Math 362: Mathematical Statistics II

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Chapter 11. Regression

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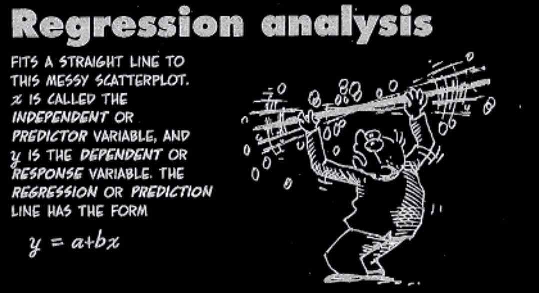
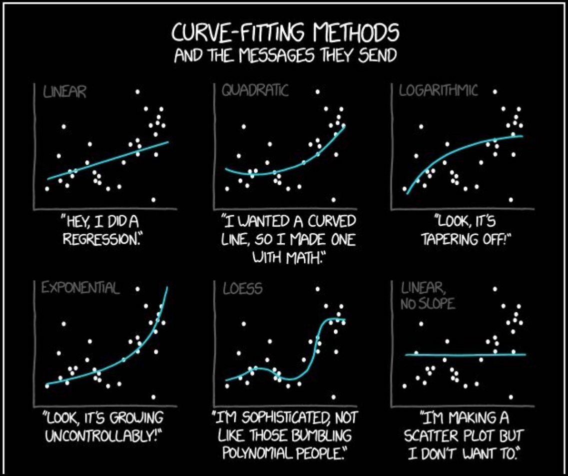


https://madhureshkumar.wordpress.com/



https://xkcd.com/

Three ways to view the same thing

$$(x_1, y_1), \cdots, (x_n, y_n)$$

1. Purely data, no probability structure assumed.

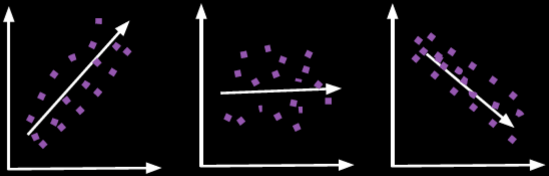
$$(x_1, Y_1), \cdots, (x_n, Y_n)$$

2. A random sample of size n, where Y_i follows a distribution depending on x_i which is deterministic.

$$(X_1, Y_1), \cdots, (X_n, Y_n)$$

3. A random sample of size n, where (X_i, Y_i) follow some joint distribution.

CORRELATION



Positive Correlation

Zero Correlation

Negative Correlation

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$$

Covarianced normalized by Standard Deviation

Correlation between X and Y

Standard deviation of X

Standard deviation of Y

Notation: $\mathit{Corr}(X, Y) = \rho(X, Y) = \rho_{XY}$

Computing: $\mathrm{Var}(X) = \sigma_X^2$, $\mathrm{Var}(Y) = \sigma_Y^2$, $\mathrm{Cov}(X, Y) = \sigma_{XY}$

$$\Downarrow$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Thm. For any two random variables X and Y,

a. $|\rho(X, Y)| \leq 1$

b. $\rho(X, Y) = 1$ if and only if $Y = aX + b$ for some $a > 0$ and $b \in \mathbb{R}$;
$\rho(X, Y) = -1$ if and only if $Y = aX + b$ for some $a < 0$ and $b \in \mathbb{R}$.

Proof. (a)

$$|\rho(X, Y)| \leq 1$$

$$\Updownarrow$$

$$\begin{aligned} |\mathbb{E}\left((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\right)| &\leq \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} \\ &= \sqrt{\mathbb{E}\left((X - \mathbb{E}(X))^2\right)}\sqrt{\mathbb{E}\left((Y - \mathbb{E}(Y))^2\right)} \end{aligned}$$

which is nothing but the Cauchy-Schwartz inequality.

(b) In the Cauchy-Schwartz inequality, the equality holds if and only if for some $a \neq 0$,

$$X - \mathbb{E}(X) = a[Y - E(Y)]$$

namely,

$$X = aY + b, \qquad \text{with} \quad b = \mathbb{E}(X) - a\mathbb{E}(Y).$$

In particular, $a > 0$ corresponds to the case $\rho(X, Y) = 1$ and $a < 0$ to $\rho(X, Y) = -1$. $\square$

Estimating $\rho(X, Y)$ – Sample correlation coefficient

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

$$= \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\mathbb{E}[X^2] - \mathbb{E}[X]^2}\sqrt{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2}}$$

$$\Downarrow$$

$$R = \frac{n\sum_{i=1}^n X_i Y_i - \left(\sum_{i=1}^n X_i\right)\left(\sum_{i=1}^n Y_i\right)}{\sqrt{n\sum_{i=1}^n X_i^2 - \left(\sum_{i=1}^n X_i\right)^2}\sqrt{n\sum_{i=1}^n Y_i^2 - \left(\sum_{i=1}^n Y_i\right)^2}}$$

Pearson product-moment correlation coefficient

or

Sample correlation coefficient

Thm.

$$R^2 = 1 - \frac{SSE}{SST} = \frac{SST - SSE}{SST} = \frac{SSTR}{SST}$$

where

$$SSE = \sum_{i=1}^{n} \left(Y_i - \widehat{Y}_i\right)^2, \quad \widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

$$SST = \sum_{i=1}^{n} \left(Y_i - \overline{Y}_i\right)^2, \quad \text{and} \quad SSTR = SST - SSE.$$

Remark SSE: sum of square errors $\sim$ the variation in y_i's not explained by L.M.

SST: Total sum of squares $\sim$ total variability.

SSTR: Treatment sum of sqrs. $\sim$ the variation in y_i's explained by L.M.

R^2 (or r^2 when X_i and Y_i are replaced by x_i and y_i) $\sim$ proportion of total variation in the y_i's that can be attributed to L.M.

Coefficient of determination or simply *R squared*

Proof

□

Adjusted R-squared

Def. The adjusted R-squareed:

$$R^2_{adj} := 1 - \frac{MSE}{MST}$$

where

$$MSE = \frac{SSE}{n-q} \qquad \text{and} \qquad MST = \frac{SST}{n-1}$$

and q is number of parameters in the model.

Relation:

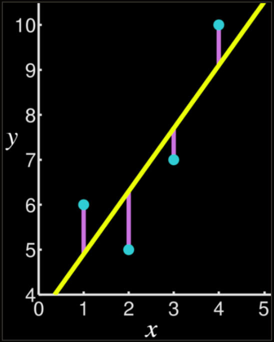
$$R^2_{adj} = 1 - \left(1 - R^2\right) \frac{n-1}{n-q}$$

MSE: Mean squared error.

MST: Mean squared total.

MSR = MSTR: Mean square for treatment (or regresssion).

$$MSR = MSTR = \frac{SSTR}{q-1}$$



In linear regression, the observations (**red**) are assumed to be the result of random deviations (**green**) from an underlying relationship (**blue**) between a dependent variable (*y*) and an independent variable (*x*).

Goal: Find a blue line that minimizes the sum of the square of the green lines

Thm. Given n points $(x_1, y_1), \cdots, (x_n, y_n)$, the straight line $y = a + bx$ minimizing

$$L(a,b) = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

when

$$b = \frac{n \sum_{i=1}^n x_i y_i - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{n\left(\sum_{i=1}^n x_i^2\right) - \left(\sum_{i=1}^n x_i\right)^2}$$

and

$$a = \frac{\sum_{i=1}^n y_i - b \sum_{i=1}^n x_i}{n} = \bar{y} - b\bar{x}.$$

Proof.

$$\begin{cases} \dfrac{\partial}{\partial a} L(a,b) = \displaystyle\sum_{i=1}^{n} (-2)\left[y_i - (a + bx_i)\right] = 0 \\ \dfrac{\partial}{\partial b} L(a,b) = \displaystyle\sum_{i=1}^{n} (-2x_i)\left[y_i - (a + bx_i)\right] = 0 \end{cases} \qquad \text{(Normal equations)}$$

$$\iff \quad \begin{cases} \sum_{i=1}^{n} y_i - na - b\sum_{i=1}^{n} x_i = 0 & (1) \\ \sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 = 0 & (2) \end{cases}$$

$$(1) \quad \Longrightarrow \quad a = \bar{y} - b\bar{x}$$

$$(1) \times \sum_{i=1}^{n} x_i - (2) \times n \quad \Longrightarrow \quad b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$\square$

(Moore-Penrose) Pseudoinverse

1. Well determined system

$$Ax = b \quad \Longrightarrow \quad x = A^{-1}y.$$

2. Overdetermined system

$$\begin{aligned}
Ax &= y \\
A^T A x &= A^T y \\
\underbrace{(A^T A)^{-1} A^T A}_{=I} x &= (A^T A)^{-1} A^T y \\
x &= \underbrace{(A^T A)^{-1} A^T}_{=:A^+} y
\end{aligned}$$

3. Under determined system

$$Ax = y \quad \Longrightarrow \quad x = \underbrace{A^T (AA^T)^{-1}}_{=:A^+} y.$$

Proof. (Another proof based on pseudoinverse)

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}_{n\times 2}, \qquad x = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}_{2\times 1}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}_{1\times n}$$

$$A^T A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \begin{pmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{pmatrix}$$

$$A^T y = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} = x =& (A^T A)^{-1} A^T y \\ =& \frac{1}{n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \begin{pmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix} \\ =& \begin{pmatrix} \dfrac{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i y_i\right)}{n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \\ \dfrac{n \sum_{i=1}^n x_i y_i - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \end{pmatrix} \end{aligned}$$

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}.$$

$$a = \frac{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left[\left(\sum_{i=1}^{n} x_i y_i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\right]}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$- \frac{\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} y_i - b\frac{1}{n}\sum_{i=1}^{n} x_i = \bar{y} - b\bar{x}.$$

$\square$

A probabilistic view ...

Def. The function $f(X)$ for which

$$\mathbb{E}\left[(Y - f(X))^2\right]$$

is minimized is called the **regression curve of Y on X**.

Thm. Let (X, Y) be two random variables such that $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ both exist. Then the regression cure of Y on X is given (for all x) by

$$f(x) = \mathbb{E}\left[Y|X = x\right].$$

Proof. Let $f(x) = \mathbb{E}[Y|X=x]$ and let $\phi(x)$ be a general function. Then

$$\begin{aligned}\mathbb{E}\left[(Y-\phi(X))^2\right] =& \mathbb{E}\left[([Y-f(X)]+[f(X)-\phi(X)])^2\right] \\ =& \mathbb{E}\left[(Y-f(X))^2\right] + \mathbb{E}\left[(f(X)-\phi(X))^2\right] \\ & + \mathbb{E}\left[(Y-f(X))\,(f(X)-\phi(X))\right].\end{aligned}$$

Let $\psi(x)$ be either $f(x)$ or $\phi(x)$. We claim that

$$\mathbb{E}\left[(Y-f(X))\,\psi(X)\right] = 0.$$

Indeed,

$$\begin{aligned}\mathbb{E}[Y\psi(X)] &= \iint_{\mathbb{R}^2} f_{X,Y}(x,y) y \psi(x) \mathrm{d}y \mathrm{d}x \\ &= \int_{\mathbb{R}} \mathrm{d}x \psi(x) f_X(x) \underbrace{\int_{\mathbb{R}} \mathrm{d}y \frac{f_{X,Y}(x,y)}{f_X(x)} y}_{=\mathbb{E}[Y|X=x]} \\ &= \mathbb{E}[f(X)\psi(X)].\end{aligned}$$

Hence,

$$\mathbb{E}\left[(Y-\phi(X))^2\right] = \mathbb{E}\left[(Y-f(X))^2\right] + \mathbb{E}\left[(f(X)-\phi(X))^2\right]$$

which is minimized when $\phi(x) = f(x)$. $\square$

If one imposes that $f(x) = a + bx$, then

Thm. The following squared error:

$$\mathbb{E}\left[\{Y - (a + bX)\}^2\right]$$

is minimized at

$$b = \rho_{XY}\frac{\sigma_Y}{\sigma_X} = \frac{\sigma_{XY}}{\sigma_X^2} \quad \text{and} \quad a = \mathbb{E}[Y] - b\mathbb{E}[X]$$

with the mean squared error

$$\mathbb{E}\left[\{Y - (a + bX)\}^2\right] = \left(1 - \rho_{XY}^2\right)\sigma_Y^2.$$

Proof.

$$
\begin{aligned}
&\mathbb{E}\left[\{Y-(a+bX)\}^2\right] \\
=&\mathbb{E}\left[\left\{[Y-\mathbb{E}(Y)]-b[X-\mathbb{E}(X)]-[a-\mathbb{E}[Y]+b\mathbb{E}(X)]\right\}^2\right]
\end{aligned}
$$

$$
\begin{array}{ccc}
|| & & \\
\mathbb{E}\left[[Y-\mathbb{E}(Y)]^2\right] & & \mathrm{Var}(Y) \\
+b^2\mathbb{E}\left[[X-\mathbb{E}(X)]^2\right] & & +b^2\mathrm{Var}(X) \\
+\left[a-\mathbb{E}[Y]+b\mathbb{E}(X)\right]^2 & = & +\left[a-\mathbb{E}[Y]+b\mathbb{E}(X)\right]^2 \\
-2b\mathbb{E}\left[[Y-\mathbb{E}(Y)][X-\mathbb{E}(X)]\right] & & -2b\,\mathrm{Cov}(X,Y) \\
-2\left[a-\mathbb{E}[Y]+b\mathbb{E}(X)\right]\mathbb{E}\left[Y-\mathbb{E}(Y)\right] & & +0 \\
+2b\left[a-\mathbb{E}[Y]+b\mathbb{E}(X)\right]\mathbb{E}\left[X-\mathbb{E}(X)\right] & & +0
\end{array}
$$

$$\Downarrow$$

$$\mathbb{E}\left[\{Y-(a+bX)\}^2\right]$$

$$||$$

$$\mathrm{Var}(Y)+b^2\mathrm{Var}(X)+\Big[a-\mathbb{E}[Y]+b\mathbb{E}(X)\Big]^2-2b\,\mathrm{Cov}(X,Y)$$

The best a, called a^*, should be such that

$$\Big[a^*-\mathbb{E}[Y]+b\mathbb{E}(X)\Big]^2=0 \quad \Longleftrightarrow \quad a^*=\mathbb{E}[Y]-b\mathbb{E}[X]$$

$$\Downarrow$$

$$\mathbb{E}\left[\left\{Y - (a^* + bX)\right\}^2\right]$$

$$||$$

$$\mathrm{Var}(Y) + b^2\mathrm{Var}(X) - 2b\,\mathrm{Cov}(X, Y)$$

$$||$$

$$\sigma_Y^2 + b^2\sigma_X^2 - 2b\rho_{XY}\sigma_X\sigma_Y$$

$$||$$

$$\left(1 - \rho_{XY}^2\right)\sigma_Y^2 + \Big(b\sigma_X - \rho_{XY}\sigma_Y\Big)^2$$

The best b, called b^*, should be

$$\left(b^*\sigma_X - \rho_{XY}\sigma_Y\right)^2 = 0 \quad \Longleftrightarrow \quad b^* = \rho_{XY}\frac{\sigma_Y}{\sigma_X}$$

$$\Downarrow$$

$$\mathbb{E}\left[\{Y - (a^* + b^* X)\}^2\right]$$

$$||$$

$$\left(1 - \rho_{XY}^2\right) \sigma_Y^2$$

with

$$b^* = \rho_{XY} \frac{\sigma_Y}{\sigma_X} = \frac{\sigma_{XY}}{\sigma_X^2} \qquad \text{and} \qquad a^* = \mathbb{E}[Y] - b\mathbb{E}[X]$$

$$\square$$

Remark In practice, we have data $(x_1, y_1), \cdots, (x_n, y_n)$ instead of the joint law of (X, Y)

$$\Downarrow$$

Replace

$$\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{XY}, \sigma_{XY}$$

by their maximum likelihood estimates

$$\bar{x}, \bar{y}, \hat{\sigma}_X^2, \hat{\sigma}_Y^2, r_{XY}, \hat{\sigma}_{XY}$$

1. $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

2. $$\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n^2}$$

$$\hat{\sigma}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \bar{y}^2 = \frac{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}{n^2}$$

3. $$\hat{\sigma}_{XY} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}$$

$$= \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n^2}$$

4. $$r_{XY} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X \hat{\sigma}_Y}$$

$$\Downarrow$$

$$\boxed{b = r_{XY}\frac{\hat{\sigma}_Y}{\hat{\sigma}_X} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X^2}, \qquad a = \bar{y} - b\bar{x}}$$

Maximum likelihood estimates

$$\hat{\sigma}_X^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\hat{\sigma}_Y^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$\hat{\sigma}_{XY} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

Sample (co)variances

$$s_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$s_Y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

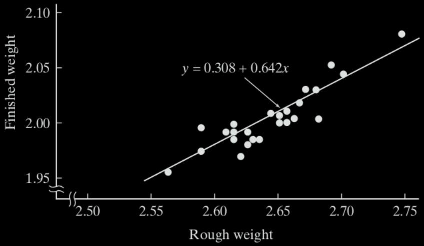
$$s_{XY} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

E.g. 1 Producing air conditioners. x = rough weight of a rod. y = finished weight. Find the best linear approximation of xy-relationship. Predict the weight when $x = 2.71$

Table 11.2.1

Rod Number	Rough Weight, x	Finished Weight, y	Rod Number	Rough Weight, x	Finished Weight, y
1	2.745	2.080	14	2.635	1.990
2	2.700	2.045	15	2.630	1.990
3	2.690	2.050	16	2.625	1.995
4	2.680	2.005	17	2.625	1.985
5	2.675	2.035	18	2.620	1.970
6	2.670	2.035	19	2.615	1.985
7	2.665	2.020	20	2.615	1.990
8	2.660	2.005	21	2.615	1.995
9	2.655	2.010	22	2.610	1.990
10	2.655	2.000	23	2.590	1.975
11	2.650	2.000	24	2.590	1.995
12	2.650	2.005	25	2.565	1.955
13	2.645	2.015			

Sol. ...



2.10
2.05
2.00
1.95
Finished weight
y = 0.308 + 0.642x
2.50
2.55
2.60
2.65
2.70
2.75
Rough weight

...

□

Def. Let a and b be the least squares coefficients with the sample $(x_1, y_1), \cdots, (x_n, y_n)$.

$\hat{y} = a + bx$: **predicted value** of y

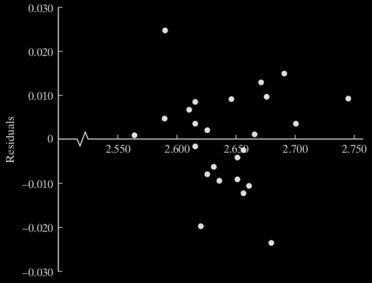
$y_i - \hat{y}_i = y_i - (a + bx_i)$: ***i*th residual**

Remark Use the residual plots to assessing the model.

E.g. 1' Here are the residues and their plots:

Table 11.2.2

x_i	y_i	$\hat{y}_i$	$y_i - \hat{y}_i$
2.745	2.080	2.070	0.010
2.700	2.045	2.041	0.004
2.690	2.050	2.035	0.015
2.680	2.005	2.029	−0.024
2.675	2.035	2.025	0.010
2.670	2.035	2.022	0.013
2.665	2.020	2.019	0.001
2.660	2.005	2.016	−0.011
2.655	2.010	2.013	−0.003
2.655	2.000	2.013	−0.013
2.650	2.000	2.009	−0.009
2.650	2.005	2.009	−0.004
2.645	2.015	2.006	0.009
2.635	1.990	2.000	−0.010
2.630	1.990	1.996	−0.006
2.625	1.995	1.993	0.002
2.625	1.985	1.993	−0.008
2.620	1.970	1.990	−0.020
2.615	1.985	1.987	−0.002
2.615	1.990	1.987	0.003
2.615	1.995	1.987	0.008
2.610	1.990	1.984	0.006
2.590	1.975	1.971	0.004
2.590	1.995	1.971	0.024
2.565	1.955	1.955	0.000



E.g. 2 Predict the Social Security expenditures.

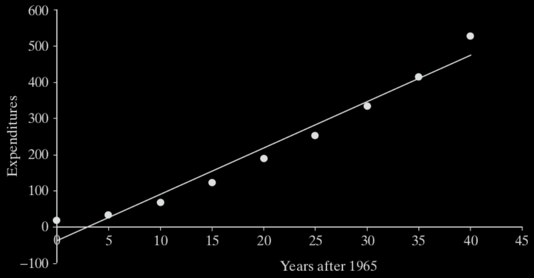
Table 11.2.3

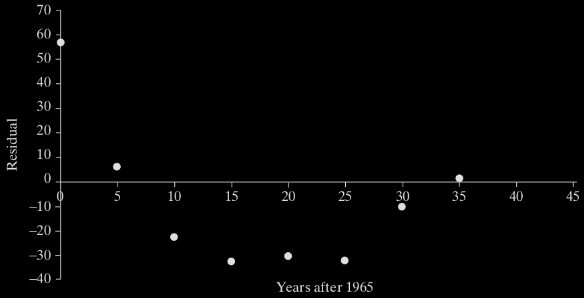
Year	Years after 1965, x	Social Security Expenditures ($ billions), y
1965	0	19.2
1970	5	33.1
1975	10	69.2
1980	15	123.6
1985	20	190.6
1990	25	253.1
1995	30	339.8
2000	35	415.1
2005	40	529.9

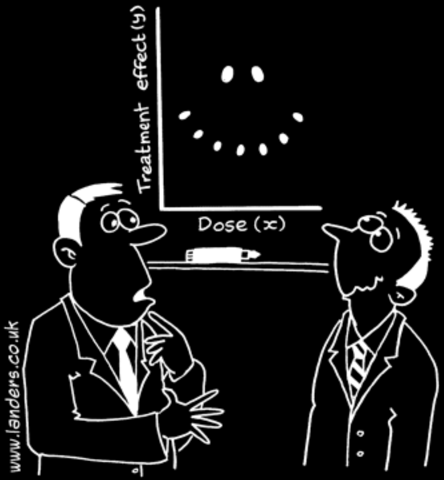
Source: www.socialsecurity.gov/history/trustfunds.html.

Does the the least squares line $y = -38.0 + 12.9x$ a good model to predict the cost in 2010 would be \$543, i.e., the case $x = 45$?

Sol.

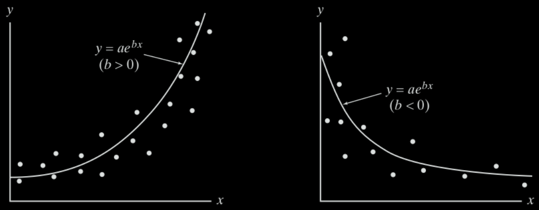

600
500
400
300
200
100
0
−100
Expenditures
0
5
10
15
20
25
30
35
40
45
Years after 1965


70
60
50
40
30
20
10
0
−10
−20
−30
−40
Residual
0
5
10
15
20
25
30
35
40
45
Years after 1965



"It's a non-linear pattern with outliers.....but for some reason I'm very happy with the data."

Exponential Regression



$$y = ae^{bx} \quad \Longleftrightarrow \quad \ln y = \ln a + bx$$

$$b = \frac{n\sum_{i=1}^{n} x_i \ln y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \ln y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} \qquad \ln a = \frac{\sum_{i=1}^{n} \ln y_i - b\sum_{i=1}^{n} x_i}{n}$$

E.g. Moore's law:

Gordon Moore predicted in 1965 that the number of transistors per chip would double every 18 months.

Based on the real data, check:

1) Whether is the chip capacity doubling at a fixed rate?

2) Find out the rate.

Table 11.2.5

Chip	Year	Years after 1975, x	Transistors per Chip, y
8080	1975	0	4,500
8086	1978	3	29,000
80286	1982	7	90,000
80386	1985	10	229,000
80486	1989	14	1,200,000
Pentium	1993	18	3,100,000
Pentium Pro	1995	20	5,500,000

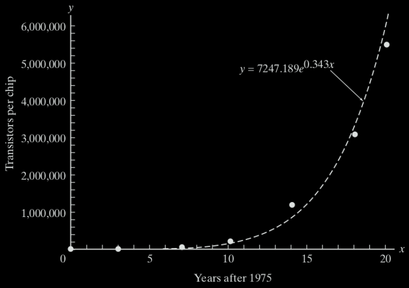
Source: en.wikipedia.org/wiki/Transistor—count.

Sol. To check whether chip capacity doubles in a fixed rate, one needs to carry out exponential regression:

Table 11.2.6

Years after 1975, x_i	x_i^2	Transistors per Chip, y_i	$\ln y_i$	$x_i \cdot \ln y_i$
0	0	4,500	8.41183	0
3	9	29,000	10.27505	30.82515
7	49	90,000	11.40756	79.85292
10	100	229,000	12.34148	123.41480
14	196	1,200,000	13.99783	195.96962
18	324	3,100,000	14.94691	269.04438
20	400	5,500,000	15.52026	310.40520
72	1078		86.90093	1009.51207

$$\Longrightarrow \quad b = \cdots = 0.342810, \quad a = \cdots = e^{\ln a} = e^{8.89} = 7247.189.$$



Finally, to find out the rate:

$$e^{0.343x} = e^{\ln 2 \times \frac{0.343}{\ln 2} x} = 2^{\frac{0.343}{\ln 2} x}$$

$$\frac{0.343}{\ln 2} x = 1 \quad \Longrightarrow \quad x = \frac{\ln 2}{0.343} = 2.020837.$$

$\square$

Other curvilinear models

Table 11.2.10

a. If $y = ae^{bx}$, then $\ln y$ is linear with x.

b. If $y = ax^{b}$, then $\log y$ is linear with $\log x$.

c. If $y = L/(1 + e^{a+bx})$, then $\ln\left(\frac{L-y}{y}\right)$ is linear with x.

d. If $y = \frac{1}{a+bx}$, then $\frac{1}{y}$ is linear with x.

e. If $y = \frac{x}{a+bx}$, then $\frac{1}{y}$ is linear with $\frac{1}{x}$.

f. If $y = 1 - e^{-x^{b}/a}$, then $\ln\ln\left(\frac{1}{1-y}\right)$ is linear with $\ln x$.

Recall For any two random variables X and Y, the regression curve of Y on X, namely,

$$f(x) = \mathbb{E}\left[Y|X = x\right].$$

minimizes the squared error

$$\mathbb{E}\left[(Y - f(X))^2\right]$$

Difficulties The regression curve $y = \mathbb{E}[Y|x]$ is complicated and hard to obtain.

Compromise Assume that $f(x) = a + bx$ (i.e., the first order approximation)

Def. **(Simple) linear model**:

1. $f_{Y|x}(y)$ is a normal pdf for any x given.

2. The standard deviation, σ, of $Y|x$ is the same for all x, i.e.,

$$\sigma^2 \equiv \mathbb{E}[Y^2|x] - \mathbb{E}[Y|x]^2.$$

3. The mean of $Y|x$ is collinear, i.e.,

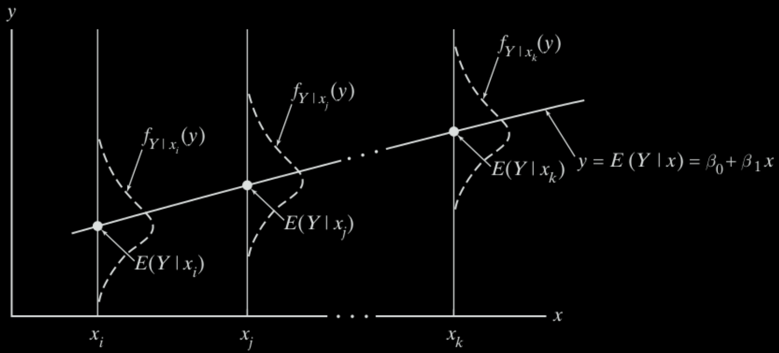
$$y = \mathbb{E}[Y|x] = \beta_0 + \beta_1 x.$$

4. All of the conditional distributions represnt indep. random variables.

Summary Let $Y_1, \cdots, Y_n$ be independent r.v.'s where $Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ with x_i are known and β_0, β_1 and σ^2 are unknown.

$$\Updownarrow$$

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad \epsilon_i \text{ are indep. and } \epsilon_i \sim N(0, \sigma^2).$$



y
$f_{Y|x_k}(y)$
$f_{Y|x_j}(y)$
$f_{Y|x_i}(y)$
$E(Y|x_k)$
$y = E(Y|x) = \beta_0 + \beta_1 x$
$E(Y|x_j)$
$E(Y|x_i)$
x
x_i
x_j
x_k

MLE for linear model

Thm. Let $(x_1, Y_1), \cdots, (x_n, Y_n)$ be a set of points satisfying the linear model, $\mathbb{E}[Y|x] = \beta_0 + \beta_1 x$.

($\iff$ let $Y_1, \cdots, Y_n$ be independent r.v.'s where $Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ with x_i are known and β_0, β_1 and σ^2 are unknown.)

The maximum likelihood estimators for β_0, β_1 and σ^2 are given by

$$\hat{\beta}_1 = \frac{n \sum_{i=1}^n x_i Y_i - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n Y_i\right)}{n\left(\sum_{i=1}^n x_i^2\right) - \left(\sum_{i=1}^n x_i\right)^2}$$

$$\hat{\beta}_0 = \frac{\sum_{i=1}^n Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i}{n} = \overline{Y} - \hat{\beta}_1 \overline{x}$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \left(Y_i - \widehat{Y}_i\right)^2, \qquad \widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i.$$

Proof. Since $Y_i \sim N\left(\beta_0 + \beta_1 x_i, \sigma^2\right)$,

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} f_{Y_i|x_i}(y_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\right).$$

Then take partial derivatives and set them to zero:

$$\frac{\partial \ln L}{\partial \beta_0} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\frac{\partial \ln L}{\partial \beta_1} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) x_i = 0$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{(\sigma^2)^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 = 0$$

Once β_0 and β_1 are solved from the first relations, then the third relation shows that

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2.$$

The first two relations give

$$\begin{aligned} \left(\sum_{i=1}^{n} y_i\right) - \beta_0 n - \beta_1\left(\sum_{i=1}^{n} x_i\right) &= 0 \\ \left(\sum_{i=1}^{n} x_i y_i\right) - \beta_0\left(\sum_{i=1}^{n} x_i\right) - \beta_1\left(\sum_{i=1}^{n} x_i^2\right) &= 0 \end{aligned}$$

or

$$\begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix}\begin{pmatrix}\beta_0 \\ \beta_1\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i\end{pmatrix}$$

Hence,

$$\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix}$$

$$= \frac{1}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2} \begin{pmatrix} \sum_{i=1}^n x_i^2 & -\sum_{i=1}^n x_i \\ -\sum_{i=1}^n x_i & n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix}$$

$$= \frac{1}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2} \begin{pmatrix} \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i y_i\right) \\ -\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) + n\left(\sum_{i=1}^n x_i y_i\right) \end{pmatrix}$$

$$\Downarrow$$

$$\beta_0 = \frac{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

$$\beta_1 = \frac{-\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) + n\left(\sum_{i=1}^n x_i y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

Recall

$$\beta_1 = \frac{n\left(\sum_{i=1}^n x_i y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

Let's simply β_0:

$$\beta_0 = \frac{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

$$= \frac{\left[\left(\sum_{i=1}^n x_i^2\right) - \frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2\right]\left(\sum_{i=1}^n y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

$$+ \frac{\frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2\left(\sum_{i=1}^n y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

$$= \frac{1}{n}\sum_{i=1}^n y_i + \frac{1}{n}\beta_1 \sum_{i=1}^n x_i$$

Finally, replacing β_0, β_1, σ^2 and y_i by $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$ and Y_i, respectively, proves the theorem. $\square$

Properties of linear model estimators

Theorem:

1. $\hat{\beta}_0$ and $\hat{\beta}_1$ are both normally distributed.
2. $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased: $\mathbb{E}[\hat{\beta}_0] = \beta_0$ and $\mathbb{E}[\hat{\beta}_1] = \beta_1$.
3. Variances are eqal to

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (x - \bar{x})^2}$$

$$\mathrm{Var}(\hat{\beta}_0) = \frac{\sigma^2 \sum_{i=1}^n x_i^2}{n \sum_{i=1}^n (x_i - \bar{x})^2} = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right]$$

4. $\hat{\beta}_1$, $\overline{Y}$ and $\hat{\sigma}^2$ are mutually independent.
5. $\frac{n\hat{\sigma}^2}{\sigma^2} \sim$ Chi Square with $n-2$ degrees of freedom. $\Longrightarrow \mathbb{E}[\hat{\sigma}^2] = \frac{n-2}{n}\sigma^2$

Remark 1 Because

$$\widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i = \overline{Y} - \overline{x}\hat{\beta}_1 + \hat{\beta}_1 x_i = \overline{Y} + (x_i - \overline{x})\hat{\beta}_1,$$

(4) implies that, for all $i = 1, \cdots, n$,

$$\widehat{Y}_i \perp \hat{\sigma}^2$$

Remark 2 By (5)

$$\mathbb{E}\left[\frac{n\hat{\sigma}^2}{\sigma^2}\right] = n - 2 \quad \Longleftrightarrow \quad \mathbb{E}[\hat{\sigma}^2] = \frac{n-2}{n}\sigma^2$$

$$\Longleftrightarrow \quad \mathbb{E}\left[\frac{n}{n-2}\hat{\sigma}^2\right] = \sigma^2$$

Or equivalently,

$\hat{\sigma}^2$ is a biased, but asymptotically unbiased, estimator for σ^2

$\frac{n}{n-2}\hat{\sigma}^2$ is an unbiased estimator for σ^2.

Proof. (1) Notice that both

$$\hat{\beta}_1 = \frac{n \sum_{i=1}^n x_i Y_i - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n Y_i\right)}{n\left(\sum_{i=1}^n x_i^2\right) - \left(\sum_{i=1}^n x_i\right)^2}$$

and

$$\hat{\beta}_0 = \frac{\sum_{i=1}^n Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i}{n}$$

are linear combinations for normal random variables, we see that both β_0 and β_1 are normal.

(2) Because $\mathbb{E}[Y|x] = \beta_0 + \beta_1 x$, we see that

$$\begin{aligned}
\mathbb{E}[\hat{\beta}_1] &= \frac{n\sum_{i=1}^n x_i \mathbb{E}[Y_i] - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n \mathbb{E}[Y_i]\right)}{n\left(\sum_{i=1}^n x_i^2\right) - \left(\sum_{i=1}^n x_i\right)^2} \\
&= \frac{n\sum_{i=1}^n x_i(\beta_0 + \beta_1 x_i) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n (\beta_0 + \beta_1 x_i)\right)}{n\left(\sum_{i=1}^n x_i^2\right) - \left(\sum_{i=1}^n x_i\right)^2} \\
&= \frac{n\beta_0 \sum_{i=1}^n x_i + \beta_1 \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)\left(n\beta_0 + \beta_1 \sum_{i=1}^n x_i\right)}{n\left(\sum_{i=1}^n x_i^2\right) - \left(\sum_{i=1}^n x_i\right)^2} \\
&= \beta_1,
\end{aligned}$$

and then

$$\begin{aligned}
\mathbb{E}[\hat{\beta}_0] &= \frac{\sum_{i=1}^n \mathbb{E}[Y_i] - \mathbb{E}[\hat{\beta}_1]\sum_{i=1}^n x_i}{n} \\
&= \frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i) - \beta_1 \sum_{i=1}^n x_i}{n} \\
&= \beta_0.
\end{aligned}$$

Hence, both $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimators for β_0 and β_1, respectively.

(3) Notice that

$$\hat{\beta}_1 = \frac{n\sum_{i=1}^{n} x_i Y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{\sum_{i=1}^{n} x_i Y_i - \overline{x}\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i - \overline{x})Y_i}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} = \sum_{i=1}^{n} \frac{(x_i - \overline{x})}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} Y_i$$

By independence of Y_i, we see that

$$\mathrm{Var}\left(\hat{\beta}_1\right) = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)^2} \mathrm{Var}\left(Y_i\right) = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)^2}\sigma^2$$

Because $\sum_{i=1}^{n}(x_i - \overline{x})^2 = \sum_{i=1}^{n} x_i^2 - n\overline{x}^2$, we see that

$$\mathrm{Var}\left(\hat{\beta}_1\right) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}.$$

- As for $\hat{\beta}_0$, notice that

$$\hat{\beta}_0 = \frac{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n Y_i\right) - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i Y_i\right)}{n(\sum_{i=1}^n x_i^2) - (\sum_{i=1}^n x_i)^2}$$

$$= \frac{\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n Y_i\right) - \overline{x}\left(\sum_{i=1}^n x_i Y_i\right)}{\sum_{i=1}^n x_i^2 - n\overline{x}^2}$$

$$= \sum_{j=1}^n \frac{\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right) - \overline{x}x_j}{\sum_{i=1}^n x_i^2 - n\overline{x}^2} Y_j$$

Hence,

$$\operatorname{Var}\left(\hat{\beta}_0\right) = \sum_{j=1}^n \left[\frac{\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right) - \overline{x}x_j}{\sum_{i=1}^n x_i^2 - n\overline{x}^2}\right]^2 \sigma^2$$

$$\mathrm{Var}\left(\hat{\beta}_0\right) = \sum_{j=1}^{n} \left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \overline{x} x_j}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} \right]^2 \sigma^2$$

$$= \sigma^2 \frac{\sum_{j=1}^{n} \left[\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \overline{x} x_j\right]^2}{\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]^2}$$

$$= \sigma^2 \frac{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right)^2 - \overline{x}^2 \sum_{j=1}^{n} x_j^2}{\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]^2}$$

$$= \sigma^2 \frac{\frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]^2 + 2\overline{x}^2\left(\sum_{i=1}^{n} x_i^2\right) - n\overline{x}^4 - \overline{x}^2 \sum_{j=1}^{n} x_j^2}{\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]^2}$$

$$= \sigma^2 \frac{\frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]^2 + \overline{x}^2\left(\sum_{i=1}^{n} x_i^2\right) - n\overline{x}^4}{\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]^2}$$

$$= \sigma^2 \left[\frac{1}{n} + \frac{\overline{x}^2}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}\right] = \sigma^2 \left[\frac{1}{n} + \frac{\overline{x}^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2}\right]$$

(4) Since both $\hat{\beta}_1$ and $\overline{Y}$ are Gaussian, to show that they are independent, we need only to show that

$$\mathbb{E}[\hat{\beta}_1 \overline{Y}] = \mathbb{E}[\hat{\beta}_1]\mathbb{E}[\overline{Y}]$$

One can compute separately left- and right-hand sides and compare them. The computations are long and tedious but there is no fundamental difficulties.

The independence with $\hat{\sigma}^2$ is deeper and out of the scope of the book.

(5) See Appendix 11.A.1. $\square$

Estimating σ^2

1. MLE:

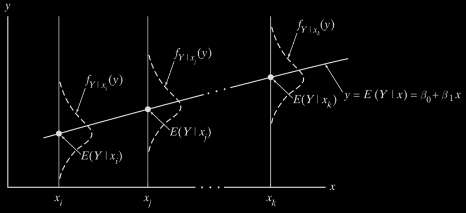
$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \widehat{Y}_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2.$$

2. The unbiased estimator:

$$MSE = S^2 = \frac{n}{n-2}\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2.$$

Notation

Parameter	Estimator	Estimate
β_1	$\hat{\beta}_1$	β_{1e}
β_0	$\hat{\beta}_0$	β_{0e}
σ	S	s
σ^2	S^2	s^2
σ^2	$\hat{\sigma}^2$	σ_e^2
	$\overline{Y}$	$\bar{y}$
	$\widehat{Y}_i$	$\hat{y}_i = \beta_{0e} + \beta_{1e}x_i$



Drawing inferences on

1. the slope β_1
2. the intercept β_0
3. shape parameter σ^2
4. the regresion line itself
   $y = \mathbb{E}[Y|x] = \beta_0 + \beta_1 x$
5. the future observations
6. testing two slopes.

1. Drawing inferences on β_1

Thm. $$T_{n-2} = \frac{\hat{\beta}_1 - \beta_1}{s \Big/ \sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \sim \text{Student t distribution with df} = n-2.$$

1. Hypothesis test $H_0 : \beta_1 = \beta_1'$ vs. ....
2. C.I. for β_1: $\quad \beta_{1e} \pm t_{\alpha/2,n-2} \frac{s}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})}}$

2. Drawing inferences on β_0

The GLRT procedure for assessing the credibility of $H_0 : \beta_0 = \beta_{0_o}$ is based on a Student t random variable with $n-2$ degrees of freedom:

$$T_{n-2} = \frac{(\hat{\boldsymbol{\beta}}_0 - \beta_{0_o})\sqrt{n}\sqrt{\sum_{i=i}^{n}(x_i - \bar{x})^2}}{S\sqrt{\sum_{i=1}^{n}x_i^2}} = \frac{\hat{\boldsymbol{\beta}}_0 - \beta_{0_o}}{\sqrt{\widehat{\text{Var}}(\hat{\boldsymbol{\beta}}_0)}} \tag{11.3.6}$$

"Inverting" Equation 11.3.6 (recall the proof of Theorem 11.3.6) yields

$$\left[\hat{\beta}_0 - t_{\alpha/2,n-2} \cdot \frac{s\sqrt{\sum_{i=1}^{n}x_i^2}}{\sqrt{n}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}, \hat{\beta}_0 + t_{\alpha/2,n-2} \cdot \frac{s\sqrt{\sum_{i=1}^{n}x_i^2}}{\sqrt{n}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}\right]$$

as the formula for a $100(1-\alpha)\%$ confidence interval for β_0.

3. Drawing inferences on σ^2

Since $(n-2)S^2/\sigma^2$ has a χ^2 pdf with $n-2$ df (if the n observations satisfy the stipulations implicit in the simple linear model), it follows that

$$P\left[\chi^2_{\alpha/2,n-2} \leq \frac{(n-2)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2,n-2}\right] = 1-\alpha$$

Equivalently,

$$P\left[\frac{(n-2)S^2}{\chi^2_{1-a/2,n-2}} \leq \sigma^2 \leq \frac{(n-2)S^2}{\chi^2_{\alpha/2,n-2}}\right] = 1-\alpha$$

in which case

$$\left[\frac{(n-2)s^2}{\chi^2_{1-\alpha/2,n-2}}, \frac{(n-2)s^2}{\chi^2_{\alpha/2,n-2}}\right]$$

becomes the *100(1 − α)% confidence interval for* σ^2 (recall Theorem 7.5.1). Testing $H_0: \sigma^2 = \sigma_o^2$ is done by calculating the ratio

$$\chi^2 = \frac{(n-2)s^2}{\sigma_o^2}$$

which has a χ^2 distribution with $n-2$ df when the null hypothesis is true. Except for the degrees of freedom ($n-2$ rather than $n-1$), the appropriate decision rules for one-sided and two-sided H_1's are similar to those given in Theorem 7.5.2.

4. Drawing inference on the regression line

Intuition tells us that a reasonable point estimator for $E(Y \mid x)$ is the height of the regression line at x—that is, $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 x$. By Theorem 11.3.2, the latter is unbiased:

$$E(\hat{Y}) = E(\hat{\beta}_0 + \hat{\beta}_1 x) = E(\hat{\beta}_0) + x\,E(\hat{\beta}_1) = \beta_0 + \beta_1 x$$

Of course, to use $\hat{Y}$ in any inference procedure requires that we know its variance. But

$$\begin{aligned}
\text{Var}(\hat{Y}) &= \text{Var}(\hat{\beta}_0 + \hat{\beta}_1 x) = \text{Var}(\bar{Y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x) \\
&= \text{Var}[\bar{Y} + \hat{\beta}_1 (x - \bar{x})] \\
&= \text{Var}(\bar{Y}) + (x - \bar{x})^2 \text{Var}(\hat{\beta}_1) \quad \text{(why?)} \\
&= \frac{1}{n}\sigma^2 + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sigma^2 \\
&= \sigma^2 \left[\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right]
\end{aligned}$$

An application of Definition 7.3.3, then, allows us to construct a Student t random variable based on $\hat{Y}$. Specifically,

$$T_{n-2} = \frac{\hat{Y} - (\beta_0 + \beta_1 x)}{\sigma \sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}} \Bigg/ \sqrt{\frac{(n-2)S^2}{\frac{\sigma^2}{n-2}}} = \frac{\hat{Y} - (\beta_0 + \beta_1 x)}{S \sqrt{\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}}$$

has a Student t distribution with $n-2$ degrees of freedom. Isolating $\beta_0 + \beta_1 x = E(Y \mid x)$ in the center of the inequalities $P(-t_{\alpha/2,n-2} \le T_{n-2} \le t_{\alpha/2,n-2}) = 1 - \alpha$ produces a $100(1-\alpha)\%$ confidence interval for $E(Y \mid x)$.

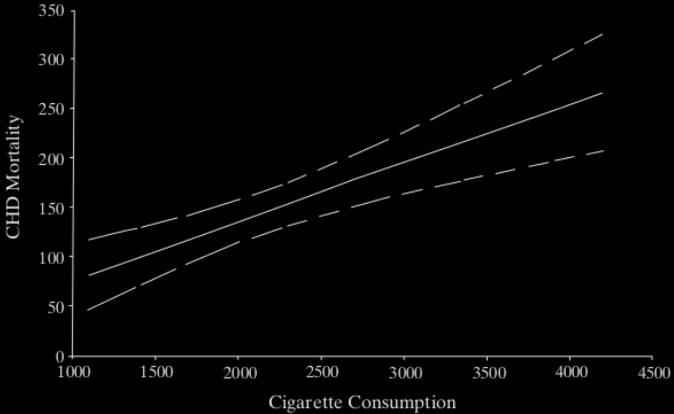

350
300
250
200
150
100
50
0
CHD Mortality
1000
1500
2000
2500
3000
3500
4000
4500
Cigarette Consumption

Figure 11.3.4

Let $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ be a set of n points that satisfy the assumptions of the simple linear model, and let (x, Y) be a hypothetical future observation, where Y is independent of the n Y_i's. A *prediction interval* is a range of numbers that contains Y with a specified probability.

Consider the difference $\hat{Y} - Y$. Clearly,

$$E(\hat{Y} - Y) = E(\hat{Y}) - E(Y) = (\beta_0 + \beta_1 x) - (\beta_0 + \beta_1 x) = 0$$

and

$$\begin{aligned}
\mathrm{Var}(\hat{Y} - Y) &= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(Y) \\
&= \sigma^2 \left[\frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right] + \sigma^2 \\
&= \sigma^2 \left[1 + \frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right]
\end{aligned}$$

Following exactly the same steps that were taken in the derivation of Theorem 11.3.7, a Student t random variable with $n-2$ degrees of freedom can be constructed from $\hat{Y}-Y$ (using Definition 7.3.3). Inverting the equation $P(-t_{\alpha/2,n-2} \leq T_{n-2} \leq t_{\alpha/2,n-2}) = 1-\alpha$ will then yield the prediction interval $(\hat{y}-w, \hat{y}+w)$ given in Theorem 11.3.8.

Theorem 11.3.8 *Let $(x_1, Y_1), (x_2, Y_2), \ldots,$ and (x_n, Y_n) be a set of n points that satisfy the assumptions of the simple linear model. A* $100(1-\alpha)\%$ prediction interval for Y *at the fixed value x is given by $(\hat{y}-w, \hat{y}+w)$, where*

$$w = t_{\alpha/2,n-2} \cdot s \sqrt{1 + \frac{1}{n} + \frac{(x-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

and $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$. □

E.g. 1 Does smoking contribute to coronary heat disease?

Table 11.3.1

Country	Cigarette Consumption per Adult per Year, x	CHD Mortality per 100,000 (ages 35–64), y
United States	3900	256.9
Canada	3350	211.6
Australia	3220	238.1
New Zealand	3220	211.8
United Kingdom	2790	194.1
Switzerland	2780	124.5
Ireland	2770	187.3
Iceland	2290	110.5
Finland	2160	233.1
West Germany	1890	150.3
Netherlands	1810	124.7
Greece	1800	41.2
Austria	1770	182.1
Belgium	1700	118.1
Mexico	1680	31.9
Italy	1510	114.3
Denmark	1500	144.9
France	1410	59.7
Sweden	1270	126.9
Spain	1200	43.9
Norway	1090	136.3

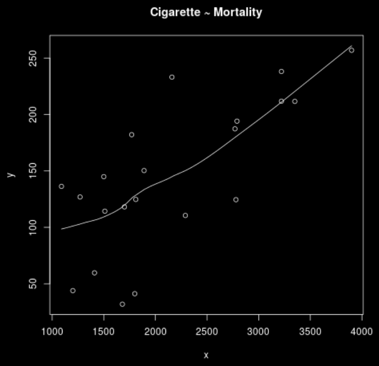
1) Test $H_0 : \beta_1 = 0$ v.s. $H_1 : \beta_1 > 0$ at $\alpha = 0.05$.

2) Find C.I. for β_1 with the same α.

Sol. http://r-statistics.co/Linear-Regression.html

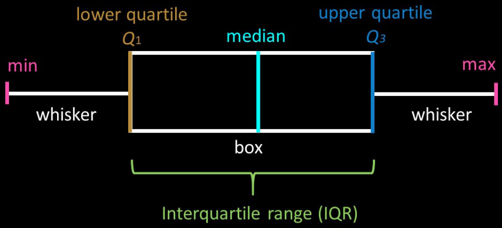
1. Let's first take of look of the data by scatter plot:

```
scatter.smooth(x=x, y=y, main="Cigarette ~ Mortality")
```



Suggests a linearly increasing relationship between x and y.

2. Check outliers using boxplot.

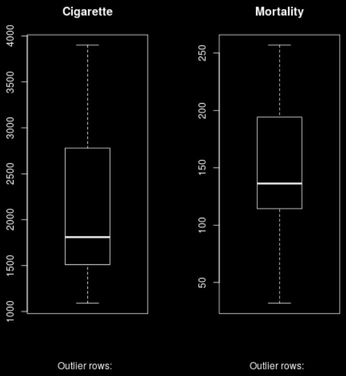


Any datapoint that lies outside the $r \times \mathrm{IQR}$ is considered an outlier.

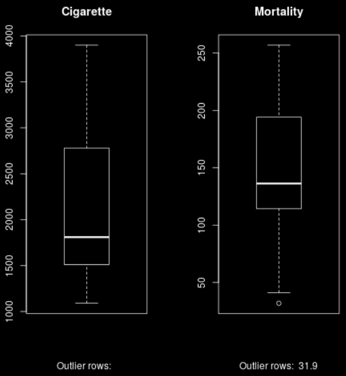
Generally, $r = 1.5$.

```r
r <- 1.5
par(mfrow=c(1, 2)) # divide graph area in 2 columns
boxplot(x, main="Cigarette", range=r, sub=paste("Outlier rows: ", boxplot.stats(x,
    coef=r)$out)) # box plot for 'Cigarette'
boxplot(y, main="Mortality", range=r, sub=paste("Outlier rows: ", boxplot.stats(y,
    coef=r)$out)) # box plot for 'Mortality'
```

$r = 1.5$

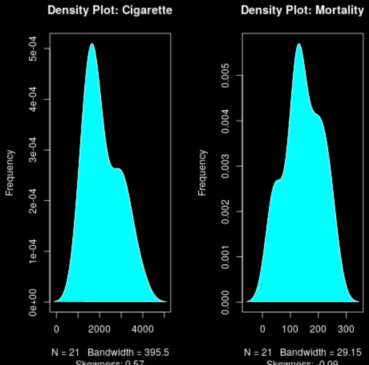


$r = 1$



3. Compute kernel density estimates

```r
library(e1071)
plot(density(x), main="Density Plot: Cigarette", ylab="Frequency",
      sub=paste("Skewness:", round(e1071::skewness(x), 2))) # density plot for 'Cigarette'
polygon(density(x), col="red")
plot(density(y), main="Density Plot: Mortality", ylab="Frequency",
      sub=paste("Skewness:", round(e1071::skewness(y), 2))) # density plot for 'Mortality'
polygon(density(y), col="red")
```



4. Compute correlation coeficient.

Correlation is a statistical measure with values in $[-1, 1]$ that suggests the level of linear dependence between two variables.

A value closer to 0 suggests a weak relationship between the variables. A low correlation $(-0.2, 0.2)$ probably suggests that much of variation of the response variable Y is unexplained by the predictor X, in which case, we should probably look for better explanatory variables.

```
> cor(x,y)
[1] 0.7295154
```

5. Compute linear regression.

```
> CigMort <- data.frame("Cigarette" = x, "Mortality" = y) # Build the data frame
> linearMod <- lm(Mortality ~ Cigarette, data=CigMort) # linear regression
> print(linearMod) # Print out the result

Call:
lm(formula = Mortality ~ Cigarette, data = CigMort)

Coefficients:
(Intercept)    Cigarette
    15.7711       0.0601
```

$$y = 15.7711 + 0.0601x$$

6. Check statistical significance of the linear model

```
> summary(linearMod)

Call:
lm(formula = Mortality ~ Cigarette, data = CigMort)

Residuals:
    Min      1Q  Median      3Q     Max
-84.835 -40.809   5.058  28.814  87.518

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 15.77115   29.57889   0.533 0.600085
Cigarette    0.06010    0.01293   4.649 0.000175 ***
---
Signif. codes:  0 "***" 0.001 "**" 0.01 "*" 0.05 "." 0.1 " " 1

Residual standard error: 46.71 on 19 degrees of freedom
Multiple R-squared: 0.5322, Adjusted R-squared: 0.5076
F-statistic: 21.62 on 1 and 19 DF, p-value: 0.0001749
```

0.1 By default, p-values are computed for $H_0 : \beta_i = 0$ vs. $H_1 : \beta_i \neq 0$, $i = 0, 1$.
0.2 The more stars by the variable's p-Value, the more significant the variable.

Testing $H_0 : \beta_1 = 0$ v.s. $H_1 : \beta_1 \neq 0$

t-score is 4.4649.

p-value= 0.000175

Conclusion: reject at $\alpha = 0.05$.

95% C.I. for β_1:

Testing $H_0 : \beta_0 = 0$ v.s. $H_1 : \beta_0 \neq 0$

t-score is 0.533.

p-value= 0.600

Conclusion: fail to reject at $\alpha = 0.05$.

95% C.I. for β_0:

```
> # 95% C.I. for slope parameter beta_1
> alpha <- 0.05
> for (i in c(1,0)) {
+  coef <- summary(linearMod)$coefficient
+  df <- linearMod$df.residual
+  lbd <- coef[i+1,1] - pt(1-alpha/2,df) * coef[i+1,2]
+  ubd <- coef[i+1,1] + pt(1-alpha/2,df) * coef[i+1,2]
+  print(paste("95% C.I. for the slope is beta_",i,
+              " is (", round(lbd,3), ",", round(ubd,3),")"))
+ }
[1] "95% C.I. for the slope is beta_ 1 is ( 0.049 , 0.071 )"
[1] "95% C.I. for the slope is beta_ 0 is ( -8.753 , 40.295 )"
```

7. Compute R-Squared and the adjusted R-Squared.

$$R^2 = 1 - \frac{SSE}{SST} \quad \text{and} \quad R^2_{adj} = 1 - \frac{MSE}{MST}$$

```
> names(summary(linearMod))
 [1] "call"          "terms"         "residuals"     "coefficients"
 [5] "aliased"       "sigma"         "df"            "r.squared"
 [9] "adj.r.squared" "fstatistic"    "cov.unscaled"
> summary(linearMod)$r.squared
[1] 0.5321927
> summary(linearMod)$adj.r.squared
[1] 0.5075712
```

The large r^2 or r^2_{adj} the better, the more powerful or expressive is the L.M.

8. Residue standard error and F-statistic

$$\text{Residue standard error} = \sqrt{MSE} = \sqrt{\frac{SSE}{n-q}}$$

$$F = \frac{MSR}{MSE} = \frac{SSR/(q-1)}{SSE/(n-q)} \sim \text{F-distribution } (df_1 = q-1, df_2 = n-q)$$

```
> names(summary(linearMod))
 [1] "call"          "terms"         "residuals"     "coefficients"
 [5] "aliased"       "sigma"         "df"            "r.squared"
 [9] "adj.r.squared" "fstatistic"    "cov.unscaled"
> summary(linearMod)$sigma
[1] 46.70826
> summary(linearMod)$fstatistic
   value    numdf   dendf
21.61501 1.00000 19.00000
> f <- summary(linearMod)$fstatistic
> pf(f[1], f[2], f[3], lower=FALSE)
       value
0.0001748805
```

9. Model selection:

Akaike's information criterion — AIC (Akaike, 1974)	Bayesian information criterion — BIC (Schwarz, 1978)
$AIC = -2\ln(\widehat{L}) + 2q$	$BIC = -2\ln(\widehat{L}) + q\ln(n)$

$\widehat{L}$: the maximum of likelihood.

q: the number of parameters in the model.

n: the sample size.

```
> AIC(linearMod)
[1] 224.9383
> BIC(linearMod)
[1] 228.0719
```

The lower the better!

10. Does L.M. fit our model?

Statistic	criterion	our case
R^2	Higher the better (>0.7)	0.53
R^2_{adj}	Higher the better	0.51
AIC	Lower the better	225
BIC	Lower the better	228
⋮	⋮	⋮

11. Drawing inference on $\mathbb{E}(Y|x)$

Find 95% C.I. for Y at $x = 4200$.

Here, $n = 21$, $t_{.025,19} = 2.0930$, $\sum_{i=1}^{21}(x_i - \bar{x})^2 = 13{,}056{,}523.81$, $s = 46.707$, $\hat{\beta}_0 = 15.7661$, $\hat{\beta}_1 = 0.0601$, and $\bar{x} = 2148.095$. From Theorem 11.3.7, then,

$$\hat{y} = 15.7661 + 0.0601(4200) = 268.1861$$

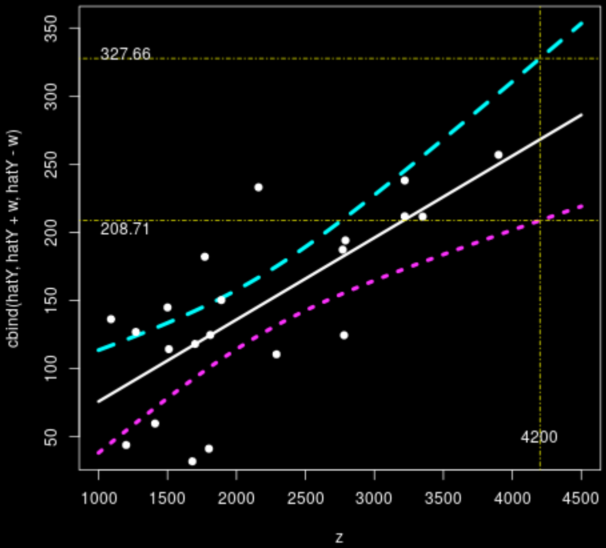
$$w = 2.0930(46.707)\sqrt{\frac{1}{21} + \frac{(4200 - 2148.095)^2}{13{,}056{,}523.81}} = 59.4714$$

and the 95% confidence interval for $E(Y|4200)$ is

$$(268.1861 - 59.4714,\ 268.1861 - 59.4714)$$

which rounded to two decimal places is

$$(208.71, 327.66)$$


327.66
208.71
4200
cbind(hatY, hatY + w, hatY - w)
z
50
100
150
200
250
300
350
1000
1500
2000
2500
3000
3500
4000
4500

```r
s <- summary(linearMod)$sigma
beta <- linearMod$coefficients
z <- seq(1000,4500,1)
hatY <- beta[1]+beta[2]*z
w <- qt(0.975,19) * s * sqrt(1/21+(z-mean(x))^2/(sum((x-mean(x))^2)))
matplot(z,cbind(hatY,hatY+w,hatY-w),type = c("l","l","l"),lwd=c(3,4,4))
points(x, y, pch = 19)
abline(v=4200,col = "blue", lty = 4)
abline(h=208.71,col = "blue", lty = 4)
abline(h=327.66,col = "blue", lty = 4)
text(4200,50,4200)
text(1200,203,208.71)
text(1200,331,327.66)
```

12. Drawing inference on future observations.

Find 95% prediction interval for Y at $x = 4200$.

When $x = 4200$, $\hat{y} = 268.1861$ for both intervals. From Theorem 11.3.8, the width of the 95% prediction interval for Y is:

$$w = 2.0930(46.707)\sqrt{1 + \frac{1}{21} + \frac{(4200 - 2148.095)^2}{13{,}056{,}523.81}} = 114.4725$$

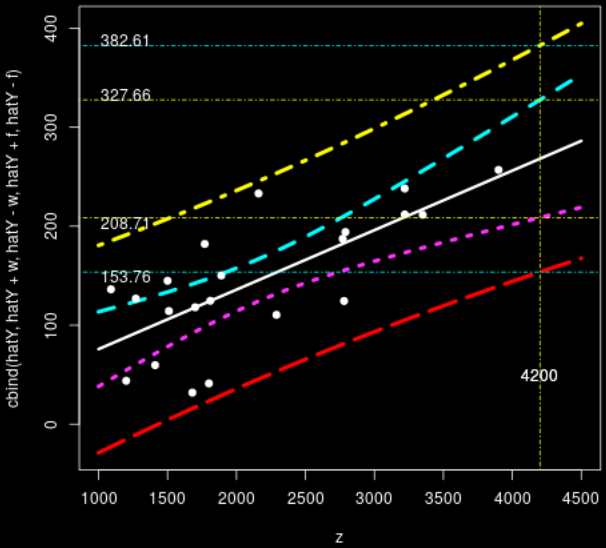
The 95% prediction interval, then, is

$$(268.1861 - 114.4725, 268.1861 + 114.4725)$$

which rounded to two decimal places is

$$(153.76, 382.61)$$

which makes it 92% wider than the 95% confidence interval for $E(Y|4200)$. ■

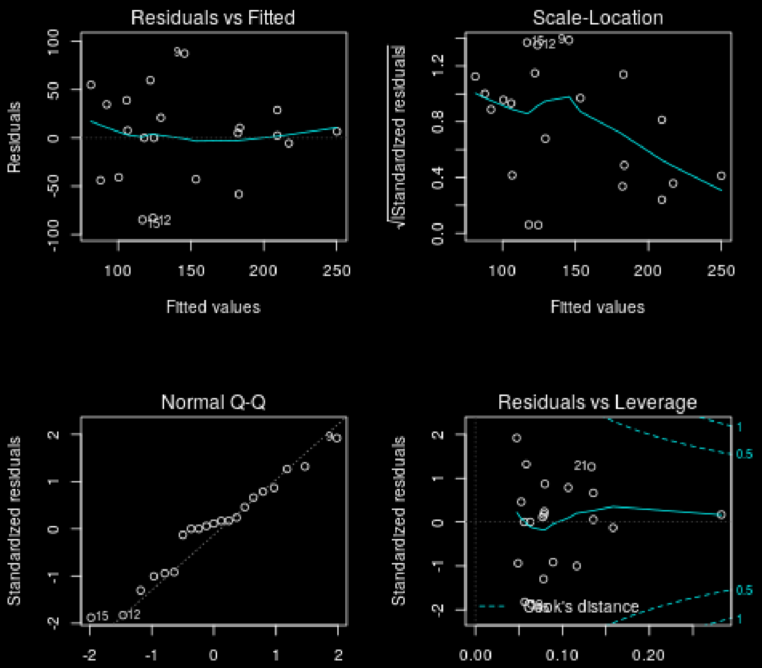


382.61
327.66
208.71
153.76
4200
400
300
200
100
0
cbind(hatY, hatY + w, hatY - w, hatY + f, hatY - f)
1000
1500
2000
2500
3000
3500
4000
4500
z

```r
s <- summary(linearMod)$sigma
beta <- linearMod$coefficients
z <- seq(1000,4500,1)
hatY <- beta[1]+beta[2]*z
w <- qt(0.975,19) * s * sqrt(1/21+(z-mean(x))^2/(sum((x-mean(x))^2)))
f <- qt(0.975,19) * s * sqrt(1+1/21+(z-mean(x))^2/(sum((x-mean(x))^2)))
matplot(z,cbind(hatY,hatY+w,hatY-w,hatY+f,hatY-f),
        type = c("l","l","l","l","l"),lwd=c(3,4,4,4,4))
points(x, y, pch = 19)
abline(v=4200,col = "blue", lty = 4)
abline(h=208.71,col = "blue", lty = 4)
abline(h=327.66,col = "blue", lty = 4)
text(4200,50,4200)
text(1200,208.71-5,208.71)
text(1200,327.66+5,327.66)
abline(h=153.76,col = "red", lty = 4)
abline(h=382.61,col = "red", lty = 4)
text(4200,50,4200)
text(1200,153.76-5,153.76)
text(1200,382.61+5,382.61)
```

13. More about diagnozing the linear model:

```
# diagnostic plots
layout(matrix(c(1,2,3,4),2,2)) # optional 4 graphs/page
plot(linearMod)
```


Residuals vs Fitted
Residuals
Fitted values
Scale-Location
√|Standardized residuals|
Fitted values
Normal Q-Q
Standardized residuals
Residuals vs Leverage
Standardized residuals
Cook's distance

E.g. 2 Find 95% C.I. for the amount of increas year-by-year in the cost of Toyota Camry sedan.

Table 11.3.2

Year	Year after 2005	Suggested Retail Price ($)
2005	0	7,935
2006	1	8,495
2007	2	10,160
2008	3	10,817
2009	4	11,078
2010	5	11,967
2011	6	12,658
2012	7	13,844
2013	8	13,982
2014	9	14,629

Data from: kbb.com

Sol. We first find the regression:

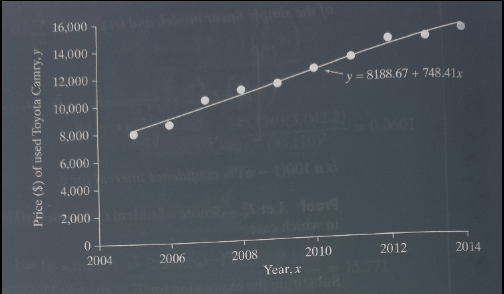


Figure 11.3.3

The slope of the line, $\hat{\beta}_1$, represents the amount of increase year-by-year in the cost of an older model. Often a range of values is better than a single estimate, so a good way to provide this is using a confidence interval for the true value β_1.

Here, $\sqrt{\sum_{i=0}^{9}(x_i - \bar{x})^2} = \sqrt{82.5} = 9.083$

and from Equation 11.3.5, $s^2 = \frac{1}{10-2}\left(\sum_{i=0}^{9} y_i^2 - \hat{\beta}_0 \sum_{i=0}^{9} y_i - \hat{\beta}_1 \sum_{i=0}^{9} x_i y_i\right)$

$$\frac{1}{8}[1{,}382{,}678{,}777 - (8188.67)(115{,}565) - (748.41)(581{,}786)] = 117{,}727.98$$

so $s = \sqrt{117{,}727.98} = 343.11$.

Using $t_{.025,8} = 2.3060$, the expression given in Theorem 11.3.6 reduces to $\left(748.41 - 2.3060\frac{343.11}{9.083}, 748.41 + 2.3060\frac{343.11}{9.083}\right) = (\$661.30, \$835.52)$

7. Testing the equality of two slopes

Table 11.3.3

Date	Day no., $x (= x^*)$	Strain A pop^n, y	Strain B pop^n, y^*
Feb 2	0	100	100
May 13	100	250	203
Aug 21	200	304	214
Nov 29	300	403	295
Mar 8	400	446	330
Jun 16	500	482	324

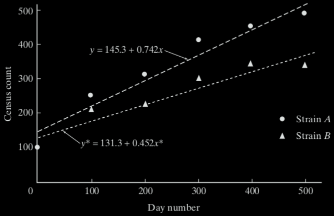


Figure 11.3.5

Do you believe that $\beta_1 = \beta_1^*$?

Or is $\beta_1 > \beta_1^*$ statisically significantly?

Theorem 11.3.9 *Let $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ and $(x_1^*, Y_1^*), (x_2^*, Y_2^*), \ldots, (x_m^*, Y_m^*)$ be two independent sets of points, each satisfying the assumptions of the simple linear model—that is, $E(Y \mid x) = \beta_0 + \beta_1 x$ and $E(Y^* \mid x^*) = \beta_0^* + \beta_1^* x^*$.*

a. *Let*

$$T = \frac{\hat{\boldsymbol{\beta}}_1 - \hat{\boldsymbol{\beta}}_1^* - (\beta_1 - \beta_1^*)}{S\sqrt{\frac{1}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} + \frac{1}{\sum\limits_{i=1}^{m}(x_i^* - \bar{x}^*)^2}}}$$

where

$$S = \sqrt{\frac{\sum\limits_{i=1}^{n}[Y_i - (\hat{\boldsymbol{\beta}}_0 + \hat{\boldsymbol{\beta}}_1 x_i)]^2 + \sum\limits_{i=1}^{m}[Y_i^* - (\hat{\boldsymbol{\beta}}_0^* + \hat{\boldsymbol{\beta}}_1^* x_i)]^2}{n + m - 4}}$$

Then T has a Student t distribution with $n + m - 4$ degrees of freedom.

b. *To test $H_0 : \beta_1 = \beta_1^*$ versus $H_1 : \beta_1 \neq \beta_1^*$ at the α level of significance, reject H_0 if t is either (1) $\leq -t_{\alpha/2, n+m-4}$ or (2) $\geq t_{\alpha/2, n+m-4}$, where*

$$t = \frac{\hat{\beta}_1 - \hat{\beta}_1^*}{s\sqrt{\frac{1}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} + \frac{1}{\sum\limits_{i=1}^{m}(x_i^* - \bar{x}^*)^2}}}$$

(One-sided tests are defined in the usual way by replacing $\pm t_{\alpha/2, n+m-4}$ with either $t_{\alpha, n+m-4}$ or $-t_{\alpha, n+m-4}$.)

$$S^2 = SSE \text{ and } q = 4.$$

Sol. Test

$$H_0 : \beta_1 = \beta_1^* \quad \text{v.s.} \quad H_1 : \beta_1 > \beta_1^*.$$

Long computations ... $t = 2.50$.

`http://math.emory.edu/~lchen41/teaching/2020_Spring/Ex_11-3-4.R`

Critical region: $t > t_{0.05,8} = 1.8595$.

Reject. $\square$

```
> # Example 11.3.4
> # Read data first
> Input <- ("
+ x     yA    yB
+ 0     100   100
+ 100   250   203
+ 200   304   214
+ 300   403   295
+ 400   446   330
+ 500   482   324
+ ")
> Data = read.table(textConnection(Input),
+                   header=TRUE)
> Data
    x  yA  yB
1   0 100 100
2 100 250 203
3 200 304 214
4 300 403 295
5 400 446 330
6 500 482 324
```

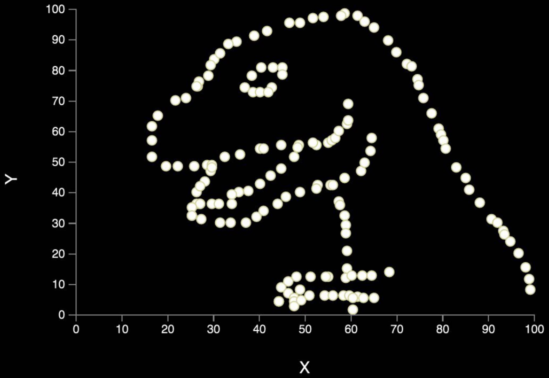
```
> #fit the first model ...
> DataA <- data.frame(x = Data$x,yA = Data$yA)
> fitA <- lm(yA~x, DataA)
> summary(fitA)

Call:
lm(formula = yA ~ x, data = DataA)

Residuals:
      1       2       3       4       5       6
-45.333  30.467  10.267  35.067   3.867 -34.333

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 145.33333   26.86684   5.409  0.00566 **
x             0.74200    0.08874   8.362  0.00112 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 37.12 on 4 degrees of freedom
Multiple R-squared: 0.9459, Adjusted R-squared: 0.9324
F-statistic: 69.92 on 1 and 4 DF, p-value: 0.001119
```

```
> #fit the second model ...
> DataB <- data.frame(x = Data$x,yB = Data$yB)
> fitB <- lm(yB~x, DataB)
> summary(fitB)

Call:
lm(formula = yB ~ x, data = DataB)

Residuals:
      1       2       3       4       5       6
-31.333  26.467  -7.733  28.067  17.867 -33.333

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 131.33333   22.77255   5.767  0.00449 **
x             0.45200    0.07522   6.009  0.00386 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 31.46 on 4 degrees of freedom
Multiple R-squared: 0.9003, Adjusted R-squared: 0.8754
F-statistic: 36.11 on 1 and 4 DF, p-value: 0.00386
```

```r
> # Now compute t-score and p-value
> sA <- summary(fitA)$coefficients
> sA
             Estimate Std. Error t value    Pr(>|t|)
(Intercept) 145.3333 26.86683800 5.409395 0.005656733
x             0.7420  0.08873825 8.361671 0.001118570
> sB <- summary(fitB)$coefficients
> sB
             Estimate Std. Error t value    Pr(>|t|)
(Intercept) 131.3333 22.77254682 5.767178 0.004486443
x             0.4520  0.07521525 6.009420 0.003860274
> db <- (sA[2,1]-sB[2,1]) # difference of beta_1's
> db
[1] 0.29
> sd <- sqrt(sB[2,2]^2+sA[2,2]^2) # standard deviation
> sd
[1] 0.1163263
> df <- (fitA$df.residual+fitB$df.residual) # degrees of freedom
> df
[1] 8
> td <- db/sd # t-score
> pv <- 2*pt(-abs(td), df) # two-sided p-value
> print(paste("t-score is ", round(td,3),
+             "and p-value is", round(pv,3)))
[1] "t-score is 2.493 and p-value is 0.037"
```

You should always visualize your data before any analysis

N = 157 ; X mean = 50.7333 ; X SD = 19.5661 ; Y mean = 46.495 ; Y SD = 27.2828 ;
Pearson correlation = -0.1772



	Indep. variables			Dependent variables		
Sample 1	x_{11}	$\cdots$	x_{1m}	y_{11}	$\cdots$	y_{1d}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sample n	x_{n1}	$\cdots$	x_{nm}	y_{n1}	$\cdots$	y_{nd}

$$Y_{ij} = \sum_{k=1}^{m} \beta_{kj} X_{ik} + \epsilon_{ij}, \quad 1 \leq i \leq n, 1 \leq j \leq d, \;\; \epsilon_{ij} \; i.i.d. \sim N(0, \sigma^2).$$

$m = d = 1$	(Simple) linear regression
$m \geq 2$	Multiple linear regression
$d \geq 2$	Multivariate linear regression

1. Overdetermined system: $Y = XB$.

2. The least square solutions are (provided that $X^T X$ is nonsignular)

$$B = (X^T X)^{-1} X^T Y$$

E.g. Broadway shows[1]

```
> # This is an example of multimple regression.
> # Dataset is explained here:
> # https://dasl.datadescription.com/datafile/broadway-shows/?_sfm_methods
    =Multiple+Regression&_sfm_cases=4+59943&sort_order=title+asc
>
> # Read data from the URL link
> library(data.table)
> mydat <- fread('https://dasl.datadescription.com/download/data/3087')
 [100*] Downloaded 965 bytes...
> head(mydat)
   Season Gross($M) Attendance Playing weeks New Productions Mean ticket Pct
      .sold LogGross
1:   1984       209       7.26           1078              33    28.78788 0.04714286
      2.320146
2:   1985       190       6.54           1041              34    29.05199 0.04397695
      2.278754
3:   1986       208       7.04           1039              41    29.54546 0.04743022
      2.318063
4:   1987       253       8.14           1113              30    31.08108 0.05119497
      2.403120
5:   1988       262       7.96           1108              33    32.91457 0.05028881
      2.418301
6:   1989       282       8.04           1070              39    35.07463 0.05259813
      2.450249
```

[1] https://dasl.datadescription.com/datafile/broadway-shows/?_sfm_methods=Multiple+Regression&_sfm_cases=4+59943&sort_order=title+asc

```
> # Multiple Linear Regression Example with intercept
> fit <- lm(‘Gross($M)‘ ~ Season + Attendance + ‘Playing weeks‘ + ‘New Productions‘ + ‘Mean ticket‘ + ‘Pct.sold‘ + LogGross, data=mydat)
> summary(fit) # show results

Call:
lm(formula = ‘Gross($M)‘ ~ Season + Attendance + ‘Playing weeks‘ +
    ‘New Productions‘ + ‘Mean ticket‘ + Pct.sold + LogGross,
    data = mydat)

Residuals:
    Min      1Q  Median     3Q    Max
-31.925 -5.756 -0.055 7.172 14.040

Coefficients:
                     Estimate Std. Error t value Pr(>|t|)
(Intercept)       -2.053e+04 7.348e+03 -2.795 0.00983 **
Season             1.132e+01 3.829e+00 2.957 0.00670 **
Attendance         9.745e+01 3.537e+01 2.755 0.01079 *
‘Playing weeks‘    4.566e-02 3.084e-01 0.148 0.88348
‘New Productions‘ -9.560e-01 5.982e-01 -1.598 0.12255
‘Mean ticket‘      1.680e+01 8.306e-01 20.221 < 2e-16 *
Pct.sold           1.779e+03 6.811e+03 0.261 0.79604
LogGross          -1.301e+03 1.610e+02 -8.085 1.94e-08 *
---
Signif. codes: 0 '*' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 10.61 on 25 degrees of freedom
Multiple R-squared: 0.9994, Adjusted R-squared: 0.9992
F-statistic: 6068 on 7 and 25 DF, p-value: < 2.2e-16
```

```
> # Compute the coefficients using the generalized inverse (with intercept)
> library(matlib)
> m <-length(mydat)-1
> M <- data.matrix(mydat, rownames.force = NA)
> n <- nrow(M)
> m <- ncol(M)
> X<- cbind(rep(1,n),M[1:n,c(1,3:m)])
> Y <- M[1:n,2]
> inv((t(X)*X)) * t(X) * Y
                          [,1]
                 -2.053451e+04
Season            1.132227e+01
Attendance        9.745043e+01
Playing weeks     4.565847e-02
New Productions  -9.560446e-01
Mean ticket       1.679521e+01
Pct.sold          1.779471e+03
LogGross         -1.301463e+03
> # Or you can compute the generalized inverse use the package pracma
> library(pracma)
> pinv(X) *Y
              [,1]
[1,] -2.053451e+04
[2,]  1.132227e+01
[3,]  9.745043e+01
[4,]  4.565847e-02
[5,] -9.560446e-01
[6,]  1.679521e+01
[7,]  1.779471e+03
[8,] -1.301463e+03
```

```
> # Multiple Linear Regression Example without intercept
> fit2 <− lm(‘Gross($M)‘ ~ Season + Attendance + ‘Playing weeks‘ + ‘New
      Productions‘ + ‘Mean ticket‘ + ‘Pct.sold‘ + LogGross −1, data=mydat)
> summary(fit2) # show results

Call:
lm(formula = ‘Gross($M)‘ ~ Season + Attendance + ‘Playing weeks‘ +
    ‘New Productions‘ + ‘Mean ticket‘ + Pct.sold + LogGross −
    1, data = mydat)

Residuals:
    Min      1Q Median     3Q     Max
−36.334 −3.758 2.570  6.282  18.324

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
Season              0.62744   0.15089  4.158 0.000309 *
Attendance         91.28669  39.65848  2.302 0.029610 *
‘Playing weeks‘     0.04173   0.34641  0.120 0.905047
‘New Productions‘ −0.74486 0.66658 −1.117 0.274032
‘Mean ticket‘      18.09840   0.77213 23.440 < 2e−16 *
Pct.sold         1369.35407 7649.90823 0.179 0.859323
LogGross         −990.63826 130.72506 −7.578 4.81e−08 *
−−−
Signif. codes: 0 ’*’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1

Residual standard error: 11.92 on 26 degrees of freedom
Multiple R−squared: 0.9998, Adjusted R−squared: 0.9998
F−statistic: 2.069e+04 on 7 and 26 DF, p−value: < 2.2e−16
```

```
> # Compute the coefficients using the generalized inverse (without intercept)
> library(matlib)
> m <-length(mydat)-1
> M <- data.matrix(mydat, rownames.force = NA)
> n <- nrow(M)
> m <- ncol(M)
> X <- M[1:n,c(1,3:m)]
> Y <- M[1:n,2]
> inv((t(X)*X)) * t(X) * Y
                         [,1]
Season             0.62744066
Attendance        91.28668689
Playing weeks      0.04172758
New Productions   -0.74485881
Mean ticket       18.09839993
Pct.sold        1369.35406937
LogGross        -990.63826155
> # Or you can compute the generalized inverse use the package pracma
> library(pracma)
> pinv(X) *Y
              [,1]
[1,]    0.62744066
[2,]   91.28668689
[3,]    0.04172758
[4,]   -0.74485881
[5,]   18.09839993
[6,] 1369.35406890
[7,] -990.63826154
```

Chapter 11. Regression

§ 11.5 The Bivariate Normal Distribution