Math 362: Mathematical Statistics II

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Chapter 6. Hypothesis Testing

Plan

Difficulties

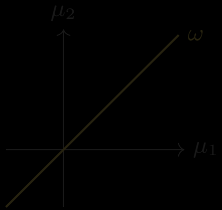
Scalar parameter		Vector parameter
Simple-vs-Composite test	$\Rightarrow$	Composite-vs-Composite test
$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$		$H_0 : \theta \in \omega$ vs $H_1 : \theta \in \Omega \cap \omega^c$

E.g. Two normal populations $N(\mu_i, \sigma_i)$, $i = 1, 2$. σ_i are known, μ_i unknown.

$$H_0 : \mu_1 = \mu_2 \quad \text{vs} \quad H_1 : \mu_1 \neq \mu_2.$$

Equivalently,

$$H_0 : (\mu_1, \mu_2) \in \omega \quad \text{vs} \quad H_1 : (\mu_1, \mu_2) \notin \omega.$$



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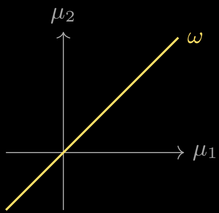

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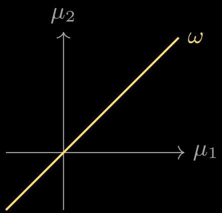

- Let $Y_1, \cdots, Y_n$ be a random sample of size n from $f_Y(y; \theta_1, \cdots, \theta_k)$
- Let Ω be all possible values of the parameter vector $(\theta_1, \cdots, \theta_k)$
- Let $\omega \subseteq \Omega$ be a subset of Ω.
- Test:

$$H_0 : \theta \in \omega \quad \text{vs} \quad H_1 : \theta \in \Omega \setminus \omega.$$

- The **generalized likelihood ratio**, λ, is defined as

$$\lambda := \frac{\max_{(\theta_1, \cdots, \theta_k) \in \omega} L(\theta_1, \cdots, \theta_k)}{\max_{(\theta_1, \cdots, \theta_k) \in \Omega} L(\theta_1, \cdots, \theta_k)}$$

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$$\lambda \in (0, 1]$$

λ close to zero
data NOT compatible with H_0
reject H_0

λ close to one
data compatible with H_0
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- **Generalized likelihood ratio test (GLRT)**: Use the following critical region

$$C = \{\lambda : \lambda \in (0, \lambda^*]\}$$

to reject H_0 with either α or y^* being determined through

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Remarks:

1. Maximization over Ω instead of $\Omega \setminus \omega$ in denominator:

   In practice, little effect on this change.

   In theory, much easier/nicer: $L(\theta_1, \cdots, \theta_k)$ is maximized over the whole space Ω by the max. likelihood estimates: $\Omega_e := (\theta_{e,1}, \cdots, \theta_{e,k}) \in \Omega$.

2. Suppose the maximization over ω is achieved at $\omega_e \in \omega$.

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5. Working with Λ is hard since $f_\Lambda(\lambda|H_0)$ is hard to obtain.

   If Λ is a *(monotonic) function* of some r.v. W, whose pdf is known.

   Suggesting testing procedure

   Test based on λ $\iff$ Test based on w.

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$$H_0 : \theta = \theta_0 \quad \text{v.s.} \quad H_1 : \theta < \theta_0 \qquad \text{with given } \alpha.$$

Sol. 1) The null hypothesis is simple, and hence

$$L(\omega_e) = L(\theta_0) = \theta_0^{-n} \prod_{i=1}^{n} I_{[0,\theta_0]}(y_i) = \theta^{-n} I_{[0,\theta_0]}(y_{max}).$$

2) The MLE for θ is y_{max} and hence,

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\alpha &= \mathbb{P}(0 < \Lambda \leq \lambda^* | H_0 \text{ is true}) \\
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$$\begin{aligned} f_{Y_{max}}(y;\theta_0) &= nF_Y(y;\theta_0)^{n-1} f_Y(y;\theta_0) \\ &= \frac{ny^{n-1}}{\theta_0^n}, \quad y \in [0, \theta_0]. \end{aligned}$$

6) Finally, by setting $y^* := \theta_0(\lambda^*)^{1/n}$, we see that

$$\begin{aligned} \alpha &= \mathbb{P}\left(Y_{max} \leq y^* \middle| H_0 \text{ is true}\right) \\ &= \int_0^{y^*} \frac{ny^{n-1}}{\theta_0^n} \mathrm{d}y \\ &= \frac{(y^*)^n}{\theta_0^n} \iff y^* = \theta_0 \alpha^{1/n}. \end{aligned}$$

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Find a test statistic Λ for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

Sol. Let $\overline{X}$ and $\bar{k}$ be the sample mean. Because the null hypothesis is simple,

$$L(\omega_e) = L(p_0) = \prod_{i=1}^{n} (1 - p_0)^{k_i - 1} p_0 = (1 - p_0)^{n\bar{k} - n} p_0^n,$$

which shows that $\bar{k}$ is a sufficient estimator.

On the other hand, the MLE for the parameter p is $1/\bar{k}$. So

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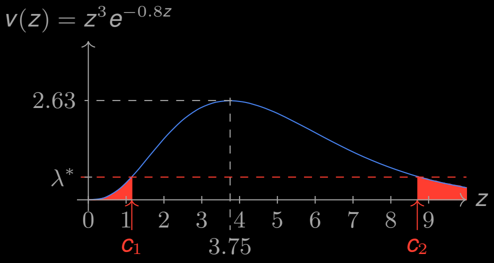

This suggests that the critical region in terms of z should be of the form:

$$(0, c_1) \cup (c_2, \infty)$$

For convenience, we put $\alpha/2$ mass on each tails of the density of Z:

Find c_1 and c_2 such that

$$\int_0^{c_1} f_Z(z)dz = \int_{c_2}^{\infty} f_Z(z)dz = \frac{\alpha}{2}.$$

	using V	using Z
Critical region	$(0, v^*]$	$(0, z_1] \cup [z_2, \infty)$
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