Math 362: Mathematical Statistics II

Le Chen
le.chen@emory.edu
chenle02@gmail.com

Emory University
Atlanta, GA

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Chapter 5. Estimation

Plan

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§ 5.6 Sufficient Estimators

Rationale: Let $\widehat{\theta}$ be an estimator to the unknown parameter θ. Whether does $\widehat{\theta}$ contain all information about θ?

Equivalently, how can one reduce the random sample of size n, denoted by $(X_1, \cdots, X_n)$, to a function without losing any information about θ?

E.g., let's choose the function $h(X_1, \cdots, X_n) := \frac{1}{n}\sum_{i=1}^{n} X_i$, the sample mean. In many cases, $h(X_1, \cdots, X_n)$ contains all relevant information about the true mean $\mathbb{E}(X)$. In that case, $h(X_1, \cdots, X_n)$, as an estimator, is sufficient.

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Definition. Let $(X_1, \cdots, X_n)$ be a random sample of size n from a discrete population with a unknown parameter θ, of which $\widehat{\theta}$ (resp. θ_e) be an estimator (resp. estimate). We call $\widehat{\theta}$ and θ_e **sufficient** if

$$\mathbb{P}\left(X_1 = k_1, \cdots, X_n = k_n \,\middle|\, \widehat{\theta} = \theta_e\right) = b(k_1, \cdots, k_n) \qquad \text{(Sufficency-1)}$$

is a function that does not depend on θ.

In case for random sample $(Y_1, \cdots, Y_n)$ from the continuous population, (Sufficency-1) should be

$$f_{Y_1,\cdots,Y_n|\widehat{\theta}=\theta_e}\left(y_1, \cdots, y_n \,\middle|\, \widehat{\theta} = \theta_e\right) = b(y_1, \cdots, y_n)$$

Note: $\widehat{\theta} = h(X_1, \cdots, X_n)$ and $\theta_e = h(k_1, \cdots, k_n)$.
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Equivalently,

Definition. ... $\widehat{\theta}$ (or θ_e) is **sufficient** if the likelihood function can be factorized as:

$$L(\theta) = \begin{cases} \prod_{i=1}^{n} p_X(k_i;\theta) = g(\theta_e,\theta)\, b(k_1,\cdots,k_n) & \text{Discrete} \\ \prod_{i=1}^{n} f_Y(y_i;\theta) = g(\theta_e,\theta)\, b(y_1,\cdots,y_n) & \text{Continous} \end{cases} \qquad \text{(Sufficency-2)}$$

where g is a function of two arguments only and b is a function that does not depend on θ.

E.g. 1. A random sample of size n from Bernoulli(P). $\widehat{p} = \sum_{i=1}^{n} X_i$. Check sufficiency of $\widehat{p}$ for p by (Sufficency-1):

Case I: If $k_1, \cdots, k_n \in \{0, 1\}$ such that $\sum_{i=1}^{n} k_i \neq c$, then

$$\mathbb{P}\left(X_1 = k_1, \cdots, X_n = k_n \,\middle|\, \widehat{p} = c\right) = 0.$$

Case II: If $k_1, \cdots, k_n \in \{0, 1\}$ such that $\sum_{i=1}^{n} k_i = c$, then

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&\quad = \frac{\left(\prod_{i=1}^{n-1} p^{k_i}(1-p)^{1-k_i}\right) \times p^{c-\sum_{i=1}^{n-1} k_i}(1-p)^{1-c+\sum_{i=1}^{n-1} k_i}}{\binom{n}{c} p^c (1-p)^{n-c}} \\
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Notice that $p_e = \sum_{i=1}^{n} k_i$. Then

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L(p) = \prod_{i=1}^{n} p_X(k_i; p) &= \prod_{i=1}^{n} p^{k_i}(1-p)^{1-k_i} \\
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Therefore, p_e (or $\widehat{p}$) is sufficient since (Sufficency-2) is satisfied with

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g(p_e, p) = p^{p_e}(1-p)^{n-p_e} \quad \text{and} \quad b(k_1, \cdots, k_n) = 1.
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Comment 1. The estimator $\widehat{p}$ is sufficient but not unbiased since $\mathbb{E}(\widehat{p}) = np \neq p$.

2. Any one-to-one function of a sufficient estimator is again a sufficient estimator. E.g., $\widehat{p}_2 := \frac{1}{n}\widehat{p}$, which is a unbiased, sufficient, and MVE.

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Comment 1. The estimator $\widehat{p}$ is sufficient but not unbiased since $\mathbb{E}(\widehat{p}) = np \neq p$.

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Notice that $p_e = \sum_{i=1}^{n} k_i$. Then

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\begin{aligned}
L(p) = \prod_{i=1}^{n} p_X(k_i; p) &= \prod_{i=1}^{n} p^{k_i}(1-p)^{1-k_i} \\
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E.g. 2. Poisson(λ), $p_X(k;\lambda) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \cdots$. Show that $\widehat{\lambda} = (\sum_{i=1}^n X_i)^2$ is sufficient for λ for a sample of size n.

Sol: The Corresponding estimate is $\lambda_e = (\sum_{i=1}^n k_i)^2$.

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\begin{aligned}
L(\lambda) &= \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{k_i}}{k_i!} \\
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E.g. 3. Let $Y_1, \cdots, Y_n$ be a random sample from $f_Y(y;\theta) = \frac{2y}{\theta^2}$ for $y \in [0,\theta]$. Whether is the MLE $\widehat{\theta} = Y_{max}$ sufficient for θ?

Sol: The corresponding estimate is $\theta_e = y_{max}$.

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