

Financial Mathematics

MATH 5870/6870¹
Fall 2021

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¹Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 20. Brownian Motion and Ito Lemma

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§ 20.1 The Black-Scholes assumption about stock prices

§ 20.2 Brownian motion

§ 20.3 Geometric Brownian motion

§ 20.4 The Ito formula

§ 20.5 The Sharpe ratio

§ 20.6 Risk-neutral valuation

§ 20.7 Problems

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0. \quad (1)$$

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- ▶ $S(t)$ is the **stock price**. $dS(t)$ is the instantaneous change in the stock price. S_0 is the initial asset value.
- ▶ α is the continuously compound expected return on the stock;
- ▶ σ is the volatility, i.e., the standard deviation of the instantaneous return;
- ▶ $Z(t)$ is the standard Brownian motion.
- ▶ $dZ(t)$ requires rigorous justification.

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 - ▶ Solution to this specific equation is the geometric Brownian motion.

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- ▶ Equation of this type is called **stochastic differential equation**.
 - ▶ Solution to this specific equation is the **geometric Brownian motion**.

Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(\mathcal{S}(t)) \sim N \left(\ln(\mathcal{S}_0) + \left(\alpha - \delta - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right), \quad \text{for all } t > 0.$$

Remark 20.1-2 Note that Remark 20.1-1 is valid for all $t > 0$. It works for the terminal time $t = T$. It can also help us solve path-dependent options.

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Definition 20.2-1 A real-valued stochastic process $Z(t)$ is called a **Brownian motion** or **Wiener process** if

1. It starts at 0:

$$Z(0) = 0.$$

2. For $0 \leq s < t$, the increment $Z(t) - Z(s)$ is normally distributed with mean zero and variance $t - s$:

$$Z(t) - Z(s) \sim N(0, t - s).$$

3. Its increments are independent: if

$$0 \leq t_0 \leq t_1 \leq \cdots \leq t_k,$$

then

$$\mathbb{P}(Z(t_i) - Z(t_{i-1}) \in H_i, 1 \leq i \leq k) = \prod_{i=1}^k \mathbb{P}(Z(t_i) - Z(t_{i-1}) \in H_i).$$

Remark 20.2-1 One can always construct a continuous version of the Brownian motion; from now on, we always assume that Brownian motion is a continuous process.

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Remark 20.2-1 One can always construct a **continuous version** of the Brownian motion; from now on, we always assume that Brownian motion is a continuous process.

Theorem 20.2-1 (Some properties of Brownian motion)

1. $Z(t)$ is nowhere differentiable.

(Hence, $dZ(t)$ requires some special treatment.)

2. $Z(t)$ satisfies the scaling property:

$$\tilde{Z}(t) := \frac{1}{\sqrt{c}}Z(ct) \text{ is also a B.M. for all } c > 0.$$

3. $Z(t)$ is a martingale, namely,

$$\mathbb{E}(Z(t+s)|Z(t)) = Z(t).$$

4. For any $t > 0$, $Z(t) \sim N(0, t)$ and

$$\mathbb{E}(Z(t)Z(s)) = \min(t, s) \quad \text{for all } t, s \geq 0.$$

5. $Z(t)$ is translation invariant, namely,

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Proof. Part (1) goes beyond this course. All the rest could be proved using our current knowledge.



Arithmetic Brownian motion

Definition 20.2-2 Let $Z(t)$ be a B.M. Then the process $X(t)$ given by

$$dX(t) = \alpha dt + \sigma dZ(t)$$

is called an **arithmetic Brownian motion**. Equivalently, $X(t)$ can be written in the following integral representation:

$$X(t) = X(0) + \int_0^t \alpha ds + \int_0^t \sigma dZ(s).$$

Remark 20.2-2

1. $X(t)$ is normally distributed:

$$X(t) = \sigma t + \sigma Z(t) \sim N(\sigma t, \sigma^2 t).$$

2. $X(t)$ takes both positive and negative values almost surely.
3. αt is a drift term.

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3. σt is a drift term.

The Ornstein-Uhlenbeck process

Definition 20.2-3 Let $Z(t)$ be a B.M. Then the process $X(t)$ given by

$$dX(t) = \lambda (\alpha - X(t)) dt + \sigma dZ(t)$$

is called the **Ornstein-Uhlenbeck process**.

Remark 20.2-3 Equivalently, $X(t)$ can be written in the following integral representation:

$$X(t) = X(0) + \lambda \int_0^t (\alpha - X(s)) ds + \int_0^t \sigma dZ(s),$$

which is an integral equation (unknown X appears on both sides).

Remark 20.2-4 We have introduced **mean reversion** in the drift term.

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Problems: 20.1, 20.2, 20.3, 20.4, 20.5, 20.6, 20.7, 20.8, 20.9, 20.10, 20.11, 20.12.

Due Date: TBA