Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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Last updated on

November 8, 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 19. Monte Carlo Valuation

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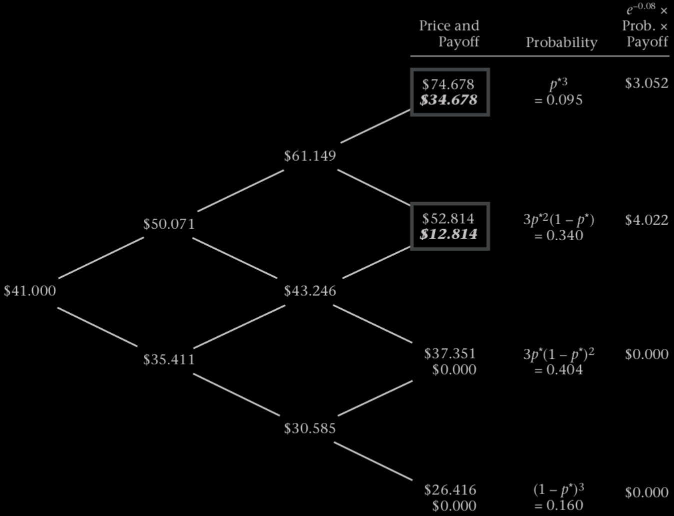
For European call, if one use risk-neutral probability[2], then

$$C = e^{-rT} \sum_{i=0}^{n} \max(Su^{n-i} d^{i} - K, 0) \binom{n}{i} (p^*)^{n-i} (1 - p^*)^{i}$$

[2] One cannot have this simple expression if one uses the true probability.

FIGURE 19.1

Binomial tree (the same as in Figure 10.5) showing stock price paths, along with risk-neutral probabilities of reaching the various terminal prices. Assumes $S = \$41.00$, $K = \$40.00$, $\sigma = 0.30$, $r = 0.08$, $t = 1.00$ years, $\delta = 0.00$, and $h = 0.333$. The risk-neutral probability of going up is $p^* = 0.4568$. At the final node the stock price and terminal option payoff (beneath the price) are given.



Instead of using the formula to compute the option price, one can simulate ...

Example 19.1-1 Write a piece of code to simulate the binomial tree and compute the corresponding average payoff.

Solution. Check

codes/Section_19-1.py

□

Chapter 19. Monte Carlo Valuation

Check out the numpy.random reference[3] :

`https://numpy.org/doc/1.16/reference/routines.random.html`

[3] There is no need to build the wheels by ourselves.

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§ 19.1 Computing the option price as a discounted expected value

§ 19.2 Computing random numbers

§ 19.3 Simulating lognormal stock prices

§ 19.4 Monte Carlo valuation

§ 19.5 Efficient Monte Carlo valuation

§ 19.6 Valuation of American options

$$S_T = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z}$$

$$\begin{aligned}
S_h &= S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}Z_1} \\
S_{2h} &= S_h e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}Z_2} \\
\vdots \quad & \quad \vdots \\
S_{nh} &= S_{(n-1)h} e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}Z_n}
\end{aligned}$$

$$\Downarrow$$

$$S_{nh} = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{h}\sum_{i=1}^{n} Z_i} = S_0 e^{\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)h+\sigma\sqrt{T}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i\right]}$$

where

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i \sim N(0,1)$$

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$$V(S_0, 0) = \frac{1}{n} e^{-rT} \sum_{n=1}^{n} V\left(S_T^i, T\right)$$

where

- $S_T^1, \cdots, S_T^n$ are n randomly drawn time-T stock prices.
- For European Call:

$$V(S_T^i, T) = \max\left(0, S_T^i - K\right)$$

Similarly one finds the expression for European put.

Example 19.4-1 Carry out the Monte Carlo valuation of the European call under the setting of the following table:

TABLE 19.2 Results of Monte Carlo valuation of European call with $S = \$40$, $K = \$40$, $\sigma = 30\%$, $r = 8\%$, $t = 91$ days, and $\delta = 0$. The Black-Scholes price is \$2.78. Each trial uses 500 random draws.

Trial	Computed Price (\$)
1	2.98
2	2.75
3	2.63
4	2.75
5	2.91
Average	2.804

Solution. Check

codes/Table_19-2.py

□

Example 19.4-2 Carry out the Monte Carlo valuation of the Asian call under the setting of the following table:

TABLE 19.3

Prices of arithmetic average-price Asian options estimated using Monte Carlo and exact prices of geometric average price options. Assumes option has 3 months to expiration and average is computed using equal intervals over the period. Each price is computed using 10,000 trials, assuming $S = \$40$, $K = \$40$, $\sigma = 30\%$, $r = 8\%$, $T = 0.25$, and $\delta = 0$. In each row, the same random numbers were used to compute both the geometric and arithmetic average price options. σ_n is the standard deviation of the estimated arithmetic option prices, divided by $\sqrt{10{,}000}$.

Number of	Monte Carlo Prices ($)		Exact	
Averages	Arithmetic	Geometric	Geometric Price ($)	σ_n
1	2.79	2.79	2.78	0.0408
3	2.03	1.99	1.94	0.0291
5	1.78	1.74	1.77	0.0259
10	1.70	1.66	1.65	0.0241
20	1.66	1.61	1.59	0.0231
40	1.63	1.58	1.56	0.0226

Solution. Check

codes/Table_19-3.py

□

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This section will be skipped for the interested students.

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