Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 11. Binomial Option Pricing: Selected Topics

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§ 11.1 Understanding Early Exercise

§ 11.2 Understanding risk-neutral pricing

§ 11.3 The Binomial tree and lognormality

§ 11.4 Problems

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§ 11.3 The Binomial tree and lognormality

§ 11.4 Problems

Options may be rationally exercised prior to expiration

By exercising, the option holder

+ Receives the stock and thus receives dividends

– Pays the strike price prior to expiration (this has an interest cost)

– Loses the insurance implicit in the call against the possibility that the stock price will be less than the strike price at expiration

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Example 11.1-1 For a call option, let $K = 100$, $r = 0.05$, $\delta = 0.05$, $\sigma = 0$ and the stock price today is $S = 200$. Shall we exercise the call?

Solution.

+ Receives the stock and thus receives dividends:

$$S \times \delta = 200 \times 0.05 = \$10.00.$$

− Pays the strike price prior to expiration (this has an interest cost)

$$K \times r = 100 \times 0.05 = \$5.00.$$

− Loses the insurance: $\$0$ because $\delta = 0$.

Hence, we need to early exercise! □

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If volatility is zero, the value of insurance is zero. Then, it is optimal to defer exercise as long as interest savings on the strike exceed dividends lost

$$rK > \delta S$$

$$\Downarrow$$

$$\text{It is optimal to exercise} \quad \Longleftrightarrow \quad S > \frac{rK}{\delta}$$

E.g. If $r = \delta$, any in-the-money option should be exercised immediately.

If $r = 3\delta$, we exercise when the stock price is 3 times of the strike price.

When volatility is positive, the implicit insurance has value that varies with time to expiration.

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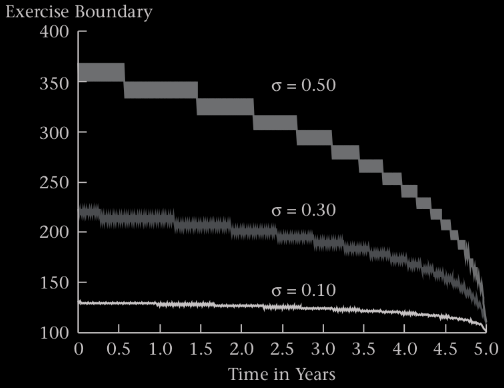
Early-exercise boundary – American call

FIGURE 11.1

Early-exercise boundaries for volatilities of 10%, 30%, and 50% for a 5-year American call option. In all cases, $K = \$100$, $r = 5\%$, and $\delta = 5\%$.



- Curve computed using 500 binomial steps.
- When $\sigma = 0$, the boundary should be $S = K = \$100$.
- The value of insurance diminishes in time.

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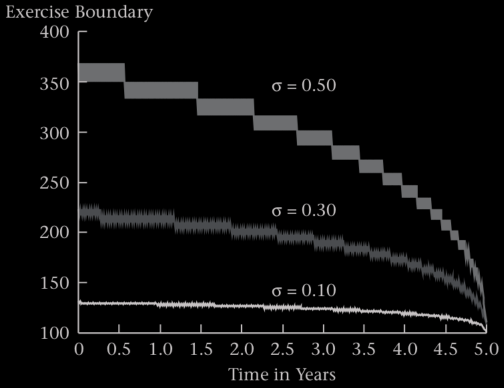

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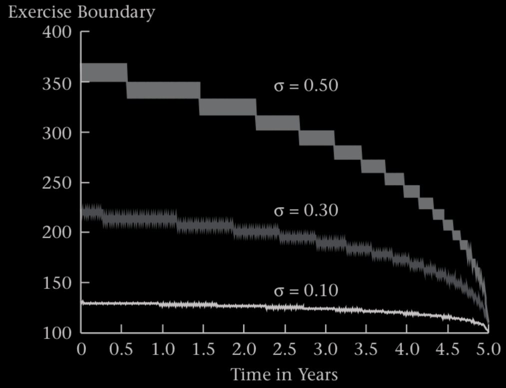

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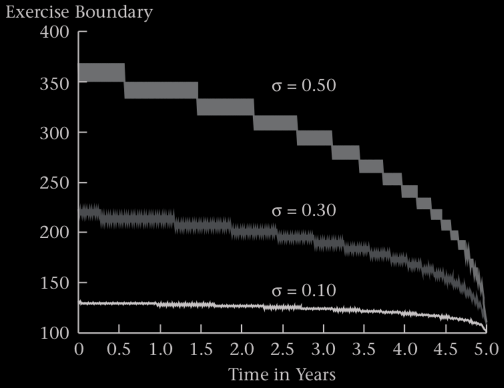

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Early-exercise boundary – American put

FIGURE 11.2

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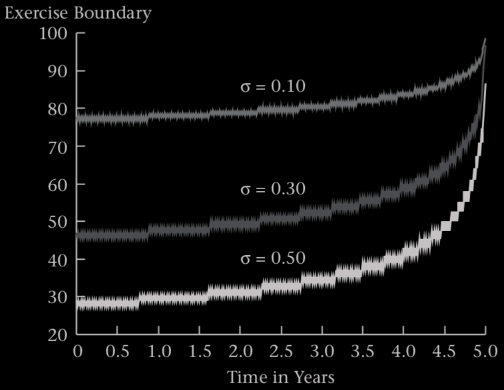

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Risk-Neutral Probability

Recall the binomial option pricing formula:

$$C = \Delta S + B = e^{-rh}\left[p^* C_u + (1 - p^*) C_d\right]$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \quad \sim$$ risk-neutral probability that the stock will go up

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$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \quad \Longleftrightarrow \quad p^* uSe^{\delta h} + (1 - p^*) dSe^{\delta h} = e^{rh} S$$

Two offers:

(a) \$1000 cash

(b) \$2000 or \$0 cash with probability 1/2 for each

Both offers have the same expected return, while (b) bears risk and (a) does not.

A risk-averse investor prefers (a).

A risk-neutral investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing. Hence, he/she prefers equally to (a) and (b).

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The option pricing formula can be said to price options
as if investors are risk-neutral

Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return.

Pricing an option using real probability

- Suppose that the continuously compounded expected return on the stock is α and that the stock does not pay dividends.
- If p is the true probability of the stock going up, p must be consistent with u, d and α

$$puS + (1-p)dS = e^{\alpha h}S$$

- Solving for p gives us

$$p = \frac{e^{\alpha h} - d}{u - d}$$

- For p to be a probability, we have to have $u \geq e^{\alpha h} \geq d$.
- Using this p, the actual expected payoff to the option one period is

$$pC_u + (1-p)C_d = \frac{e^{\alpha h} - d}{u - d}C_u + \frac{u - e^{\alpha h}}{u - d}C_d.$$

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At what rate do we discount this expected payoff?

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- Since an option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B}e^{\alpha h} + \frac{B}{S\Delta + B}e^{rh}$$

- Hence, the discounted at this appropriate discount rate, the price for the option should be

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u-d}C_u + \frac{u - e^{\alpha h}}{u-d}C_d\right]$$

- By setting $\alpha = r$, one obtains the simplest pricing procedure.
- This gives an alternative way to compute the option price, instead of $\Delta S + B$.

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One can use either

$$C = \Delta S + B$$

or

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d\right]$$

to compute the option price

- **First equation** is more efficient
- For the **second one**, in order to compute γ, one needs to computer Δ and B first and then obtains γ via

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Given the continuously compounded expected return of the stock α

1. Compute the probability that stock goes up

$$p = \frac{e^{\alpha h} - d}{u - d}$$

2. Compute the actual expected payoff (to be discounted)

$$X := pC_u + (1 - p)C_d$$

3. Using r and δ to compute Δ and B:

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S(u - d)} \qquad \text{and} \qquad B = e^{-rh}\frac{uC_d - dC_u}{u - d}.$$

4. Compute the discounted rate γ:

$$\gamma = \frac{1}{h}\log\left(\frac{S\Delta}{S\Delta + B}e^{\alpha h} + \frac{B}{S\Delta + B}e^{rh}\right)$$

5. Finally, the option price should be the discounted value:

$$Xe^{-\gamma h}.$$

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2. Compute the actual expected payoff (to be discounted)

$$X := pC_u + (1 - p)C_d$$

3. Using r and δ to compute Δ and B:

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u - d)} \qquad \text{and} \qquad B = e^{-rh} \frac{uC_d - dC_u}{u - d}.$$

4. Compute the discounted rate γ:

$$\gamma = \frac{1}{h} \log \left(\frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{rh} \right)$$

5. Finally, the option price should be the discounted value:

$$Xe^{-\gamma h}.$$

Given the continuously compounded expected return of the stock α

1. Compute the probability that stock goes up

$$p = \frac{e^{\alpha h} - d}{u - d}$$

2. Compute the actual expected payoff (to be discounted)

$$X := pC_u + (1 - p)C_d$$

3. Using r and δ to compute Δ and B:

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S(u - d)} \qquad \text{and} \qquad B = e^{-rh}\frac{uC_d - dC_u}{u - d}.$$

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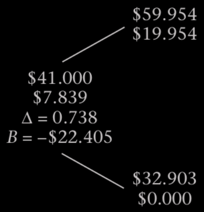
$$Xe^{-\gamma h}.$$

An one-period example

FIGURE 11.3

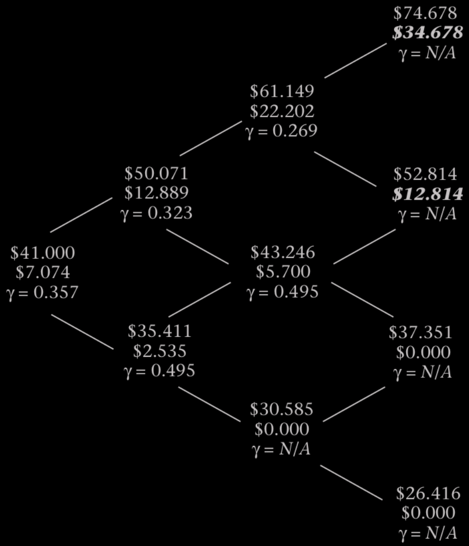
Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 1.000$. This is the same as Figure 10.3.



A multi-period example

FIGURE 11.4

Binomial tree for pricing an American call option; assumes S = \$41.00, K = \$40.00, σ = 0.30, r = 0.08, T = 1.00 years, δ = 0.00, and h = 0.333. The continuously compounded true expected return on the stock, α, is 15%. At each node the stock price, option price, and continuously compounded true discount rate for the option, γ, are given. Option price in ***bold italic*** signify that exercise is optimal at that node.



Chapter 11. Binomial Option Pricing: Selected Topics

§ 11.1 Understanding Early Exercise

§ 11.2 Understanding risk-neutral pricing

§ 11.3 The Binomial tree and lognormality

§ 11.4 Problems

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§ 11.1 Understanding Early Exercise

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§ 11.4 Problems

The usefulness of the binomial pricing model hinges on

the binomial tree providing

a reasonable representation of

the stock price distribution

The binomial tree approximates a lognormal distribution

Random Walk

- Let Y_i be a sequence of i.i.d. random variables, each following

$$Y_i = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

- Random walk Z_n is defined to be

$$Z_n = \sum_{i=1}^{n} Y_i.$$

- From the walk Z_n, one can also retrieve the values of Y_n

$$Y_n = Z_n - Z_{n-1}$$

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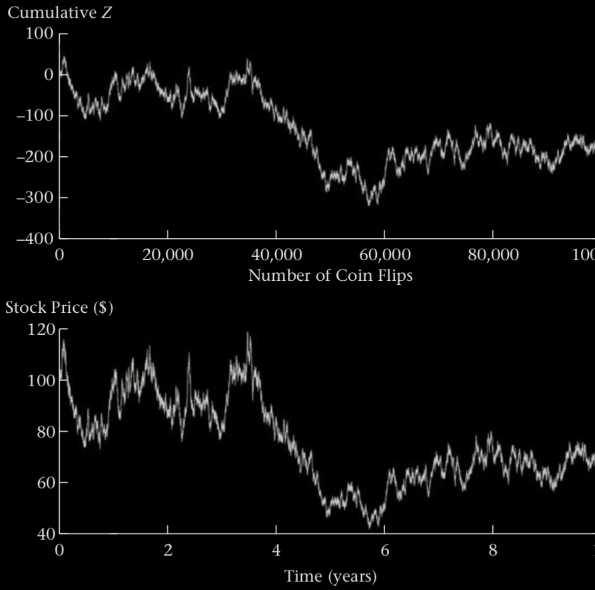
- From the walk Z_n, one can also retrieve the values of Y_n

$$Y_n = Z_n - Z_{n-1}$$

FIGURE 11.5

In the top panel is an illustration of a random walk, where the counter, Z, increases by 1 when a fair coin flip comes up heads, and decreases by 1 with tails. In the bottom panel is a particular path through a 10,000-step binomial tree, where the up and down moves are the same as in the top panel. Assumes $S_0 = \$100$, $r = 6\%$, $\sigma = 30\%$, $T = 10$ years, and $h = 0.0001$.



Modelling Stock prices as a random walk

The idea that asset prices should follow a random walk was articulated in Samuelson (1965)

In efficient markets, an asset price should reflect all available information. In response to new information the price is equally likely to move up or down, as with the coin flip.

The price after a period of time is the initial price plus the cumulative up and down movements due to informational surprises

Modelling Stock prices as a random walk – Issues and Binomial Model

If by chance we get enough cumulative down movements, the stock price will become negative

The stock, on average, should have a positive return. The random walk model taken literally does not permit this

The magnitude of the move ($1) should depend upon how quickly the coin flips occur and the level of the stock price

---

The binomial model is a variant of the random walk model that solves all of these problems at once:

$$S_{t+h} = S_t e^{(r-\delta)h \pm \sigma h},$$

which says, instead of the prices jumping like a random walk, the compound rate follows a random walk.

Lognormality of the binomial model

The binomial tree approximates a lognormal distribution, which is commonly used to model stock prices

The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed

With the lognormal distribution, the stock price is positive, and the distribution is skewed to the right, that is, there is a chance that extremely high stock prices will occur

FIGURE 11.6

Construction of a binomial tree depicting stock price paths, along with risk-neutral probabilities of reaching the various terminal prices.

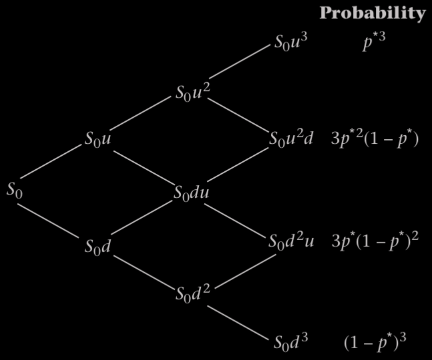


FIGURE 11.7

Comparison of lognormal distribution with three-period binomial approximation.

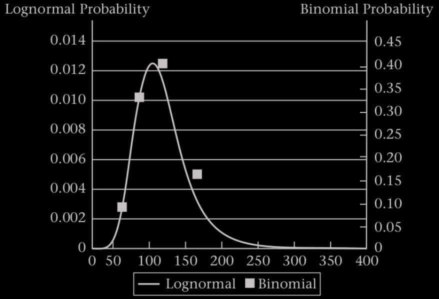
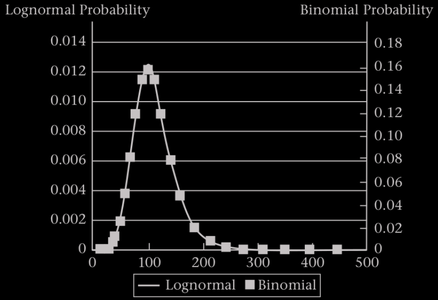


FIGURE 11.8

Comparison of lognormal distribution with 25-period binomial approximation.



Alternative Binomial trees

Binomial Tree	Cox-Ross-Rubinstein binomial tree	Lognormal tree
$u = e^{(r-\delta)h+\sigma\sqrt{h}}$	$u = e^{+\sigma\sqrt{h}}$	$u = e^{(r-\delta-0.5\delta^2)h+\sigma\sqrt{h}}$
$d = e^{(r-\delta)h-\sigma\sqrt{h}}$	$d = e^{-\sigma\sqrt{h}}$	$d = e^{(r-\delta-0.5\delta^2)h-\sigma\sqrt{h}}$

- Even though the values of u and d are deferent, the ratio, which measures volatility, remains the same:

$$u/d = e^{2\sigma\sqrt{h}}.$$

- Once u and d are determined, the rest computations for option price remain the same.
- All three methods of constructing a binomial tree yield different option prices for finite n, but they approach the same price as $n \to \infty$.

Alternative Binomial trees

Binomial Tree	Cox-Ross-Rubinstein binomial tree	Lognormal tree
$u = e^{(r-\delta)h+\sigma\sqrt{h}}$	$u = e^{+\sigma\sqrt{h}}$	$u = e^{(r-\delta-0.5\delta^2)h+\sigma\sqrt{h}}$
$d = e^{(r-\delta)h-\sigma\sqrt{h}}$	$d = e^{-\sigma\sqrt{h}}$	$d = e^{(r-\delta-0.5\delta^2)h-\sigma\sqrt{h}}$

- Even though the values of u and d are deferent, the ratio, which measures volatility, remains the same:

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- Even though the values of u and d are deferent, the ratio, which measures volatility, remains the same:

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Alternative Binomial trees

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$u = e^{(r-\delta)h+\sigma\sqrt{h}}$	$u = e^{+\sigma\sqrt{h}}$	$u = e^{(r-\delta-0.5\delta^2)h+\sigma\sqrt{h}}$
$d = e^{(r-\delta)h-\sigma\sqrt{h}}$	$d = e^{-\sigma\sqrt{h}}$	$d = e^{(r-\delta-0.5\delta^2)h-\sigma\sqrt{h}}$

- Even though the values of u and d are deferent, the ratio, which measures volatility, remains the same:

$$u/d = e^{2\sigma\sqrt{h}}.$$

- Once u and d are determined, the rest computations for option price remain the same.
- All three methods of constructing a binomial tree yield different option prices for finite n, but they approach the same price as $n \to \infty$.

Is the Binomial model realistic?

The binomial model is a form of the random walk model, adapted to modeling stock prices. The lognormal random walk model in this section assumes among other things, that

Volatility is constant

"Large" stock price movements do not occur

Returns are independent over time

All of these assumptions appear to be violated in the data

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§ 11.4 Problems

Problems: 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.17, 11.18,

Due Date: TBA