Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry
§4-2. Projections and Planes

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Ilijas Farah, York University**

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A new topic
A new example or problem
A new or better proof to an existing theorem
Any other suggestions to improve the material

Contact Lyryx at info@lyryx.com with your ideas.

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

The Dot Product and Angles

Definition

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2,$$

i.e., $\vec{u} \cdot \vec{v}$ is a scalar.

Remark

Another way to think about the dot product is as the 1×1 matrix

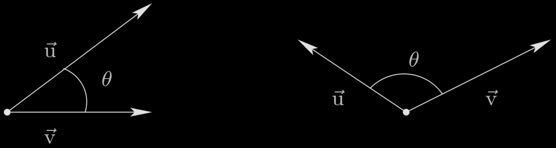
$$\vec{u}^T\vec{v} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 + y_1y_2 + z_1z_2 \end{bmatrix}.$$

Theorem (Properties of the Dot Product)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v}$ is a real number.
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. (commutative property)
3. $\vec{u} \cdot \vec{0} = 0$.
4. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.
5. $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$. (associative property)
6. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$. (distributive properties)
   $\vec{u} \cdot (\vec{v} - \vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$.

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$). There is a unique angle θ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$.



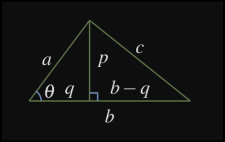
Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta.$$

Proof.

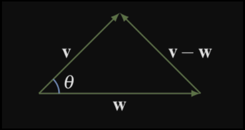
We first prove the Law of Cosines – a generalization of the Pythagorean theorem:



$$
\begin{aligned}
c^2 = p^2 + (b - q)^2 &= a^2 \sin^2\theta + (b - a\cos\theta)^2 \\
&= a^2\left(\sin^2\theta + \cos^2\theta\right) + b^2 - 2ab\cos\theta \\
&= a^2 + b^2 - 2ab\cos\theta.
\end{aligned}
$$

Proof. (continued)

In terms of vectors, we see that



$$
\begin{gathered}
||\vec{v} - \vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||\,||\vec{w}||\cos\theta \\
|| \\
(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = ||\vec{v}||^2 - 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2 \\
\Downarrow \\
||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||\,||\vec{w}||\cos\theta = ||\vec{v}||^2 - 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2 \\
\Downarrow \\
\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta.
\end{gathered}
$$

$\blacksquare$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.
- If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$.

Therefore, for nonzero vectors $\vec{u}$ and $\vec{v}$,

- $\vec{u} \cdot \vec{v} > 0$ if and only if $0 \leq \theta < \frac{\pi}{2}$.
- $\vec{u} \cdot \vec{v} = 0$ if and only if $\theta = \frac{\pi}{2}$.
- $\vec{u} \cdot \vec{v} < 0$ if and only if $\frac{\pi}{2} < \theta \leq \pi$.

Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$.

Theorem

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

Problem

Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Solution

$\vec{u} \cdot \vec{v} = 1$, $||\vec{u}|| = \sqrt{2}$ and $||\vec{v}|| = \sqrt{2}$.
Therefore,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, $\theta = \frac{\pi}{3}$.

Therefore, the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$.

Problem

Find the angle between $\vec{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

Solution

$\vec{u} \cdot \vec{v} = 0$, and therefore the angle between the vectors is $\frac{\pi}{2}$.

Problem

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both $\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Solution

There are infinitely many such vectors. Since $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$,

$$\begin{aligned} \vec{v} \cdot \vec{u} &= -x - 3y + 2z = 0 \\ \vec{v} \cdot \vec{w} &= y + z = 0 \end{aligned}$$

Solution (continued)

This is a homogeneous system of two linear equation in three variables.

$$\left[\begin{array}{rrr|r} -1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Therefore, $\vec{v} = t \left[\begin{array}{r} 5 \\ -1 \\ 1 \end{array}\right]$ for all $t \in \mathbb{R}$.

Problem

Are A$(4, -7, 9)$, B$(6, 4, 4)$ and C$(7, 10, -6)$ the vertices of a right angle triangle?

Solution

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \quad \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \quad \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

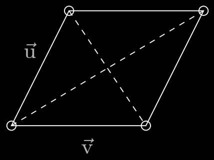
- $\overrightarrow{AB} \cdot \overrightarrow{AC} = 6 + 187 + 75 \neq 0.$
- $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\overrightarrow{AB}) \cdot \overrightarrow{BC} = -2 - 66 - 50 \neq 0.$
- $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-\overrightarrow{AC}) \cdot (-\overrightarrow{BC}) = \overrightarrow{AC} \cdot \overrightarrow{BC} = 3 + 102 + 150 \neq 0.$

Because none of the angles is $\frac{\pi}{2}$, the triangle is not a right angle triangle.

Problem

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Solution



Define the parallelogram (rhombus) by vectors $\vec{u}$ and $\vec{v}$.

Then the diagonals are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

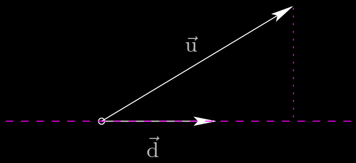
Show that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular.

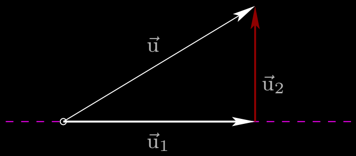
$$\begin{aligned}
(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v} \\
&= ||\vec{u}||^2-\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{v}-||\vec{v}||^2 \\
&= ||\vec{u}||^2-||\vec{v}||^2 \\
&= 0, \qquad \text{since } ||\vec{u}||=||\vec{v}||.
\end{aligned}$$

Therefore, the diagonals are perpendicular.

Projections

Given two nonzero vectors $\vec{u}$ and $\vec{d}$, one can always express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.

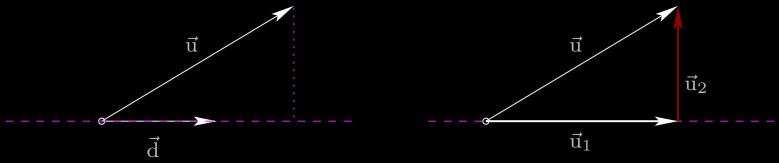




$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \mathrm{proj}_{\vec{d}}\vec{u}$.

How to find $\vec{u}_1 = \mathrm{proj}_{\vec{d}}\vec{u}$?

$$
\begin{aligned}
\vec{\mathrm{u}}_2 \cdot \vec{\mathrm{u}}_1 &= 0 && (\vec{\mathrm{u}}_1 \perp \vec{\mathrm{u}}_2)\\
\vec{\mathrm{u}}_2 \cdot (\mathrm{t}\vec{\mathrm{d}}) &= 0 && (\vec{\mathrm{u}}_1 = \mathrm{t}\vec{\mathrm{d}})\\
\mathrm{t}(\vec{\mathrm{u}}_2 \cdot \vec{\mathrm{d}}) &= 0 &&\\
\vec{\mathrm{u}}_2 \cdot \vec{\mathrm{d}} &= 0 && (\mathrm{t} \neq 0 \text{ b.c. } \vec{\mathrm{u}} \neq \vec{0})\\
(\vec{\mathrm{u}} - \vec{\mathrm{u}_1}) \cdot \vec{\mathrm{d}} &= 0 && (\vec{\mathrm{u}}_1 + \vec{\mathrm{u}}_2 = \vec{\mathrm{u}})\\
\vec{\mathrm{u}} \cdot \vec{\mathrm{d}} - \vec{\mathrm{u}_1} \cdot \vec{\mathrm{d}} &= 0 &&\\
\vec{\mathrm{u}} \cdot \vec{\mathrm{d}} - (\mathrm{t}\vec{\mathrm{d}}) \cdot \vec{\mathrm{d}} &= 0 && (\vec{\mathrm{u}}_1 = \mathrm{t}\vec{\mathrm{d}})\\
\vec{\mathrm{u}} \cdot \vec{\mathrm{d}} - \mathrm{t}(\vec{\mathrm{d}} \cdot \vec{\mathrm{d}}) &= 0 &&\\
\vec{\mathrm{u}} \cdot \vec{\mathrm{d}} - \mathrm{t}||\vec{\mathrm{d}}||^2 &= 0 &&\\
\vec{\mathrm{u}} \cdot \vec{\mathrm{d}} &= \mathrm{t}||\vec{\mathrm{d}}||^2 &&\\
\mathrm{t} &= \frac{\vec{\mathrm{u}} \cdot \vec{\mathrm{d}}}{||\vec{\mathrm{d}}||^2} && (\vec{\mathrm{d}} \neq \vec{0})\\
\vec{\mathrm{u}}_1 &= \frac{\vec{\mathrm{u}} \cdot \vec{\mathrm{d}}}{||\vec{\mathrm{d}}||^2}\vec{\mathrm{d}} && (\vec{\mathrm{u}}_1 = \mathrm{t}\vec{\mathrm{d}})
\end{aligned}
$$



Theorem

Let $\vec{u}$ and $\vec{d}$ be vectors with $\vec{d} \neq \vec{0}$.

1. The projection of $\vec{u}$ onto $\vec{d}$ is

$$\vec{u}_1 = \mathrm{proj}_{\vec{d}}\vec{u} = \frac{\vec{u} \cdot \vec{d}}{||\vec{d}||^2}\vec{d}.$$

2.

$$\vec{u}_2 = \vec{u} - \frac{\vec{u} \cdot \vec{d}}{||\vec{d}||^2}\vec{d}$$

is orthogonal to $\vec{d}$.

unit vector

$$\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u} = \left(\underbrace{\vec{u} \cdot \frac{\vec{d}}{||\vec{d}||}}_{\text{length}}\right) \frac{\vec{d}}{||\vec{d}||} = \frac{\vec{u}\cdot\vec{d}}{||\vec{d}||^2}\vec{d}$$

direction

Problem

Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{u}_1$ and $\vec{u}_2$ so that $\vec{u} = \vec{u}_1 + \vec{u}_2$, with $\vec{u}_1$ parallel to $\vec{v}$ and $\vec{u}_2$ orthogonal to $\vec{v}$.

Solution

$$\vec{u}_1 = \mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\vec{v} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 5/11 \\ -5/11 \end{bmatrix}.$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 \\ -16 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -16/11 \\ 5/11 \end{bmatrix}.$$

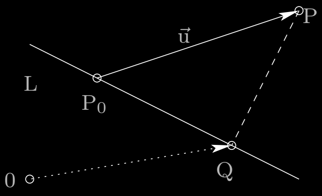
Problem

Let $P(3, 2, -1)$ be a point in $\mathbb{R}^3$ and L a line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Find the shortest distance from P to L, and find the point Q on L that is closest to P.

Solution



Let $P_0 = P_0(2, 1, 3)$ be a point on L,

and let $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$.

Then $\overrightarrow{P_0Q} = \mathrm{proj}_{\vec{d}} \overrightarrow{P_0P}$, $\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q}$,

and the shortest distance from P to L is

the length of $\overrightarrow{QP}$, where $\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q}$.

Solution (continued)

$\overrightarrow{P_0P} = \begin{bmatrix} 1 & 1 & -4 \end{bmatrix}^T$, $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$.

$$\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{d}}{||\vec{d}||^2}\vec{d} = \frac{10}{14}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix}.$$

Therefore,

$$\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 29 \\ 2 \\ 11 \end{bmatrix},$$

so $Q = Q\left(\frac{29}{7}, \frac{2}{7}, \frac{11}{7}\right)$.

Solution (continued)

Finally, the shortest distance from $P(3, 2, -1)$ to L is the length of $\overrightarrow{QP}$, where

$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

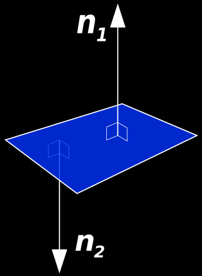
Therefore the shortest distance from P to L is

$$||\overrightarrow{QP}|| = \frac{2}{7}\sqrt{(-4)^2 + 6^2 + (-9)^2} = \frac{2}{7}\sqrt{133}.$$

Planes

Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \cdot \vec{v} = 0$ for every vector $\vec{v}$ in the plane.



Given a point P_0 and a nonzero vector $\vec{n}$, there is a unique plane containing P_0 and orthogonal to $\vec{n}$.

Consider a plane containing a point P_0 and orthogonal to vector $\vec{n}$, and let P be an arbitrary point on this plane.

Then

$$\vec{n} \cdot \overrightarrow{P_0P} = 0,$$

or, equivalently,

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0,$$

and is a vector equation of the plane.

$$\vec{n}\cdot(\overrightarrow{0P}-\overrightarrow{0P_0})=0 \quad \Longleftrightarrow \quad \vec{n}\cdot\overrightarrow{0P}=\vec{n}\cdot\overrightarrow{0P_0}$$

by setting $P_0 = P_0(x_0, y_0, z_0)$, $P = P(x, y, z)$, $\vec{n} = \begin{bmatrix} a & b & c \end{bmatrix}^T$

$$\Longleftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\cdot\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$\Longleftrightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0,$$

setting $d = ax_0 + by_0 + cz_0$ – a scalar

$$\Longleftrightarrow \quad \boxed{ax + by + cz = d}, \text{ where } a, b, c, d \in \mathbb{R}.$$

This is the scalar equation of the plane.

Problem

Find an equation of the plane containing $P_0(1,-1,0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

Solution

A vector equation of this plane is

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix} = 0.$$

A scalar equation of this plane is

$$-3x + 5y + 2z = -3(1) + 5(-1) + 2(0) = -8,$$

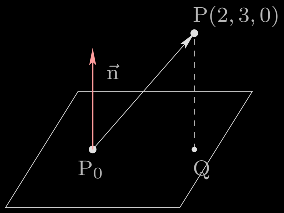
i.e., the plane has scalar equation

$$-3x + 5y + 2z = -8.$$

Problem

Find the shortest distance from the point $P(2,3,0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

Solution



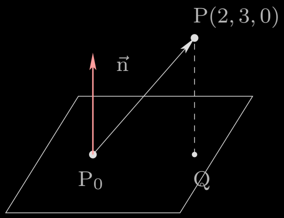
Pick an arbitrary point P_0 on the plane.

Then $\overrightarrow{QP} = \mathrm{proj}_{\vec{n}} \overrightarrow{P_0P}$,
$||\overrightarrow{QP}||$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T$. Choose $P_0 = P_0(0,0,-1)$. Then

$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T$$

Solution (continued)



$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

$$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

$$\overrightarrow{QP} = \mathrm{proj}_{\vec{n}}\overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

Since $||\overrightarrow{QP}|| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from P to the plane is $\frac{14\sqrt{3}}{9}$.

To find Q, we have

$$\begin{aligned} \overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} &= \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T - \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T \\ &= \frac{1}{27}\begin{bmatrix} -16 & 67 & -14 \end{bmatrix}^T. \end{aligned}$$

Therefore $Q = Q\left(-\frac{16}{27}, \frac{67}{27}, -\frac{14}{27}\right)$.

Remark

Here is a general answer: the distance from $P\,(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$ is

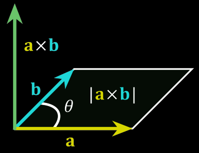
$$\text{distance} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

The Cross Product

Definition

Let $\vec{u} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$



Remark

$\vec{u} \times \vec{v}$ is a vector:

- Direction: orthogonal to both $\vec{u}$ and $\vec{v}$.
- Size: the area of the corresponding parallelogram.

Remark

A mnemonic device:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & x_1 & x_2 \\ \vec{j} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Or equivalently,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

Theorem

Let $\vec{\mathrm{v}}, \vec{\mathrm{w}} \in \mathbb{R}^3$.

1. $\vec{\mathrm{v}} \times \vec{\mathrm{w}}$ is orthogonal to both $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$.
2. If $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$ are both nonzero, then $\vec{\mathrm{u}} \times \vec{\mathrm{w}} = \vec{0}$ if and only if $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$ are parallel.

Problem

Find all vectors orthogonal to both $\vec{u} = \begin{bmatrix} -1 & -3 & 2 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$. (We previously solved this using the dot product.)

Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & -1 & 0 \\ \vec{j} & -3 & 1 \\ \vec{k} & 2 & 1 \end{vmatrix} = -5\vec{i} + \vec{j} - \vec{k} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}.$$

Any scalar multiple of $\vec{u} \times \vec{v}$ is also orthogonal to both $\vec{u}$ and $\vec{v}$, so

$$t \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \quad \forall t \in \mathbb{R},$$

gives all vectors orthogonal to both $\vec{u}$ and $\vec{v}$.

(Compare this with our earlier answer.)

Problem

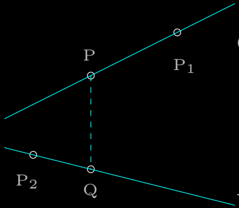
Given two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

A. Find the shortest distance between L_1 and L_2.

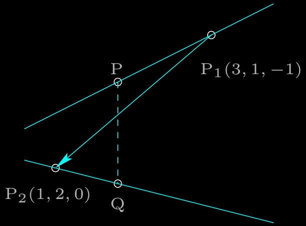
B. Find the points P on L_1 and Q on L_2 that are closest together.

Solution



Choose $P_1(3, 1, -1)$ on L_1 and $P_2(1, 2, 0)$ on L_2.

Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ denote direction vectors for L_1 and L_2, respectively.



$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

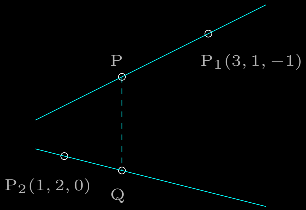
The shortest distance between L_1 and L_2 is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

$$\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\overrightarrow{P_1P_2} = \frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{||\vec{n}||^2}\vec{n}, \quad \text{and} \quad ||\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}|| = \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{||\vec{n}||}.$$

Therefore, the shortest distance between L_1 and L_2 is $\frac{|-8|}{\sqrt{14}} = \frac{4}{7}\sqrt{14}$.

Solution B.



$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$$

$$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix} \text{ for some } s \in \mathbb{R};$$

$$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix} \text{ for some } t \in \mathbb{R}.$$

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both L_1 and L_2, so

$$\overrightarrow{PQ} \cdot \vec{d}_1 = 0 \quad \text{and} \quad \overrightarrow{PQ} \cdot \vec{d}_2 = 0,$$

i.e.,

$$\begin{aligned} -2-3s-t &= 0 \\ s+5t &= 0. \end{aligned}$$

This system has unique solution $s = -\frac{5}{7}$ and $t = \frac{1}{7}$. Therefore,

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \quad \text{and} \quad Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right).$$

The shortest distance between L_1 and L_2 is $||\overrightarrow{PQ}||$. Since

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \quad \text{and} \quad Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

$$\overrightarrow{PQ} = \frac{1}{7}\begin{bmatrix} 8 \\ 14 \\ 2 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 16 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix},$$

and

$$||\overrightarrow{PQ}|| = \frac{1}{7}\sqrt{224} = \frac{4}{7}\sqrt{14}.$$

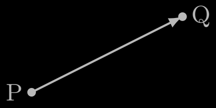
Therefore the shortest distance between L_1 and L_2 is $\frac{4}{7}\sqrt{14}$.

Problem (Challenge Problem)

Write yourself a plan to find the shortest distance in $\mathbb{R}^3$ between either a point, line or plane, to either a point, line or plane.

Point-point distance

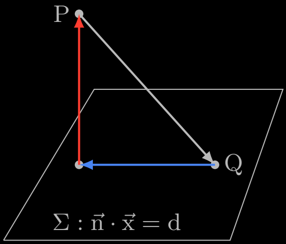
If P and Q are two points, then $d\left(P, Q\right) = |\overrightarrow{PQ}|$.



Point-plane distance

If P is a point and $\Sigma : \vec{\mathrm{n}} \cdot \vec{\mathrm{x}} = \mathrm{d}$ is a plane containing a point Q, then

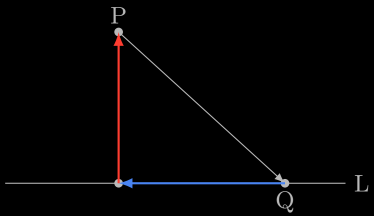
$$\mathrm{d}\left(\mathrm{P}, \Sigma\right) = \frac{\left|\overrightarrow{\mathrm{PQ}} \cdot \vec{\mathrm{n}}\right|}{|\vec{\mathrm{n}}|}$$



Point-line distance

If P is a point and L is a line $\vec{r}(t) = Q + t\vec{u}$, then

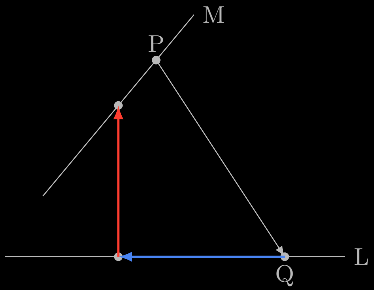
$$d\left(P, L\right) = \frac{\left|\overrightarrow{PQ} \times \vec{u}\right|}{\left|\vec{u}\right|}$$



Line-line distance

If L is a line $\vec{r}(t) = Q + t\vec{u}$ and M is another line $\vec{s} = P + t\vec{v}$, then

$$d\left(L, M\right) = \frac{\left|\overrightarrow{PQ} \cdot (\vec{u} \times \vec{v})\right|}{|\vec{u} \times \vec{v}|}$$



Plane-plane distance

If $\Sigma : \vec{n} \cdot \vec{x} = d$ and $\Theta : \vec{n} \cdot \vec{x} = e$ are two parallel planes, then

$$d(\Sigma, \Theta) = \frac{|e - d|}{|\vec{n}|}$$