Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-1. Subspaces and Spanning

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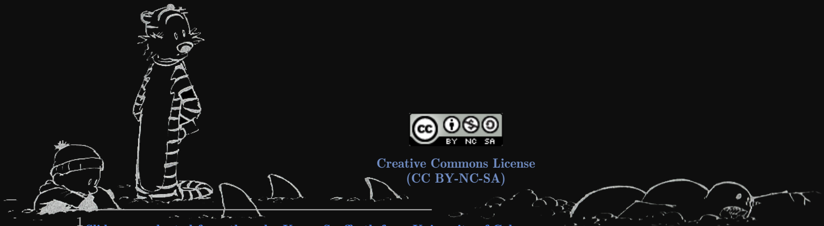

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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Subspaces of $\mathbb{R}^n$

Definitions

1. $\mathbb{R}$ denotes the set of real numbers, and is an example of a set of scalars.
2. $\mathbb{R}^n$ is the set of all n-tuples of real numbers, i.e.,

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, 1 \le i \le n\}.$$

3. The vector space $\mathbb{R}^n$ consists of the set $\mathbb{R}^n$ written as column matrices, along with the (matrix) operations of addition and scalar multiplication. Unless stated otherwise, $\mathbb{R}^n$ means the vector space $\mathbb{R}^n$.

Remark

$\mathbb{R}^n$ is a concrete example of the abstract vector space will be studied in the next chapter.

A vectors is denoted by a lower case letter with an arrow written over it; for example, $\vec{u}$, $\vec{v}$, and $\vec{x}$ denote vectors.

Another example: $\vec{u} = \begin{bmatrix} -2 \\ 3 \\ 0.7 \\ 5 \\ \pi \end{bmatrix}$ is a vector in $\mathbb{R}^5$, written $\vec{u} \in \mathbb{R}^5$.

To save space on the page, the same vector $\vec{u}$ may be written instead as a row matrix by taking the transpose of the column:

$$\vec{u} = \begin{bmatrix} -2, & 3, & 0.7, & 5, & \pi \end{bmatrix}^T.$$

We are interested in nice subsets of $\mathbb{R}^n$, defined as follows.

Definition (Subspaces)

A subset U of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

S1. The zero vector of $\mathbb{R}^n$, $\vec{0}_n$, is in U;

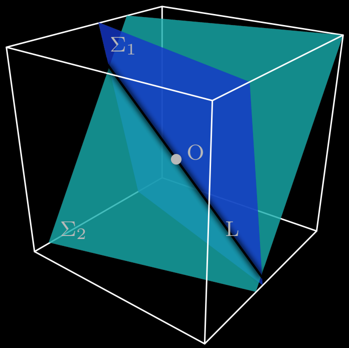
S2. U is closed under addition, i.e., for all $\vec{u}, \vec{w} \in U$, $\vec{u} + \vec{w} \in U$;

S3. U is closed under scalar multiplication, i.e., for all $\vec{u} \in U$ and $k \in \mathbb{R}$, $k\vec{u} \in U$.

Both subset $U = \left\{ \vec{0}_n \right\}$ and R^n itself are subspaces of $\mathbb{R}^n$. Any other subspace of $\mathbb{R}^n$ is called a proper subspace of $\mathbb{R}^n$.

Notation

If U is a subset of $\mathbb{R}^n$, we write $U \subseteq \mathbb{R}^n$.



Σ_1
O
Σ_2
L

Example

In $\mathbb{R}^3$, the line L through the origin that is parallel to the vector $\vec{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$ has (vector) equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$, $t \in \mathbb{R}$, so

$$L = \left\{ t\vec{d} \mid t \in \mathbb{R} \right\}.$$

Claim. L is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in L$ since $0\vec{d} = \vec{0}_3$.
- Suppose $\vec{u}, \vec{v} \in L$. Then by definition, $\vec{u} = s\vec{d}$ and $\vec{v} = t\vec{d}$, for some $s, t \in \mathbb{R}$. Thus
$$\vec{u} + \vec{v} = s\vec{d} + t\vec{d} = (s + t)\vec{d}.$$
Since $s + t \in \mathbb{R}$, $\vec{u} + \vec{v} \in L$; i.e., L is closed under addition.

Example (continued)

- Suppose $\vec{u} \in L$ and $k \in \mathbb{R}$ (k is a scalar). Then $\vec{u} = t\vec{d}$, for some $t \in \mathbb{R}$, so
$$k\vec{u} = k(t\vec{d}) = (kt)\vec{d}.$$
Since $kt \in \mathbb{R}$, $k\vec{u} \in L$; i.e., L is closed under scalar multiplication.
- Therefore, L is a subspace of $\mathbb{R}^3$.

Remark

Note that there is nothing special about the vector $\vec{d}$ used in this example; the same proof works for any nonzero vector $\vec{d} \in \mathbb{R}^3$, so any line through the origin is a subspace of $\mathbb{R}^3$.

Example

In $\mathbb{R}^3$, let M denote the plane through the origin having equation $3x - 2y + z = 0$; then M has normal vector $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. If $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$M = \left\{ \vec{u} \in \mathbb{R}^3 \mid \vec{n} \cdot \vec{u} = 0 \right\},$$

where $\vec{n} \cdot \vec{u}$ is the dot product of vectors $\vec{n}$ and $\vec{u}$.

Claim. M is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in M$ since $\vec{n} \cdot \vec{0}_3 = 0$.
- Suppose $\vec{u}, \vec{v} \in M$. Then by definition, $\vec{n} \cdot \vec{u} = 0$ and $\vec{n} \cdot \vec{v} = 0$, so

  $$\vec{n} \cdot (\vec{u} + \vec{v}) = n \cdot \vec{u} + n \cdot \vec{v} = 0 + 0 = 0,$$

  and thus $(\vec{u} + \vec{v}) \in M$; i.e., M is closed under addition.

Example (continued)

- Suppose $\vec{u} \in M$ and $k \in \mathbb{R}$. Then $\vec{n} \cdot \vec{u} = 0$, so
$$\vec{n} \cdot (k\vec{u}) = k(\vec{n} \cdot \vec{u}) = k(0) = 0,$$
and thus $k\vec{u} \in M$; i.e., M is closed under scalar multiplication.
- Therefore, M is a subspace of $\mathbb{R}^3$.

Remark

As in the previous example, there is nothing special about the plane chosen for this example; any plane through the origin is a subspace of $\mathbb{R}^3$.

Problem

Is $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{R} \quad \text{and} \quad 2a - b = c + 2d \right\}$ a subspace of $\mathbb{R}^4$?

Justify your answer.

Solution

The zero vector of $\mathbb{R}^4$ is the vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ with $a = b = c = d = 0$.

In this case, $2a - b = 2(0) + 0 = 0$ and $c + 2d = 0 + 2(0) = 0$, so $2a - b = c + 2d$. Therefore, $\vec{0}_4 \in U$. $\blacksquare$

Solution (continued)

Suppose

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \text{ are in } U.$$

Then $2a_1 - b_1 = c_1 + 2d_1$ and $2a_2 - b_2 = c_2 + 2d_2$. Now

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{bmatrix},$$

and

$$\begin{aligned} 2(a_1 + a_2) - (b_1 + b_2) &= (2a_1 - b_1) + (2a_2 - b_2) \\ &= (c_1 + 2d_1) + (c_2 + 2d_2) \\ &= (c_1 + c_2) + 2(d_1 + d_2). \end{aligned}$$

Therefore, $\vec{v}_1 + \vec{v}_2 \in U$.

Solution (continued)

Finally, suppose

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U \quad \text{and} \quad k \in \mathbb{R}.$$

Then $2a - b = c + 2d$. Now

$$k\vec{v} = k \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix},$$

and

$$2ka - kb = k(2a - b) = k(c + 2d) = kc + 2kd.$$

Therefore, $k\vec{v} \in U$.

It follows from the Subspace Test that U is a subspace of $\mathbb{R}^4$.

Problem

Is $U = \left\{ \begin{bmatrix} 1 \\ s \\ t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

Solution

Note that $\vec{0}_3 \notin U$, and thus U is not a subspace of $\mathbb{R}^3$.

(You could also show that U is not closed under addition, or not closed under scalar multiplication.)

Problem

Is $U = \left\{ \begin{bmatrix} r \\ 0 \\ s \end{bmatrix} \;\middle|\; r, s \in \mathbb{R} \quad \text{and} \quad r^2 + s^2 = 0 \right\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

Solution

Since $r \in \mathbb{R}$, $r^2 \geq 0$ with equality if and only if $r = 0$. Similarly, $s \in \mathbb{R}$ implies $s^2 \geq 0$, and $s^2 = 0$ if and only if $s = 0$. This means $r^2 + s^2 = 0$ if and only if $r^2 = s^2 = 0$; thus $r^2 + s^2 = 0$ if and only if $r = s = 0$. Therefore U contains only $\vec{0}_3$, the zero vector, i.e., $U = \{\vec{0}_3\}$. As we already observed, $\{\vec{0}_n\}$ is a subspace of $\mathbb{R}^n$, and therefore U is a subspace of $\mathbb{R}^3$. ■

The null space and the image space

Definitions (Null Space and Image Space)

Let A be an $m \times n$ matrix. The null space of A is defined as

$$\mathrm{null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}_m\},$$

and the image space of A is defined as

$$\mathrm{im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$

Remark

1. Since A is $m \times n$ and $\vec{x} \in \mathbb{R}^n$, $A\vec{x} \in \mathbb{R}^m$, so $\mathrm{im}(A) \subseteq \mathbb{R}^m$ while $\mathrm{null}(A) \subseteq \mathbb{R}^n$.
2. Image space is also called column space of A, denoted as $\mathrm{col}(A)$:

$$\mathrm{col}(A) = \mathrm{span}\left(\vec{a}_1, \cdots, \vec{a}_n\right) = \mathrm{im}(A).$$

Problem

Prove that if A is an $m \times n$ matrix, then $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$.

Proof.

S1. Since $A\vec{0}_n = \vec{0}_m$, $\vec{0}_n \in \mathrm{null}(A)$.

S2. Let $\vec{x}, \vec{y} \in \mathrm{null}(A)$. Then $A\vec{x} = \vec{0}_m$ and $A\vec{y} = \vec{0}_m$, so

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

and thus $\vec{x} + \vec{y} \in \mathrm{null}(A)$.

S3. Let $\vec{x} \in \mathrm{null}(A)$ and $k \in \mathbb{R}$. Then $A\vec{x} = \vec{0}_m$, so

$$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$

and thus $k\vec{x} \in \mathrm{null}(A)$.

Therefore, $\mathrm{null}(A)$ is a subspace of $\mathbb{R}^n$. $\blacksquare$

Problem

Prove that if A is an $m \times n$ matrix, then $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^m$.

Proof.

S1. Since $\vec{0}_n \in \mathbb{R}^n$ and $A\vec{0}_n = \vec{0}_m$, $\vec{0}_m \in \mathrm{im}(A)$.

S2. Let $\vec{x}, \vec{y} \in \mathrm{im}(A)$. Then $\vec{x} = A\vec{u}$ and $\vec{y} = A\vec{v}$ for some $\vec{u}, \vec{v} \in \mathbb{R}^n$, so

$$\vec{x} + \vec{y} = A\vec{u} + A\vec{v} = A(\vec{u} + \vec{v}).$$

Since $\vec{u} + \vec{v} \in \mathbb{R}^n$, it follows that $\vec{x} + \vec{y} \in \mathrm{im}(A)$.

S3. Let $\vec{x} \in \mathrm{im}(A)$ and $k \in \mathbb{R}$. Then $\vec{x} = A\vec{u}$ for some $\vec{u} \in \mathbb{R}^n$, and thus

$$k\vec{x} = k(A\vec{u}) = A(k\vec{u}).$$

Since $k\vec{u} \in \mathbb{R}^n$, it follows that $k\vec{x} \in \mathrm{im}(A)$.

Therefore, $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^m$. $\blacksquare$

The Eigenspace

Definition (Eigenspace)

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ is the set

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

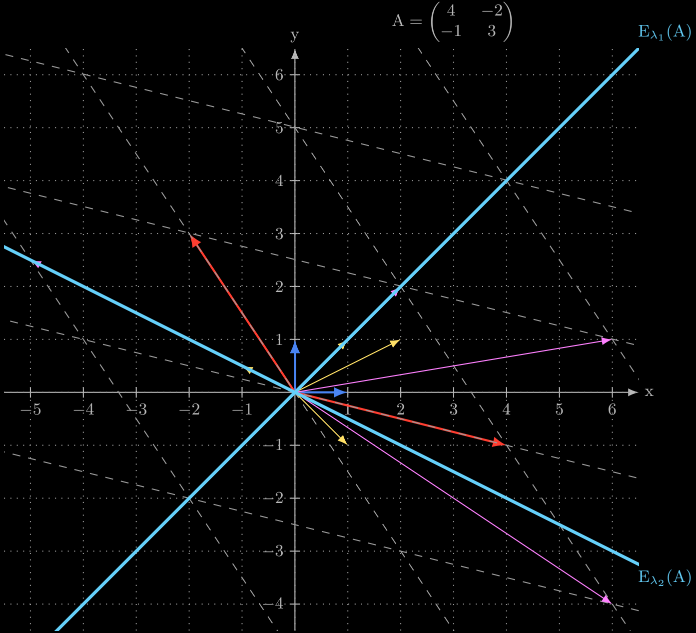
Example

$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$ has two eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 5$ with corresponding eigenvectors

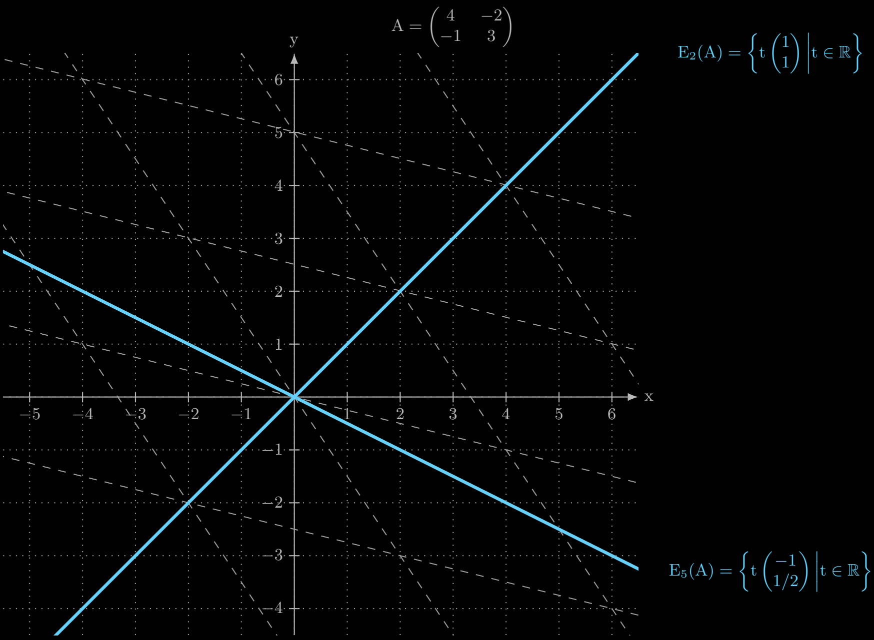
$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

Hence,

$$E_{\lambda_1}(A) = E_2(A) = \{t\vec{v}_1 | t \in \mathbb{R}\}$$
$$E_{\lambda_2}(A) = E_5(A) = \{t\vec{v}_2 | t \in \mathbb{R}\}$$



$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
$E_{\lambda_1}(A)$
$E_{\lambda_2}(A)$
x
y
−5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4


$A = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$
$E_2(A) = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$
$E_5(A) = \left\{ t \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$
x
y

Note that

$$\begin{aligned} E_\lambda(A) &= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} \\ &= \left\{\vec{x} \in \mathbb{R}^n \mid \lambda\vec{x} - A\vec{x} = \vec{0}_n\right\} \\ &= \left\{\vec{x} \in \mathbb{R}^n \mid (\lambda I - A)\vec{x} = \vec{0}_n\right\} \end{aligned}$$

showing that

$$E_\lambda(A) = \text{null}(\lambda I - A).$$

It follows that

- if λ is not an eigenvalue of A, then $E_\lambda(A) = \{\vec{0}_n\}$;
- the nonzero vectors of $E_\lambda(A)$ are the eigenvectors of A corresponding to λ;
- the eigenspace of A corresponding to λ is a subspace of $\mathbb{R}^n$.

Linear Combinations and Spanning Sets

Definition (Linear Combinations and Spanning)

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$ and $t_1, t_2, \ldots, t_k \in \mathbb{R}$. Then the vector

$$\vec{x} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$$

is called a linear combination of the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$; the (scalars) $t_1, t_2, \ldots, t_k \in \mathbb{R}$ are the coefficients. The set of all linear combinations of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ is called the span of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$, and is written

$$\mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} = \{t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k \mid t_1, t_2, \ldots, t_k \in \mathbb{R}\}.$$

Additional Terminology. If $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$, then

- U is spanned by the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$.
- the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ span U.
- the set of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a spanning set for U.

Example

Let $\vec{x} \in \mathbb{R}^3$ be a nonzero vector. Then $\mathrm{span}\{\vec{x}\} = \{k\vec{x} \mid k \in \mathbb{R}\}$ is a line through the origin having direction vector $\vec{x}$.

Example

Let $\vec{x}, \vec{y} \in \mathbb{R}^3$ be nonzero vectors that are not parallel. Then

$$\mathrm{span}\{\vec{x}, \vec{y}\} = \{k\vec{x} + t\vec{y} \mid k, t \in \mathbb{R}\}$$

is a plane through the origin containing $\vec{x}$ and $\vec{y}$.

How would you find the equation of this plane?

Problem

Let $\vec{x} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$. Is $\vec{x} \in \text{span}\{\vec{y}, \vec{z}\}$?

Solution

An equivalent question is: can $\vec{x}$ be expressed as a linear combination of $\vec{y}$ and $\vec{z}$?

Suppose there exist $a, b \in \mathbb{R}$ so that $\vec{x} = a\vec{y} + b\vec{z}$. Then

$$\begin{bmatrix} 8 \\ 3 \\ -13 \\ 20 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solve this system of four linear equations in the two variables a and b.

Solution (continued)

$$\left[\begin{array}{rr|r} 2 & -1 & 8 \\ 1 & 0 & 3 \\ -3 & 2 & -13 \\ 5 & -3 & 20 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

Since the system has no solutions, $\vec{x} \notin \operatorname{span}\{\vec{y}, \vec{z}\}$. ■

Problem

Let $\vec{w} = \begin{bmatrix} 8 \\ 3 \\ -13 \\ 21 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 5 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$. Is $\vec{w} \in \text{span}\{\vec{y}, \vec{z}\}$?

This is almost identical to a previous problem, except that $\vec{w}$ (above) has one entry that is different from the vector $\vec{x}$ of that problem.

Solution

In this case, the system of linear equations is consistent, and gives us $\vec{w} = 3\vec{y} - 2\vec{z}$, so $\vec{w} \in \text{span}\{\vec{y}, \vec{z}\}$.

Theorem

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$ and let $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. Then

1. U is a subspace of $\mathbb{R}^n$ containing each $\vec{x}_i$, $1 \leq i \leq k$;
2. if W is a subspace of $\mathbb{R}^n$ and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in W$, then $U \subseteq W$.

Remark

Property 2 is saying that U is the "smallest" subspace of $\mathbb{R}^n$ that contains $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$.

Proof. (Part 1 of Theorem)

Since $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ and $0\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_k = \vec{0}_n$, $\vec{0}_n \in U$.

Suppose $\vec{x}, \vec{y} \in U$. Then for some $s_i, t_i \in \mathbb{R}$, $1 \leq i \leq k$,

$$\begin{aligned}\vec{x} &= s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k \\ \vec{y} &= t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k\end{aligned}$$

Thus

$$\begin{aligned}\vec{x} + \vec{y} &= (s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k) + (t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k) \\ &= (s_1 + t_1)\vec{x}_1 + (s_2 + t_2)\vec{x}_2 + \cdots + (s_k + t_k)\vec{x}_k.\end{aligned}$$

Since $s_i + t_i \in \mathbb{R}$ for all $1 \leq i \leq k$, $\vec{x} + \vec{y} \in U$, i.e., U is closed under addition.

Proof. (Part 1 of Theorem – continued)

Suppose $\vec{x} \in U$ and $a \in \mathbb{R}$. Then for some $s_i \in \mathbb{R}$, $1 \leq i \leq k$,

$$\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$$

Thus

$$\begin{aligned} a\vec{x} &= a(s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k) \\ &= (as_1)\vec{x}_1 + (as_2)\vec{x}_2 + \cdots + (as_k)\vec{x}_k. \end{aligned}$$

Since $as_i \in \mathbb{R}$ for all $1 \leq i \leq k$, $a\vec{x} \in U$. Hence, U is closed under scalar multiplication.

Therefore, U is a subspace of $\mathbb{R}^n$. Furthermore, since

$$\vec{x}_i = \sum_{j=1}^{i-1} 0\vec{x}_j + 1\vec{x}_i + \sum_{j=i+1}^{k} 0\vec{x}_j,$$

it follows that $\vec{x}_i \in U$ for all i, $1 \leq i \leq k$.

Proof. (Part 2 of Theorem)

Let $W \subset \mathbb{R}^n$ be a subspace that contains $\vec{x}_1, \cdots, \vec{x}_n$. We need to prove that $U \subseteq W$.

Suppose $\vec{x} \in U$. Then $\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$ for some $s_i \in \mathbb{R}$, $1 \leq i \leq k$. Since W contain each $\vec{x}_i$ and W is closed under scalar multiplication, it follows that $s_i\vec{x}_i \in W$ for each i, $1 \leq i \leq k$. Furthermore, since W is closed under addition, $\vec{x} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k \in W$. Therefore, $U \subseteq W$.

Problem (revisited)

Is $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \quad \text{and} \quad 2a - b = c + 2d \right\}$ a subspace of $\mathbb{R}^4$? Justify your answer.

Solution (Another)

Let $\vec{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U$. Since $2a - b = c + 2d$, $c = 2a - b - 2d$, and thus

$$U = \left\{ \begin{bmatrix} a \\ b \\ 2a - b - 2d \\ d \end{bmatrix} \middle| \; a, b, d \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

By a previous Theorem, U is a subspace of $\mathbb{R}^4$.

Problem

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, $U_1 = \text{span}\{\vec{x}, \vec{y}\}$, and $U_2 = \text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\}$. Prove that $U_1 = U_2$.

Solution

To show that $U_1 = U_2$, prove that $U_1 \subseteq U_2$, and $U_2 \subseteq U_1$. We begin by noting that, by the first part of the previous Theorem, U_1 and U_2 are subspaces of $\mathbb{R}^n$.

Since $2\vec{x} - \vec{y}, 2\vec{y} + \vec{x} \in U_1$, it follows from the second part of the previous Theorem that $\text{span}\{2\vec{x} - \vec{y}, 2\vec{y} + \vec{x}\} \subseteq U_1$, i.e., $U_2 \subseteq U_1$.

Also, since

$$\begin{aligned}
\vec{x} &= \frac{2}{5}(2\vec{x} - \vec{y}) + \frac{1}{5}(2\vec{y} + \vec{x}), \\
\vec{y} &= -\frac{1}{5}(2\vec{x} - \vec{y}) + \frac{2}{5}(2\vec{y} + \vec{x}),
\end{aligned}$$

$\vec{x}, \vec{y} \in U_2$. Therefore, by the second part of the previous Theorem, $\text{span}\{\vec{x}, \vec{y}\} \subseteq U_2$, i.e., $U_1 \subseteq U_2$. The result now follows.

Problem

Show that $\mathbb{R}^n = \mathrm{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, where $\vec{e}_j$ denote the j^{th} column of I_n.

Solution

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then $\vec{x} = x_1\vec{e_1} + x_2\vec{e_2} + \cdots + x_n\vec{e_n}$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Therefore, $\vec{x} \in \mathrm{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$, and thus $\mathbb{R}^n \subseteq \mathrm{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

Conversely, since $\vec{e}_i \in \mathbb{R}^n$ for each i, $1 \leq i \leq n$ (and $\mathbb{R}^n$ is a vector space), it follows that $\mathrm{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} \subseteq \mathbb{R}^n$. The equality now follows.

Problem

Let $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.
Does $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ span $\mathbb{R}^4$? (Equivalently, is $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \mathbb{R}^4$?)

Solution

To prove $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} = \mathbb{R}^4$, we need to prove two directions:

$$\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq \mathbb{R}^4 \quad \text{and} \quad \mathbb{R}^4 \subseteq \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}.$$

For the first relation, since $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4 \in \mathbb{R}^4$ (and $\mathbb{R}^4$ is a vector space), $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\} \subseteq \mathbb{R}^4$.

Solution (continued)

For the second relation, notice that

$$\begin{aligned} \vec{e}_1 &= \vec{x}_1 - \vec{x}_2 \\ \vec{e}_2 &= \vec{x}_2 - \vec{x}_3 \\ \vec{e}_3 &= \vec{x}_3 - \vec{x}_4 \\ \vec{e}_4 &= \vec{x}_4, \end{aligned}$$

showing that $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \in \mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$. Therefore, since $\mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ is a vector space,

$$\mathbb{R}^4 = \mathrm{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathrm{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\},$$

and the equality follows.

Problem

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\operatorname{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

If you check, you'll find that $\vec{e}_2$ can not be written as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and $\vec{u}_4$.

Spanning sets of null(A) and im(A)

Lemma

Let A be an $m \times n$ matrix, and let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ denote a set of basic solutions to $A\vec{x} = \vec{0}_m$. Then

$$\mathrm{null}(A) = \mathrm{span}\{\vec{x}_1, \cdots, \vec{x}_k\}.$$

Lemma

Let A be an $m \times n$ matrix with columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$. Then

$$\mathrm{im}(A) = \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Proof. (of $\mathrm{null}(A) = \mathrm{span}\{\vec{x}_1, \cdots, \vec{x}_k\}$)

"$\supseteq$:" Because $\vec{x}_i \in \mathrm{null}(A)$ for each i, $1 \leq i \leq k$, it follows that

$$\mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathrm{null}(A).$$

"$\subseteq$:" Every solution to $A\vec{x} = \vec{0}_m$ can be expressed as a linear combination of basic solutions, implying that

$$\mathrm{null}(A) \subseteq \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}.$$

Therefore, $\mathrm{null}(A) = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. ■

Proof. (of $\mathrm{im}(A) = \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$)

"$\subseteq$:" Suppose $\vec{y} \in \mathrm{im}(A)$. Then (by definition) there is a vector $\vec{x} \in \mathbb{R}^n$ so that $\vec{y} = A\vec{x}$. Write $\vec{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$. Then

$$\vec{y} = A\vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \ldots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n.$$

Therefore, $\vec{y} \in \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$, implying that

$$\mathrm{im}(A) \subseteq \mathrm{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}.$$

Proof. (continued)

Notice that for each j, $1 \le j \le n$,

$$
\begin{array}{rcl}
A\vec{e}_j & = & \left[\begin{array}{cccc} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \leftarrow \text{ jth row} \\ \\ \\ \\ \end{array} \\
& = & 0\vec{c}_1 + 0\vec{c}_2 + \cdots + 0\vec{c}_{j-1} + 1\vec{c}_j + 0\vec{c}_{j+1} + \cdots + 0\vec{c}_n \\
& = & \vec{c}_j.
\end{array}
$$

Thus $\vec{c}_j \in \mathrm{im}(A)$ for each j, $1 \le j \le n$. It follows that

$$\mathrm{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\} \subseteq \mathrm{im}(A),$$

and therefore

$$\mathrm{im}(A) = \mathrm{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\}.$$

$\blacksquare$