Math 362: Mathematical Statistics II

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Chapter 6. Hypothesis Testing

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Plan

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Go over the example first....

Suppose our friend Jory claims that he has some magic power to predict the side of a randomly tossed fair-coin.

Jory claims that he could do more than

$1/2$

of the time on average.

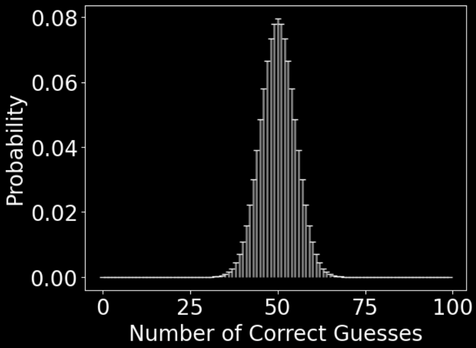
Let's test Jory to see if we believe his claim.

We made Jory guess a repeatedly tossed coin for 100 times.

He guesses correctly 54 times.

Question:
Does this provide strong evidence that Jory has the proclaimed magic power?

If Jory is guessing randomly, the number of correct guesses would follow a binomial distribution with parameters $n = 100$ and $p = 1/2$.



What is probability that Jory gets 54 or more correct when guessing randomly?

$$\mathbb{P}(X \geq 54) = \sum_{n=54}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{100-n} = 0.2421.$$

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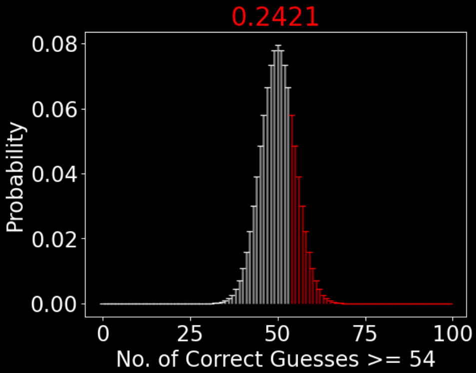

$$\mathbb{P}\left(X \geq 54\right) = \sum_{n=54}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{100-n} = 0.2421.$$

It is not unlikely to get this many correct guesses due to chance.

Conclusion:

There is No strong evidence that Jory has better than a $1/2$ chance of correctly guessing the coin.

What is probability that Jory gets 60 or more correct when guessing randomly?

$$\mathbb{P}(X \geq 60) = \sum_{n=60}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{100-n} = 0.0284.$$

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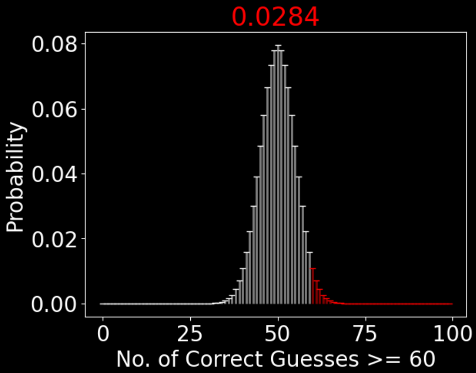

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Either

Jory is purely guessing with probability of success of $\frac{1}{2}$, and we witnessed a very unusual event due to chance.

Or

Jory is truly having the magic power to guess the coin.

Conclusion:

We have strong evidence against
Red Hypothesis

Or the test is in favor of
Green Hypothesis

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Before testing Jory, could you set up a threshold above which we will believe Jory's super power?

Find smallest m such that

$$\mathbb{P}(X \geq m) = \sum_{n=m}^{100} \binom{100}{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{100-n} \leq 0.05$$

$$\Downarrow$$

$$\boxed{m = 59}$$

b.c. $\mathbb{P}(X \geq 58) = 0.067$ & $\mathbb{P}(X \geq 59) = 0.044$

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We have just informally conducted a
hypothesis test with the
null hypothesis

$$H_0 : p = \frac{1}{2}$$

against the
alternative hypothesis

$$H_1 : p > \frac{1}{2}$$

under the
significance level $\alpha = 0.05$

which is equivalent to either

producing the
critical region
$m \geq 59$

or

comparing with
the p-value.

- Test statistic: Any function of the observed data whose numerical value dictates whether H_0 is accepted or rejected.

- Critical region C: The set of values for the test statistic that result in the null hypothesis being rejected.

  Critical value: The particular point in C that separates the rejection region from the acceptance region.

- Level of significance α: The probability that the test statistic lies in the critical region C under H_0.

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Test Normal mean $H_0 : \mu = \mu_0$ (σ known)

Setup:

1. Let $Y_1 = y_1, \cdots, Y_n = y_n$ be a random sample of size n from $N(\mu, \sigma^2)$ with σ known.
2. Set $\bar{y} = \frac{1}{n}(y_1 + \cdots + y_n)$ and $z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}}$.
3. The level of significance is α.

Test:

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu > \mu_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu < \mu_0 \end{cases}$$

reject H_0 if $z \leq -z_\alpha$.

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reject H_0 if $|z| \geq z_{\alpha/2}$.

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reject H_0 if $z \geq z_\alpha$. reject H_0 if $z \leq -z_\alpha$. reject H_0 if $|z| \geq z_{\alpha/2}$.

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- Simple hypothesis: Any hypothesis which specifies the population distribution completely.

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Definition. The P-value associated with an observed test statistic is the probability of getting a value for that test statistic as extreme as or more extreme than what was actually observed (relative to H_1) given that H_0 is true.

Note: Test statistics that yield small P-values should be interpreted as evidence against H_0.

E.g. Suppose that test statistic $z = 0.60$. Find P-value for

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$\mathbb{P}(Z \geq 0.60) = 0.2743.$ $\quad$ $\mathbb{P}(Z \leq 0.60) = 0.7257.$ $\quad$ $\mathbb{P}(|Z| \geq 0.60) = 2 \times 0.2743 = 0.5486.$

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