Math 221: LINEAR ALGEBRA

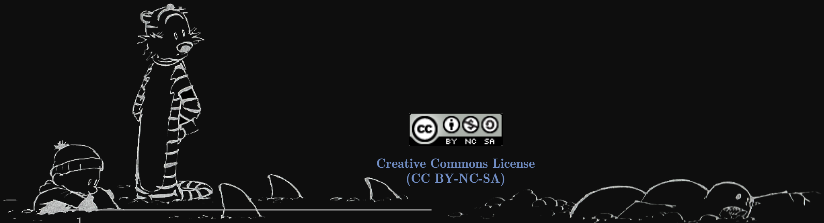
Chapter 5. Vector Space $\mathbb{R}^n$

§5-5. Similarity and Diagonalization

Le Chen[1]

Emory University, 2021 Spring

(last updated on 01/12/2023)



[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Ilijas Farah, York University**

BE A CHAMPION OF OER!

Contribute suggestions for improvements, new content, or errata:
A new topic
A new example or problem
A new or better proof to an existing theorem
Any other suggestions to improve the material

Contact Lyryx at info@lyryx.com with your ideas.

Similar Matrices

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. A is similar to B, written $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.

Similar Matrices

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. A is similar to B, written $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.

Lemma

Similarity is an equivalence relation, i.e., for $n \times n$ matrices A, B and C

1. $A \sim A$ (reflexive);

Similar Matrices

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. A is similar to B, written $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.

Lemma

Similarity is an equivalence relation, i.e., for $n \times n$ matrices A, B and C

1. $A \sim A$ (reflexive);
2. if $A \sim B$, then $B \sim A$ (symmetric);

Similar Matrices

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. A is similar to B, written $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.

Lemma

Similarity is an equivalence relation, i.e., for $n \times n$ matrices A, B and C

1. $A \sim A$ (reflexive);
2. if $A \sim B$, then $B \sim A$ (symmetric);
3. if $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive).

Similar Matrices

Definition (Similar Matrices)

Let A and B be $n \times n$ matrices. A is similar to B, written $A \sim B$, if there exists an invertible matrix P such that $B = P^{-1}AP$.

Lemma

Similarity is an equivalence relation, i.e., for $n \times n$ matrices A, B and C

1. $A \sim A$ (reflexive);
2. if $A \sim B$, then $B \sim A$ (symmetric);
3. if $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive).

Proof.

1. Since $A = I_n A I_n$ and $I_n^{-1} = I_n$, $A = I_n^{-1} A I_n$. Therefore, $A \sim A$.
2. Suppose $A \sim B$. Then there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Multiplying both sides on the left by P, on the right by P^{-1}, and simplifying gives us $PBP^{-1} = A$. Therefore, $A = (P^{-1})^{-1}A(P^{-1})$, so $A \sim B$.

Proof. (continued)

3. Since $A \sim B$ and $B \sim C$, there exist invertible $n \times n$ matrices P and Q such that
$$B = P^{-1}AP \quad \text{and} \quad C = Q^{-1}BQ.$$
Thus
$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ),$$
where PQ is invertible, and hence $A \sim C$.

■

Definition

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the trace of A is

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Definition

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the trace of A is

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Lemma (Properties of trace)

For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$;
2. $\mathrm{tr}(kA) = k \cdot \mathrm{tr}(A)$;
3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Proof.

The proofs of (1) and (2) are trivial. As for (3), ...

Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is

$$c_A(x) = \det(xI - A),$$

and is a polynomial of degree n.

Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is

$$c_A(x) = \det(xI - A),$$

and is a polynomial of degree n.

Theorem (Properties of Similar Matrices)

If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$;
2. $\text{rank}\,(A) = \text{rank}\,(B)$;
3. $\text{tr}(A) = \text{tr}(B)$;
4. $c_A(x) = c_B(x)$;
5. A and B have the same eigenvalues.

Proof.

Since $A \sim B$, there exists an $n \times n$ invertible matrix P so that $B = P^{-1}AP$.

Proof.

Since $A \sim B$, there exists an $n \times n$ invertible matrix P so that $B = P^{-1}AP$.

1. $\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \cdot \det(A) \cdot \det(P)$.
   Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

   $$\det(B) = \frac{1}{\det(P)} \cdot \det(A) \cdot \det(P) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(A) = \det(A).$$

   Therefore, $\det(B) = \det(A)$.

Proof.

Since $A \sim B$, there exists an $n \times n$ invertible matrix P so that $B = P^{-1}AP$.

1. $\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \cdot \det(A) \cdot \det(P)$.
   Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so
   $$\det(B) = \frac{1}{\det(P)} \cdot \det(A) \cdot \det(P) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(A) = \det(A).$$
   Therefore, $\det(B) = \det(A)$.

2. $\mathrm{rank}\ (B) = \mathrm{rank}\ (P^{-1}AP)$.
   Since P is invertible, $\mathrm{rank}\ (P^{-1}AP) = \mathrm{rank}\ (P^{-1}A)$,
   since P^{-1} is invertible, $\mathrm{rank}\ (P^{-1}A) = \mathrm{rank}\ (A)$.
   Therefore, $\mathrm{rank}\ (B) = \mathrm{rank}\ (A)$.

Proof.

Since $A \sim B$, there exists an $n \times n$ invertible matrix P so that $B = P^{-1}AP$.

1. $\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \cdot \det(A) \cdot \det(P)$.

   Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

   $$\det(B) = \frac{1}{\det(P)} \cdot \det(A) \cdot \det(P) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(A) = \det(A).$$

   Therefore, $\det(B) = \det(A)$.

2. $\text{rank }(B) = \text{rank }(P^{-1}AP)$.

   Since P is invertible, $\text{rank }(P^{-1}AP) = \text{rank }(P^{-1}A)$,

   since P^{-1} is invertible, $\text{rank }(P^{-1}A) = \text{rank }(A)$.

   Therefore, $\text{rank }(B) = \text{rank }(A)$.

3. $\text{tr}(B) = \text{tr}[(P^{-1}A)P] = \text{tr}[P(P^{-1}A)] = \text{tr}[(PP^{-1})A] = \text{tr}(IA) = \text{tr}(A)$.

Proof. (continued)

4.

$$\begin{aligned} c_B(x) = \det(xI - B) &= \det(xI - P^{-1}AP) \\ &= \det(xP^{-1}P - P^{-1}AP) \\ &= \det(P^{-1}xP - P^{-1}AP) \\ &= \det[P^{-1}(xI - A)P] \\ &= \det(P^{-1}) \cdot \det(xI - A) \cdot \det(P) \\ &= \det(P^{-1}) \cdot \det(P) \cdot \det(xI - A) \end{aligned}$$

Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

$$c_B(x) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(xI - A) = \det(xI - A) = c_A(x).$$

Proof. (continued)

4.

$$\begin{aligned} c_B(x) = \det(xI - B) &= \det(xI - P^{-1}AP) \\ &= \det(xP^{-1}P - P^{-1}AP) \\ &= \det(P^{-1}xP - P^{-1}AP) \\ &= \det[P^{-1}(xI - A)P] \\ &= \det(P^{-1}) \cdot \det(xI - A) \cdot \det(P) \\ &= \det(P^{-1}) \cdot \det(P) \cdot \det(xI - A) \end{aligned}$$

Since P is invertible, $\det(P^{-1}) = \frac{1}{\det(P)}$, so

$$c_B(x) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(xI - A) = \det(xI - A) = c_A(x).$$

5. Since the eigenvalues of a matrix are the roots of the characteristic polynomial, $c_B(x) = c_A(x)$ implies that A and B have the same eigenvalues. ∎

Copyright

Similar Matrices

Diagonalization Revisited

Algebraic and Geometric Multiplicities

Characterizing Diagonalizable Matrices

Complex Eigenvalues

Eigenvalues of Real Symmetric Matrices

Diagonalization Revisited

Diagonalization Revisited

Recall that if λ is an eigenvalue of A, then $A\vec{x} = \lambda\vec{x}$ for some nonzero vector $\vec{x}$ in $\mathbb{R}^n$. Such a vector $\vec{x}$ is called a λ-eigenvector of A or an eigenvector of A corresponding to λ.

Diagonalization Revisited

Recall that if λ is an eigenvalue of A, then $A\vec{x} = \lambda\vec{x}$ for some nonzero vector $\vec{x}$ in $\mathbb{R}^n$. Such a vector $\vec{x}$ is called a λ-eigenvector of A or an eigenvector of A corresponding to λ.

Definition (Diagonalizable – rephrased)

An $n \times n$ matrix A is diagonalizable if $A \sim D$ for some diagonal matrix D.

Diagonalization Revisited

Recall that if λ is an eigenvalue of A, then $A\vec{x} = \lambda\vec{x}$ for some nonzero vector $\vec{x}$ in $\mathbb{R}^n$. Such a vector $\vec{x}$ is called a λ-eigenvector of A or an eigenvector of A corresponding to λ.

Definition (Diagonalizable – rephrased)

An $n \times n$ matrix A is diagonalizable if $A \sim D$ for some diagonal matrix D.

Remark (Diagonalizability)

Determining whether or not a square matrix A is diagonalizable is done by checking whether

the number of linearly independent eigenvectors
– geometric multiplicity

$$||?$$

the multiplicity of each eigenvalue
– algebraic multiplicity

Example

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\lambda = -1$ is an eigenvalue of A, and $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a (-1)-eigenvector of A since

$$A\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\lambda = -1$ is an eigenvalue of A, and $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a (-1)-eigenvector of A since

$$A\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Theorem

Suppose A is an $n \times n$ matrix.

1. The eigenvalues of A are the roots of $c_A(x)$.
2. The λ-eigenvectors of A are all the nonzero solutions to $(\lambda I - A)\vec{x} = \vec{0}_n$.

Problem

Determine all eigenvalues of $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 6 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$.

Problem

Determine all eigenvalues of $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 6 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$.

Solution

$$\det(xI - A) = \begin{vmatrix} x+2 & 0 & 0 & 0 \\ -3 & x-6 & 0 & 0 \\ 1 & 0 & x-6 & 0 \\ -4 & -2 & 1 & x-1 \end{vmatrix} = (x+2)(x-6)(x-6)(x-1).$$

Thus, the eigenvalues of A are $-2, 6, 6$ and 1, precisely the elements on the main diagonal of A. ■

Problem

Determine all eigenvalues of $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 6 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$.

Solution

$$\det(xI - A) = \begin{vmatrix} x+2 & 0 & 0 & 0 \\ -3 & x-6 & 0 & 0 \\ 1 & 0 & x-6 & 0 \\ -4 & -2 & 1 & x-1 \end{vmatrix} = (x+2)(x-6)(x-6)(x-1).$$

Thus, the eigenvalues of A are $-2, 6, 6$ and 1, precisely the elements on the main diagonal of A. ■

Remark

In general, the eigenvalues of any triangular matrix are the entries on its main diagonal.

Theorem

Let A be an $n \times n$ matrix.

1. A is diagonalizable if and only if $\mathbb{R}^n$ has a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ of eigenvectors of A.
2. If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ are eigenvectors of A and form a basis of $\mathbb{R}^n$, then

$$P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right]$$

is an invertible matrix such that

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where λ_i is the eigenvalue of A corresponding to $\vec{x}_i$.

This result was covered earlier, but without the use of term basis.

Theorem

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. For each i, let $\vec{x}_i$ be a λ_i-eigenvector of A. Then $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

Theorem

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. For each i, let $\vec{x}_i$ be a λ_i-eigenvector of A. Then $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

Proof.

We need to show that $t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}$ only has trivial solution $t_1 = \cdots = t_k = 0$. Notice that

$$\begin{aligned}
t_1A\vec{x}_1 + t_2A\vec{x}_2 + \cdots + t_kA\vec{x}_k &= t_1\lambda_1\vec{x}_1 + t_2\lambda_2\vec{x}_2 + \cdots + t_k\lambda_k\vec{x}_k = \vec{0} \\
t_1A^2\vec{x}_1 + t_2A^2\vec{x}_2 + \cdots + t_kA^2\vec{x}_k &= t_1\lambda_1^2\vec{x}_1 + t_2\lambda_2^2\vec{x}_2 + \cdots + t_k\lambda_k^2\vec{x}_k = \vec{0} \\
\vdots \qquad & \qquad \vdots \\
t_1A^{k-1}\vec{x}_1 + \cdots\cdots + t_kA^{k-1}\vec{x}_k &= t_1\lambda_1^{k-1}\vec{x}_1 + \cdots\cdots + t_k\lambda_k^{k-1}\vec{x}_k = \vec{0}
\end{aligned}$$

Proof.

$$\begin{array}{ccccccccc} t_1\lambda_1\vec{x}_1 & + & t_2\lambda_2\vec{x}_2 & + & \cdots & + & t_k\lambda_k\vec{x}_k & = & \vec{0} \\ t_1\lambda_1^2\vec{x}_1 & + & t_2\lambda_2^2\vec{x}_2 & + & \cdots & + & t_k\lambda_k^2\vec{x}_k & = & \vec{0} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ t_1\lambda_1^{k-1}\vec{x}_1 & + & t_2\lambda_2^{k-1}\vec{x}_2 & + & \cdots & + & t_k\lambda_k^{k-1}\vec{x}_k & = & \vec{0} \end{array}$$

$$\Updownarrow$$

$$\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{pmatrix} \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_k \end{bmatrix} \begin{pmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^{k-1} \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_k^0 & \lambda_k^1 & \cdots & \lambda_k^{k-1} \end{pmatrix} = O_{k\times k}.$$

Proof.

Since λ_i are distinct, the Vandermonde matrix is invertible, hence,

$$\begin{pmatrix}\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k\end{pmatrix}\begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_k \end{bmatrix} = O_{k\times k}.$$

$$\Updownarrow$$

$$t_i\vec{x}_i = 0 \quad \text{for all } i = 1, \cdots, k$$

$$\Downarrow$$

$$t_i = 0 \quad \text{for all } i = 1, \cdots, k$$

Only trivial solution is found. Hence, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is independent. ∎

Proof. (Another proof left for you to study)

The proof is by induction on k, the number of distinct eigenvalues.

Basis. If $k = 1$, then $\{\vec{x}_1\}$ is an independent set because $\vec{x}_1 \neq \vec{0}_n$.
Suppose that for some $k \geq 1$, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is independent, where $\vec{x}_i$ is an eigenvector of A corresponding to λ_i, $1 \leq i \leq k$, and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. (This is the Inductive Hypothesis.) Now suppose $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are distinct eigenvalues of A that have corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{k+1}$, respectively. Consider

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_{k+1}\vec{x}_{k+1} = \vec{0}_n, \text{ for } t_1, t_2, \ldots, t_{k+1} \in \mathbb{R}. \tag{1}$$

Multiplying equation (1) by A (on the left) gives us

Proof. (continued)

$$t_1 A\vec{x}_1 + t_2 A\vec{x}_2 + \cdots + t_{k+1} A\vec{x}_{k+1} = \vec{0}_n,$$

$$\Downarrow$$

$$t_1 \lambda_1 \vec{x}_1 + t_2 \lambda_2 \vec{x}_2 + \cdots + t_{k+1} \lambda_{k+1} \vec{x}_{k+1} = \vec{0}_n. \tag{2}$$

Also, multiplying (1) by λ_{k+1} gives us

$$t_1 \lambda_{k+1} \vec{x}_1 + t_2 \lambda_{k+1} \vec{x}_2 + \cdots + t_{k+1} \lambda_{k+1} \vec{x}_{k+1} = \vec{0}_n, \tag{3}$$

and subtracting (3) from (2) results in

$$t_1(\lambda_1 - \lambda_{k+1})\vec{x}_1 + t_2(\lambda_2 - \lambda_{k+1})\vec{x}_2 + \cdots + t_k(\lambda_k - \lambda_{k+1})\vec{x}_k = \vec{0}_n.$$

Proof. (continued)

By the inductive hypothesis, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is independent, so

$$t_i(\lambda_i - \lambda_{k+1}) = 0 \text{ for } i = 1, 2, \ldots k.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, $(\lambda_i - \lambda_{k+1}) \neq 0$ for $i = 1, 2, \ldots, k$, and thus $t_i = 0$ for $i = 1, 2, \ldots, k$. Substituting these values into (1) yields

$$t_{k+1}\vec{x}_{k+1} = \vec{0}_n,$$

implying that $t_{k+1} = 0$, since $\vec{x}_{k+1} \neq \vec{0}_n$.

Therefore, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{k+1}\}$ is an independent set, and the result follows by induction. ∎

The next result is an easy consequence of the previous Theorem.

The next result is an easy consequence of the previous Theorem.

Theorem (Covered earlier, but now with a proof)

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

The next result is an easy consequence of the previous Theorem.

Theorem (Covered earlier, but now with a proof)

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Proof.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ denote the n (distinct) eigenvalues of A, and let $\vec{x}_i$ be an eigenvector of A corresponding to λ_i, $1 \leq i \leq n$. By the previous Theorem, $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is an independent set. A subset of n linearly independent vectors of $\mathbb{R}^n$ also spans $\mathbb{R}^n$, and thus $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$. Thus A is diagonalizable. ■

Problem

Is the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 8 & 6 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

diagonalizable?

Problem

Is the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 8 & 6 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$

diagonalizable?

Solution

Because A has characteristic polynomial

$$c_A(x) = (x+3)(x-2)(x-4),$$

A has distinct eigenvalues $-3, 2$ and 4.

Since A has three distinct eigenvalues, A is diagonalizable. ■

Problem (Covered earlier, but with different wording)

Is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ diagonalizable? Explain.

Problem (Covered earlier, but with different wording)

Is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ diagonalizable? Explain.

Solution

First, $c_A(x) = (x-2)(x+1)^2$, so the eigenvalues of A are $\lambda_1 = 2, \lambda_2 = -1$, and $\lambda_3 = -1$. Since the eigenvalues are not distinct, it isn't immediately obvious that A is diagonalizable. The general solution to $(-I - A)\vec{x} = \vec{0}_3$:

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is $x_1 = -s - t, x_2 = s$, and $x_3 = t$ for $s, t \in \mathbb{R}$, leading to basic solutions

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

that are linearly independent. Therefore, there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A, so A is diagonalizable. ■

Algebraic and Geometric Multiplicities

Lemma (Technical but useful)

Let A be an $n \times n$ matrix, with independent eigenvectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. Extend $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ to a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k, \ldots, \vec{x}_n\}$ of $\mathbb{R}^n$, and let $P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right]$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the (not necessarily distinct) eigenvalues corresponding to $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$, then

$$P^{-1}AP = \left[\begin{array}{cc} \mathrm{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{array}\right],$$

where B is an $k \times (n-k)$ matrix and A_1 is an $(n-k) \times (n-k)$ matrix.

Proof.

$$\left[\; A\vec{x}_1 \;\middle|\; \cdots \;\middle|\; A\vec{x}_k \;\middle|\; A\vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \;\right] = \left[\; \lambda_1\vec{x}_1 \;\middle|\; \cdots \;\middle|\; \lambda_k\vec{x}_k \;\middle|\; A\vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \;\right]$$

$$\|$$

$$A\left[\; \vec{x}_1 \;\middle|\; \vec{x}_2 \;\middle|\; \cdots \;\middle|\; \vec{x}_n \;\right] \qquad \|$$

Proof.

$$\left[\ A\vec{x}_1 \mid \cdots \mid A\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n\ \right] = \left[\ \lambda_1\vec{x}_1 \mid \cdots \mid \lambda_k\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n\ \right]$$

$$\|$$

$$A\left[\ \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n\ \right]$$

$$\|$$

$$\left[\ \vec{x}_1 \mid \cdots \mid \vec{x}_k \mid \vec{x}_{k+1} \mid \cdots \mid \vec{x}_n\ \right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right]$$

Proof.

$$\left[\ A\vec{x}_1 \ \middle|\ \cdots \ \middle|\ A\vec{x}_k \ \middle|\ A\vec{x}_{k+1} \ \middle|\ \cdots \ \middle|\ A\vec{x}_n \ \right] = \left[\ \lambda_1\vec{x}_1 \ \middle|\ \cdots \ \middle|\ \lambda_k\vec{x}_k \ \middle|\ A\vec{x}_{k+1} \ \middle|\ \cdots \ \middle|\ A\vec{x}_n \ \right]$$

$$\|$$

$$A\left[\ \vec{x}_1 \ \middle|\ \vec{x}_2 \ \middle|\ \cdots \ \middle|\ \vec{x}_n \ \right] \qquad\qquad \|$$

$$\left[\ \vec{x}_1 \ \middle|\ \cdots \ \middle|\ \vec{x}_k \ \middle|\ \vec{x}_{k+1} \ \middle|\ \cdots \ \middle|\ \vec{x}_n \ \right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right]$$

$$\begin{array}{ccc} \uparrow & \cdots & \uparrow \\ P^{-1}A\vec{x}_{k+1} & \cdots & P^{-1}A\vec{x}_n \end{array}$$

Proof.

$$\left[\, A\vec{x}_1 \mid \cdots \mid A\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right] = \left[\, \lambda_1\vec{x}_1 \mid \cdots \mid \lambda_k\vec{x}_k \mid A\vec{x}_{k+1} \mid \cdots \mid A\vec{x}_n \,\right]$$

$$\begin{array}{ccc} \| & & \| \\ A\left[\, \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n \,\right] & & \end{array}$$

$$\left[\, \vec{x}_1 \mid \cdots \mid \vec{x}_k \mid \vec{x}_{k+1} \mid \cdots \mid \vec{x}_n \,\right] \begin{array}{c} \left[\begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & 0 & & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array}\right] \\ \begin{array}{ccc} \uparrow & \cdots & \uparrow \\ P^{-1}A\vec{x}_{k+1} & \cdots & P^{-1}A\vec{x}_n \end{array} \end{array}$$

$$\Updownarrow$$

$$AP = P\begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{bmatrix}$$

Proof.

$$\left[\, A\vec{x}_1 \;\middle|\; \cdots \;\middle|\; A\vec{x}_k \;\middle|\; A\vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \,\right] = \left[\, \lambda_1\vec{x}_1 \;\middle|\; \cdots \;\middle|\; \lambda_k\vec{x}_k \;\middle|\; A\vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; A\vec{x}_n \,\right]$$

$$\|$$

$$A\left[\, \vec{x}_1 \;\middle|\; \vec{x}_2 \;\middle|\; \cdots \;\middle|\; \vec{x}_n \,\right] \qquad\qquad \|$$

$$\left[\, \vec{x}_1 \;\middle|\; \cdots \;\middle|\; \vec{x}_k \;\middle|\; \vec{x}_{k+1} \;\middle|\; \cdots \;\middle|\; \vec{x}_n \,\right]
\left[\begin{array}{ccc|ccc}
\lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\
& \ddots & & \vdots & \vdots & \vdots \\
& & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\
\hline
& & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\
& 0 & & \vdots & \vdots & \vdots \\
& & & a_{n,k+1} & \cdots & a_{n,k+1}
\end{array}\right]$$

$$\begin{array}{ccc}
\uparrow & \cdots & \uparrow \\
P^{-1}A\vec{x}_{k+1} & \cdots & P^{-1}A\vec{x}_n
\end{array}$$

$$\Updownarrow$$

$$AP = P\begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{bmatrix}$$

$$\Updownarrow$$

$$P^{-1}AP = \begin{bmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_k) & B \\ 0_{(n-k)\times k} & A_1 \end{bmatrix}$$

$\blacksquare$

Proof. (Another proof)

Recall that $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis of $\mathbb{R}_n$. Since $I_n = P^{-1}P$,

$$\begin{aligned} \left[\begin{array}{cccc} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{array}\right] = P^{-1}P &= P^{-1}\left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right] \\ &= \left[\begin{array}{cccc} P^{-1}\vec{x}_1 & P^{-1}\vec{x}_2 & \cdots & P^{-1}\vec{x}_n \end{array}\right] \end{aligned}$$

Thus for each j, $1 \leq j \leq n$, $P^{-1}\vec{x}_j = \vec{e}_j$. Also,

$$\begin{aligned} P^{-1}AP &= P^{-1}A\left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right] \\ &= \left[\begin{array}{cccc} P^{-1}A\vec{x}_1 & P^{-1}A\vec{x}_2 & \cdots & P^{-1}A\vec{x}_n \end{array}\right], \end{aligned}$$

so the j^th column of $P^{-1}AP$, $1 \leq j \leq k$, is equal to

$$P^{-1}(A\vec{x}_j) = P^{-1}(\lambda_j\vec{x}_j) = \lambda_j(P^{-1}\vec{x}_j) = \lambda_j\vec{e}_j.$$

This gives us the first k columns of $P^{-1}AP$, and the result follows. ■

Definition

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

Definition

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

Remark

1. The eigenspace $E_\lambda(A)$ is indeed a subspace of $\mathbb{R}^n$ because

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} = \{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\} = \mathrm{null}(\lambda I - A).$$

2. If λ is not an eigenvalue of A, then $E_\lambda(A) = \{0\}$.

Definition

1. If A is an $n \times n$ matrix and λ is an eigenvalue of A, then the (algebraic) multiplicity of λ is the largest value of m for which

$$c_A(x) = (x - \lambda)^m g(x)$$

for some polynomial $g(x)$, i.e., the multiplicity of λ is the number of times that λ occurs as a root of $c_A(x)$.

2. $\dim(E_\lambda(A))$ is called the geometric multiplicity of λ.

Definition

1. If A is an $n \times n$ matrix and λ is an eigenvalue of A, then the (algebraic) multiplicity of λ is the largest value of m for which

$$c_A(x) = (x - \lambda)^m g(x)$$

for some polynomial $g(x)$, i.e., the multiplicity of λ is the number of times that λ occurs as a root of $c_A(x)$.

2. $\dim(E_\lambda(A))$ is called the geometric multiplicity of λ.

Lemma

If A is an $n \times n$ matrix, and λ is an eigenvalue of A of multiplicity m, then

$$\dim(E_\lambda(A)) \leq m,$$

Definition

1. If A is an $n \times n$ matrix and λ is an eigenvalue of A, then the (algebraic) multiplicity of λ is the largest value of m for which
$$c_A(x) = (x - \lambda)^m g(x)$$
for some polynomial $g(x)$, i.e., the multiplicity of λ is the number of times that λ occurs as a root of $c_A(x)$.
2. $\dim(E_\lambda(A))$ is called the geometric multiplicity of λ.

Lemma

If A is an $n \times n$ matrix, and λ is an eigenvalue of A of multiplicity m, then
$$\dim(E_\lambda(A)) \leq m,$$
that is,
$$\text{Geometric multiplicity} \leq \text{Algebraic multiplicity}.$$

Proof.

Let $d = \dim(E_\lambda(A))$, and let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d\}$ be a basis of $E_\lambda(A)$. As a consequence, we know that there exists an invertible $n \times n$ matrix P so that

$$P^{-1}AP = \left[\begin{array}{cc} \mathrm{diag}(\lambda, \ldots, \lambda) & B \\ 0_{(n-d)\times d} & A_1 \end{array}\right] = \left[\begin{array}{cc} \lambda I_d & B \\ 0_{(n-d)\times d} & A_1 \end{array}\right]$$

where B is $d \times (n-d)$ and A_1 is $(n-d) \times (n-d)$.

Define $A' = P^{-1}AP$. Then $A \sim A'$, so A and A' have the same characteristic polynomial. Thus

$$\begin{aligned} c_A(x) = c_{A'}(x) = \det(xI - A') &= \det \left[\begin{array}{cc} (x-\lambda)I_d & -B \\ 0_{(n-d)\times d} & xI_{n-d} - A_1 \end{array}\right] \\ &= \det[(x-\lambda)I_d] \det(xI_{n-d} - A_1) \\ &= (x-\lambda)^d c_{A_1}(x) \\ &= (x-\lambda)^d g(x). \end{aligned}$$

Since λ has multiplicity m, $d \leq m$, and therefore $\dim(E_\lambda(A)) \leq m$ as required. $\blacksquare$

Characterizing Diagonalizable Matrices

Characterizing Diagonalizable Matrices

The crucial consequence of this Lemma is the characterization of matrices that are diagonalizable.

Characterizing Diagonalizable Matrices

The crucial consequence of this Lemma is the characterization of matrices that are diagonalizable.

Theorem (Covered earlier, here with new terminology)

For an $n \times n$ matrix A, the following two conditions are equivalent.

1. A is diagonalizable.
2. For each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the multiplicity of λ, i.e.,

Characterizing Diagonalizable Matrices

The crucial consequence of this Lemma is the characterization of matrices that are diagonalizable.

Theorem (Covered earlier, here with new terminology)

For an $n \times n$ matrix A, the following two conditions are equivalent.

1. A is diagonalizable.
2. For each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the multiplicity of λ, i.e.,

$$\text{Diagonalizable} \quad \Updownarrow \quad \text{Geometric multiplicity} = \text{Algebraic multiplicity}, \quad \text{for all } \lambda.$$

Problem (Covered earlier, here with new terminology)

If possible, diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$. Otherwise, explain why A is not diagonalizable.

Problem (Covered earlier, here with new terminology)

If possible, diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$. Otherwise, explain why A is not diagonalizable.

Solution

$c_A(x) = (x-3)(x+1)^2$, so A has eigenvalues $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -1$. Find the dimension of $E_{-1}(A)$ by solving the linear system $(-I - A)\vec{x} = \vec{0}_3$.

$$\left[\begin{array}{rrr|r} 4 & -1 & -6 & 0 \\ -2 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this, we see that $\dim(E_{-1}(A)) = 1$. Since -1 is an eigenvalue of multiplicity two, A is not diagonalizable. ■

Problem (Covered earlier, here with new terminology)

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Show that A is diagonalizable, and that B is not diagonalizable.

Problem (Covered earlier, here with new terminology)

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Show that A is diagonalizable, and that B is not diagonalizable.

Solution

Both A and B are triangular matrices, so we immediately see that A and B have the same eigenvalues: $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Thus for each matrix, 1 is an eigenvalue of multiplicity two.

Solving the system $(I - A)\vec{x} = \vec{0}_3$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that there are two parameters in the general solution, so $\dim(E_1(A)) = 2$. Therefore, A is diagonalizable.

Solution (continued)

Solving the system $(I - B)\vec{x} = \vec{0}_3$:

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that the general solution has only one parameter, so $\dim(E_1(B)) = 1$. However, the algebraic multiplicity of $\lambda = 1$ is 2. Therefore, B is not diagonalizable. ■

Complex Eigenvalues

Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

Problem

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & -1 \\ 1 & x-1 \end{vmatrix} = x^2 - 2x + 2.$$

The roots of $c_A(x)$ are distinct complex numbers: $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, so A is diagonalizable. Corresponding eigenvectors are

$$\vec{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix},$$

respectively.

Solution (continued)

A diagonalizing matrix for A is

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix},$$

and

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

$\blacksquare$

Solution (continued)

A diagonalizing matrix for A is

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix},$$

and

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

■

Remark

Notice that A is a real matrix, but has complex eigenvalues (and eigenvectors).

Eigenvalues of Real Symmetric Matrices

Eigenvalues of Real Symmetric Matrices

Theorem

The eigenvalues of any real symmetric matrix are real.

Eigenvalues of Real Symmetric Matrices

Theorem

The eigenvalues of any real symmetric matrix are real.

Proof.

Let A be an $n \times n$ real symmetric matrix, and let λ be an eigenvalue of A. To prove that λ is real, it is enough to prove that $\overline{\lambda} = \lambda$, i.e., λ is equal to its (complex) conjugate.

We use $\overline{A}$ to denote the matrix obtained from A by replacing each entry by its conjugate. Since A is real, $\overline{A} = A$.

Suppose

$$\vec{x} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

is a λ-eigenvector of A. Then $A\vec{x} = \lambda\vec{x}$.

Proof. (continued)

Let $c = \vec{x}^T \overline{\vec{x}} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \\ \vdots \\ \overline{z}_n \end{bmatrix}$.

Then $c = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$; since $\vec{x} \neq \vec{0}$, c is a positive real number. Now

$$\begin{aligned}
\lambda c &= \lambda(\vec{x}^T \overline{\vec{x}}) = (\lambda \vec{x}^T)\overline{\vec{x}} = (\lambda \vec{x})^T \overline{\vec{x}} \\
&= (A\vec{x})^T \overline{\vec{x}} = \vec{x}^T A^T \overline{\vec{x}} \\
&= \vec{x}^T A \overline{\vec{x}} \qquad \text{(since A is symmetric)} \\
&= \vec{x}^T \ \overline{A} \ \overline{\vec{x}} \qquad \text{(since A is real)} \\
&= \vec{x}^T (\overline{A\vec{x}}) = \vec{x}^T (\overline{\lambda \vec{x}}) = \vec{x}^T \ \overline{\lambda} \ \overline{\vec{x}} \\
&= \overline{\lambda}(\vec{x}^T \overline{\vec{x}}) \\
&= \overline{\lambda} c.
\end{aligned}$$

Thus, $\lambda c = \overline{\lambda} c$. Since $c \neq 0$, it follows that $\lambda = \overline{\lambda}$, and therefore λ is real. ■