Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 10. Binomial Option Pricing: Basic Concepts

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Binomial option pricing

The

binomial option pricing model

or

Cox-Ross-Rubinstein pricing model

assumes that

the price of the underlying asset follows a binomial distribution,

that is,

the asset price in each period can
move only up or down by a specified amount.

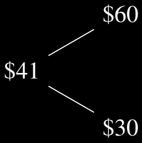
The binomial option pricing model enables us to

determine the price of an option,

given the characteristics of the stock or other underlying asset.

Example 10.1-1 Consider an European call option on the stock of XYZ, with a \$40 strike price and one year expiration. XYZ does not pay dividends and its current price is \$41.

Assume that, in a year, the price can be either \$60 or \$30.



Can one determine the call premium?

(Let the continuously compounded risk free interest rate be 8%.)

Law of one price

Positions that have the same payoff should have the same cost!

Two portfolios (positions)

- Portfolio A: Buy one 40-strike call option.
- Portfolio B: Buy $\Delta \in (0, 1)$ share of stock and borrow B at the risk-free rate.

These two positions should have the same cost.

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These two positions should have the same cost.

Solution. The cost for Portfolio B at day zero is

$$\Delta \times S_0 - B.$$

and its payoff at expiration is

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} & \text{if the stock price is 30} \\ \Delta \times 60 - B \times e^{0.08} & \text{if the stock price is 60} \end{cases}$$

On the other hand, the payoff for Portfolio A should be

$$\begin{cases} 0 & \text{if the stock price is 30} \\ (60 - 40) & \text{if the stock price is 60} \end{cases}$$

By equating the two payoffs, one obtains that

$$\begin{cases} \Delta \times 30 - B \times e^{0.08} = 0 \\ \Delta \times 60 - B \times e^{0.08} = 60 - 40 \end{cases}$$

Solution. Hence,

$$B = 20 \times e^{-0.08} \quad \text{and} \quad \Delta = 2/3.$$

Finally, since the cost of Portfolio A has to be equal to that of Portfolio B, we find the cost of Portfolio A:

$$\Delta \times S_0 - B = \frac{2}{3} S_0 - 20 \times e^{-0.08}.$$

If we plug in $S_0 = \$41$, we have

$$B = \$18.462 \quad \text{and the cost is \$ 8.871.}$$

$\square$

More generally, suppose the stock change its value over a period of time h as

$$\begin{tikzcd}[column sep=small, row sep=small] & u \times S \\ S \arrow[ur, dash] \arrow[dr, dash] & \\ & d \times S \end{tikzcd}$$

Portfolio A

Payoff	$d \times S$	$u \times S$
Option	0	$u \times S - K$
Total	$C_d = 0$	$C_u = u \cdot S - K$

Portfolio B

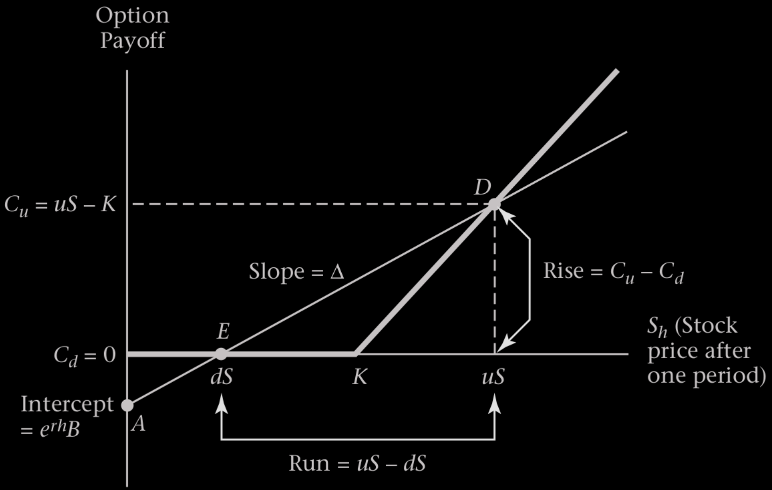
Payoff	$d \times S$	$u \times S$
Δ share	$\Delta \cdot d \cdot S \cdot e^{\delta h}$	$\Delta \cdot u \cdot S \cdot e^{\delta h}$
B bond	Be^{rh}	Be^{rh}
Total	$\Delta \cdot d \cdot S \cdot e^{\delta h} + Be^{rh}$	$\Delta \cdot u \cdot S \cdot e^{\delta h} + Be^{rh}$

For two unknowns: Δ and B, solve:

$$\begin{cases} \Delta dSe^{\delta h} + Be^{rh} = C_d \\ \Delta uSe^{\delta h} + Be^{rh} = C_u \end{cases}$$

Set S_h be either dS or uS and
C_h be either C_u or C_d.
Plot S_h (x-axis) versus C_h (y-axis).

$$\Delta S_h e^{\delta h} + Be^{rh} = C_h$$



Option Payoff
$C_u = uS - K$
D
Slope = Δ
Rise = $C_u - C_d$
E
$C_d = 0$
S_h (Stock price after one period)
dS
K
uS
Intercept = $e^{rh}B$
A
Run = $uS - dS$

$$\Delta = e^{-\delta h} \frac{C_h - C_d}{S(u-d)} \qquad \text{and} \qquad B = e^{-rh} \frac{uC_d - dC_u}{u-d}$$

$$\Delta S + B = e^{-rh} \left(C_u \underbrace{\frac{e^{(r-\delta)h} - d}{u-d}}_{:=p^*} + C_d \underbrace{\frac{u - e^{(r-\delta)h}}{u-d}}_{:=1-p^*} \right)$$

p^* the **risk-neutral probability** of
an increase in the stock price.

Arbitraging a mispriced option

Example 10.1-2 Find arbitrage opportunities in Example 10.1-1 with

- the option price being overpriced with \$9.00;
- the option price being underpriced with \$8.25,

instead of the risk-neutral pricing \$8.871.

Solution. One can buy the synthetic option which cost \$8.25 and sell the real one by earning \$9.00. Hence, the present value of the profit is

$$\$9 - \$8.871 = \$0.129.$$

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Chapter 10. Binomial Option Pricing: Basic Concepts

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

- r: continuously compounded annual interest rate.
- δ: continuously dividend yield.
- σ: annual volatility.
- h: the length of a binomial period in years.

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Continuously Compounded Returns

$$r_{t,t+h} = \ln\left(S_{t_h}/S_t\right)$$

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$

Go over 3 examples on p. 301

Volatility

The volatility of an asset is the standard deviation of continuously compounded returns.

- A year is dividend into n periods (say, $n = 12$) of length $h = 1/n$.
- Let σ^2 be the annual continuously compounded return.
- Assuming that the continuously compounded returns are independent and identically distributed
- We have

$$\sigma^2 = 12 \times \sigma^2_{\text{monthly}}$$

and

$$\sigma_h = \sigma\sqrt{h} \quad \text{or} \quad \sigma = \frac{\sigma_h}{\sqrt{h}}.$$

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Constructing u and d

With no volatility

$$S_{t+h} = F_{t,t+h} = S_t e^{(r-\delta)h}$$

With volatility

$$uS_t = F_{t,t+h} e^{+\sigma\sqrt{h}}$$

$$dS_t = F_{t,t+h} e^{-\sigma\sqrt{h}}$$

$$\Downarrow$$

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

Estimating Historical Volatility

TABLE 10.1 Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

	S&P 500		IBM	
Date	Price	$\ln(S_t/S_{t-1})$	Price	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	−0.02363	125.27	−0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	−0.03408	129.83	−0.01103
8/18/2010	1094.16	0.00430	129.39	−0.00338
8/25/2010	1055.33	−0.03613	125.27	−0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
Standard deviation	0.02800		0.02486	
Standard deviation $\times \sqrt{52}$	0.20194		0.17926	

- Volatility computation should exclude dividend.
- But since dividends are small and infrequent; the standard deviation will be similar whether you exclude dividends or not when computing the standard deviation.

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One-period Example with a Forward Tree

Example 10.2-1 Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%. The continuously compounded risk-free interest rate is 8%.

Use these inputs to calculate the followings:

1. the final stock prices uS and dS
2. the final option values C_u and C_d
3. Δ and B
4. the option price: $\Delta S + B$.

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Solution. In summary:

$$S = 41,\ K = 40,\ r = 0.08,\ \delta = 0,\ \sigma = 0.30,\ h = 1.$$

$uS = \$59.954$
$C_u = \$19.954$

$S = \$41.000$
Option price $= \$7.839$
$\Delta = 0.738$
$B = -\$22.405$

$dS = \$32.903$
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Questions

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- How to price put options?
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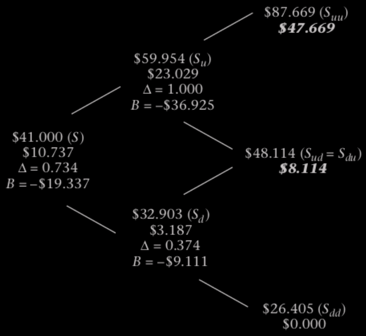
Chapter 10. Binomial Option Pricing: Basic Concepts

FIGURE 10.4

Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 2.00$ years, $\delta = 0.00$, and $h = 1.000$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



Some observations:

- The option price is greater for the 2-year than for the 1-year option
- The option was priced by working backward through the binomial tree.
- The option's Δ and B are different at different nodes. At a given point in time, Δ increases to 1 as we go further into the money
- Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than S–K; hence, we would not exercise even if the option had been American.

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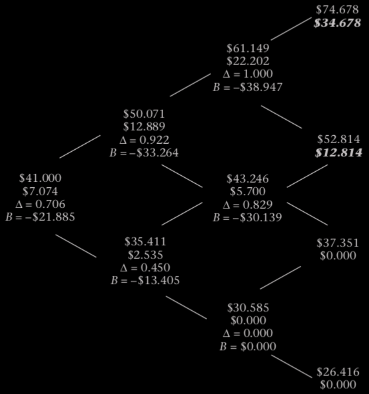
Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration.

FIGURE 10.5

Binomial tree for pricing a European call option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



Chapter 10. Binomial Option Pricing: Basic Concepts

We compute put option prices using the same stock price tree and in almost the same way as call option prices

The only difference with a European put option occurs at expiration
Instead of computing the price as

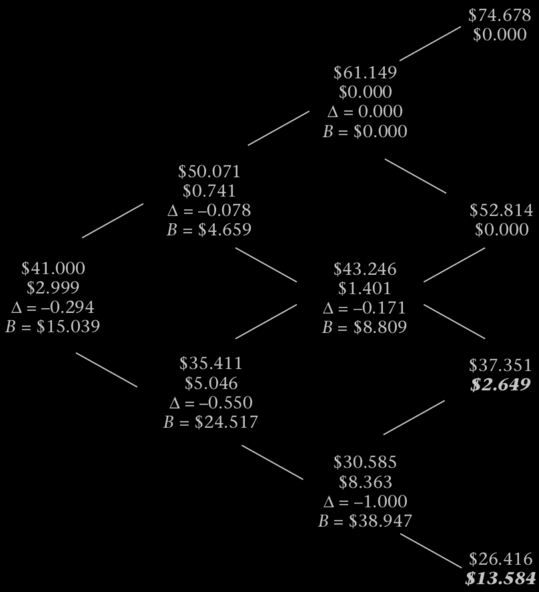
$$\max(0, S - K)$$

we use

$$\max(0, K - S)$$

FIGURE 10.6

Binomial tree for pricing a European put option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



Chapter 10. Binomial Option Pricing: Basic Concepts

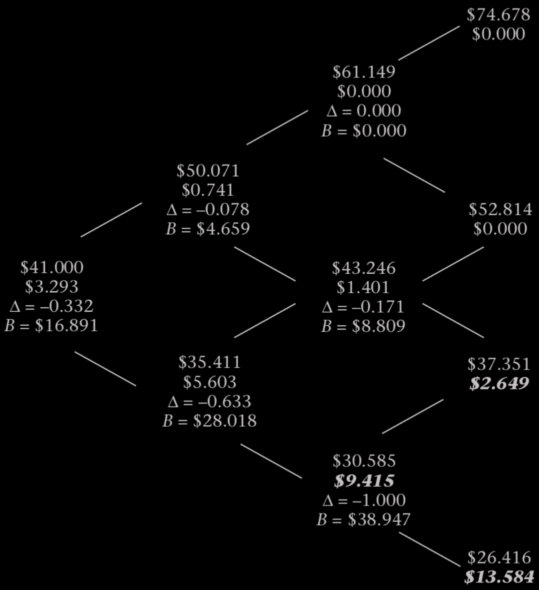
At each node we use the following formula to compute the price:

$$P(S, K, t) = \max\left(K - S, e^{-rh}\left[P(uS, K, t+h)p^* + P(dS, K, t+h)(1 - p^*)\right]\right)$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

FIGURE 10.7

Binomial tree for pricing an American put option; assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $T = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. At each node the stock price, option price, Δ, and B are given. Option prices in ***bold italic*** signify that exercise is optimal at that node.



Chapter 10. Binomial Option Pricing: Basic Concepts

This section is left for motivated students.

Chapter 10. Binomial Option Pricing: Basic Concepts

Problems: 10.1, 10.2, 10.3, 10.6, 10.7, 10.8, 10.10, 10.12,

Due Date: TBA