Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-3. Orthogonality

Le Chen[1]

Emory University, 2021 Spring

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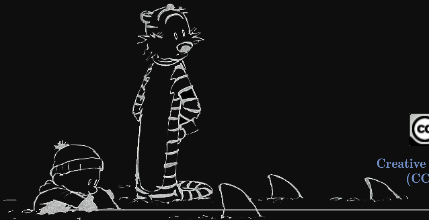

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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Dot Product

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Definitions

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$.

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1. The dot product of $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \vec{x}^T \vec{y}.$$

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2. The length or norm of $\vec{x}$, denoted $||\vec{x}||$ is

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 \cdots + x_n^2} = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^T\vec{x}}.$$

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3. $\vec{x}$ is called a unit vector if $||\vec{x}|| = 1$.

Theorem (Properties of length and the dot product)

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6. $||a\vec{x}|| = |a| \; ||\vec{x}||$.

Example

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$\begin{aligned}
||\vec{x}+\vec{y}||^2 &= (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y} \\
&= \vec{x}\cdot\vec{x} + 2(\vec{x}\cdot\vec{y}) + \vec{y}\cdot\vec{y} \\
&= ||\vec{x}||^2 + 2(\vec{x}\cdot\vec{y}) + ||\vec{y}||^2.
\end{aligned}$$

Problem

Let $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\} \in \mathbb{R}^n$ and suppose $\mathbb{R}^n = \mathrm{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\}$. Furthermore, suppose that there exists a vector $\vec{x} \in \mathbb{R}^n$ for which $\vec{x} \cdot \vec{f}_j = 0$ for all j, $1 \leq j \leq k$. Show that $\vec{x} = \vec{0}_n$.

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Proof.

Write $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + \cdots + t_k\vec{f}_k$ for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$ (this is possible because $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k$ span $\mathbb{R}^n$, is this representation unique?).

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$$||\vec{x}||^2 = \vec{x} \cdot \vec{x}$$

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$$\begin{aligned} ||\vec{x}||^2 &= \vec{x} \cdot \vec{x} \\ &= \vec{x} \cdot (t_1\vec{f}_1 + t_2\vec{f}_2 + \cdots + t_k\vec{f}_k) \end{aligned}$$

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&= t_1(\vec{x} \cdot \vec{f}_1) + t_2(\vec{x} \cdot \vec{f}_2) + \cdots + t_k(\vec{x} \cdot \vec{f}_k) \\
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&= t_1(0) + t_2(0) + \cdots + t_k(0) = 0.
\end{aligned}$$

Since $||\vec{x}||^2 = 0$, $||\vec{x}|| = 0$. By the previous theorem, $||\vec{x}|| = 0$ if and only if $\vec{x} = \vec{0}_n$. Therefore, $\vec{x} = \vec{0}_n$. ■

Cauchy-Schwartz Inequality

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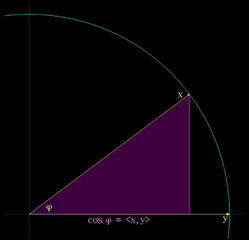
Theorem (Cauchy-Schwartz Inequality)

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||\ ||\vec{y}||$ with equality if and only if $\{\vec{x}, \vec{y}\}$ is linearly dependent.



$$\left| \frac{\vec{x}}{||\vec{x}||} \cdot \frac{\vec{y}}{||\vec{y}||} \right| \leq 1$$

$$\{\vec{x}, \vec{y}\} \text{ is linearly dependent} \quad \Leftrightarrow \quad \vec{x} = t\vec{y}, \quad \text{for some } t \in \mathbb{R}.$$

Proof.

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then

$$\begin{aligned} 0 \leq ||t\vec{x} + \vec{y}||^2 &= (t\vec{x} + \vec{y}) \cdot (t\vec{x} + \vec{y}) \\ &= t^2 \vec{x} \cdot \vec{x} + 2t\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\ &= t^2 ||\vec{x}||^2 + 2t(\vec{x} \cdot \vec{y}) + ||\vec{y}||^2. \end{aligned}$$

The quadratic $t^2||\vec{x}||^2 + 2t(\vec{x} \cdot \vec{y}) + ||\vec{y}||^2$ in t is always nonnegative, so it does not have distinct real roots. Thus, if we use the quadratic formula to solve for t, the discriminant must be non-positive, i.e.,

$$\Delta = (2\vec{x} \cdot \vec{y})^2 - 4||\vec{x}||^2||\vec{y}||^2 \leq 0$$

Therefore, $(2\vec{x} \cdot \vec{y})^2 \leq 4||\vec{x}||^2||\vec{y}||^2$. Since both sides of the inequality are nonnegative, we can take (positive) square roots of both sides:

$$|2\vec{x} \cdot \vec{y}| \leq 2||\vec{x}||\ ||\vec{y}||$$

Therefore, $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||\ ||\vec{y}||$. What remains is to show that $|\vec{x} \cdot \vec{y}| = ||\vec{x}||\ ||\vec{y}||$ if and only if $\{\vec{x}, \vec{y}\}$ is linearly dependent. $\blacksquare$

Proof. (continued)

First suppose that $\{\vec{x}, \vec{y}\}$ is dependent. Then by symmetry (of $\vec{x}$ and $\vec{y}$), $\vec{x} = k\vec{y}$ for some $k \in \mathbb{R}$. Hence

$$|\vec{x}\cdot\vec{y}| = |(k\vec{y})\cdot\vec{y}| = |k|\,|\vec{y}\cdot\vec{y}| = |k|\,||\vec{y}||^2, \quad \text{and} \quad ||\vec{x}||\,||\vec{y}|| = ||k\vec{y}||\,||\vec{y}|| = |k|\,||\vec{y}||^2,$$

so $|\vec{x}\cdot\vec{y}| = ||\vec{x}||\,||\vec{y}||$.

Conversely, suppose $\{\vec{x}, \vec{y}\}$ is independent; then $t\vec{x} + \vec{y} \neq \vec{0}_n$ for all $t \in \mathbb{R}$, so $||t\vec{x} + \vec{y}||^2 > 0$ for all $t \in \mathbb{R}$. Thus the quadratic

$$t^2||\vec{x}||^2 + 2t(\vec{x}\cdot\vec{y}) + ||\vec{y}||^2 > 0$$

so has no real roots. It follows that the the discriminant is negative, i.e.,

$$(2\vec{x}\cdot\vec{y})^2 - 4||\vec{x}||^2||\vec{y}||^2 < 0.$$

Therefore, $(2\vec{x}\cdot\vec{y})^2 < 4||\vec{x}||^2||\vec{y}||^2$; taking square roots of both sides (they are both nonnegative) and dividing by two gives us

$$|\vec{x}\cdot\vec{y}| < ||\vec{x}||\,||\vec{y}||,$$

showing that equality is impossible. ■

Corollary (Triangle Inequality I)

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$.

Corollary (Triangle Inequality I)

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$.

Proof.

$$||\vec{x}+\vec{y}||^2 \quad = \quad (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y})$$

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$$\begin{aligned} ||\vec{x} + \vec{y}||^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \end{aligned}$$

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&\leq ||\vec{x}||^2 + 2||\vec{x}||\ ||\vec{y}|| + ||\vec{y}||^2 \text{ by the Cauchy Inequality}
\end{aligned}$$

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&= (||\vec{x}|| + ||\vec{y}||)^2.
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&\leq ||\vec{x}||^2 + 2||\vec{x}|| \, ||\vec{y}|| + ||\vec{y}||^2 \text{ by the Cauchy Inequality} \\
&= (||\vec{x}|| + ||\vec{y}||)^2.
\end{aligned}$$

Since both sides of the inequality are nonnegative, we take (positive) square roots of both sides:

$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||.$$

∎

Definition

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the distance between $\vec{x}$ and $\vec{y}$ is defined as

$$d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||.$$

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Proof. (Proof of the Triangle Inequality II)

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$\blacksquare$

Orthogonality

Orthogonality

Definitions

- Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. We say that two vectors $\vec{x}$ and $\vec{y}$ are orthogonal if $\vec{x} \cdot \vec{y} = 0$.

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- More generally, $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is an orthogonal set if each $\vec{x}_i$ is nonzero, and every pair of distinct vectors of X is orthogonal, i.e., $\vec{x}_i \cdot \vec{x}_j = 0$ for all $i \neq j$, $1 \leq i, j \leq k$.

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- A set $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is an orthonormal set if X is an orthogonal set of unit vectors, i.e., $||\vec{x}_i|| = 1$ for all i, $1 \leq i \leq k$.

Examples

1. The standard basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$ of $\mathbb{R}^n$ is an orthonormal set (and hence an orthogonal set).

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3. If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is an orthogonal subset of $\mathbb{R}^n$ and $p \neq 0$, then $\{p\vec{x}_1, p\vec{x}_2, \ldots, p\vec{x}_k\}$ is an orthogonal subset of $\mathbb{R}^n$.

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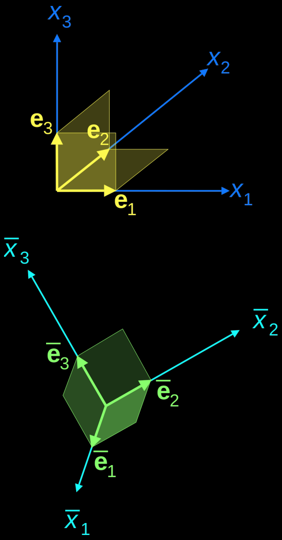
4. $$\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$
   is an orthonormal subset of $\mathbb{R}^4$.

Definition

Normalizing an orthogonal set is the process of turning an orthogonal (but not orthonormal) set into an orthonormal set. If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is an orthogonal subset of $\mathbb{R}^n$, then

$$\left\{ \frac{1}{||\vec{x}_1||}\vec{x}_1, \frac{1}{||\vec{x}_2||}\vec{x}_2, \ldots, \frac{1}{||\vec{x}_k||}\vec{x}_k \right\}$$

is an orthonormal set.



x_3
x_2
$\mathbf{e}_3$
$\mathbf{e}_2$
x_1
$\mathbf{e}_1$
$\bar{x}_3$
$\bar{x}_2$
$\bar{\mathbf{e}}_3$
$\bar{\mathbf{e}}_2$
$\bar{\mathbf{e}}_1$
$\bar{x}_1$

Problem

Verify that

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \right\}$$

is an orthogonal set, and normalize this set.

Solution

$$
\begin{aligned}
\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} &= 0 - 2 + 2 = 0, \\
\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} &= 0 + 2 - 2 = 0, \\
\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} &= 5 - 1 - 4 = 0,
\end{aligned}
$$

proving that the set is orthogonal. Normalizing gives us the orthonormal set

$$
\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \right\}.
$$

$\blacksquare$

Theorem (Pythagoras' Theorem)

If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is orthogonal, then

$$||\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k||^2 = ||\vec{x}_1||^2 + ||\vec{x}_2||^2 + \cdots + ||\vec{x}_k||^2.$$

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Proof.

Start with

$$||\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k||^2 \quad = \quad (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k) \cdot (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k)$$

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$$\begin{aligned}
||\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k||^2 &= (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k) \cdot (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k) \\
&= \ \ (\vec{x}_1 \cdot \vec{x}_1 + \vec{x}_1 \cdot \vec{x}_2 + \cdots + \vec{x}_1 \cdot \vec{x}_k) \\
&\quad + (\vec{x}_2 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 + \cdots + \vec{x}_2 \cdot \vec{x}_k) \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad + (\vec{x}_k \cdot \vec{x}_1 + \vec{x}_k \cdot \vec{x}_2 + \cdots + \vec{x}_k \cdot \vec{x}_k) \\
&= \vec{x}_1 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 + \cdots + \vec{x}_k \cdot \vec{x}_k
\end{aligned}$$

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$$\begin{array}{rcl}
||\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k||^2 & = & (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k) \cdot (\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k) \\
& = & \quad (\vec{x}_1 \cdot \vec{x}_1 + \vec{x}_1 \cdot \vec{x}_2 + \cdots + \vec{x}_1 \cdot \vec{x}_k) \\
& & +(\vec{x}_2 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 + \cdots + \vec{x}_2 \cdot \vec{x}_k) \\
& & \quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& & +(\vec{x}_k \cdot \vec{x}_1 + \vec{x}_k \cdot \vec{x}_2 + \cdots + \vec{x}_k \cdot \vec{x}_k) \\
& = & \vec{x}_1 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 + \cdots + \vec{x}_k \cdot \vec{x}_k \\
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&= ||\vec{x}_1||^2 + ||\vec{x}_2||^2 + \cdots + ||\vec{x}_k||^2.
\end{aligned}$$

The second last equality follows from the fact that the set is orthogonal, so for all i and j, $i \neq j$ and $1 \leq i, j \leq k$, $\vec{x}_i \cdot \vec{x}_j = 0$. Thus, the only nonzero terms are the ones of the form $\vec{x}_i \cdot \vec{x}_i$, $1 \leq i \leq k$. ■

Orthogonality and Independence

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If $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is an orthogonal set, then S is independent.

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Form the linear equation: $t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}$. We need to check whether there is only trivial solution.

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Form the linear equation: $t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}$. We need to check whether there is only trivial solution. Notice that for all i, $1 \leq i \leq k$,

$$0 = (t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k) \cdot \vec{x}_i = t_i\vec{x}_i \cdot \vec{x}_i = t_i||\vec{x}_i||^2,$$

since $t_j\vec{x}_j \cdot \vec{x}_i = 0$ for all j, $1 \leq j \leq k$ where $j \neq i$.

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since $t_j\vec{x}_j \cdot \vec{x}_i = 0$ for all j, $1 \leq j \leq k$ where $j \neq i$. Since $\vec{x}_i \neq \vec{0}_n$ and $t_i||\vec{x}_i||^2 = 0$, it follows that $t_i = 0$ for all i, $1 \leq i \leq k$. Therefore, S is linearly independent. ∎

Example

Given an arbitrary vector

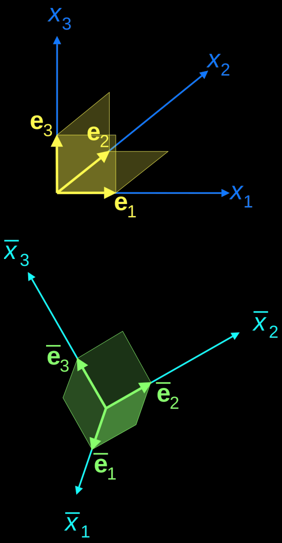
$$\vec{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n,$$

it is trivial to express $\vec{x}$ as a linear combination of the standard basis vectors of $\mathbb{R}^n$, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$:

$$\vec{x} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n.$$

Problem

Given any orthogonal basis B of $\mathbb{R}^n$ (so not necessarily the standard basis), and an arbitrary vector $\vec{x} \in \mathbb{R}^n$, how do we express $\vec{x}$ as a linear combination of the vectors in B?



Copyright

The Dot Product

The Cauchy Inequality

Orthogonality

Orthogonality and Independence

Fourier Expansion

Fourier Expansion

Fourier Expansion

Theorem (Fourier Expansion)

Let $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ be an orthogonal basis of a subspace U of $\mathbb{R}^n$. Then for any $\vec{x} \in U$,

$$\vec{x} = \left(\frac{\vec{x}\cdot\vec{f}_1}{||\vec{f}_1||^2}\right)\vec{f}_1 + \left(\frac{\vec{x}\cdot\vec{f}_2}{||\vec{f}_2||^2}\right)\vec{f}_2 + \cdots + \left(\frac{\vec{x}\cdot\vec{f}_m}{||\vec{f}_m||^2}\right)\vec{f}_m.$$

This expression is called the Fourier expansion of $\vec{x}$, and

$$\frac{\vec{x}\cdot\vec{f}_j}{||\vec{f}_j||^2}, \quad j = 1, 2, \ldots, m$$

are called the Fourier coefficients.

Example

Let $\vec{f}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{f}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{f}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$, and let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

We have seen that $B = \{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ is an orthogonal subset of $\mathbb{R}^3$.

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To express $\vec{x}$ as a linear combination of the vectors of B, apply the Fourier Expansion Theorem. Assume $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + t_3\vec{f}_3$.

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$$t_1 = \frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2} = \frac{2}{6}, \quad t_2 = \frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2} = \frac{3}{5}, \quad \text{and} \quad t_3 = \frac{\vec{x} \cdot \vec{f}_3}{||\vec{f}_3||^2} = \frac{4}{30}.$$

Example

Let $\vec{f}_1 = \left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right]$, $\vec{f}_2 = \left[\begin{array}{r} 0 \\ 2 \\ 1 \end{array}\right]$, and $\vec{f}_3 = \left[\begin{array}{r} 5 \\ 1 \\ -2 \end{array}\right]$, and let $\vec{x} = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right]$.
We have seen that $B = \{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ is an orthogonal subset of $\mathbb{R}^3$.

It follows that B is an orthogonal basis of $\mathbb{R}^3$. (Why?)

To express $\vec{x}$ as a linear combination of the vectors of B, apply the Fourier Expansion Theorem. Assume $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + t_3\vec{f}_3$. Then

$$t_1 = \frac{\vec{x}\cdot\vec{f}_1}{||\vec{f}_1||^2} = \frac{2}{6}, \quad t_2 = \frac{\vec{x}\cdot\vec{f}_2}{||\vec{f}_2||^2} = \frac{3}{5}, \quad \text{and} \quad t_3 = \frac{\vec{x}\cdot\vec{f}_3}{||\vec{f}_3||^2} = \frac{4}{30}.$$

Therefore,

$$\left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] = \frac{1}{3}\left[\begin{array}{r} 1 \\ -1 \\ 2 \end{array}\right] + \frac{3}{5}\left[\begin{array}{r} 0 \\ 2 \\ 1 \end{array}\right] + \frac{2}{15}\left[\begin{array}{r} 5 \\ 1 \\ -2 \end{array}\right].$$

Proof. (Fourier Expansion)

Let $\vec{x} \in U$. Since $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ is a basis of U, $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + \cdots + t_m\vec{f}_m$ for some $t_1, t_2, \ldots, t_m \in \mathbb{R}$.

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$$\vec{x} \cdot \vec{f}_i = (t_1\vec{f}_1 + t_2\vec{f}_2 + \cdots + t_m\vec{f}_m) \cdot \vec{f}_i$$

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\vec{x} \cdot \vec{f}_i &= (t_1\vec{f}_1 + t_2\vec{f}_2 + \cdots + t_m\vec{f}_m) \cdot \vec{f}_i \\
&= t_i\vec{f}_i \cdot \vec{f}_i \quad \text{since } \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\} \text{ is orthogonal} \\
&= t_i||\vec{f}_i||^2.
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Since $\vec{f}_i$ is nonzero, we obtain

$$t_i = \frac{\vec{x} \cdot \vec{f}_i}{||\vec{f}_i||^2}.$$

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Remark

If $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ is an orthonormal basis, then the Fourier coefficients are simply $t_j = \vec{x} \cdot \vec{f}_j$, $j = 1, 2, \ldots, m$.

Problem

$$\text{Let}\quad \vec{f}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{f}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{f}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{f}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Show that $B = \{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4\}$ is an orthogonal basis of $\mathbb{R}^4$, and express $\vec{x} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ as a linear combination of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ and $\vec{f}_4$.

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Solution

Computing $\vec{f}_i \cdot \vec{f}_j$ for $1 \leq i < j \leq 4$ gives us

$$\begin{array}{lllllllll} \vec{f}_1 \cdot \vec{f}_2 & = & 0, & \vec{f}_1 \cdot \vec{f}_3 & = & 0, & \vec{f}_1 \cdot \vec{f}_4 & = & 0, \\ \vec{f}_2 \cdot \vec{f}_3 & = & 0, & \vec{f}_2 \cdot \vec{f}_4 & = & 0, & \vec{f}_3 \cdot \vec{f}_4 & = & 0. \end{array}$$

Hence, B is an orthogonal set.

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Hence, B is an orthogonal set. It follows that B is independent, and since $|B| = 4 = \dim(\mathbb{R}^4)$, B also spans $\mathbb{R}^4$. Therefore, B is an orthogonal basis of $\mathbb{R}^4$.

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Hence, B is an orthogonal set. It follows that B is independent, and since $|B| = 4 = \dim(\mathbb{R}^4)$, B also spans $\mathbb{R}^4$. Therefore, B is an orthogonal basis of $\mathbb{R}^4$. By the Fourier Expansion Theorem,

$$\vec{x} = \left(\frac{a+b}{2}\right)\vec{f}_1 + \left(\frac{a-b}{2}\right)\vec{f}_2 + \left(\frac{c+d}{2}\right)\vec{f}_3 + \left(\frac{c-d}{2}\right)\vec{f}_4.$$

$\blacksquare$