# Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 12. The Black-Scholes Formula

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The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.

**TABLE 12.1** Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

| Number of Steps (n) | Binomial Call Price ($) |
|---|---|
| 1 | 7.839 |
| 4 | 7.160 |
| 10 | 7.065 |
| 50 | 6.969 |
| 100 | 6.966 |
| 500 | 6.960 |
| $\infty$ | 6.961 |

Check Python code Figure12-1.py

- Consider an European call (or put) option written on a stock
- Assume that the stock pays dividend at the continuous rate $\delta$

| Call options | Put options |
|---|---|
| $C(S, K, \sigma, r, T, \delta)$ | $P(S, K, \sigma, r, T, \delta)$ |
| $\vert\vert$ | $\vert\vert$ |
| $Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$ | $Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$ |

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Put-call Parity

$$P = C + Ke^{-rT} - Se^{-\delta T}$$

$$d_1 - d_2 = \sigma\sqrt{T}$$

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$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$

Example 12.1-1 Verify that the Black-Scholes formula for call and put

$$C := C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$
$$P := P(S, K, \sigma, r, T, \delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

with

$$d_i = \frac{\ln(S/K) + (r - \delta - (-1)^i \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad i = 1, 2$$

satisfies the call-put parity: $C - P = Se^{-\delta T} - Ke^{-rT}$.

Solution.

$\square$

Example 12.1-2 Let $S = \$41$, $K = \$40$, $\sigma = 0.3$, $r = 8\%$, $T = 0.25$ (3 months), and $\delta = 0$. Compute the Black-Scholes call and put prices. Compare what you obtained with the results obtained from the binomial tree.

Check code

Example12-1.py

# When is the Black-Scholes formula valid?

Assumptions about stock return distribution

- Continuously compounded returns on the stock are normally distributed and independent over time (no "jumps")
- The volatility of continuously compounded returns is known and constant
- Future dividends are known, either as dollar amount or as a fixed dividend yield

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Assumptions about the economic environment

- The risk-free rate is known and constant
- There are no transaction costs or taxes
- It is possible to short-sell costlessly and to borrow at the risk-free rate

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