Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-7. LU Factorization

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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Definition

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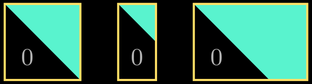

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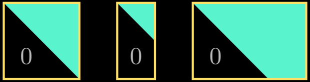

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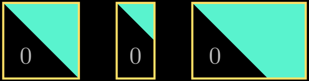

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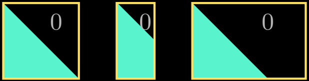

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$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

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Consider the following reduction:

$$\begin{aligned} A\vec{x} &= B \\ (LU)\vec{x} &= B \\ L(U\vec{x}) &= B \\ L\vec{y} &= B \end{aligned}$$

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Therefore, if we can solve $L\vec{y} = B$ for $\vec{y}$, then all that remains is to solve $U\vec{x} = \vec{y}$ for $\vec{x}$.

Example

Find all solutions to

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 10 & 5 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

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Solution

Using a method of your choice, verify that the LU factorization of A gives

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

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Now we solve $U\vec{x} = \vec{y}$.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

Solution (continued)

Multiplying and solving (or finding the reduced row-echelon form), the general solution is given by

$$\vec{x} = \begin{bmatrix} -12 \\ 2 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 13 \\ -3 \\ -2 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

$\blacksquare$

Finding the LU Factorization

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Condition for the existence of LU factorization: A matrix A has LU factorization provided that A can be lower reduced, namely, the row-echelon form of A can be calculated without interchanging rows.

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Determine if the LU factorization of A exists, and if so, find it.

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Because the row-echelon form can be obtained without interchanging rows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_3 + r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

the LU factorization exists, or A can be lower reduced.

Solution (continued)

We proceed to finding L and U. Assign variables to the unknown entries and multiply.

$$
\begin{aligned}
A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \\
&= \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}
\end{aligned}
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&= \quad \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}
\end{aligned}
$$

Solving each entry will give us values for the unknown entries.

Solution (continued)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{e} \\ \mathrm{ax} & \mathrm{dx} + \mathrm{b} & \mathrm{ex} + \mathrm{f} \\ \mathrm{ay} & \mathrm{dy} + \mathrm{bz} & \mathrm{ey} + \mathrm{fz} + \mathrm{c} \end{bmatrix}$$

Solution (continued)

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We see easily that $a = 1, d = 1$, and $e = 2$. Continuing to solve the first column gives $x = 2, y = 1$. The other values are calculated as follows.

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$$\begin{aligned} dx + b &= 3 \\ (1)(2) + b &= 3 \\ b &= 1 \end{aligned}$$

$$\begin{aligned} ex + f &= 0 \\ (2)(2) + f &= 0 \\ f &= -4 \end{aligned}$$

$$\begin{aligned} dy + bz &= 0 \\ (1)(1) + (1)z &= 0 \\ z &= -1 \end{aligned}$$

$$\begin{aligned} ey + fz + c &= 5 \\ (2)(1) + (-4)(-1) + c &= 5 \\ c &= -1 \end{aligned}$$

Solution (continued)

Therefore,

$$\begin{array}{c} L \\ \| \\ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \\ \| \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \end{array}$$

Solution (continued)

Therefore,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

$\blacksquare$

Remark

If you want the diagonal terms of U to be all 1's:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\|$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -0 \\ 2 & 1 & -0 \\ 1 & -1 & -1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ -0 & -0 & 1 \end{bmatrix}}_{U}$$

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Find the LU factorization of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

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First, write A as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

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Solution (continued)

To do so, we use row operations to remove the entries of A below the main diagonal. For every operation we apply to A (the matrix on the right), we apply the inverse operation to the identity matrix (on the left). This ensures the product remains the same.

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The first step is to add (-2) times the first row of A to the second row. To preserve the product, add (2) times the second column to the first column, for the matrix on the left.

$$\|$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\|$$

$$c_1 + 2c_2 \to c_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} \quad r_2 - 2r_1 \to r_2$$

Solution (continued)

We proceed in the same way.

$$c_1 + c_3 \to c_1 \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} \quad r_3 - r_1 \to r_3$$

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At this point we have a lower triangular matrix L on the left, and an upper triangular matrix U on the right so we are done. You can (and should!) check that this product equals A.

If you want the diagonal terms of U to be all 1's:

$$-1 \times c_3 \to c_3 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} -1 \times r_3 \to r_3$$

∎

Problem

Use the multiplier method to verify the LU factorization for

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix}$$

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Solution

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

■

Theorem (LU-Algorithm)

Let A be an $m \times n$ matrix of rank r, and suppose that A can be lower reduced to a row-echelon matrix U. Then $A = LU$ where the lower triangular, invertible matrix L is constructed as follows:

1. If $A = 0$, take $L = I_m$ and $U = 0$.
2. If $A \neq 0$, write $A_1 = A$ and let $\vec{c}_1$ be the leading column of A_1. Use $\vec{c}_1$ to create the first leading 1 and make its below all zeros. When this is completed, let A_2 denote the matrix consisting of rows 2 to m of the matrix just created.
3. If $A_2 \neq 0$, let $\vec{c}_2$ be the leading column of A_2 and repeat Step 2 on A_2 to create A_3.
4. Continue in this way until U is reached, where all rows below the last leading 1 consist of zeros. This will happen after r steps.
5. Create L by placing $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_r$ at the bottom of the first r columns of I_m.

Problem

Find an LU-factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

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$$\begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -1 \end{matrix}} & \begin{matrix} 4 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 2 \\ 2 \\ 2 \end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} & \begin{matrix} 1 \\ 3 \end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \boxed{5} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 5 \end{bmatrix} = L$$

■

Problem

Find an LU-factorization for $A = \begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 5 & -5 & 10 & 0 & 5 \\ -3 & 3 & 2 & 2 & 1 \\ -2 & 2 & 0 & -1 & 0 \\ 1 & -1 & 10 & 2 & 5 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 8 & 2 & 4 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 8 & 2 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$||$$

$$U$$

Solution

$$\begin{bmatrix} \boxed{\begin{matrix} 5 \\ -3 \\ -2 \\ 1 \end{matrix}} & \begin{matrix} -5 & 10 & 0 & 5 \\ 3 & 2 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ -1 & 10 & 2 & 5 \end{matrix} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & \boxed{8} & 2 & 4 \\ 0 & 0 & \boxed{4} & -1 & 2 \\ 0 & 0 & \boxed{8} & 2 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & \boxed{-2} & 0 \\ 0 & 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\Downarrow$$

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 \\ -3 & 8 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & 8 & 0 & 1 \end{bmatrix}$$

$\blacksquare$