

# Math 221: LINEAR ALGEBRA

## Chapter 3. Determinants and Diagonalization

### §3-3. Diagonalization and Eigenvalues

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.



# Linear Algebra with Applications

## Lecture Notes

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Why Diagonalization?

Eigenvalues and Eigenvectors

Geometric Interpretation of Eigenvalues and Eigenvectors

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## Why Diagonalization?

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# Why Diagonalization?

## Example

Let  $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ . Find  $A^{100}$ .

How can we do this efficiently?

Consider the matrix  $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ . Observe that  $P$  is invertible (why?), and that

$$P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Furthermore,

$$P^{-1}AP = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D,$$

where  $D$  is a **diagonal** matrix.

## Example (continued)

This is significant, because

$$\begin{aligned}P^{-1}AP &= D \\P(P^{-1}AP)P^{-1} &= PDP^{-1} \\(PP^{-1})A(PP^{-1}) &= PDP^{-1} \\IAI &= PDP^{-1} \\A &= PDP^{-1},\end{aligned}$$

and so

$$\begin{aligned}A^{100} &= (PDP^{-1})^{100} \\&= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) \\&= PD(P^{-1}P)D(P^{-1}P)D(P^{-1} \cdots P)DP^{-1} \\&= PDIDIDI \cdots IDP^{-1} \\&= PD^{100}P^{-1}.\end{aligned}$$

## Example (continued)

Now,

$$D^{100} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^{100} &= P D^{100} P^{-1} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2 \cdot 2^{100} + 5^{100} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2^{101} + 5^{100} \end{bmatrix}. \end{aligned}$$





## Theorem (Diagonalization and Matrix Powers)

If  $A = PDP^{-1}$ , then  $A^k = PD^kP^{-1}$  for each  $k = 1, 2, 3, \dots$

The process of finding an **invertible** matrix  $P$  and a **diagonal** matrix  $D$  so that  $A = PDP^{-1}$  is referred to as **diagonalizing** the matrix  $A$ , and  $P$  is called the **diagonalizing** matrix for  $A$ .

## Problem

- ▶ When is it possible to diagonalize a matrix?
- ▶ How do we find a diagonalizing matrix?

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Why Diagonalization?

**Eigenvalues and Eigenvectors**

Geometric Interpretation of Eigenvalues and Eigenvectors

Diagonalization

Linear Dynamical Systems

# Eigenvalues and Eigenvectors

## Definition

Let  $A$  be an  $n \times n$  matrix,  $\lambda$  a real number, and  $\vec{x} \neq \vec{0}$  an  $n$ -vector. If  $A\vec{x} = \lambda\vec{x}$ , then  $\lambda$  is an **eigenvalue** of  $A$ , and  $\vec{x}$  is an **eigenvector** of  $A$  corresponding to  $\lambda$ , or a  **$\lambda$ -eigenvector**.

## Example

Let  $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Then

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\vec{x}.$$

This means that 3 is an **eigenvalue** of  $A$ , and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an **eigenvector of  $A$  corresponding to 3** (or a 3-eigenvector of  $A$ ).

Suppose that  $A$  is an  $n \times n$  matrix,  $\vec{x} \neq \vec{0}$  an  $n$ -vector,  $\lambda \in \mathbb{R}$ , and that  $A\vec{x} = \lambda\vec{x}$ .

Then

$$\begin{aligned}\lambda\vec{x} - A\vec{x} &= \vec{0} \\ \lambda I\vec{x} - A\vec{x} &= \vec{0} \\ (\lambda I - A)\vec{x} &= \vec{0}\end{aligned}$$

Since  $\vec{x} \neq \vec{0}$ , the matrix  $\lambda I - A$  has no inverse, and thus

$$\det(\lambda I - A) = 0.$$

## Definition

The **characteristic polynomial** of an  $n \times n$  matrix  $A$  is

$$c_A(x) = \det(xI - A).$$

## Example

The characteristic polynomial of  $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$  is

$$\begin{aligned} c_A(x) &= \det \left( \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3) - 2 \\ &= x^2 - 7x + 10 \end{aligned}$$

## Theorem (Eigenvalues and Eigenvectors of a Matrix)

Let  $A$  be an  $n \times n$  matrix.

1. The eigenvalues of  $A$  are the **roots** of  $c_A(x)$ .
2. The  $\lambda$ -eigenvectors  $\vec{x}$  are the **nontrivial solutions** to  $(\lambda I - A)\vec{x} = \vec{0}$ .

### Example (continued)

For  $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ , we have

$$c_A(x) = x^2 - 7x + 10 = (x - 2)(x - 5),$$

so  $A$  has eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = 5$ .

To find the 2-eigenvectors of  $A$ , solve  $(2I - A)\vec{x} = \vec{0}$ :

$$\left[ \begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

### Example (continued)

The general solution, in parametric form, is

$$\vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

To find the 5-eigenvectors of A, solve  $(5I - A)\vec{x} = \vec{0}$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

The general solution, in parametric form, is

$$\vec{x} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R}.$$

## Definition

A **basic eigenvector** of an  $n \times n$  matrix  $A$  is any nonzero multiple of a basic solution to  $(\lambda I - A)\vec{x} = \vec{0}$ , where  $\lambda$  is an eigenvalue of  $A$ .

## Example (continued)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$  are basic eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

corresponding to eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = 5$ , respectively.



## Problem

For  $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$ , find  $c_A(x)$ , the eigenvalues of  $A$ , and the corresponding basic eigenvectors.

## Solution

$$\begin{aligned} \det(xI - A) &= \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix} \\ &= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3) \begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix} \\ &= (x-3)(x^2 - 3x + 2) = (x-3)(x-2)(x-1) = c_A(x). \end{aligned}$$

### Solution (continued)

Therefore, the eigenvalues of A are  $\lambda_1 = 3$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = 1$ .

Basic eigenvectors corresponding to  $\lambda_1 = 3$ : solve  $(3\mathbf{I} - \mathbf{A})\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Thus } \vec{x} = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

Choosing  $t = 2$  gives us  $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$  as a basic eigenvector corresponding to  $\lambda_1 = 3$ .

### Solution (continued)

Basic eigenvectors corresponding to  $\lambda_2 = 2$ : solve  $(2\mathbf{I} - \mathbf{A})\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Thus } \vec{x} = \begin{bmatrix} 2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

Choosing  $s = 1$  gives us  $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$  as a basic eigenvector corresponding to  $\lambda_2 = 2$ .

### Solution (continued)

Basic eigenvectors corresponding to  $\lambda_3 = 1$ : solve  $(I - A)\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Thus } \vec{x} = \begin{bmatrix} r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, r \in \mathbb{R}.$$

Choosing  $r = 1$  gives us  $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  as a basic eigenvector corresponding to  $\lambda_3 = 1$ . ■

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Why Diagonalization?

Eigenvalues and Eigenvectors

**Geometric Interpretation of Eigenvalues and Eigenvectors**

Diagonalization

Linear Dynamical Systems

# Geometric Interpretation of Eigenvalues and Eigenvectors

Let  $A$  be a  $2 \times 2$  matrix. Then  $A$  can be interpreted as a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ .

## Problem

How does the linear transformation affect the eigenvectors of the matrix?

## Definition

Let  $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$  be a nonzero vector in  $\mathbb{R}^2$ . Then  $L_{\vec{v}}$  is the set of all scalar multiples of  $\vec{v}$ , i.e.,

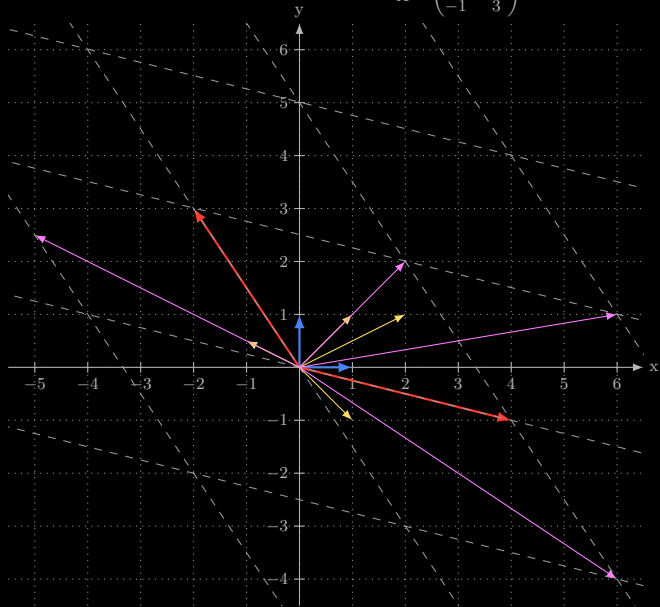
$$L_{\vec{v}} = \mathbb{R}\vec{v} = \{t\vec{v} \mid t \in \mathbb{R}\}.$$

Example (revisited)

$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$  has two eigenvalues:  $\lambda_1 = 2$  and  $\lambda_2 = 5$  with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$





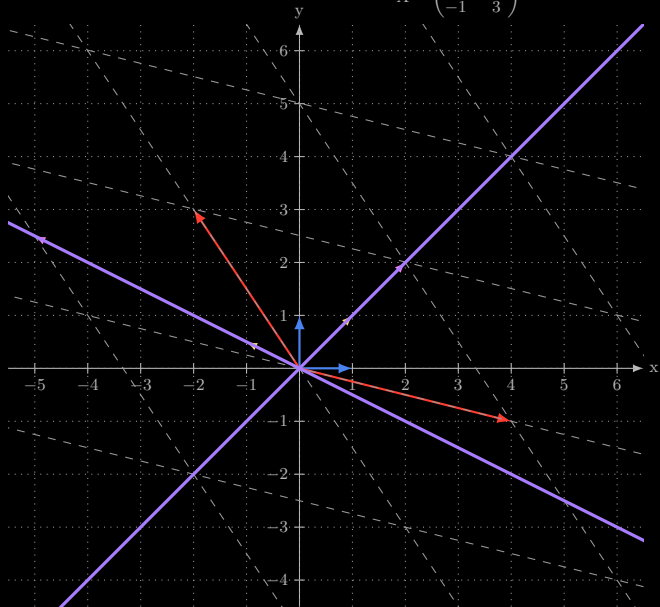
## Definition

Let  $A$  be a  $2 \times 2$  matrix and  $L$  a line in  $\mathbb{R}^2$  through the origin. Then  $L$  is said to be **A-invariant** if the vector  $A\vec{x}$  lies in  $L$  whenever  $\vec{x}$  lies in  $L$ ,  
i.e.,  $A\vec{x}$  is a scalar multiple of  $\vec{x}$ ,  
i.e.,  $A\vec{x} = \lambda\vec{x}$  for some scalar  $\lambda \in \mathbb{R}$ ,  
i.e.,  $\vec{x}$  is an eigenvector of  $A$ .

## Theorem (A-Invariance)

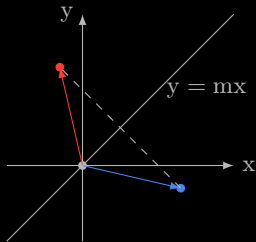
Let  $A$  be a  $2 \times 2$  matrix and let  $\vec{v} \neq 0$  be a vector in  $\mathbb{R}^2$ . Then  $L_{\vec{v}}$  is  $A$ -invariant if and only if  $\vec{v}$  is an eigenvector of  $A$ .

$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$



## Problem

Let  $m \in \mathbb{R}$  and consider the linear transformation  $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , i.e., reflection in the line  $y = mx$ .

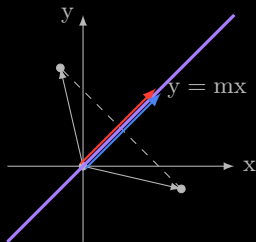


Recall that this is a matrix transformation induced by

$$A = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Find the lines that pass through origin and are  $A$ -invariant. Determine corresponding eigenvalues.

## Solution

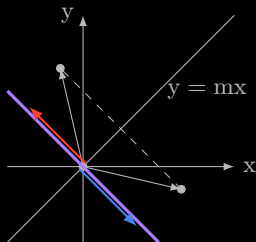


Let  $\vec{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$ . Then  $L_{\vec{x}_1}$  is  $A$ -invariant, that is,  $\vec{x}_1$  is an eigenvector.

Since the vector won't change, its eigenvalue should be 1. Indeed, one can verify that

$$A\vec{x}_1 = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \dots = \begin{pmatrix} 1 \\ m \end{pmatrix} = \vec{x}_1.$$

Solution (continued)



Let  $\vec{x}_2 = \begin{bmatrix} -m \\ 1 \end{bmatrix}$ . Then  $L_{\vec{x}_2}$  is  $A$ -invariant, that is,  $\vec{x}_2$  is an eigenvector.

Since the vector won't change the size, only flip the direction, its eigenvalue should be  $-1$ . Indeed, one can verify that

$$A\vec{x}_2 = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \begin{pmatrix} -m \\ 1 \end{pmatrix} = \cdots = \begin{pmatrix} m \\ -1 \end{pmatrix} = -\vec{x}_2.$$



### Example

Let  $\theta$  be a real number, and  $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  rotation through an angle of  $\theta$ , induced by the matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

**Claim:**  $A$  has no real eigenvalues unless  $\theta$  is an integer multiple of  $\pi$ , i.e.,  $\pm\pi, \pm2\pi, \pm3\pi$ , etc.

Consequence: a line  $L$  in  $\mathbb{R}^2$  is  $A$  invariant if and only if  $\theta$  is an integer multiple of  $\pi$ .

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Why Diagonalization?

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# Diagonalization

Denote an  $n \times n$  diagonal matrix by

$$\text{diag}(a_1, a_2, \dots, a_n) = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

Recall that if  $A$  is an  $n \times n$  matrix and  $P$  is an invertible  $n \times n$  matrix so that  $P^{-1}AP$  is diagonal, then  $P$  is called a **diagonalizing matrix** of  $A$ , and  $A$  is **diagonalizable**.



- Suppose we have n eigenvalue-eigenvector pairs:

$$A\vec{x}_j = \lambda_j\vec{x}_j, \quad j = 1, 2, \dots, n$$

- Pack the above n columns vectors into a matrix:

$$\begin{aligned} \left[ \begin{array}{c|c|c|c} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{array} \right] &= \left[ \begin{array}{c|c|c|c} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{array} \right] \\ \parallel & \\ A \left[ \begin{array}{c|c|c|c} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right] &\parallel \\ & \left[ \begin{array}{c|c|c|c} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right] \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \ddots \\ \lambda_n \end{array} \right] \end{aligned}$$

► By denoting:

$$\mathbf{P} = \left[ \begin{array}{c|c|c|c} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right] \quad \text{and} \quad \mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n)$$

we see that

$$\mathbf{A}\mathbf{P} = \mathbf{P}\mathbf{D}$$

► Hence, provided  $\mathbf{P}$  is invertible, we have

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad \text{or equivalently} \quad \mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

that is,  $\mathbf{A}$  is diagonalizable.

## Theorem (Matrix Diagonalization)

Let  $A$  be an  $n \times n$  matrix.

1.  $A$  is diagonalizable if and only if it has eigenvectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  so that

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

is invertible.

2. If  $P$  is invertible, then

$$P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

where  $\lambda_i$  is the eigenvalue of  $A$  corresponding to the eigenvector  $\vec{x}_i$ , i.e.,  $A\vec{x}_i = \lambda_i\vec{x}_i$ .

### Example

$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$  has eigenvalues and corresponding basic eigenvectors

$$\lambda_1 = 3 \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix};$$

$$\lambda_2 = 2 \quad \text{and} \quad \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix};$$

$$\lambda_3 = 1 \quad \text{and} \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

### Example (continued)

Let  $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ . Then  $P$  is invertible, so by the above Theorem,

$$P^{-1}AP = \text{diag}(3, 2, 1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



### Remark

It is not always possible to find  $n$  eigenvectors so that  $P$  is invertible.

### Example

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}.$$

Then

$$c_A(x) = \begin{vmatrix} x-1 & 2 & -3 \\ -2 & x-6 & 6 \\ -1 & -2 & x+1 \end{vmatrix} = \cdots = (x-2)^3.$$

$A$  has only one eigenvalue,  $\lambda_1 = 2$ , with multiplicity three. Sometimes, one writes

$$\lambda_1 = \lambda_2 = \lambda_3 = 2.$$

### Example (continued)

To find the 2-eigenvectors of  $A$ , solve the system  $(2I - A)\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The general solution in parametric form is

$$\vec{x} = \begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Since the system has only two basic solutions, there are only two basic eigenvectors, implying that the matrix  $A$  is not diagonalizable. ■

### Problem

Diagonalize, if possible, the matrix  $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ .

### Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

A has eigenvalues  $\lambda_1 = 1$  of multiplicity two;  $\lambda_2 = -3$  of multiplicity one.



### Solution (continued)

Eigenvectors for  $\lambda_1 = 1$ : solve  $(I - A)\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\vec{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$ ,  $s, t \in \mathbb{R}$  so basic eigenvectors corresponding to  $\lambda_1 = 1$  are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

### Solution (continued)

Eigenvectors for  $\lambda_2 = -3$ : solve  $(-3\mathbf{I} - \mathbf{A})\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{x} = \begin{bmatrix} -\frac{1}{4}t \\ 0 \\ t \end{bmatrix}, t \in \mathbb{R} \text{ so a basic eigenvector corresponding to } \lambda_2 = -3 \text{ is}$$

$$\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

## Solution (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then  $P$  is invertible, and

$$P^{-1}AP = \text{diag}(-3, 1, 1) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



### Theorem (Matrix Diagonalization Test)

A square matrix  $A$  is diagonalizable if and only if every eigenvalue  $\lambda$  of multiplicity  $m$  yields exactly  $m$  basic eigenvectors, i.e., the solution to  $(\lambda I - A)\vec{x} = \vec{0}$  has  $m$  parameters.

A special case of this is:

### Theorem (Distinct Eigenvalues and Diagonalization)

An  $n \times n$  matrix with distinct eigenvalues is diagonalizable.

### Problem

Show that  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$  is not diagonalizable.

### Solution

First,

$$c_A(x) = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = (x-1)^2(x-2),$$


so  $A$  has eigenvalues  $\lambda_1 = 1$  of multiplicity two;  $\lambda_2 = 2$  (of multiplicity one).

### Solution (continued)

Eigenvectors for  $\lambda_1 = 1$ : solve  $(I - A)\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore,  $\vec{x} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$ ,  $s \in \mathbb{R}$ .

Since  $\lambda_1 = 1$  has multiplicity two, but has only one basic eigenvector, we can conclude that  $A$  is NOT diagonalizable. 

Copyright

Why Diagonalization?

Eigenvalues and Eigenvectors

Geometric Interpretation of Eigenvalues and Eigenvectors

Diagonalization

**Linear Dynamical Systems**

# Linear Dynamical Systems

## Definition

A **linear dynamical system** consists of

- an  $n \times n$  matrix  $A$  and an  $n$ -vector  $\vec{v}_0$ ;
- a **matrix recursion** defining  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots$  by  $\vec{v}_{k+1} = A\vec{v}_k$ ; i.e.,

$$\begin{aligned}\vec{v}_1 &= A\vec{v}_0 \\ \vec{v}_2 &= A\vec{v}_1 = A(A\vec{v}_0) = A^2\vec{v}_0 \\ \vec{v}_3 &= A\vec{v}_2 = A(A^2\vec{v}_0) = A^3\vec{v}_0 \\ &\vdots \\ \vec{v}_k &= A^k\vec{v}_0.\end{aligned}$$

## Remark

Linear dynamical systems are used, for example, to model the evolution of populations over time.



If  $A$  is diagonalizable, then

$$P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the (not necessarily distinct) eigenvalues of  $A$ .

Thus  $A = PDP^{-1}$ , and  $A^k = PD^kP^{-1}$ . Therefore,

$$\vec{v}_k = A^k \vec{v}_0 = PD^kP^{-1} \vec{v}_0.$$

## Problem

Consider the linear dynamical system  $\vec{v}_{k+1} = A\vec{v}_k$  with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for  $\vec{v}_k$ .

## Solution

First,  $c_A(x) = (x - 2)(x + 1)$ , so  $A$  has eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = -1$ , and thus is diagonalizable.

Solve  $(2I - A)\vec{x} = \vec{0}$ :

$$\left[ \begin{array}{cc|c} 0 & 0 & 0 \\ -3 & 3 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

has general solution  $\vec{x} = \begin{bmatrix} s \\ s \end{bmatrix}$ ,  $s \in \mathbb{R}$ , and basic solution  $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

### Solution (continued)

Solve  $(-I - A)\vec{x} = \vec{0}$ :

$$\left[ \begin{array}{cc|c} -3 & 0 & 0 \\ -3 & 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

has general solution  $\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ ,  $t \in \mathbb{R}$ , and basic solution  $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

Thus,  $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  is a diagonalizing matrix for  $A$ ,

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

## Solution (continued)

Therefore,

$$\begin{aligned}\vec{v}_k &= A^k \vec{v}_0 \\ &= PD^k P^{-1} \vec{v}_0 \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2^k & 0 \\ 2^k & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2^k \\ 2^k - 2(-1)^k \end{bmatrix}.\end{aligned}$$



## Remark

Often, instead of finding an exact formula for  $\vec{v}_k$ , it suffices to estimate  $\vec{v}_k$  as  $k$  gets large.

This can easily be done if  $A$  has a **dominant eigenvalue with multiplicity one**: an eigenvalue  $\lambda_1$  with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \dots, n.$$

Suppose that

$$\vec{v}_k = PD^kP^{-1}\vec{v}_0,$$

and assume that  $A$  has a dominant eigenvalue,  $\lambda_1$ , with corresponding basic eigenvector  $\vec{x}_1$  as the first column of  $P$ .

For convenience, write  $P^{-1}\vec{v}_0 = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$ .

Then

$$\begin{aligned}
 \vec{v}_k &= PD^kP^{-1}\vec{v}_0 \\
 &= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
 &= b_1\lambda_1^k\vec{x}_1 + b_2\lambda_2^k\vec{x}_2 + \cdots + b_n\lambda_n^k\vec{x}_n \\
 &= \lambda_1^k \left( b_1\vec{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k \vec{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k \vec{x}_n \right)
 \end{aligned}$$

Now,  $\left| \frac{\lambda_j}{\lambda_1} \right| < 1$  for  $j = 2, 3, \dots, n$ , and thus  $\left( \frac{\lambda_j}{\lambda_1} \right)^k \rightarrow 0$  as  $k \rightarrow \infty$ .

Therefore, for large values of  $k$ ,  $\vec{v}_k \approx \lambda_1^k b_1 \vec{x}_1$ .

## Problem

If

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

estimate  $\vec{v}_k$  for large values of  $k$ .

## Solution

In our previous example, we found that  $A$  has eigenvalues 2 and  $-1$ . This means that  $\lambda_1 = 2$  is a **dominant** eigenvalue; let  $\lambda_2 = -1$ .

As before  $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is a basic eigenvector for  $\lambda_1 = 2$ , and  $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  is a basic eigenvector for  $\lambda_2 = -1$ , giving us

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

### Solution (continued)

$$P^{-1}\vec{v}_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of  $k$ ,

$$\vec{v}_k \approx \lambda_1^k b_1 \vec{x}_1 = 2^k(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}.$$



### Remark

Let's compare this to the exact formula for  $\vec{v}_k$  that we obtained earlier:

$$\vec{v}_k = \begin{bmatrix} 2^k \\ 2^k - 2(-1/2)^k \end{bmatrix} \approx \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}.$$