

# Math 221: LINEAR ALGEBRA

## Chapter 5. Vector Space $\mathbb{R}^n$

### §5-3. Orthogonality

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.



# Linear Algebra with Applications

## Lecture Notes

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## The Dot Product

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# Dot Product

## Definitions

Let  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  and  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$  be vectors in  $\mathbb{R}^n$ .

1. The **dot product** of  $\vec{x}$  and  $\vec{y}$  is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots x_n y_n = \vec{x}^T \vec{y}.$$

2. The **length** or **norm** of  $\vec{x}$ , denoted  $||\vec{x}||$  is

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^T \vec{x}}.$$

3.  $\vec{x}$  is called a **unit vector** if  $||\vec{x}|| = 1$ .

## Theorem (Properties of length and the dot product)

Let  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ , and let  $a \in \mathbb{R}$ . Then

1.  $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$  (the dot product is commutative)
2.  $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$  (the dot product distributes over addition)
3.  $(a\vec{x}) \cdot \vec{y} = a(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (a\vec{y})$
4.  $\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$ .
5.  $\|\vec{x}\| \geq 0$  with equality if and only if  $\vec{x} = \vec{0}_n$ .
6.  $\|a\vec{x}\| = |a| \|\vec{x}\|$ .

## Example

Let  $\vec{x}, \vec{y} \in \mathbb{R}^n$ . Then

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{x} + \vec{y} \cdot \vec{y} \\ &= \vec{x} \cdot \vec{x} + 2(\vec{x} \cdot \vec{y}) + \vec{y} \cdot \vec{y} \\ &= \|\vec{x}\|^2 + 2(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2. \end{aligned}$$



## Problem

Let  $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_k\} \in \mathbb{R}^n$  and suppose  $\mathbb{R}^n = \text{span}\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_k\}$ . Furthermore, suppose that there exists a vector  $\vec{x} \in \mathbb{R}^n$  for which  $\vec{x} \cdot \vec{f}_j = 0$  for all  $j$ ,  $1 \leq j \leq k$ . Show that  $\vec{x} = \vec{0}_n$ .

## Proof.

Write  $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + \dots + t_k\vec{f}_k$  for some  $t_1, t_2, \dots, t_k \in \mathbb{R}$  (this is possible because  $\vec{f}_1, \vec{f}_2, \dots, \vec{f}_k$  span  $\mathbb{R}^n$ , is this representation unique?). Then

$$\begin{aligned} \|\vec{x}\|^2 &= \vec{x} \cdot \vec{x} \\ &= \vec{x} \cdot (t_1\vec{f}_1 + t_2\vec{f}_2 + \dots + t_k\vec{f}_k) \\ &= \vec{x} \cdot (t_1\vec{f}_1) + \vec{x} \cdot (t_2\vec{f}_2) + \dots + \vec{x} \cdot (t_k\vec{f}_k) \\ &= t_1(\vec{x} \cdot \vec{f}_1) + t_2(\vec{x} \cdot \vec{f}_2) + \dots + t_k(\vec{x} \cdot \vec{f}_k) \\ &= t_1(0) + t_2(0) + \dots + t_k(0) = 0. \end{aligned}$$

Since  $\|\vec{x}\|^2 = 0$ ,  $\|\vec{x}\| = 0$ . By the previous theorem,  $\|\vec{x}\| = 0$  if and only if  $\vec{x} = \vec{0}_n$ . Therefore,  $\vec{x} = \vec{0}_n$ . ■

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**The Cauchy Inequality**

Orthogonality

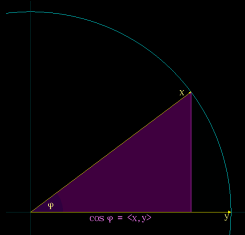
Orthogonality and Independence

Fourier Expansion

# Cauchy-Schwartz Inequality

## Theorem (Cauchy-Schwartz Inequality)

If  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , then  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$  with equality if and only if  $\{\vec{x}, \vec{y}\}$  is linearly dependent.



$$\left| \frac{\vec{x}}{\|\vec{x}\|} \cdot \frac{\vec{y}}{\|\vec{y}\|} \right| \leq 1$$

$\{\vec{x}, \vec{y}\}$  is linearly dependent  $\Leftrightarrow \vec{x} = t\vec{y}$ , for some  $t \in \mathbb{R}$ .

**Proof.**

Let  $\vec{x}, \vec{y} \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . Then

$$\begin{aligned} 0 \leq \|t\vec{x} + \vec{y}\|^2 &= (t\vec{x} + \vec{y}) \cdot (t\vec{x} + \vec{y}) \\ &= t^2\vec{x} \cdot \vec{x} + 2t\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\ &= t^2\|\vec{x}\|^2 + 2t(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2. \end{aligned}$$

The quadratic  $t^2\|\vec{x}\|^2 + 2t(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2$  in  $t$  is always nonnegative, so it does not have distinct real roots. Thus, if we use the quadratic formula to solve for  $t$ , the discriminant must be non-positive, i.e.,

$$\Delta = (2\vec{x} \cdot \vec{y})^2 - 4\|\vec{x}\|^2\|\vec{y}\|^2 \leq 0$$

Therefore,  $(2\vec{x} \cdot \vec{y})^2 \leq 4\|\vec{x}\|^2\|\vec{y}\|^2$ . Since both sides of the inequality are nonnegative, we can take (positive) square roots of both sides:

$$|2\vec{x} \cdot \vec{y}| \leq 2\|\vec{x}\| \|\vec{y}\|$$

Therefore,  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$ . What remains is to show that  $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$  if and only if  $\{\vec{x}, \vec{y}\}$  is linearly dependent. ■

Proof. (continued)

First suppose that  $\{\vec{x}, \vec{y}\}$  is dependent. Then by symmetry (of  $\vec{x}$  and  $\vec{y}$ ),  $\vec{x} = k\vec{y}$  for some  $k \in \mathbb{R}$ . Hence

$$|\vec{x} \cdot \vec{y}| = |(k\vec{y}) \cdot \vec{y}| = |k| |\vec{y} \cdot \vec{y}| = |k| \|\vec{y}\|^2, \quad \text{and} \quad \|\vec{x}\| \|\vec{y}\| = \|k\vec{y}\| \|\vec{y}\| = |k| \|\vec{y}\|^2,$$

so  $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$ .

Conversely, suppose  $\{\vec{x}, \vec{y}\}$  is independent; then  $t\vec{x} + \vec{y} \neq \vec{0}_n$  for all  $t \in \mathbb{R}$ , so  $\|t\vec{x} + \vec{y}\|^2 > 0$  for all  $t \in \mathbb{R}$ . Thus the quadratic

$$t^2 \|\vec{x}\|^2 + 2t(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2 > 0$$

so has no real roots. It follows that the the discriminant is negative, i.e.,

$$(2\vec{x} \cdot \vec{y})^2 - 4\|\vec{x}\|^2 \|\vec{y}\|^2 < 0.$$

Therefore,  $(2\vec{x} \cdot \vec{y})^2 < 4\|\vec{x}\|^2 \|\vec{y}\|^2$ ; taking square roots of both sides (they are both nonnegative) and dividing by two gives us

$$|\vec{x} \cdot \vec{y}| < \|\vec{x}\| \|\vec{y}\|,$$

showing that equality is impossible. ■

### Corollary (Triangle Inequality I )

If  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , then  $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ .

Proof.

$$\begin{aligned}\|\vec{x} + \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\ &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\|\vec{x}\| \|\vec{y}\| + \|\vec{y}\|^2 \text{ by the Cauchy Inequality} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2.\end{aligned}$$

Since both sides of the inequality are nonnegative, we take (positive) square roots of both sides:

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$



## Definition

If  $\vec{x}, \vec{y} \in \mathbb{R}^n$ , then the **distance** between  $\vec{x}$  and  $\vec{y}$  is defined as

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|.$$

## Theorem (Properties of the distance function)

Let  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ . Then

1.  $d(\vec{x}, \vec{y}) \geq 0$ .
2.  $d(\vec{x}, \vec{y}) = 0$  if and only if  $\vec{x} = \vec{y}$ .
3.  $d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$ .
4.  $d(\vec{x}, \vec{z}) \leq d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$  (Triangle Inequality II).

## Proof. (Proof of the Triangle Inequality II)

$$\begin{aligned} d(\vec{x}, \vec{z}) = \|\vec{x} - \vec{z}\| &= \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \\ &\leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| \text{ by Triangle Inequality I} \\ &= d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}). \end{aligned}$$



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# Orthogonality

## Definitions

- ▶ Let  $\vec{x}, \vec{y} \in \mathbb{R}^n$ . We say that two vectors  $\vec{x}$  and  $\vec{y}$  are **orthogonal** if  $\vec{x} \cdot \vec{y} = 0$ .
- ▶ More generally,  $X = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \mathbb{R}^n$  is an **orthogonal set** if each  $\vec{x}_i$  is nonzero, and every pair of **distinct** vectors of  $X$  is orthogonal, i.e.,  $\vec{x}_i \cdot \vec{x}_j = 0$  for all  $i \neq j$ ,  $1 \leq i, j \leq k$ .
- ▶ A set  $X = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \mathbb{R}^n$  is an **orthonormal** set if  $X$  is an orthogonal set of **unit vectors**, i.e.,  $\|\vec{x}_i\| = 1$  for all  $i$ ,  $1 \leq i \leq k$ .

## Examples

1. The standard basis  $\{\vec{e}_1, \dots, \vec{e}_n\}$  of  $\mathbb{R}^n$  is an orthonormal set (and hence an orthogonal set).

2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal (but not orthonormal) subset of  $\mathbb{R}^4$ .

3. If  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is an orthogonal subset of  $\mathbb{R}^n$  and  $p \neq 0$ , then  $\{p\vec{x}_1, p\vec{x}_2, \dots, p\vec{x}_k\}$  is an orthogonal subset of  $\mathbb{R}^n$ .

4.

$$\left\{ \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

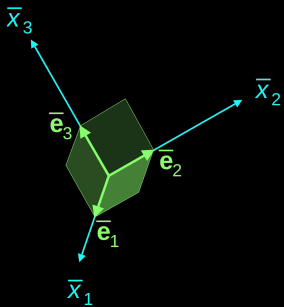
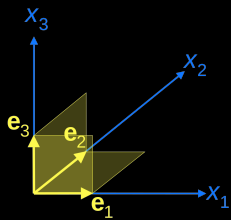
is an orthonormal subset of  $\mathbb{R}^4$ .

## Definition

**Normalizing an orthogonal set** is the process of turning an orthogonal (but not orthonormal) set into an orthonormal set. If  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is an orthogonal subset of  $\mathbb{R}^n$ , then

$$\left\{ \frac{1}{\|\vec{x}_1\|} \vec{x}_1, \frac{1}{\|\vec{x}_2\|} \vec{x}_2, \dots, \frac{1}{\|\vec{x}_k\|} \vec{x}_k \right\}$$

is an orthonormal set.



### Problem

Verify that

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \right\}$$

is an orthogonal set, and normalize this set.

## Solution

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0 - 2 + 2 = 0,$$

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} = 0 + 2 - 2 = 0,$$

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} = 5 - 1 - 4 = 0,$$

proving that the set is orthogonal. Normalizing gives us the orthonormal set

$$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \right\}.$$



## Theorem (Pythagoras' Theorem)

If  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \mathbb{R}^n$  is orthogonal, then

$$\|\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k\|^2 = \|\vec{x}_1\|^2 + \|\vec{x}_2\|^2 + \dots + \|\vec{x}_k\|^2.$$

Proof.

Start with

$$\begin{aligned} \|\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k\|^2 &= (\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k) \cdot (\vec{x}_1 + \vec{x}_2 + \dots + \vec{x}_k) \\ &= (\vec{x}_1 \cdot \vec{x}_1 + \vec{x}_1 \cdot \vec{x}_2 + \dots + \vec{x}_1 \cdot \vec{x}_k) \\ &\quad + (\vec{x}_2 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 + \dots + \vec{x}_2 \cdot \vec{x}_k) \\ &\quad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ &\quad + (\vec{x}_k \cdot \vec{x}_1 + \vec{x}_k \cdot \vec{x}_2 + \dots + \vec{x}_k \cdot \vec{x}_k) \\ &= \vec{x}_1 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 + \dots + \vec{x}_k \cdot \vec{x}_k \\ &= \|\vec{x}_1\|^2 + \|\vec{x}_2\|^2 + \dots + \|\vec{x}_k\|^2. \end{aligned}$$

The second last equality follows from the fact that the set is orthogonal, so for all  $i$  and  $j$ ,  $i \neq j$  and  $1 \leq i, j \leq k$ ,  $\vec{x}_i \cdot \vec{x}_j = 0$ . Thus, the only nonzero terms are the ones of the form  $\vec{x}_i \cdot \vec{x}_i$ ,  $1 \leq i \leq k$ . ■

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Fourier Expansion



# Orthogonality and Independence


## Theorem

If  $S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\} \subseteq \mathbb{R}^n$  is an orthogonal set, then  $S$  is independent.

## Proof.

Form the linear equation:  $t_1\vec{x}_1 + t_2\vec{x}_2 + \dots + t_k\vec{x}_k = \vec{0}$ . We need to check whether there is only trivial solution. Notice that for all  $i$ ,  $1 \leq i \leq k$ ,

$$0 = (t_1\vec{x}_1 + t_2\vec{x}_2 + \dots + t_k\vec{x}_k) \cdot \vec{x}_i = t_i\vec{x}_i \cdot \vec{x}_i = t_i\|\vec{x}_i\|^2,$$

since  $t_j\vec{x}_j \cdot \vec{x}_i = 0$  for all  $j$ ,  $1 \leq j \leq k$  where  $j \neq i$ . Since  $\vec{x}_i \neq \vec{0}_n$  and  $t_i\|\vec{x}_i\|^2 = 0$ , it follows that  $t_i = 0$  for all  $i$ ,  $1 \leq i \leq k$ . Therefore,  $S$  is linearly independent. 

## Example

Given an arbitrary vector

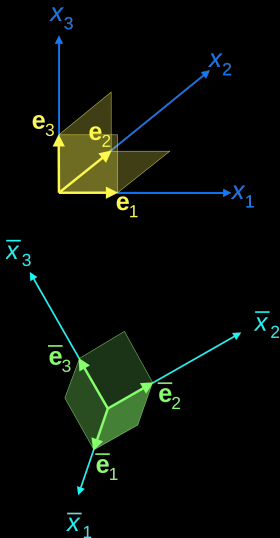
$$\vec{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n,$$

it is trivial to express  $\vec{x}$  as a linear combination of the standard basis vectors of  $\mathbb{R}^n$ ,  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ :

$$\vec{x} = a_1\vec{e}_1 + a_2\vec{e}_2 + \dots + a_n\vec{e}_n.$$

## Problem

Given any orthogonal basis  $B$  of  $\mathbb{R}^n$  (so not necessarily the standard basis), and an arbitrary vector  $\vec{x} \in \mathbb{R}^n$ , how do we express  $\vec{x}$  as a linear combination of the vectors in  $B$ ?



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**Fourier Expansion**

# Fourier Expansion

## Theorem (Fourier Expansion)

Let  $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$  be an orthogonal basis of a subspace  $U$  of  $\mathbb{R}^n$ . Then for any  $\vec{x} \in U$ ,

$$\vec{x} = \left( \frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2} \right) \vec{f}_1 + \left( \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2} \right) \vec{f}_2 + \dots + \left( \frac{\vec{x} \cdot \vec{f}_m}{\|\vec{f}_m\|^2} \right) \vec{f}_m.$$

This expression is called the **Fourier expansion** of  $\vec{x}$ , and

$$\frac{\vec{x} \cdot \vec{f}_j}{\|\vec{f}_j\|^2}, \quad j = 1, 2, \dots, m$$

are called the **Fourier coefficients**.

### Example

$$\text{Let } \vec{f}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{f}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \vec{f}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}, \text{ and let } \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We have seen that  $B = \{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$  is an orthogonal subset of  $\mathbb{R}^3$ .

It follows that  $B$  is an orthogonal basis of  $\mathbb{R}^3$ . (Why?)

To express  $\vec{x}$  as a linear combination of the vectors of  $B$ , apply the Fourier Expansion Theorem. Assume  $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + t_3\vec{f}_3$ . Then

$$t_1 = \frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2} = \frac{2}{6}, \quad t_2 = \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2} = \frac{3}{5}, \quad \text{and} \quad t_3 = \frac{\vec{x} \cdot \vec{f}_3}{\|\vec{f}_3\|^2} = \frac{4}{30}.$$

Therefore,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \frac{2}{15} \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}.$$

## Proof. (Fourier Expansion)

Let  $\vec{x} \in U$ . Since  $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$  is a basis of  $U$ ,  $\vec{x} = t_1\vec{f}_1 + t_2\vec{f}_2 + \dots + t_m\vec{f}_m$  for some  $t_1, t_2, \dots, t_m \in \mathbb{R}$ . Notice that for any  $i$ ,  $1 \leq i \leq m$ ,

$$\begin{aligned}\vec{x} \cdot \vec{f}_i &= (t_1\vec{f}_1 + t_2\vec{f}_2 + \dots + t_m\vec{f}_m) \cdot \vec{f}_i \\ &= t_i\vec{f}_i \cdot \vec{f}_i \quad \text{since } \{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\} \text{ is orthogonal} \\ &= t_i \|\vec{f}_i\|^2.\end{aligned}$$

Since  $\vec{f}_i$  is nonzero, we obtain

$$t_i = \frac{\vec{x} \cdot \vec{f}_i}{\|\vec{f}_i\|^2}.$$

The result now follows. ■

## Remark

If  $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$  is an orthonormal basis, then the Fourier coefficients are simply  $t_j = \vec{x} \cdot \vec{f}_j$ ,  $j = 1, 2, \dots, m$ .

### Problem

$$\text{Let } \vec{f}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{f}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{f}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{f}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Show that  $B = \{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4\}$  is an orthogonal basis of  $\mathbb{R}^4$ , and express  $\vec{x} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$  as a linear combination of  $\vec{f}_1, \vec{f}_2, \vec{f}_3$  and  $\vec{f}_4$ .

### Solution

Computing  $\vec{f}_i \cdot \vec{f}_j$  for  $1 \leq i < j \leq 4$  gives us

$$\begin{aligned} \vec{f}_1 \cdot \vec{f}_2 &= 0, & \vec{f}_1 \cdot \vec{f}_3 &= 0, & \vec{f}_1 \cdot \vec{f}_4 &= 0, \\ \vec{f}_2 \cdot \vec{f}_3 &= 0, & \vec{f}_2 \cdot \vec{f}_4 &= 0, & \vec{f}_3 \cdot \vec{f}_4 &= 0. \end{aligned}$$

Hence,  $B$  is an orthogonal set. It follows that  $B$  is independent, and since  $|B| = 4 = \dim(\mathbb{R}^4)$ ,  $B$  also spans  $\mathbb{R}^4$ . Therefore,  $B$  is an orthogonal basis of  $\mathbb{R}^4$ . By the Fourier Expansion Theorem,

$$\vec{x} = \left( \frac{a+b}{2} \right) \vec{f}_1 + \left( \frac{a-b}{2} \right) \vec{f}_2 + \left( \frac{c+d}{2} \right) \vec{f}_3 + \left( \frac{c-d}{2} \right) \vec{f}_4.$$

