Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations
§1-2. Gaussian Elimination

Le Chen[1]

Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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Row-Echelon Matrix

Definition

A matrix is called a row-echelon matrix if

- All rows consisting entirely of zeros are at the bottom.
- The first nonzero entry in each nonzero row is a 1 (called the leading 1 for that row).
- Each leading 1 is to the right of all leading 1's in rows above it.

A matrix is said to be in the row-echelon form (REF) if it a row-echelon matrix.

Example

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Definition

A matrix is called a reduced row-echelon matrix if

- Row-echelon matrix.
- Each leading 1 is the only nonzero entry in its column.

A matrix is said to be in the reduced row-echelon form (RREF) if it a reduced row-echelon matrix.

Example

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number.

Examples

Which of the following matrices are in the REF?

Which ones are in the RREF?

(a) $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example

Suppose that the following matrix is the augmented matrix of a system of linear equations. We see from this matrix that the system of linear equations has four equations and seven variables.

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & -3 & 4 & -2 & 5 & -7 & 0 & 4 \\ 0 & 0 & 1 & 8 & 0 & 3 & -7 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Note that the matrix is a row-echelon matrix.

- Each column of the matrix corresponds to a variable, and the leading variables are the variables that correspond to columns containing leading ones.
- The remaining variables are called non-leading variables.

We will use elementary row operations to transform a matrix to row-echelon (REF) or reduced row-echelon form (RREF).

Solving Systems of Linear Equations – Gaussian Elimination

Theorem

Every matrix can be brought to (reduced) row-echelon form by a sequence of elementary row operations.

Gaussian Elimination

To solve a system of linear equations proceed as follows:

1. Carry the augmented matrix to a reduced row-echelon matrix using elementary row operations.
2. If a row of the form $[0\ 0\ \cdots 0 \mid 1]$ occurs, the system is inconsistent.
3. Otherwise assign the nonleading variables (if any) parameters and use the equations corresponding to the reduced row-echelon matrix to solve for the leading variables in terms of the parameters.

Problem

Solve the system

$$\begin{cases} 2x & + & y & + & 3z & = & 1 \\ 2y & - & z & + & x & = & 0 \\ 9z & + & x & - & 4y & = & 2 \end{cases}$$

Solution

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & -4 & 9 & 2 \end{array}\right] \to^{r_1 \leftrightarrow r_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & -4 & 9 & 2 \end{array}\right]$$

$$\to^{-2r_1+r_2, -r_1+r_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & -6 & 10 & 2 \end{array}\right] \to^{-2r_2+r_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to^{-\frac{1}{3}r_2} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & 1 & -5/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to^{-2r_2+r_1} \left[\begin{array}{rrr|r} 1 & 0 & 7/3 & 2/3 \\ 0 & 1 & -5/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

Given the reduced row-echelon matrix

$$\left[\begin{array}{rrr|r} 1 & 0 & 7/3 & 2/3 \\ 0 & 1 & -5/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x and y are leading variables; z is a non-leading variable and so assign a parameter to z. Thus the solution to the original system is given by

$$\left.\begin{array}{rcrcr} x & = & \frac{2}{3} & - & \frac{7}{3}s \\ y & = & -\frac{1}{3} & + & \frac{5}{3}s \\ z & = & s & & \end{array}\right\} \text{ for all } s \in \mathbb{R}.$$

Problem

Solve the system
$$\left\{\begin{array}{rcrcrcr} x & + & y & + & 2z & = & -1 \\ y & + & 2x & + & 3z & = & 0 \\ z & - & 2y & & & = & 2 \end{array}\right.$$

Solution

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & 0 \\ 0 & -2 & 1 & 2 \end{array}\right] \rightarrow^{-2r_1+r_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & 1 & 2 \end{array}\right]$$

$$\rightarrow^{-1\cdot r_2} \left[\begin{array}{rrr|r} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 1 & 2 \end{array}\right] \rightarrow^{2r_2+r_3} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

$$\rightarrow^{\frac{1}{3}r_3} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \rightarrow^{-r_3+r_2,-r_3+r_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$$

The unique solution is $x = 5/3$, $y = -4/3$, $z = -2/3$.

Check your answer!

Problem

Solve the system $\left\{\begin{array}{rcrcrcr} -3x_1 & - & 9x_2 & + & x_3 & = & -9 \\ 2x_1 & + & 6x_2 & - & x_3 & = & 6 \\ x_1 & + & 3x_2 & - & x_3 & = & 2 \end{array}\right.$

Solution

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 2 \\ 2 & 6 & -1 & 6 \\ -3 & -9 & 1 & -9 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row of the final matrix corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 = 1$$

which is impossible!

Therefore, this system is inconsistent, i.e., it has no solutions.

Problem (General Patterns for Systems of Linear Equations)

Find all values of a, b and c (or conditions on a, b and c) so that the system

$$\begin{array}{rcrcrcl} 2x & + & 3y & + & az & = & b \\ & - & y & + & 2z & = & c \\ x & + & 3y & - & 2z & = & 1 \end{array}$$

has (i) a unique solution, (ii) no solutions, and (iii) infinitely many solutions. In (i) and (iii), find the solution(s).

Solution

$$\left[\begin{array}{rrr|r} 2 & 3 & a & b \\ 0 & -1 & 2 & c \\ 1 & 3 & -2 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & c \\ 2 & 3 & a & b \end{array}\right]$$

Solution (continued)

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & c \\ 2 & 3 & a & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & -1 & 2 & c \\ 0 & -3 & a+4 & b-2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & -c \\ 0 & -3 & a+4 & b-2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & a-2 & b-2-3c \end{array}\right]$$

Case 1. $a-2 \neq 0$, i.e., $a \neq 2$. In this case,

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 1 & \frac{b-2-3c}{a-2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1+3c-4\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 1 & 0 & -c+2\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 0 & 1 & \frac{b-2-3c}{a-2} \end{array}\right]$$

Solution (continued)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1+3c-4\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 1 & 0 & -c+2\left(\frac{b-2-3c}{a-2}\right) \\ 0 & 0 & 1 & \frac{b-2-3c}{a-2} \end{array}\right]$$

(i) When $a \neq 2$, the unique solution is

$$x = 1 + 3c - 4\left(\frac{b-2-3c}{a-2}\right)$$

$$y = -c + 2\left(\frac{b-2-3c}{a-2}\right)$$

$$z = \frac{b-2-3c}{a-2}$$

Solution (continued)

Case 2. If $a = 2$, then the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & a-2 & b-2-3c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 0 & b-2-3c \end{array}\right]$$

From this we see that the system has no solutions when $b - 2 - 3c \neq 0$.

(ii) When $a = 2$ and $b - 3c \neq 2$, the system has no solutions.

Solution (continued)

Finally when $a = 2$ and $b - 3c = 2$, the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 0 & b-2-3c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1+3c \\ 0 & 1 & -2 & -c \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and the system has infinitely many solutions.

(iii) When $a = 2$ and $b - 3c = 2$, the system has infinitely many solutions:

$$\left.\begin{array}{rcrcr} x & = & 1+3c & - & 4s \\ y & = & -c & + & 2s \\ z & = & s & & \end{array}\right\} \text{ for all } s \in \mathbb{R}.$$

$\blacksquare$

Rank

Definition

The rank of a matrix A, denoted rank A, is the number of leading 1's in any row-echelon matrix obtained from A by performing elementary row operations.

Suppose A is the augmented matrix of a consistent system of m linear equations in n variables, and rank $A = r$.

$$m\left\{\underbrace{\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]}_{n}\right. \rightarrow \underbrace{\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{r \text{ leading } 1'\text{s}}$$

Then the set of solutions to the system has $n - r$ parameters, so

- if $r < n$, there is at least one parameter, and the system has infinitely many solutions;
- if $r = n$, there are no parameters, and the system has a unique solution.

Problem

Find the rank of $A = \begin{bmatrix} a & b & 5 \\ 1 & -2 & 1 \end{bmatrix}$.

Solution

$$\begin{bmatrix} a & b & 5 \\ 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ a & b & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & b+2a & 5-a \end{bmatrix}$$

If $b + 2a = 0$ and $5 - a = 0$, i.e., $a = 5$ and $b = -10$, then rank $A = 1$.
Otherwise, rank $A = 2$.

For any system of linear equations, exactly one of the following holds:

1. the system is inconsistent;
2. the system has a unique solution, i.e., exactly one solution;
3. the system has infinitely many solutions.

One can see what case applies by looking at the RREF matrix equivalent to the augmented matrix of the system and distinguishing three cases:

1. The last nonzero row is $[0, \cdots, 0, 1]$: no solution.
2. The last nonzero row is not $[0, \cdots, 0, 1]$ and all variables are leading: unique solution.
3. The last nonzero row is not $[0, \cdots, 0, 1]$ and there are non-leading variables: infinitely many solutions.

Problem

Solve the system

$$\begin{array}{rcrcrcrcrcr} -3x_1 & + & 6x_2 & - & 4x_3 & - & 9x_4 & + & 3x_5 & = & -1 \\ -x_1 & + & 2x_2 & - & 2x_3 & - & 4x_4 & - & 3x_5 & = & 3 \\ x_1 & - & 2x_2 & + & 2x_3 & + & 2x_4 & - & 5x_5 & = & 1 \\ x_1 & - & 2x_2 & + & x_3 & + & 3x_4 & - & x_5 & = & 1 \end{array}$$

Solution

Begin by putting the augmented matrix in reduced row-echelon form.

$$\left[\begin{array}{rrrrr|r} 1 & -2 & 2 & 2 & -5 & 1 \\ -3 & 6 & -4 & -9 & 3 & -1 \\ -1 & 2 & -2 & -4 & -3 & 3 \\ 1 & -2 & 1 & 3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent. The rank of the augmented matrix is 3.
Since the system is consistent, the set of solutions has $5 - 3 = 2$ parameters.

Solution (continued)

From the reduced row-echelon matrix

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & -13 & 9 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

we obtain the general solution

$$\left.\begin{array}{rcl} x_1 & = & 9 + 2r + 13s \\ x_2 & = & r \\ x_3 & = & -2 \\ x_4 & = & -2 - 4s \\ x_5 & = & s \end{array}\right\} \quad \forall r, s \in \mathbb{R}$$

The solution has two parameters (r and s) as we expected.

Uniqueness of the Reduced Row-Echelon Form

Theorem

Systems of linear equations that correspond to row equivalent augmented matrices have exactly the same solutions.

Theorem

Every matrix A is row equivalent to a unique reduced row-echelon matrix.

Problem

Solve the system

$$\begin{array}{ccccccc} 2x & + & y & + & 3z & = & 1 \\ 2y & - & z & + & x & = & 0 \\ 9z & + & x & - & 4y & = & 2 \end{array}$$

Solution

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 0 \\ 1 & -4 & 9 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ 1 & -4 & 9 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & -6 & 10 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution (continued)

This row-echelon matrix corresponds to the system

$$\begin{array}{rcrcrcr} x & + & 0y & + & \frac{7}{3}z & = & -\frac{2}{3} \\ & & y & - & \frac{5}{3}z & = & -\frac{1}{3} \end{array},$$

and thus

$$\begin{array}{rcl} x & = & \frac{2}{3} - \frac{7}{3}z \\ y & = & -\frac{1}{3} + \frac{5}{3}z \end{array}$$

Setting $z = s$, where $s \in \mathbb{R}$, gives us (as before):

$$\begin{array}{rcrcr} x & = & \frac{2}{3} & - & \frac{7}{3}s \\ y & = & -\frac{1}{3} & + & \frac{5}{3}s \\ z & = & s & & \end{array}$$

Always check your answer! ■

One Application

Problem

Derive the formula for $1^r + 2^r + \cdots + n^r$ for $r = 3$.

Solution

We know that $1^3 + 2^3 + \cdots + n^3$ is a polynomial in n of oder 4, namely,

$$1^3 + 2^3 + \cdots + n^3 = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4.$$

It is easy to see that when $n = 0$, both sides should be equal to zero. Hence, $a_0 = 0$. Now we have 4 unknowns, $a_1, \cdots, a_4$. We can let $n = 1, \cdots, 4$ to form 4 equations in order to find these unknowns:

$$\begin{array}{ccccccccccc}
1^1 a_1 & + & 1^2 a_2 & + & 1^3 a_3 & + & 1^4 a_4 & = & 1^3 & & (n = 1) \\
2^1 a_1 & + & 2^2 a_2 & + & 2^3 a_3 & + & 2^4 a_4 & = & 1^3 + 2^3 & & (n = 2) \\
3^1 a_1 & + & 3^2 a_2 & + & 3^3 a_3 & + & 3^4 a_4 & = & 1^3 + 2^3 + 3^3 & & (n = 3) \\
4^1 a_1 & + & 4^2 a_2 & + & 4^3 a_3 & + & 4^4 a_4 & = & 1^3 + 2^3 + 3^3 + 4^3 & & (n = 4)
\end{array}$$

Solution (continued)

Hence, we have the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 & 9 \\ 3 & 9 & 27 & 81 & 36 \\ 4 & 16 & 64 & 256 & 100 \end{array}\right)$$

You can use Octave or Matlab to compute the reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right)$$

Therefore, we have that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2}{4} + \frac{n^3}{2} + \frac{n^4}{4} = \frac{1}{4}n^2(n+1)^2.$$

$\blacksquare$