

Financial Mathematics

MATH 5870/6870¹
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¹Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 20. Brownian Motion and Ito Lemma

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§ 20.1 The Black-Scholes assumption about stock prices

§ 20.2 Brownian motion

§ 20.3 Geometric Brownian motion

§ 20.4 The Ito formula

§ 20.5 The Sharpe ratio

§ 20.6 Risk-neutral valuation

§ 20.7 Problems

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0. \quad (1)$$

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- ▶ $S(t)$ is the **stock price**. $dS(t)$ is the instantaneous change in the stock price. S_0 is the initial asset value.
- ▶ α is the continuously compound expected return on the stock;
- ▶ σ is the volatility, i.e., the standard deviation of the instantaneous return;
- ▶ $Z(t)$ is the standard Brownian motion.
- ▶ $dZ(t)$ requires rigorous justification.

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- ▶ Equation of this type is called stochastic differential equation.
 - ▶ Solution to this specific equation is the geometric Brownian motion.

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Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(\mathcal{S}(t)) \sim N \left(\ln(\mathcal{S}_0) + \left(\alpha - \delta - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right), \quad \text{for all } t > 0.$$

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