Remark 18.2-1 If Y is lognormally distributed with parameters μ and σ, then

$$\mathbb{E}\left(Y^t\right) = e^{\mu t + \sigma^2 t^2/2}, \qquad \text{for all } t \in \mathbb{R}.$$

Remark 18.2-2 By *Jensen's inequality*, if g is a convex function, then

$$\mathbb{E}\left(g(X)\right) \leq g\left(\mathbb{E}(X)\right).$$

Hence, for $g(x) = e^x$, we see that

$$\mathbb{E}\left(e^X\right) \leq e^{\mathbb{E}(X)}.$$

This is consistent with our computations above because

$$LHS = e^{\mu + \sigma^2/2} \geq e^{\mu} = RHS.$$

Theorem 18.2-3 If Y is lognormally distributed such that $\ln(Y) \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2} \quad \text{and} \quad \mathrm{Var}(Y) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

Proof. Let $X = \ln(Y)$. By (1),

$$\mathbb{E}(Y) = \mathbb{E}\left(e^X\right) = e^{\mu + \frac{1}{2}\sigma^2}$$

and

$$\mathbb{E}(Y^2) = \mathbb{E}\left(e^{2X}\right) = e^{2\mu+2\sigma^2}.$$

Therefore,

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = e^{2\mu+2\sigma^2} - e^{2\left(\mu+\frac{1}{2}\sigma^2\right)} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

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$\square$