

# Math 362: Mathematical Statistics II

Le Chen

le.chen@emory.edu  
chenle02@gmail.com

Emory University  
Atlanta, GA

Last updated on Spring 2021  
Last compiled on January 15, 2023

2021 Spring

Creative Commons License  
(CC By-NC-SA)

# Chapter 5. Estimation

§ 5.1 Introduction

§ 5.2 Estimating parameters: MLE and MME

§ 5.3 Interval Estimation

§ 5.4 Properties of Estimators

§ 5.5 Minimum-Variance Estimators: The Cramér-Rao Lower Bound

§ 5.6 Sufficient Estimators

§ 5.7 Consistency

§ 5.8 Bayesian Estimation

# Chapter 5. Estimation

§ 5.1 Introduction

§ 5.2 Estimating parameters: MLE and MME

§ 5.3 Interval Estimation

§ 5.4 Properties of Estimators

§ 5.5 Minimum-Variance Estimators: The Cramér-Rao Lower Bound

§ 5.6 Sufficient Estimators

**§ 5.7 Consistency**

§ 5.8 Bayesian Estimation

**Definition.** An estimator  $\hat{\theta}_n = h(W_1, \dots, W_n)$  is said to be **consistent** if it converges to  $\theta$  *in probability*, i.e., for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( |\hat{\theta}_n - \theta| < \epsilon \right) = 1.$$

Comment: In the  $\epsilon$ - $\delta$  language, the above **convergence in probability** says

$$\forall \epsilon > 0, \forall \delta > 0, \exists n(\epsilon, \delta) > 0, \text{ s.t. } \forall n \geq n(\epsilon, \delta),$$

$$\mathbb{P} \left( |\hat{\theta}_n - \theta| < \epsilon \right) > 1 - \delta.$$

A useful tool to check convergence in probability is

**Theorem.** (Chebyshev's inequality) Let  $W$  be any r.v. with finite mean  $\mu$  and variance  $\sigma^2$ . Then for any  $\epsilon > 0$

$$\mathbb{P}(|W - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2},$$

or, equivalently,

$$\mathbb{P}(|W - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

Proof. ...



As a consequence of Chebyshev's inequality, we have

**Proposition.** The sample mean  $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n W_i$  is consistent for  $\mathbb{E}(W) = \mu$ , provided that the population  $W$  has finite mean  $\mu$  and variance  $\sigma^2$ .

Proof.

$$\mathbb{E}(\hat{\mu}_n) = \mu \quad \text{and} \quad \text{Var}(\hat{\mu}_n) = \frac{\sigma^2}{n}.$$

$$\forall \epsilon > 0, \quad \mathbb{P}(|\hat{\mu}_n - \mu| \leq \epsilon) \geq 1 - \frac{\sigma^2}{n\epsilon^2} \rightarrow 1.$$

□

E.g. 1. Let  $Y_1, \dots, Y_n$  be a random sample of size  $n$  from the uniform pdf  $f_Y(y; \theta) = 1/\theta$ ,  $y \in [0, \theta]$ . Let  $\hat{\theta}_n = Y_{\max}$ . We know that  $Y_{\max}$  is biased. Is it consistent?

**Sol.** The c.d.f. of  $Y$  is equal to  $F_Y(y) = y/\theta$  for  $y \in [0, \theta]$ . Hence,

$$f_{Y_{\max}}(y) = nF_Y(y)^{n-1}f_Y(y) = \frac{ny^{n-1}}{\theta^n}, \quad y \in [0, \theta].$$

Therefore,

$$\begin{aligned}\mathbb{P}(|\hat{\theta}_n - \theta| < \epsilon) &= \mathbb{P}(\theta - \epsilon < \hat{\theta}_n < \theta + \epsilon) \\ &= \int_{\theta-\epsilon}^{\theta} \frac{ny^{n-1}}{\theta^n} dy + \int_{\theta}^{\theta+\epsilon} 0 dy \\ &= 1 - \left(\frac{\theta - \epsilon}{\theta}\right)^n \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

□

E.g. 2. Suppose  $Y_1, Y_2, \dots, Y_n$  is a random sample from the exponential pdf,  $f_Y(y; \lambda) = \lambda e^{-\lambda y}$ ,  $y > 0$ . Show that  $\hat{\lambda}_n = Y_1$  is not consistent for  $\lambda$ .

**Sol.** To prove  $\hat{\lambda}_n$  is not consistent for  $\lambda$ , we need only to find out one  $\epsilon > 0$  such that the following limit does not hold:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( |\hat{\lambda}_n - \lambda| < \epsilon \right) = 1. \quad (3)$$

We can choose  $\epsilon = \lambda/m$  for any  $m \geq 1$ . Then

$$\begin{aligned} |\hat{\lambda}_n - \lambda| \leq \frac{\lambda}{m} &\iff \left(1 - \frac{1}{m}\right) \lambda \leq \hat{\lambda}_n \leq \left(1 + \frac{1}{m}\right) \lambda \\ &\implies \hat{\lambda}_n \geq \left(1 - \frac{1}{m}\right) \lambda. \end{aligned}$$



Hence,

$$\begin{aligned}\mathbb{P}\left(|\hat{\lambda}_n - \lambda| < \frac{\lambda}{m}\right) &\leq \mathbb{P}\left(\hat{\lambda}_n \geq \left(1 - \frac{1}{m}\right)\lambda\right) \\ &= \mathbb{P}\left(Y_1 \geq \left(1 - \frac{1}{m}\right)\lambda\right) \\ &= \int_{\left(1 - \frac{1}{m}\right)\lambda}^{\infty} \lambda e^{-\lambda y} dy \\ &= e^{-\left(1 - \frac{1}{m}\right)\lambda^2} < 1.\end{aligned}$$

Therefore, the limit in (3) cannot hold. □