Math 221: LINEAR ALGEBRA

Chapter 8. Orthogonality
§8-4. QR Factorization

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

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QR Factorization

Algorithm for the QR Factorization

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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QR Factorization

Algorithm for the QR Factorization

The QR Factorization

The QR Factorization

Definition

Let A be a real m × n matrix. Then a QR factorization of A can be written as

$$A = QR$$

where Q is an orthogonal matrix and R is an upper (or right) triangular matrix.

$$\overset{\mathbf{A}}{\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix}} = \underbrace{\overset{\mathbf{Q}}{\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix}}}_{\substack{\text{orthogonal} \\ \text{unit vector}}} \underbrace{\overset{\mathbf{R}}{\begin{bmatrix} \mathbf{e}_1^T \cdot \mathbf{a}_1 & \mathbf{e}_1^T \cdot \mathbf{a}_2 & \mathbf{e}_1^T \cdot \mathbf{a}_3 \\ 0 & \mathbf{e}_2^T \cdot \mathbf{a}_2 & \mathbf{e}_2^T \cdot \mathbf{a}_3 \\ 0 & 0 & \mathbf{e}_3^T \cdot \mathbf{a}_3 \end{bmatrix}}}_{\substack{\text{upper diagonal} \\ \text{matrix}}}$$

Theorem

Let A be a real $m \times n$ matrix with linearly independent columns. Then A can be written

$$A = QR$$

with Q orthogonal and R upper triangular with positive entries on the main diagonal.

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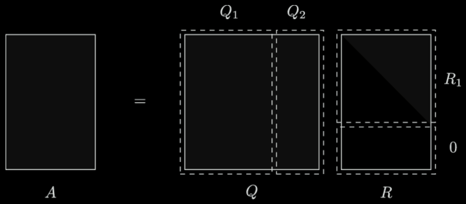
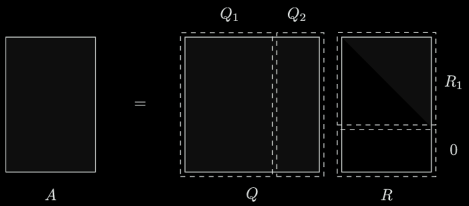
Proof.

Using columns of A to carry out the Gram-Schmidt algorithm to find an orthonormal basis for $\mathrm{im}(A)$ or $\mathrm{col}(A) \subseteq \mathbb{R}^m$ – columns of Q_1. One may further extend this basis to an orthonormal basis for the whole space $\mathbb{R}^m$ – columns of $Q = [Q_1, Q_2]$.



The Gram-Schmidt algorithm guarantees that the ith column of A is linear combinations of all jth columns of Q with $j = 1, \cdots, i$, which gives the upper triangular structure of R. ∎



Remark

$$A = QR = [Q_1, Q_2] \begin{bmatrix} R_1 \\ O \end{bmatrix} = Q_1 R_1 + Q_2 O = Q_1 R_1.$$

Both QR and $Q_1 R_1$ are called QR decompositions of A. The textbook refers $Q_1 R_1$.

Remark

Q is orthogonal matrix, namely, $QQ^T = Q^T Q = I_m$.
However, Q_1 is not orthogonal matrix (not a square matrix). But We have $Q_1^T Q_1 = I_n$ and $Q_1 Q_1^T \neq I_m$ (in general).

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QR Factorization

Algorithm for the QR Factorization

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Algorithm 2: QR Factorization Algorithm

Input : Independent columns of A: $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\} \in \mathrm{col}(A) \subseteq \mathbb{R}^m$

for $j \leftarrow 1$ **to** n **do**

$$\vec{f}_j \leftarrow \vec{c}_j - \frac{\vec{c}_j \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{c}_j \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots - \frac{\vec{c}_j \cdot \vec{f}_{j-1}}{||\vec{f}_{j-1}||^2}\vec{f}_{j-1}.$$

$$\vec{q}_j \leftarrow \frac{\vec{f}_j}{||\vec{f}_j||}$$

  for $i \leftarrow 1$ **to** j **do**

$$r_{ij} \leftarrow \vec{q}_i \cdot \vec{c}_j$$

  end

end

Output: $Q = [\vec{q}_1, \cdots, \vec{q}_n]$ and $R = [r_{ij}]$

Problem

Let

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the QR factorization of A.

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Solution

Set $A = [\vec{c}_1, \vec{c}_2]$. When $j = 1$,

$$\vec{f}_1 = \vec{c}_1 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{q}_1 = \frac{\vec{f}_1}{||\vec{f}_1||} = \begin{bmatrix} \frac{4}{\sqrt{20}} \\ \frac{2}{\sqrt{20}} \\ 0 \end{bmatrix}.$$

For $i = 1$,

$$r_{11} = \vec{q}_1 \cdot \vec{c}_1 = \frac{\vec{f}_1}{||\vec{f}_1||} \cdot \vec{f}_1 = ||\vec{f}_1|| = \sqrt{20}.$$

Solution (continued)

When j = 2,

$$\vec{f}_2 = \vec{c}_2 - \frac{\vec{c}_2 \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - \frac{10}{20}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{q}_2 = \frac{\vec{f}_2}{||\vec{f}_2||} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

For i = 1,

$$r_{12} = \vec{q}_1 \cdot \vec{c}_2 = \begin{bmatrix} \frac{4}{\sqrt{20}} \\ \frac{2}{\sqrt{20}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \sqrt{5}.$$

and for i = 2,

$$r_{22} = \vec{q}_2 \cdot \vec{c}_2 = \frac{\vec{f}_2}{||\vec{f}_2||} \cdot \left(\vec{f}_2 + \frac{\vec{c}_2 \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1\right) = \frac{\vec{f}_2}{||\vec{f}_2||} \cdot \vec{f}_2 = ||\vec{f}_2|| = \sqrt{6}.$$

Solution (continued)

Therefore,

$$A = QR = [\vec{q}_1, \vec{q}_2] \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{20} & \sqrt{5} \\ 0 & \sqrt{6} \end{bmatrix}$$

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