Math 362: Mathematical Statistics II

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Chapter 9. Two-Sample Inferences

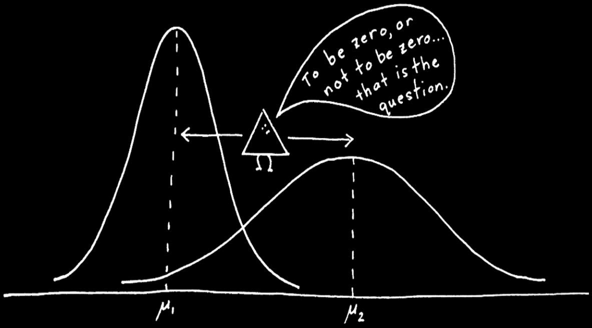
§ 9.1 Introduction

§ 9.2 Testing $H_0 : \mu_X = \mu_Y$

§ 9.3 Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

§ 9.4 Binomial Data: Testing $H_0 : p_X = p_Y$

§ 9.5 Confidence Intervals for the Two-Sample Problem



Multilevel designs:

1. Two methods applied to two independent sets of similar subjects.

   E.g., comparing two products.

2. Same method applied to two different kinds of subjects.

   E.g., comparing bones of European kids and American kids.

Test for normal parameters (two sample test)

1. Let $X_1, \cdots, X_n$ be a random sample of size n from $N(\mu_X, \sigma_X^2)$.
2. Let $Y_1, \cdots, Y_m$ be a random sample of size m from $N(\mu_Y, \sigma_Y^2)$.

Prob. 1 Find a test statistic Λ in order to test $H_0 : \mu_X = \mu_Y$ v.s. $H_1 : \mu_X \neq \mu_Y$.

- 1-1 When σ_X^2 and σ_Y^2 are known
- 1-2 When $\sigma_X^2 = \sigma_Y^2$ is unknown
- 1-3 When $\sigma_X^2 \neq \sigma_Y^2$, both are unknown

Prob. 2 Find a test statistic Λ in order to test $H_0 : \sigma_X^2 = \sigma_Y^2$ v.s. $H_1 : \sigma_X^2 \neq \sigma_Y^2$.

Prob. 1-1 Find a test statistic for $H_0 : \mu_X = \mu_Y$ v.s. $H_1 : \mu_X \neq \mu_Y$, with σ_X^2 and σ_Y^2 known.

Sol.

$$\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1)$$

Test statistics: $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$.

Critical region $|z| \geq z_{\alpha/2}$. $\square$

Prob. 1-2 Find a test statistic for $H_0 : \mu_X = \mu_Y$ v.s. $H_1 : \mu_X \neq \mu_Y$,

with $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ but unknown.

Sol. Composite-vs-composite test with:

$$\omega = \left\{(\mu_X, \mu_Y, \sigma^2) : \mu_X = \mu_Y \in \mathbb{R}, \quad \sigma^2 > 0\right\}$$

$$\Omega = \left\{(\mu_X, \mu_Y, \sigma^2) : \mu_X \in \mathbb{R},\ \mu_Y \in \mathbb{R},\ \sigma^2 > 0\right\}$$

The likelihood function

$$L(\omega) = \prod_{i=1}^{n} f_X(x_i) \prod_{j=1}^{m} f_Y(y_j)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{m+n} \exp\left(-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \mu_X)^2 + \sum_{j=1}^{m}(y_i - \mu_Y)^2\right]\right)$$

Under ω, the MLE $\omega_e = (\mu_{\omega_e}, \mu_{\omega_e}, \sigma^2_{\omega_e})$ is

$$\mu_{\omega_e} = \frac{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j}{n+m}$$

$$\sigma^2_{\omega_e} = \frac{\sum_{i=1}^{n}(x_i - \mu_{\omega_e})^2 + \sum_{j=1}^{m}(y_j - \mu_{\omega_e})^2}{n+m}$$

Hence,

$$L(\omega_e) = \left(\frac{e^{-1}}{2\pi\sigma^2_{\omega_e}}\right)^{\frac{n+m}{2}}$$

Under Ω, the MLE $\omega_e = (\mu_{X_e}, \mu_{Y_e}, \sigma^2_{\Omega_e})$ is

$$\mu_{X_e} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \mu_{Y_e} = \frac{1}{m}\sum_{j=1}^{m} y_j$$

$$\sigma^2_{\Omega_e} = \frac{\sum_{i=1}^{n}(x_i - \mu_{X_e})^2 + \sum_{j=1}^{m}(y_j - \mu_{Y_e})^2}{n+m}$$

Hence,

$$L(\Omega_e) = \left(\frac{e^{-1}}{2\pi\sigma^2_{\Omega_e}}\right)^{\frac{n+m}{2}}$$

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \left(\frac{\sigma^2_{\Omega_e}}{\sigma^2_{\omega_e}}\right)^{\frac{m+n}{2}}$$

$$\lambda^{\frac{2}{n+m}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{n}(y_j - \bar{y})^2}{\sum_{i=1}^{n}\left(x_i - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2 + \sum_{j=1}^{n}\left(y_j - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2}$$

$$\sum_{i=1}^{n}\left(x_i - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{m^2 n}{(m+n)^2}(\bar{x}-\bar{y})^2$$

$$\sum_{j=1}^{m}\left(y_j - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2 = \sum_{j=1}^{m}(y_i-\bar{y})^2 + \frac{mn^2}{(m+n)^2}(\bar{x}-\bar{y})^2$$

$$\Downarrow$$

$$\sum_{i=1}^{n}\left(x_i - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2 + \sum_{j=1}^{n}\left(y_j - \frac{n\bar{x}+m\bar{y}}{m+n}\right)^2$$

$$||$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 + \sum_{j=1}^{m}(y_i-\bar{y})^2 + \frac{mn}{m+n}(\bar{x}-\bar{y})^2$$

$$\lambda^{\frac{2}{m+n}} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{j=1}^m (y_i - \bar{y})^2}{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{j=1}^m (y_i - \bar{y})^2 + \frac{mn}{m+n}(\bar{x} - \bar{y})^2}$$

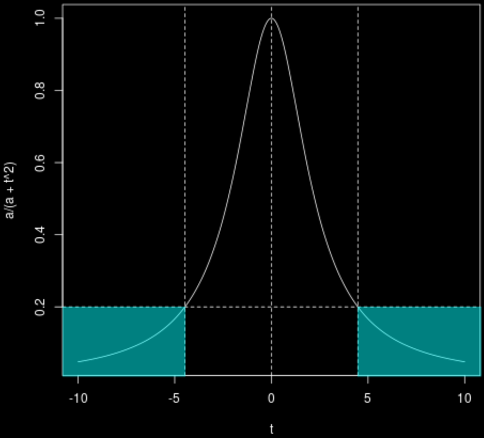
$$= \frac{1}{1 + \frac{(\bar{x} - \bar{y})^2}{\left[\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{j=1}^m (y_i - \bar{y})^2\right]\left(\frac{1}{m} + \frac{1}{n}\right)}}$$

$$= \frac{n+m-2}{n+m-2 + \frac{(\bar{x} - \bar{y})^2}{\frac{1}{n+m-2}\left[\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{j=1}^m (y_i - \bar{y})^2\right]\left(\frac{1}{m} + \frac{1}{n}\right)}}$$

$$= \frac{n+m-2}{n+m-2 + \frac{(\bar{x} - \bar{y})^2}{s_p^2\left(\frac{1}{m} + \frac{1}{n}\right)}} = \frac{n+m-2}{n+m-2+t^2}.$$

$$\boxed{t := \frac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{m} + \frac{1}{n}}}}$$

$$t \mapsto \frac{a}{a + t^2}$$



One can use the following statistic

$$T = \frac{\overline{X} - \overline{Y}}{S_p \sqrt{\frac{1}{m} + \frac{1}{n}}}$$

where S_p^2 is called the *pooled sample variance*

$$S_p^2 = \frac{1}{n+m-2} \left[\sum_{i=1}^{n} \left(X_i - \overline{X} \right)^2 + \sum_{i=1}^{m} \left(Y_j - \overline{Y} \right)^2 \right]$$

$$= \frac{1}{n+m-2} \left[(n-1) S_X^2 + (m-1) S_Y^2 \right]$$

Three observations:

1. $\mathbb{E}[\overline{X} - \overline{Y}] = 0$ and

$$\mathrm{Var}(\overline{X} - \overline{Y}) = \mathrm{Var}(\overline{X}) + \mathrm{Var}(\overline{Y}) = \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m} = \sigma^2\left(\frac{1}{n} + \frac{1}{m}\right)$$

Hence, $\frac{\overline{X}-\overline{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}} \sim N(0,1)$

2. $\frac{n+m-2}{\sigma^2} S_p^2 = \sum_{i=1}^n \left(\frac{X_i - \overline{X}}{\sigma}\right)^2 + \sum_{j=1}^m \left(\frac{Y_j - \overline{Y}}{\sigma}\right)^2 \sim \text{Chi square}(n+m-2)$

3. $\frac{\overline{X}-\overline{Y}}{\sigma\sqrt{\frac{1}{n}+\frac{1}{m}}} \perp \frac{n+m-2}{\sigma^2} S_p^2$

$$\Longrightarrow T = \frac{\dfrac{\overline{X} - \overline{Y}}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\dfrac{n+m-2}{\sigma^2} S_p^2 \times \dfrac{1}{n+m-2}}} = \frac{\overline{X}-\overline{Y}}{S_p\sqrt{\frac{1}{m}+\frac{1}{n}}} \sim \text{t distr.}(n+m-2)$$

Finally,

Test statistics: $t = \frac{\bar{x}-\bar{y}}{s_p\sqrt{\frac{1}{m}+\frac{1}{n}}}$

Critical region: $|t| \geq t_{\alpha/2,n+m-2}$. $\square$

Prob. 1-3 Find a test statistic for $H_0 : \mu_X = \mu_Y$ v.s. $H_1 : \mu_X \neq \mu_Y$,

with $\sigma_X^2 \neq \sigma_Y^2$, both unknown.

Remark: 1. Known as the *Behrens-Fisher problem*.

2. No exact solutions!

3. We will derive a widely used approximation by

Bernard Lewis Welch (1911–1989)

Sol.

$$W = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \Bigg/ \frac{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

$$U := \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0, 1)$$

$$\frac{V}{\nu} := \frac{\dfrac{S_X^2}{n} + \dfrac{S_Y^2}{m}}{\dfrac{\sigma_X^2}{n} + \dfrac{\sigma_Y^2}{m}}$$

!! **Assumption/Approximation:**

Assume that V follows Chi Square(ν) and assume that $V \perp U$.

$\Longrightarrow$ Then, $W \sim$ Student's t-distribution of freedom ν.

? It remains to estimate ν: Suppose we have

$$\nu = \frac{\left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)^2}{\frac{\sigma_X^4}{n^2(n-1)} + \frac{\sigma_Y^4}{m^2(m-1)}} = \frac{\left(\theta + \frac{n}{m}\right)^2}{\frac{1}{n-1}\theta^2 + \frac{1}{m-1}\left(\frac{n}{m}\right)^2}, \quad \theta = \frac{\sigma_X^2}{\sigma_Y^2}.$$

!! Still need to know $\theta = \sigma_X^2/\sigma_Y^2$... Another approximation $\hat{\theta} = S_X^2/S_Y^2$, i.e.,

$$\nu \approx \frac{\left(\frac{s_X^2}{n} + \frac{s_Y^2}{m}\right)^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}} = \frac{\left(\hat{\theta} + \frac{n}{m}\right)^2}{\frac{1}{n-1}\hat{\theta}^2 + \frac{1}{m-1}\left(\frac{n}{m}\right)^2}, \quad \hat{\theta} = \frac{S_X^2}{S_Y^2}.$$

In summary:

$$W = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \sim \text{Student's t of freedom } \nu$$

$$\nu = \left[\frac{\left(\frac{s_X^2}{n} + \frac{s_Y^2}{m} \right)^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}} \right] = \left[\frac{\left(\hat{\theta} + \frac{n}{m} \right)^2}{\frac{1}{n-1}\hat{\theta}^2 + \frac{1}{m-1}\left(\frac{n}{m}\right)^2} \right], \quad \hat{\theta} = \frac{s_X^2}{s_Y^2}.$$

Test statistic: $t = \frac{\bar{x} - \bar{y} - (\mu_X - \mu_Y)}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}$

Critical region: $|t| \geq t_{\alpha/2,\nu}$. □

Remark If $\nu \geq 100$, replace the t-score, e.g., $t_{\alpha/2,\nu}$ by the z-score, e.g., $z_{\alpha/2}$.

Thm The moment estimate for ν

$$\nu = \frac{\left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)^2}{\frac{\sigma_X^4}{n^2(n-1)} + \frac{\sigma_Y^4}{m^2(m-1)} + \frac{\sigma_X^2\sigma_Y^2}{mn}}$$

$$\approx \frac{\left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)^2}{\frac{\sigma_X^4}{n^2(n-1)} + \frac{\sigma_Y^4}{m^2(m-1)}} = \frac{\left(\theta + \frac{n}{m}\right)^2}{\frac{1}{n-1}\theta^2 + \frac{1}{m-1}\left(\frac{n}{m}\right)^2}, \quad \theta = \frac{\sigma_X^2}{\sigma_Y^2}.$$

Proof.

$$\frac{V}{\nu}\left(\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right) = \frac{S_X^2}{n} + \frac{S_Y^2}{m}$$

$$(n-1)S_X^2/\sigma_X^2 \sim \text{Chi Sqr}(n-1) \Longrightarrow \mathbb{E}(S_X^2) = \sigma_X^2. \text{ Similarly, } \mathbb{E}(S_Y^2) = \sigma_Y^2.$$

First moment gives identity. Need to consider second moment.

Second moments for Chi sqr(r) is $2r$. Hence, $\mathbb{E}(S_X^4) = \frac{\sigma_X^4}{n-1}$.

$$\frac{2\nu}{\nu^2}\left(\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}\right)^2 = 2\frac{\sigma_X^4}{n^2(n-1)} + 2\frac{\sigma_Y^4}{m^2(m-1)} + 2\frac{\sigma_X^2\sigma_Y^2}{mn}$$

... $\square$

Remark Welch (1938) approximation is more involved, which actually assumes that V follows the *Type III Pearson distribution*.

https://en.wikipedia.org/wiki/Behrens-Fisher_problem

Prob. 2 Find a test statistic Λ in order to test $H_0 : \sigma_X^2 = \sigma_Y^2$ v.s. $H_1 : \sigma_X^2 \neq \sigma_Y^2$.

Sol.

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim \text{F-disribution } (n-1, m-1)$$

Test statistic: $f = \frac{s_X^2/\sigma_X^2}{s_Y^2/\sigma_Y^2} = \frac{s_X^2}{s_Y^2}$

Critical regions: $f \leq F_{\alpha/2,n-1,m-1}$ or $f \geq F_{1-\alpha/2,n-1,m-1}$. $\square$

- Let $X_1, \cdots, X_n$ be a random sample of size n from $N(\mu_X, \sigma_X^2)$.
- Let $Y_1, \cdots, Y_m$ be a random sample of size m from $N(\mu_Y, \sigma_Y^2)$.

Prob. 1 Testing $H_0 : \mu_X = \mu_Y$ if $\sigma_X^2 = \sigma_Y^2$.

Prob. 2 Testing $H_0 : \mu_X = \mu_Y$ if $\sigma_X^2 \neq \sigma_Y^2$.

- True means: μ_X, μ_Y
- True std. dev.'s: σ_X, σ_Y
- True variances: σ_X^2, σ_Y^2

- Sample means: $\overline{X}$, $\overline{Y}$
- Sample std. dev.'s: S_X, S_Y
- Sample variances: S_X^2, S_Y^2

When $\sigma_X^2 = \sigma_Y^2 = \sigma^2$

Def. The **pooled variance**: $$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$$

$$= \frac{\sum_{i=1}^n (X_i - \overline{X})^2 + \sum_{j=1}^n (Y_j - \overline{Y})^2}{n+m-2}$$

Thm. $$T_{n+m-2} = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim \text{Student t distr. of } n+m-2 \text{ dgs of fd.}$$

Proof. (See slides on Section 9.1) $\square$

When $\sigma_X^2 = \sigma_Y^2 = \sigma^2$

Testing $H_0 : \mu_X = \mu_Y$ v.s.

(at the α level of significance)

$$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$$

$H_1 : \mu_X < \mu_Y$:	$H_1 : \mu_X \neq \mu_Y$:	$H_1 : \mu_X > \mu_Y$:
Reject H_0 if	Reject H_0 if	Reject H_0 if
$t \leq -t_{\alpha,n+m-2}$	$\lvert t \rvert \geq t_{\alpha/2,n+m-2}$	$t \geq t_{\alpha,n+m-2}$

E.g. Test whether Mark Twain and Snodgrass are the same person by checking the proportion of three-letter words at the 99% level of significance.

Table 9.2.1 Proportion of Three-Letter Words

Twain	Proportion	QCS	Proportion
Sergeant Fathom letter	0.225	Letter I	0.209
Madame Caprell letter	0.262	Letter II	0.205
Mark Twain letters in		Letter III	0.196
Territorial Enterprise		Letter IV	0.210
First letter	0.217	Letter V	0.202
Second letter	0.240	Letter VI	0.207
Third letter	0.230	Letter VII	0.224
Fourth letter	0.229	Letter VIII	0.223
First *Innocents Abroad* letter		Letter IX	0.220
First half	0.235	Letter X	0.201
Second half	0.217		

Sol. We need to test

$$H_0 : \mu_X = \mu_Y \quad v.s. \quad H_1 : \mu_X \neq \mu_Y.$$

Since we are tesing whether they are the same person, one can assume that $\sigma_X^2 = \sigma_Y^2$.

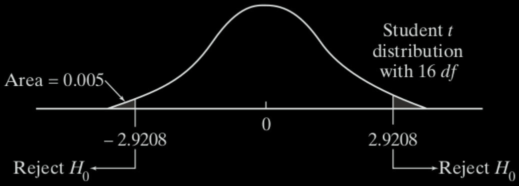
1. $n = 8,\ m = 10,$

$$\sum_{i=1}^{n} x_i = 1.855, \quad \sum_{i=1}^{n} x_i^2 = 0.4316$$
$$\sum_{i=1}^{m} y_i = 2.097, \quad \sum_{i=1}^{m} y_i^2 = 0.4406$$

2. Hence,

$$\bar{x} = 1.855/8 = 02319 \quad \bar{y} = 2.097/10 = 0.2097$$
$$s_X^2 = \frac{8 \times 0.4316 - 1.855^2}{8 \times 7} = 0.0002103$$
$$s_Y^2 = \frac{10 \times 0.4406 - 2.097^2}{10 \times 9} = 0.0000955$$
$$s_p^2 = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2} = ... = 0.0001457$$
$$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}} = ... = 3.88$$

3. Critical region: $|t| \geq t_{0.005,n+m-2} = t_{0.005,16} = 2.9208.$



4. Conclusion: Rejection! □

E.g. Comparing large-scales and small-scales companies:

Based on the data below, can we say that the return o equity differs between the two types of companies?

Table 9.2.4

Large-Sales Companies	Return on Equity (%)	Small-Sales Companies	Return on Equity (%)
Deckers Outdoor	21	NVE	21
Jos. A. Bank Clothiers	23	Hi-Shear Technology	21
National Instruments	13	Bovie Medical	14
Dolby Laboratories	22	Rocky Mountain Chocolate Factory	31
Quest Software	7	Rochester Medical	19
Green Mountain Coffee Roasters	17	Anika Therapeutics	19
Lufkin Industries	19	Nathan's Famous	11
Red Hat	11	Somanetics	29
Matrix Service	2	Bolt Technology	20
DXP Enterprises	30	Energy Recovery	27
Franklin Electric	15	Transcend Services	27
LSB Industries	43	IEC Electronics	24

Sol. Let μ_X and μ_Y be the average returns. We are asked to test

$$H_0 : \mu_X = \mu_Y \qquad v.s. \qquad H_1 : \mu_X \neq \mu_Y.$$

1.

$$n = 12, \qquad \sum_{i=1}^{n} x_i = 223 \qquad \sum_{i=1}^{n} x_i^2 = 5421$$

$$m = 12, \qquad \sum_{i=1}^{m} y_i = 263 \qquad \sum_{i=1}^{m} y_i^2 = 6157$$

2.

$$\bar{x} = 18.5833, \qquad s_X^2 = 116.0833$$

$$\bar{y} = 21.9167, \qquad s_Y^2 = 35.7197$$

$$w = \frac{18.5833 - 21.9167}{\sqrt{\frac{116.0833}{12} + \frac{35.7197}{12}}} = -0.9371932.$$

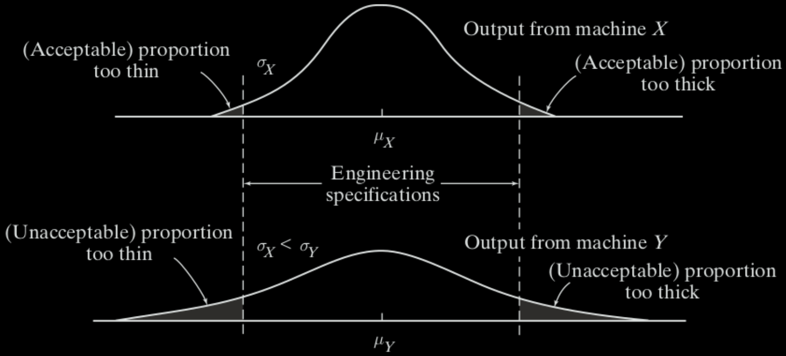
$$\hat{\theta} = \frac{116.0833}{35.7179} = 3.250 \quad \Rightarrow \quad \nu = \left[\frac{(3.250 + 1)^2}{\frac{1}{11} 3.250^2 + \frac{1}{11} 1^2} \right] = [17.18403] = 17.$$

3. The critical region is $|w| \geq t_{\alpha/2,17} = 2.1098$.

4. Conclusion:

   Since $w = -0.94$ is not in the critical region, we fail to reject H_0. $\square$

Mot. 1



Mot. 2 To test $H_0 : \mu_X = \mu_Y$ under the assumption $\sigma_X^2 = \sigma_Y^2$, we need to first test $\sigma_X^2 = \sigma_Y^2$.

Testing $H_0 : \sigma_X^2 = \sigma_Y^2$

v.s.

(at the α level of significance)

$H_1 : \sigma_X^2 < \sigma_Y^2$:	$H_1 : \sigma_X^2 \neq \sigma_Y^2$:	$H_1 : \sigma_X^2 > \sigma_Y^2$:
Reject H_0 if	Reject H_0 if	Reject H_0 if
$s_Y^2/s_X^2 \leq F_{\alpha,m-1,n-1}$	$s_Y^2/s_X^2 \geq F_{1-\alpha/2,m-1,n-1}$ or $s_Y^2/s_X^2 \leq F_{\alpha/2,m-1,n-1}$	$s_Y^2/s_X^2 \geq F_{1-\alpha,m-1,n-1}$

E.g. Electroencephalograms (EEG).

Twenty inmates in a Canadian prison, randomly split into two groups of equal size: one in solitary confinement, one in their own cells.

Measure the alpha waves. Whether the observed difference in variability is significant (set $\alpha = 0.05$.)

Table 9.3.1 Alpha-Wave Frequencies (CPS)

Nonconfined, x_i	Solitary Confinement, y_i
10.7	9.6
10.7	10.4
10.4	9.7
10.9	10.3
10.5	9.2
10.3	9.3
9.6	9.9
11.1	9.5
11.2	9.0
10.4	10.9

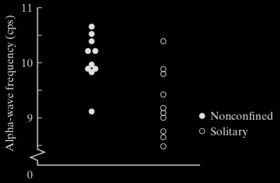


Figure 9.3.2 Alpha-wave frequencies (cps).

Sol. … □

Another example here:

https://www.itl.nist.gov/div898/handbook/eda/section3/eda359.htm

By the central limit theorem, when n and m are large

$$\frac{\frac{X}{n} - \frac{Y}{m} - \mathbb{E}\left(\frac{X}{n} - \frac{Y}{m}\right)}{\sqrt{\operatorname{Var}\left(\frac{X}{n} - \frac{Y}{m}\right)}} \overset{approx.}{\sim} N(0,1)$$

Under $H_0 : p_X = p_Y$,

$$\mathbb{E}\left(\frac{X}{n} - \frac{Y}{m}\right) = 0$$

$$\operatorname{Var}\left(\frac{X}{n} - \frac{Y}{m}\right) = \frac{p(1-p)}{n} + \frac{p(1-p)}{m}$$

The MLE for p under H_0 is

$$p_e = \frac{x+y}{n+m}$$

Testing $H_0 : p_X = p_Y$

v.s.

(at the α level of significance)

$$z = \frac{\frac{x}{n} - \frac{y}{m}}{\sqrt{p_e(1-p_e)\left(\frac{1}{n}+\frac{1}{m}\right)}}, \qquad p_e = \frac{x+y}{n+m}$$

$H_1 : p_X < p_Y$:	$H_1 : p_X \neq p_Y$:	$H_1 : p_X > p_Y$:
Reject H_0 if	Reject H_0 if	Reject H_0 if
$z \leq -z_\alpha$	$\lvert z\rvert \geq z_{\alpha/2}$	$z \geq z_\alpha$

E.g. Nightmares among men and women:

Table 9.4.1 Frequency of Nightmares			
	Men	Women	Total
Nightmares often	55	60	115
Nightmares seldom	105	132	237
Totals	160	192	
% often:	34.4	31.3	

Is 34.4% significantly different from 31.1% ($\alpha = 0.05$)?

Sol. ... $\square$

Similar to the hypothesis test ...

1. Let $X_1, \cdots, X_n$ be a random sample of size n from $N(\mu_X, \sigma_X^2)$.
2. Let $Y_1, \cdots, Y_m$ be a random sample of size m from $N(\mu_Y, \sigma_Y^2)$.

Prob. 1 Find the $100(1-\alpha)\%$ C.I. for $\mu_X - \mu_Y$

When both σ_X^2 and σ_Y^2 are known

When $\sigma_X^2 = \sigma_Y^2 = \sigma^2$, but is unknown

When $\sigma_X^2 \neq \sigma_Y^2$, both are unknown

Prob. 2 Find the $100(1-\alpha)\%$ C.I. for σ_X^2/σ_Y^2, or σ_X/σ_Y

Prob. 1-1 Find the $100(1-\alpha)\%$ C.I. for $\mu_X - \mu_Y$ with σ_X^2 and σ_Y^2 known.

Sol.

$$\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1)$$

$$\mathbb{P}\left(-z_{\alpha/2} \le \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \le z_{\alpha/2}\right) = 1 - \alpha$$

$$||$$

$$\mathbb{P}\left((\overline{X} - \overline{Y}) - z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \le \mu_X - \mu_Y \le (\overline{X} - \overline{Y}) + z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right)$$

$$\left((\overline{x} - \overline{y}) - z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \quad , \quad (\overline{x} - \overline{y}) + z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}\right)$$

$\square$

Prob. 1-2 Find the $100(1-\alpha)\%$ C.I. for $\mu_X - \mu_Y$ when $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ unknown

Sol.

$$\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim \text{Student t-distribution } (n + m - 2)$$

$$\mathbb{P}\left(-t_{\alpha/2,n+m-2} \leq \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \leq t_{\alpha/2,n+m-2}\right) = 1 - \alpha$$

$$||$$

$$\mathbb{P}\left((\overline{X} - \overline{Y}) - t_{\alpha/2,n+m-2}S_p\sqrt{\frac{1}{n} + \frac{1}{m}} \leq \mu_X - \mu_Y \leq (\overline{X} - \overline{Y}) + t_{\alpha/2,n+m-2}S_p\sqrt{\frac{1}{n} + \frac{1}{m}}\right)$$

$$\left((\overline{x} - \overline{y}) - t_{\alpha/2,n+m-2}s_p\sqrt{\frac{1}{n} + \frac{1}{m}} \quad , \quad (\overline{x} - \overline{y}) + t_{\alpha/2,n+m-2}s_p\sqrt{\frac{1}{n} + \frac{1}{m}}\right)$$

$\square$

Prob. 1-3 Find the $100(1-\alpha)\%$ C.I. for $\mu_X - \mu_Y$ when $\sigma_X^2 \neq \sigma_Y^2$ unknown.

Sol.

$$\frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \sim \text{Student t-distribution } (\nu)$$

$$\mathbb{P}\left(-t_{\alpha/2,\nu} \leq \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \leq t_{\alpha/2,\nu}\right) \approx 1 - \alpha$$

$$||$$

$$\mathbb{P}\left((\overline{X} - \overline{Y}) - t_{\alpha/2,\nu}\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}} \leq \mu_X - \mu_Y \leq (\overline{X} - \overline{Y}) + t_{\alpha/2,\nu}\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}\right)$$

$$\left((\overline{x} - \overline{y}) - t_{\alpha/2,\nu}\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}} \quad , \quad (\overline{x} - \overline{y}) + t_{\alpha/2,\nu}\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}\right)$$

$\square$

Prob. 2 Find the $100(1-\alpha)\%$ C.I. for σ_X^2/σ_Y^2

Sol 1.

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim \text{F-disribution } (n-1, m-1)$$

$$\mathbb{P}\left(F_{\alpha/2,n-1,m-1} \le \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \le F_{1-\alpha/2,n-1,m-1}\right) = 1-\alpha$$

$$||$$

$$\mathbb{P}\left(\frac{S_X^2}{S_Y^2}\frac{1}{F_{1-\alpha/2,n-1,m-1}} \le \frac{\sigma_X^2}{\sigma_Y^2} \le \frac{S_X^2}{S_Y^2}\frac{1}{F_{\alpha/2,n-1,m-1}}\right)$$

$$\left(\frac{s_X^2}{s_Y^2}\frac{1}{F_{1-\alpha/2,n-1,m-1}} \quad , \quad \frac{s_X^2}{s_Y^2}\frac{1}{F_{\alpha/2,n-1,m-1}}\right)$$

$\square$

Sol 2. Or equivalently,

$$\frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} \sim \text{F-disribution } (m-1, n-1)$$

$$\mathbb{P}\left(F_{\alpha/2,m-1,n-1} \leq \frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} \leq F_{1-\alpha/2,m-1,n-1}\right) = 1-\alpha$$

$$||$$

$$\mathbb{P}\left(\frac{S_X^2}{S_Y^2}F_{\alpha/2,m-1,n-1} \leq \frac{\sigma_X^2}{\sigma_Y^2} \leq \frac{S_X^2}{S_Y^2}F_{1-\alpha/2,m-1,n-1}\right)$$

$$\left(\frac{s_X^2}{s_Y^2}F_{\alpha/2,m-1,n-1} \quad , \quad \frac{s_X^2}{s_Y^2}F_{1-\alpha/2,m-1,n-1}\right)$$

$\square$

Recall: $$F_{\alpha,m,n} = \frac{1}{F_{1-\alpha,n,m}}$$

Examples from the book...