Math 221: LINEAR ALGEBRA

§Review session for test II

Le Chen[1]

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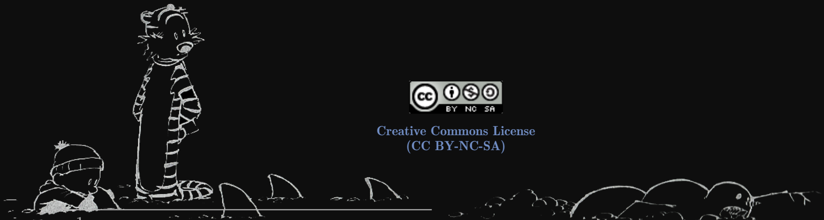

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

3.1.17 (H)

Problem

Show that $\det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix} = -3x^2.$

3.1.18 (H)

Problem

Show that

$$\det \begin{bmatrix} 1 & x & x^2 & x^3 \\ a & 1 & x & x^2 \\ p & b & 1 & x \\ q & r & c & 1 \end{bmatrix} = (1 - ax)(1 - bx)(1 - cx)\,.$$

3.1.19 (H)

Problem

Given the polynomial $p(x) = a + bx + cx^2 + dx^3 + x^4$, the matrix

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix}$$

is called the companion matrix of $p(x)$. Show that

$$\det(xI - C) = p(x).$$

3.1.20 (P)

Problem

Show that

$$\det\begin{pmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{pmatrix} = (a+b+c+3x)\left[(ab+ac+bc)-\left(a^2+b^2+c^2\right)\right]$$

3.2.30 (P)

Problem

S1ow that $\det \begin{pmatrix} O & A \\ B & X \end{pmatrix} = \det A \det B$ when A and B are 2×2. What if A and B are $n \times n$ for general $n \geq 2$?

3.2.32 (P)

Problem

If A is 3×3 and invertible, compute $\det\left(-A^2\,(\operatorname{adj}A)^{-1}\right)$.

3.2.33 (H)

Problem

Show that $\mathrm{adj}\,(uA) = u^{n-1}\mathrm{adj}(A)$ for all $n \times n$ matrices A.

3.2.34 (P)

Problem

Let A and B denote invertible $n \times n$ matrices. Show that

1. $\operatorname{adj}(\operatorname{adj}(A)) = \det(A)^{n-2} A$ for $n \geq 2$.
2. $\operatorname{adj}\left(A^{-1}\right) = \operatorname{adj}(A)^{-1}$.
3. $\operatorname{adj}\left(A^{T}\right) = \operatorname{adj}(A)^{T}$
4. $\operatorname{adj}(AB) = \operatorname{adj}(B)\operatorname{adj}(A)$

3.3.6 (L)

Problem

Find the characteristic polynomial of the $n \times n$ identity matrix I. Show that I has exactly one eigenvalue and find the eigenvectors.

3.3.14 (H)

Problem

If A is diagonalizable and 0 and 1 are the only eigenvalues, show that $A^2 = A$.

3.3.16 (H)

Problem

If $P^{-1}AP$ and $P^{-1}BP$ are both diagonalizable, show that $AB = BA$.

3.3.20 (P)

Problem

Let A be an invertible $n \times n$ matrix.

1. Show that the eigenvalues of A are nonzero.
2. Show that the eigenvalues of A^{-1} are precisely the numbers $1/\lambda$, where λ is an eigenvalue of A.
3. Show that $c_{A^{-1}} = \frac{(-x)^n}{\det A} c_A\left(\frac{1}{x}\right)$.

3.3.21 (P, L)

Problem

Suppose λ is an eigenvalue of a square matrix A with eigenvector $\vec{x} \neq \vec{0}$.

1. Show that λ^2 is an eigenvalue of A^2 with the same eigenvector $\vec{x}$.
2. Show that $\lambda^3 - 2\lambda + 3$ is an eigenvalue of $A^3 - 2A + 3I$.
3. Show that $p(\lambda)$ is an eigenvalue of $p(A)$ for any nonzero polynomial $p(x)$.

3.3.22 (P)

Problem

If A is an $n \times n$ matrix, show that

$$c_{A^2}(x^2) = (-1)^n c_A(x) c_A(-x).$$

3.3.24 (P)

Problem

Let A be diagonalizable with real eigenvalues and assume that $A^m = I$ for some $m \geq 1$.

1. Show that $A^2 = I$.
2. If m is odd, show that $A = I$.

3.3.25 (P)

Problem

Let $A^2 = I$, and assume that $A \neq I$ and $A \neq -I$.

1. Show that the only eigenvalues of A are $\lambda = 1$ and $\lambda = -1$.
2. Show that A is diagonalizable.
   (Hint: Verify that $A(A+I) = A+I$ and $A(A-I) = -(A-I)$, and then look at nonzero columns of $A+I$ and of $A-I$.)

3.3.29 (P, L)

Problem

Let $A = \begin{bmatrix} B & O \\ O & C \end{bmatrix}$ where B and C are square matrices.

1. If B and C are diagonalizable via Q and R, that is $Q^{-1}BQ$ and $R^{-1}CR$ are diagonal, show that A is diagonalizable via $\begin{bmatrix} Q & O \\ O & R \end{bmatrix}$.
2. Use (1) to diagonalize $A = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 7 & -1 \\ 0 & 0 & -1 & 7 \end{bmatrix}$

3.3.30 (P)

Problem

Let $A = \begin{bmatrix} B & O \\ O & C \end{bmatrix}$ where B and C are square matrices.

1. Show that $c_A(x) = c_B(x)c_C(x)$.
2. If $\vec{x}$ and $\vec{y}$ are eigenvectors of B and C, respectively, show that $\begin{bmatrix} \vec{x} \\ \vec{0} \end{bmatrix}$ and $\begin{bmatrix} \vec{0} \\ \vec{y} \end{bmatrix}$ are eigenvectors of A, and show how every eigenvector of A arises from such eigenvectors.

Problem

112. Let λ be an eigenvalue of the matrix A. Select the correct statements:

1. λ^2 is an eigenvalue of A^2
2. $A^2 = \lambda I$
3. λA is invertible
4. $\lambda - 3$ is an eigenvalue of $A = 3I$
5. $AX = \lambda X$ for every column $X \neq 0$

Problem

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5. $AX = \lambda X$ for every column $X \neq 0$

Answer: 1,4.

Problem

124. For $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, find the eigenvalues of A and determine whether A is diagonalizable.

1. $3, -1, -1$; not diagonalizable
2. $1, 1, 1$; not diagonalizable
3. $3, 1, 1$; not diagonalizable
4. $3, 1, 1$; diagonalizable
5. $-3, 1, 1$; diagonalizable
6. $3, -1, -1$; diagonalizable

Problem

124. For $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, find the eigenvalues of A and determine whether A is diagonalizable.

1. $3, -1, -1$; not diagonalizable
2. $1, 1, 1$; not diagonalizable
3. $3, 1, 1$; not diagonalizable
4. $3, 1, 1$; diagonalizable
5. $-3, 1, 1$; diagonalizable
6. $3, -1, -1$; diagonalizable

Answer: 4

Problem

99. If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ then $P^{-1}AP$ is diagonal if P is (choose the correct answers):

1. $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$
4. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$
5. $\begin{bmatrix} -2 & 3 \\ 4 & 3 \end{bmatrix}$
6. $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

Problem

99. If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ then $P^{-1}AP$ is diagonal if P is (choose the correct answers):

1. $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$
4. $\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$
5. $\begin{bmatrix} -2 & 3 \\ 4 & 3 \end{bmatrix}$
6. $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

Answer: 2, 3, 5.

Problem

38. Compute $\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}^{1001}$.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$
4. $\begin{bmatrix} -1001 & -3003 \\ 0 & 1001 \end{bmatrix}$
5. $\begin{bmatrix} 1001 & 3003 \\ 0 & -1001 \end{bmatrix}$
6. $\begin{bmatrix} -1 & 3^{1001} \\ 0 & 1 \end{bmatrix}$

Problem

38. Compute $\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}^{1001}$.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$
4. $\begin{bmatrix} -1001 & -3003 \\ 0 & 1001 \end{bmatrix}$
5. $\begin{bmatrix} 1001 & 3003 \\ 0 & -1001 \end{bmatrix}$
6. $\begin{bmatrix} -1 & 3^{1001} \\ 0 & 1 \end{bmatrix}$

Answer: 3.

Problem

118. Suppose A is 3×3 and has 2 and 3 as its only eigenvalues. Then (select the correct answers):

1. A is not diagonalizable
2. $A^2 = 0$
3. A is invertible
4. A is not invertible
5. A has an eigenvalue of multiplicity 2.

Problem

118. Suppose A is 3×3 and has 2 and 3 as its only eigenvalues. Then (select the correct answers):

1. A is not diagonalizable
2. $A^2 = 0$
3. A is invertible
4. A is not invertible
5. A has an eigenvalue of multiplicity 2.

Answer: 3, 5

Problem

110. If a 2×2, invertible matrix A has eigenvalues 2 and 5, find the correct statements:

1. A is invertible
2. A is diagonalizable
3. A is symmetric
4. $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$
5. $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ for some invertible P
6. $A^2 = 10I$.

Problem

110. If a 2×2, invertible matrix A has eigenvalues 2 and 5, find the correct statements:

1. A is invertible
2. A is diagonalizable
3. A is symmetric
4. $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$
5. $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ for some invertible P
6. $A^2 = 10I$.

Answer: 1, 2, 5.

Problem

56. If A and B are symmetric (that is $A^T = A$), which of the following are true:

(i) $A - B$ is symmetric.

(ii) AB is symmetric.

(iii) If A is invertible, A^{-1} is symmetric.

(iv) B^2 is symmetric.

(v) $AB^T = BA^T$.

1. (i) and (ii) only
2. (i), (iii) and (iv) only
3. (iii) and (iv) only
4. (ii) and (iv) only
5. none of them are true
6. all of them are true

Problem

56. If A and B are symmetric (that is $A^T = A$), which of the following are true:

(i) $A - B$ is symmetric.

(ii) AB is symmetric.

(iii) If A is invertible, A^{-1} is symmetric.

(iv) B^2 is symmetric.

(v) $AB^T = BA^T$.

1. (i) and (ii) only
2. (i), (iii) and (iv) only
3. (iii) and (iv) only
4. (ii) and (iv) only
5. none of them are true
6. all of them are true

Answer: 2.

Problem

20. For $A = \begin{bmatrix} -9 & -8 & -4 \\ 18 & 17 & 9 \\ -14 & -14 & -8 \end{bmatrix}$, find the eigenvalues of A and determine whether A is diagonalizable.

1. $2, -1, -1$; not diagonalizable
2. $2, 2, 1$; diagonalizable
3. $2, 2, -1$; not diagonalizable
4. $2, 2, -2$; diagonalizable
5. $-2, 1, 1$; diagonalizable
6. $2, -1, -1$; diagonalizable

Problem

20. For $A = \begin{bmatrix} -9 & -8 & -4 \\ 18 & 17 & 9 \\ -14 & -14 & -8 \end{bmatrix}$, find the eigenvalues of A and determine whether A is diagonalizable.

1. $2, -1, -1$; not diagonalizable
2. $2, 2, 1$; diagonalizable
3. $2, 2, -1$; not diagonalizable
4. $2, 2, -2$; diagonalizable
5. $-2, 1, 1$; diagonalizable
6. $2, -1, -1$; diagonalizable

Answer: 6.

Problem

16. Let A be a diagonalizable matrix. If $\lambda^3 = \lambda$ for each eigenvalue λ of A, then (select the correct answers):

1. A is not invertible
2. $A = I, -I$ or 0
3. $A^T = -A$
4. $A = -I$
5. $A^2 = A$
6. $A^3 = A$.

Problem

16. Let A be a diagonalizable matrix. If $\lambda^3 = \lambda$ for each eigenvalue λ of A, then (select the correct answers):

1. A is not invertible
2. $A = I, -I$ or 0
3. $A^T = -A$
4. $A = -I$
5. $A^2 = A$
6. $A^3 = A$.

Answer: 6.

Problem

126. For $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, find the eigenvalues of A and determine whether A is diagonalizable.

1. $3, 1, 1$; not diagonalizable
2. $3, -1, -1$; diagonalizable
3. $-3, 1, 1$; not diagonalizable
4. $3, -1, -1$; not diagonalizable
5. $-3, 1, 1$; diagonalizable
6. $3, 1, 1$; diagonalizable

Problem

126. For $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, find the eigenvalues of A and determine whether A is diagonalizable.

1. $3, 1, 1$; not diagonalizable
2. $3, -1, -1$; diagonalizable
3. $-3, 1, 1$; not diagonalizable
4. $3, -1, -1$; not diagonalizable
5. $-3, 1, 1$; diagonalizable
6. $3, 1, 1$; diagonalizable

Answer: 6.