Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 5. Financial Forwards and Futures

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§ 5.1 Alternative ways to buy a stock

§ 5.2 Prepaid forward contracts on stock

§ 5.3 Forward contracts on stock

§ 5.4 Futures contracts

§ 5.5 Problems

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Chapter 5. Financial Forwards and Futures

Four different payment and receipt timing combinations

1. Outright purchase: ordinary transaction
2. Fully leveraged purchase: investor borrows the full amount
3. Prepaid forward contract: pay today, receive the share later
4. Forward contract: agree on price now, pay/receive later

	Day 0	Day T	Payment
Outright purchase	pay+receive	—	S_0
Fully leveraged purchase	receive	pay	$S_0 e^{rT}$
Prepaid forward contract	pay	receive	?
Forward contract	—	pay+receive	$? \times e^{rT}$

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Three ways to determine the payment for the prepaid forward contracts (no dividend case)

- Pricing the prepaid forward by analogy
- Pricing the prepaid forward by discounted present value
- Pricing the prepaid forward by arbitrage

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Pricing the prepaid forward by analogy

In the absence of dividends, whether you receive physical possession today or at time T is irrelevant: In either case you own the stock, and at time T it will be exactly as if you had owned the stock the whole time. Hence,

$$F^p_{0,T} = S_0$$

Pricing the prepaid forward by discounted present value

Let α be risk-adjusted discount rate.
Let $\mathbb{E}_0(S_T)$ be the expected stock price at time T.
Hence,

$$F^p_{0,T} = \underbrace{\mathbb{E}_0(S_T)}_{=S_0 \times e^{\alpha T}} \times e^{-\alpha T} = S_0$$

Pricing the prepaid forward by arbitrage

Arbitrage = Free money

The price of a derivative should be such that

no arbitrage is possible.

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1. If $F^p_{0,T} > S_0$: find the arbitrage.

2. If $F^p_{0,T} < S_0$: find the arbitrage.

   Hence, $F^p_{0,T} = S_0$.

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Pricing prepaid forwards with dividends – Discrete dividends

Suppose a stock is expected to make dividend payments of D_{t_i} at time t_i, $i = 1, \cdots, n$. Then

$$F^P_{0,T} = S_0 - \sum_{i=1}^{n} \mathrm{PV}_{0,t_i}\left(D_{t_i}\right),$$

where $\mathrm{PV}_{0,t}(\cdot)$ is the present value at time zero of a time t_j payment.

Example 5.2-1 Suppose XYZ stock costs \$100 today and is expected to pay a \$1.25 quarterly dividend, with the first coming 3 months from today and the last just prior to the delivery of the stock. Suppose the annual continuously compounded risk-free rate is 10%. The quarterly continuously compounded rate is therefore 2.5%. Find a 1-year prepaid forward contract for the stock would cost.

Solution.

$$F_{0,1}^{T} = \$100 - \sum_{i=1}^{4} \$1.25 \times e^{-0.025i} = \$93.30.$$

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Pricing prepaid forwards with dividends – Continuous dividends

Let δ be the compounded dividend yield. Then

$$F^P_{0,T} = S_0 e^{-\delta T}$$

Example 5.2-2 Suppose that the index is \$125 and the annualized daily compounded dividend yield is 3%. Find the prepaid forward price at one year.

Solution.

$$F^p_{0,1} = \$125 e^{-0.03\times 1} = \$121.306.$$

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Problems: 5.2, 5.3, 5.4, 5.5, 5.8, 5.10, 5.11, 5.12, 5.16, 5.20.

Due Date: TBD