Financial Mathematics

MATH 5870/6870[1]
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Le Chen

lzc0090@auburn.edu

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Auburn University
Auburn AL

[1] Based on Robert L. McDonald's *Derivatives Markets*, 3rd Ed, Pearson, 2013.

Chapter 20. Brownian Motion and Ito Lemma

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§ 20.1 The Black-Scholes assumption about stock prices

§ 20.2 Brownian motion

§ 20.3 Geometric Brownian motion

§ 20.4 The Ito formula

§ 20.5 The Sharpe ratio

§ 20.6 Risk-neutral valuation

§ 20.7 Problems

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0. \tag{1}$$

$$\boxed{dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0}$$

- $S(t)$ is the stock price. $dS(t)$ is the instantaneous change in the stock price. S_0 is the initial asset value.
- α is the continuously compound expected return on the stock;
- σ is the volatility, i.e., the standard deviation of the instantaneous return;
- $Z(t)$ is the standard Brownian motion.
- $dZ(t)$ requires rigorous justification.

- Equation of this type is called stochastic differential equation.
- Solution to this specific equation is the geometric Brownian motion.

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- Equation of this type is called stochastic differential equation.
- Solution to this specific equation is the geometric Brownian motion.

Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(S(t)) \sim N\left(\ln(S_0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t,\ \sigma^2\, t\right), \quad \text{for all } t > 0.$$

Remark 20.1-2 Note that Remark 20.1-1 is valid for all $t > 0$. It works for the terminal time $t = T$. It can also help us solve path-dependent options.

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Definition 20.2-1 A real-valued stochastic process $Z(t)$ is called a Brownian motion or Wiener process if

1. It starts at 0:

$$Z(0) = 0.$$

2. For $0 \le s < t$, the increment $Z(t) - Z(s)$ is normally distributed with mean zero and variance $t - s$:

$$Z(t) - Z(s) \sim N(0, t - s).$$

3. Its increments are independent: if

$$0 \le t_0 \le t_1 \le \cdots \le t_k,$$

then

$$\mathbb{P}\left(Z(t_i) - Z(t_{i-1}) \in H_i,\ 1 \le i \le k\right) = \prod_{i=1}^{k} \mathbb{P}\left(Z(t_i) - Z(t_{i-1}) \in H_i\right).$$

Remark 20.2-1 One can always construct a continuous version of the Brownian motion; from now on, we always assume that Brownian motion is a continuous process.

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Remark 20.2-1 One can always construct a continuous version of the Brownian motion; from now on, we always assume that Brownian motion is a continuous process.

Theorem 20.2-1 (Some properties of Brownian motion)

1. $Z(t)$ is nowhere differentiable.

   (Hence, $dZ(t)$ requires some special treatment.)

2. $Z(t)$ satisfies the scaling property:

$$\widetilde{Z}(t) := \frac{1}{\sqrt{c}} Z(ct) \text{ is also a B.M. for all } c > 0.$$

3. $Z(t)$ is a martingale, namely,

$$\mathbb{E}\left(Z(t+s) | Z(t)\right) = Z(t).$$

4. For any $t > 0$, $Z(t) \sim N(0, t)$ and

$$\mathbb{E}(Z(t)Z(s)) = \min(t, s) \quad \text{for all } t, s \geq 0.$$

5. $Z(t)$ is translation invariant, namely,

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Proof. Part (1) goes beyond this course. All the rest could be proved using our current knowledge.

$\square$

Arithmetic Brownian motion

Definition 20.2-2 Let $Z(t)$ be a B.M. Then the process $X(t)$ given by

$$dX(t) = \alpha dt + \sigma dZ(t)$$

is called an arithmetic Brownian motion. Equivalently, $X(t)$ can be written in the following integral representation:

$$X(t) = X(0) + \int_0^t \alpha ds + \int_0^t \sigma dZ(s).$$

Remark 20.2-2

1. $X(t)$ is normally distributed:

$$X(t) = \sigma t + \sigma Z(t) \sim N\left(\sigma t, \sigma^2 t\right).$$

2. $X(t)$ takes both positive and negative values almost surely.
3. αt is a drift term.

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Definition 20.2-3 Let $Z(t)$ be a B.M. Then the process $X(t)$ given by

$$dX(t) = \lambda\left(\alpha - X(t)\right)dt + \sigma dZ(t)$$

is called the Ornstein-Uhlenbeck process.

Remark 20.2-3 Equivalently, $X(t)$ can be written in the following integral representation:

$$X(t) = X(0) + \lambda \int_0^t (\alpha - X(s))\, ds + \int_0^t \sigma dZ(s),$$

which is an integral equation (unknown X appears on both sides).

Remark 20.2-4 We have introduced mean reversion in the drift term.

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