Math 362: Mathematical Statistics II

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Chapter 7. Inference Based on The Normal Distribution

§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.4 Drawing Inferences About μ

§ 7.5 Drawing Inferences About σ^2

Plan

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For a random sample of size n from $N(\mu, \sigma^2)$:

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(Y_i - \overline{Y}\right)^2$$

$$\Downarrow$$

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(Y_i - \overline{Y}\right)^2 \sim \text{Chi Square}(n-1)$$

$$\mathbb{P}\left(\chi^2_{\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{1-\alpha/2,n-1}\right) = 1 - \alpha.$$

$100(1-\alpha)\%$ C.I. for σ^2:

$$\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right)$$

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Testing $H_0 : \sigma^2 = \sigma_0^2$

v.s.

(at the α level of significance)

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

$H_1 : \sigma^2 < \sigma_0^2$:

Reject H_0 if

$\chi^2 \leq \chi^2_{\alpha,n-1}$

$H_1 : \sigma^2 \neq \sigma_0^2$:

Reject H_0 if

$\chi^2 \leq \chi^2_{\alpha/2,n-1}$ or

$\chi^2 \geq \chi^2_{1-\alpha/2,n-1}$

$H_1 : \sigma^2 > \sigma_0^2$:

Reject H_0 if

$\chi^2 \geq \chi^2_{1-\alpha,n-1}$

E.g. 1. The width of a confidence interval for σ^2 is a function of n and S^2:

$$W = \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} - \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

Find the smallest n such that the average width of a 95% C.I. for σ^2 is no greater than $0.8\sigma^2$.

Sol. Notice that $\mathbb{E}[S^2] = \sigma^2$. Hence, we need to find n s.t.

$$(n-1)\left(\frac{1}{\chi^2_{0.025,n-1}} - \frac{1}{\chi^2_{0.975,n-1}}\right) \leq 0.8.$$

Trial and error (numerics on R) gives $n = 57$.

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```
> # Example 7.5.1
> n=seq(45,60,1)
> l=qchisq(0.025,n-1)
> u=qchisq(0.975,n-1)
> e=(n-1)* (1/l-1/u)
> m=cbind(n,l,u,e)
> colnames(m) = c("n",
+                  "chi(0.025,n-1)",
+                  "chi(0.975,n-1)",
+                  "error")
> m
       n chi(0.025,n-1) chi(0.975,n-1)     error
 [1,] 45       27.57457       64.20146 0.9103307
 [2,] 46       28.36615       65.41016 0.8984312
 [3,] 47       29.16005       66.61653 0.8869812
 [4,] 48       29.95620       67.82065 0.8759533
 [5,] 49       30.75451       69.02259 0.8653224
 [6,] 50       31.55492       70.22241 0.8550654
 [7,] 51       32.35736       71.42020 0.8451612
 [8,] 52       33.16179       72.61599 0.8355901
 [9,] 53       33.96813       73.80986 0.8263340
[10,] 54       34.77633       75.00186 0.8173761
[11,] 55       35.58634       76.19205 0.8087008
[12,] 56       36.39811       77.38047 0.8002937

[13,] 57       37.21159       78.56716 0.7921414
[14,] 58       38.02674       79.75219 0.7842313
[15,] 59       38.84351       80.93559 0.7765517
[16,] 60       39.66186       82.11741 0.7690918
```

Case Study 7.5.2

Mutual funds are investment vehicles consisting of a portfolio of various types of investments. If such an investment is to meet annual spending needs, the owner of shares in the fund is interested in the average of the annual returns of the fund. Investors are also concerned with the volatility of the annual returns, measured by the variance or standard deviation. One common method of evaluating a mutual fund is to compare it to a benchmark, the Lipper Average being one of these. This index number is the average of returns from a universe of mutual funds.

The Global Rock Fund is a typical mutual fund, with heavy investments in international funds. It claimed to best the Lipper Average in terms of volatility over the period from 1989 through 2007. Its returns are given in the table below.

Year	Investment Return %	Year	Investment Return %
1989	15.32	1999	27.43
1990	1.62	2000	8.57
1991	28.43	2001	1.88
1992	11.91	2002	−7.96
1993	20.71	2003	35.98
1994	−2.15	2004	14.27
1995	23.29	2005	10.33
1996	15.96	2006	15.94
1997	11.12	2007	16.71
1998	0.37		

The standard deviation for these returns is 11.28%, while the corresponding figure for the Lipper Average is 11.67%. Now, clearly, the Global Rock Fund has a smaller standard deviation than the Lipper Average, but is this small difference due just to random variation? The hypothesis test is meant to answer such questions.

$$H_0: \sigma^2 = (11.67)^2$$

versus

$$H_1: \sigma^2 < (11.67)^2$$

Let $\alpha = 0.05$. With $n = 19$, the critical value for the chi square ratio [from part (b) of Theorem 7.5.2] is $\chi^2_{1-\alpha,n-1} = \chi^2_{.05,18} = 9.390$ (see Figure 7.5.3). But

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(19-1)(11.28)^2}{(11.67)^2} = 16.82$$

so our decision is clear: Do not reject H_0.