### Financial Mathematics

MATH 5870/6870<sup>1</sup> Fall 2021

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Last updated on August 15, 2021

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<sup>&</sup>lt;sup>1</sup>Based on Robert L. McDonald's *Derivatives Markets*. 3rd Ed. Pearson. 2013.

- § 20.1 The Black-Scholes assumption about stock prices
- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- § 20.7 Problems

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The vast majority of technical option pricing discussions, including the original paper by Black and Scholes, assume that the price of the underlying asset follows a process determined by

$$dS(t) = (\alpha - \delta)dt + \sigma dZ(t), \quad S(0) = S_0.$$
 (1)

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- ▶ S(t) is the stock price. dS(t) is the instantaneous change in the stock price.  $S_0$  is the initial asset value.
- $\triangleright$   $\alpha$  is the continuously compound expected return on the stock:
- $\triangleright$   $\sigma$  is the volatility, i.e., the standard deviation of the instantaneous return;
- $\triangleright$  Z(t) is the standard Brownian motion
- ightharpoonup dZ(t) requires rigorous justification.

- Equation of this type is called stochastic differential equation
- ▶ Solution to this specific equation is the geometric Brownian motion

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Remark 20.1-1 We will see in this chapter that solution to this equation is lognormally distributed:

$$\ln(\mathcal{S}(t)) \sim \mathcal{N}\left(\ln(\mathcal{S}_0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t, \ \sigma^2 \ t\right), \quad ext{for all } t>0.$$

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1. It starts at 0:

$$Z(0)=0.$$

2. For  $0 \le s < t$ , the increment Z(t) - Z(s) is normally distributed with mean zero and variance t - s:

$$Z(t) - Z(s) \sim N(0, t - s)$$

3. Its increments are independent: if

$$0 \leq t_0 \leq t_1 \leq \cdots \leq t_k$$

then

$$\mathbb{P}(Z(t_i) - Z(t_{i-1}) \in H_i, \ 1 \le i \le k) = \prod_{i=1}^k \mathbb{P}(Z(t_i) - Z(t_{i-1}) \in H_i)$$

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### 1. Z(t) is nowhere differentiable.

(Hence, dZ(t) requires some special treatment.)

2. Z(t) satisfies the scaling property.

$$\widetilde{Z}(t) := \frac{1}{\sqrt{c}} Z(ct)$$
 is also a B.M. for all  $c>0$ 

3. Z(t) is a martingale, namely,

$$\mathbb{E}\left(Z(t+s)|Z(t)\right)=Z(t).$$

4. For any t > 0,  $Z(t) \sim N(0, t)$  and

$$\mathbb{E}(Z(t)Z(s)) = \min(t, s)$$
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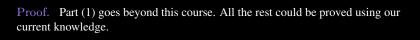
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#### Arithmetic Brownian motion

Definition 20.2-2 Let Z(t) be a B.M. Then the process X(t) given by

$$dX(t) = \alpha dt + \sigma dZ(t)$$

is called an arithmetic Brownian motion. Equivalently, X(t) can be written in the following integral representation:

$$X(t) = X(0) + \int_0^t \alpha ds + \int_0^t \sigma dZ(s).$$

#### Remark 20.2-2

1. X(t) is normally distributed:

$$X(t) = \sigma t + \sigma Z(t) \sim N(\sigma t, \sigma^2 t).$$

- 2. X(t) takes both positive and negative values almost surely.
- 3.  $\alpha t$  is a drift term

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## The Ornstein-Uhlenbeck process

Definition 20.2-3 Let Z(t) be a B.M. Then the process X(t) given by

$$dX(t) = \lambda \left(\alpha - X(t)\right) dt + \sigma dZ(t)$$

is called the Ornstein-Uhlenbeck process.

Remark 20.2-3 Equivalently, X(t) can be written in the following integral representation:

$$X(t) = X(0) + \lambda \int_0^t (\alpha - X(s)) ds + \int_0^t \sigma dZ(s),$$

which is an integral equation (unknown X appears on both sides).

Remark 20.2-4 We have introduced mean reversion in the drift term.

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- § 20.7 Problems

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- § 20.7 Problems

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- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- § 20.7 Problems

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- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- § 20.7 Problems

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- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- § 20.7 Problems

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- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- 8 20.7 Problems

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- § 20.2 Brownian motion
- § 20.3 Geometric Brownian motion
- § 20.4 The Ito formula
- § 20.5 The Sharpe ratio
- § 20.6 Risk-neutral valuation
- § 20.7 Problems

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- § 20.2 Brownian motion
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- § 20.7 Problems

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- § 20.5 The Sharpe ratio
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- § 20.7 Problems