Math 221: LINEAR ALGEBRA

Chapter 3. Determinants and Diagonalization

§3-4. Application to Linear Recurrences

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Emory University, 2021 Spring

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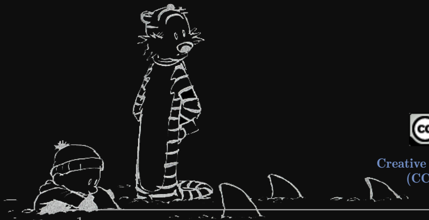

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

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Linear Recurrences

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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Linear Recurrences

Linear Recurrences

Example

The Fibonacci Numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

and can be defined by the linear recurrence relation

$$f_{n+2} = f_{n+1} + f_n \text{ for all } n \geq 0,$$

with the initial conditions $f_0 = 1$ and $f_1 = 1$.

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Problem

Find f_{100}.

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Problem

Find f_{100}.

Instead of using the recurrence to compute f_{100}, we'd like to find a formula for f_n that holds for all $n \geq 0$.

Definitions

A sequence of numbers $x_0, x_1, x_2, x_3, \ldots$ is defined recursively if each number in the sequence is determined by the numbers that occur before it in the sequence.

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A linear recurrence of length k has the form

$$x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_k x_n, n \geq 0,$$

for some real numbers $a_1, a_2, \ldots, a_k$.

Example

The simplest linear recurrence has length one, so has the form

$$x_{n+1} = ax_n \text{ for } n \geq 0,$$

with $a \in \mathbb{R}$ and some initial value x_0.

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In this case,

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = a^2 x_0 \\ x_3 &= ax_2 = a^3 x_0 \\ \vdots &\quad \vdots \quad \vdots \\ x_n &= ax_{n-1} = a^n x_0 \end{aligned}$$

Therefore, $x_n = a^n x_0$.

Example

Find a formula for x_n if

$$x_{n+2} = 2x_{n+1} + 3x_n \text{ for } n \geq 0,$$

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Solution. Define $V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for each $n \geq 0$. Then

$$V_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and for $n \geq 0$,

$$V_{n+1} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix}$$

Example (continued)

Now express $V_{n+1} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix}$ as a matrix product:

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This is a linear dynamical system, so we can apply the techniques from §3.3, provided that A is diagonalizable.

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x & -1 \\ -3 & x-2 \end{vmatrix} = x^2 - 2x - 3 = (x-3)(x+1)$$

Therefore A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$, and is diagonalizable.

Example (continued)

$\vec{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a basic eigenvector corresponding to $\lambda_1 = 3$, and

$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a basic eigenvector corresponding to $\lambda_2 = -1$.

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Furthermore $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ is invertible and is the diagonalizing matrix for A, and $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

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Writing $P^{-1}V_0 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we get

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Example (continued)

Therefore,

$$\begin{aligned} V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} &= b_1\lambda_1^n\vec{x}_1 + b_2\lambda_2^n\vec{x}_2 \\ &= \frac{1}{4}3^n \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{4}(-1)^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \end{aligned}$$

and so

$$x_n = \frac{1}{4}3^n - \frac{1}{4}(-1)^n.$$

Example

Solve the recurrence relation

$$\mathrm{x_{k+2} = 5x_{k+1} - 6x_k, k \geq 0}$$

with $\mathrm{x_0 = 0}$ and $\mathrm{x_1 = 1}$.

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Solution. Write

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ 5x_{k+1} - 6x_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

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Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

Example (continued)

A has eigenvalues $\lambda_1 = 2$ with corresponding eigenvector $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\lambda_2 = 3$ with corresponding eigenvector $\vec{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

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$$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}V_0 = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

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Finally,

$$V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = b_1\lambda_1^k\vec{x}_1 + b_2\lambda_2^k\vec{x}_2 = (-1)2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Example

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = (-1)2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and therefore

$$x_k = 3^k - 2^k.$$