Math 362: Mathematical Statistics II

Le Chen
le.chen@emory.edu
chenle02@gmail.com

Emory University
Atlanta, GA

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Chapter 5. Estimation

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Rationale: Let W be an estimator dependent on a parameter θ.

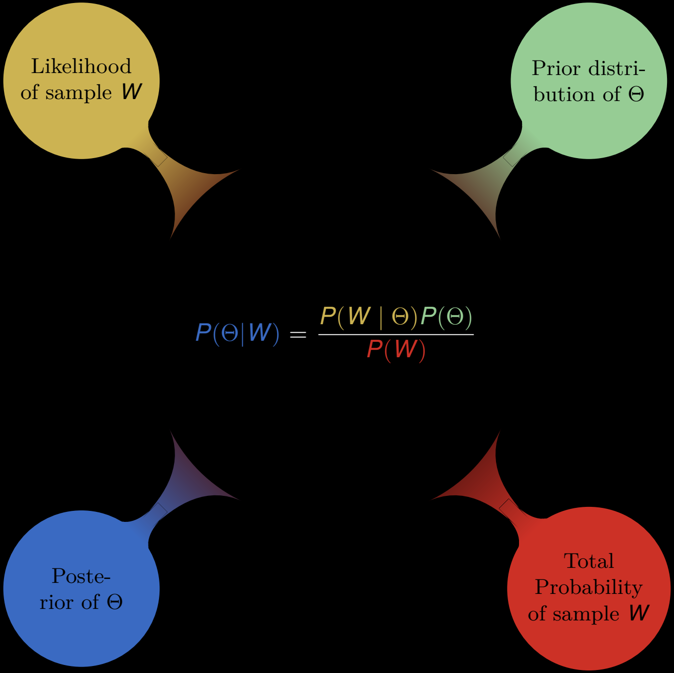
1. Frequentists view θ as a parameter whose exact value is to be estimated.

2. Bayesians view θ is the value of a random variable Θ.

   One can incorporate our knowledge on Θ — the **prior distribution** $p_\Theta(\theta)$ if Θ is discrete and $f_\Theta(\theta)$ if Θ is continuous — and use Bayes' formula to update our knowledge on Θ upon new observation $W = w$:

$$g_\Theta(\theta|W = w) = \begin{cases} \dfrac{p_W(w|\Theta = \theta)p_\Theta(\theta)}{\mathbb{P}(W = w)} & \text{if } W \text{ is discrete} \\[2em] \dfrac{f_W(w|\Theta = \theta)f_\Theta(\theta)}{f_W(w)} & \text{if } W \text{ is continuous} \end{cases}$$

   where $g_\Theta(\theta|W = w)$ is called **posterior distribution** of Θ.



Likelihood of sample W

Prior distribution of Θ

$$P(\Theta|W) = \frac{P(W \mid \Theta)P(\Theta)}{P(W)}$$

Posterior of Θ

Total Probability of sample W

Four cases for computing posterior distribution

$g_\Theta(\theta \mid W = w)$	W discrete	W continuous
Θ discrete	$\dfrac{p_W(w \mid \Theta = \theta) p_\Theta(\theta)}{\sum_i p_W(w \mid \Theta = \theta_i) p_\Theta(\theta_i)}$	$\dfrac{f_W(w \mid \Theta = \theta) p_\Theta(\theta)}{\sum_i f_W(w \mid \Theta = \theta_i) p_\Theta(\theta_i)}$
Θ continuous	$\dfrac{p_W(w \mid \Theta = \theta) f_\Theta(\theta)}{\int_{\mathbb{R}} p_W(w \mid \Theta = \theta') f_\Theta(\theta') \mathrm{d}\theta'}$	$\dfrac{f_W(w \mid \Theta = \theta) f_\Theta(\theta)}{\int_{\mathbb{R}} f_W(w \mid \Theta = \theta') f_\Theta(\theta') \mathrm{d}\theta'}$

Gamma distributions

$$\Gamma(r) := \int_0^\infty y^{r-1} e^{-y} \mathrm{d}y, \quad r > 0.$$

Two parametrizations for **Gamma distributions**:

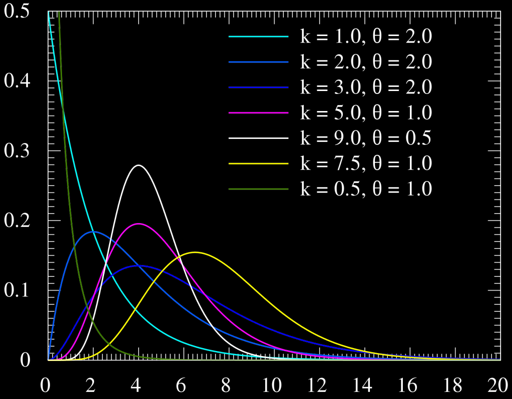
1. With a shape parameter r and a scale parameter θ:

$$f_Y(y; r, \theta) = \frac{y^{r-1} e^{-y/\theta}}{\theta^r \Gamma(r)}, \qquad y > 0, r, \theta > 0.$$

2. With a shape parameter r and a rate parameter $\lambda = 1/\theta$,

$$f_Y(y; r, \lambda) = \frac{\lambda^r y^{r-1} e^{-\lambda y}}{\Gamma(r)}, \qquad y > 0, r, \lambda > 0.$$

$$\mathbb{E}[Y] = \frac{r}{\lambda} = r\theta \quad \text{and} \quad \mathrm{Var}(Y) = \frac{r}{\lambda^2} = r\theta^2$$



```
# Plot  gamma distributions
x  = seq(0,20,0.01)
k= 3 # Shape parameter
theta  = 0.5 # Scale parameter
plot(x,dgamma(x, k, scale = theta),
       type="l",
       col="red")
```

Beta distributions

$$B(\alpha,\beta) := \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\mathrm{d}y, \quad \alpha,\beta > 0.$$

$$\vdots \qquad \vdots$$

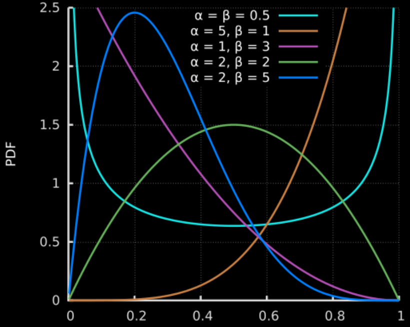
$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

(see Appendix)

Beta distribution

$$f_Y(y;\alpha,\beta) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, \quad y \in [0,1], \alpha,\beta > 0.$$

$$\mathbb{E}[Y] = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \mathrm{Var}(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$



```
    # Plot Beta distributions
x = seq(0,1,0.01)
a = 13
b = 2
plot (x,dbeta(x,a,b),
      type="l",
      col="red")
```

E.g. 1. Let $X_1, \cdots, X_n$ be a random sample from Bernoulli(θ): $p_{X_i}(k;\theta) = \theta^k(1-\theta)^{1-k}$ for $k = 0, 1$.

Let $X = \sum_{i=1}^n X_i$. Then X follows binomial(n,θ).

Prior distribution: Θ ~beta(r,s), i.e., $f_\Theta(\theta) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1}$ for $\theta \in [0,1]$.

$$\begin{array}{rcl} X_1,\cdots,X_n \,\big|\theta & \sim & \text{Bernoulli}(\theta) \\ \Theta & \sim & \text{Beta}(r,s) \\ & & r \;\&\; s \text{ are known} \end{array}$$

$$\begin{array}{rcl} X = \sum_{i=1}^n X_i \,\Big|\theta & \sim & \text{Binomial}(n,\theta) \\ \Theta & \sim & \text{Beta}(r,s) \\ & & r \;\&\; s \text{ are known} \end{array}$$

Example 5.8.2

Max, a video game pirate (and Bayesian), is trying to decide how many illegal copies of *Zombie Beach Party* to have on hand for the upcoming holiday season. To get a rough idea of what the demand might be, he talks with n potential customers and finds that $X = k$ would buy a copy for a present (or for themselves). The obvious choice for a probability model for X, of course, would be the binomial pdf. Given n potential customers, the probability that k would actually buy one of Max's illegal copies is the familiar

$$p_X(k \mid \theta) = \binom{n}{k} \theta^k (1-\theta)^{n-k}, \quad k = 0, 1, \ldots, n$$

where the maximum likelihood estimate for θ is given by $\theta_e = \frac{k}{n}$.

It may very well be the case, though, that Max has some additional insight about the value of θ on the basis of similar video games that he illegally marketed in previous years. Suppose he suspects, for example, that the percentage of potential customers who will buy *Zombie Beach Party* is likely to be between 3% and 4% and probably will not exceed 7%. A reasonable prior distribution for Θ, then, would be a pdf mostly concentrated over the interval 0 to 0.07 with a mean or median in the 0.035 range.

One such probability model whose shape would comply with the restraints that Max is imposing is the *beta pdf*. Written with Θ as the random variable, the (two-parameter) beta pdf is given by

$$f_\Theta(\theta) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1}, \quad 0 \le \theta \le 1$$

The beta distribution with $r = 2$ and $s = 4$ is pictured in Figure 5.8.1. By choosing different values for r and s, $f_\Theta(\theta)$ can be skewed more sharply to the right or to the left, and the bulk of the distribution can be concentrated close to zero or close to one. The question is, if an appropriate beta pdf is used as a *prior* distribution for Θ, and if a random sample of k potential customers (out of n) said they would buy the video game, what would be a reasonable *posterior* distribution for Θ?

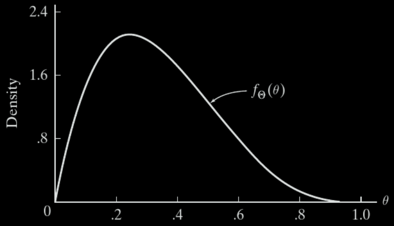


Figure 5.8.1

X is discrete and Θ is continuous.

$$g_\Theta(\theta|X=k) = \frac{p_X(k|\Theta=\theta)f_\Theta(\theta)}{\int_\mathbb{R} p_X(k|\Theta=\theta')f_\Theta(\theta')\mathrm{d}\theta'}$$

$$\begin{aligned} p_X(k|\Theta=\theta)f_\Theta(\theta) &= \binom{n}{k}\theta^k(1-\theta)^{n-k} \times \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1} \\ &= \binom{n}{k}\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{k+r-1}(1-\theta)^{n-k+s-1}, \quad \theta \in [0,1]. \end{aligned}$$

$$\begin{aligned} p_X(k) &= \int_\mathbb{R} p_X(k|\Theta=\theta')f_\Theta(\theta')\mathrm{d}\theta' \\ &= \binom{n}{k}\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\int_0^1 \theta'^{k+r-1}(1-\theta')^{n-k+s-1}\mathrm{d}\theta' \\ &= \binom{n}{k}\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \times \frac{\Gamma(k+r)\Gamma(n-k+s)}{\Gamma((k+r)+(n-k+s))} \end{aligned}$$

$$
\begin{aligned}
g_\Theta(\theta|X=k) =& \frac{\binom{n}{k}\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \times \theta^{k+r-1}(1-\theta)^{n-k+s-1}}{\binom{n}{k}\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \times \frac{\Gamma(k+r)\Gamma(n-k+s)}{\Gamma((k+r)+(n-k+s))}} \\
=& \frac{\Gamma(n+r+s)}{\Gamma(k+r)\Gamma(n-k+s)}\theta^{k+r-1}(1-\theta)^{n-k+s-1}, \qquad \theta \in [0,1]
\end{aligned}
$$

Conclusion: the posterior $\sim$ beta distribution$(k+r, n-k+s)$.

Recall that the prior $\sim$ beta distribution(r, s).

It remains to determine the values of r and s to incorporate the prior knowledge:

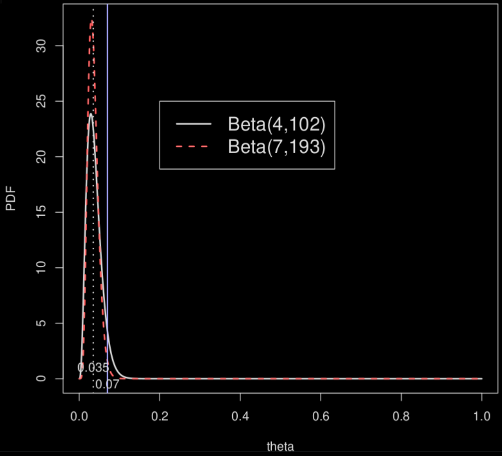
PK 1. Mean is about 0.035.

$$\mathbb{E}(\Theta) = 0.035 \quad \Longrightarrow \quad \frac{r}{r+s} = 0.035 \quad \Longleftrightarrow \quad \frac{r}{s} = \frac{7}{193}$$

PK 2. The pdf mostly concentrated over $[0, 0.07]$. ... trial ...

```
x <- seq(0, 1, length = 1025)
plot(x,dbeta(x,4,102),
     type="l")
plot(x,dbeta(x,7,193),
     type="l")
dev.off()

pdf=cbind(dbeta(x,4,102),dbeta(x,7,193))
matplot(x,pdf,
        type="l",
        lty = 1:2,
        xlab = "theta", ylab = "PDF",
        lwd = 2 # Line width
        )
legend(0.2, 25, # Position of legend
       c("Beta(4,102)", "Beta(7,193)"),
       col = 1:2, lty = 1:2,
       ncol = 1, # Number of columns
       cex = 1.5, # Fontsize
       lwd=2 # Line width
       )
abline(v=0.07, col="blue", lty=1,lwd=1.5)
text(0.07, -0.5, "0.07")
abline(v=0.035, col="gray60", lty=3,lwd=2)
text(0.035, 1, "0.035")
```



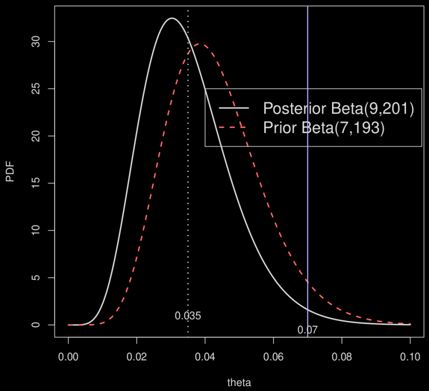
If we choose $r = 7$ and $s = 193$:

$$g_\Theta(\theta|X = k) = \frac{\Gamma(n + 200)}{\Gamma(k + 7)\Gamma(n - k + 193)} \theta^{k+6} (1 - \theta)^{n-k+192}, \qquad \theta \in [0, 1]$$

Moreover, if $n = 10$ and $k = 2$,

$$g_\Theta(\theta|X = k) = \frac{\Gamma(210)}{\Gamma(9)\Gamma(201)} \theta^{8} (1 - \theta)^{200}, \qquad \theta \in [0, 1]$$

```r
x <- seq(0, 0.1, length = 1025)
pdf=cbind(dbeta(x,7,193),dbeta(x,9,201))
matplot(x,pdf,
          type="l",
           lty  = 1:2,
          xlab = "theta", ylab = "PDF",
          lwd = 2 # Line width
)
legend(0.05, 25, # Position of legend
        c("Posterior Beta(9,201)", "Prior Beta(7,193)"),
        col  = 1:2,  lty  = 1:2,
        ncol  = 1, # Number of columns
        cex = 1.5, # Fontsize
        lwd=2 # Line width
)
abline(v=0.07,col="blue", lty =1,lwd=1.5)
text(0.07, -0.5, "0.07")
abline(v=0.035,col="black", lty =3,lwd=2)
text(0.035, 1, "0.035")
```



Definition. If the posterior distributions $p(\Theta|X)$ are in the same probability distribution family as the prior probability distribution $p(\Theta)$, the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function.

1. Beta distributions are conjugate priors for Bernoulli, binomial, nega. binomial, geometric likelihood.

2. Gamma distributions are conjugate priors for Poisson and exponential likelihood.

E.g. 2. Let $X_1, \cdots, X_n$ be a random sample from Poisson(θ): $p_X(k;\theta) = \frac{e^{-\theta}\theta^k}{k!}$ for $k = 0, 1, \cdots$.

Let $W = \sum_{i=1}^n X_i$. Then W follows Poisson$(n\theta)$.

Prior distribution: $\Theta \sim$ Gamma(s, μ), i.e., $f_\Theta(\theta) = \frac{\mu^s}{\Gamma(s)}\theta^{s-1}e^{-\mu\theta}$ for $\theta > 0$.

$$\begin{aligned} X_1, \cdots, X_n \,\big|\, \theta &\sim \text{Poisson}(\theta) \\ \Theta &\sim \text{Gamma}(s, \mu) \\ & \quad s \text{ \& } \mu \text{ are known} \end{aligned}$$

$$\begin{aligned} W = \sum_{i=1}^n X_i \,\Big|\, \theta &\sim \text{Poisson}(n\theta) \\ \Theta &\sim \text{Gamma}(s, \mu) \\ & \quad s \text{ \& } \mu \text{ are known} \end{aligned}$$

$$g_\Theta(\theta|W=w)=\frac{p_W(w|\Theta=\theta)f_\Theta(\theta)}{\int_{\mathbb{R}}p_W(w|\Theta=\theta')f_\Theta(\theta')\mathrm{d}\theta'}$$

$$\begin{aligned}p_W(w|\Theta=\theta)f_\Theta(\theta)&=\frac{e^{-n\theta}(n\theta)^w}{w!}\times\frac{\mu^s}{\Gamma(s)}\theta^{s-1}e^{-\mu\theta}\\&=\frac{n^w}{w!}\frac{\mu^s}{\Gamma(s)}\times\theta^{w+s-1}e^{-(\mu+n)\theta},\quad\theta>0.\end{aligned}$$

$$\begin{aligned}p_W(w)=&\int_{\mathbb{R}}p_W(w|\Theta=\theta')f_\Theta(\theta')\mathrm{d}\theta'\\=&\frac{n^w}{w!}\frac{\mu^s}{\Gamma(s)}\int_0^\infty\theta'^{w+s-1}e^{-(\mu+n)\theta'}\mathrm{d}\theta'\\=&\frac{n^w}{w!}\frac{\mu^s}{\Gamma(s)}\times\frac{\Gamma(w+s)}{(\mu+n)^{w+s}}\end{aligned}$$

$$\begin{aligned} g_\Theta(\theta|X=k) &= \frac{\dfrac{n^w}{w!}\dfrac{\mu^s}{\Gamma(s)} \times \theta^{w+s-1} e^{-(\mu+n)\theta}}{\dfrac{n^w}{w!}\dfrac{\mu^s}{\Gamma(s)} \times \dfrac{\Gamma(w+s)}{(\mu+n)^{w+s}}} \\ &= \frac{(\mu+n)^{w+s}}{\Gamma(w+s)} \theta^{w+s-1} e^{-(\mu+n)\theta}, \qquad \theta > 0. \end{aligned}$$

Conclusion: the posterior of $\Theta \sim$ gamma distribution$(w+s, n+\mu)$.

Recall that the prior of $\Theta \sim$ gamma distribution(s, μ).

Case Study 5.8.1

```r
x <- seq(0, 4, length = 1025)
pdf=cbind(dgamma(x, shape=88, rate=50),
          dgamma(x, shape=88+92, 100),
          dgamma(x, 88+92+72, 150))
matplot(x,pdf,
        type="l",
         lty  = 1:3,
        xlab = "theta", ylab = "PDF",
        lwd = 2 # Line width
)
legend(2, 3.5, # Position of legend
       c("Prior Gamma(88,50)",
         "Posterior1 Beta(180,100)",
         "Posterior2 Beta(252,150)"),
       col = 1:3, lty = 1:3,
       ncol = 1, # Number of columns
       cex = 1.5, # Fontsize
       lwd=2 # Line width
)
```

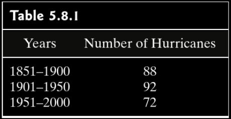
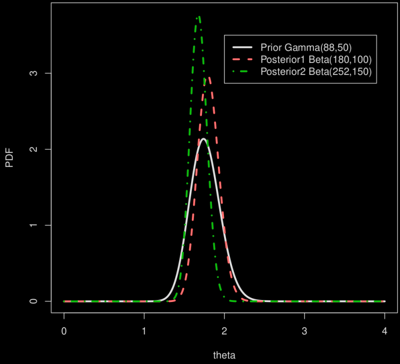


Table 5.8.1

Years	Number of Hurricanes
1851–1900	88
1901–1950	92
1951–2000	72



Bayesian Point Estimation

Question. Can one calculate an appropriate *point estimate* θ_e given the posterior $g_\Theta(\theta|W = w)$?

Definitions. Let θ_e be an estimate for θ based on a statistic W. The loss function associated with θ_e is denoted $L(\theta_e, \theta)$, where $L(\theta_e, \theta) \geq 0$ and $L(\theta, \theta) = 0$.

Let $g_\Theta(\theta|W = w)$ be the posterior distribution of the random variable Θ. Then the risk associated with $\widehat{\theta}$ is the expected value of the loss function with respect to the posterior distribution of Θ:

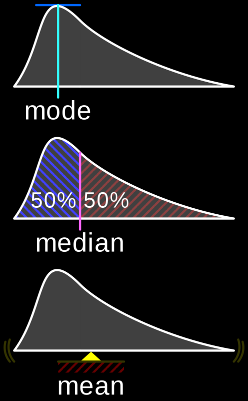
$$\text{risk} = \begin{cases} \displaystyle\int_{\mathbb{R}} L(\widehat{\theta}, \theta) g_\Theta(\theta|W = w)\,\mathrm{d}\theta & \text{if } \Theta \text{ is continuous} \\ \displaystyle\sum_i L(\widehat{\theta}, \theta_i) g_\Theta(\theta_i|W = w) & \text{if } \Theta \text{ is discrete} \end{cases}$$

Theorem. Let $g_\Theta(\theta|W = w)$ be the posterior distribution of the random variable Θ.

1. If $L(\theta_e, \theta) = |\theta_e - \theta|$, then the Bayes point estimate for θ is the median of $g_\Theta(\theta|W = w)$.
2. If $L(\theta_e, \theta) = (\theta_e - \theta)^2$, then the Bayes point estimate for θ is the mean of $g_\Theta(\theta|W = w)$.

Remarks

1. Median usually does not have a closed form formula.
2. Mean usually has a closed formula.



https://en.wikipedia.org

Proof. (of Part 1.)

Let m be the median of the random variable W. We first claim that

$$\mathbb{E}(|W - m|) \le \mathbb{E}(|W|). \tag{$\star$}$$

For any constant $b \in \mathbb{R}$, because

$$\frac{1}{2} = \mathbb{P}(W \le m) = \mathbb{P}(W - b \le m - b)$$

we see that $m - b$ is the median of $W - b$. Hence, by ($\star$),

$$\mathbb{E}\left(|W - m|\right) = \mathbb{E}\left(|(W - b) - (m - b)|\right) \le \mathbb{E}\left(|W - b|\right), \quad \textit{for all } b \in \mathbb{R},$$

which proves the statement.

Proof. (of Part 1. continued)

It remains to prove ($\star$)*. Without loss of generality, we may assume* $m > 0$*. Then*

$$\begin{aligned}
\mathbb{E}(|W-m|) &= \int_{\mathbb{R}} |w-m| f_W(w)dw \\
&= \int_{-\infty}^{m} (m-w) f_W(w)dw + \int_{m}^{\infty} (w-m) f_W(w)dw \\
&= -\int_{-\infty}^{m} w f_W(w)dw + \int_{m}^{\infty} w\, f_W(w)dw + \frac{1}{2}(m-m) \\
&= -\int_{-\infty}^{0} w f_W(w)dw - \underbrace{\int_{0}^{m} w f_W(w)dw}_{\geq 0} + \int_{m}^{\infty} w\, f_W(w)dw \\
&\leq -\int_{-\infty}^{0} w f_W(w)dw + \int_{0}^{\infty} w\, f_W(w)dw \\
&= \int_{\mathbb{R}} |w| f_W(w)dw \\
&= \mathbb{E}(|W|).
\end{aligned}$$

■

Proof. (of Part 2.)

Let μ be the mean of W. Then for any $b \in \mathbb{R}$, we see that

$$\begin{aligned}\mathbb{E}\left[(W-b)^2\right] &= \mathbb{E}\left[([W-\mu]+[\mu-b])^2\right] \\ &= \mathbb{E}\left[(W-\mu)^2\right] + 2(\mu-b)\underbrace{\mathbb{E}(W-\mu)}_{=0} + [\mu-b]^2 \\ &= \mathbb{E}\left[(W-\mu)^2\right] + [\mu-b]^2 \\ &\geq \mathbb{E}\left[(W-\mu)^2\right],\end{aligned}$$

that is,

$$\mathbb{E}\left[(W-\mu)^2\right] \leq \mathbb{E}\left[(W-b)^2\right], \quad \textit{for all } b \in \mathbb{R}.$$

■

E.g. 1'.

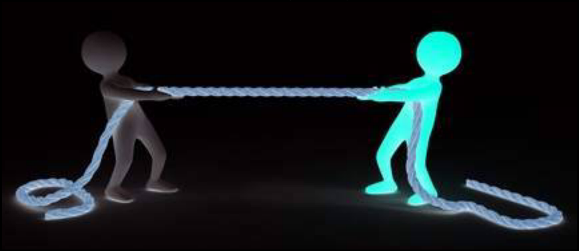
$$\begin{aligned} X_1, \cdots, X_n \,\Big|\theta &\sim \text{Bernoulli}(\theta) \\ \Theta &\sim \text{Beta}(r, s) \\ & r \;\&\; s \text{ are known} \end{aligned} \qquad \begin{aligned} X = \sum_{i=1}^{n} X_i \,\Big|\theta &\sim \text{Binomial}(n, \theta) \\ \Theta &\sim \text{Beta}(r, s) \\ & r \;\&\; s \text{ are known} \end{aligned}$$

Prior Beta$(r, s) \rightarrow$ posterior Beta$(k + r, n - k + s)$
upon observing $X = k$ for a random sample of size n.

Consider the L^2 loss function.

$$\begin{aligned} \theta_e &= \text{mean of Beta}(k + r, n - k + s) \\ &= \frac{k + r}{n + r + s} \\ &= \frac{n}{n + r + s} \times \underbrace{\left(\frac{k}{n}\right)}_{\text{MLE}} + \frac{r + s}{n + r + s} \times \underbrace{\left(\frac{r}{r + s}\right)}_{\text{Mean of Prior}} \end{aligned}$$

MLE vs. Prior



$$\theta_e$$

$$||$$

$$\frac{n}{n+r+s} \times \underbrace{\left(\frac{k}{n}\right)}_{\text{MLE}} + \frac{r+s}{n+r+s} \times \underbrace{\left(\frac{r}{r+s}\right)}_{\text{Mean of Prior}}$$

E.g. 2'.

$$\begin{aligned} X_1, \cdots, X_n \,\big|\, \theta &\sim \text{Poisson}(\theta) \\ \Theta &\sim \text{Gamma}(s, \mu) \\ & \quad s \ \&\ \mu \text{ are known} \end{aligned}$$

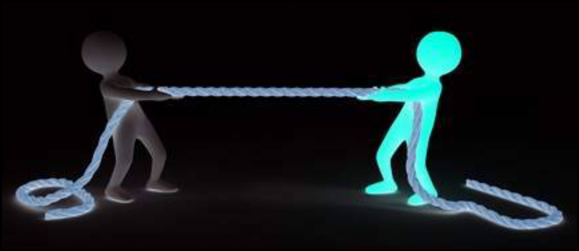
$$\begin{aligned} W = \sum_{i=1}^{n} X_i \,\Big|\, \theta &\sim \text{Poisson}(n\theta) \\ \Theta &\sim \text{Gamma}(s, \mu) \\ & \quad s \ \&\ \mu \text{ are known} \end{aligned}$$

Prior Gamma$(s, \mu) \to$ Posterior Gamma$(w + s, \mu + n)$
upon observing $W = w$ for a random sample of size n.

Consider the L^2 loss function.

$$\begin{aligned} \theta_e &= \text{mean of Gamma}(w + s, \mu + n) \\ &= \frac{w + s}{\mu + n} \\ &= \frac{n}{\mu + n} \times \underbrace{\left(\frac{w}{n}\right)}_{\text{MLE}} + \frac{\mu}{\mu + n} \times \underbrace{\left(\frac{s}{\mu}\right)}_{\text{Mean of Prior}} \end{aligned}$$

MLE vs. Prior



$$\theta_e$$

$$||$$

$$\frac{n}{\mu+n}\times\underbrace{\left(\frac{w}{n}\right)}_{\text{MLE}}+\frac{\mu}{\mu+n}\times\underbrace{\left(\frac{s}{\mu}\right)}_{\text{Mean of Prior}}$$

Appendix: Beta integral

Lemma. $$B(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Proof. Notice that

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx \quad \text{and} \quad \Gamma(\beta) = \int_0^\infty y^{\beta-1}e^{-y}dy.$$

Hence,

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty x^{\alpha-1}y^{\beta-1}e^{-(x+y)}dxdy.$$

The key in the proof is the following change of variables:

$$\begin{cases} x = r^2\cos^2(\theta) \\ y = r^2\sin^2(\theta) \end{cases}$$

$$\implies \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} 2r\cos^2(\theta) & 2r\sin^2(\theta) \\ -2r^2\cos(\theta)\sin(\theta) & 2r^2\cos(\theta)\sin(\theta) \end{pmatrix}$$

$$\implies \quad \left|\det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)\right| = 4r^3\sin(\theta)\cos(\theta).$$

Therefore,

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} dr\, r^{2(\alpha+\beta)-4} e^{-r^2} \cos^{2\alpha-2}(\theta) \sin^{2\beta-2}(\theta) \times \underbrace{4r^3 \sin(\theta)\cos(\theta)}_{\text{Jacobian}}$$

$$=4 \left(\int_0^{\frac{\pi}{2}} \cos^{2\alpha-1}(\theta) \sin^{2\beta-1}(\theta) d\theta \right) \left(\int_0^{\infty} r^{2(\alpha+\beta)-1} e^{-r^2} dr \right).$$

Now let us compute the following two integrals separately:

$$I_1 := \int_0^{\frac{\pi}{2}} \cos^{2\alpha-1}(\theta) \sin^{2\beta-1}(\theta) d\theta$$

$$I_2 := \int_0^{\infty} r^{2(\alpha+\beta)-1} e^{-r^2} dr$$

For I_2, by change of variable $r^2 = u$ (so that $2rdr = du$),

$$
\begin{aligned}
I_2 &= \int_0^\infty r^{2(\alpha+\beta)-1} e^{-r^2} dr \\
&= \frac{1}{2} \int_0^\infty r^{2(\alpha+\beta)-2} e^{-r^2} \underbrace{2rdr}_{=du} \\
&= \frac{1}{2} \int_0^\infty u^{\alpha+\beta-1} e^{-u} du \\
&= \frac{1}{2} \Gamma(\alpha+\beta).
\end{aligned}
$$

For I_1, by the change of variables $\sqrt{x} = \cos(\theta)$ (so that $-\sin(\theta)d\theta = \frac{1}{2\sqrt{x}}dx$),

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \cos^{2\alpha-1}(\theta)\sin^{2\beta-1}(\theta)d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos^{2\alpha-1}(\theta)\sin^{2\beta-2}(\theta) \times \underbrace{\sin(\theta)d\theta}_{=-\frac{1}{2\sqrt{x}}dx} \\
&= \int_1^0 x^{\alpha-\frac{1}{2}}(1-x)^{\beta-1}\,\frac{-1}{2\sqrt{x}}dx \\
&= \frac{1}{2}\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \\
&= \frac{1}{2}B(\alpha,\beta)
\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= 4I_1 \times I_2 \\ &= 4 \times \frac{1}{2}\Gamma(\alpha+\beta) \times \frac{1}{2}B(\alpha,\beta)\end{aligned}$$

i.e.,

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

□