Math 362: Mathematical Statistics II

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Last updated on Spring 2021
Last compiled on January 15, 2023

2021 Spring

Chapter 7. Inference Based on The Normal Distribution

§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.4 Drawing Inferences About μ

§ 7.5 Drawing Inferences About σ^2

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Def. **Sampling distributions**

Distributions of $\underbrace{\textit{functions of random sample}}_{\text{statistics / estimators}}$ of given size.

E.g. A random sample of size n from $N(\mu, \sigma^2)$ with σ^2 known.

Sample mean $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N(\mu, \sigma^2/n)$

Aim: Determine distributions for

Sample variance $S^2 := \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2$	*Chi square distr.*
$T := \dfrac{\overline{Y} - \mu}{S/\sqrt{n}}$	*Student t distr.*
$\dfrac{S_1^2}{\sigma_1^2} \Big/ \dfrac{S_2^2}{\sigma_2^2}$	*F distr.*

Thm 7.3.1. Let $U = \sum_{i=1}^{m} Z_j^2$, where Z_j are independent $N(0,1)$ normal r.v.s. Then

$$U \sim \text{Gamma(shape=}m/2\text{, rate=}1/2\text{)}.$$

namely,

$$f_U(u) = \frac{1}{2^{m/2}\Gamma(m/2)} u^{\frac{m}{2}-1} e^{-u/2}, \qquad u \geq 0.$$

Def 7.3.1. U in Thm 7.3.1 is called **chi square distribution** with m dgs of freedom.

Proof. We first consider the case when $m = 1$. In this case,

$$
\begin{aligned}
F_{Z^2}(u) &= \mathbb{P}\left(Z^2 \le u\right) \\
&= \mathbb{P}\left(-\sqrt{u} \le Z \le \sqrt{u}\right) \\
&= 2\mathbb{P}(0 \le Z \le \sqrt{u}) \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{2\pi} e^{-z^2/2} \mathrm{d}z
\end{aligned}
$$

Differentiating both sides of the above eq. in order to obtain the pdf:

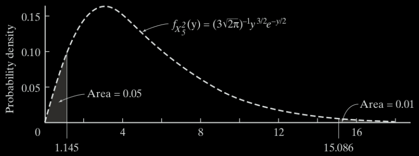
$$
\begin{aligned}
f_{Z^2}(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_{Z^2}(u) \\
&= \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{u}} e^{-u/2} \\
&= \frac{1}{\sqrt{2}\Gamma(1/2)} u^{(1/2)-1} e^{-u/2},
\end{aligned}
$$

which is the pdf of a gamma distribution with $r = \lambda = 1/2$.

Then adding m independent copies of gamma distributions gives anther gamma distribution with $r = m/2$ and $\lambda = 1/2$ (See Theorem 4.6.4). $\square$

Chi Square Table

	p							
df	.01	.025	.05	.10	.90	.95	.975	.99
1	0.000157	0.000982	0.00393	0.0158	2.706	3.841	5.024	6.635
2	0.0201	0.0506	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.336	26.217



$$\mathbb{P}(\chi^2_5 \leq 1.145) = 0.05 \iff \chi^2_{0.05,5} = 1.145$$

$$\mathbb{P}(\chi^2_5 \leq 15.086) = 0.99 \iff \chi^2_{0.99,5} = 15.086$$

```
> pchisq(1.145, df = 5)
[1] 0.04995622
> pchisq(15.086, df = 5)
[1] 0.9899989
```

```
> qchisq(0.05, df = 5)
[1] 1.145476
> qchisq(0.99, df = 5)
[1] 15.08627
```

Chi Square Table

	p							
df	.01	.025	.05	.10	.90	.95	.975	.99
1	0.000157	0.000982	0.00393	0.0158	2.706	3.841	5.024	6.635
2	0.0201	0.0506	0.103	0.211	4.605	5.991	7.378	9.210
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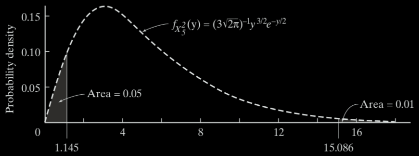


$$\mathbb{P}(\chi_5^2 \leq 1.145) = 0.05 \quad \Longleftrightarrow \quad \chi^2_{0.05,5} = 1.145$$

$$\mathbb{P}(\chi_5^2 \leq 15.086) = 0.99 \quad \Longleftrightarrow \quad \chi^2_{0.99,5} = 15.086$$

```
> scipy.stats.chi2.cdf(1.145, 5)
 [1]: 0.04995622155207728
> scipy.stats.chi2.cdf(15.086, 5)
 [1]: 0.9899988752378142
```

```
> scipy.stats.chi2.ppf(0.05, 5)
 [1]: 1.1454762260617692
> scipy.stats.chi2.ppf(0.99, 5)
 [1]: 15.08627246938899
```

Thm 7.3.2. Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

(a) S^2 and $\overline{Y}$ are independent.

(b) $\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2 \quad \sim \quad$ Chi Square$(n-1)$.

Proof. We will prove the case $n = 2$.

$$\overline{Y} = \frac{Y_1 + Y_2}{2}, \qquad Y_1 - \overline{Y} = \frac{Y_1 - Y_2}{2}, \qquad Y_2 - \overline{Y} = \frac{Y_2 - Y_1}{2}$$

$$S^2 = ... = \frac{1}{2}\left(Y_1 - Y_2\right)^2$$

(a) It is equivalanet to show $Y_1 + Y_2 \perp Y_1 - Y_2$. Since they are normal, it suffices to show that

$$\mathbb{E}[(Y_1 + Y_2)(Y_1 - Y_2)] = \mathbb{E}[Y_1 + Y_2]\mathbb{E}[Y_1 - Y_2]$$

(b) $\frac{(n-1)S^2}{\sigma^2} = \left(\frac{Y_1 - Y_2}{\sqrt{2}\sigma}\right)^2$ and $\frac{Y_1 - Y_2}{\sqrt{2}\sigma} \sim N(0, 1)$ ... $\square$

Def 7.3.2. If $U \sim$ Chi Square(n) and $V \sim$ Chi Square(m), and $U \perp V$, then

$$F := \frac{V/m}{U/n}$$

follows the **(Snedecor's) F distribution** with m and n degrees of freedom.

Thm 7.3.3. Let $F_{m,n} = \frac{V/m}{U/n}$ be an F r.v. with m and n degrees of freedom. Then

$$f_{F_{m,n}}(w) = \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \times \frac{w^{m/2-1}}{(n+mw)^{(m+n)/2}}, \quad w \geq 0$$

Equivalently,

$$f_{F_{m,n}}(w) = B(m/2, n/2)^{-1} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n} w\right)^{-\frac{m+n}{2}}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

Recall

Thm 3.8.4 Let X and Y be independent continuous random variables, with pdf $f_X(x)$ and $f_Y(y)$, respectively.

Assume that X is zero for at most a set of isolated points.

Then $W = Y/X$ follows a distribution with pdf:

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx) \mathrm{d}x.$$

Thm 3.8.2 Suppose X is a continuous random variable and $a \neq 0$.

Then $Y = aX + b$ follows a distribution with pdf:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Proof. Let us first find the pdf for $W := V/U$. By Theorem 7.3.1,

$$f_V(v) = \frac{1}{2^{m/2}\Gamma(m/2)} v^{(m/2)-1} e^{-v/2},$$
$$f_U(u) = \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2}.$$

Then by Theorem 3.8.4, we see that the pdf of W is

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} |u| f_U(u)\, f_V(uw) \mathrm{d}u \\ &= \int_0^{\infty} u \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2} \frac{1}{2^{m/2}\Gamma(m/2)} (uw)^{(m/2)-1} e^{-uw/2} \mathrm{d}u \\ &= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \int_0^{\infty} u^{\frac{n+m}{2}-1} e^{-\frac{1+w}{2}u} \mathrm{d}u \end{aligned}$$

Then by the change of variables, $y = \frac{1+w}{2}u$, we see that

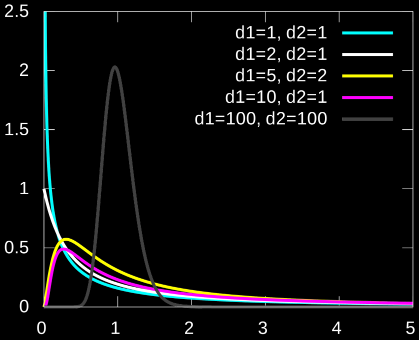
$$f_W(w) = \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \left(\frac{2}{1+w}\right)^{\frac{n+m}{2}} \int_0^\infty y^{\frac{n+m}{2}-1} e^{-y} \mathrm{d}y$$
$$= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \left(\frac{2}{1+w}\right)^{\frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)$$

where the last equality is due to the definition of the Gamma function.

Finally, by Theorem 3.8.2, we see that $F = \frac{V/m}{U/n} = \frac{n}{m}W$ follows a distribution with pdf

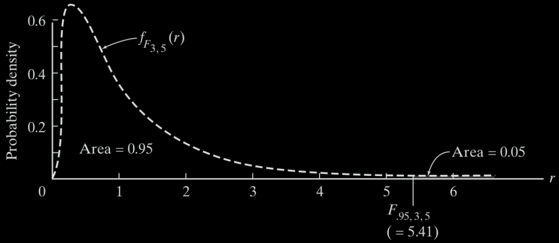
$$f_F(y) = \frac{m}{n} f_W\left(\frac{m}{n}y\right)$$
$$= \frac{m}{n} \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \left(\frac{m}{n}y\right)^{(m/2)-1} \left(\frac{2}{1+\frac{m}{n}y}\right)^{\frac{n+m}{2}} \Gamma\left(\frac{n+m}{2}\right)$$
$$= \cdots \qquad y \geq 0.$$

$\square$




```
# Draw F density
x=seq(0,5,0.01)
pdf= cbind(df(x, df1 = 1, df2 = 1),
df(x, df1 = 2, df2 = 1),
df(x, df1 = 5, df2 = 2),
df(x, df1 = 10, df2 = 1),
df(x, df1 = 100, df2 = 100))
matplot(x,pdf, type = "l")
 title ("F with various dgrs of freedom")
```

F- Table



$$\mathbb{P}(F_{3,5} \leq 5.41) = 0.95 \quad \Longleftrightarrow \quad F_{0.95,3,5} = 5.41$$

```
> pf(5.41, df1 = 3, df2 = 5)
[1] 0.9500093
```

```
> qf(0.95, df1 = 3, df2 = 5)
[1] 5.409451
```

```
> scipy.stats.f.cdf(5.41, 3, 5)
[1] 0.9500092950699683
```

```
> scipy.stats.f.ppf(0.95, 3, 5)
[1] 5.40945131805649
```

Def 7.3.3. Suppose $Z \sim N(0,1)$, $U \sim$ Chi Square(n), and $Z \perp U$. Then

$$T_n = \frac{Z}{\sqrt{U/n}}$$

follows the **Student's t-distribution** of n degrees of freedom.

Remark $T_n^2 \sim F$-distribution with 1 and n degrees of freedom.

Thm 7.3.4. The pdf of the Student t of degree n is

$$f_{T_n}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \times \left(1 + \frac{t^2}{n}\right)^{-\frac{n+2}{2}}, \quad t \in \mathbb{R}.$$

Proof. Note that $T_n^2 = \frac{Z^2}{U/n}$ follows an $F(1, n)$ distribution. Hence,

$$f_{T_n^2}(t) = \frac{n^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})} t^{-\frac{1}{2}} \frac{1}{(n+t)^{\frac{n+1}{2}}}, \quad t > 0.$$

Therefore,

$$F_{T_n}(t) = \mathbb{P}(T_n \leq t) = \mathbb{P}(-\infty < T_n \leq 0) + \mathbb{P}(0 \leq T_n \leq t).$$

The term $\mathbb{P}(-\infty < T_n \leq 0)$ is a constant which will disappear upon differentiation.

Notice that

$$\begin{aligned} \left\{T_n^2 \leq t^2\right\} &= \{-t \leq T_n \leq t\} = \{-t \leq T_n \leq 0\} \cup \{0 \leq T_n \leq t\} \\ &= \left\{-t\sqrt{U/n} \leq Z \leq 0\right\} \cup \left\{0 \leq Z \leq t\sqrt{U/n}\right\} \end{aligned}$$

By symmetry of the distribution of Z,

$$\mathbb{P}\left(-t\sqrt{U/n} \leq Z \leq 0\right) = \mathbb{P}\left(0 \leq Z \leq t\sqrt{U/n}\right)$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(T_n^2 \leq t^2\right) &= \mathbb{P}\left(-t\sqrt{U/n} \leq Z \leq 0\right) + \mathbb{P}\left(0 \leq Z \leq t\sqrt{U/n}\right) \\ &= 2\mathbb{P}\left(0 \leq Z \leq t\sqrt{U/n}\right) \\ &= 2\mathbb{P}(0 \leq T_n \leq t).\end{aligned}$$

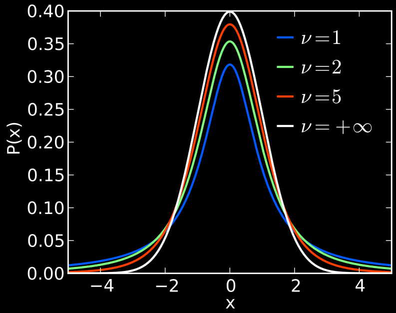
Hence,

$$F_{T_n}(t) = const. + \frac{1}{2}\mathbb{P}\left(T_n^2 \leq t^2\right)$$

Finally, differentiation gives the density:

$$f_{T_n}(t) = \frac{d}{dt}F_{T_n}(t) = \frac{d}{dt}\frac{1}{2}F_{T_n^2}(t^2) = t \cdot f_{T_n^2}(t^2) = \cdots .$$

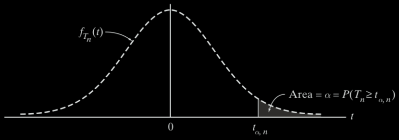
$\square$




```r
# Draw Student t-density
x=seq(-5,5,0.01)
pdf= cbind(dt(x, df = 1),
           dt(x, df = 2),
           dt(x, df = 5),
           dt(x, df = 100))
matplot(x,pdf, type = "l")
 title ("Student's t- distributions ")
```

t Table

	α						
df	.20	.15	.10	.05	.025	.01	.005
1	1.376	1.963	3.078	6.3138	12.706	31.821	63.657
2	1.061	1.386	1.886	2.9200	4.3027	6.965	9.9248
3	0.978	1.250	1.638	2.3534	3.1825	4.541	5.8409
4	0.941	1.190	1.533	2.1318	2.7764	3.747	4.6041
5	0.920	1.156	1.476	2.0150	2.5706	3.365	4.0321
6	0.906	1.134	1.440	1.9432	2.4469	3.143	3.7074
⋮			⋮				
30	0.854	1.055	1.310	1.6973	2.0423	2.457	2.7500
∞	0.84	1.04	1.28	1.64	1.96	2.33	2.58



$$\mathbb{P}\left(T_3 > 4.541\right) = 0.01 \quad \Longleftrightarrow \quad t_{0.01,3} = 4.541$$

```
> 1-pt(4.541, df =3)
[1] 0.009998238
```

```
> alpha = 0.01
> qt(1-alpha, df = 3)
[1] 4.540703
```

```
> 1 - scipy.stats.t.cdf(4.541, 3)
[1] 0.00999823806449407
```

```
> scipy.stats.t.ppf(1-0.01, 3)
[1] 4.540702858698419
```

Thm 7.3.5. Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

$$T_{n-1} = \frac{\overline{Y} - \mu}{S/\sqrt{n}} \quad \sim \quad \text{Student's t of degree } n-1.$$

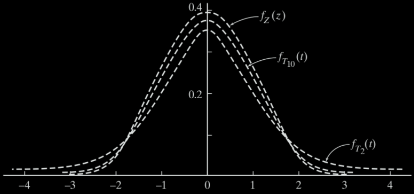
Proof.

$$\frac{\overline{Y} - \mu}{S/\sqrt{n}} = \frac{\dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2(n-1)}}}$$

$$\frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \qquad \perp \qquad \frac{(n-1)S^2}{\sigma^2} \sim \text{Chi Square}(n-1)$$

By Def. 7.3.3 ... $\square$

As $n \to \infty$, Students' t distribution will converge to $N(0,1)$:



Thm 7.3.6. $f_{T_n}(x) \to f_Z(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ as $n \to \infty$, where $Z \sim N(0,1)$.

Proof By Stirling's formula:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z)) \qquad \text{as } z \to \infty$$

$$\Longrightarrow \quad \lim_{n\to\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} = \frac{1}{\sqrt{2\pi}}$$

...... $\square$