Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations
§7-1. Examples and Elementary Properties

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Emory University, 2021 Spring

(last updated on 01/12/2023)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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What is a Linear Transformation?

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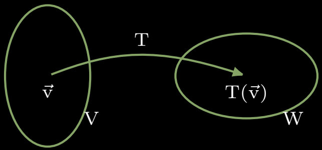

Definition

Let V and W be vector spaces, and $T : V \to W$ a function. Then T is called a linear transformation if it satisfies the following two properties.

1. T preserves addition.
   For all $\vec{v}_1, \vec{v}_2 \in V$, $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$.
2. T preserves scalar multiplication.
   For all $\vec{v} \in V$ and $r \in \mathbb{R}$, $T(r\vec{v}) = rT(\vec{v})$.

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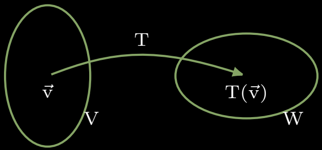

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2. T preserves scalar multiplication.
   For all $\vec{v} \in V$ and $r \in \mathbb{R}$, $T(r\vec{v}) = rT(\vec{v})$.

Remark

Note that the sum $\vec{v}_1 + \vec{v}_2$ is in V, while the sum $T(\vec{v}_1) + T(\vec{v}_2)$ is in W. Similarly, $r\vec{v}$ is scalar multiplication in V, while $rT(\vec{v})$ is scalar multiplication in W.

Theorem (Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$)

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is induced by an $m \times n$ matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix},$$

where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis of $\mathbb{R}^n$, and thus for each $\vec{x} \in \mathbb{R}^n$

$$T(\vec{x}) = A\vec{x}.$$

Example

$\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\mathrm{T}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-z \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$.

One can show that T preserves addition and scalar multiplication, and hence is a linear transformation. Therefore, the matrix that induces T is

$$A = \begin{bmatrix} \mathrm{T}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \mathrm{T}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \mathrm{T}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Remark (Notation and Terminology)

1. If A is an $m \times n$ matrix, then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n$$

   is the linear (or matrix) transformation induced by A.

2. Let V be a vector space. A linear transformation $T : V \to V$ is called a linear operator on V.

Example

Let V and W be vector spaces.

1. The zero transformation.

   $0 : \mathrm{V} \to \mathrm{W}$ is defined by $0(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathrm{V}$.

Examples and Problems

Example

Let V and W be vector spaces.

1. The zero transformation.

   $0 : \mathrm{V} \to \mathrm{W}$ is defined by $0(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathrm{V}$.

2. The identity operator on V.

   $1_{\mathrm{V}} : \mathrm{V} \to \mathrm{V}$ is defined by $1_{\mathrm{V}}(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathrm{V}$.

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1. The zero transformation.

   $0 : V \to W$ is defined by $0(\vec{x}) = \vec{0}$ for all $\vec{x} \in V$.

2. The identity operator on V.

   $1_V : V \to V$ is defined by $1_V(\vec{x}) = \vec{x}$ for all $\vec{x} \in V$.

3. The scalar operator on V.

   Let $a \in \mathbb{R}$. $s_a : V \to V$ is defined by $s_a(\vec{x}) = a\vec{x}$ for all $\vec{x} \in V$.

Problem

For vector spaces V and W, prove that the zero transformation 0, the identity operator 1_V, and the scalar operator s_a are linear transformations.

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Solution (the scalar operator)

Let V be a vector space and let $a \in \mathbb{R}$.

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Solution (the scalar operator)

Let V be a vector space and let $a \in \mathbb{R}$.

1. Let $\vec{u}, \vec{w} \in V$. Then $s_a(\vec{u}) = a\vec{u}$ and $s_a(\vec{w}) = a\vec{w}$. Now

$$s_a(\vec{u} + \vec{w}) = a(\vec{u} + \vec{w}) = a\vec{u} + a\vec{w} = s_a(\vec{u}) + s_a(\vec{w}),$$

   and thus s_a preserves addition.

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2. Let $\vec{u} \in V$ and $k \in \mathbb{R}$. Then $s_a(\vec{u}) = a\vec{u}$. Now

$$s_a(k\vec{u}) = ak\vec{u} = ka\vec{u} = ks_a(\vec{u}),$$

   and thus s_a preserves scalar multiplication.

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For vector spaces V and W, prove that the zero transformation 0, the identity operator 1_V, and the scalar operator s_a are linear transformations.

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   and thus s_a preserves scalar multiplication.

Since s_a preserves addition and scalar multiplication, s_a is a linear transformation. ■

Problem (Matrix transposition)

Let $R : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ be a transformation defined by

$$R(A) = A^T \text{ for all } A \in \mathbf{M}_{nn}.$$

Show that R is a linear transformation.

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1. Let $A, B \in \mathbf{M}_{nn}$. Then $R(A) = A^T$ and $R(B) = B^T$, so

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$$R(kA) = (kA)^T = kA^T = kR(A).$$

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Since R preserves addition and scalar multiplication, R is a linear transformation. ■

Problem (Evaluation at a point)

For each $a \in \mathbb{R}$, the transformation $E_a : \mathcal{P}_n \to \mathbb{R}$ is defined by

$$E_a(p) = p(a) \text{ for all } p \in \mathcal{P}_n.$$

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1. Let $p, q \in \mathcal{P}_n$. Then $E_a(p) = p(a)$ and $E_a(q) = q(a)$, so

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Since E_a preserves addition and scalar multiplication, E_a is a linear transformation. ∎

Problem

Let $S : \mathbf{M}_{nn} \to \mathbb{R}$ be a transformation defined by

$$S(A) = \mathrm{tr}(A) \text{ for all } A \in \mathbf{M}_{nn}.$$

Prove that S is a linear transformation.

Solution

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

$$S(A) = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad S(B) = \sum_{i=1}^{n} b_{ii}.$$

Solution

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

$$S(A) = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad S(B) = \sum_{i=1}^{n} b_{ii}.$$

1. Since $A + B = [a_{ij} + b_{ij}]$,

$$S(A+B) = \mathrm{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \left(\sum_{i=1}^{n} a_{ii}\right)+\left(\sum_{i=1}^{n} b_{ii}\right) = S(A)+S(B).$$

Solution

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

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2. Let $k \in \mathbb{R}$. Since $kA = [ka_{ij}]$,

$$S(kA) = \mathrm{tr}(kA) = \sum_{i=1}^{n} ka_{ii} = k\sum_{i=1}^{n} a_{ii} = kS(A).$$

Solution

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

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Therefore, S preserves addition and scalar multiplication, and thus is a linear transformation. ■

Problem

Show that the differentiation and integration operations on $\mathbf{P}_n$ are linear transformations. More precisely,

$$D : \mathbf{P}_n \to \mathbf{P}_{n-1} \quad \text{where } D[p(x)] = p'(x) \text{ for all } p(x) \text{ in } \mathbf{P}_n$$

$$I : \mathbf{P}_n \to \mathbf{P}_{n+1} \quad \text{where } I[p(x)] = \int_0^x p(t)dt \text{ for all } p(x) \text{ in } \mathbf{P}_n$$

are linear transformations.

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are linear transformations.

Solution (Sketch)

$$[p(x) + q(x)]' = p'(x) + q'(x), \qquad [rp(x)]' = (rp)'(x)$$

$$\int_0^x [p(t) + q(t)]\, dt = \int_0^x p(t)dt + \int_0^x q(t)dt, \quad \int_0^x rp(t)dt = r\int_0^x p(t)dt$$

■

Properties of Linear Transformations

Properties of Linear Transformations

Theorem

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. Then

1. T preserves the zero vector. $T(\vec{0}) = \vec{0}$.
2. T preserves additive inverses. For all $\vec{v} \in V$, $T(-\vec{v}) = -T(\vec{v})$.
3. T preserves linear combinations.
   For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ and all $k_1, k_2, \ldots, k_m \in \mathbb{R}$,

$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_mT(\vec{v}_m).$$

Proof.

1. Let $\vec{0}_V$ denote the zero vector of V and let $\vec{0}_W$ denote the zero vector of W. We want to prove that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{x} \in V$. Then $0\vec{x} = \vec{0}_V$ and

$$T(\vec{0}_V) = T(0\vec{x}) = 0T(\vec{x}) = \vec{0}_W.$$

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$$T(\vec{0}_V) = T(0\vec{x}) = 0T(\vec{x}) = \vec{0}_W.$$

2. Let $\vec{v} \in V$; then $-\vec{v} \in V$ is the additive inverse of $\vec{v}$, so $\vec{v} + (-\vec{v}) = \vec{0}_V$. Thus
$$\begin{aligned} T(\vec{v} + (-\vec{v})) &= T(\vec{0}_V) \\ T(\vec{v}) + T(-\vec{v})) &= \vec{0}_W \\ T(-\vec{v}) &= \vec{0}_W - T(\vec{v}) = -T(\vec{v}). \end{aligned}$$

Proof.

1. Let $\vec{0}_V$ denote the zero vector of V and let $\vec{0}_W$ denote the zero vector of W. We want to prove that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{x} \in V$. Then $0\vec{x} = \vec{0}_V$ and
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3. This result follows from preservation of addition and preservation of scalar multiplication. A formal proof would be by induction on m.

$\blacksquare$

Problem

Let $T : \mathcal{P}_2 \to \mathbb{R}$ be a linear transformation such that

$$T(x^2 + x) = -1; \quad T(x^2 - x) = 1; \quad T(x^2 + 1) = 3.$$

Find $T(4x^2 + 5x - 3)$.

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Find $T(4x^2 + 5x - 3)$.

Solution (first)

Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then

$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$

Problem

Let $T : \mathcal{P}_2 \to \mathbb{R}$ be a linear transformation such that

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Solution (first)

Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then

$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$

Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$.

Problem

Let $T : \mathcal{P}_2 \to \mathbb{R}$ be a linear transformation such that

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Find $T(4x^2+5x-3)$.

Solution (first)

Suppose $a(x^2+x)+b(x^2-x)+c(x^2+1) = 4x^2+5x-3$. Then

$$(a+b+c)x^2+(a-b)x+c = 4x^2+5x-3.$$

Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$. Thus

$$\begin{aligned} T(4x^2+5x-3) &= T\left(6(x^2+x)+(x^2-x)-3(x^2+1)\right) \\ &= 6T(x^2+x)+T(x^2-x)-3T(x^2+1) \\ &= 6(-1)+1-3(3) = -14. \end{aligned}$$

■

Solution (second)

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

Solution (second)

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

$$\begin{aligned}
x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\
x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\
1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x).
\end{aligned}$$

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$$\Downarrow$$

$$\begin{aligned} T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) + \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \end{aligned}$$

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$$\begin{aligned}
x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\
x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\
1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x).
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) + \tfrac{1}{2}T(x^2 - x) \\
&= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\
T(x) &= T\left(\tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\
&= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1. \\
T(1) &= T\left((x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) \\
&= T(x^2 + 1) - \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\
&= 3 - \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = 3.
\end{aligned}$$

Solution (second)

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

$$\begin{aligned} x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\ x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\ 1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x). \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} T(x^2) &= T\left(\tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) + \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\ T(x) &= T\left(\tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\ &= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1. \\ T(1) &= T\left((x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) \\ &= T(x^2 + 1) - \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\ &= 3 - \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = 3. \end{aligned}$$

$$\Downarrow$$

$$T(4x^2 + 5x - 3) = 4T(x^2) + 5T(x) - 3T(1) = 4(0) + 5(-1) - 3(3) = -14.$$

$\blacksquare$

Remark

The advantage of the second solution over the first is that if you were now asked to find $T(-6x^2 - 13x + 9)$, it is easy to use $T(x^2) = 0$, $T(x) = -1$ and $T(1) = 3$:

$$\begin{aligned} T(-6x^2 - 13x + 9) &= -6T(x^2) - 13T(x) + 9T(1) \\ &= -6(0) - 13(-1) + 9(3) = 13 + 27 = 40. \end{aligned}$$

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More generally,

$$\begin{aligned} T(ax^2 + bx + c) &= aT(x^2) + bT(x) + cT(1) \\ &= a(0) + b(-1) + c(3) = -b + 3c. \end{aligned}$$

Definition (Equality of linear transformations)

Let V and W be vector spaces, and let S and T be linear transformations from V to W. Then $S = T$ if and only if,

$$S(\vec{v}) = T(\vec{v}) \qquad \text{for every } \vec{v} \in V.$$

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Theorem

Let V and W be vector spaces, where

$$V = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}.$$

Suppose that S and T are linear transformations from V to W. If $S(\vec{v}_i) = T(\vec{v}_i)$ for all i, $1 \leq i \leq n$, then $S = T$.

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Remark

This theorem tells us that a linear transformation is completely determined by its actions on a spanning set.

Proof.

We must show that $S(\vec{v}) = T(\vec{v})$ for each $\vec{v} \in V$. Let $\vec{v} \in V$. Then (since V is spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$), there exist $k_1, k_2, \ldots, k_n \in \mathbb{R}$ so that

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n.$$

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$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n.$$

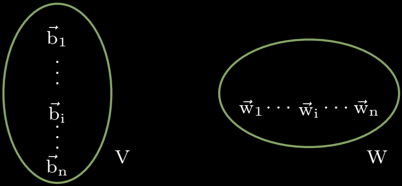
It follows that

$$\begin{aligned}
S(\vec{v}) &= S(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n) \\
&= k_1S(\vec{v}_1) + k_2S(\vec{v}_2) + \cdots + k_nS(\vec{v}_n) \\
&= k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_nT(\vec{v}_n) \\
&= T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n) \\
&= T(\vec{v}).
\end{aligned}$$

Therefore, $S = T$. $\blacksquare$

Constructing Linear Transformations

Constructing Linear Transformations



Theorem

Let V and W be vector spaces, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis of V, and let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ be (not necessarily distinct) vectors of W.

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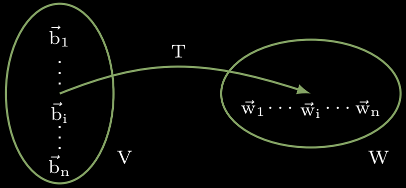

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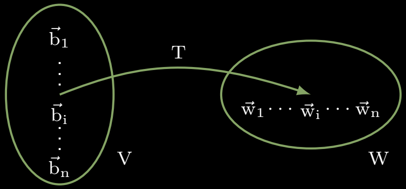
1. There exists a linear transformation $T : V \to W$ such that $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$;

Constructing Linear Transformations



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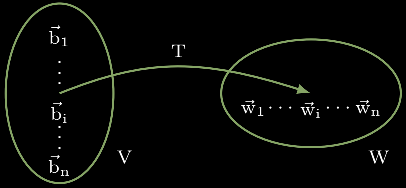
1. There exists a linear transformation $T : V \to W$ such that $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \le i \le n$;
2. This transformation is unique;

Constructing Linear Transformations



Theorem

Let V and W be vector spaces, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis of V, and let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ be (not necessarily distinct) vectors of W. Then

1. There exists a linear transformation $T : V \to W$ such that $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$;
2. This transformation is unique;
3. If
$$\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n$$
is a vector of V, then
$$T(\vec{v}) = k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n.$$

Proof.

Suppose $\vec{v} \in V$. Since B is a basis, there exist unique scalars $k_1, k_2, \dots, k_n \in \mathbb{R}$ so that $\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n$. We now define $T : V \to W$ by

$$T(\vec{v}) = k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n$$

for each $\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n$ in V. From this definition, $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$.

To prove that T is a linear transformation, prove that T preserves addition and scalar multiplication. Let $\vec{v}, \vec{u} \in V$. Then

$$\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n \quad \text{and} \quad \vec{u} = \ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n$$

for some $k_1, k_2, \dots, k_n \in \mathbb{R}$ and $\ell_1, \ell_2, \dots, \ell_n \in \mathbb{R}$.

Proof. (continued)

$$
\begin{aligned}
T(\vec{v} + \vec{u}) &= T[(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + (\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n)] \\
&= T[(k_1 + \ell_1)\vec{b}_1 + (k_2 + \ell_2)\vec{b}_2 + \cdots + (k_n + \ell_n)\vec{b}_n] \\
&= (k_1 + \ell_1)\vec{w}_1 + (k_2 + \ell_2)\vec{w}_2 + \cdots + (k_n + \ell_n)\vec{w}_n \\
&= (k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n) + (\ell_1\vec{w}_1 + \ell_2\vec{w}_2 + \cdots + \ell_n\vec{w}_n) \\
&= T(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + T(\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n) \\
&= T(\vec{v}) + T(\vec{u}).
\end{aligned}
$$

Therefore, T preserves addition.

Proof. (continued)

$$\begin{aligned}
T(\vec{v}+\vec{u}) &= T[(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + (\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n)] \\
&= T[(k_1+\ell_1)\vec{b}_1 + (k_2+\ell_2)\vec{b}_2 + \cdots + (k_n+\ell_n)\vec{b}_n] \\
&= (k_1+\ell_1)\vec{w}_1 + (k_2+\ell_2)\vec{w}_2 + \cdots + (k_n+\ell_n)\vec{w}_n \\
&= (k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n) + (\ell_1\vec{w}_1 + \ell_2\vec{w}_2 + \cdots + \ell_n\vec{w}_n) \\
&= T(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + T(\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n) \\
&= T(\vec{v}) + T(\vec{u}).
\end{aligned}$$

Therefore, T preserves addition. Let $\vec{v}$ be as already defined and let $r \in \mathbb{R}$. Then

$$\begin{aligned}
T(r\vec{v}) &= T[r(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n)] \\
&= T[(rk_1)\vec{b}_1 + (rk_2)\vec{b}_2 + \cdots + (rk_n)\vec{b}_n] \\
&= (rk_1)\vec{w}_1 + (rk_2)\vec{w}_2 + \cdots + (rk_n)\vec{w}_n \\
&= r(k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n) \\
&= rT(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) \\
&= rT(\vec{v}).
\end{aligned}$$

Therefore, T preserves scalar multiplication.

Proof. (continued)

Finally, the previous Theorem guarantees that T is unique: since B is a basis (and hence a spanning set), the action of T is completely determined by the fact that $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$. This completes the proof of the theorem. $\blacksquare$

Proof. (continued)

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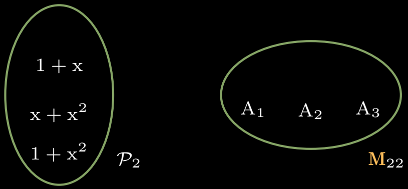
Remark

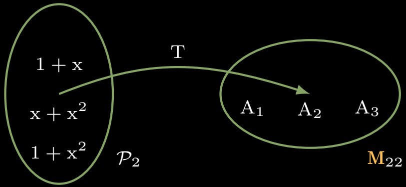
The significance of this Theorem is that it gives us the ability to define linear transformations between vector spaces, a useful tool in what follows.



Problem

$B = \{1 + x, x + x^2, 1 + x^2\}$ is a basis of $\mathcal{P}_2$. Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$



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Find a linear transformation $T : \mathcal{P}_2 \to \mathbf{M}_{22}$ so the

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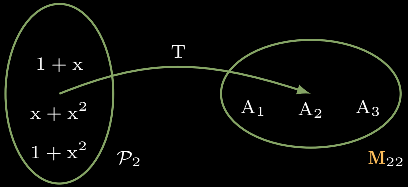

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Find a linear transformation $T : \mathcal{P}_2 \to \mathbf{M}_{22}$ so the

$$T(1+x) = A_1, \quad T(x+x^2) = A_2, \quad \text{and} \quad T(1+x^2) = A_3,$$

by specifying $T(a+bx+cx^2)$ for any $a+bx+cx^2 \in \mathcal{P}_2$.

Solution

Notice that $(1+x)+(x+x^2)-(1+x^2)=2x$, and thus

$$\begin{aligned} x &= \tfrac{1}{2}(1+x)+\tfrac{1}{2}(x+x^2)-\tfrac{1}{2}(1+x^2), \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} T(x) &= \tfrac{1}{2}T(1+x)+\tfrac{1}{2}T(x+x^2)-\tfrac{1}{2}T(1+x^2) \\ &= \tfrac{1}{2}A_1+\tfrac{1}{2}A_2-\tfrac{1}{2}A_3 \\ &= \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+\tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}-\tfrac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

Solution (continued)

Next, $1 = (1 + x) - x$, so $T(1) = T(1 + x) - T(x)$, and thus

$$T(1) = A_1 - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

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Finally, $x^2 = (x + x^2) - x$, so $T(x^2) = T(x + x^2) - T(x)$, and thus

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Therefore,

$$\begin{aligned} T(a+bx+cx^2) &= aT(1) + bT(x) + cT(x^2) \\ &= \frac{a}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{b}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{c}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} a+b-c & -a+b+c \\ -a+b+c & a-b+c \end{bmatrix}. \end{aligned}$$

■

Solution (Two – sketch)

Since the set $\{1+x, x+x^2, 1+x^2\}$ is a basis of $\mathcal{P}_2$, there exits unique representation:

$$\begin{aligned} a + bx + cx^2 =& \ell_1(1+x) + \ell_2(x+x^2) + \ell_3(1+x^2) \\ =& (\ell_1+\ell_3) + (\ell_1+\ell_2)x + (\ell_2+\ell_3)x^2 \end{aligned}$$

$$\Downarrow$$

$$\begin{cases} \ell_1 + \ell_3 = a \\ \ell_1 + \ell_2 = b \\ \ell_2 + \ell_3 = c \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \ell_1 = \frac{1}{2}(a+b-c) \\ \ell_2 = \frac{1}{2}(-a+b+c) \\ \ell_3 = \frac{1}{2}(a-b-c) \end{cases}$$

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Solution (Two – continued)

Hence,

$$\mathrm{T}\left[\mathrm{a}+\mathrm{bx}+\mathrm{cx}^2\right]$$

$$\|$$

$$\mathrm{T}\left[\ell_1(1+\mathrm{x})+\ell_2(\mathrm{x}+\mathrm{x}^2)+\ell_3(1+\mathrm{x}^2)\right]$$

$$\|$$

$$\ell_1\mathrm{T}[1+\mathrm{x}]+\ell_2\mathrm{T}[\mathrm{x}+\mathrm{x}^2]+\ell_3\mathrm{T}[1+\mathrm{x}^2]$$

$$\|$$

$$\ell_1\left[\begin{array}{cc}1 & 0\\ 0 & 0\end{array}\right]+\ell_2\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]+\ell_3\left[\begin{array}{cc}0 & 0\\ 0 & 1\end{array}\right]$$

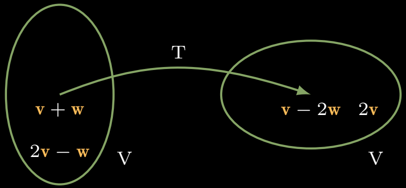
$$\|$$

$$\frac{1}{2}(\mathrm{a}+\mathrm{b}-\mathrm{c})\left[\begin{array}{cc}1 & 0\\ 0 & 0\end{array}\right]+\frac{1}{2}(-\mathrm{a}+\mathrm{b}+\mathrm{c})\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]+\frac{1}{2}(\mathrm{a}-\mathrm{b}+\mathrm{c})\left[\begin{array}{cc}0 & 0\\ 0 & 1\end{array}\right]$$

$$\|$$

$$=\frac{1}{2}\left[\begin{array}{cc}\mathrm{a}+\mathrm{b}-\mathrm{c} & -\mathrm{a}+\mathrm{b}+\mathrm{c}\\ -\mathrm{a}+\mathrm{b}+\mathrm{c} & \mathrm{a}-\mathrm{b}+\mathrm{c}\end{array}\right]$$

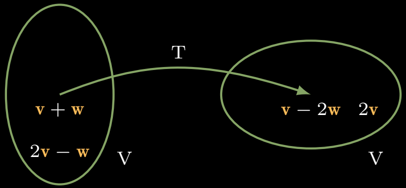
$\blacksquare$



Problem

Let V be a vector space, and T be a linear operator on V, and $\mathbf{v}, \mathbf{w} \in V$ such that

$$T(\mathbf{v} + \mathbf{w}) = \mathbf{v} - 2\mathbf{w} \quad \text{and} \quad T(2\mathbf{v} - \mathbf{w}) = 2\mathbf{v}.$$



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Find $\mathrm{T}(\mathbf{v})$ and $\mathrm{T}(\mathbf{w})$.

Solution

$$\begin{aligned} \mathrm{T}(\mathbf{v}) &= \mathrm{T}\left[\frac{1}{3}\left([\mathbf{v}+\mathbf{w}]+[2\mathbf{v}-\mathbf{w}]\right)\right] \\ &= \frac{1}{3}\mathrm{T}\left[\mathbf{v}+\mathbf{w}\right]+\frac{1}{3}\mathrm{T}\left[2\mathbf{v}-\mathbf{w}\right] \\ &= \frac{1}{3}\left(\mathbf{v}-2\mathbf{w}\right)+\frac{2}{3}\mathbf{v} \\ &= \mathbf{v}-\frac{2}{3}\mathbf{w}. \end{aligned}$$

Similarly, as an exercise, $\mathrm{T}(\mathbf{w}) = -\frac{4}{3}\mathbf{w}$. ■