Math 362: Mathematical Statistics II

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Chapter 12. The Analysis of Variance

Model assumptions

1. Independence of observations
2. Normality
3. Homogeneity of variances

Table 12.1.1

	Treatment Level			
	1	2	...	k
	Y_{11}	Y_{12}		Y_{1k}
	Y_{21}	Y_{22}		
	$\vdots$	$\vdots$	...	$\vdots$
	$Y_{n_1 1}$	$Y_{n_2 2}$		$Y_{n_k k}$
Sample sizes:	n_1	n_2	...	n_k
Sample totals:	$T_{.1}$	$T_{.2}$		$T_{.k}$
Sample means:	$\overline{Y}_{.1}$	$\overline{Y}_{.2}$		$\overline{Y}_{.k}$
True means:	μ_1	μ_2		μ_k

$\Updownarrow$

Assume:
$\forall j = 1, \cdots, k, \forall j = 1, \cdots, n_i,$

1. Y_{ij} are independent.
2. $Y_{ij} \sim N(\mu_j, \sigma^2)$

$\Longleftrightarrow$

Assume:
$\forall j = 1, \cdots, k, \forall j = 1, \cdots, n_i,$
$$Y_{ij} = \mu_j + \epsilon_{ij}$$

1. ϵ_{ij} are independent.
2. $\epsilon_{ij} \sim N(0, \sigma^2)$

Likelihood ratio test

1. The parameter spaces are

$$\Omega = \{(\mu_1, \cdots, \mu_k, \sigma^2) : -\infty < \mu_1, \cdots, \mu_k < \infty, \sigma^2 > 0\}$$

$$\omega = \{(\mu_1, \cdots, \mu_k, \sigma^2) : -\infty < \mu_1 = \cdots = \mu_k < \infty, \sigma^2 > 0\}$$

2. The likelihood functions are

$$L(\omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{k}\sum_{i=1}^{n_j}(y_{ij} - \mu)^2\right\}$$

$$L(\Omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{k}\sum_{i=1}^{n_j}(y_{ij} - \mu_j)^2\right\}$$

3. Now

$$\frac{\partial \ln L(\omega)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \mu)$$

$$\frac{\partial \ln L(\omega)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \mu)^2$$

Setting the above derivatives to zero, the solutsions for μ and σ^2 are,

$$\frac{1}{n} \sum_{j=1}^{k} \sum_{i=1}^{n_j} y_{ij} = \bar{y}_{..}$$

$$\frac{1}{n} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2 = v$$

3' Similarly,

$$\frac{\partial \ln L(\Omega)}{\partial \mu_j} = \frac{1}{\sigma^2} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \mu_j), \quad j = 1, \cdots, k$$

$$\frac{\partial \ln L(\Omega)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \mu_j)^2$$

Setting the above derivatives to zero, the solutsions for μ_j and σ^2 are,

$$\frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij} = \bar{y}_{\cdot j}$$

$$\frac{1}{n} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2 = w$$

4. Hence,

$$L(\hat{\omega}) = \left(\frac{n}{2\pi \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2} \right)^{n/2} \exp\left\{ -\frac{n \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2}{2 \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2} \right\}$$

$$||$$

$$\left(\frac{n}{2\pi \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{..})^2} \right)^{n/2} e^{-n/2}$$

Similarly,

$$L(\hat{\Omega}) = \left(\frac{n}{2\pi \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2} \right)^{n/2} \exp\left\{ -\frac{n \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2}{2 \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2} \right\}$$

$$||$$

$$\left(\frac{n}{2\pi \sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{.j})^2} \right)^{n/2} e^{-n/2}$$

5. Finally,

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left(\frac{\sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2}{\sum_{j=1}^{k} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot\cdot})^2} \right)^{n/2}$$

$\Rightarrow$ Test statistic:

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left(\frac{\sum_{j=1}^{k} \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{\cdot j})^2}{\sum_{j=1}^{k} \sum_{i=1}^{n_j} (Y_{ij} - \bar{Y}_{\cdot\cdot})^2} \right)^{n/2}$$

$$
\begin{aligned}
SSTOT &:= \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot\cdot} \right)^2 \\
&= \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left[\left(Y_{ij} - \overline{Y}_{\cdot j} \right) + \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot} \right) \right]^2 \\
&= \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot j} \right)^2 + \text{zero cross term} + \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot} \right)^2 \\
&= \underbrace{\sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot j} \right)^2}_{SSE} + \underbrace{\sum_{j=1}^{k} n_j \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot} \right)^2}_{SSTR}
\end{aligned}
$$

$$\Downarrow$$

$$
\Lambda = \left(\frac{SSE}{SSTOT} \right)^{n/2} = \left(\frac{SSE}{SSE + SSTR} \right)^{n/2} = \left(\frac{1}{1 + SSTR/SSE} \right)^{n/2}
$$

6. Critical regions: for some $\lambda_* \in (0,1)$ close to 0,

$$\alpha = \mathbb{P}\left(\Lambda \le \lambda_*\right)$$

$$= \mathbb{P}\left(\frac{1}{1 + \mathit{SSTR}/\mathit{SSE}} \le \lambda_*^{2/n}\right)$$

$$= \mathbb{P}\left(\frac{\mathit{SSTR}}{\mathit{SSE}} \le \lambda_*^{-2/n} - 1\right)$$

$$= \mathbb{P}\left(\frac{\mathit{SSTR}/(k-1)}{\mathit{SSE}/(n-k)} \le \left(\lambda_*^{-2/n} - 1\right)\frac{n-k}{k-1}\right)$$

7. We will prove that under H_0, $\frac{\mathit{SSTR}/(k-1)}{\mathit{SSE}/(n-k)} \sim$ F-distr. $df_1 = k-1, df_2 = n-k$

$$\Rightarrow \quad \left(\lambda_*^{-2/n} - 1\right)\frac{n-k}{k-1} = F_{1-\alpha,k-1,n-k}.$$

□

Treatment sum of squares: SSTR

Sample size: (Weights)	n_1	n_2	$\cdots$	n_k	$n = \sum_{j=1}^{k} n_j$ *Weighted average*
Sample means:	$\overline{Y}_{\cdot 1}$	$\overline{Y}_{\cdot 2}$	$\cdots$	$\overline{Y}_{\cdot k}$	$\overline{Y}_{\cdot\cdot} = \frac{1}{n} \sum_{j=1}^{k} n_j \overline{Y}_{\cdot j}$
True means:	μ_1	μ_2	$\cdots$	μ_k	$\mu = \frac{1}{n} \sum_{j=1}^{k} n_j \mu_j$
Squares:	$\left(\overline{Y}_{\cdot 1} - \overline{Y}_{\cdot\cdot}\right)^2$	$\left(\overline{Y}_{\cdot 2} - \overline{Y}_{\cdot\cdot}\right)^2$	$\cdots$	$\left(\overline{Y}_{\cdot k} - \overline{Y}_{\cdot\cdot}\right)^2$	*SSTR*

$$SSTR := \sum_{j=1}^{k} n_j \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}\right)^2$$

1. When $k = 1$, $SSTR \equiv 0$.
2. When $k = 2$, say $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$:

$$\overline{Y_{..}} = \frac{1}{m+n}\left(n\overline{X} + m\overline{Y}\right)$$

$$\begin{aligned}
SSTR &= n\left[\overline{X} - \frac{1}{n+m}\left(n\overline{X} + m\overline{Y}\right)\right]^2 + m\left[\overline{Y} - \frac{1}{n+m}\left(n\overline{X} + m\overline{Y}\right)\right]^2 \\
&= n\left[\frac{m(\overline{X} - \overline{Y})}{n+m}\right]^2 + m\left[\frac{n(\overline{X} - \overline{Y})}{n+m}\right]^2 \\
&= \left[\frac{nm^2}{(n+m)^2} + \frac{n^2 m}{(n+m)^2}\right]\left(\overline{X} + \overline{Y}\right)^2 \\
&= \frac{nm}{n+m}\left(\overline{X} - \overline{Y}\right)^2
\end{aligned}$$

$$SSTR = \frac{\left(\overline{X} - \overline{Y}\right)^2}{\dfrac{1}{m} + \dfrac{1}{n}}$$

$$
\begin{aligned}
SSTR &= \sum_{j=1}^{k} n_j \left(\overline{Y}_{.j} - \overline{Y}_{..}\right)^2 = \sum_{j=1}^{k} n_j \left[\left(\overline{Y}_{.j} - \mu\right) - \left(\overline{Y}_{..} - \mu\right)\right]^2 \\
&= \sum_{j=1}^{k} n_j \left[\left(\overline{Y}_{.j} - \mu\right)^2 + \left(\overline{Y}_{..} - \mu\right)^2 - 2\left(\overline{Y}_{.j} - \mu\right)\left(\overline{Y}_{..} - \mu\right)\right] \\
&= \sum_{j=1}^{k} n_j \left(\overline{Y}_{.j} - \mu\right)^2 + \sum_{j=1}^{k} n_j \left(\overline{Y}_{..} - \mu\right)^2 - 2\left(\overline{Y}_{..} - \mu\right) \sum_{j=1}^{k} n_j \left(\overline{Y}_{.j} - \mu\right) \\
&= \sum_{j=1}^{k} n_j \left(\overline{Y}_{.j} - \mu\right)^2 + n\left(\overline{Y}_{..} - \mu\right)^2 - 2\left(\overline{Y}_{..} - \mu\right) n \left(\overline{Y}_{..} - \mu\right) \\
&= \sum_{j=1}^{k} n_j \left(\overline{Y}_{.j} - \mu\right)^2 - n\left(\overline{Y}_{..} - \mu\right)^2 \qquad (12.2.1)
\end{aligned}
$$

$$\Downarrow$$

$$SSTR = \sum_{j=1}^{k} n_j \left[\left(\overline{Y}_{.j} - \mu_j\right)^2 - 2\left(\overline{Y}_{.j} - \mu_j\right)(\mu - \mu_j) + (\mu - \mu_j)^2\right] - n\left(\overline{Y}_{..} - \mu\right)^2$$

Notice that

$$\overline{Y}_{\cdot j} \sim N(\mu_j, \sigma^2/n_j) \qquad \text{and} \qquad \overline{Y}_{\cdot\cdot} \sim N(\mu, \sigma^2/n)$$

$\Longrightarrow$

$$\mathbb{E}[SSTR] = \sum_{j=1}^{k} n_j \left[\frac{\sigma^2}{n_j} - 2 \times 0 + (\mu - \mu_j)^2 \right] - n\frac{\sigma^2}{n}$$

$$= (k-1)\sigma^2 + \sum_{j=1}^{k} n_j (\mu - \mu_j)^2$$

Remark When $\mu_1 = \cdots = \mu_j$ then

0.1 $\mathbb{E}[SSTR] = (k-1)\sigma^2$

0.2 $MSTR := \frac{SSTR}{k-1}$ is an unbiased estimator for σ^2.

0.3 $SSTR/\sigma^2 \sim$ Chi square $(df = k-1)$. (Homework)

Test $H_0 : \mu_1 = \cdots = \mu_k$ v.s. μ_j are not the same.

Case I. when σ^2 is known.

Reject H_0 if $SSTR/\sigma^2 \geq \chi^2_{1-\alpha,k-1}$.

Case II. when σ^2 is unknown.

......

Sum of Squared Errors: SSE

1. Sum of squred error:

$$SSE := \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot j} \right)^2$$

$$= \sum_{j=1}^{k} (n_j - 1) \left[\frac{1}{n_j - 1} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot j} \right)^2 \right]$$

$$= \sum_{j=1}^{k} (n_j - 1) S_j^2$$

2. Pooled variance S_p^2:

$$S_p^2 = \frac{SSE}{\sum_{j=1}^{k} (n_j - 1)} = \frac{SSE}{n - k}$$

Mean square of error $MSE = S_p^2$

Notice that

1. $(n_j - 1)S_j^2/\sigma^2$ ~Chi square ($df = n_j - 1$)
2. S_j^2's are independent
3. $SSE/\sigma^2 = (n-k)S_p^2/\sigma^2 = \sum_{j=1}^{k}(n_j - 1)S_j^2/\sigma^2$,

   Sum of independent of Chi squares

$$\Downarrow$$

Thm. No mattter $H_0 : \mu_1 = \cdots = \mu_k$ is true or not

a. $SSE/\sigma^2 = (n-k)S_p^2/\sigma^2 \sim$ Chi square ($df = \sum_{j=1}^{k}(n_j - 1) = n - k$)

b. $SSTR \perp SSE$.

Proof. We have shown part (a). Part (b) is trickier. Indeed, both parts are a consequence of **Cochran's theorem**[1] ... $\square$

[1] https://en.wikipedia.org/wiki/Cochran%27s_theorem

Let's see two special cases of

Thm. No mattter $H_0 : \mu_1 = \cdots = \mu_k$ is true or not

a. $SSE/\sigma^2 = (n-k)S_p^2/\sigma^2 \sim$ Chi square $(df = \sum_{j=1}^{k}(n_j - 1) = n - k)$

b. $SSTR \perp SSE$.

Cases

1. $k = 1$, one sample case, S_p^2 is sample variance — Chapter 7

   a. $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$

   b. $SSTR \equiv 0$

2. $k = 2$, two sample case — Chapter 9

   a. $(n-2)S_p^2/\sigma^2 \sim \chi^2(n-2)$

   b. $\overline{X} - \overline{Y} \perp S_p^2 \quad \Longleftrightarrow \quad SSTR \perp SSE$

Under $H_0 : \mu_1 = \cdots = \mu_k$

1. $SSTR/\sigma^2 \sim \chi^2(k-1)$
2. $SSE/\sigma^2 \sim \chi^2(n-k)$
3. $SSTR \perp SSE$

$$\Longrightarrow \qquad F = \frac{SSTR/(k-1)}{SSE/(n-k)} \sim F(df_1 = k-1, df_2 = n-k)$$

Reject H_0 if $F \geq F_{1-\alpha,k-1,n-k}$

Total Sum of Squares: SSTOT

SSTOT=SSE+SSTR

$$SSTOT := \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot\cdot}\right)^2$$

$$||$$

$$\sum_{j=1}^{k} \sum_{i=1}^{n_j} \left[\left(Y_{ij} - \overline{Y}_{j\cdot}\right) + \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}\right)\right]^2$$

$$||$$

$$\sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{j\cdot}\right)^2 + 2\sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot j}\right)\left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}\right) + \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}\right)^2$$

$$||$$

$$\sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{j\cdot}\right)^2 + 2\sum_{j=1}^{k} \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}\right) \sum_{i=1}^{n_j} \left(Y_{ij} - \overline{Y}_{\cdot j}\right) + \sum_{j=1}^{k} n_j \left(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}\right)^2$$

$$||$$

$$SSE + 0 + SSTR$$

$$\begin{array}{ccccc}
SSTOT & = & SSE & + & SSTR \\
& \Downarrow & & & \\
\dfrac{SSTOT}{\sigma^2} & = & \dfrac{SSE}{\sigma^2} & + & \dfrac{SSTR}{\sigma^2} \\
\wr & & \wr & & \wr \\
\chi^2(n-1) & & \chi^2(n-k) & \perp & \chi^2(k-1) \\
\text{Under } H_0 & & \checkmark & & \text{Under } H_0
\end{array}$$

One-way ANOVA Table

Source of Variance	Degree of Freedom (df)	Sum Square (SS)	Mean Square (MS)	F-ratio
Between Groups (Treatment)	k-1	$SSB = \sum_{j=1}^{k}\left(\frac{T_j^2}{n_j}\right) - \frac{T^2}{n}$ $\quad SSB = \sum_{j=1}^{k} n_j(\bar{X}_j - \bar{X}_t)^2$	$MSB = \frac{SSB}{k-1}$	$F = \frac{MSB}{MSW}$
Within Groups (Error)	n-k	$SSW = \sum_{j=1}^{K}\sum_{i=1}^{n} X_{ij}^2 - \sum_{j=1}^{K}\left(\frac{T_j^2}{n_j}\right)$ $\quad SSW = \sum_{j=1}^{k}\sum_{i=1}^{n}(X_{ij} - \bar{X}_j)^2$	$MSW = \frac{SSW}{n-k}$	
Total	n-1	$SST = \sum_{j=1}^{K}\sum_{i=1}^{n} X_{ij}^2 - \frac{T^2}{n}$ $\quad SST = \sum_{j=1}^{k}\sum_{i=1}^{n}(X_{ij} - \bar{X}_t)^2$		

- SST = SSB + SSW

k: number of groups n: number of samples
df: degree of freedom

Source	df	SS	MS	F	P
Treatment	$k-1$	$SSTR$	$MSTR$	$\frac{MSTR}{MSE}$	$P(F_{k-1,n-k} \geq \text{observed}F)$
Error	$n-k$	SSE	MSE		
Total	$n-1$	$SSTOT$			

Common notation

d.f.

k-1	Error sum of squares	$SSE = SSW = SS_{within}$
	Mean square of error	$MSE = MSW = MS_{within} = S_p^2$
	(Pooled sample variance)	
n-k	Treatment sum of squares	$SSTR = SSB = SS_{between}$
	Mean square of treatment	$MSTR = MSB = MS_{between}$
n-1	Total sum of squares:	$SST = SSTOT$

One way ANOVA v.s. Two sample t-test

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ be samples from $N(\mu_X, \sigma^2)$ and $N(\mu_Y, \sigma^2)$, respectively.

Recall

1. $SSTR/\sigma^2 = \dfrac{\left(\overline{X} - \overline{Y}\right)^2}{\sigma^2 \left(\frac{1}{n} + \frac{1}{m}\right)} \qquad \sim \chi^2(1)$
2. $SSE/\sigma^2 = (n+m-2)S_p^2/\sigma^2 \qquad \sim \chi^2(n+m-2)$

$$\Longrightarrow F = \frac{SSTR/1}{SSE/(n+m-2)} = \underbrace{\frac{\left(\overline{X} - \overline{Y}\right)^2}{S_p^2 \left(\frac{1}{n} + \frac{1}{m}\right)}}_{\| \atop T^2} \sim F(df_1 = 1, df_2 = n+m-2)$$

$$\Longrightarrow \alpha = \mathbb{P}\left(|T| \geq t_{\alpha/2, n+m-2}\right) = \mathbb{P}\left(T^2 \geq t^2_{\alpha/2, n+m-2}\right) = \mathbb{P}\left(F \geq F_{1-\alpha, 1, n+m-2}\right)$$

Equivalent!

E.g. 1 Study the relation between smoking and heart rates.

Generations of athletes have been cautioned that cigarette smoking impedes performance. One measure of the truth of that warning is the effect of smoking on heart rate. In one study, six nonsmokers, six light smokers, six moderate smokers, and six heavy smokers each engaged in sustained physical exercise. Table 8.1.1 lists their heart rates after they had rested for three minutes.

Table 8.1.1 Heart Rates

	Nonsmokers	Light Smokers	Moderate Smokers	Heavy Smokers
	69	55	66	91
	52	60	81	72
	71	78	70	81
	58	58	77	67
	59	62	57	95
	65	66	79	84
Averages:	62.3	63.2	71.7	81.7

Show whether smoking affects heart rates at $\alpha = 0.05$.

Sol. Let $\mu_1, \cdots, \mu_4$ be the true heart rates.

Test $H_0 : \mu_0 = \cdots = \mu_4$ or not.

Critical region:

Let $\alpha = 0.05$. For these data, $k = 4$ and $n = 24$, so $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ should be rejected if

$$F = \frac{SSTR/(4-1)}{SSE/(24-4)} \geq F_{1-0.05,4-1,24-4} = F_{.95,3,20} = 3.10$$

(see Figure 12.2.2).

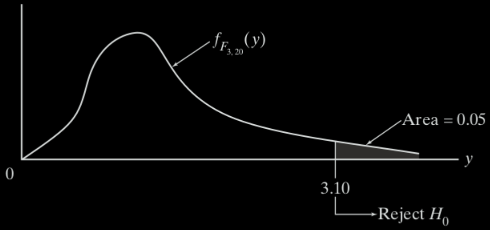


Figure 12.2.2

Computing....

Table 12.2.1

	Nonsmokers	Light Smokers	Moderate Smokers	Heavy Smokers
	69	55	66	91
	52	60	81	72
	71	78	70	81
	58	58	77	67
	59	62	57	95
	65	66	79	84
$T_{.j}$	374	379	430	490
$\overline{Y}_{.j}$	62.3	63.2	71.7	81.7

The overall sample mean, $\overline{Y}_{..}$, is given by

$$\overline{Y}_{..} = \frac{1}{n}\sum_{j=1}^{k} T_{.j} = \frac{374+379+430+490}{24}$$

$$= 69.7$$

Therefore,

$$SSTR = \sum_{j=1}^{4} n_j\left(\overline{Y}_{.j} - \overline{Y}_{..}\right)^2 = 6[(62.3-69.7)^2 + \cdots + (81.7-69.7)^2]$$

$$= 1464.125$$

Similarly,

$$SSE = \sum_{j=1}^{4}\sum_{i=1}^{6}\left(Y_{ij} - \overline{Y}_{.j}\right)^2 = [(69-62.3)^2 + \cdots + (65-62.3)^2] + \cdots + [(91-81.7)^2 + \cdots + (84-81.7)^2]$$

$$= 1594.833$$

The observed test statistic, then, equals *6.12*:

$$F = \frac{1464.125/(4-1)}{1594.833/(24-4)} = 6.12$$

Since $6.12 > F_{.95,3,20} = 3.10$, H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$ should be rejected. These data support the contention that smoking influences a person's heart rate.

Figure 12.2.3 shows the analysis of these data summarized in the ANOVA table format. Notice that the small P-value ($= 0.004$) is consistent with the conclusion that H_0 should be rejected.

Source	df	SS	MS	F	P
Treatment	3	1464.125	488.04	6.12	0.004
Error	20	1594.833	79.74		
Total	23	3058.958			

Figure 12.2.3

□

http://www.sthda.com/english/wiki/one-way-anova-test-in-r

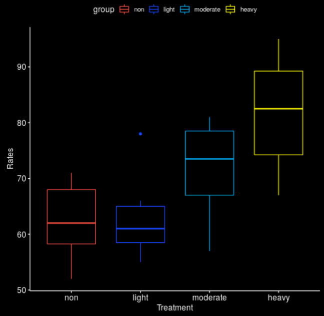
```
> Input <-c("
+ rates group
+ 69  non
+ 52  non
+ 71  non
+ 58  non
+ 59  non
+ 65  non
+ 55  light
+ 60  light
+ 78  light
+ 58  light
+ 62  light
+ 66  light
+ 66  moderate
+ 81  moderate
+ 70  moderate
+ 77  moderate
+ 57  moderate
+ 79  moderate
+ 91  heavy
+ 72  heavy
+ 81  heavy
+ 67  heavy
+ 95  heavy
+ 84  heavy
+ ")
> Data = read.table(textConnection(Input),
+                   header=TRUE)
```

```
> Data
   rates    group
1     69      non
2     52      non
3     71      non
4     58      non
5     59      non
6     65      non
7     55    light
8     60    light
9     78    light
10    58    light
11    62    light
12    66    light
13    66 moderate
14    81 moderate
15    70 moderate
16    77 moderate
17    57 moderate
18    79 moderate
19    91    heavy
20    72    heavy
21    81    heavy
22    67    heavy
23    95    heavy
24    84    heavy
```

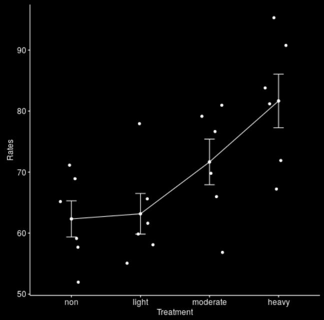
```
> # Check the levels
> levels (Data$group)
[1]  "heavy"     " light "      "moderate" "non"
> # Order the groups
> Data$group <- ordered(Data$group, levels = c("non", "light" , "moderate", "heavy"))
> levels (Data$group)
[1]  "non"         " light "      "moderate" "heavy"
```

```
> # Compute summary statistics by groups
> # including  count,  mean, sd:
> library (dplyr)  # a grammar of data manipulation
> group_by(Data, group) %>%
+    summarise(
+       count = n() ,
+       mean = mean(rates, na.rm = TRUE),
+       sd = sd(rates,  na.rm = TRUE)
+    )
# A  tibble :  4  x  4
  group      count  mean    sd
  <ord>      <int> <dbl> <dbl>
1 non            6   62.3   7.26
2  light         6   63.2   8.16
3 moderate       6   71.7   9.16
4 heavy          6   81.7 10.8
```

```r
# Box plots
# ++++++++++++++++++++
# Plot  rates  by group and color by group
 library (ggpubr)
png("Case_12-2-1-ggboxplot.png")
ggboxplot(Data, x = "group", y = "rates",
              color  = "group",  palette  = c("#00AFBB", "#E7B800", "#FC4E07", "blue"),
              order = c("non",  " light ",  "moderate","heavy"),
              ylab  = "Rates", xlab  = "Treatment")
dev.off ()
```



```r
# Mean plots
# ++++++++++++++++++++
# Plot rates by group
# Add error bars: mean_se
# (other values include: mean_sd, mean_ci, median_iqr, ....)
png("Case_12-2-1-ggline.png")
 library (ggpubr)
ggline(Data, x = "group", y = "rates",
       add = c("mean_se", "jitter"),
       order = c("non", "light", "moderate","heavy"),
       ylab = "Rates", xlab = "Treatment")
dev.off ()
```



```
> # Compute the analysis of variance
> res.aov <- aov(rates ~ group, data = Data)
> # Summary of the analysis
> summary(res.aov)
            Df Sum Sq Mean Sq F value  Pr(>F)
group        3   1464   488.0    6.12 0.00398 **
Residuals   20   1595    79.7
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
> # Tukey multiple  multiple-comparisons
> TukeyHSD(res.aov)
  Tukey multiple  comparisons of means
    95% family-wise confidence level

 Fit :  aov(formula = rates ~ group, data = Data)

$group
                       diff          lwr        upr       p adj
 light -non         0.8333333 -13.596955 15.26362 0.9984448
moderate-non   9.3333333 -5.096955 23.76362 0.2978123
heavy-non      19.3333333  4.903045 33.76362 0.0063659
moderate-light  8.5000000 -5.930289 22.93029 0.3755571
heavy-light     18.5000000  4.069711 32.93029 0.0091463
heavy-moderate 10.0000000 -4.430289 24.43029 0.2438158
```

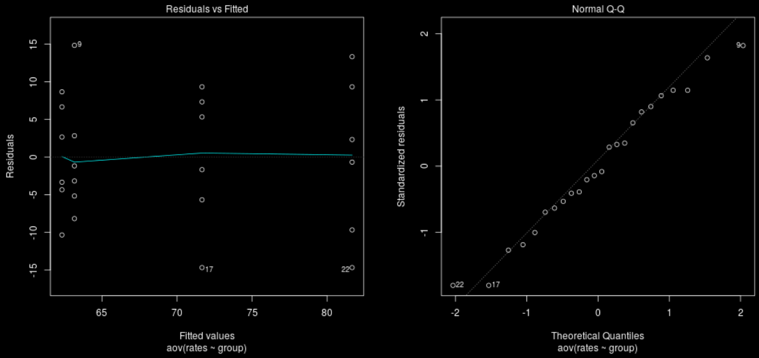
1. diff: difference between means of the two groups
2. lwr, upr: the lower and the upper end point of the C.I. at 95% (default)
3. p adj: p-value after adjustment for the multiple comparisons

Inferences

if p-value ≤ 0.05 $\iff$ if zero is in the C.I.

```
> # Or one may use multcomp package or multiple comparisons
>  library (multcomp)
> summary(glht(res.aov,  linfct  = mcp(group = "Tukey")))

     Simultaneous Tests for General Linear Hypotheses

Multiple  Comparisons of Means: Tukey Contrasts


Fit :  aov(formula = rates ~ group, data = Data)

Linear  Hypotheses:
                          Estimate Std. Error  t  value Pr(>|t|)
light   – non == 0          0.8333      5.1556   0.162  0.99844
moderate – non == 0         9.3333      5.1556   1.810  0.29776
heavy – non == 0           19.3333      5.1556   3.750  0.00629 **
moderate – light  == 0      8.5000      5.1556   1.649  0.37544
heavy –  light   == 0      18.5000      5.1556   3.588  0.00901 **
heavy – moderate == 0 10.0000      5.1556   1.940  0.24382
–––
Signif .  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1  ' '  1
(Adjusted p values reported –– single–step method)
```

```r
# Check ANOVA assumptions: test validity?
# diagnostic  plots
layout(matrix(c(1,2) ,1,2) )  # optional  1x2 graphs/page
plot(res.aov,c(1,2))
```



1. Residuals vs Fitted: test homogeneity of variances
   One can also use Levene's test for this purpose:

```
> # Use Levene's test to gest homogeneity of variances
> library (car)
> leveneTest(rates ~ group, data = Data)
Levene's Test for Homogeneity of Variance (center = median)
      Df F value Pr(>F)
group  3  0.3885 0.7625
      20
```

2. Normal Q-Q plot: Test normality. (It should be close to diagonal line.)
   One can also use Shapiro-Wilk test:

```
# Extract the residuals
> aov_residuals <- residuals(object = res.aov )
> # Run Shapiro-Wilk test
> shapiro.test (x = aov_residuals )

  Shapiro-Wilk normality test

data:  aov_residuals
W = 0.9741, p-value = 0.7677
```

Non-parametric alternative to one-way ANOVA test

```
> # Non–parametric alternative to one–way ANOVA test
> # a non–parametric alternative to one–way ANOVA
> # is Kruskal–Wallis rank sum test, which can be
> # used when ANNOVA assumptions are not met.
> kruskal.test(rates ~ group, data = Data)

    Kruskal–Wallis rank sum test

data:  rates by group
Kruskal–Wallis chi–squared = 10.729, df = 3, p–value = 0.01329
```

See Section 4 of Chapter 14 for more details.