Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-4. Rank of a Matrix

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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Row Space and Column Spaces

Definitions

Let A be an $m \times n$ matrix.

- The column space of A, denoted col(A) is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

- The row space of A, denoted row(A) is the subspace of $\mathbb{R}^n$ spanned by the rows of A (or the columns of A^T).

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

We saw earlier that $\mathrm{col}(A) = \mathrm{im}(A)$.

Remark (Notation)

Let A and B be $m \times n$ matrices. We write $A \to B$ if B can be obtained from A by a sequence of elementary row (column) operations. Note that $A \to B$ if and only if $B \to A$.

Lemma

Let A and B be $m \times n$ matrices.

1. If $A \to B$ by elementary row operations, then $\mathrm{row}(A) = \mathrm{row}(B)$.
2. If $A \to B$ by elementary column operations, then $\mathrm{col}(A) = \mathrm{col}(B)$.

Proof.

It suffices to prove only part one, and only for a single row operation. (Why?)

Thus let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of A.

- If B is obtained from A by interchanging two rows of A, then A and B have exactly the same rows, so $\mathrm{row}(B) = \mathrm{row}(A)$.

Proof. (continued)

- Suppose $p \neq 0$, and suppose that for some j, $1 \leq j \leq m$, B is obtained from A by multiplying row j by p. Then
$$\mathrm{row}(B) = \mathrm{span}\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\}.$$
Since
$$\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(A),$$
it follows that $\mathrm{row}(B) \subseteq \mathrm{row}(A)$. Conversely, since
$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(B),$$
it follows that $\mathrm{row}(A) \subseteq \mathrm{row}(B)$. Therefore, $\mathrm{row}(B) = \mathrm{row}(A)$.

Proof. (continued)

- Suppose $p \neq 0$, and suppose that for some i and j, $1 \leq i, j \leq m$, B is obtained from A by adding p time row j to row i. Without loss of generality, we may assume $i < j$. Then

$$\mathrm{row}(B) = \mathrm{span}\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_j, \ldots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(A),$$

it follows that $\mathrm{row}(B) \subseteq \mathrm{row}(A)$. Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(B),$$

it follows that $\mathrm{row}(A) \subseteq \mathrm{row}(B)$. Therefore, $\mathrm{row}(B) = \mathrm{row}(A)$.

$\blacksquare$

Corollary

Let A be an $m \times n$ matrix, U an invertible $m \times m$ matrix, and V an invertible $n \times n$ matrix. Then $\text{row}(UA) = \text{row}(A)$ and $\text{col}(AV) = \text{col}(A)$,

Proof.

Since U is invertible, U is a product of elementary matrices, implying that $A \to UA$ by a sequence of elementary row operations. By Lemma 2, $\text{row}(UA) = \text{row}(A)$.

Now consider AV: $\text{col}(AV) = \text{row}((AV)^T) = \text{row}(V^T A^T)$ and V^T is invertible (a matrix is invertible if and only if its transpose is invertible). It follows from the first part of this Corollary that

$$\text{row}(V^T A^T) = \text{row}(A^T).$$

But $\text{row}(A^T) = \text{col}(A)$, and therefore $\text{col}(AV) = \text{col}(A)$. $\blacksquare$

Lemma

If R is a row-echelon matrix then

1. the nonzero rows of R are a basis of row(R);
2. the columns of R containing the leading ones are a basis of col(R).

Example

Let

$$R = \begin{bmatrix} 1 & 2 & 2 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Since the nonzero rows of R are linearly independent, they form a basis of row(R).
2. Let $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathbb{R}^5$. Then B is linearly independent and spans col(R), and thus is a basis of col(R). This tells us that $\dim(\mathrm{col}(R)) = 4$. Now let X denote the set of columns of R that contain the leading ones. Then X is a linearly independent subset of col(R) with $4 = \dim(\mathrm{col}(R))$ vectors. It follows that X spans col(R), and therefore is a basis of col(R).

Problem

Find a basis of $U = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$ and find $\dim(U)$.

Solution

Let A the the 3×4 matrix whose rows are the three columns listed. Then $U = \text{row}(A)$, so it suffices to find a basis of $\text{row}(A)$.

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix}.$$

Find R, a row-echelon form of A. Then the nonzero rows of R are a basis of $\text{row}(R)$. Since $\text{row}(A) = \text{row}(R)$, the nonzero rows of R are a basis of $\text{row}(A)$.

Solution (continued)

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 5/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ -5 \end{bmatrix} \right\}$ is a basis of U and $\dim(U) = 2$.

Solution (Another solution – usually more work.)

Take a linear combination of the three given vectors and set it equal to $\vec{0}_4$. If the vectors are independent, then they form a basis of U. Otherwise, delete vectors to cut the given set of vectors down to a basis.

The Rank Theorem

$$\dim(\mathsf{row}(A)) = \dim(\mathsf{col}(A)) = \mathsf{rank}\,(A)$$

Remark

Recall that rank (A) is defined to be the nonzero rows in the row echelon form of A. From what we just learned, the rank of A can be equivalently defined as rank $(A) = \dim(\text{row}(A))$.

Theorem (Rank Theorem)

Let $A = \begin{bmatrix} \vec{A}_1 & \vec{A}_2 & \cdots & \vec{A}_n \end{bmatrix}$ be an $m \times n$ matrix with columns $\{\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n\}$, and suppose that rank $(A) = r$. Then

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = r.$$

Furthermore, if R is a row-echelon form of A then

1. the r nonzero rows of R are a basis of $\text{row}(A)$;
2. if $S = \{\vec{A}_{j_1}, \vec{A}_{j_2}, \ldots, \vec{A}_{j_r}\}$ are the r columns of A corresponding to the columns of R containing leading ones, then S is basis of $\text{col}(A)$.

Problem

For the following matrix A, find rank (A) and bases for row(A) and col(A).

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- rank $(A) = 2$.
- $\{\begin{bmatrix} 1 & -2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 2 \end{bmatrix}\}$ is a basis of row(A).
- $\left\{\begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -5 \\ -1 \end{bmatrix}\right\}$ is a basis of col(A). ■

Problem (revisited)

Find a basis of $U = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$ and find $\dim(U)$.

Solution

Let A denote the matrix whose columns are the three vectors listed, and let R denote a row-echelon form of A. Then

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & -1 \\ 0 & 5 & 5 \\ 3 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

By the Rank Theorem, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis of $U = \text{col}(A)$, so $\dim(U) = 2$. ■

Compare this to the basis found earlier.

Corollary

1. For any matrix A, rank $(A) = $ rank (A^T).
2. For any $m \times n$ matrix A, rank $(A) \leq m$ and rank $(A) \leq n$.
3. Let A be an $m \times n$ matrix. If U and V are invertible matrices (of sizes $m \times m$ and $n \times n$, respectively), then

$$\text{rank }(A) = \text{rank }(UA) = \text{rank }(AV).$$

Lemma

Let A be an $m \times n$ matrix, U a $p \times m$ matrix, and V an $n \times q$ matrix.

1. $\mathrm{col}(AV) \subseteq \mathrm{col}(A)$ with equality if $VV' = I_n$ for some V'.
2. $\mathrm{row}(UA) \subseteq \mathrm{row}(A)$ with equality if $U'U = I_m$ for some U'.

Proof.

(1) Write $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_q \end{bmatrix}$, where $\vec{v}_j$ denotes column j of V, $1 \leq j \leq q$. Then $AV = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_q \end{bmatrix}$, where $A\vec{v}_j$ is column j of AV. By the definition of matrix-vector multiplication, $A\vec{v}_j$ is a linear combination of the columns of A, and thus $A\vec{v}_j \in \mathrm{col}(A)$ for each j. Since $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_q \in \mathrm{col}(A)$,

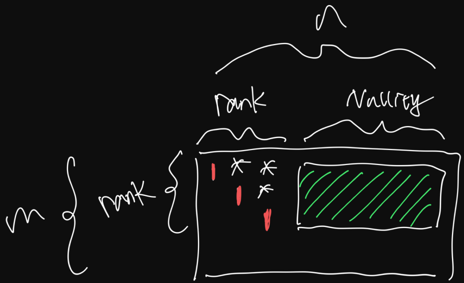
$$\mathrm{span}\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_q\} \subseteq \mathrm{col}(A),$$

i.e., $\mathrm{col}(AV) \subseteq \mathrm{col}(A)$. If for some V' we have $VV' = I_n$, then

$$\mathrm{col}(A) = \mathrm{col}(AVV') \subseteq \mathrm{col}(AV) \subseteq \mathrm{col}(A).$$

(2) This can be proved by part (1) and the fact that $\mathrm{row}(A) = \mathrm{col}(A^T)$. ■

Rank-Nullity Theorem



Theorem (Rank-Nullity Theorem)

Let A denote an $m \times n$ matrix of rank r. Then

1. The $n - r$ basic solutions to the system $A\vec{x} = \vec{0}_m$ provided by the Gaussian algorithm are a basis of $\mathrm{null}(A)$, so

$$\dim(\mathrm{null}(A)) = n - r.$$

2. The rank theorem provides a basis of $\mathrm{im}(A) = \mathrm{col}(A)$, and $\dim(\mathrm{im}(A)) = r$.

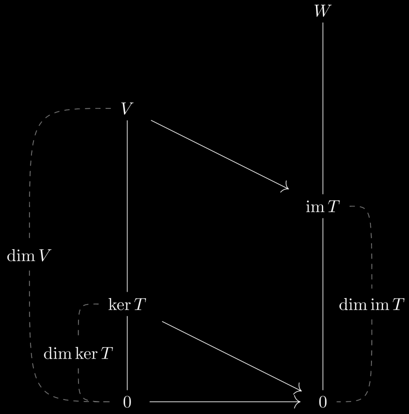
Remark (Common notation)

The nullspace A is also called kernel space of A, written as $\ker(A)$, i.e., $\ker(A) = \mathrm{null}(A)$. Usually, the nullity of A is defined to be

$$\mathrm{Nullity}(A) = \dim(\mathrm{null}(A)) = \dim(\ker(A))$$

Let $T : V \mapsto W$ be the linear map from space V to W. Suppose $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ and let A be the induced matrix.

$$
\begin{array}{ccccc}
\mathrm{Rank}(T) & + & \mathrm{Nullity}(T) & = & \dim(V) \\
|| & & || & & || \\
\mathrm{Rank}(A) & & \mathrm{Nullity}(A) & & \dim(\mathbb{R}^n) \\
|| & & || & & || \\
\dim(\mathrm{im}(A)) & & \dim(\mathrm{null}(A)) & & n \\
|| & & || & & \\
r & & \dim(\ker(A)) & &
\end{array}
$$


W
V
$\operatorname{im} T$
$\dim V$
$\ker T$
$\dim \operatorname{im} T$
$\dim \ker T$
0
0

Proof. (Outline)

- We have already seen that $\mathrm{null}(A)$ is spanned by any set of basic solutions to $A\vec{x} = \vec{0}_m$, so it is enough to prove that $\dim(\mathrm{null}(A)) = n - r$, which will implies that the set of basic solutions is independent, hence this set forms a basis.
- Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a basis of $\mathrm{null}(A)$
- Extend $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ to a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k, \ldots \vec{x}_n\}$ of $\mathbb{R}^n$.
- Consider the set $\{A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_k, \ldots A\vec{x}_n\} \subseteq \mathbb{R}^m$
- Then $A\vec{x}_j = \vec{0}_m$ for $1 \leq j \leq k$ since $\vec{x}_1, \ldots, \vec{x}_k \in \mathrm{null}(A)$.
- To complete the proof, show $S = \{A\vec{x}_{k+1}, \ldots A\vec{x}_n\}$ is a basis of $\mathrm{im}(A)$, by showing that (exercise!)
  (1) S is independent
  (2) S spans $\mathrm{im}(A)$
- Since $\mathrm{im}(A) = \mathrm{col}(A)$, $\dim(\mathrm{im}(A)) = r$, implying $n - k = r$. Hence $k = n - r$. ∎

Problem

For the following matrix A, find bases for null(A) and im(A), and find their dimensions.

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

Solution

Find the basic solutions to $A\vec{x} = \vec{0}_4$.

$$\left[\begin{array}{rrrr|r} 2 & -4 & 6 & 8 & 0 \\ 2 & -1 & 3 & 2 & 0 \\ 4 & -5 & 9 & 10 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 3 & 4 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence,

$$\vec{x} = \left[\begin{array}{c} -s \\ s + 2t \\ s \\ t \end{array}\right] \quad s, t \in \mathbb{R}.$$

Therefore,

$$\left\{ \left[\begin{array}{r} -1 \\ 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} 0 \\ 2 \\ 0 \\ 1 \end{array}\right] \right\} \quad \text{and} \quad \left\{ \left[\begin{array}{r} 2 \\ 2 \\ 4 \\ 0 \end{array}\right], \left[\begin{array}{r} -4 \\ -1 \\ -5 \\ -1 \end{array}\right] \right\}$$

are bases of $\mathrm{null}(A)$ and $\mathrm{im}(A)$, respectively, so

$$\dim(\mathrm{null}(A)) = 2 \quad \text{and} \quad \dim(\mathrm{im}(A)) = 2.$$

Problem

Can a 5×6 matrix have independent columns? Independent rows? Justify your answer.

Solution

The rank of the matrix is at most five; since there are six columns, the columns can not be independent. However, the rows could be independent: take a 5×6 matrix whose first five columns are the columns of the 5×5 identity matrix.

Problem

Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = m$. Prove that $m \leq n$.

Proof.

As a consequence of the Rank Theorem, we have

$$\operatorname{rank}(A) \leq m \quad \text{and} \quad \operatorname{rank}(A) \leq n.$$

Since $\operatorname{rank}(A) = m$, it follows that $m \leq n$. $\blacksquare$

Problem

Let A be an 5×9 matrix. Is it possible that $\dim(\text{null}(A)) = 3$? Justify your answer.

Solution

As a consequence of the Rank Theorem, we have rank $(A) \leq 5$, so $\dim(\text{im}(A)) \leq 5$. Since $\dim(\text{null}(A)) = 9 - \dim(\text{im}(A))$, it follows that

$$\dim(\text{null}(A)) \geq 9 - 5 = 4.$$

Therefore, it is not possible that $\dim(\text{null}(A)) = 3$. ■

Full Rank Cases

Theorem

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}\,(A) = n$.
2. $\text{row}(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$.
3. The columns of A are independent in $\mathbb{R}^m$.
4. The $n \times n$ matrix $A^T A$ is invertible.
5. There exists and $n \times m$ matrix C so that $CA = I_n$.
6. If $A\vec{x} = \vec{0}_m$ for some $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \vec{0}_n$.

Theorem

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}\,(A) = m$.
2. $\text{col}(A) = \mathbb{R}^m$, i.e., the columns of A span $\mathbb{R}^m$.
3. The rows of A are independent in $\mathbb{R}^n$.
4. The $m \times m$ matrix AA^T is invertible.
5. There exists and $n \times m$ matrix C so that $AC = I_m$.
6. The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

Problem

Let $\vec{x} = (x_1, \cdots, x_k)^T \in \mathbb{R}^k$. Show that the following matrix is invertible if and only if $\{x_i, i = 1, \cdots, k\}$ are not all equal:

$$\begin{pmatrix} k & x_1 + \cdots + x_k \\ x_1 + \cdots + x_k & ||x||^2 \end{pmatrix}$$

Solution

Notice that

$$\begin{pmatrix} k & x_1 + \cdots + x_k \\ x_1 + \cdots + x_k & ||x||^2 \end{pmatrix} = A^T A$$

with

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_k \end{bmatrix}.$$

Now $A^T A$ is invertible iff the two columns of A are independent iff $\{x_i, i = 1, \cdots, k\}$ are not all equal. $\blacksquare$