

# Math 221: LINEAR ALGEBRA

## Chapter 5. Vector Space $\mathbb{R}^n$

### §5-1. Subspaces and Spanning

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Emory University, 2021 Spring

(last updated on 01/12/2023)



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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.



# Linear Algebra with Applications

## Lecture Notes

### Current Lecture Notes Revision: Version 2018 — Revision B

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The null space and the image space

The Eigenspace

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## Subspaces of $\mathbb{R}^n$

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## Definitions

1.  $\mathbb{R}$  denotes the set of **real** numbers, and is an example of a set of **scalars**.
2.  $\mathbb{R}^n$  is the set of all n-tuples of real numbers, i.e.,

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}, 1 \leq i \leq n\}.$$

3. The **vector space**  $\mathbb{R}^n$  consists of the set  $\mathbb{R}^n$  written as **column matrices**, along with the (matrix) operations of addition and scalar multiplication. Unless stated otherwise,  $\mathbb{R}^n$  means the vector space  $\mathbb{R}^n$ .

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## Remark

$\mathbb{R}^n$  is a concrete example of the abstract vector space will be studied in the next chapter.



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Another example:  $\vec{u} = \begin{bmatrix} -2 \\ 3 \\ 0.7 \\ 5 \\ \pi \end{bmatrix}$  is a vector in  $\mathbb{R}^5$ , written  $\vec{u} \in \mathbb{R}^5$ .

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To save space on the page, the same vector  $\vec{u}$  may be written instead as a row matrix by taking the transpose of the column:

$$\vec{u} = \begin{bmatrix} -2, & 3, & 0.7, & 5, & \pi \end{bmatrix}^T.$$

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### Definition (Subspaces)

A subset  $U$  of  $\mathbb{R}^n$  is a **subspace** of  $\mathbb{R}^n$  if

- S1. The zero vector of  $\mathbb{R}^n$ ,  $\vec{0}_n$ , is in  $U$ ;
- S2.  $U$  is closed under addition, i.e., for all  $\vec{u}, \vec{w} \in U$ ,  $\vec{u} + \vec{w} \in U$ ;
- S3.  $U$  is closed under scalar multiplication, i.e., for all  $\vec{u} \in U$  and  $k \in \mathbb{R}$ ,  $k\vec{u} \in U$ .

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Both subset  $U = \{\vec{0}_n\}$  and  $\mathbb{R}^n$  itself are subspaces of  $\mathbb{R}^n$ . Any other subspace of  $\mathbb{R}^n$  is called a **proper** subspace of  $\mathbb{R}^n$ .

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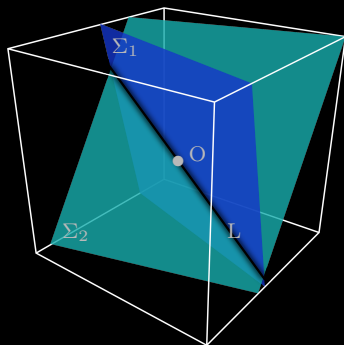
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### Notation

If  $U$  is a subset of  $\mathbb{R}^n$ , we write  $U \subseteq \mathbb{R}^n$ .





## Example

In  $\mathbb{R}^3$ , the line  $L$  through the origin that is parallel to the vector  $\vec{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$  has (vector) equation  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}, t \in \mathbb{R}$ , so

$$L = \left\{ t\vec{d} \mid t \in \mathbb{R} \right\}.$$

**Claim.**  $L$  is a subspace of  $\mathbb{R}^3$ .

- First:  $\vec{0}_3 \in L$  since  $0\vec{d} = \vec{0}_3$ .
- Suppose  $\vec{u}, \vec{v} \in L$ . Then by definition,  $\vec{u} = s\vec{d}$  and  $\vec{v} = t\vec{d}$ , for some  $s, t \in \mathbb{R}$ . Thus

$$\vec{u} + \vec{v} = s\vec{d} + t\vec{d} = (s + t)\vec{d}.$$

Since  $s + t \in \mathbb{R}$ ,  $\vec{u} + \vec{v} \in L$ ; i.e.,  $L$  is closed under addition.

### Example (continued)

- Suppose  $\vec{u} \in L$  and  $k \in \mathbb{R}$  ( $k$  is a scalar). Then  $\vec{u} = t\vec{d}$ , for some  $t \in \mathbb{R}$ , so

$$k\vec{u} = k(t\vec{d}) = (kt)\vec{d}.$$

Since  $kt \in \mathbb{R}$ ,  $k\vec{u} \in L$ ; i.e.,  $L$  is closed under scalar multiplication.

- Therefore,  $L$  is a subspace of  $\mathbb{R}^3$ .

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- Suppose  $\vec{u} \in L$  and  $k \in \mathbb{R}$  ( $k$  is a scalar). Then  $\vec{u} = t\vec{d}$ , for some  $t \in \mathbb{R}$ , so

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Since  $kt \in \mathbb{R}$ ,  $k\vec{u} \in L$ ; i.e.,  $L$  is closed under scalar multiplication.

- Therefore,  $L$  is a subspace of  $\mathbb{R}^3$ .

### Remark

Note that there is nothing special about the vector  $\vec{d}$  used in this example; the same proof works for any **nonzero** vector  $\vec{d} \in \mathbb{R}^3$ , so any line through the origin is a subspace of  $\mathbb{R}^3$ .

## Example

In  $\mathbb{R}^3$ , let  $M$  denote the plane through the origin having equation

$3x - 2y + z = 0$ ; then  $M$  has normal vector  $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ . If  $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ , then

$$M = \{ \vec{u} \in \mathbb{R}^3 \mid \vec{n} \cdot \vec{u} = 0 \},$$

where  $\vec{n} \cdot \vec{u}$  is the dot product of vectors  $\vec{n}$  and  $\vec{u}$ .

**Claim.**  $M$  is a subspace of  $\mathbb{R}^3$ .

- First:  $\vec{0}_3 \in M$  since  $\vec{n} \cdot \vec{0}_3 = 0$ .
- Suppose  $\vec{u}, \vec{v} \in M$ . Then by definition,  $\vec{n} \cdot \vec{u} = 0$  and  $\vec{n} \cdot \vec{v} = 0$ , so

$$\vec{n} \cdot (\vec{u} + \vec{v}) = \vec{n} \cdot \vec{u} + \vec{n} \cdot \vec{v} = 0 + 0 = 0,$$

and thus  $(\vec{u} + \vec{v}) \in M$ ; i.e.,  $M$  is closed under addition.

### Example (continued)

- Suppose  $\vec{u} \in M$  and  $k \in \mathbb{R}$ . Then  $\vec{n} \cdot \vec{u} = 0$ , so

$$\vec{n} \cdot (k\vec{u}) = k(\vec{n} \cdot \vec{u}) = k(0) = 0,$$

and thus  $k\vec{u} \in M$ ; i.e.,  $M$  is closed under scalar multiplication.

- Therefore,  $M$  is a subspace of  $\mathbb{R}^3$ .

### Example (continued)

- Suppose  $\vec{u} \in M$  and  $k \in \mathbb{R}$ . Then  $\vec{n} \cdot \vec{u} = 0$ , so

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and thus  $k\vec{u} \in M$ ; i.e.,  $M$  is closed under scalar multiplication.

- Therefore,  $M$  is a subspace of  $\mathbb{R}^3$ .

### Remark

As in the previous example, there is nothing special about the plane chosen for this example; any plane through the origin is a subspace of  $\mathbb{R}^3$ .

### Problem

$$\text{Is } U = \left\{ \left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right] \mid a, b, c, d \in \mathbb{R} \quad \text{and} \quad 2a - b = c + 2d \right\} \text{ a subspace of } \mathbb{R}^4?$$

Justify your answer.

### Problem

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Justify your answer.

### Solution

The zero vector of  $\mathbb{R}^4$  is the vector  $\left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right]$  with  $a = b = c = d = 0$ .

In this case,  $2a - b = 2(0) + 0 = 0$  and  $c + 2d = 0 + 2(0) = 0$ , so  $2a - b = c + 2d$ . Therefore,  $\vec{0}_4 \in U$ . ■



## Solution (continued)

Suppose

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \quad \text{are in } U.$$

Then  $2a_1 - b_1 = c_1 + 2d_1$  and  $2a_2 - b_2 = c_2 + 2d_2$ . Now

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix},$$

and

$$\begin{aligned} 2(a_1 + a_2) - (b_1 + b_2) &= (2a_1 - b_1) + (2a_2 - b_2) \\ &= (c_1 + 2d_1) + (c_2 + 2d_2) \\ &= (c_1 + c_2) + 2(d_1 + d_2). \end{aligned}$$

Therefore,  $\vec{v}_1 + \vec{v}_2 \in U$ .

### Solution (continued)

Finally, suppose

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U \quad \text{and} \quad k \in \mathbb{R}.$$

Then  $2a - b = c + 2d$ . Now

$$k\vec{v} = k \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix},$$

and

$$2ka - kb = k(2a - b) = k(c + 2d) = kc + 2kd.$$

Therefore,  $k\vec{v} \in U$ .

It follows from the **Subspace Test** that  $U$  is a subspace of  $\mathbb{R}^4$ .

### Problem

Is  $U = \left\{ \begin{bmatrix} 1 \\ s \\ t \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$  a subspace of  $\mathbb{R}^3$ ? Justify your answer.

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Is  $U = \left\{ \begin{bmatrix} 1 \\ s \\ t \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$  a subspace of  $\mathbb{R}^3$ ? Justify your answer.

### Solution

Note that  $\vec{0}_3 \notin U$ , and thus  $U$  is not a subspace of  $\mathbb{R}^3$ .

(You could also show that  $U$  is not closed under addition, or not closed under scalar multiplication.)

### Problem

Is  $U = \left\{ \begin{bmatrix} r \\ 0 \\ s \end{bmatrix} \mid r, s \in \mathbb{R} \text{ and } r^2 + s^2 = 0 \right\}$  a subspace of  $\mathbb{R}^3$ ?

Justify your answer.

### Problem

$$\text{Is } U = \left\{ \begin{bmatrix} r \\ 0 \\ s \end{bmatrix} \mid r, s \in \mathbb{R} \text{ and } r^2 + s^2 = 0 \right\} \text{ a subspace of } \mathbb{R}^3?$$

Justify your answer.

### Solution

Since  $r \in \mathbb{R}$ ,  $r^2 \geq 0$  with equality if and only if  $r = 0$ . Similarly,  $s \in \mathbb{R}$  implies  $s^2 \geq 0$ , and  $s^2 = 0$  if and only if  $s = 0$ . This means  $r^2 + s^2 = 0$  if and only if  $r^2 = s^2 = 0$ ; thus  $r^2 + s^2 = 0$  if and only if  $r = s = 0$ . Therefore  $U$  contains only  $\vec{0}_3$ , the zero vector, i.e.,  $U = \{\vec{0}_3\}$ . As we already observed,  $\{\vec{0}_n\}$  is a subspace of  $\mathbb{R}^n$ , and therefore  $U$  is a subspace of  $\mathbb{R}^3$ . ■

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# The null space and the image space

## Definitions (Null Space and Image Space)

Let  $A$  be an  $m \times n$  matrix. The **null space** of  $A$  is defined as

$$\text{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}_m\},$$

and the **image space** of  $A$  is defined as

$$\text{im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

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and the **image space** of  $A$  is defined as

$$\text{im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

## Remark

1. Since  $A$  is  $m \times n$  and  $\vec{x} \in \mathbb{R}^n$ ,  $A\vec{x} \in \mathbb{R}^m$ , so  $\text{im}(A) \subseteq \mathbb{R}^m$  while  $\text{null}(A) \subseteq \mathbb{R}^n$ .
2. Image space is also called **column space** of  $A$ , denoted as  $\text{col}(A)$ :

$$\text{col}(A) = \text{span}(\vec{a}_1, \dots, \vec{a}_n) = \text{im}(A).$$

## Problem

Prove that if  $A$  is an  $m \times n$  matrix, then  $\text{null}(A)$  is a subspace of  $\mathbb{R}^n$ .

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S2. Let  $\vec{x}, \vec{y} \in \text{null}(A)$ . Then  $A\vec{x} = \vec{0}_m$  and  $A\vec{y} = \vec{0}_m$ , so

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

and thus  $\vec{x} + \vec{y} \in \text{null}(A)$ .

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$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

and thus  $\vec{x} + \vec{y} \in \text{null}(A)$ .

S3. Let  $\vec{x} \in \text{null}(A)$  and  $k \in \mathbb{R}$ . Then  $A\vec{x} = \vec{0}_m$ , so

$$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$

and thus  $k\vec{x} \in \text{null}(A)$ .

Therefore,  $\text{null}(A)$  is a subspace of  $\mathbb{R}^n$ . ■

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Prove that if  $A$  is an  $m \times n$  matrix, then  $\text{im}(A)$  is a subspace of  $\mathbb{R}^m$ .

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### Proof.

S1. Since  $\vec{0}_n \in \mathbb{R}^n$  and  $A\vec{0}_n = \vec{0}_m$ ,  $\vec{0}_m \in \text{im}(A)$ .



## Problem

Prove that if  $A$  is an  $m \times n$  matrix, then  $\text{im}(A)$  is a subspace of  $\mathbb{R}^m$ .

## Proof.

S1. Since  $\vec{0}_n \in \mathbb{R}^n$  and  $A\vec{0}_n = \vec{0}_m$ ,  $\vec{0}_m \in \text{im}(A)$ .

S2. Let  $\vec{x}, \vec{y} \in \text{im}(A)$ . Then  $\vec{x} = A\vec{u}$  and  $\vec{y} = A\vec{v}$  for some  $\vec{u}, \vec{v} \in \mathbb{R}^n$ , so

$$\vec{x} + \vec{y} = A\vec{u} + A\vec{v} = A(\vec{u} + \vec{v}).$$

Since  $\vec{u} + \vec{v} \in \mathbb{R}^n$ , it follows that  $\vec{x} + \vec{y} \in \text{im}(A)$ .

## Problem

Prove that if  $A$  is an  $m \times n$  matrix, then  $\text{im}(A)$  is a subspace of  $\mathbb{R}^m$ .

## Proof.

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$$\vec{x} + \vec{y} = A\vec{u} + A\vec{v} = A(\vec{u} + \vec{v}).$$

Since  $\vec{u} + \vec{v} \in \mathbb{R}^n$ , it follows that  $\vec{x} + \vec{y} \in \text{im}(A)$ .

S3. Let  $\vec{x} \in \text{im}(A)$  and  $k \in \mathbb{R}$ . Then  $\vec{x} = A\vec{u}$  for some  $\vec{u} \in \mathbb{R}^n$ , and thus

$$k\vec{x} = k(A\vec{u}) = A(k\vec{u}).$$

Since  $k\vec{u} \in \mathbb{R}^n$ , it follows that  $k\vec{x} \in \text{im}(A)$ .

Therefore,  $\text{im}(A)$  is a subspace of  $\mathbb{R}^m$ . ■

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# The Eigenspace

## Definition (Eigenspace)

Let  $A$  be an  $n \times n$  matrix and  $\lambda \in \mathbb{R}$ . The eigenspace of  $A$  corresponding to  $\lambda$  is the set

$$E_{\lambda}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

### Example

$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$  has two eigenvalues:  $\lambda_1 = 2$  and  $\lambda_2 = 5$  with corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

Hence,

$$E_{\lambda_1}(A) = E_2(A) = \{t\vec{v}_1 | t \in \mathbb{R}\}$$

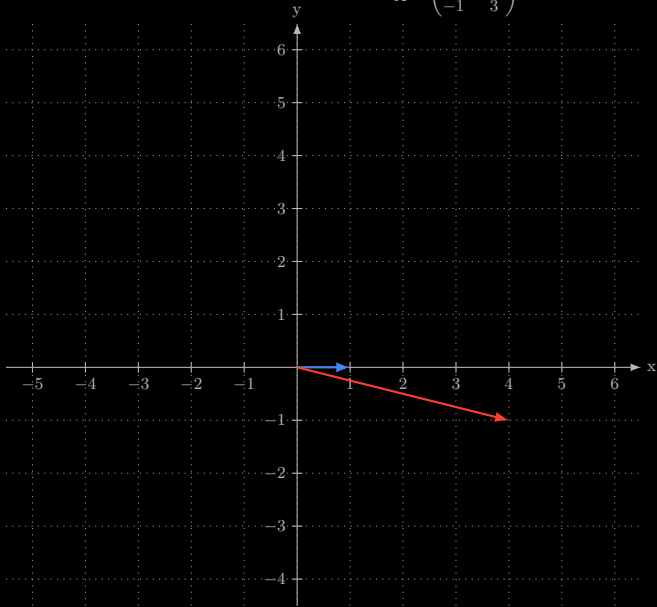
$$E_{\lambda_2}(A) = E_5(A) = \{t\vec{v}_2 | t \in \mathbb{R}\}$$



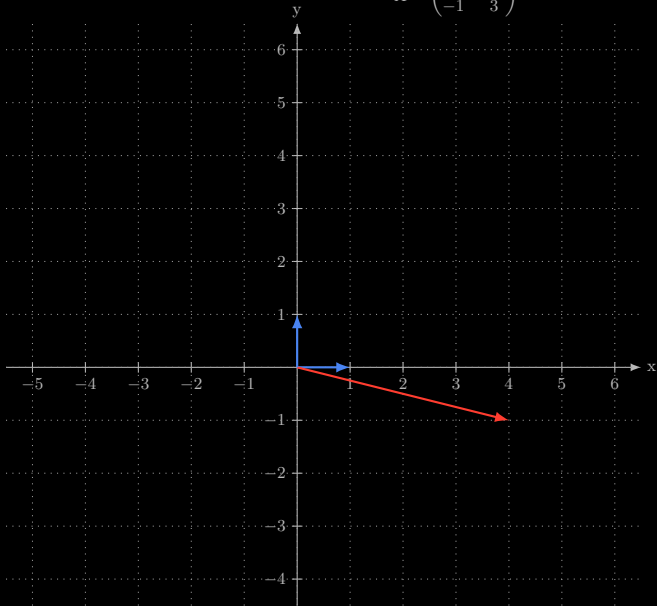




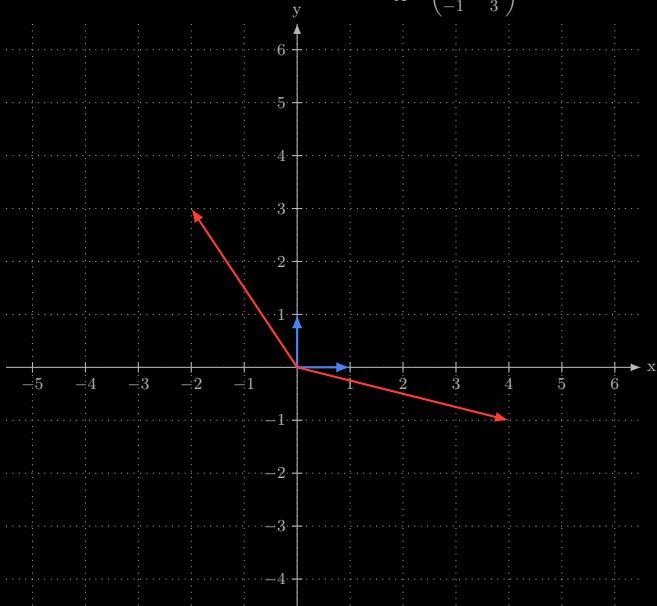
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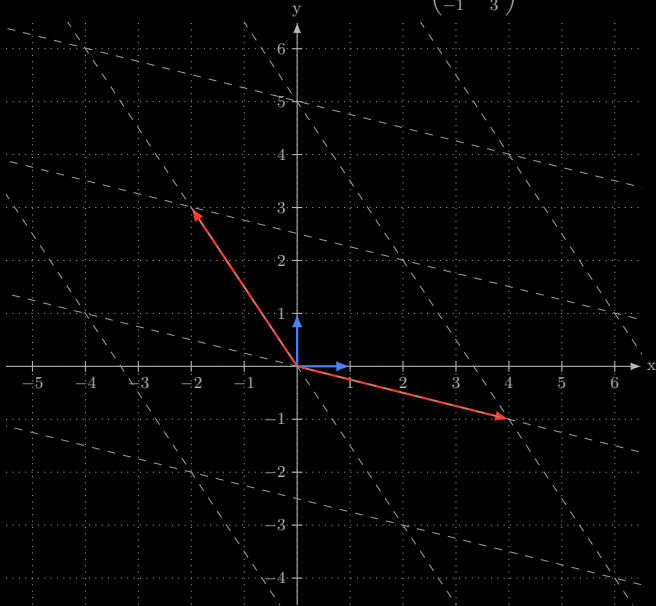
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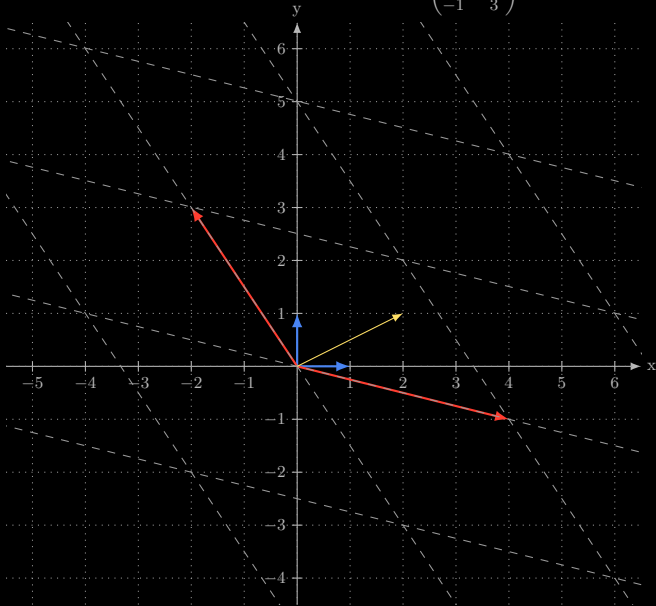
$$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$$



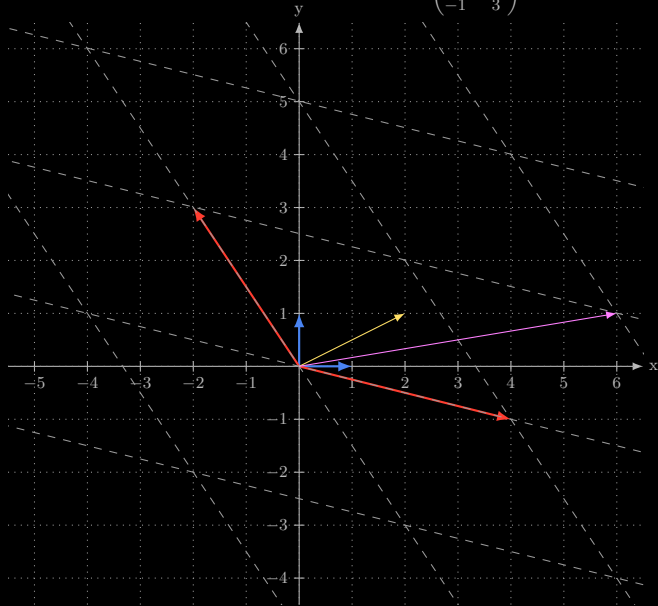
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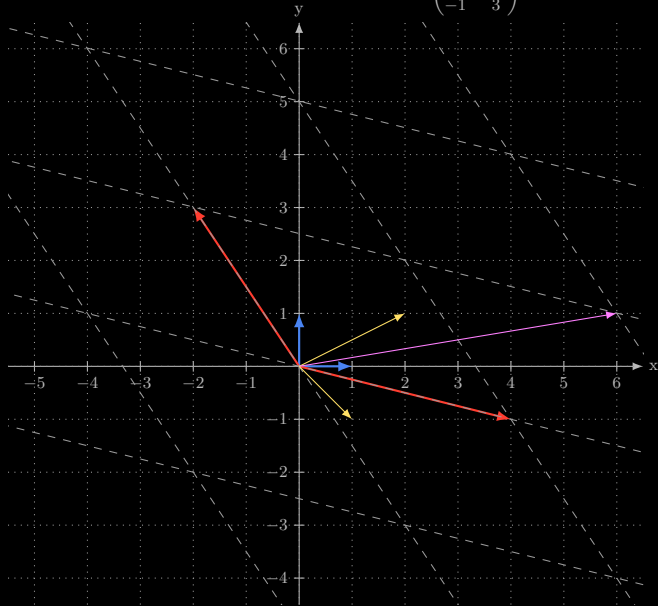
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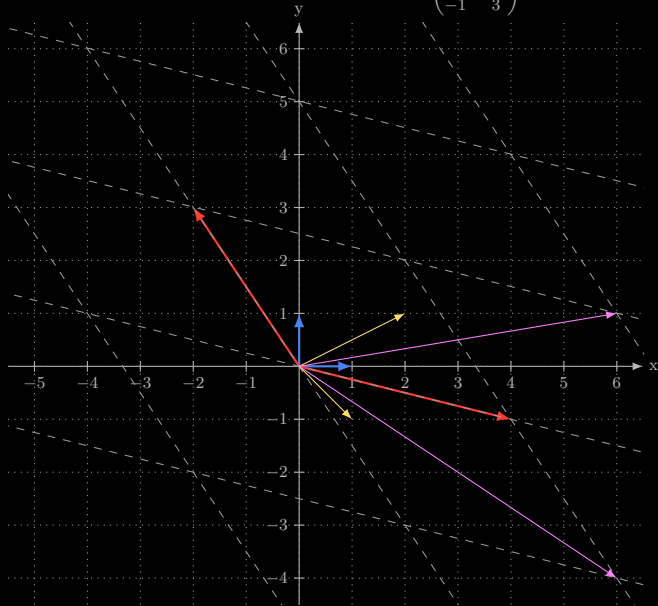
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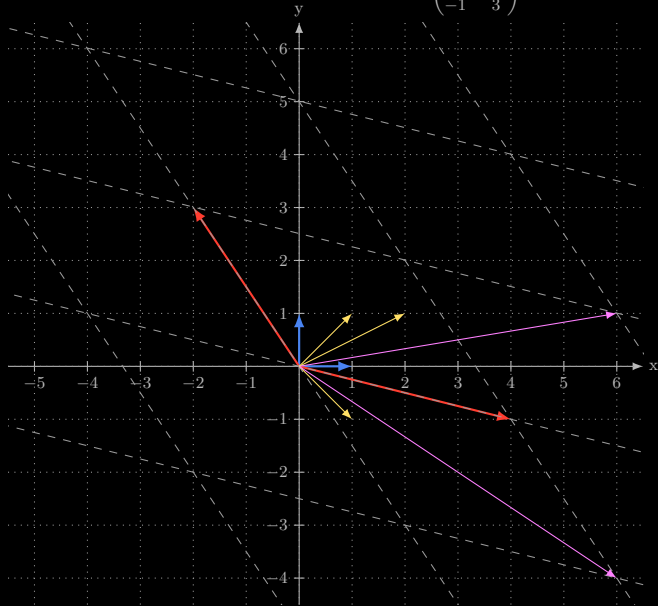


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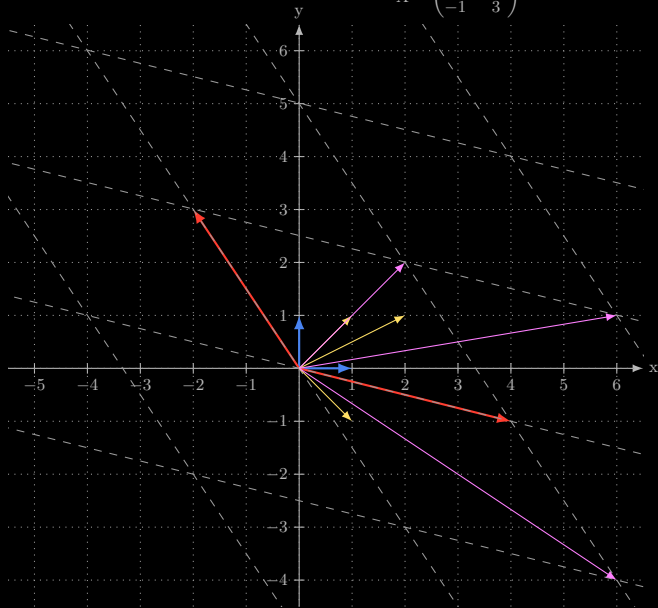


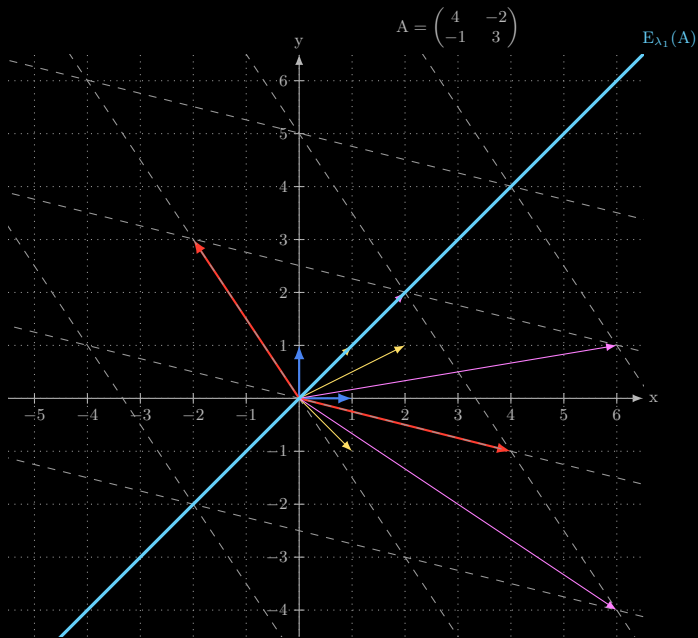


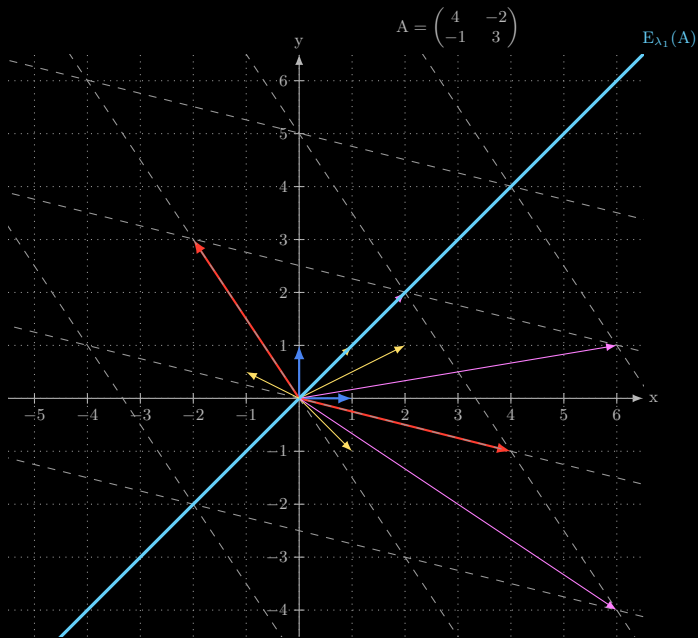
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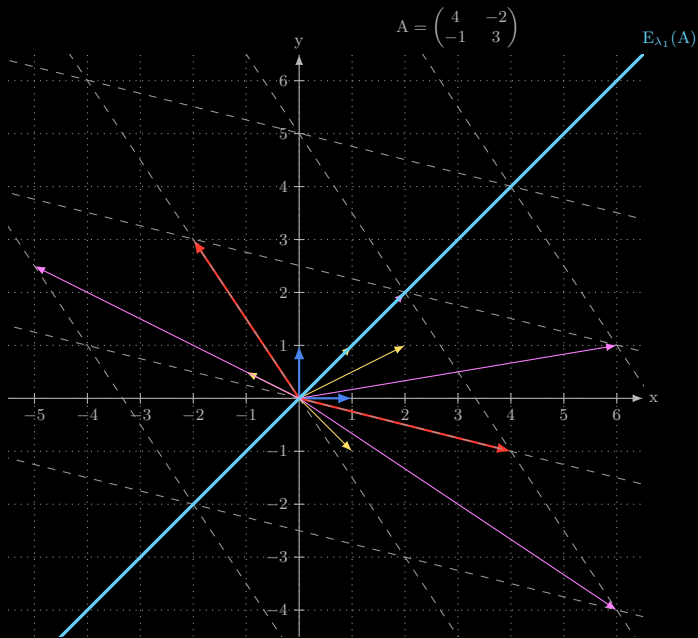


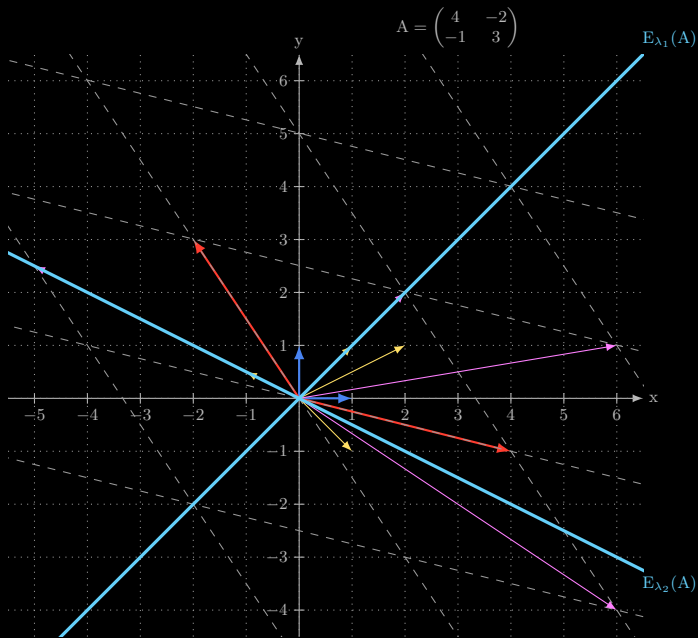
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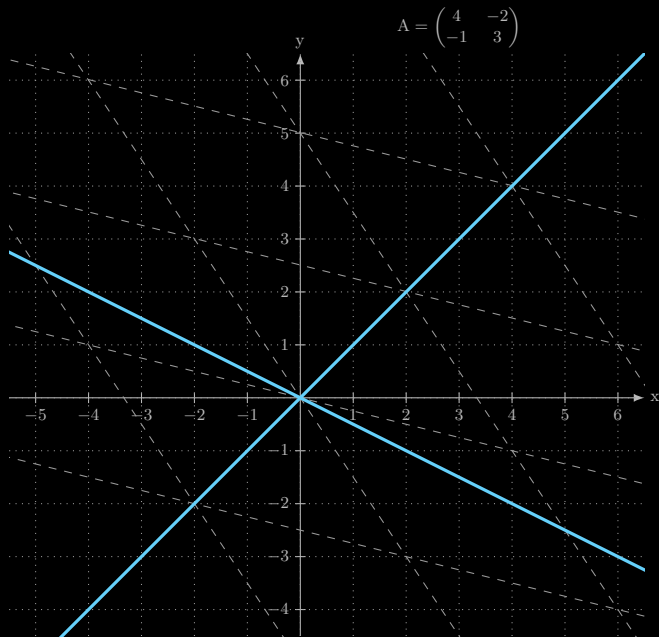












$$E_2(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$E_5(A) = \left\{ t \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

Note that

$$\begin{aligned}E_{\lambda}(A) &= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} \\&= \left\{\vec{x} \in \mathbb{R}^n \mid \lambda\vec{x} - A\vec{x} = \vec{0}_n\right\} \\&= \left\{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\right\}\end{aligned}$$

showing that

$$E_{\lambda}(A) = \text{null}(\lambda I - A).$$



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showing that

$$E_\lambda(A) = \text{null}(\lambda I - A).$$

It follows that

- ▶ if  $\lambda$  is **not** an eigenvalue of  $A$ , then  $E_\lambda(A) = \{\vec{0}_n\}$ ;
- ▶ the nonzero vectors of  $E_\lambda(A)$  are the eigenvectors of  $A$  corresponding to  $\lambda$ ;
- ▶ the eigenspace of  $A$  corresponding to  $\lambda$  is a subspace of  $\mathbb{R}^n$ .

Copyright

Subspaces of  $\mathbb{R}^n$

The null space and the image space

The Eigenspace

**Linear Combinations and Spanning Sets**

Spanning sets of  $\text{null}(A)$  and  $\text{im}(A)$

# Linear Combinations and Spanning Sets

# Linear Combinations and Spanning Sets

## Definition (Linear Combinations and Spanning)

Let  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$  and  $t_1, t_2, \dots, t_k \in \mathbb{R}$ . Then the vector

$$\vec{x} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$$

is called a **linear combination** of the vectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ ; the (scalars)  $t_1, t_2, \dots, t_k \in \mathbb{R}$  are the coefficients. The set of all linear combinations of  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$  is called **the span of  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$** , and is written

$$\text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} = \{t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k \mid t_1, t_2, \dots, t_k \in \mathbb{R}\}.$$

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Additional Terminology. If  $U = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ , then

- ▶ **U is spanned by** the vectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ .
- ▶ the vectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$  **span U**.
- ▶ the set of vectors  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is a **spanning set** for U.

### Example

Let  $\vec{x} \in \mathbb{R}^3$  be a nonzero vector. Then  $\text{span}\{\vec{x}\} = \{k\vec{x} \mid k \in \mathbb{R}\}$  is a line through the origin having direction vector  $\vec{x}$ .

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### Example

Let  $\vec{x}, \vec{y} \in \mathbb{R}^3$  be nonzero vectors that are not parallel. Then

$$\text{span}\{\vec{x}, \vec{y}\} = \{k\vec{x} + t\vec{y} \mid k, t \in \mathbb{R}\}$$

is a plane through the origin containing  $\vec{x}$  and  $\vec{y}$ .

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is a plane through the origin containing  $\vec{x}$  and  $\vec{y}$ .

How would you find the equation of this plane?



### Problem

$$\text{Let } \vec{x} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}. \text{ Is } \vec{x} \in \text{span}\{\vec{y}, \vec{z}\}?$$

### Problem

$$\text{Let } \vec{x} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}. \text{ Is } \vec{x} \in \text{span}\{\vec{y}, \vec{z}\}?$$

### Solution

An equivalent question is: can  $\vec{x}$  be expressed as a linear combination of  $\vec{y}$  and  $\vec{z}$ ?

Suppose there exist  $a, b \in \mathbb{R}$  so that  $\vec{x} = a\vec{y} + b\vec{z}$ . Then

$$\begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solve this system of four linear equations in the two variables  $a$  and  $b$ .

Solution (continued)

$$\left[ \begin{array}{cc|c} 2 & -1 & 8 \\ 1 & 0 & 3 \\ -3 & 2 & -13 \\ 5 & -3 & 20 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right]$$

Since the system has no solutions,  $\vec{x} \notin \text{span}\{\vec{y}, \vec{z}\}$ .



## Problem

$$\text{Let } \vec{w} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 21 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}. \text{ Is } \vec{w} \in \text{span}\{\vec{y}, \vec{z}\}?$$

This is almost identical to a previous problem, except that  $\vec{w}$  (above) has one entry that is different from the vector  $\vec{x}$  of that problem.

## Problem

$$\text{Let } \vec{w} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ \textcolor{red}{21} \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}. \text{ Is } \vec{w} \in \text{span}\{\vec{y}, \vec{z}\}?$$

This is almost identical to a previous problem, except that  $\vec{w}$  (above) has one entry that is different from the vector  $\vec{x}$  of that problem.

## Solution

In this case, the system of linear equations is consistent, and gives us  $\vec{w} = 3\vec{y} - 2\vec{z}$ , so  $\vec{w} \in \text{span}\{\vec{y}, \vec{z}\}$ .

## Theorem

Let  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in \mathbb{R}^n$  and let  $U = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ . Then

1.  $U$  is a subspace of  $\mathbb{R}^n$  containing each  $\vec{x}_i$ ,  $1 \leq i \leq k$ ;
2. if  $W$  is a subspace of  $\mathbb{R}^n$  and  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k \in W$ , then  $U \subseteq W$ .

## Theorem

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## Remark

Property 2 is saying that  $U$  is the “smallest” subspace of  $\mathbb{R}^n$  that contains  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ .

Proof. ( Part 1 of Theorem )

Since  $U = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  and  $0\vec{x}_1 + 0\vec{x}_2 + \dots + 0\vec{x}_k = \vec{0}_n$ ,  $\vec{0}_n \in U$ .



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Since  $U = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  and  $0\vec{x}_1 + 0\vec{x}_2 + \dots + 0\vec{x}_k = \vec{0}_n$ ,  $\vec{0}_n \in U$ .

Suppose  $\vec{x}, \vec{y} \in U$ . Then for some  $s_i, t_i \in \mathbb{R}$ ,  $1 \leq i \leq k$ ,

$$\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \dots + s_k\vec{x}_k$$

$$\vec{y} = t_1\vec{x}_1 + t_2\vec{x}_2 + \dots + t_k\vec{x}_k$$

Thus

$$\begin{aligned}\vec{x} + \vec{y} &= (s_1\vec{x}_1 + s_2\vec{x}_2 + \dots + s_k\vec{x}_k) + (t_1\vec{x}_1 + t_2\vec{x}_2 + \dots + t_k\vec{x}_k) \\ &= (s_1 + t_1)\vec{x}_1 + (s_2 + t_2)\vec{x}_2 + \dots + (s_k + t_k)\vec{x}_k.\end{aligned}$$

Since  $s_i + t_i \in \mathbb{R}$  for all  $1 \leq i \leq k$ ,  $\vec{x} + \vec{y} \in U$ , i.e.,  $U$  is closed under addition.

Proof. ( Part 1 of Theorem – continued)

Suppose  $\vec{x} \in U$  and  $a \in \mathbb{R}$ . Then for some  $s_i \in \mathbb{R}$ ,  $1 \leq i \leq k$ ,

$$\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$$

Thus

$$\begin{aligned} a\vec{x} &= a(s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k) \\ &= (as_1)\vec{x}_1 + (as_2)\vec{x}_2 + \cdots + (as_k)\vec{x}_k. \end{aligned}$$

Since  $as_i \in \mathbb{R}$  for all  $1 \leq i \leq k$ ,  $a\vec{x} \in U$ . Hence, **U is closed under scalar multiplication.**

Proof. ( Part 1 of Theorem – continued)

Suppose  $\vec{x} \in U$  and  $a \in \mathbb{R}$ . Then for some  $s_i \in \mathbb{R}$ ,  $1 \leq i \leq k$ ,

$$\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$$

Thus

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Since  $as_i \in \mathbb{R}$  for all  $1 \leq i \leq k$ ,  $a\vec{x} \in U$ . Hence, **U is closed under scalar multiplication.**

Therefore,  $U$  is a subspace of  $\mathbb{R}^n$ . Furthermore, since

$$\vec{x}_i = \sum_{j=1}^{i-1} 0\vec{x}_j + 1\vec{x}_i + \sum_{j=i+1}^k 0\vec{x}_j,$$

it follows that  $\vec{x}_i \in U$  for all  $i$ ,  $1 \leq i \leq k$ .

Proof. ( Part 1 of Theorem – continued)

Suppose  $\vec{x} \in U$  and  $a \in \mathbb{R}$ . Then for some  $s_i \in \mathbb{R}$ ,  $1 \leq i \leq k$ ,

$$\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$$

Thus

$$\begin{aligned} a\vec{x} &= a(s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k) \\ &= (as_1)\vec{x}_1 + (as_2)\vec{x}_2 + \cdots + (as_k)\vec{x}_k. \end{aligned}$$

Since  $as_i \in \mathbb{R}$  for all  $1 \leq i \leq k$ ,  $a\vec{x} \in U$ . Hence, **U is closed under scalar multiplication.**

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Proof. (Part 2 of Theorem)

Let  $W \subset \mathbb{R}^n$  be a subspace that contains  $\vec{x}_1, \dots, \vec{x}_n$ . We need to prove that  $U \subseteq W$ .

### Proof. (Part 2 of Theorem)

Let  $W \subset \mathbb{R}^n$  be a subspace that contains  $\vec{x}_1, \dots, \vec{x}_n$ . We need to prove that  $U \subseteq W$ .

Suppose  $\vec{x} \in U$ . Then  $\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \dots + s_k\vec{x}_k$  for some  $s_i \in \mathbb{R}$ ,  $1 \leq i \leq k$ . Since  $W$  contain each  $\vec{x}_i$  and  $W$  is closed under scalar multiplication, it follows that  $s_i\vec{x}_i \in W$  for each  $i$ ,  $1 \leq i \leq k$ . Furthermore, since  $W$  is closed under addition,  $\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \dots + s_k\vec{x}_k \in W$ . Therefore,  $U \subseteq W$ .

Problem (revisited)

$$\text{Is } U = \left\{ \left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right] \mid a, b, c, d \in \mathbb{R} \quad \text{and} \quad 2a - b = c + 2d \right\} \text{ a subspace of } \mathbb{R}^4?$$

Justify your answer.

### Problem (revisited)

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Justify your answer.

### Solution (Another)

$$\text{Let } \vec{v} = \left[ \begin{array}{c} a \\ b \\ c \\ d \end{array} \right] \in U. \text{ Since } 2a - b = c + 2d, c = 2a - b - 2d, \text{ and thus}$$

$$U = \left\{ \left[ \begin{array}{c} a \\ b \\ 2a - b - 2d \\ d \end{array} \right] \mid a, b, d \in \mathbb{R} \right\} = \text{span} \left\{ \left[ \begin{array}{c} 1 \\ 0 \\ 2 \\ 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 1 \\ -1 \\ 0 \end{array} \right], \left[ \begin{array}{c} 0 \\ 0 \\ -2 \\ 1 \end{array} \right] \right\}.$$

By a previous Theorem,  $U$  is a subspace of  $\mathbb{R}^4$ .



### Problem

Let  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,  $U_1 = \text{span}\{\vec{x}, \vec{y}\}$ , and  $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$ . Prove that  $U_1 = U_2$ .

## Problem

Let  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,  $U_1 = \text{span}\{\vec{x}, \vec{y}\}$ , and  $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$ . Prove that  $U_1 = U_2$ .

## Solution

To show that  $U_1 = U_2$ , prove that  $U_1 \subseteq U_2$ , and  $U_2 \subseteq U_1$ . We begin by noting that, by the first part of the previous Theorem,  $U_1$  and  $U_2$  are subspaces of  $\mathbb{R}^n$ .

## Problem

Let  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,  $U_1 = \text{span}\{\vec{x}, \vec{y}\}$ , and  $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$ . Prove that  $U_1 = U_2$ .

## Solution

To show that  $U_1 = U_2$ , prove that  $U_1 \subseteq U_2$ , and  $U_2 \subseteq U_1$ . We begin by noting that, by the first part of the previous Theorem,  $U_1$  and  $U_2$  are subspaces of  $\mathbb{R}^n$ .

Since  $2\vec{x} - \vec{y}, 2\vec{y} + \vec{x} \in U_1$ , it follows from the second part of the previous Theorem that  $\text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\} \subseteq U_1$ , i.e.,  $U_2 \subseteq U_1$ .

Also, since

$$\begin{aligned}\vec{x} &= \frac{2}{5}(2\vec{x} - \vec{y}) + \frac{1}{5}(2\vec{y} + \vec{x}), \\ \vec{y} &= -\frac{1}{5}(2\vec{x} - \vec{y}) + \frac{2}{5}(2\vec{y} + \vec{x}),\end{aligned}$$

$\vec{x}, \vec{y} \in U_2$ . Therefore, by the second part of the previous Theorem,  $\text{span}\{\vec{x}, \vec{y}\} \subseteq U_2$ , i.e.,  $U_1 \subseteq U_2$ . The result now follows.

## Problem

Show that  $\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ , where  $\vec{e}_j$  denote the  $j^{\text{th}}$  column of  $I_n$ .

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## Solution

Let  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ . Then  $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$ , where

$x_1, x_2, \dots, x_n \in \mathbb{R}$ . Therefore,  $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ , and thus  $\mathbb{R}^n \subseteq \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ .

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## Solution

Let  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ . Then  $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$ , where

$x_1, x_2, \dots, x_n \in \mathbb{R}$ . Therefore,  $\vec{x} \in \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ , and thus  $\mathbb{R}^n \subseteq \text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ .

Conversely, since  $\vec{e}_i \in \mathbb{R}^n$  for each  $i$ ,  $1 \leq i \leq n$  (and  $\mathbb{R}^n$  is a vector space), it follows that  $\text{span}\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\} \subseteq \mathbb{R}^n$ . The equality now follows.

## Problem

$$\text{Let } \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Does  $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$  span  $\mathbb{R}^4$ ? (Equivalently, is  $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \mathbb{R}^4$ ?)

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## Solution

To prove  $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \mathbb{R}^4$ , we need to prove two directions:

$$\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq \mathbb{R}^4 \quad \text{and} \quad \mathbb{R}^4 \subseteq \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}.$$



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For the first relation, since  $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4 \in \mathbb{R}^4$  (and  $\mathbb{R}^4$  is a vector space),  $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq \mathbb{R}^4$ .

### Solution (continued)

For the second relation, notice that

$$\vec{e}_1 = \vec{x}_1 - \vec{x}_2$$

$$\vec{e}_2 = \vec{x}_2 - \vec{x}_3$$

$$\vec{e}_3 = \vec{x}_3 - \vec{x}_4$$

$$\vec{e}_4 = \vec{x}_4,$$

showing that  $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \in \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ . Therefore, since  $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$  is a vector space,

$$\mathbb{R}^4 = \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\},$$

and the equality follows.

### Problem

$$\text{Let } \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Show that  $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$ .

### Problem

$$\text{Let } \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Show that  $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$ .

### Solution

If you check, you'll find that  $\vec{e}_2$  can not be written as a linear combination of  $\vec{u}_1, \vec{u}_2, \vec{u}_3$ , and  $\vec{u}_4$ .

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Subspaces of  $\mathbb{R}^n$

The null space and the image space

The Eigenspace

Linear Combinations and Spanning Sets

Spanning sets of  $\text{null}(A)$  and  $\text{im}(A)$

Spanning sets of  $\text{null}(A)$  and  $\text{im}(A)$

## Spanning sets of $\text{null}(A)$ and $\text{im}(A)$

### Lemma

Let  $A$  be an  $m \times n$  matrix, and let  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  denote a set of basic solutions to  $A\vec{x} = \vec{0}_m$ . Then

$$\text{null}(A) = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}.$$

### Lemma

Let  $A$  be an  $m \times n$  matrix with columns  $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n$ . Then

$$\text{im}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}.$$

Proof. (of  $\text{null}(A) = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$ )

" $\supseteq$ :" Because  $\vec{x}_i \in \text{null}(A)$  for each  $i$ ,  $1 \leq i \leq k$ , it follows that

$$\text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \text{null}(A).$$




Proof. (of  $\text{null}(A) = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$ )

" $\supseteq$ :" Because  $\vec{x}_i \in \text{null}(A)$  for each  $i$ ,  $1 \leq i \leq k$ , it follows that

$$\text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \text{null}(A).$$

" $\subseteq$ :" Every solution to  $A\vec{x} = \vec{0}_m$  can be expressed as a linear combination of basic solutions, implying that

$$\text{null}(A) \subseteq \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}.$$

Therefore,  $\text{null}(A) = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ . 

Proof. (of  $\text{im}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$ )

" $\subseteq$ :" Suppose  $\vec{y} \in \text{im}(A)$ . Then (by definition) there is a vector  $\vec{x} \in \mathbb{R}^n$  so that  $\vec{y} = A\vec{x}$ . Write  $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ . Then

$$\vec{y} = A\vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n.$$

Therefore,  $\vec{y} \in \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}$ , implying that

$$\text{im}(A) \subseteq \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}.$$

Proof. (continued)

Notice that for each  $j$ ,  $1 \leq j \leq n$ ,

$$\begin{aligned} A\vec{e}_j &= \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th row} \\ &= 0\vec{c}_1 + 0\vec{c}_2 + \dots + 0\vec{c}_{j-1} + 1\vec{c}_j + 0\vec{c}_{j+1} + \dots + 0\vec{c}_n \\ &= \vec{c}_j. \end{aligned}$$

Thus  $\vec{c}_j \in \text{im}(A)$  for each  $j$ ,  $1 \leq j \leq n$ . It follows that

$$\text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\} \subseteq \text{im}(A),$$

and therefore

$$\text{im}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}.$$

