

# Financial Mathematics

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<sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

## Chapter 20. Brownian Motion and Ito Lemma

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§ 20.1 The Black-Scholes assumption about stock prices

§ 20.2 Brownian motion

§ 20.3 Geometric Brownian motion

§ 20.4 The Ito formula

§ 20.5 The Sharpe ratio

§ 20.6 Risk-neutral valuation

§ 20.7 Problems

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0. \quad (1)$$

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- ▶  $S(t)$  is the **stock price**.  $dS(t)$  is the instantaneous change in the stock price.  $S_0$  is the initial asset value.
- ▶  $\alpha$  is the **continuously compound expected return** on the stock;
- ▶  $\sigma$  is the **volatility**, i.e., the standard deviation of the instantaneous return;
- ▶  $Z(t)$  is the **standard Brownian motion**.
- ▶  $dZ(t)$  requires rigorous justification.

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- ▶ Equation of this type is called **stochastic differential equation**.
  - ▶ Solution to this specific equation is the **geometric Brownian motion**.

**Remark 20.1-1** We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(S(t)) \sim N \left( \ln(S_0) + \left( \alpha - \delta - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right), \quad \text{for all } t > 0.$$

**Remark 20.1-2** Note that Remark 20.1-1 is valid for all  $t > 0$ . It works for the terminal time  $t = T$ . It can also help us solve path-dependent options.

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