Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 18. The Lognormal Distribution

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Definition 18.1-1 A random variable X is said to have the *normal distribution* (or *normally distributed*) with mean μ and variance σ^2, if the probability density function (pdf) is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

We write

$$X \sim N(\mu, \sigma^2).$$

$N(0, 1)$ is called the *standard normal distribution*.

Definition 18.1-2 The *cumulative distribution function (cdf) of the standard normal distribution* is denoted by $\Phi(\cdot)$, namely,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy.$$

Let $Z \sim N(0,1)$ and $a > 0$. By symmetry of the density, we have the following useful formulas:

1.

$$\mathbb{P}(-a \leq Z \leq a) = 2\Phi(a) - 1.$$

For example,

$$\mathbb{P}(|Z| \leq 0.3) = 2 \cdot \Phi(0.3) - 1 = 2 \times 0.6179 - 1 = 0.2358.$$

2.

$$\mathbb{P}(Z > a) = \mathbb{P}(Z < -a) = \Phi(-a) = 1 - \Phi(a)$$

3.

$$\mathbb{P}(0 < Z \leq a) = \mathbb{P}(Z < a) - \mathbb{P}(Z \leq 0) = \Phi(a) - \frac{1}{2}.$$

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Standardization

$$X \sim N\left(\mu, \sigma^2\right) \quad \Longleftrightarrow \quad Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

or equivalently

$$Z \sim N\left(0, 1\right) \quad \Longleftrightarrow \quad X = \mu + \sigma Z \sim N(\mu, \sigma^2)$$

Example 18.1-1 Assume the lifetime of a brand of light bulb follows a normal distribution with mean of 4 years and standard deviation of 0.4 years. What is the probability that it stop working before 5 years.

Solution. Let X be the lifetime of the light bulb. Then

$$X \sim N\left(4, 0.4^2\right) \quad \Longrightarrow \quad \begin{aligned} \mathbb{P}(X \le 5) &= \mathbb{P}\left(\frac{X-4}{0.4} \le 2.5\right) \\ &= \Phi(2.5) \\ &= 0.99379. \end{aligned}$$

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Sums of normal random variables

Suppose we have n jointly distributed random variables X_i, $i = 1, \ldots, n$, with mean and variance $\mathbb{E}(X_i) = \mu_i$, $\mathrm{Var}(X_i) = \sigma_i^2$, and covariance $\mathrm{Cov}(X_i, X_j) = \sigma_{ij}$.

Remark 18.1-1

1. The covariance between two random variables measures their tendency to move together.
2. $\mathrm{Var}(X_i) = \mathrm{Cov}(X_i, X_i)$.
3. Let ρ_{ij} be the *correlation coefficient* of X_i and X_j, namely,

$$\rho_{ij} := \frac{\sigma_{ij}}{\sigma_i \sigma_j}.$$

   Hence, $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

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Theorem 18.1-1 The weighted random variable $\sum_{i=1}^{n} a_i X_i$ have the following mean and variance:

$$\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \mu_i,$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}.$$

Theorem 18.1-2 If $X_i \sim N\left(\mu_i, \sigma_i^2\right)$, then

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}\right)$$

Example 18.1-2 If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, give the distribution of $aX_1 + bX_2$.

Solution.

$$aX_1 + bX_2 \sim N\left(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\,\sigma_{12}\right).$$

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Central Limit Theorem

Measurements lead to error. The error tends to be independent and is usually modeled by zero mean normal random variables thanks to the central limit theorem (CLT).

CLT usually can be phrased under different conditions. Here is one example:

Theorem 18.1-3 (Linderberg-Lévy CLT) Suppose $\{X_1, \cdots, X_n\}$ is a sequence of independent random variables having the same distribution (i.e., i.i.d.) with $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Then

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1), \qquad \text{as } n \to \infty,$$

where

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n}.$$

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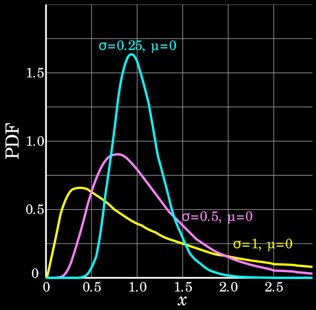
Chapter 18. The Lognormal Distribution

Definition 18.2-1 A random variable Y is *lognormally distributed* with parameters μ and $\sigma > 0$ if $\ln(Y) \sim N(\mu, \sigma^2)$.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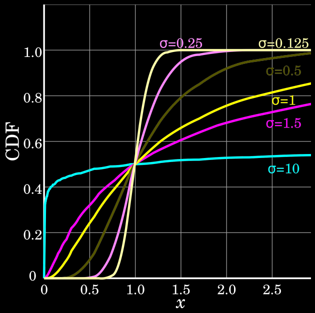
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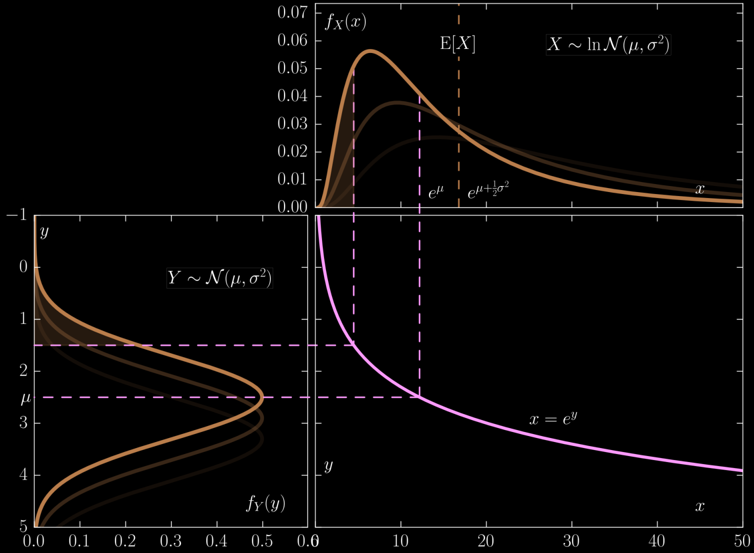

[2]

Image from Wikipedia.[2]

Theorem 18.2-1 If Y_1 and Y_2 are lognormally distributed, so is $Y_1 Y_2$.

Proof

Since Y_1 and Y_2 are lognormally distributed, $\ln(Y_1)$ and $\ln(Y_2)$ are normally distributed. Hence,

$$\ln(Y_1) + \ln(Y_2) = \ln(Y_1 Y_2)$$

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Theorem 18.2-2 If $\ln(Y) \sim N(\mu, \sigma^2)$, then the density of Y is given by

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.$$

Proof. For $y > 0$,

$$\begin{aligned}
\mathbb{P}(Y \le y) &= \mathbb{P}\left(\ln(Y) \le \ln(y)\right) \\
&= \mathbb{P}\left(\frac{\ln(Y) - \mu}{\sigma} \le \frac{\ln(y) - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy}\Phi\left(\frac{\ln(y) - \mu}{\sigma}\right) \\
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\end{aligned}$$

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Recall that the moment generating function of the normal random variable $X \sim N(\mu, \sigma^2)$ is

$$\mathbb{E}\left(e^{tX}\right) = e^{\mu t + \sigma^2 t^2/2}, \qquad \text{for all } t \in \mathbb{R}. \tag{1}$$

Remark 18.2-1 If Y is lognormally distributed with parameters μ and σ, then

$$\mathbb{E}\left(Y^t\right) = e^{\mu t + \sigma^2 t^2/2}, \qquad \text{for all } t \in \mathbb{R}.$$

Remark 18.2-2 By *Jensen's inequality*, if g is a convex function, then

$$\mathbb{E}\left(g(X)\right) \leq g\left(\mathbb{E}(X)\right).$$

Hence, for $g(x) = e^x$, we see that

$$\mathbb{E}\left(e^X\right) \leq e^{\mathbb{E}(X)}.$$

This is consistent with our computations above because

$$LHS = e^{\mu + \sigma^2/2} \geq e^{\mu} = RHS.$$

Theorem 18.2-3 If Y is lognormally distributed such that $\ln(Y) \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2} \quad \text{and} \quad \mathrm{Var}(Y) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

Proof. Let $X = \ln(Y)$. By (1),

$$\mathbb{E}\left(Y\right) = \mathbb{E}\left(e^{X}\right) = e^{\mu+\frac{1}{2}\sigma^2}$$

and

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Chapter 18. The Lognormal Distribution

- Let $R(t, s)$ be the continuously compounded return from time t to a later time s.
- For $t_0 < t_1 < t_2$, $R(\cdot, \cdot)$ has to satisfy the additivity property:

$$R(t_0, t_2) = R(t_0, t_1) + R(t_1, t_2)$$

- For time interval $[0, T]$ divided into n subintervals of equal length T/n, we have

$$R(0, T) = R(0, h) + R(h, 2h) + \cdots + R((n-1)h, T)$$

Assume that

$$\mathbb{E}\left(R((i-1)h, ih)\right) = \alpha_h \quad \text{and} \quad \mathrm{Var}\left(R((i-1)h, ih)\right) = \sigma_h^2$$

Then

$$\mathbb{E}\left(R(0, T)\right) = n\alpha_h \quad \text{and} \quad \mathrm{Var}\left(R(0, T)\right) = n\sigma_h^2$$

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Then

$$\mathbb{E}\left(R(0, T)\right) = n\alpha_h \quad \text{and} \quad \mathrm{Var}\left(R(0, T)\right) = n\sigma_h^2$$

- By central limit limit theorem, as $n \to \infty$, one can assume that

$$R(0, T) \sim N$$

$$\ln(S_t/S_0) \sim N\left([\alpha - \delta - 0.5\sigma^2]t, \sigma^2 t\right)$$

$$\ln(S_t/S_0) = [\alpha - \delta - 0.5\sigma^2]t + \sigma\sqrt{t}\, Z$$

$$S_t = S_0 e^{[\alpha-\delta-0.5\sigma^2]t} e^{\sigma\sqrt{t}\, Z}$$

$$\mathbb{E}[S_t] = S_0 e^{[\alpha-\delta]t} \quad \text{and} \quad \text{Median stock price} = e^{[\alpha-\delta-0.5\sigma^2]t}$$

$$\text{One standard deviation} \begin{cases} \text{move up} = e^{[\alpha-\delta-0.5\sigma^2]t+\sigma\sqrt{t}\times 1} \\ \text{move down} = e^{[\alpha-\delta-0.5\sigma^2]t-\sigma\sqrt{t}\times 1} \end{cases}$$

Go over examples 18.4 and 18.5 on textbook on p. 555.

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Theorem 18.4-1

$$\mathbb{P}\left(S_t < K\right) = N\left(-d_2\right) \qquad \text{with} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Or equivalently, $\mathbb{P}(S_t > d_2) = N(d_2)$.

Theorem 18.4-2 The $(1-p)\times 100\%$ prediction interval for S_t is $\left(S_t^L, S_t^U\right)$ with

$$S_t^L = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}\,N^{-1}(p/2)}$$

$$S_t^U = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}\,N^{-1}(1-p/2)}$$

Go over Examples 18.6 and 18.7 on P. 558-559.

Theorem 18.4-3 It holds that

$$\mathbb{E}(S_t | S_t < K) = S_0 e^{(\alpha-\delta)t} \frac{N(-d_1)}{N(-d_2)}$$

$$\mathbb{E}(S_t | S_t > K) = S_0 e^{(\alpha-\delta)t} \frac{N(+d_1)}{N(+d_2)}$$

where recall that

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Now we are ready to derive the Black-Scholes formula:

$$
\begin{aligned}
C(S, K, \sigma, r, t, \delta) &= e^{-rt}\mathbb{E}\left([S_t - K]\, 1_{\{S_t > K\}}\right) \\
&= e^{-rt}\mathbb{E}\left([S_t - K]\,|S_t > K\right)\mathbb{P}(S_t > K) \\
&= e^{-rt}\mathbb{E}\left(S_t|S_t > K\right)\mathbb{P}(S_t > K) + e^{-rt}\mathbb{E}\left(K|S_t > K\right)\mathbb{P}(S_t > K) \\
&= e^{-rt}\mathbb{E}\left(S_t|S_t > K\right)\mathbb{P}(S_t > K) + e^{-rt}K\mathbb{P}(S_t > K) \\
&\ \ \vdots
\end{aligned}
$$

Similar for put.

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Chapter 18. The Lognormal Distribution

Problems: 18.3, 18.4, 18.5, 18.6, 18.7, 18.8, 18.9, 18.10, 18.11, 18.12.

Due Date: TBA