Financial Mathematics

MATH 5870/6870[1]
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[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 20. Brownian Motion and Ito Lemma

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§ 20.1 The Black-Scholes assumption about stock prices

§ 20.2 Brownian motion

§ 20.3 Geometric Brownian motion

§ 20.4 The Ito formula

§ 20.5 The Sharpe ratio

§ 20.6 Risk-neutral valuation

§ 20.7 Problems

Chapter 20. Brownian Motion and Ito Lemma

The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0. \tag{1}$$

$$\boxed{dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0}$$

- $S(t)$ is the stock price. $dS(t)$ is the instantaneous change in the stock price. S_0 is the initial asset value.
- α is the continuously compound expected return on the stock;
- σ is the volatility, i.e., the standard deviation of the instantaneous return;
- $Z(t)$ is the standard Brownian motion.
- $dZ(t)$ requires rigorous justification.

- Equation of this type is called stochastic differential equation.
- Solution to this specific equation is the geometric Brownian motion.

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Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(S(t)) \sim N\left(\ln(S_0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t,\ \sigma^2\ t\right), \quad \text{for all } t > 0.$$

Remark 20.1-2 Note that Remark 20.1-1 is valid for all $t > 0$. It works for the terminal time $t = T$. It can also help us solve path-dependent options.

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