Financial Mathematics

MATH 5870/6870¹ Fall 2021

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¹Based on Robert L. McDonald's *Derivatives Markets*. 3rd Ed. Pearson. 2013.

Chapter 18. The Lognormal Distribution

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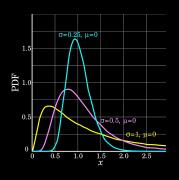
- § 18.1 The normal distribution
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- § 18.4 Lognormal probability calculations
- § 18.5 A. The expectation of a lognormal variable
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- § 18.7 Problems

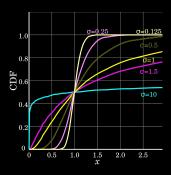
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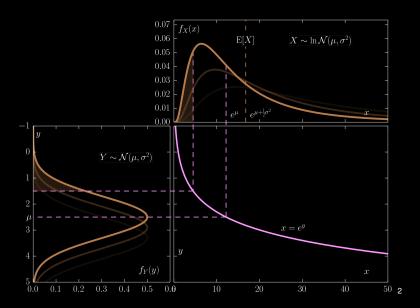
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- § 18.1 The normal distribution
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Definition 18.2-1 A random variable Y is *lognormally distributed* with parameters μ and $\sigma > 0$ if $\ln(Y) \sim N(\mu, \sigma^2)$.







²Image from Wikipedia.

Theorem 18.2-1 If Y_1 and Y_2 are lognormally distributed, so is $Y_1 Y_2$.

Since Y_1 and Y_2 are lognormally distributed, $\ln(Y_1)$ and $\ln(Y_2)$ are normally distributed. Hence,

Theorem 18.2-1 If Y_1 and Y_2 are lognormally distributed, so is $Y_1 Y_2$.

Proof.

Since Y_1 and Y_2 are lognormally distributed, $\ln(Y_1)$ and $\ln(Y_2)$ are normally distributed. Hence,

$$\ln(\mathbf{Y}_1) + \ln(\mathbf{Y}_2) = \ln(\mathbf{Y}_1 \mathbf{Y}_2)$$

is normally distributed too. Therefore, $Y_1 Y_2$ is lognormally distributed.

Theorem 18.2-2 If $\ln(Y) \sim N(\mu, \sigma^2)$, then the density of Y is given by

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.$$

Proof. For y > 0,

$$\begin{split} \mathbb{P}(Y \leq y) &= \mathbb{P}\left(\ln(Y) \leq \ln(y)\right) \\ &= \mathbb{P}\left(\frac{\ln(Y) - \mu}{\sigma} \leq \frac{\ln(y) - \mu}{\sigma}\right) \\ &= \tilde{\Phi}\left(\frac{\ln(y) - \mu}{\sigma}\right). \end{split}$$

Hence,

$$egin{aligned} f_{Y}(y) &= rac{d}{dy} \Phi\left(rac{\ln(y) - \mu}{\sigma}
ight) \ &= rac{1}{\sqrt{2\pi}} e^{-rac{1}{2}\left(rac{\ln(y) - \mu}{\sigma}
ight)^{2}} rac{d}{dy} rac{\ln(y) - \mu}{\sigma} \ &= rac{1}{v\sigma\sqrt{2\pi}} e^{-rac{1}{2}\left(rac{\ln(y) - \mu}{\sigma}
ight)^{2}}. \end{aligned}$$

Theorem 18.2-2 If $\ln(Y) \sim N(\mu, \sigma^2)$, then the density of Y is given by

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.$$

Proof. For y > 0,

$$\begin{split} \mathbb{P}(Y \le y) &= \mathbb{P}\left(\ln(Y) \le \ln(y)\right) \\ &= \mathbb{P}\left(\frac{\ln(Y) - \mu}{\sigma} \le \frac{\ln(y) - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right). \end{split}$$

Hence

$$egin{aligned} f_{V}(y) &= rac{d}{dy} \Phi\left(rac{\ln(y) - \mu}{\sigma}
ight) \ &= rac{1}{\sqrt{2\pi}} e^{-rac{1}{2}\left(rac{\ln(y) - \mu}{\sigma}
ight)^2} rac{d}{dy} rac{\ln(y) - \mu}{\sigma} \ &= rac{1}{V\sigma\sqrt{2\pi}} e^{-rac{1}{2}\left(rac{\ln(y) - \mu}{\sigma}
ight)^2}. \end{aligned}$$

Theorem 18.2-2 If $\ln(Y) \sim N(\mu, \sigma^2)$, then the density of Y is given by

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}.$$

Proof. For v > 0.

$$\begin{split} \mathbb{P}(Y \leq y) &= \mathbb{P}\left(\ln(Y) \leq \ln(y)\right) \\ &= \mathbb{P}\left(\frac{\ln(Y) - \mu}{\sigma} \leq \frac{\ln(y) - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right). \end{split}$$

Hence,

$$f_{Y}(y) = \frac{d}{dy} \Phi\left(\frac{\ln(y) - \mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y) - \mu}{\sigma}\right)^{2}} \frac{d}{dy} \frac{\ln(y) - \mu}{\sigma}$$

$$= \frac{1}{v\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y) - \mu}{\sigma}\right)^{2}}.$$

Recall that the moment generating function of the normal random variable $X \sim N(\mu, \sigma^2)$ is

$$\mathbb{E}\left(\boldsymbol{e}^{t\,X}\right) = \boldsymbol{e}^{\mu t + \sigma^2 t^2/2}, \quad \text{for all } t \in \mathbb{R}.$$

Remark 18.2-1 If Y is lognormally distributed with parameters μ and σ , then

$$\mathbb{E}\left(Y^{t}\right)=e^{\mu t+\sigma^{2}t^{2}/2},\qquad ext{for all }t\in\mathbb{R}.$$

Remark 18.2-2 By Jensen's inequality, if g is a convex function, then

$$\mathbb{E}\left(g(X)\right) \leq g\left(\mathbb{E}(X)\right).$$

Hence, for $g(x) = e^x$, we see that

$$\mathbb{E}\left(e^{X}\right) \leq e^{\mathbb{E}(X)}.$$

This is consistent with our computations above because

$$LHS = e^{\mu + \sigma^2/2} \ge e^{\mu} = RHS.$$

Theorem 18.2-3 If Y is lognormally distributed such that $\ln(Y) \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}(\mathbf{Y}) = \mathbf{e}^{\mu + \frac{1}{2}\sigma^2} \quad \text{and} \quad \mathrm{Var}(\mathbf{Y}) = \mathbf{e}^{2\mu + \sigma^2} \left(\mathbf{e}^{\sigma^2} - 1 \right).$$

Proof. Let X = ln(Y). By (1),

$$\mathbb{E}\left(Y\right) = \mathbb{E}\left(e^{X}\right) = e^{\mu + \frac{1}{2}\sigma^{2}}$$

and

$$\mathbb{E}(Y^2) = \mathbb{E}\left(e^{2X}\right) = e^{2\mu + 2\sigma^2}$$

Therefore.

$$\operatorname{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = e^{2\mu + 2\sigma^2} - e^{2(\mu + \frac{1}{2}\sigma^2)} = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1).$$

Theorem 18.2-3 If Y is lognormally distributed such that $\ln(Y) \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2}$$
 and $\operatorname{Var}(Y) = e^{2\mu + \sigma^2} \left(e^{\sigma^2} - 1 \right)$.

Proof. Let $X = \ln(Y)$. By (1),

$$\mathbb{E}(Y) = \mathbb{E}\left(e^{X}\right) = e^{\mu + \frac{1}{2}\sigma^{2}}$$

and

$$\mathbb{E}(\mathbf{Y}^2) = \mathbb{E}\left(\mathbf{e}^{2X}\right) = \mathbf{e}^{2\mu + 2\sigma^2}.$$

Therefore,

$$\operatorname{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = e^{2\mu + 2\sigma^2} - e^{2\left(\mu + \frac{1}{2}\sigma^2\right)} = e^{2\mu + \sigma^2} \left(e^{\sigma^2} - 1\right).$$

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