Math 362: Mathematical Statistics II

Le Chen
le.chen@emory.edu
chenle02@gmail.com

Emory University
Atlanta, GA

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Chapter 10. Goodness-of-fit Tests

Plan

Def. Suppose one does an experiment of extracting n balls of t different colors from a jar, replacing the extracted ball after each draw. Balls from the same color are equivalent. Denote the variable which is the number of extracted balls of color i ($i = 1, ..., t$) as X_i, and denote as p_i the probability that a given extraction will be in color i. The probability distribution function of the vector $(X_1, \cdots, X_t)$ is called the multinomial distribution, which is equal to

$$\begin{aligned} p_{X_1,\cdots,X_t}(k_1, \cdots, k_t) &= \mathbb{P}(X_1 = k_1, \cdots, X_t = k_t) \\ &= \binom{n}{k_1, \cdots, k_t} p_1^{k_1} \cdots p_t^{k_t} \end{aligned}$$

where $k_i \in \{0, 1, \cdots, n\}$, $1 \leq i \leq t$, $\sum_{i=1}^{t} k_i = n$, and $p_1 + \cdots + p_t = 1$.

Thm Suppose $(X_1, \cdots, X_t)$ follows the multinomial distribution with parameters n and $(p_1, \cdots, p_t)$ with $p_i \geq 0$ and $\sum_i p_i = 1$. Then

1. $X_i \sim$Binomail(n, p_i) and hence

   $$\mathbb{E}[X_i] = np_i$$
   $$\text{Var}(X_i) = np_i(1 - p_i)$$

2. $\text{Cov}(X_i, X_j) = -np_ip_j$, $i \neq j$. (negative correlated)

3. $M_{X_1,\cdots,X_t}(s_1, \cdots, s_t) = (p_1e^{s_1} + \cdots + p_te^{s_t})^n$.

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Proof

(3)

$$
\begin{aligned}
M_{X_1,\cdots,X_t}(s_1,\cdots,s_t) &= \mathbb{E}\left[e^{X_1 s_1+\cdots+X_t s_t}\right] \\
&= \sum_{\substack{k_1,\cdots,k_t=0 \\ k_1+\cdots+k_t=n}}^{n} \binom{n}{k_1,\cdots,k_t} p_1^{k_1}\cdots p_t^{k_t} e^{k_1 s_1+\cdots+k_t s_t} \\
&= \sum_{\substack{k_1,\cdots,k_t=0 \\ k_1+\cdots+k_t=n}}^{n} \binom{n}{k_1,\cdots,k_t} (p_1 e^{s_1})^{k_1}\cdots(p_t e^{s_t})^{k_t} \\
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(1) To find $M_{X_i}(s_i)$, we simply set $s_j \equiv 0$ for $j \neq i$. Hence

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(2) Set $M := M_{X_1,\cdots,X_t}(s_1,\cdots,s_t)$. Then for $i \neq j$,

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From a continuous pdf to a multinomial distribution:

E.g. Let Y_i be a random sample of size n from $f_Y(y) = 6y(1-y)$, $y \in [0,1]$. Define

$$X_i = \begin{cases} 1 & Y_i \in [0, 0.25) \\ 2 & Y_i \in [0.25, 0.5) \\ 3 & Y_i \in [0.5, 0.75) \\ 4 & Y_i \in [0.75, 1) \end{cases}$$

Find the distribution of $(X_1, \cdots, X_n)$.

Sol. (X_1, X_2, X_3, X_4) follows multinomial distribution with parameters (p_1, p_2, p_3, p_4) where

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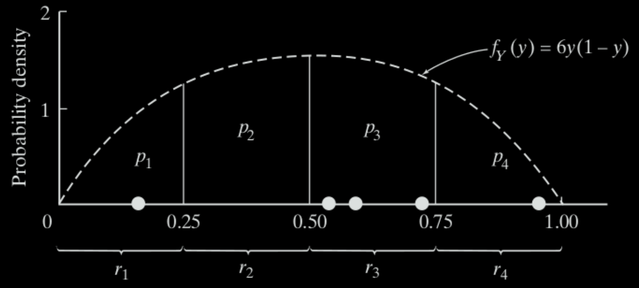

and by symmetry,

$$p_4 = p_1 = \frac{5}{32} \quad \text{and} \quad p_2 = p_3 = \frac{1}{2}(1 - p_1 - p_4) = \frac{11}{32}.$$

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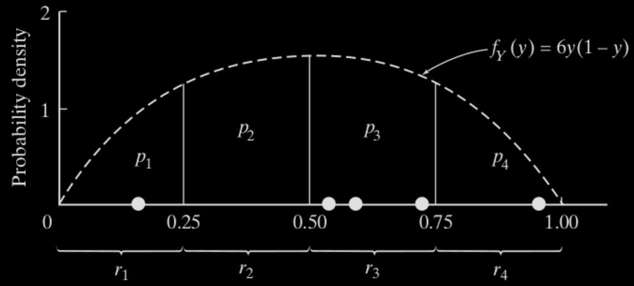
Remark In this way, we transform the outcomes, any values between $[0, 1]$, into **categorical data**. This chapter is about

Analysis of Categorical Data



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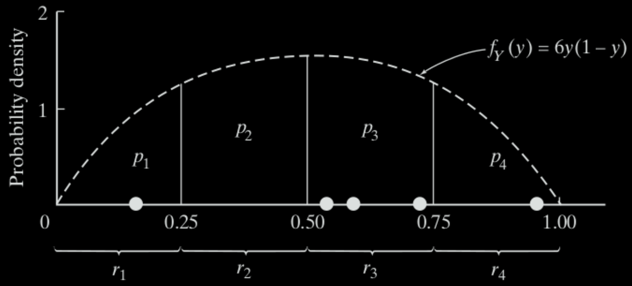

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