Math 362: Mathematical Statistics II

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§ 5.6 Sufficient Estimators

Rationale: Let $\widehat{\theta}$ be an estimator to the unknown parameter θ . Whether does $\widehat{\theta}$ contain all information about θ ?

Equivalently, how can one reduce the random sample of size n, denoted by (X_1, \dots, X_n) , to a function without losing any information about θ ?

E.g., let's choose the function $h(X_1, \dots, X_n) := \frac{1}{n} \sum_{i=1}^n X_i$, the sample mean. In many cases, $h(X_1, \dots, X_n)$ contains all relevant information about the true mean $\mathbb{E}(X)$. In that case, $h(X_1, \dots, X_n)$, as an estimator, is sufficient.

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Definition. Let (X_1, \cdots, X_n) be a random sample of size n from a discrete population with a unknown parameter θ , of which $\widehat{\theta}$ (resp. θ_e) be an estimator (resp. estimate). We call $\widehat{\theta}$ and θ_e **sufficient** if

$$\mathbb{P}\left(X_1=k_1,\cdots,X_n=k_n\ \middle|\ \widehat{\theta}=\theta_{\theta}\right)=b(k_1,\cdots,k_n) \tag{Sufficency-1}$$

is a function that does not depend on θ .

In case for random sample (Y_1, \cdots, Y_n) from the continuous population, (Sufficency-1) should be

$$f_{Y_1,\dots,Y_n\mid\widehat{\theta}=\theta_e}\left(y_1,\dots,y_n\mid\widehat{\theta}=\theta_e\right)=b(y_1,\dots,y_n)$$

Note:
$$\widehat{\theta} = h(X_1, \dots, X_n)$$
 and $\theta_e = h(k_1, \dots, k_n)$.
or $\widehat{\theta} = h(Y_1, \dots, Y_n)$ and $\theta_e = h(y_1, \dots, y_n)$.

Equivalently,

Definition. ... $\widehat{\theta}$ (or θ_e) is **sufficient** if the likelihood function can be factorized as:

$$L(\theta) = \begin{cases} \prod_{i=1}^n p_X(k_i; \theta) = g(\theta_\theta, \theta) \ b(k_1, \cdots, k_n) & \text{Discrete} \\ \prod_{i=1}^n f_Y(y_i; \theta) = g(\theta_\theta, \theta) \ b(y_1, \cdots, y_n) & \text{Continous} \end{cases}$$
(Sufficency-2)

where g is a function of two arguments only and b is a function that does not depend on θ .

E.g. 1. A random sample of size n from Bernoulli(P). $\widehat{p} = \sum_{i=1}^{n} X_i$. Check sufficiency of \widehat{p} for p by (Sufficency-1):

Case I: If
$$k_1,\cdots,k_n\in\{0,1\}$$
 such that $\sum_{i=1}^n k_i\neq c$, then
$$\mathbb{P}\left(X_1=k_1,\cdots,X_n=k_n\ \middle|\ \widehat{p}=c\right)=0.$$

Case II: If $k_1, \dots, k_n \in \{0, 1\}$ such that $\sum_{i=1}^n k_i = c$, then

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= \frac{\mathbb{P}\left(X_{1} = k_{1}, \dots, X_{n-1} = k_{n-1}, X_{n} = c - \sum_{i=1}^{n-1} k_{i}\right)}{\mathbb{P}\left(\sum_{i=1}^{n} X_{i} = c\right)} \\
= \frac{\left(\prod_{i=1}^{n-1} p^{k_{i}} (1 - p)^{1 - k_{i}}\right) \times p^{c - \sum_{i=1}^{n-1} k_{i}} (1 - p)^{1 - c + \sum_{i=1}^{n-1} k_{i}}}{\binom{n}{c} p^{c} (1 - p)^{n - c}} \\
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In summary,

$$\mathbb{P}\left(X_1=k_1,\cdots,X_n=k_n\ \middle|\ \widehat{p}=c\right)=\begin{cases} \frac{1}{\binom{n}{c}} & \text{if } k_i\in\{0,1\} \text{ s.t. } \sum_{i=1}^n k_i=c,\\ 0 & \text{otherwise}. \end{cases}$$

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Notice that $p_e = \sum_{i=1}^n k_i$. Then

$$L(p) = \prod_{i=1}^{n} p_X(k_i; p) = \prod_{i=1}^{n} p^{k_i} (1-p)^{1-k_i}$$

$$= p^{\sum_{i=1}^{n} k_i} (1-p)^{n-\sum_{i=1}^{n} k_i}$$

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Therefore, p_e (or \hat{p}) is sufficient since (Sufficency-2) is satisfied with

$$g(p_e, p) = p^{p_e}(1-p)^{n-p_e}$$
 and $b(k_1, \dots, k_n) = 1$.

Comment 1. The estimator \hat{p} is sufficient but not unbiased since $\mathbb{E}(\hat{p}) = np
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2. Any one-to-one function of a sufficient estimator is again a sufficient estimator. E.g., 6: = 5 which is a unbiased, sufficient, and MVE.

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- Comment 1. The estimator \hat{p} is sufficient but not unbiased since $\mathbb{E}(\hat{p}) = np \neq p$.
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 - 3. $\widehat{p}_3 := X_1$ is not sufficient!

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