Math 362: Mathematical Statistics II

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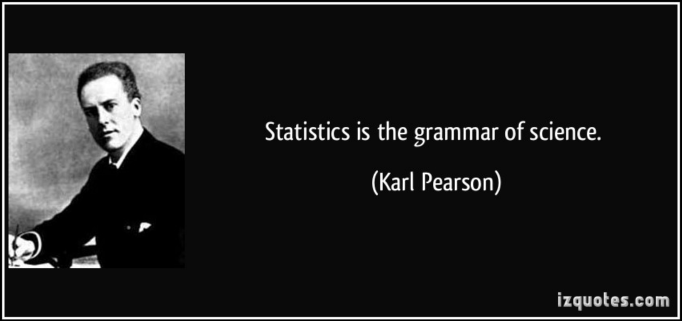
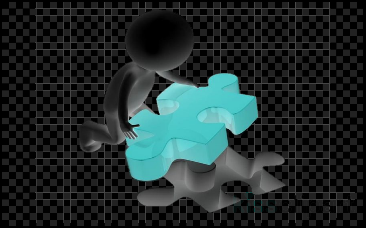
Chapter 10. Goodness-of-fit Tests



1. Karl Pearson, 1857 – 1936.
2. English mathematician and biostatistician.
3. He has been credited with establishing the discipline of mathematical statistics
4. Method of moments; p-Value; Chi-square test; Foundations of statistical hypothesis testing theory; principle component analysis ...

Pearson's chi-squared test
in one shot



$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}} \sim \text{Chi Square of } df$$

df = numer of classes − number of estimated parameters − 1

All expected ≥ 5

Def. Suppose one does an experiment of extracting n balls of t different colors from a jar, replacing the extracted ball after each draw. Balls from the same color are equivalent. Denote the variable which is the number of extracted balls of color i ($i = 1, ..., t$) as X_i, and denote as p_i the probability that a given extraction will be in color i. The probability distribution function of the vector $(X_1, \cdots, X_t)$ is called the multinomial distribution, which is equal to

$$\begin{aligned} p_{X_1,\cdots,X_t}(k_1,\cdots,k_t) &= \mathbb{P}\left(X_1 = k_1, \cdots, X_t = k_t\right) \\ &= \binom{n}{k_1,\cdots,k_t} p_1^{k_1}\cdots p_t^{k_t} \end{aligned}$$

where $k_i \in \{0, 1, \cdots, n\}$, $1 \le i \le t$, $\sum_{i=1}^{t} k_i = n$, and $p_1 + \cdots + p_t = 1$.

Thm Suppose $(X_1, \cdots, X_t)$ follows the multinomial distribution with parameters n and $(p_1, \cdots, p_t)$ with $p_i \geq 0$ and $\sum_i p_i = 1$. Then

1. $X_i \sim$Binomail(n, p_i) and hence

$$\mathbb{E}[X_i] = np_i$$
$$\mathrm{Var}(X_i) = np_i(1 - p_i)$$

2. $\mathrm{Cov}(X_i, X_j) = -np_ip_j$, $i \neq j$. (negative correlated)

3. $M_{X_1,\cdots,X_t}(s_1, \cdots, s_t) = (p_1e^{s_1} + \cdots + p_te^{s_t})^n$.

Proof

(3)

$$
\begin{aligned}
M_{X_1,\cdots,X_t}(s_1,\cdots,s_t) &= \mathbb{E}\left[e^{X_1 s_1+\cdots+X_t s_t}\right] \\
&= \sum_{\substack{k_1,\cdots,k_t=0 \\ k_1+\cdots+k_t=n}}^{n} \binom{n}{k_1,\cdots,k_t} p_1^{k_1}\cdots p_t^{k_t} e^{k_1 s_1+\cdots+k_t s_t} \\
&= \sum_{\substack{k_1,\cdots,k_t=0 \\ k_1+\cdots+k_t=n}}^{n} \binom{n}{k_1,\cdots,k_t} (p_1 e^{s_1})^{k_1}\cdots(p_t e^{s_t})^{k_t} \\
&= \left(p_1 e^{s_1}+\cdots+p_t e^{s_t}\right)^n
\end{aligned}
$$

(1) To find $M_{X_i}(s_i)$, we simply set $s_j \equiv 0$ for $j \neq i$. Hence

$$
M_{X_i}(s_i) = \big(\underbrace{p_1+\cdots+p_{i-1}+p_{i+1}+\cdots+p_t}_{=1-p_i} + p_i e^{s_i}\big)^n \Longrightarrow X_i \sim \text{Binomial}(n, p_i)
$$

(2) Set $M := M_{X_1,\cdots,X_t}(s_1,\cdots,s_t)$. Then for $i \neq j$,

$$\frac{\partial M}{\partial s_i} = n\left(p_1 e^{s_1} + \cdots + p_t e^{s_t}\right)^{n-1} p_i e^{s_i}$$

$$\frac{\partial^2 M}{\partial s_i \partial s_j} = n(n-1)\left(p_1 e^{s_1} + \cdots + p_t e^{s_t}\right)^{n-2} p_i e^{s_i} p_j e^{s_j}$$

$$\Downarrow$$

$$\mathbb{E}[X_i X_j] = \left.\frac{\partial^2 M}{\partial s_i \partial s_j}\right|_{s_1=\cdots=s_t=0} = n(n-1)(p_1+\cdots+p_t)^{n-2} p_i p_j = n(n-1) p_i p_j$$

$$\Downarrow$$

$$\begin{aligned}\mathrm{Cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \\ &= n(n-1)p_i p_j - np_i \times np_j \\ &= -np_i p_j\end{aligned}$$

$\square$

From a continuous pdf to a multinomial distribution:

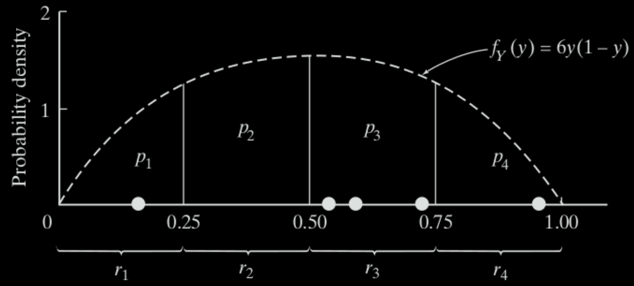
E.g. Let Y_i be a random sample of size n from $f_Y(y) = 6y(1-y)$, $y \in [0,1]$. Define

$$X_i = \begin{cases} 1 & Y_i \in [0, 0.25) \\ 2 & Y_i \in [0.25, 0.5) \\ 3 & Y_i \in [0.5, 0.75) \\ 4 & Y_i \in [0.75, 1) \end{cases}$$

Find the distribution of $(X_1, \cdots, X_n)$.

Sol. (X_1, X_2, X_3, X_4) follows multinomial distribution with parameters (p_1, p_2, p_3, p_4) where

$$p_1 = \int_0^{\frac{1}{4}} 6y(1-y)dy = \cdots = \frac{5}{32},$$



and by symmetry,

$$p_4 = p_1 = \frac{5}{32} \quad \text{and} \quad p_2 = p_3 = \frac{1}{2}(1 - p_1 - p_4) = \frac{11}{32}.$$

□

Remark In this way, we transform the outcomes, any values between $[0, 1]$, into **categorical data**. This chapter is about

Analysis of Categorical Data

Rationale

! We want to test if the c.d.f. $F_Y(\cdot)$ is given by the true c.d.f. $F_0(\cdot)$, i.e.,

$$H_0 : F_Y(y) = F_0(y) \quad v.s. \quad H_1 : F_Y(y) \neq F_0(y)$$

$\sim$ By properly partitioning the domain, the random sample should follow *an induced multinomial distribution*.

$\Longrightarrow$ Then testing $F_Y(\cdot) = F_0(\cdot)$ reduces to testing the induced multinomial distribution of the following form:

$$H_0 : p_1 = p_1', \cdots, p_n = p_n'$$

$$v.s.$$

$$H_1 : p_i \neq p_i' \quad \text{for at least one } i$$

How

1. Suppose we are sampling from the c.d.f. $F(y)$

2. Divide the range of the distribution into k mutually exclusive and exhausive intervals, say $I_1, \cdots, I_k$.

3. Let $\pi_i = \mathbb{P}(X \in I_i)$, $i = 1, \cdots, k$.

4. Let $O_1, \cdots, O_k$ be the respective observed numbers of the observations $X_1, \cdots, X_n$ in the intervals $I_1, \cdots, I_k$.

5. Then $O = (O_1, \cdots, O_k) \sim$ multinomial distribution with $(\pi_1, \cdots, \pi_k)$, i.e.,

$$\mathbb{P}\left(O_1 = o_1, \cdots, O_k = o_k\right) = \frac{n!}{\prod_{i=1}^k o_i!} \prod_{i=1}^k \pi_i^{o_i}$$

   with $\sum_{i=1}^k \pi_i = 1$, $\sum_{i=1}^k o_i = n$, and

$$\mathbb{E}[O_i] = n\pi_i =: e_i, \quad \mathrm{Var}(O_i) = n\pi_i(1 - \pi_i)$$

6. When $k = 2$, by CLT, as $n \to \infty$,

$$\frac{O_1 - n\pi_1}{\sqrt{n\pi_1(1-\pi_1)}} \xrightarrow{d} N(0,1) \quad \Longrightarrow \quad \frac{(O_1 - n\pi_1)^2}{n\pi_1(1-\pi_1)} \xrightarrow{d} \chi_1^2$$

$$\|$$

$$\frac{(O_1 - n\pi_1)^2}{n\pi_1} + \frac{(O_2 - n\pi_2)^2}{n\pi_2}$$

$$\|$$

$$\frac{(O_1 - e_1)^2}{e_1} + \frac{(O_2 - e_2)^2}{e_2}$$

Hence, as $n \to \infty$,

$$\sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i} \xrightarrow{d} \chi_{k-1}^2$$

7. For general k,

$$\sum_{i=1}^{k} \frac{(O_i - n\pi_i)^2}{n\pi_i} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

follows a complicated, but exact, distribution, from which, one can show

$$\sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i} \xrightarrow{d} \chi^2_{k-1}$$

$$\Downarrow$$

Thm. When n is large enough, namely, when $n\pi_i \geq 5$ for all i,

$$D = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i} \overset{appr.}{\sim} \chi^2_{k-1}.$$

Rmk: The above is called Pearson's chi-square test. It is asymptotically equivalent to the generalized likelihood ratio test.

Alternative: G-test

– the likelihood ration test for multinomial model

1. Under $H_0 : \pi_i = p_i$, $i = 1, \cdots, k$, the MLE of π_i are

$$\widetilde{\pi}_i = p_i = \frac{np_i}{n} = \frac{e_i}{n}, \qquad \forall i.$$

2. When there are no constraints, for $i = 1, \cdots, k-1$,

$$\frac{\partial}{\partial \pi_i} \ln L(\pi_1, \cdots, \pi_{k-1} | o_1, \cdots, o_k) = 0, \quad 1 \le i \le k-1$$

$$\Updownarrow$$

$$\frac{o_i}{\widehat{\pi}_i} = \frac{o_k}{1 - \widehat{\pi}_1 - \cdots - \widehat{\pi}_{k-1}}, \quad 1 \le i \le k-1$$

$$\Updownarrow$$

$$\widehat{\pi}_i = \frac{o_i}{n}, \quad 1 \le i \le k.$$

$\Rightarrow$

$$\lambda := \ln\left(\frac{L(\widetilde{\pi}_1, \cdots, \widetilde{\pi}_{k-1} | o_1, \cdots, o_k)}{L(\widehat{\pi}_1, \cdots, \widehat{\pi}_{k-1} | o_1, \cdots, o_k)}\right) = \log\left(\frac{\prod_{i=1}^k \widetilde{\pi}_i^{o_i}}{\prod_{i=1}^k \widehat{\pi}_i^{o_i}}\right)$$

$$= \sum_{i=1}^k o_i \ln\left(\frac{\widetilde{\pi}_i}{\widehat{\pi}_i}\right)$$

$$= \sum_{i=1}^k o_i \ln\left(\frac{e_i}{o_i}\right)$$

Critical region: $\lambda < \lambda_* < 0$.

Def.

$$G := -2\lambda = -2\sum_{i=1}^k o_i \ln\left(\frac{e_i}{o_i}\right) = 2\sum_{i=1}^k o_i \ln\left(\frac{o_i}{e_i}\right)$$

$G \overset{approx.}{\sim} \chi^2_{k-1}$ for large n.
Critical region: $G \geq G_* = \chi^2_{1-\alpha, k-1}$.

Relation G-test and Pearson's Chi square test

By second order Taylor expanson around 1,

$$
\begin{aligned}
G &= -2\sum_{i=1}^{k} o_i \ln\left(\frac{e_i}{o_i}\right) \\
&\approx -2\sum_{i=1}^{k} o_i \left[\left(\frac{e_i}{o_i}-1\right)-\frac{1}{2}\left(\frac{e_i}{o_i}-1\right)^2\right] \\
&= -2\sum_{i=1}^{k}(e_i-o_i)+\sum_{i=1}^{k} o_i\left(\left(1-\frac{o_i}{e_i}\right)+\frac{o_i}{e_i}\right)\left(\frac{e_i}{o_i}-1\right)^2 \\
&= 0+\sum_{i=1}^{n}\frac{o_i^2}{e_i}\left(1-\frac{o_i}{e_i}\right)^3+\sum_{i=1}^{k}\frac{(e_i-o_i)^2}{e_i} \\
&\approx \underbrace{\sum_{i=1}^{k}\frac{(e_i-o_i)^2}{e_i}}_{\textstyle || \atop \textstyle D}
\end{aligned}
$$

$\therefore$ Pearson's Chi-square test is an approximation of G-test.

E.g. 1 *Benford's law*:

Table 10.3.1

Digit, i	$\log_{10}(i+1) - \log_{10}(i)$
1	0.301
2	0.176
3	0.125
4	0.097
5	0.079
6	0.067
7	0.058
8	0.051
9	0.046

Initial digits

Digit	Observed, k_i
1	111
2	60
3	46
4	29
5	26
6	22
7	21
8	20
9	20
	355

Use this law to check whether the bookkeepers have made up entries.

Assume that bookkeepers are not aware of Benford's law.

Sol. The test should be

$$H_0 : p_1 = p_{10}, \cdots, p_9 = p_{90}$$
$$v.s.$$
$$H_1 : p_i \neq p_{i0} \quad \text{for at least one } i = 1, \cdots, 9.$$

Critical region: $\left(\chi^2_{.95,8}, \infty\right) = (15.507, \infty)$.

Compute the D and G scores:

Digit	o_i	p_i	e_i	$(o_i - e_i)^2/e_i$	$2o_i \ln(e_i/o_i)$
1	111	0.301			
2	60	0.176			
3	46	0.125			
4	29	0.097			
5	26	0.079			
6	22	0.067			
7	21	0.058			
8	20	0.051			
9	20	0.046			
sum	355	1	355	$d =$ ____	$g =$ ____

Digit	o_i	p_i	e_i	$(o_i - e_i)^2/e_i$	$2o_i \ln(e_i/o_i)$
1	111	0.301	106.9	0.16	8.449
2	60	0.176	62.5	0.10	-4.860
3	46	0.125	44.4	0.06	3.309
4	29	0.097	34.4	0.86	-9.963
5	26	0.079	28.0	0.15	-3.937
6	22	0.067	23.8	0.13	-3.433
7	21	0.058	20.6	0.01	0.828
8	20	0.051	18.1	0.20	3.982
9	20	0.046	16.3	0.82	8.109
sum	355	1	355	$d = \underline{2.49}$	$g = \underline{2.48}$

Conclusion: Fail to reject.

```r
> # EX 10.3.2
> library(data.table)
> mydat <- fread('http://math.emory.edu/~lchen41/teaching/2020_Spring/Case_10-3-2.data')
trying URL 'http://math.emory.edu/~lchen41/teaching/2020_Spring/Case_10-3-2.data'
Content type 'unknown' length 153 bytes
==================================================
downloaded 153 bytes

> head(mydat)
   Digit  Oi    Pi
1:     1 111 0.301
2:     2  60 0.176
3:     3  46 0.125
4:     4  29 0.097
> pi = mydat[,3]
> oi = mydat[,2]
> n = sum(oi)
> ei = n*pi
> di = (ei-oi)^2/ei
> gi = 2*oi*log(oi/ei)
> print(paste("Using Pearson's test, D value is equal to ", round(sum(di),3)))
[1] "Using Pearson's test, D value is equal to 2.491"
> print(paste("Using the G-test, G value is equal to ", round(sum(gi),3)))
[1] "Using the G-test, G value is equal to 2.484"
```

Codes available
http://math.emory.edu/~lchen41/teaching/2020_Spring/Case_10-3-2.R

E.g. 2 Test for randomness

Is the following sample of size 40 from $f_Y(y) = 6y(1-y)$, $y \in [0,1]$?

Table 10.3.4

0.18	0.06	0.27	0.58	0.98
0.55	0.24	0.58	0.97	0.36
0.48	0.11	0.59	0.15	0.53
0.29	0.46	0.21	0.39	0.89
0.34	0.09	0.64	0.52	0.64
0.71	0.56	0.48	0.44	0.40
0.80	0.83	0.02	0.10	0.51
0.43	0.14	0.74	0.75	0.22

Sol. Test continuous pdf $\rightarrow$ reduce to a set of classes:

Table 10.3.5

Class	Observed Frequency, k_i	P_{i_o}	$40p_{i_o}$
$0 \leq y < 0.20$	8	0.104	4.16
$0.20 \leq y < 0.40$	8	0.248	9.92
$0.40 \leq y < 0.60$	14	0.296	11.84
$0.60 \leq y < 0.80$	5	0.248	9.92
$0.80 \leq y < 1.00$	5	0.104	4.16

Table 10.3.6

Class	Observed Frequency, k_i	P_{i_o}	$40p_{i_o}$
$0 \leq y < 0.40$	16	0.352	14.08
$0.40 \leq y < 0.60$	14	0.296	11.84
$0.60 \leq y \leq 1.00$	10	0.352	14.08

$$d = \cdots = 1.84.$$

Critical region: $\left(\chi^2_{.95,2}, \infty\right) = (5.992, \infty)$.

Conclusion: Fail to reject.

```
> # Case Study 10.3.2
> # Read data from the URL link
> library(data.table)
> mydat <- fread('http://math.emory.edu/~lchen41/teaching/2020_Spring/EX_
    10-3-1.data')
trying URL 'http://math.emory.edu/~lchen41/teaching/2020_Spring/EX_10-3-1.
    data'
Content type 'unknown' length 234 bytes
==================================================
downloaded 234 bytes

>d(mydat)
   Col1 Col2 Col3 Col4 Col5
   1: 0.18 0.06 0.27 0.58 0.98
   2: 0.55 0.24 0.58 0.97 0.36
   3: 0.48 0.11 0.59 0.15 0.53
   4: 0.29 0.46 0.21 0.39 0.89
   5: 0.34 0.09 0.64 0.52 0.64
   6: 0.71 0.56 0.48 0.44 0.40
# Conditions for lower bounds
> lb=c(0,0.40,0.60)
> # Conditions for upper bounds
> up=c(0.40,0.60,1.00)
> # Store the results in d
> oi <- seq(1:length(lb))
> pi <- seq(1:length(lb))
> integrand <- function(y) {6*y*(1-y)}
> for (i in c(1:length(lb))) {
+  oi[i] <- table(mydat>=lb[i] & mydat<up[i])[2]
+  pi[i] <- integrate(integrand, lb[i], up[i])$value[1]
+  print(paste("the", i,"th bin has", oi[i],
+       "entries and pi is equal to", pi[i]))
+ }
```

```
[1] "the 1 th bin has 16 entries and pi is equal to 0.352"
[1] "the 2 th bin has 14 entries and pi is equal to 0.296"
[1] "the 3 th bin has 10 entries and pi is equal to 0.352"
> pi <- unlist(pi)
> n <- sum(oi)
> ei <- n*pi
> di <- (ei-oi)^2/ei
> gi <- 2*oi*log(oi/ei)
> rbind(oi,pi,ei,di,gi)
          [,1]       [,2]      [,3]
oi 16.0000000 14.0000000 10.000000
pi  0.3520000  0.2960000  0.352000
ei 14.0800000 11.8400000 14.080000
di  0.2618182  0.3940541  1.182273
gi  4.0906679  4.6920636 -6.843405
> print(paste("Using Pearson's test, D value is equal to ",round(sum(di),3)))
[1] "Using Pearson's test, D value is equal to 1.838"
> print(paste("Using the G-test, G value is equal to ", round(sum(gi),3)))
[1] "Using the G-test, G value is equal to 1.939"<Paste>
```

http://math.emory.edu/~lchen41/teaching/2020_Spring/EX_10-3-1.R

E.g. 3 Fisher's suspicion on Mendel's experiments on 1866:

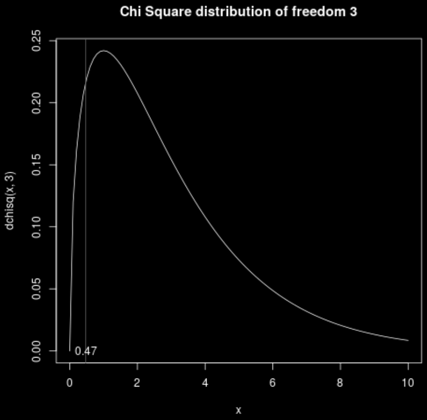
Table 10.3.7

Phenotype	Obs. Freq.	Mendel's Model	Exp. Freq.
(round, yellow)	315	9/16	312.75
(round, green)	108	3/16	104.25
(angular, yellow)	101	3/16	104.25
(angular, green)	32	1/16	34.75

$$d = \ldots = 0.47$$

$$P\text{-value} = \mathbb{P}(\chi_3^2 \leq 0.47) = 0.0746.$$

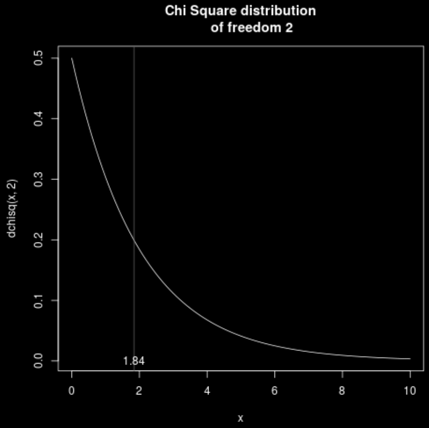
```
> # Case Study 10.3.3
> x=seq(0,10,0.1)
> plot(x,dchisq(x,3),type = ”l”)
> abline(v=0.47,col = ”gray60”)
> text(0.47,0,”0.47”)
> title(”Chi Square distribution
+       of freedom 3”)
> pchisq(0.47,3)
[1] 0.07456892
```



E.g. 2' A second look at the random generator in E.g. 2.

Does it fit the model too well? Find the P-value.

```
> # Example 10.3.1
> x=seq(0,10,0.1)
> plot(x,dchisq(x,2),type = "l")
> abline(v=1.84,col = "gray60")
> text(1.84,0,"1.84")
> title("Chi Square distribution
+        of freedom 2")
> pchisq(1.84,2)
[1] 0.601481
```



$$P\text{-value} = 0.601 \quad \Longrightarrow \quad \text{No.}$$

p_i are known	p_i are unknown
$D = \sum_{i=1}^{t} \frac{(X_i - np_i)^2}{np_i}$	$D_1 = \sum_{i=1}^{t} \frac{(X_i - n\hat{p}_i)^2}{n\hat{p}_i}$
χ^2 with f.d. $t-1$	χ^2 with f.d. $t-1-s$
$d = \sum_{i=1}^{t} \frac{(k_i - np_{i0})^2}{np_{i0}}$	$d_1 = \sum_{i=1}^{t} \frac{(k_i - n\hat{p}_{i0})^2}{n\hat{p}_{i0}}$
$np_{i0} \geq 5$	$n\hat{p}_{i0} \geq 5$
$d > \chi^2_{1-\alpha,t-1}$	$d_1 > \chi^2_{1-\alpha,t-1-s}$

† s is the number of unknown parameters.

$$\text{df} = \underline{\text{number of classes}} - 1 - \underline{\text{number of unknown parameters.}}$$

E.g. 1 Binomial data: 4096 students, each shots basketball 4 times. Let X_i be the number of hits for the ith student.

Number of Hits, i	Obs. Freq., k_i
$r_i's$ 0	1280
1	1717
2	915
3	167
4	17

People believe that X_i should following binomial$(4, p)$, that is, shotting basketball should be something like trying to get red chocolate beans from a jar of beans of two colors.

Find the MLE for p. Use the data to make a conclusion.

Sol. 1) $H_0 : X_i \sim$binomal$(4, p)$.

2) Under H_0, the MLE for p is $p_e = \ldots = 0.251$

3) Compute the expected frequenies:

Table 10.4.1

Number of Hits, i		Obs. Freq., k_i	Estimated Exp. Freq., $n\hat{p}_{i_o}$
$r_i's$	0	1280	1289.1
	1	1717	1728.0
	2	915	868.6
	3	167	194.0
	4	17	16.3

$$\Longrightarrow \quad d_1 = \cdots = 6.401.$$

4) Critical region: $(\chi^2_{.95,5-1-1}, +\infty) = (7.815, +\infty)$

5) Conclusion: Fail to reject.

6) Alternatively, P-value $= \mathbb{P}(\chi^2_3 \geq 6.401) = 0.094$, ... discuss... $\square$

E.g. 2 Does the number of death per day follow the Poisson distribution?

Number of Deaths, i	Obs. Freq., k_i
0	162
1	267
2	271
3	185
4	111
5	61
6	27
7	8
8	3
9	1
10+	0
	1096

Sol. 1) Let X_i be the number of death in ith day, $1 \leq i \leq 1096$.

2) H_0 : X_i follow Poisson(λ).

3) The MLE for λ is: $\lambda_e = \cdots = 2.157$.

4) Compute the expected frequencies:

Table 10.4.2

Number of Deaths, i	Obs. Freq., k_i	Est. Exp. Freq., $n\hat{p}_{i_o}$
0	162	126.8
1	267	273.5
2	271	294.9
3	185	212.1
4	111	114.3
5	61	49.3
6	27	17.8
7	8	5.5
8	3	1.4
9	1	0.3
10+	0	0.1
	1096	1096

Table 10.4.3

Number of Deaths, i		Obs. Freq., k_i	Est. Exp. Freq., $n\hat{p}_{i_o}$
$r_1, r_2, \ldots, r_8$	0	162	126.8
	1	267	273.5
	2	271	294.9
	3	185	212.1
	4	111	114.3
	5	61	49.3
	6	27	17.8
	7+	12	7.3
		1096	1096

$$\Longrightarrow \quad d_1 = \cdots = 25.98.$$

5) P-value $= \mathbb{P}(\chi^2_{1,8-1-1} \geq 25.98) = 0.00022$. Reject! $\square$

E.g. 1 Whether are the two ratings independent?

Table 10.5.5

		Ebert Ratings			
		Down	Sideways	Up	Total
	Down	24	8	13	45
Siskel	Sideways	8	13	11	32
Ratings	Up	10	9	64	83
	Total	42	30	88	160

E.g. 2 Whether is the suicide rate independent of the mobility factor?

Table 10.5.7

City	Suicides per 100,000, x_i	Mobility Index, y_i	City	Suicides per 100,000, x_i	Mobility Index, y_i
New York	19.3	54.3	Washington	22.5	37.1
Chicago	17.0	51.5	Minneapolis	23.8	56.3
Philadelphia	17.5	64.6	New Orleans	17.2	82.9
Detroit	16.5	42.5	Cincinnati	23.9	62.2
Los Angeles	23.8	20.3	Newark	21.4	51.9
Cleveland	20.1	52.2	Kansas City	24.5	49.4
St. Louis	24.8	62.4	Seattle	31.7	30.7
Baltimore	18.0	72.0	Indianapolis	21.0	66.1
Boston	14.8	59.4	Rochester	17.2	68.0
Pittsburgh	14.9	70.0	Jersey City	10.1	56.5
San Francisco	40.0	43.8	Louisville	16.6	78.7
Milwaukee	19.3	66.2	Portland	29.3	33.2
Buffalo	13.8	67.6			

$$\bar{x} = 20.8 \quad \text{and} \quad \bar{y} = 56.0$$

Table 10.5.8

		Mobility Index	
		Low (<56.0)	High ($\geq$56.0)
Suicide Rate	High ($\geq$20.8)	7	4
	Low (<20.8)	3	11

Thm 10.4.1 Suppose that n observations are taken on a sample space partitioned by the events $A_1, \cdots, A_r$ and $B_1, \cdots, B_c$.

Let $p_i = \mathbb{P}(A_i)$, $q_j = \mathbb{P}(B_j)$, $p_{ij} = \mathbb{P}(A_i \cap B_j)$.

Let X_{ij} be the number of observations belonging to $A_i \cap B_j$.

a) Provided that $np_{ij} \geq 5$ for all i, j, the r.v.

$$D_2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(X_{ij} - np_{ij})^2}{np_{ij}} \sim \text{Chi square of f.d. } rc - 1$$

b) To test H_0 : A_i's are independent of B_i's, calculate the test statistic

$$d_2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(k_{ij} - n\hat{p}_i \hat{q}_j)^2}{n\hat{p}_i \hat{q}_j}$$

where $\hat{p}_i$ and $\hat{q}_j$ are MLE's for p_i and q_j, respectively.

Provided that $n\hat{p}_i \hat{q}_j \geq 5$ for all i, j, the critical region is

$$(\chi^2_{1-\alpha,(r-1)(c-1)}, +\infty)$$

E.g. 1 Sol: Compute the expected frequencies:

Table 10.5.6

		Ebert Ratings			
		Down	Sideways	Up	Total
Siskel Ratings	Down	24 (11.8)	8 (8.4)	13 (24.8)	45
	Sideways	8 (8.4)	13 (6.0)	11 (17.6)	32
	Up	10 (21.8)	9 (15.6)	64 (45.6)	83
	Total	42	30	88	160

$$\Longrightarrow \quad d_2 = \cdots = 45.37$$

Critical region is

$$(\chi^2_{0.99,(3-1)\times(3-1)}, +\infty) = (13.277, +\infty)$$

Alternatively P-value $= \mathbb{P}(\chi^2_4 \geq 45.37) = 3.33 \times 10^{-9}$.

Rejection at $\alpha = 0.01$. $\square$

E.g. 2 Sol: Compute the expected frequencies:

Table 10.5.9

		Mobility Index	
		Low (<56.0)	High (≥56.0)
Suicide Rate	High (≥20.8)	4.4*	6.6
	Low (<20.8)	5.6	8.4

*$\hat{E}(X_{11}) = 4.4$ does not quite satisfy the "$n\hat{p}_i\hat{q}_j \geq 5$" restriction stated in Theorem 10.5.1, but 4.4 is close enough to 5 to maintain the integrity of the χ^2 approximation.

$$\Longrightarrow \quad d_2 = \cdots = 4.57$$

Critical region is

$$(\chi^2_{0.95,(2-1)\times(2-1)}, +\infty) = (3.41, +\infty)$$

Alternatively P-value $= \mathbb{P}(\chi^2_1 \geq 4.57) = 0.033$

Rejection at $\alpha = 0.05$. $\square$