# Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 10. Binomial Option Pricing: Basic Concepts

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$$d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

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- $\delta$: continuously dividend yield.
- $\sigma$: annual volatility.
- $h$: the length of a binomial period in years.

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# Continuously Compounded Returns

$$r_{t,t+h} = \ln\left(S_{t_h}/S_t\right)$$

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$

---

Go over 3 examples on p. 301

# Volatility

The volatility of an asset is the standard deviation of continuously compounded returns.

- A year is dividend into $n$ periods (say, $n = 12$) of length $h = 1/n$.
- Let $\sigma^2$ be the annual continuously compounded return.
- Assuming that the continuously compounded returns are independent and identically distributed
- We have

$$\sigma^2 = 12 \times \sigma^2_{\text{monthly}}$$

and

$$\sigma_h = \sigma\sqrt{h} \quad \text{or} \quad \sigma = \frac{\sigma_h}{\sqrt{h}}.$$

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# Constructing $u$ and $d$

With no volatility

$$S_{t+h} = F_{t,t+h} = S_t e^{(r-\delta)h}$$

---

With volatility

$$uS_t = F_{t,t+h} e^{+\sigma\sqrt{h}}$$

$$dS_t = F_{t,t+h} e^{-\sigma\sqrt{h}}$$

$$\Downarrow$$

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$

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# Estimating Historical Volatility

**TABLE 10.1** Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

| | S&P 500 | | IBM | |
|---|---|---|---|---|
| Date | Price | $\ln(S_t/S_{t-1})$ | Price | $\ln(S_t/S_{t-1})$ |
| 7/7/2010 | 1060.27 | | 127 | |
| 7/14/2010 | 1095.17 | 0.03239 | 130.72 | 0.02887 |
| 7/21/2010 | 1069.59 | −0.02363 | 125.27 | −0.04259 |
| 7/28/2010 | 1106.13 | 0.03359 | 128.43 | 0.02491 |
| 8/4/2010 | 1127.24 | 0.01890 | 131.27 | 0.02187 |
| 8/11/2010 | 1089.47 | −0.03408 | 129.83 | −0.01103 |
| 8/18/2010 | 1094.16 | 0.00430 | 129.39 | −0.00338 |
| 8/25/2010 | 1055.33 | −0.03613 | 125.27 | −0.03238 |
| 9/1/2010 | 1080.29 | 0.02338 | 125.77 | 0.00398 |
| 9/8/2010 | 1098.87 | 0.01705 | 126.08 | 0.00246 |
| Standard deviation | 0.02800 | | 0.02486 | |
| Standard deviation $\times \sqrt{52}$ | 0.20194 | | 0.17926 | |

- Volatility computation should exclude dividend.
- But since dividends are small and infrequent; the standard deviation will be similar whether you exclude dividends or not when computing the standard deviation.

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# One-period Example with a Forward Tree

Example 10.2-1 Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%. The continuously compounded risk-free interest rate is 8%.

Use these inputs to calculate the followings:

1. the final stock prices $uS$ and $dS$
2. the final option values $C_u$ and $C_d$
3. $\Delta$ and $B$
4. the option price: $\Delta S + B$.

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Solution. In summary:

$$S = 41,\ K = 40,\ r = 0.08,\ \delta = 0,\ \sigma = 0.30,\ h = 1.$$

$uS = \$59.954$
$C_u = \$19.954$

$S = \$41.000$
Option price $= \$7.839$
$\Delta = 0.738$
$B = -\$22.405$

$dS = \$32.903$
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