Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-4. Matrix Inverses

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Emory University, 2021 Spring

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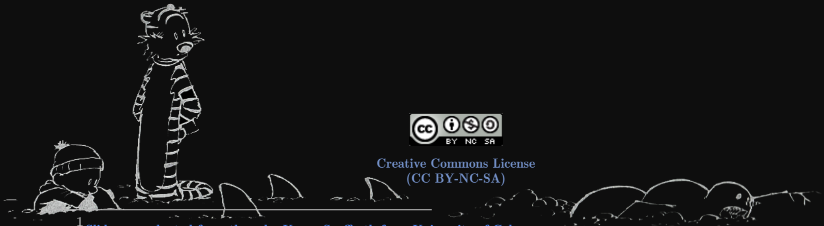

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications
Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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The Identity and Inverse Matrices

Definition

For each $n \geq 2$, the $n \times n$ identity matrix, denoted I_n, is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

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Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Let $n \geq 2$. For each j, $1 \leq j \leq n$, we denote by $\vec{e}_j$ the j^{th} column of I_n.

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Example

When $n = 3$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Theorem

Let A be an $m \times n$ matrix. Then $AI_n = A$ and $I_mA = A$.

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Proof.

The (i,j)-entry of AI_n is the product of the i^{th} row of $A = [a_{ij}]$, namely $\left[\begin{array}{cccccc} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{array} \right]$ with the j^{th} column of I_n, namely $\vec{e}_j$. Since $\vec{e}_j$ has a one in row j and zeros elsewhere,

$$\left[\begin{array}{cccccc} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{array} \right] \vec{e}_j = a_{ij}$$

Since this is true for all $i \leq m$ and all $j \leq n$, $AI_n = A$.

The proof of $I_m A = A$ is analogous—work it out! $\blacksquare$

Instead of AI_n and I_mA we often write AI and IA, respectively, since the size of the identity matrix is clear from the context: the sizes of A and I must be compatible for matrix multiplication.

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Thus

$$AI = A \quad \text{and} \quad IA = A$$

which is why I is called an identity matrix – it is an identity for matrix multiplication.

Definition (Matrix Inverses)

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Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, B is an inverse of A.

Problem

Does every square matrix have an inverse?

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Solution

No! Take e.g. the zero matrix $\mathbf{O_n}$ (all entries of $\mathbf{O_n}$ are equal to 0)

$$A\mathbf{O_n} = \mathbf{O_n}A = \mathbf{O_n}$$

for all $n \times n$ matrices A:

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$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$$

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Theorem (Uniqueness of an Inverse)

If A is a square matrix and B and C are inverses of A, then B = C.

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If A is a square matrix and B and C are inverses of A, then $B = C$.

Proof.

Since B and C are inverses of A, $AB = I = BA$ and $AC = I = CA$. Then

$$C = CI = C(AB) = CAB$$

and

$$B = IB = (CA)B = CAB$$

so $B = C$. ■

Example (revisited)

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, we saw that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The preceding theorem tells us that B is the inverse of A, rather than just an inverse of A.

Remark (notation)

Let A be a square matrix, i.e., an $n \times n$ matrix.

- The inverse of A, if it exists, is denoted A^{-1}, and

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- If A has an inverse, then we say that A is invertible.

Copyright

The Identity and Inverse Matrices

Finding the Inverse of a Matrix

Properties of the Inverse

Inverse of Transformations

Finding the inverse of a 2×2 matrix

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Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then there is a formula for A^{-1}:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

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Showing that $A^{-1}A = I_2$ is left as an exercise.

Remark

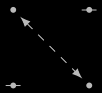
Here are some terminology related to this example:

1. Determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - cd$$

2. Adjugate:

$$\operatorname{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$



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Although the formula for the inverse of a 2×2 matrix is quicker and easier to use than the matrix inversion algorithm, the general formula for the inverse an $n \times n$ matrix, $n \geq 3$ (which we will see later), is more complicated and difficult to use than the matrix inversion algorithm. To find inverses of square matrices that are not 2×2, the matrix inversion algorithm is the most efficient method to use.

The Matrix Inversion Algorithm

Let A be an $n \times n$ matrix. To find A^{-1}, if it exists,

Step 1 take the $n \times 2n$ matrix

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$

obtained by augmenting A with the $n \times n$ identity matrix, I_n.

Step 2 Perform elementary row operations to transform $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ into a reduced row-echelon matrix.

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Theorem (Matrix Inverses)

Let A be an $n \times n$ matrix. Then the following conditions are equivalent.

1. A is invertible.
2. the reduced row-echelon form on A is I.
3. $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ can be transformed into $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$ using the Matrix Inversion Algorithm.

Problem

Find, if possible, the inverse of $\left[\begin{array}{rrr} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{array}\right]$.

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$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

From this, we see that A has no inverse.

Problem

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.

Solution

Using the matrix inversion algorithm

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$$[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] = [\,I \mid A^{-1}\,]$$

Solution (continued)

Therefore, A^{-1} exists, and

$$A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}.$$

$\blacksquare$

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■

Remark

It is always a good habit to check your answer by computing AA^{-1} and $A^{-1}A$.

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Example

The system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

can be written in matrix form as $A\vec{x} = \vec{b}$:

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You can check that $A^{-1} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$.

Example (continued)

Since A^{-1} exists and has the property that $A^{-1}A = I$, we obtain the following.

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i.e., $A\vec{x} = \vec{b}$ has the unique solution given by $\vec{x} = A^{-1}\vec{b}$. Therefore,

$$\vec{x} = A^{-1}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

is the unique solution to the system. $\blacksquare$

Remark

The last example illustrates another method for solving a system of linear equations when the coefficient matrix is square and invertible.

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The last example illustrates another method for solving a system of linear equations when the coefficient matrix is square and invertible. Unless that coefficient matrix is 2×2, this is generally NOT an efficient method for solving a system of linear equations.

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Problem

Can you find square matrices A, B and C for which $AB = AC$ but $B \neq C$?

Properties of the Inverse

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4. If A is invertible and $p \in \mathbb{R}$ is nonzero, then pA is invertible, and $(pA)^{-1} = \frac{1}{p}A^{-1}$.

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Given $(3\mathrm{I} - \mathrm{A}^{\mathrm{T}})^{-1} = 2\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, we wish to find the matrix A.

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\mathrm{A} &= \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix}
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$\blacksquare$

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Therefore, A is invertible, and $\boxed{???} = \frac{1}{4}A^2$, i.e., $A^{-1} = \frac{1}{4}A^2$. ■

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2. The rank of A is n.
3. The reduced row echelon form of A is I_n.
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Proof.

(1), (2), (4), (5) and (6) are all equivalent.

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Hence, (8) holds with $C = [\vec{c}_1, \cdots, \vec{c}_n]$.

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(9) $\Rightarrow$ (1): By reversing the roles of A and C and apply (8) to see that C is invertible. Thus A is the inverse of C, and hence A is itself invertible. ∎

Corollary

If A and B are $n \times n$ matrices such that $AB = I$, then $BA = I$. Furthermore, A and B are invertible, with $B = A^{-1}$ and $A = B^{-1}$.

$= I_n$

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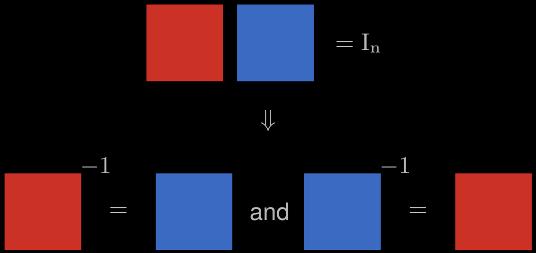
If A and B are $n \times n$ matrices such that $AB = I$, then $BA = I$. Furthermore, A and B are invertible, with $B = A^{-1}$ and $A = B^{-1}$.



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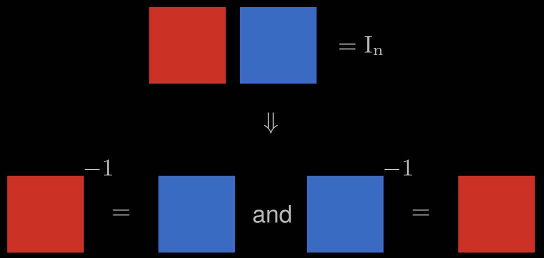

Remark

Important Fact In Corollary, it is essential that the matrices be square.

Theorem

If A and B are matrices such that $AB = I$ and $BA = I$, then A and B are square matrices (of the same size).

Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

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Remark

This example illustrates why "an inverse" of a non-square matrix doesn't make sense. If A is $m \times n$ and B is $n \times m$, where $m \neq n$, then even if $AB = I$, it will never be the case that $BA = I$.

Inverse of Transformations

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix transformation induced by matrix A. Then we have:

1. A is invertible if and only if T has an inverse.
2. In the case where T has an inverse, the inverse is unique and is denoted T^{-1}.
3. Furthermore, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is induced by the matrix A^{-1}.

Fundamental Identities relating T and T^{-1}

1. $T^{-1} \circ T = 1_{\mathbb{R}^n}$
2. $T \circ T^{-1} = 1_{\mathbb{R}^n}$

Example

Let $\mathrm{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a transformation given by

$$\mathrm{T}\begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} \mathrm{x}+\mathrm{y} \\ \mathrm{y} \end{bmatrix}$$

Then T is a linear transformation induced by $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

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Notice that the matrix A is invertible. Therefore the transformation T has an inverse, T^{-1}, induced by

$$\mathrm{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

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Therefore,

$$T^{-1}\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

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