# Math 221: LINEAR ALGEBRA

## Chapter 2. Matrix Algebra
## §2-1. Matrix Addition, Scalar Multiplication and Transposition

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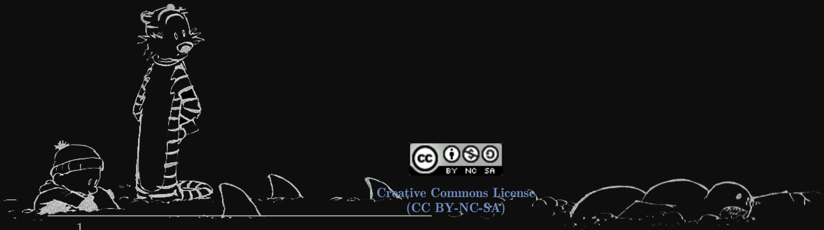



[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

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# Linear Algebra with Applications

## Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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# Matrices – Definitions and Basic Properties

## Definition

Let $m$ and $n$ be positive integers.

- An $m \times n$ matrix is a rectangular array of numbers having $m$ rows and $n$ columns. Such a matrix is said to have size $m \times n$.
- A row matrix (or row) is a $1 \times n$ matrix, and a column matrix (or column) is an $m \times 1$ matrix.
- A square matrix is an $n \times n$ matrix.
- The $(i, j)$-entry of a matrix is the entry in row $i$ and column $j$. For a matrix $A$, the $(i, j)$-entry of $A$ is often written as $a_{ij}$.

General notation for an $m \times n$ matrix, $A$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]$$

## Remark (Basic Properties)

1. Equality: two matrices are equal if and only if they have the same size and the corresponding entries are equal.
2. Zero Matrix: an $m \times n$ matrix with all entries equal to zero.
3. Addition: matrices must have the same size; add corresponding entries.
4. Scalar Multiplication: multiply each entry of the matrix by the scalar.
5. Negative of a Matrix: for an $m \times n$ matrix $A$, its negative is denoted $-A$ and $-A = (-1)A$.
6. Subtraction: for $m \times n$ matrices $A$ and $B$, $A - B = A + (-1)B$.

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# Matrix Addition

## Definition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where $C$ is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

## Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & -2 \\ 6 & 1 \end{bmatrix}$. Then,

$$\begin{aligned} A + B &= \begin{bmatrix} 1+0 & 3+-2 \\ 2+6 & 5+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 8 & 6 \end{bmatrix} \end{aligned}$$

**Theorem (Properties of Matrix Addition)**

Let $A, B$ and $C$ be $m \times n$ matrices. Then the following properties hold.

1. $A + B = B + A$ (matrix addition is commutative).
2. $(A + B) + C = A + (B + C)$ (matrix addition is associative).
3. There exists an $m \times n$ zero matrix, $0$, such that $A + 0 = A$. (existence of an additive identity).
4. There exists an $m \times n$ matrix $-A$ such that $A + (-A) = 0$. (existence of an additive inverse).

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# Scalar Multiplication

## Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $k$ be a scalar. Then $kA = [ka_{ij}]$.

## Example

Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & -2 \\ 0 & 4 & 5 \end{bmatrix}$.

Then

$$\begin{aligned} 3A &= \begin{bmatrix} 3(2) & 3(0) & 3(-1) \\ 3(3) & 3(1) & 3(-2) \\ 3(0) & 3(4) & 3(5) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & -3 \\ 9 & 3 & -6 \\ 0 & 12 & 15 \end{bmatrix} \end{aligned}$$

## Theorem (Properties of Scalar Multiplication)

Let $A, B$ be $m \times n$ matrices and let $k, p \in \mathbb{R}$ (scalars). Then the following properties hold.

1. $k(A + B) = kA + kB$.
   (scalar multiplication distributes over matrix addition).

2. $(k + p)A = kA + pA$.
   (addition distributes over scalar multiplication).

3. $k(pA) = (kp)A$. (scalar multiplication is associative).

4. $1A = A$. (existence of a multiplicative identity).

## Example

$$2\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -16 & -4 \\ 13 & 3 \end{bmatrix}$$

## Problem

Let A and B be $m \times n$ matrices. Simplify the expression

$$2[9(A - B) + 7(2B - A)] - 2[3(2B + A) - 2(A + 3B) - 5(A + B)]$$

## Solution

$$\begin{aligned} & 2[9(A - B) + 7(2B - A)] - 2[3(2B + A) - 2(A + 3B) - 5(A + B)] \\ = \; & 2(9A - 9B + 14B - 7A) - 2(6B + 3A - 2A - 6B - 5A - 5B) \\ = \; & 2(2A + 5B) - 2(-4A - 5B) \\ = \; & 12A + 20B \end{aligned}$$

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# Matrix Transpose

## Definition

If $A$ is an $m \times n$ matrix, then its transpose, denoted $A^T$, is the $n \times m$ whose $i^{th}$ row is the $i^{th}$ column of $A$, $1 \leq i \leq n$; i.e., if $A = [a_{ij}]$, then

$$A^T = [a_{ij}]^T = [a_{ji}]$$

i.e., the $(i, j)$-entry of $A^T$ is the $(j, i)$-entry of $A$.

## Theorem (Properties of the Transpose of a Matrix)

Let $A$ and $B$ be $m \times n$ matrices, $C$ be a $n \times p$ matrix, and $r \in \mathbb{R}$ a scalar. Then

1. $(A^T)^T = A$
2. $(rA)^T = rA^T$
3. $(A + B)^T = A^T + B^T$
4. $(AC)^T = C^T A^T$

To prove each these properties, you only need to compute the $(i, j)$-entries of the matrices on the left-hand side and the right-hand side. And you can do it!

## Problem

Find the matrix A if $\left(\mathrm{A}+3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}\right)^{\mathrm{T}}=\begin{bmatrix}2 & 1\\ 0 & 5\\ 3 & 8\end{bmatrix}$.

## Solution

$$\begin{aligned}
\left[\left(\mathrm{A}+3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}\right)^{\mathrm{T}}\right]^{\mathrm{T}} &= \begin{bmatrix}2 & 1\\ 0 & 5\\ 3 & 8\end{bmatrix}^{\mathrm{T}}\\
\mathrm{A}+3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix} &= \begin{bmatrix}2 & 0 & 3\\ 1 & 5 & 8\end{bmatrix}\\
\mathrm{A} &= \begin{bmatrix}2 & 0 & 3\\ 1 & 5 & 8\end{bmatrix}-3\begin{bmatrix}1 & -1 & 0\\ 1 & 2 & 4\end{bmatrix}\\
\mathrm{A} &= \begin{bmatrix}-1 & 3 & 3\\ -2 & -1 & -4\end{bmatrix}
\end{aligned}$$

### Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal of $A$.

### Definition (Symmetric Matrices)

The matrix $A$ is called symmetric if and only if $A^T = A$. Note that this immediately implies that $A$ is a square matrix.

### Examples

$$\begin{bmatrix} 2 & -3 \\ -3 & 17 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 5 \\ 0 & 2 & 11 \\ 5 & 11 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 5 & -1 \\ 2 & 1 & -3 & 0 \\ 5 & -3 & 2 & -7 \\ -1 & 0 & -7 & 4 \end{bmatrix}$$

are symmetric matrices, and each is symmetric about its main diagonal.

## Definition

An $n \times n$ matrix $A$ is said to be skew symmetric if $A^T = -A$.

## Example (Skew Symmetric Matrices)

$$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 9 & 4 \\ -9 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix}$$

## Problem

Show that if $A$ is a square matrix, then $A - A^T$ is skew-symmetric.

## Solution

We must show that $(A - A^T)^T = -(A - A^T)$. Using the properties of matrix addition, scalar multiplication, and transposition

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$