Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations
§1-3. Homogeneous Equations

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

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Homogeneous Equations

Linear Combination

Linear Algebra with Applications

Lecture Notes

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Homogeneous Equations

Definition

A homogeneous linear equation is one whose constant term is equal to zero. A system of linear equations is called homogeneous if each equation in the system is homogeneous. A homogeneous system has the form

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}\right.$$

where a_{ij} are scalars and x_i are variables, $1 \leq i \leq m$, $1 \leq j \leq n$.

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where a_{ij} are scalars and x_i are variables, $1 \leq i \leq m$, $1 \leq j \leq n$.

Remark

1. Notice that $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to a homogeneous system of equations. We call this the trivial solution.

2. We are interested in finding, if possible, nontrivial solutions (ones with at least one variable not equal to zero) to homogeneous systems.

Example

Solve the system
$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 0 \\ -x_1 & + & 4x_2 & + & 5x_3 & - & 2x_4 & = & 0 \\ x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & = & 0 \end{array}\right.$$

Example

Solve the system $\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 0 \\ -x_1 & + & 4x_2 & + & 5x_3 & - & 2x_4 & = & 0 \\ x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & = & 0 \end{array}\right.$

Solution

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 0 \\ -1 & 4 & 5 & -2 & 0 \\ 1 & 6 & 3 & 4 & 0 \end{array}\right]$$

Example

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Solution

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 0 \\ -1 & 4 & 5 & -2 & 0 \\ 1 & 6 & 3 & 4 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{rrrr|r} 1 & 0 & -9/5 & 14/5 & 0 \\ 0 & 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example

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The system has infinitely many solutions, and the general solution is

$$\left\{\begin{array}{rcl} x_1 & = & \frac{9}{5}s - \frac{14}{5}t \\ x_2 & = & -\frac{4}{5}s - \frac{1}{5}t \\ x_3 & = & s \\ x_4 & = & t \end{array}\right.$$

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Theorem

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

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Homogeneous Equations

Linear Combination

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Linear Combination

Definition

If $X_1, X_2, \ldots, X_p$ are columns with the same number of entries, and if $a_1, a_2, \ldots a_p \in \mathbb{R}$ (are scalars) then $a_1X_1 + a_2X_2 + \cdots + a_pX_p$ is a linear combination of columns $X_1, X_2, \ldots, X_p$.

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Example (continued)

In the previous example,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{9}{5}s - \frac{14}{5}t \\ -\frac{4}{5}s - \frac{1}{5}t \\ s \\ t \end{bmatrix}$$

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$$= s\begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix}$$

Example (continued)

This gives us

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} = sX_1 + tX_2,$$

$$\text{with} \quad X_1 = \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix}.$$

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The columns X_1 and X_2 are called basic solutions to the original homogeneous system.

Example (continued)

Notice that

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} &= \frac{s}{5} \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix} + \frac{t}{5} \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix} \\
&= r \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix} + q \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix} \\
&= r(5X_1) + q(5X_2)
\end{aligned}$$

where $r, q \in \mathbb{R}$.

Example (continued)

The columns $5X_1 = \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix}$ and $5X_2 = \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix}$ are also basic solutions to the original homogeneous system.

Example (continued)

The columns $5X_1 = \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix}$ and $5X_2 = \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix}$ are also basic solutions to the original homogeneous system.

Remark

In general, any nonzero multiple of a basic solution (to a homogeneous system of linear equations) is also a basic solution.

What does the rank tell us in the homogeneous case?

Suppose A is the augmented matrix of an homogeneous system of m linear equations in n variables, and rank A = r.

$$
m\left\{\begin{array}{c}\underbrace{\left[\begin{array}{cccc|c} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \end{array}\right]}_{n}\end{array}\right. \rightarrow \underbrace{\left[\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{r leading } 1's}
$$

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$$m\left\{\vphantom{\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}}\right. \underbrace{\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}}_{n}\!\!\begin{array}{|c}0\\0\\0\\0\\0\end{array}\Bigg] \to \underbrace{\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{r \text{ leading } 1's}\!\!\begin{array}{|c}0\\0\\0\\0\\0\end{array}\Bigg]$$

There is always a solution, and the set of solutions to the system has $n - r$ parameters, so

- if $r < n$, there is at least one parameter, and the system has infinitely many solutions;
- if $r = n$, there are no parameters, and the system has a unique solution, the trivial solution.

Theorem

Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. The system has exactly $n - r$ basic solutions, one for each parameter.
2. Every solution is a linear combination of these basic solutions.

Problem

Find all values of a for which the system

$$\left\{\begin{array}{ccccccc} x & + & y & & & = & 0 \\ & & ay & + & z & = & 0 \\ x & + & y & + & az & = & 0 \end{array}\right.$$

has nontrivial solutions, and determine the solutions.

Problem

Find all values of a for which the system

$$\begin{cases} x + y = 0 \\ ay + z = 0 \\ x + y + az = 0 \end{cases}$$

has nontrivial solutions, and determine the solutions.

Solution

Non-trivial solutions occur only when $a = 0$, and the solutions when $a = 0$ are given by (rank $r = 2$, $n - r = 3 - 2 = 1$ parameter)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \forall s \in \mathbb{R}.$$