# Financial Mathematics

MATH 5870/6870[1]
Fall 2021

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Last updated on

August 8, 2021

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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

# Chapter 5. Financial Forwards and Futures

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Four different payment and receipt timing combinations

1. Outright purchase: ordinary transaction
2. Fully leveraged purchase: investor borrows the full amount
3. Prepaid forward contract: pay today, receive the share later
4. Forward contract: agree on price now, pay/receive later

| | Day 0 | Day $T$ | Payment |
|---|---|---|---|
| Outright purchase | pay+receive | — | $S_0$ |
| Fully leveraged purchase | receive | pay | $S_0 e^{rT}$ |
| Prepaid forward contract | pay | receive | ? |
| Forward contract | — | pay+receive | $? \times e^{rT}$ |

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Three ways to determine the payment for the prepaid forward contracts (no dividend case)

- Pricing the prepaid forward by analogy
- Pricing the prepaid forward by discounted present value
- Pricing the prepaid forward by arbitrage

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# Pricing the prepaid forward by analogy

In the absence of dividends, whether you receive physical possession today or at time $T$ is irrelevant: In either case you own the stock, and at time $T$ it will be exactly as if you had owned the stock the whole time. Hence,

$$F^{p}_{0,T} = S_0$$

# Pricing the prepaid forward by discounted present value

Let $\alpha$ be the expected return on the stock.

Let $\mathbb{E}_0(S_T)$ be the expected stock price at time $T$.

Hence,

$$F_{0,T}^{p} = \underbrace{\mathbb{E}_0(S_T)}_{= S_0 \times e^{\alpha T}} \times e^{-\alpha T} = S_0$$

# Pricing the prepaid forward by arbitrage

Arbitrage = Free money

The price of a derivative should be such that

**no arbitrage is possible**.

---

1. If $F^{p}_{0,T} > S_0$: find the arbitrage.
2. If $F^{p}_{0,T} < S_0$: find the arbitrage.

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# Pricing prepaid forwards with dividends – Discrete dividends

Suppose a stock is expected to make dividend payments of $D_{t_i}$ at time $t_i$, $i = 1, \cdots, n$. Then

$$F_{0,T}^{P} = S_0 - \sum_{i=1}^{n} \mathrm{PV}_{0,t_i}\left(D_{t_i}\right),$$

where $\mathrm{PV}_{0,t}(\cdot)$ is the present value at time zero of a time $t_j$ payment.

Example 5.2-1 Suppose XYZ stock costs \$100 today and is expected to pay a \$1.25 quarterly dividend, with the first coming 3 months from today and the last just prior to the delivery of the stock. Suppose the annual continuously compounded risk-free rate is 10%. The quarterly continuously compounded rate is therefore 2.5%. Find a 1-year prepaid forward contract for the stock would cost.

Solution.

$$F^T_{0,1} = \$100 - \sum_{i=1}^{4} \$1.25 \times e^{-0.025i} = \$93.30.$$

$\square$

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# Pricing prepaid forwards with dividends – Continuous dividends

Let $\delta$ be the compounded dividend yield. Then

$$F_{0,T}^{P} = S_0 e^{-\delta T}$$

Example 5.2-2 Suppose that the index is $125 and the annualized daily compounded dividend yield is 3%. Find the prepaid forward price at one year.

Solution.

$$F_{0,1}^{P} = \$125e^{-0.03\times 1} = \$121.306.$$

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**Forward price** is the future value of the prepaid forward price:

- No dividends

$$F_{0,T} = \mathrm{FV}\left(F_{0,T}^{p}\right)$$

- Continuous dividends

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

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$$\text{Forward premium} = \frac{F_{0,T}}{S_0}$$

$$\text{Annualized forward premium} = \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right)$$

# Does the forward price predict the future spot price?

Buying a stock

| Compensation for | Earn | Buying a stock |
|---|---|---|
| time value of the money | interest | ✓ |
| the risk of the stock | risk premium | ✓ |

Entering a forward contract

| Compensation for | Earn | Entering a forward contract |
|---|---|---|
| time value of the money | interest | ✗ |
| the risk of the stock | risk premium | ✓ |

The forward price is the expected future spot price, discounted at the risk premium.

$$F_{0,T} = e^{rT} \times \underbrace{F^{p}_{0,T}}_{=\mathbb{E}_0(S_T)e^{-\alpha T}} = \mathbb{E}_0(S_T)e^{-(\alpha - r)T}$$

# Creating a synthetic forward contract

Assuming that the dividends are continuous and paid at the rate $\delta$.

Recall that

Payoff of a long forward position at expiration

$$||$$

$$S_T - F_{0,T}$$

$$||$$

$$S_T - S_0 e^{(r-\delta)T}$$

$$\text{Forward} = \text{Stock} - \text{Zero-coupon bond}$$

| | Cash Flows | |
|---|---|---|
| **Transaction** | **Time 0** | **Time $T$ (expiration)** |
| Buy $e^{-\delta T}$ units of the index | $-S_0 e^{-\delta T}$ | $+ S_T$ |
| Borrow $S_0 e^{-\delta T}$ | $+S_0 e^{-\delta T}$ | $- S_0 e^{(r-\delta)T}$ |
| **Total** | 0 | $S_T - S_0 e^{(r-\delta)T}$ |

$$\text{Stock} = \text{Forward} + \text{Zero-coupon bond}$$

| | Cash Flows | |
|---|---|---|
| **Transaction** | **Time 0** | **Time $T$ (expiration)** |
| Long one forward | 0 | $S_T - F_{0,T}$ |
| Lend $S_0 e^{-\delta T}$ | $-S_0 e^{-\delta T}$ | $+S_0 e^{(r-\delta)T}$ |
| **Total** | $-S_0 e^{-\delta T}$ | $S_T$ |

Zero-coupon bond = Stock − Forward

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Long one forward | 0 | $S_T - F_{0,T}$ |
| Lend $S_0 e^{-\delta T}$ | $-S_0 e^{-\delta T}$ | $+S_0 e^{(r-\delta)T}$ |
| **Total** | $-S_0 e^{-\delta T}$ | $S_T$ |

**Cash-and-carry** is a transaction in which one buys the underlying asset and short the offsetting forward contract.

A cash-and-carry has no risk because
You have an obligation to deliver the asset
that you have already owned.

| Transaction | Cash Flows | |
|---|---|---|
| | Time 0 | Time $T$ (expiration) |
| Buy tailed position in stock, paying $S_0e^{-\delta T}$ | $-S_0e^{-\delta T}$ | $+S_T$ |
| Borrow $S_0e^{-\delta T}$ | $+S_0e^{-\delta T}$ | $-S_0e^{(r-\delta)T}$ |
| Short forward | 0 | $F_{0,T} - S_T$ |
| **Total** | 0 | $F_{0,T} - S_0e^{(r-\delta)T}$ |

## Cash-and-carry

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Buy tailed position in stock, paying $S_0 e^{-\delta T}$ | $-S_0 e^{-\delta T}$ | $+S_T$ |
| Borrow $S_0 e^{-\delta T}$ | $+S_0 e^{-\delta T}$ | $-S_0 e^{(r-\delta)T}$ |
| Short forward | 0 | $F_{0,T} - S_T$ |
| **Total** | 0 | $F_{0,T} - S_0 e^{(r-\delta)T}$ |

Arbitrage when $F_{0,T} > S_0 e^{(r-\delta)T}$

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## Reverse cash-and-carry

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Short tailed position in stock, receiving $S_0 e^{-\delta T}$ | $+S_0 e^{-\delta T}$ | $-S_T$ |
| Lend $S_0 e^{-\delta T}$ | $-S_0 e^{-\delta T}$ | $+S_0 e^{(r-\delta)T}$ |
| Long forward | 0 | $S_T - F_{0,T}$ |
| **Total** | 0 | $S_0 e^{(r-\delta)T} - F_{0,T}$ |

Arbitrage when $F_{0,T} < S_0 e^{(r-\delta)T}$

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Problems: 5.2, 5.3, 5.4, 5.5, 5.8, 5.10, 5.11, 5.12, 5.16, 5.20.

Due Date: TBD