

# Financial Mathematics

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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

## Chapter 12. The Black-Scholes Formula

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§ 12.1 Introduction to the Black-Scholes formula

§ 12.2 Applying the formula to other assets

§ 12.3 Option Greeks

§ 12.4 Problems

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## What happens to the option price when one and only one input changes?

- ▶ Delta ( $\Delta$ ): change in option price when stock price increases by \$1
- ▶ Gamma ( $\Gamma$ ): change in delta when option price increases by \$1
- ▶ Vega: change in option price when volatility increases by 1%
- ▶ Theta ( $\theta$ ): change in option price when time to maturity decreases by 1 day
- ▶ Rho ( $\rho$ ): change in option price when interest rate increases by 1%
- ▶ Psi ( $\psi$ ): change in the option premium due to a change in the dividend yield

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- ▶ The Greek measure of a portfolio is weighted average of Greeks of individual portfolio components

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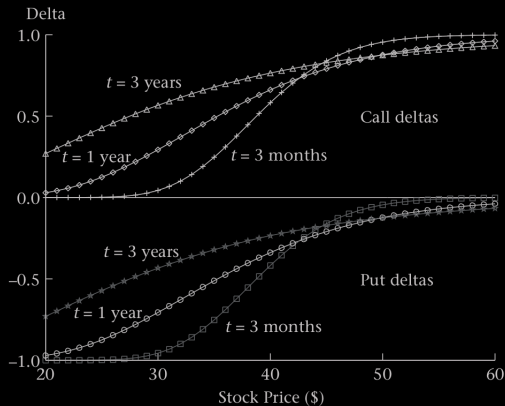
## Delta

Delta ( $\Delta$ ): change in option price when stock price increases by \$1.

$$\Delta = \begin{cases} \frac{\partial C(S, K, \sigma, T - t, \delta)}{\partial S} = +e^{-\delta(T-t)} N(+d_1) & \text{Call} \\ \frac{\partial P(S, K, \sigma, T - t, \delta)}{\partial S} = -e^{-\delta(T-t)} N(-d_1) & \text{Put} \end{cases}$$

**FIGURE 12.1**

Call (top graph) and put (bottom graph) deltas for 40-strike options with different times to expiration. Assumes  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .



## Gamma and Vega

**Gamma ( $\Gamma$ ):** change in delta when option price increases by \$1

$$\Gamma = \frac{\partial^2 C(S, K, \sigma, r, T - t, \delta)}{\partial S^2} = \frac{\partial^2 P(S, K, \sigma, r, T - t, \delta)}{\partial S^2} = \frac{e^{-d(T-t)} N'(d_1)}{S\sigma\sqrt{T-t}}$$

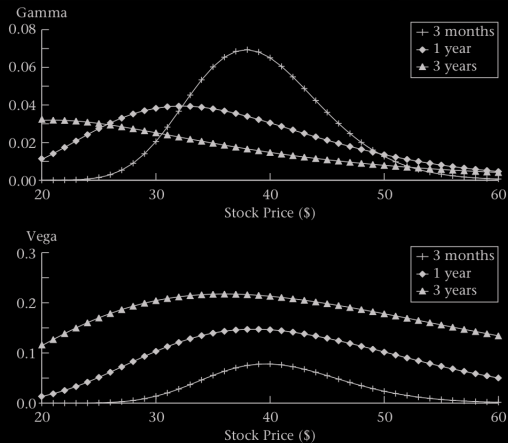
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**Vega:** change in option price when volatility increases by 1%

$$\text{Vega} = \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial \sigma} = \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial \sigma} = Se^{-\delta(T-t)} N'(d_1) \sqrt{T-t}$$

**FIGURE 12.2**

Gamma (top panel) and vega (bottom panel) for 40-strike options with different times to expiration. Assumes  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ . Vega is the sensitivity of the option price to a 1 percentage point change in volatility. Otherwise identical calls and puts have the same gamma and vega.



## Theta

Theta ( $\theta$ ): change in option price when time to maturity decreases by 1 day

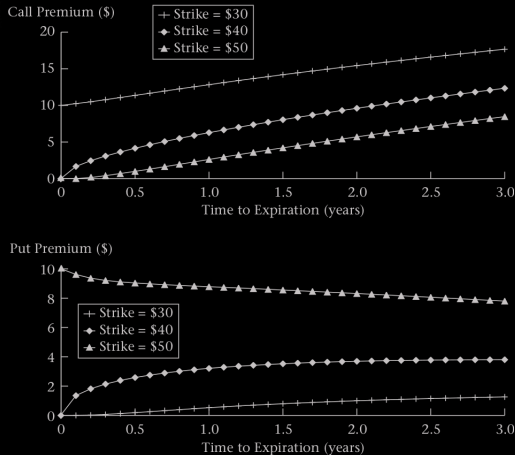
$$\begin{aligned}\text{Call } \theta &= \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial t} \\ &= \delta S e^{-\delta(T-t)} N(d_1) - r K e^{-r(T-t)} N(d_2) - \frac{K e^{r(T-r)} N'(d_2) \sigma}{2\sqrt{T-t}}\end{aligned}$$

$$\begin{aligned}\text{Put } \theta &= \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial t} \\ &= \text{Call } \theta + r K e^{-r(T-t)} + \delta S e^{-\delta(T-t)}\end{aligned}$$



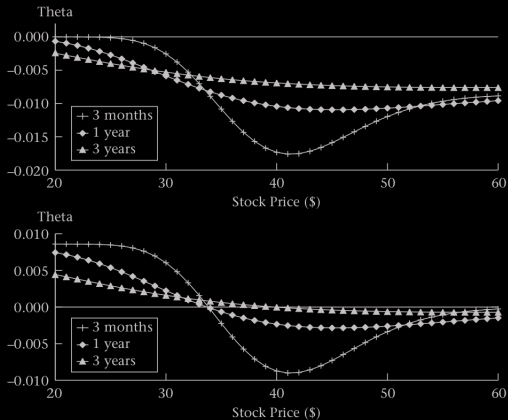
**FIGURE 12.3**

Call (top panel) and put (bottom panel) prices for options with different strikes at different times to expiration. Assumes  $S = \$40$ ,  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .



**FIGURE 12.4**

Theta for calls (top panel) and puts (bottom panel) with different expirations at different stock prices. Assumes  $K = \$40$ ,  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .



## Rho and Psi

**Rho ( $\rho$ ):** change in option price when interest rate increases by 1%

$$\text{Call } \rho = \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial r} = +(T - t)Ke^{-r(T-t)}N(+d_2)$$

$$\text{Put } \rho = \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial r} = -(T - t)Ke^{-r(T-t)}N(-d_2)$$

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**Psi ( $\psi$ ):** change in the option premium due to a change in the dividend yield

$$\text{Call } \psi = \frac{\partial C(S, K, \sigma, r, T - t, \delta)}{\partial \delta} = -(T - t)Ke^{-\delta(T-t)}N(+d_1)$$

$$\text{Put } \psi = \frac{\partial P(S, K, \sigma, r, T - t, \delta)}{\partial \delta} = +(T - t)Ke^{-\delta(T-t)}N(-d_1)$$

**FIGURE 12.5**

Rho (top panel) and psi (bottom panel) at different stock prices for call options with different maturities. Assumes  $K = \$40$ ,  $\sigma = 30\%$ ,  $r = 8\%$ , and  $\delta = 0$ .

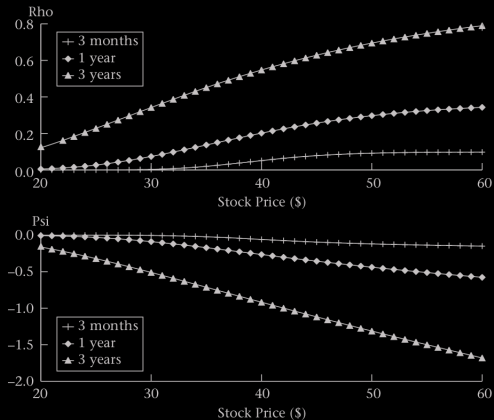


TABLE 12.1

Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price  $S = \$41$ , the strike price  $K = \$40$ , volatility  $\sigma = 0.30$ , risk-free rate  $r = 0.08$ , time to expiration  $T = 1$ , and dividend yield  $\delta = 0$ .

Number of Steps (n)	Binomial Call Price (\$)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
$\infty$	6.961

Delta ( $\Delta$ ): change in option price when stock price increases by \$1

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Option Elasticity ( $\Omega$ ): If stock price  $S$  changes by 1%, what is the percentage change in the value of the option  $C$ :

$$\Omega = \frac{\text{Percentage change in option price}}{\text{Percentage change in stock price}} = \frac{\frac{\epsilon \Delta}{C}}{\frac{\epsilon}{S}} = \frac{S \Delta}{C}.$$