

Financial Mathematics

MATH 5870/6870¹
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¹Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 10. Binomial Option Pricing: Basic Concepts

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§ 10.1 A one-period Binomial tree

§ 10.2 Constructing a Binomial tree

§ 10.3 Two or more binomial periods

§ 10.4 Put options

§ 10.5 American options

§ 10.6 Options on other assets

§ 10.7 Problems

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Binomial option pricing

The
binomial option pricing model
or
Cox-Ross-Rubinstein pricing model
assumes that

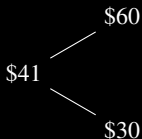
the price of the underlying asset follows a binomial distribution,
that is,

the asset price in each period can
move only up or down by a specified amount.

The binomial option pricing model enables us to
determine the price of an option,
given the characteristics of the stock or other underlying asset.

Example 10.1-1 Consider an European call option on the stock of XYZ, with a \$40 strike price and one year expiration. XYZ does not pay dividends and its current price is \$41.

Assume that, in a year, the price can be either \$60 or \$30.



Can one determine the call premium?

(Let the continuously compounded risk free interest rate be 8%.)

Law of one price

Positions that have the same payoff should have the same cost!

Two portfolios (positions)

- ▶ Portfolio A: Buy one 40-strike call option.
- ▶ Portfolio B: Buy $\Delta \in (0, 1)$ share of stock and borrow B at the risk-free rate.

These two positions should have the same cost.

Solution. The cost for Portfolio B at day zero is

$$\Delta \times S_0 - B.$$

and its payoff at expiration is

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} & \text{if the stock price is 30} \\ \Delta \times 60 - B \times e^{0.08} & \text{if the stock price is 60} \end{cases}$$

On the other hand, the payoff for Portfolio A should be

$$\begin{cases} 0 & \text{if the stock price is 30} \\ (60 - 40) & \text{if the stock price is 60} \end{cases}$$

By equating the two payoffs, one obtains that

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} = 0 \\ \Delta \times 60 - B \times e^{0.08} = 60 - 40 \end{cases}$$

Solution. Hence,

$$B = 20 \times e^{-0.08} \quad \text{and} \quad \Delta = 2/3.$$

Finally, since the cost of Portfolio A has to be equal to that of Portfolio B, we find the cost of Portfolio A:

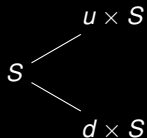
$$\Delta \times S_0 - B = \frac{2}{3} S_0 - 20 \times e^{-0.08}.$$

If we plug in $S_0 = \$41$, we have

$$B = \$18.462 \quad \text{and the cost is } \$8.871.$$



More generally, suppose the stock change its value over a period of time h as



Portfolio A

Payoff	$d \times S$	$u \times S$
Option	0	$u \times S - K$
Total	$C_d = 0$	$C_u = u \cdot S - K$

Portfolio B

Payoff	$d \times S$	$u \times S$
Δ share	$\Delta \cdot d \cdot S \cdot e^{\delta h}$	$\Delta \cdot u \cdot S \cdot e^{\delta h}$
B bond	Be^{rh}	Be^{rh}
Total	$\Delta \cdot d \cdot S \cdot e^{\delta h} + Be^{rh}$	$\Delta \cdot u \cdot S \cdot e^{\delta h} + Be^{rh}$

For two unknowns: Δ and B , solve:

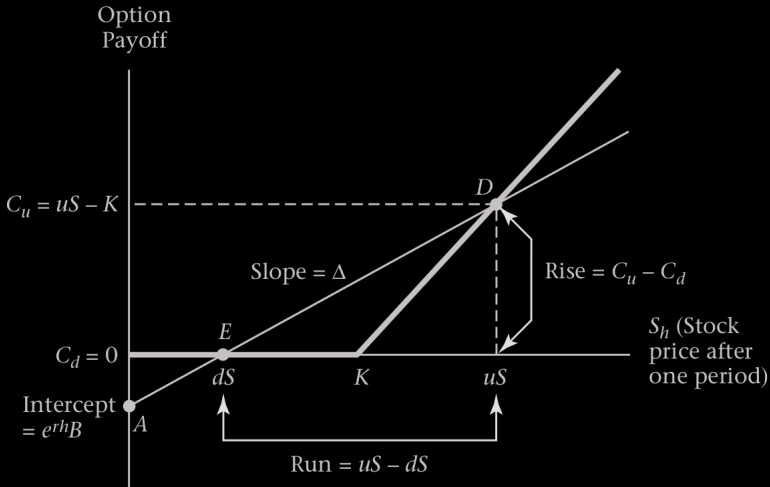
$$\begin{cases} \Delta dSe^{\delta h} + Be^{rh} = C_d \\ \Delta uSe^{\delta h} + Be^{rh} = C_u \end{cases}$$

Set S_h be either dS or uS and

C_h be either C_u or C_d .

Plot S_h (x-axis) versus C_h (y-axis).

$$\Delta S_h e^{\delta h} + Be^{rh} = C_h$$



$$\Delta = e^{-\delta h} \frac{C_h - C_d}{S(u - d)} \quad \text{and} \quad B = e^{-rh} \frac{uC_d - dC_u}{u - d}$$

$$\Delta S + B = e^{-rh} \left(C_u \underbrace{\frac{e^{(r-\delta)h} - d}{u - d}}_{:=p^*} + C_d \underbrace{\frac{u - e^{(r-\delta)h}}{u - d}}_{:=1-p^*} \right)$$

p^* the **risk-neutral probability** of
an increase in the stock price.

Arbitraging a mispriced option

Example 10.1-2 Find arbitrage opportunities in Example 10.1-1 with

- ▶ the option price being overpriced with \$9.00;
- ▶ the option price being underpriced with \$8.25,

instead of the risk-neutral pricing \$8.871.

Solution. One can buy the synthetic option which cost \$8.25 and sell the real one by earning \$9.00. Hence, the present value of the profit is

$$\$9 - \$8.871 = \$0.129.$$

