Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 11. Binomial Option Pricing: Selected Topics

Risk-Neutral Probability

Recall the binomial option pricing formula:

$$C = \Delta S + B = e^{-rh}\left[p^* C_u + (1 - p^*)C_d\right]$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$ $\sim$ risk-neutral probability that the stock will go up

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$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \iff p^* uSe^{\delta h} + (1 - p^*)dSe^{\delta h} = e^{rh}S$$

Two offers:

(a) \$1000 cash

(b) \$2000 or \$0 cash with probability 1/2 for each

Both offers have the same expected return, while (b) bears risk and (a) does not.

A risk-averse investor prefers (a).

A risk-neutral investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing. Hence, he/she prefers equally to (a) and (b).

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The option pricing formula can be said to price options
as if investors are risk-neutral

Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return.

Pricing an option using real probability

- Suppose that the continuously compounded expected return on the stock is α and that the stock does not pay dividends.
- If p is the true probability of the stock going up, p must be consistent with u, d and α

$$puS + (1-p)dS = e^{\alpha h}S$$

- Solving for p gives us

$$p = \frac{e^{\alpha h} - d}{u - d}$$

- For p to be a probability, we have to have $u \geq e^{\alpha h} \geq d$.
- Using this p, the actual expected payoff to the option one period is

$$pC_u + (1-p)C_d = \frac{e^{\alpha h} - d}{u - d}C_u + \frac{u - e^{\alpha h}}{u - d}C_d.$$

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At what rate do we discount this expected payoff?

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It is not correct to discount the option at the expected return on the stock, α, because the option is equivalent to a leveraged investment in the stock and hence is riskier than the stock

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- Denote the appropriate per-period discount rate for the option as γ
- Since an option is equivalent to holding a portfolio consisting of Δ shares of stock and B bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B}e^{\alpha h} + \frac{B}{S\Delta + B}e^{rh}$$

- Hence, the discounted at this appropriate discount rate, the price for the option should be

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u-d}C_u + \frac{u - e^{\alpha h}}{u-d}C_d\right]$$

- By setting $\alpha = r$, one obtains the simplest pricing procedure.
- This gives an alternative way to compute the option price, instead of $\Delta S + B$.

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One can use either

$$C = \Delta S + B$$

or

$$C = e^{-\gamma h}\left[\frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d\right]$$

to compute the option price

- **First equation** is more efficient
- For the **second one**, in order to compute γ, one needs to computer Δ and B first and then obtains γ via

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{rh}$$

Interested students should check out two examples on p. 328 – 330 using the second formula.

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