

Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra §2-3. Matrix Multiplication

Le Chen¹

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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Matrix Multiplication

Properties of Matrix Multiplication

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Matrix Multiplication

Properties of Matrix Multiplication

Matrix Multiplication

Definition (Product of two matrices)

Let A be an $m \times n$ matrix and let $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix}$ be an $n \times p$ matrix, whose columns are $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p$. The **product of A and B** is the matrix

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}$$

i.e., the first column of AB is $A\vec{b}_1$, the second column of AB is $A\vec{b}_2$, etc.
Note that AB has size $m \times p$.

Problem

Find the product AB of matrices

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution

AB has columns

$$A\vec{b}_1 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix},$$

$$A\vec{b}_2 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

$$A\vec{b}_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

$$\text{Thus, } AB = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}.$$

Definition

Let A and B be matrices, and suppose that A is $m \times n$.

- ▶ In order for the product AB to exist, the number of rows in B must be equal to the number of columns in A , implying that B is an $n \times p$ matrix for some p .
- ▶ When defined, AB is an $m \times p$ matrix.

If the product is defined, then A and B are said to be **compatible** for (matrix) multiplication.

Example

As we saw in the previous problem

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{matrix} 2 \times 3 \\ \\ \end{matrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \times 3 \\ \\ \end{matrix} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix} \begin{matrix} 2 \times 3 \\ \\ \end{matrix}$$

Note that the product

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix} \begin{matrix} 3 \times 3 \\ \\ \end{matrix} \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{matrix} 2 \times 3 \\ \\ \end{matrix}$$

does not exist.

Example (Multiplication by the zero matrix)

Compute the product AO for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies AO = O.$$



Definition (The (i,j)-entry of a product)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the **(i,j)-entry of AB** is given by the dot product of row i of A and column j of B :

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

$$\begin{matrix}
 & \swarrow & \searrow & & \swarrow & \searrow \\
 & m \times n & & n \times s & & m \times s
 \end{matrix}$$

$$\begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 \vdots & \vdots & & \vdots \\
 \underline{a_{i1}} & \underline{a_{i2}} & \cdots & \underline{a_{in}} \\
 \vdots & \vdots & & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{bmatrix}
 \begin{bmatrix}
 b_{11} & \cdots & \underline{b_{1j}} & \cdots & b_{1s} \\
 b_{21} & \cdots & \underline{b_{2j}} & \cdots & b_{2s} \\
 \vdots & & \vdots & & \vdots \\
 b_{n1} & \cdots & \underline{b_{nj}} & \cdots & b_{ns}
 \end{bmatrix}
 =
 \begin{bmatrix}
 c_{11} & \cdots & c_{1j} & \cdots & c_{1s} \\
 \vdots & & \vdots & & \vdots \\
 c_{i1} & \cdots & \underline{c_{ij}} & \cdots & c_{is} \\
 \vdots & & \vdots & & \vdots \\
 c_{m1} & \cdots & c_{mj} & \cdots & c_{ms}
 \end{bmatrix}$$

Example

Using the above definition, the $(2,3)$ -entry of the product

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

is computed by the dot product of the **second row** of the first matrix and the **third column** of the second matrix:

$$2 \times 2 + (-1) \times 4 + 1 \times 0 = 4 - 4 + 0 = 0.$$

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Matrix Multiplication

Properties of Matrix Multiplication

Properties of Matrix Multiplication

Given matrices A and B , is $AB = BA$?

Problem

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

Solution

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

BA does not exist!



Problem

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- ▶ Does GH exist? If so, compute it.
- ▶ Does HG exist? If so, compute it.

Solution

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$

Remark

In this example, GH and HG both exist, but they are not equal. They aren't even the same size!

Problem

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

Solution

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$



Remark

In this example, PQ and QP both exist and are the same size, but $PQ \neq QP$.

Fact

The three preceding problems illustrate an important property of matrix multiplication.

In general, matrix multiplication is **not** commutative, i.e., the order of the matrices in the product is important.

In other words, in general

$$AB \neq BA.$$

Multiplying from left or right, it MATTERS!

Problem

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

Solution

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$



Remark

In this particular example, the matrices **commute**, i.e., $UV = VU$.

Theorem (Properties of Matrix Multiplication)

Let A , B , and C be matrices of the appropriate sizes, and let $r \in \mathbb{R}$ be a scalar. Then the following properties hold.

1. $IA = A$ and $AI = A$.
2. $A(B + C) = AB + AC$.
(matrix multiplication distributes over matrix addition).
3. $(B + C)A = BA + CA$.
(matrix multiplication distributes over matrix addition).
4. $A(BC) = (AB)C$. (matrix multiplication is associative).
5. $r(AB) = (rA)B = A(rB)$.
6. $(AB)^T = B^T A^T$.

Remark

This applies to matrix-vector multiplication as well, since a vector is a row matrix or a column matrix.

Problem

Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be three $n \times n$ matrices. For $1 \leq i, j \leq n$ write down a formula for the (i, j) -entry of each of the following matrices.

1. AB

2. BA

3. $A+C$

4. $C(A+B)$

5. $A(BC)$

6. $(AB)C$

Problem

Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix. Prove that if A and B commute with C , then $A + B$ commutes with C .

Proof.

We are given that $AC = CA$ and $BC = CB$. Consider $(A + B)C$.

$$\begin{aligned}(A + B)C &= AC + BC \\ &= CA + CB \\ &= C(A + B)\end{aligned}$$

Since $(A + B)C = C(A + B)$, $A + B$ commutes with C . ■

Problem

Let A, B and C be $n \times n$ matrices, and suppose that both A and B commute with C , i.e., $AC = CA$ and $BC = CB$. Show that AB commutes with C .

Proof.

We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

$$\begin{aligned}(AB)C &= A(BC) \quad (\text{matrix multiplication is associative}) \\ &= A(CB) \quad (B \text{ commutes with } C) \\ &= (AC)B \quad (\text{matrix multiplication is associative}) \\ &= (CA)B \quad (A \text{ commutes with } C) \\ &= C(AB) \quad (\text{matrix multiplication is associative})\end{aligned}$$

Therefore, AB commutes with C . ■

Partitioned matrix and block multiplication

Observation

We can partition matrix into blocks so that each entry of the partitioned matrix is again a matrix.

Example

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array} \right] = \begin{bmatrix} I_2 & 0_{23} \\ P & Q \end{bmatrix} \quad \text{and} \quad B = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array} \right] = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Example

Let A and B be $m \times n$ and $n \times k$ matrices, respectively. We can partition then into either column vectors or row vectors: When viewed as partitioned matrices, AB can be equivalently written in one of the following four ways:

$$A_{mn} = (\vec{a}_1, \dots, \vec{a}_n) = \begin{pmatrix} \vec{\alpha}_1^T \\ \vdots \\ \vec{\alpha}_m^T \end{pmatrix} \quad \text{and} \quad B_{nk} = (\vec{b}_1, \dots, \vec{b}_k) = \begin{pmatrix} \vec{\beta}_1^T \\ \vdots \\ \vec{\beta}_n^T \end{pmatrix}$$

1.

$$AB = A (\vec{b}_1, \dots, \vec{b}_k) = (A\vec{b}_1, \dots, A\vec{b}_k)$$

2.

$$AB = \begin{pmatrix} \vec{\alpha}_1^T \\ \vdots \\ \vec{\alpha}_m^T \end{pmatrix} B = \begin{pmatrix} \vec{\alpha}_1^T B \\ \vdots \\ \vec{\alpha}_m^T B \end{pmatrix}$$

Example (continued)

3

$$AB = (\vec{a}_1, \dots, \vec{a}_n) \begin{pmatrix} \vec{\beta}_1^T \\ \vdots \\ \vec{\beta}_n^T \end{pmatrix} = \vec{a}_1 \vec{\beta}_1^T + \vec{a}_2 \vec{\beta}_2^T + \dots \vec{a}_n \vec{\beta}_n^T$$

4

$$AB = \begin{pmatrix} \vec{\alpha}_1^T \\ \vdots \\ \vec{\alpha}_m^T \end{pmatrix} (\vec{b}_1, \dots, \vec{b}_k) = \begin{pmatrix} \vec{\alpha}_1^T \vec{b}_1 & \vec{\alpha}_1^T \vec{b}_2 & \dots & \vec{\alpha}_1^T \vec{b}_k \\ \vec{\alpha}_2^T \vec{b}_1 & \vec{\alpha}_2^T \vec{b}_2 & \dots & \vec{\alpha}_2^T \vec{b}_k \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\alpha}_m^T \vec{b}_1 & \vec{\alpha}_m^T \vec{b}_m & \dots & \vec{\alpha}_m^T \vec{b}_k \end{pmatrix}$$

Example (continued)

One can also partition A and B as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

in a way that dimensions match. Then

$$AB = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

Problem

Let A be a square matrix. Compute A^k where $A = \begin{pmatrix} I & X \\ O & O \end{pmatrix}$.

Solution

$$A^2 = \cdots = A.$$

Hence, $A^k = A$ for all $k \geq 2$.