

Math 362: Mathematical Statistics II

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Chapter 10. Goodness-of-fit Tests

§ 10.1 Introduction

§ 10.2 The Multinomial Distribution

§ 10.3 Goodness-of-Fit Tests: All Parameters Known

§ 10.4 Goodness-of-Fit Tests: Parameters Unknown

§ 10.5 Contingency Tables

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§ 10.5 Contingency Tables

E.g. 1 Whether are the two ratings independent?

| Table 10.5.5 | | | | | |
|---------------------|----------|---------------|----------|----|-------|
| | | Ebert Ratings | | | Total |
| | | Down | Sideways | Up | |
| Siskel Ratings | Down | 24 | 8 | 13 | 45 |
| | Sideways | 8 | 13 | 11 | 32 |
| | Up | 10 | 9 | 64 | 83 |
| | Total | 42 | 30 | 88 | 160 |

E.g. 2 Whether is the suicide rate independent of the mobility factor?

Table 10.5.7

| City | Suicides per 100,000, x_i | Mobility Index, y_i | City | Suicides per 100,000, x_i | Mobility Index, y_i |
|---------------|--------------------------------|--------------------------|--------------|--------------------------------|--------------------------|
| New York | 19.3 | 54.3 | Washington | 22.5 | 37.1 |
| Chicago | 17.0 | 51.5 | Minneapolis | 23.8 | 56.3 |
| Philadelphia | 17.5 | 64.6 | New Orleans | 17.2 | 82.9 |
| Detroit | 16.5 | 42.5 | Cincinnati | 23.9 | 62.2 |
| Los Angeles | 23.8 | 20.3 | Newark | 21.4 | 51.9 |
| Cleveland | 20.1 | 52.2 | Kansas City | 24.5 | 49.4 |
| St. Louis | 24.8 | 62.4 | Seattle | 31.7 | 30.7 |
| Baltimore | 18.0 | 72.0 | Indianapolis | 21.0 | 66.1 |
| Boston | 14.8 | 59.4 | Rochester | 17.2 | 68.0 |
| Pittsburgh | 14.9 | 70.0 | Jersey City | 10.1 | 56.5 |
| San Francisco | 40.0 | 43.8 | Louisville | 16.6 | 78.7 |
| Milwaukee | 19.3 | 66.2 | Portland | 29.3 | 33.2 |
| Buffalo | 13.8 | 67.6 | | | |

$$\bar{x} = 20.8 \quad \text{and} \quad \bar{y} = 56.0$$

Table 10.5.8

| | | Mobility Index | |
|-----------------|----------------------|----------------|----------------------|
| | | Low (<56.0) | High (≥ 56.0) |
| Suicide Rate | High (≥ 20.8) | 7 | 4 |
| | Low (<20.8) | 3 | 11 |

Thm 10.4.1 Suppose that n observations are taken on a sample space partitioned by the events A_1, \dots, A_r and B_1, \dots, B_c .

Let $p_i = \mathbb{P}(A_i)$, $q_j = \mathbb{P}(B_j)$, $p_{ij} = \mathbb{P}(A_i \cap B_j)$.

Let X_{ij} be the number of observations belonging to $A_i \cap B_j$.

a) Provided that $np_{ij} \geq 5$ for all i, j , the r.v.

$$D_2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(X_{ij} - np_{ij})^2}{np_{ij}} \sim \text{Chi square of f.d. } rc - 1$$

b) To test $H_0 : A_i$'s are independent of B_j 's, calculate the test statistic

$$d_2 = \sum_{i=1}^r \sum_{j=1}^c \frac{(k_{ij} - n\hat{p}_i\hat{q}_j)^2}{n\hat{p}_i\hat{q}_j}$$

where \hat{p}_i and \hat{q}_j are MLE's for p_i and q_j , respectively.

Provided that $n\hat{p}_i\hat{q}_j \geq 5$ for all i, j , the critical region is

$$(\chi_{1-\alpha, (r-1)(c-1)}^2, +\infty)$$

E.g. 1 Sol: Compute the expected frequencies:

| Table 10.5.6 | | | | | |
|---------------------|----------|---------------|-------------|--------------|-------|
| | | Ebert Ratings | | | Total |
| | | Down | Sideways | Up | |
| Siskel Ratings | Down | 24 (11.8) | 8 (8.4) | 13 (24.8) | 45 |
| | Sideways | 8 (8.4) | 13 (6.0) | 11 (17.6) | 32 |
| | Up | 10 (21.8) | 9 (15.6) | 64 (45.6) | 83 |
| | Total | 42 | 30 | 88 | 160 |

$$\implies d_2 = \dots = 45.37$$

Critical region is

$$(\chi_{0.99, (3-1) \times (3-1)}^2, +\infty) = (13.277, +\infty)$$

$$\text{Alternatively } P\text{-value} = \mathbb{P}(\chi_4^2 \geq 45.37) = 3.33 \times 10^{-9}.$$

Rejection at $\alpha = 0.01$.

□

E.g. 2 Sol: Compute the expected frequencies:

| Table 10.5.9 | | | |
|---|--------------|----------------|--------------|
| | | Mobility Index | |
| | | Low (<56.0) | High (≥56.0) |
| Suicide Rate | High (≥20.8) | 4.4* | 6.6 |
| | Low (<20.8) | 5.6 | 8.4 |
| * $\hat{E}(X_{11}) = 4.4$ does not quite satisfy the " $n\hat{p}_i\hat{q}_j \geq 5$ " restriction stated in Theorem 10.5.1, but 4.4 is close enough to 5 to maintain the integrity of the χ^2 approximation. | | | |

$$\implies d_2 = \dots = 4.57$$

Critical region is

$$(\chi_{0.95, (2-1) \times (2-1)}^2, +\infty) = (3.41, +\infty)$$

Alternatively $P\text{-value} = \mathbb{P}(\chi_1^2 \geq 4.57) = 0.033$

Rejection at $\alpha = 0.05$.

