Math 362: Mathematical Statistics II

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Last updated on Spring 2021
Last compiled on January 15, 2023

2021 Spring

Chapter 6. Hypothesis Testing

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Plan

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Setup: Let $X_1 = k_1, \cdots, X_n = k_n$ be a' random sample of size n from Bernoulli(p). $X = \sum_{i=1}^{n} X_i \sim$ Binomial(n, p). We want to test $H_0 : p = p_0$.

1. When n is large, use Z score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$
\begin{gathered}
n \text{ is large} \\
\Updownarrow \\
0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n \\
\Updownarrow \\
n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).
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3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k - np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is α.

Test:

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

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Small-sample test for p

E.g. $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

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Binomial with n = 19 and p = 0.85

x	P(X = x)
6	0.000000
7	0.000002
8	0.000018
9	0.000123
10	0.000699
11	0.003242
12	0.012246
13	0.037366
14	0.090746
15	0.171409
16	0.242829
17	0.242829
18	0.152892
19	0.045599

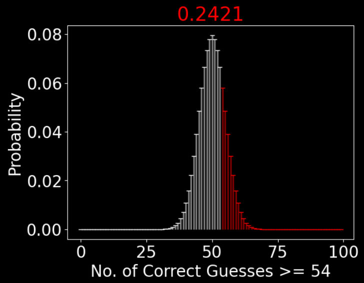
$\rightarrow P(X \leq 13) = 0.053696$

$\rightarrow P(X = 19) = 0.045599$

```python
# Eg_6-3-1.py
from scipy.stats import binom
n = 19
p = 0.85
rv = binom(n, p)
low = rv.ppf(0.05)
upper = rv.ppf(0.95)
left = round(rv.cdf(low), 6)
right = round(1-rv.cdf(upper), 6)
both = round(rv.cdf(low)+1-rv.cdf(upper), 6)
Results = """\
    The critical regions is less or equal to {low:.0f}, or strictly greater than {upper:.0f}.
    The size of the tail is {left:.6f} and that of the right tail is {right:.6f}.
    Under this critical region, the level of significance is {both:.6f}
""".format(**locals())
print(Results)
```

```
In [487]: run Eg_6-3-1.py
    The critical regions is less or equal to 13, or strictly greater than 18.
    The size of the left tail is 0.053696 and that of the right tail is 0.045599.
    Under this critical region, the level of significance is 0.099296
```

$X \sim \text{Binomial}(100, 1/2)$

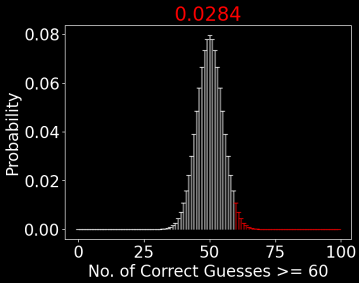


$$\mathbb{P}(X \geq 54) = \sum_{n=54}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{100-n} = 0.2421.$$

vs

$$\mathbb{P}\left(\frac{X - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}} \geq \frac{54 - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \approx \mathbb{P}\left(Z \geq \frac{4}{5}\right) = 0.2119$$

$X \sim \text{Binomial}(100, 1/2)$



$$\mathbb{P}(X \geq 60) = \sum_{n=60}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{100-n} = 0.0284.$$

vs

$$\mathbb{P}\left(\frac{X - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}} \geq \frac{60 - 50}{\sqrt{100 \times \frac{1}{2} \times \frac{1}{2}}}\right) \approx \mathbb{P}(Z \geq 2) = 0.0228$$