

# Math 221: LINEAR ALGEBRA

## Chapter 5. Vector Space $\mathbb{R}^n$

### §5-5. Similarity and Diagonalization

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.



# Linear Algebra with Applications

## Lecture Notes

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Similar Matrices

Diagonalization Revisited

Algebraic and Geometric Multiplicities

Characterizing Diagonalizable Matrices

Complex Eigenvalues

Eigenvalues of Real Symmetric Matrices

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## Similar Matrices

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Eigenvalues of Real Symmetric Matrices

# Similar Matrices

## Definition (Similar Matrices)

Let  $A$  and  $B$  be  $n \times n$  matrices.  $A$  is similar to  $B$ , written  $A \sim B$ , if there exists an invertible matrix  $P$  such that  $B = P^{-1}AP$ .

## Lemma

Similarity is an equivalence relation, i.e., for  $n \times n$  matrices  $A$ ,  $B$  and  $C$

1.  $A \sim A$  (reflexive);
2. if  $A \sim B$ , then  $B \sim A$  (symmetric);
3. if  $A \sim B$  and  $B \sim C$ , then  $A \sim C$  (transitive).

## Proof.

1. Since  $A = I_n A I_n$  and  $I_n^{-1} = I_n$ ,  $A = I_n^{-1} A I_n$ . Therefore,  $A \sim A$ .
2. Suppose  $A \sim B$ . Then there exists an invertible  $n \times n$  matrix  $P$  such that  $B = P^{-1}AP$ . Multiplying both sides on the left by  $P$ , on the right by  $P^{-1}$ , and simplifying gives us  $PBP^{-1} = A$ . Therefore,  $A = (P^{-1})^{-1}A(P^{-1})$ , so  $A \sim B$ .

Proof. (continued)

3. Since  $A \sim B$  and  $B \sim C$ , there exist invertible  $n \times n$  matrices  $P$  and  $Q$  such that

$$B = P^{-1}AP \quad \text{and} \quad C = Q^{-1}BQ.$$

Thus

$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ),$$

where  $PQ$  is invertible, and hence  $A \sim C$ .



## Definition

If  $A = [a_{ij}]$  is an  $n \times n$  matrix, then the **trace of A** is

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}.$$

## Lemma (Properties of trace)

For  $n \times n$  matrices  $A$  and  $B$ , and any  $k \in \mathbb{R}$ ,

1.  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ ;
2.  $\text{tr}(kA) = k \cdot \text{tr}(A)$ ;
3.  $\text{tr}(AB) = \text{tr}(BA)$ .



Proof.

The proofs of (1) and (2) are trivial. As for (3), ...

Recall that for any  $n \times n$  matrix  $A$ , the **characteristic polynomial** of  $A$  is

$$c_A(x) = \det(xI - A),$$

and is a polynomial of degree  $n$ .

### **Theorem (Properties of Similar Matrices)**

If  $A$  and  $B$  are  $n \times n$  matrices and  $A \sim B$ , then

1.  $\det(A) = \det(B)$ ;
2.  $\text{rank}(A) = \text{rank}(B)$ ;
3.  $\text{tr}(A) = \text{tr}(B)$ ;
4.  $c_A(x) = c_B(x)$ ;
5.  $A$  and  $B$  have the same eigenvalues.

### Proof.

Since  $A \sim B$ , there exists an  $n \times n$  invertible matrix  $P$  so that  $B = P^{-1}AP$ .

$$1. \det(B) = \det(P^{-1}AP) = \det(P^{-1}) \cdot \det(A) \cdot \det(P).$$

Since  $P$  is invertible,  $\det(P^{-1}) = \frac{1}{\det(P)}$ , so

$$\det(B) = \frac{1}{\det(P)} \cdot \det(A) \cdot \det(P) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(A) = \det(A).$$

Therefore,  $\det(B) = \det(A)$ .

$$2. \text{rank}(B) = \text{rank}(P^{-1}AP).$$

Since  $P$  is invertible,  $\text{rank}(P^{-1}AP) = \text{rank}(P^{-1}A)$ ,

since  $P^{-1}$  is invertible,  $\text{rank}(P^{-1}A) = \text{rank}(A)$ .

Therefore,  $\text{rank}(B) = \text{rank}(A)$ .

$$3. \text{tr}(B) = \text{tr}[(P^{-1}A)P] = \text{tr}[P(P^{-1}A)] = \text{tr}[(PP^{-1})A] = \text{tr}(IA) = \text{tr}(A).$$

Proof. (continued)

4.

$$\begin{aligned}c_B(x) = \det(xI - B) &= \det(xI - P^{-1}AP) \\&= \det(xP^{-1}P - P^{-1}AP) \\&= \det(P^{-1}xP - P^{-1}AP) \\&= \det[P^{-1}(xI - A)P] \\&= \det(P^{-1}) \cdot \det(xI - A) \cdot \det(P) \\&= \det(P^{-1}) \cdot \det(P) \cdot \det(xI - A)\end{aligned}$$

Since  $P$  is invertible,  $\det(P^{-1}) = \frac{1}{\det(P)}$ , so

$$c_B(x) = \frac{1}{\det(P)} \cdot \det(P) \cdot \det(xI - A) = \det(xI - A) = c_A(x).$$

5. Since the eigenvalues of a matrix are the roots of the characteristic polynomial,  $c_B(x) = c_A(x)$  implies that  $A$  and  $B$  have the same eigenvalues. ■

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Similar Matrices

**Diagonalization Revisited**

Algebraic and Geometric Multiplicities

Characterizing Diagonalizable Matrices

Complex Eigenvalues

Eigenvalues of Real Symmetric Matrices

# Diagonalization Revisited

Recall that if  $\lambda$  is an **eigenvalue** of  $A$ , then  $A\vec{x} = \lambda\vec{x}$  for some nonzero vector  $\vec{x}$  in  $\mathbb{R}^n$ . Such a vector  $\vec{x}$  is called a  **$\lambda$ -eigenvector of  $A$**  or an eigenvector of  $A$  corresponding to  $\lambda$ .

## Definition (Diagonalizable – rephrased)

An  $n \times n$  matrix  $A$  is **diagonalizable** if  $A \sim D$  for some diagonal matrix  $D$ .

## Remark ( Diagonalizability )

Determining whether or not a square matrix  $A$  is diagonalizable is done by checking whether

the number of linearly independent eigenvectors  
– **geometric multiplicity**

||?

the multiplicity of each eigenvalue  
– **algebraic multiplicity**

### Example

Let  $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . Then  $\lambda = -1$  is an eigenvalue of  $A$ , and  $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is a  $(-1)$ -eigenvector of  $A$  since

$$A\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

### Theorem

Suppose  $A$  is an  $n \times n$  matrix.


1. The eigenvalues of  $A$  are the roots of  $c_A(x)$ .
2. The  $\lambda$ -eigenvectors of  $A$  are all the nonzero solutions to  $(\lambda I - A)\vec{x} = \vec{0}_n$ .

## Problem

Determine all eigenvalues of  $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 6 & 0 \\ 4 & 2 & -1 & 1 \end{bmatrix}$ .

## Solution

$$\det(xI - A) = \begin{vmatrix} x+2 & 0 & 0 & 0 \\ -3 & x-6 & 0 & 0 \\ 1 & 0 & x-6 & 0 \\ -4 & -2 & 1 & x-1 \end{vmatrix} = (x+2)(x-6)(x-6)(x-1).$$

Thus, the eigenvalues of  $A$  are  $-2, 6, 6$  and  $1$ , precisely the elements on the main diagonal of  $A$ . 

## Remark

In general, the eigenvalues of any **triangular** matrix are the entries on its main diagonal.



## Theorem

Let  $A$  be an  $n \times n$  matrix.

1.  $A$  is diagonalizable if and only if  $\mathbb{R}^n$  has a basis  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$  of eigenvectors of  $A$ .
2. If  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$  are eigenvectors of  $A$  and form a basis of  $\mathbb{R}^n$ , then

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$$

is an invertible matrix such that

$$P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

where  $\lambda_i$  is the eigenvalue of  $A$  corresponding to  $\vec{x}_i$ .

This result was covered earlier, but without the use of term basis.

## Theorem

Let  $A$  be an  $n \times n$  matrix, and suppose that  $A$  has distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ . For each  $i$ , let  $\vec{x}_i$  be a  $\lambda_i$ -eigenvector of  $A$ . Then  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is linearly independent.

## Proof.

We need to show that  $t_1\vec{x}_1 + t_2\vec{x}_2 + \dots + t_k\vec{x}_k = \vec{0}$  only has trivial solution  $t_1 = \dots = t_k = 0$ . Notice that

$$\begin{aligned} t_1 A \vec{x}_1 + t_2 A \vec{x}_2 + \dots + t_k A \vec{x}_k &= t_1 \lambda_1 \vec{x}_1 + t_2 \lambda_2 \vec{x}_2 + \dots + t_k \lambda_k \vec{x}_k = \vec{0} \\ t_1 A^2 \vec{x}_1 + t_2 A^2 \vec{x}_2 + \dots + t_k A^2 \vec{x}_k &= t_1 \lambda_1^2 \vec{x}_1 + t_2 \lambda_2^2 \vec{x}_2 + \dots + t_k \lambda_k^2 \vec{x}_k = \vec{0} \\ &\vdots \qquad \qquad \qquad \vdots \\ t_1 A^{k-1} \vec{x}_1 + \dots + t_k A^{k-1} \vec{x}_k &= t_1 \lambda_1^{k-1} \vec{x}_1 + \dots + t_k \lambda_k^{k-1} \vec{x}_k = \vec{0} \end{aligned}$$

Proof.

$$\begin{array}{ccccccccc}
 t_1 \lambda_1 \vec{x}_1 & + & t_2 \lambda_2 \vec{x}_2 & + & \cdots & + & t_k \lambda_k \vec{x}_k & = & \vec{0} \\
 t_1 \lambda_1^2 \vec{x}_1 & + & t_2 \lambda_2^2 \vec{x}_2 & + & \cdots & + & t_k \lambda_k^2 \vec{x}_k & = & \vec{0} \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 t_1 \lambda_1^{k-1} \vec{x}_1 & + & t_2 \lambda_2^{k-1} \vec{x}_2 & + & \cdots & + & t_k \lambda_k^{k-1} \vec{x}_k & = & \vec{0}
 \end{array}$$

$$\Updownarrow$$

$$\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{pmatrix} \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_k \end{bmatrix} \begin{pmatrix} \lambda_1^0 & \lambda_1^1 & \cdots & \lambda_1^{k-1} \\ \lambda_2^0 & \lambda_2^1 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_k^0 & \lambda_k^1 & \cdots & \lambda_k^{k-1} \end{pmatrix} = O_{k \times k}.$$

Proof.

Since  $\lambda_i$  are distinct, the **Vandermonde matrix** is invertible, hence,

$$\begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{pmatrix} \begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_k \end{bmatrix} = O_{k \times k}.$$

$$\Updownarrow$$

$$t_i \vec{x}_i = 0 \quad \text{for all } i = 1, \dots, k$$

$$\Downarrow$$

$$t_i = 0 \quad \text{for all } i = 1, \dots, k$$

Only trivial solution is found. Hence,  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is independent. ■

Proof. ( Another proof left for you to study )

The proof is by induction on  $k$ , the number of distinct eigenvalues.

Basis. If  $k = 1$ , then  $\{\vec{x}_1\}$  is an independent set because  $\vec{x}_1 \neq \vec{0}_n$ .

Suppose that for some  $k \geq 1$ ,  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is independent, where  $\vec{x}_i$  is an eigenvector of  $A$  corresponding to  $\lambda_i$ ,  $1 \leq i \leq k$ , and  $\lambda_1, \lambda_2, \dots, \lambda_k$  are distinct. (This is the Inductive Hypothesis.) Now suppose  $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$  are distinct eigenvalues of  $A$  that have corresponding eigenvectors  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{k+1}$ , respectively. Consider

$$t_1 \vec{x}_1 + t_2 \vec{x}_2 + \dots + t_{k+1} \vec{x}_{k+1} = \vec{0}_n, \text{ for } t_1, t_2, \dots, t_{k+1} \in \mathbb{R}. \quad (1)$$

Multiplying equation (1) by  $A$  (on the left) gives us

Proof. (continued)

$$t_1 A \vec{x}_1 + t_2 A \vec{x}_2 + \cdots + t_{k+1} A \vec{x}_{k+1} = \vec{0}_n,$$

$$\Downarrow$$

$$t_1 \lambda_1 \vec{x}_1 + t_2 \lambda_2 \vec{x}_2 + \cdots + t_{k+1} \lambda_{k+1} \vec{x}_{k+1} = \vec{0}_n. \quad (2)$$

Also, multiplying (1) by  $\lambda_{k+1}$  gives us

$$t_1 \lambda_{k+1} \vec{x}_1 + t_2 \lambda_{k+1} \vec{x}_2 + \cdots + t_{k+1} \lambda_{k+1} \vec{x}_{k+1} = \vec{0}_n, \quad (3)$$

and subtracting (3) from (2) results in

$$t_1 (\lambda_1 - \lambda_{k+1}) \vec{x}_1 + t_2 (\lambda_2 - \lambda_{k+1}) \vec{x}_2 + \cdots + t_k (\lambda_k - \lambda_{k+1}) \vec{x}_k = \vec{0}_n.$$

Proof. (continued)

By the inductive hypothesis,  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is independent, so

$$t_i(\lambda_i - \lambda_{k+1}) = 0 \text{ for } i = 1, 2, \dots, k.$$

Since  $\lambda_1, \lambda_2, \dots, \lambda_k$  are distinct,  $(\lambda_i - \lambda_{k+1}) \neq 0$  for  $i = 1, 2, \dots, k$ , and thus  $t_i = 0$  for  $i = 1, 2, \dots, k$ . Substituting these values into (1) yields

$$t_{k+1}\vec{x}_{k+1} = \vec{0}_n,$$

implying that  $t_{k+1} = 0$ , since  $\vec{x}_{k+1} \neq \vec{0}_n$ .


Therefore,  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_{k+1}\}$  is an independent set, and the result follows by induction. ■

The next result is an easy consequence of the previous Theorem.

### Theorem (Covered earlier, but now with a proof)

If  $A$  is an  $n \times n$  matrix with  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

#### Proof.

Let  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  denote the  $n$  (distinct) eigenvalues of  $A$ , and let  $\vec{x}_i$  be an eigenvector of  $A$  corresponding to  $\lambda_i$ ,  $1 \leq i \leq n$ . By the previous Theorem,  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$  is an independent set. A subset of  $n$  linearly independent vectors of  $\mathbb{R}^n$  also spans  $\mathbb{R}^n$ , and thus  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$  is a basis of  $\mathbb{R}^n$ . Thus  $A$  is diagonalizable. 



## Problem

Is the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 8 & 6 & -2 \\ 0 & 0 & -3 \end{bmatrix}$$


diagonalizable?

## Solution

Because A has characteristic polynomial

$$c_A(x) = (x + 3)(x - 2)(x - 4),$$

A has distinct eigenvalues  $-3, 2$  and  $4$ .

Since A has three distinct eigenvalues, A is diagonalizable. 

Problem (Covered earlier, but with different wording)

Is  $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  diagonalizable? Explain.


### Solution

First,  $c_A(x) = (x - 2)(x + 1)^2$ , so the eigenvalues of  $A$  are  $\lambda_1 = 2, \lambda_2 = -1$ , and  $\lambda_3 = -1$ . Since the eigenvalues are not distinct, it isn't immediately obvious that  $A$  is diagonalizable. The general solution to  $(-I - A)\vec{x} = \vec{0}_3$ :

$$\left[ \begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

is  $x_1 = -s - t, x_2 = s$ , and  $x_3 = t$  for  $s, t \in \mathbb{R}$ , leading to basic solutions

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

that are linearly independent. Therefore, there is a basis of  $\mathbb{R}^3$  consisting of eigenvectors of  $A$ , so  $A$  is diagonalizable. 

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Diagonalization Revisited

**Algebraic and Geometric Multiplicities**

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# Algebraic and Geometric Multiplicities

## Lemma (Technical but useful)

Let  $A$  be an  $n \times n$  matrix, with independent eigenvectors  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ . Extend  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  to a basis  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \dots, \vec{x}_n\}$  of  $\mathbb{R}^n$ , and let  $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$ . If  $\lambda_1, \lambda_2, \dots, \lambda_k$  are the (not necessarily distinct) eigenvalues corresponding to  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ , then

$$P^{-1}AP = \begin{bmatrix} \text{diag}(\lambda_1, \dots, \lambda_k) & B \\ 0_{(n-k) \times k} & A_1 \end{bmatrix},$$

where  $B$  is an  $k \times (n - k)$  matrix and  $A_1$  is an  $(n - k) \times (n - k)$  matrix.

Proof.

$$\begin{bmatrix} A\vec{x}_1 & \cdots & A\vec{x}_k & A\vec{x}_{k+1} & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{x}_1 & \cdots & \lambda_k \vec{x}_k & A\vec{x}_{k+1} & \cdots & A\vec{x}_n \end{bmatrix}$$

$$\parallel$$

$$A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \parallel$$

$$\begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_k & \vec{x}_{k+1} & \cdots & \vec{x}_n \end{bmatrix} \left[ \begin{array}{ccc|ccc} \lambda_1 & & & a_{1,k+1} & \cdots & a_{1,k+1} \\ & \ddots & & \vdots & \vdots & \vdots \\ & & \lambda_k & a_{k,k+1} & \cdots & a_{k,k+1} \\ \hline & & & a_{k+1,k+1} & \cdots & a_{k+1,k+1} \\ & & 0 & \vdots & \vdots & \vdots \\ & & & a_{n,k+1} & \cdots & a_{n,k+1} \end{array} \right]$$

$$\begin{array}{ccc} \uparrow & \cdots & \uparrow \\ P^{-1}A\vec{x}_{k+1} & \cdots & P^{-1}A\vec{x}_n \end{array}$$

$$\Updownarrow$$

$$AP = P \begin{bmatrix} \text{diag}(\lambda_1, \dots, \lambda_k) & B \\ 0_{(n-k) \times k} & A_1 \end{bmatrix}$$

$$\Updownarrow$$

$$P^{-1}AP = \begin{bmatrix} \text{diag}(\lambda_1, \dots, \lambda_k) & B \\ 0_{(n-k) \times k} & A_1 \end{bmatrix}$$



Proof. (Another proof)

Recall that  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is the standard basis of  $\mathbb{R}_n$ . Since  $I_n = P^{-1}P$ ,

$$\begin{aligned} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} &= P^{-1}P = P^{-1} \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \\ &= \begin{bmatrix} P^{-1}\vec{x}_1 & P^{-1}\vec{x}_2 & \cdots & P^{-1}\vec{x}_n \end{bmatrix} \end{aligned}$$

Thus for each  $j$ ,  $1 \leq j \leq n$ ,  $P^{-1}\vec{x}_j = \vec{e}_j$ . Also,

$$\begin{aligned} P^{-1}AP &= P^{-1}A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} \\ &= \begin{bmatrix} P^{-1}A\vec{x}_1 & P^{-1}A\vec{x}_2 & \cdots & P^{-1}A\vec{x}_n \end{bmatrix}, \end{aligned}$$

so the  $j^{\text{th}}$  column of  $P^{-1}AP$ ,  $1 \leq j \leq k$ , is equal to

$$P^{-1}(A\vec{x}_j) = P^{-1}(\lambda_j\vec{x}_j) = \lambda_j(P^{-1}\vec{x}_j) = \lambda_j\vec{e}_j.$$

This gives us the first  $k$  columns of  $P^{-1}AP$ , and the result follows. ■

## Definition

Let  $A$  be an  $n \times n$  matrix and  $\lambda \in \mathbb{R}$ . The **eigenspace of  $A$  corresponding to  $\lambda$**  is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

## Remark

1. The eigenspace  $E_\lambda(A)$  is indeed a subspace of  $\mathbb{R}^n$  because

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} = \{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\} = \text{null}(\lambda I - A).$$

2. If  $\lambda$  is not an eigenvalue of  $A$ , then  $E_\lambda(A) = \{0\}$ .

## Definition

1. If  $A$  is an  $n \times n$  matrix and  $\lambda$  is an eigenvalue of  $A$ , then the **(algebraic) multiplicity of  $\lambda$**  is the largest value of  $m$  for which

$$c_A(x) = (x - \lambda)^m g(x)$$

for some polynomial  $g(x)$ , i.e., the multiplicity of  $\lambda$  is the number of times that  $\lambda$  occurs as a root of  $c_A(x)$ .

2.  $\dim(E_\lambda(A))$  is called the **geometric multiplicity** of  $\lambda$ .

## Lemma

If  $A$  is an  $n \times n$  matrix, and  $\lambda$  is an eigenvalue of  $A$  of multiplicity  $m$ , then

$$\dim(E_\lambda(A)) \leq m,$$

that is,

$$\text{Geometric multiplicity} \leq \text{Algebraic multiplicity}.$$



### Proof.

Let  $d = \dim(E_\lambda(A))$ , and let  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_d\}$  be a basis of  $E_\lambda(A)$ . As a consequence, we know that there exists an invertible  $n \times n$  matrix  $P$  so that

$$P^{-1}AP = \begin{bmatrix} \text{diag}(\lambda, \dots, \lambda) & B \\ 0_{(n-d) \times d} & A_1 \end{bmatrix} = \begin{bmatrix} \lambda I_d & B \\ 0_{(n-d) \times d} & A_1 \end{bmatrix}$$

where  $B$  is  $d \times (n - d)$  and  $A_1$  is  $(n - d) \times (n - d)$ .

Define  $A' = P^{-1}AP$ . Then  $A \sim A'$ , so  $A$  and  $A'$  have the same characteristic polynomial. Thus

$$\begin{aligned} c_A(x) = c_{A'}(x) = \det(xI - A') &= \det \begin{bmatrix} (x - \lambda)I_d & -B \\ 0_{(n-d) \times d} & xI_{n-d} - A_1 \end{bmatrix} \\ &= \det[(x - \lambda)I_d] \det(xI_{n-d} - A_1) \\ &= (x - \lambda)^d c_{A_1}(x) \\ &= (x - \lambda)^d g(x). \end{aligned}$$

Since  $\lambda$  has multiplicity  $m$ ,  $d \leq m$ , and therefore  $\dim(E_\lambda(A)) \leq m$  as required. ■

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# Characterizing Diagonalizable Matrices

The crucial consequence of this Lemma is the characterization of matrices that are diagonalizable.

**Theorem (Covered earlier, here with new terminology)**

For an  $n \times n$  matrix  $A$ , the following two conditions are equivalent.

1.  $A$  is diagonalizable.
2. For each eigenvalue  $\lambda$  of  $A$ ,  $\dim(E_\lambda(A))$  is equal to the multiplicity of  $\lambda$ , i.e.,

Diagonalizable



Geometric multiplicity = Algebraic multiplicity, for all  $\lambda$ .


Problem (Covered earlier, here with new terminology)

If possible, diagonalize the matrix  $A = \begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$ . Otherwise, explain why  $A$  is not diagonalizable.

### Solution

$c_A(x) = (x - 3)(x + 1)^2$ , so  $A$  has eigenvalues  $\lambda_1 = 3$ ,  $\lambda_2 = \lambda_3 = -1$ . Find the dimension of  $E_{-1}(A)$  by solving the linear system  $(-I - A)\vec{x} = \vec{0}_3$ .

$$\left[ \begin{array}{ccc|c} 4 & -1 & -6 & 0 \\ -2 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

From this, we see that  $\dim(E_{-1}(A)) = 1$ . Since  $-1$  is an eigenvalue of multiplicity **two**,  $A$  is not diagonalizable. 

Problem (Covered earlier, here with new terminology)

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Show that A is diagonalizable, and that B is not diagonalizable.

### Solution

Both A and B are triangular matrices, so we immediately see that A and B have the same eigenvalues:  $\lambda_1 = \lambda_2 = 1$  and  $\lambda_3 = 2$ . Thus for each matrix, 1 is an eigenvalue of multiplicity **two**.

Solving the system  $(I - A)\vec{x} = \vec{0}_3$ :


$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that there are two parameters in the general solution, so  $\dim(E_1(A)) = 2$ . Therefore, A is diagonalizable.

### Solution (continued)

Solving the system  $(I - B)\vec{x} = \vec{0}_3$ :

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

we see that the general solution has only one parameter, so  $\dim(E_1(B)) = 1$ . However, the algebraic multiplicity of  $\lambda = 1$  is 2. Therefore, B is not diagonalizable. 

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**Complex Eigenvalues**

Eigenvalues of Real Symmetric Matrices

# Complex Eigenvalues

If a matrix has eigenvalues that have imaginary parts (and aren't simply real numbers), we can still find eigenvectors and possibly diagonalize the matrix.

## Problem

Diagonalize, if possible, the matrix  $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ .

## Solution

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & -1 \\ 1 & x-1 \end{vmatrix} = x^2 - 2x + 2.$$

The roots of  $c_A(x)$  are **distinct complex numbers**:  $\lambda_1 = 1 + i$  and  $\lambda_2 = 1 - i$ , so  $A$  is diagonalizable. Corresponding eigenvectors are

$$\vec{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix},$$

respectively.



### Solution (continued)

A diagonalizing matrix for A is

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix},$$

and

$$P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$



### Remark

Notice that A is a real matrix, but has complex eigenvalues (and eigenvectors).

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**Eigenvalues of Real Symmetric Matrices**

# Eigenvalues of Real Symmetric Matrices

## Theorem

The eigenvalues of any real symmetric matrix are real.

## Proof.

Let  $A$  be an  $n \times n$  real symmetric matrix, and let  $\lambda$  be an eigenvalue of  $A$ . To prove that  $\lambda$  is real, it is enough to prove that  $\overline{\lambda} = \lambda$ , i.e.,  $\lambda$  is equal to its (complex) conjugate.

We use  $\overline{A}$  to denote the matrix obtained from  $A$  by replacing each entry by its conjugate. Since  $A$  is real,  $\overline{A} = A$ .

Suppose

$$\vec{x} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

is a  $\lambda$ -eigenvector of  $A$ . Then  $A\vec{x} = \lambda\vec{x}$ .

Proof. (continued)

$$\text{Let } c = \vec{x}^T \vec{\bar{x}} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix}.$$

Then  $c = z_1 \bar{z}_1 + z_2 \bar{z}_2 + \cdots + z_n \bar{z}_n = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$ ; since  $\vec{x} \neq \vec{0}$ ,  $c$  is a positive real number. Now

$$\begin{aligned} \lambda c &= \lambda(\vec{x}^T \vec{\bar{x}}) = (\lambda \vec{x}^T) \vec{\bar{x}} = (\lambda \vec{x})^T \vec{\bar{x}} \\ &= (A\vec{x})^T \vec{\bar{x}} = \vec{x}^T A^T \vec{\bar{x}} \\ &= \vec{x}^T A \vec{\bar{x}} \quad (\text{since } A \text{ is symmetric}) \\ &= \vec{x}^T \overline{A} \vec{\bar{x}} \quad (\text{since } A \text{ is real}) \\ &= \vec{x}^T (\overline{A\vec{x}}) = \vec{x}^T (\overline{\lambda \vec{x}}) = \vec{x}^T \overline{\lambda} \vec{\bar{x}} \\ &= \overline{\lambda} (\vec{x}^T \vec{\bar{x}}) \\ &= \overline{\lambda} c. \end{aligned}$$

Thus,  $\lambda c = \overline{\lambda} c$ . Since  $c \neq 0$ , it follows that  $\lambda = \overline{\lambda}$ , and therefore  $\lambda$  is real. ■