Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations
§1-1. Solutions and Elementary Operations

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Solutions of Linear Equations

Elementary Operations

The Augmented Matrix

Solving a System using Back Substitution

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Solutions of Linear Equations

Example

Find all solutions of the (linear) equation in one variable:

$$ax = b$$

Solution

- If $a \neq 0$, there is a unique solution $x = b/a$.
- Else if $a = 0$ and

  $b \neq 0$, there is no solution.

  $b = 0$, there are infinitely many solutions, in fact any $x \in \mathbb{R}$ is a solution.

This a complete description of all possible solutions of $ax = b$.

Objective:

Can we do the same for linear equations in more variables?

Definition

A linear equation is an expression

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $n \geq 1$, $a_1, \ldots, a_n$ are real numbers, not all of them equal to zero, and b is a real number.

A system of linear equations is a set of $m \geq 1$ linear equations. It is not required that $m = n$.

A solution to a system of m equations in n variables is an n-tuple of numbers that satisfy each of the equations.

Solve a system means 'find all solutions to the system'.

Example

A system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

- variables: x_1, x_2, x_3.
- coefficients:

$$\begin{array}{rcrcrcr} 1x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -1x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

- constant terms:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

Example (continued)

$x_1 = -3$, $x_2 = -1$, $x_3 = 0$ is a solution to the system

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

because

$$\begin{array}{rcrcrcr} (-3) & - & 2(-1) & - & 7\cdot 0 & = & -1 \\ -(-3) & + & 3(-1) & + & 6\cdot 0 & = & 0. \end{array}$$

Another solution to the system is $x_1 = 6$, $x_2 = 0$, $x_3 = 1$ (check!).

However, $x_1 = -1$, $x_2 = 0$, $x_3 = 0$ is not a solution to the system, because

$$\begin{array}{rcrcrcrl} (-1) & - & 2\cdot 0 & - & 7\cdot 0 & = & -1 & \\ -(-1) & + & 3\cdot 0 & + & 6\cdot 0 & = & 1 & \neq 0 \end{array}$$

A solution to the system must be a solution to every equation in the system.

The system above is consistent, meaning that the system has at least one solution.

Example (continued)

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & -8 \end{array}$$

is an example of an inconsistent system, meaning that it has no solutions.

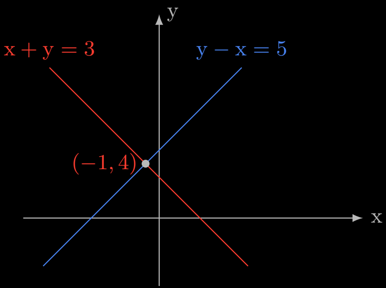
Why are there no solutions?

Example

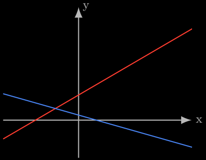
Consider the system of linear equations in two variables

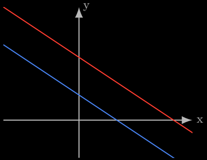
$$\begin{cases} x + y = 3 \\ y - x = 5 \end{cases}$$

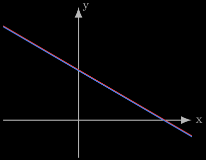
A solution to this system is a pair (x, y) satisfying both equations. Since each equation corresponds to a line, a solution to the system corresponds to a point that lies on both lines, so the solutions to the system can be found by graphing the two lines and determining where they intersect.



Given a system of two equations in two variables, graphed on the xy-coordinate plane, there are three possibilities:







intersect in one point

consistent

(unique solution)

parallel but different

inconsistent

(no solutions)

line are the same

consistent

(infinitely many solutions)

Number of Solutions

For a system of linear equations in two variables, exactly one of the following holds:

1. the system is inconsistent;
2. the system has a unique solution, i.e., exactly one solution;
3. the system has infinitely many solutions.

Remark

We will see in what follows that this generalizes to systems of linear equations in more than two variables.

Example

The system of linear equations in three variables that we saw earlier

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0, \end{array}$$

has solutions $x_1 = -3 + 9s$, $x_2 = -1 + s$, $x_3 = s$ where s is any real number (written $s \in \mathbb{R}$).

Verify this by substituting the expressions for x_1, x_2, and x_3 into the two equations.

s is called a parameter, and the expression

$$x_1 = -3 + 9s, \quad x_2 = -1 + s, \quad x_3 = s, \quad \text{where } s \in \mathbb{R}$$

is called the general solution in parametric form.

Problem

Find all solutions to a system of m linear equations in n variables, i.e., solve a system of linear equations.

Definition

Two systems of linear equations are equivalent if they have exactly the same solutions.

Example

The two systems of linear equations

$$\begin{array}{rcrcr} 2x & + & y & = & 2 \\ 3x & & & = & 3 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcr} x & + & y & = & 1 \\ & & y & = & 0 \end{array}$$

are equivalent because both systems have the unique solution $x = 1$, $y = 0$.

Elementary Operations

Any system of linear equations can be solved by using Elementary Operations to transform the system into an equivalent but simpler system from which the solution can be easily obtained.

Three types of Elementary Operations

- Type I: Interchange two equations, $r_1 \leftrightarrow r_2$.
- Type II: Multiply an equation by a nonzero number, $-2r_1$.
- Type III: Add a multiple of one equation to a different equation, $3r_3 + r_2$.

Example

Consider the system of linear eq's: $$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

1. Interchange first two equations (Type I):

$$r_1 \leftrightarrow r_2 \qquad \begin{array}{rcrcrcr} -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 3x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

2. Multiply first equation by -2 (Type II):

$$-2r_1 \qquad \begin{array}{rcrcrcr} -6x_1 & + & 4x_2 & + & 14x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

3. Add 3 time the second equation to the first equation (Type III):

$$3r_2 + r_1 \qquad \begin{array}{rcrcrcr} & & 7x_2 & + & 11x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

Theorem (Elementary Operations and Solutions)

Suppose that a sequence of elementary operations is performed on a system of linear equations. Then the resulting system has the same set of solutions as the original, so the two systems are equivalent.

As a consequence, performing a sequence of elementary operations on a system of linear equations results in an equivalent system of linear equations, with the exact same solutions.

The Augmented Matrix

Represent a system of linear equations with its augmented matrix.

Example

The system of linear equations

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

is represented by the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -2 & -7 & -1 \\ -1 & 3 & 6 & 0 \end{array}\right]$$

(A matrix is a rectangular array of numbers.)

Remark

Two other matrices associated with a system of linear equations are the coefficient matrix and the constant matrix:

$$\left[\begin{array}{rrr} 1 & -2 & -7 \\ -1 & 3 & 6 \end{array}\right], \quad \left[\begin{array}{r} -1 \\ 0 \end{array}\right].$$

For convenience, instead of performing elementary operations on a system of linear equations, perform corresponding elementary row operations on the corresponding augmented matrix.

Type I: Interchange two rows.

Example

Interchange rows 1 and 3.

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{rrrr|r} 0 & 5 & -6 & 1 & 0 \\ -2 & 0 & 3 & 3 & -1 \\ 2 & -1 & 0 & 5 & -3 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right]$$

Type II: Multiply a row by a nonzero number.

Example

Multiply row 4 by 2.

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right] \xrightarrow{2r_4} \left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 2 & -8 & 4 & 4 & 4 \end{array}\right]$$

Type III: Add a multiple of one row to a different row.

Example

Add 2 times row 4 to row 2.

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right] \xrightarrow{2\mathrm{r}_4+\mathrm{r}_2} \left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ 0 & -8 & 7 & 7 & 3 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right]$$

Definition

Two matrices A and B are row equivalent (or simply equivalent) if one can be obtained from the other by a sequence of elementary row operations.

Problem

Prove that A can be obtained from B by a sequence of elementary row operations if and only if B can be obtained from A by a sequence of elementary row operations.
Prove that row equivalence is an equivalence relation.

Solving a System using Back Substitution

Problem

Solve the system using back substitution

$$\begin{aligned} 2x + y &= 4 \\ x - 3y &= 1 \end{aligned}$$

Solution

Add (-2) times the second equation to the first equation.

$$\begin{aligned} 2x + y + (-2)x - (-2)(3)y &= 4 + (-2)1 \\ x - 3y &= 1 \end{aligned}$$

The result is an equivalent system

$$\begin{aligned} 7y &= 2 \\ x - 3y &= 1 \end{aligned}$$

Solution (continued)

The first equation of the system,

$$7y = 2$$

can be rearranged to give us

$$y = \frac{2}{7}.$$

Substituting $y = \frac{2}{7}$ into second equation:

$$x - 3y = x - 3\left(\frac{2}{7}\right) = 1,$$

and simplifying, gives us

$$x = 1 + \frac{6}{7} = \frac{13}{7}.$$

Therefore, the solution is $x = 13/7, y = 2/7$.

The method illustrated in this example is called back substitution. ■

We shall describe an algorithm for solving any given system of linear equations.