Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces

§6-2. Subspaces and Spanning Sets

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

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Subspaces and Spanning Sets

Linear Combinations and Spanning Sets

Linear Algebra with Applications

Lecture Notes

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Subspaces and spanning Sets

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Definition (Subspaces of a Vector Space)

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Theorem (Subspace Test)

Let V be a vector space and $U \subseteq V$. Then U is a subspace of V if and only if it satisfies the following three properties:

1. U contains the zero vector of V, i.e., $\mathbf{0} \in U$ where $\mathbf{0}$ is the zero vector of V.
2. U is closed under addition, i.e., if $\mathbf{u}, \mathbf{v} \in U$, then $\mathbf{u} + \mathbf{v} \in U$.
3. U is closed under scalar multiplication, i.e., if $\mathbf{u} \in U$ and $k \in \mathbb{R}$, then $k\mathbf{u} \in U$.

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3. U is closed under scalar multiplication, i.e., if $\mathbf{u} \in U$ and $k \in \mathbb{R}$, then $k\mathbf{u} \in U$.

Remark

The proof of this theorem requires one to show that if the three properties listed above hold, then all the vector space axioms hold.

Remark (Important Note)

As a consequence of the proof, any subspace U of a vector space V has the same zero vector as V, and each $\mathbf{u} \in U$ has the same additive inverse in U as in V.

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Examples (Two extreme examples)

Let V be a vector space.

1. V is a subspace of V.
2. $\{\mathbf{0}\}$ is a subspace of V, where $\mathbf{0}$ denotes the zero vector of V.

Problem

Let A be a fixed (arbitrary) $n \times n$ real matrix, and define

$$U = \{X \in \mathbf{M}_{nn} \mid AX = XA\},$$

i.e., U is the subset of matrices of $\mathbf{M}_{nn}$ that commute with A. Prove that U is a subspace of $\mathbf{M}_{nn}$.

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Solution

- Let $\mathbf{0}_{nn}$ denote the $n \times n$ matrix of all zeros. Then $A\mathbf{0}_{nn} = \mathbf{0}_{nn}$ and $\mathbf{0}_{nn}A = \mathbf{0}_{nn}$, so $A\mathbf{0}_{nn} = \mathbf{0}_{nn}A$. Thus $\mathbf{0}_{nn} \in U$.

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- Suppose $X, Y \in U$. Then $AX = XA$ and $AY = YA$, implying that

  $$A(X + Y) = AX + AY = XA + YA = (X + Y)A,$$

  and thus $X + Y \in U$, so U is closed under addition.

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- Suppose $X, Y \in U$. Then $AX = XA$ and $AY = YA$, implying that

  $$A(X+Y) = AX + AY = XA + YA = (X+Y)A,$$

  and thus $X + Y \in U$, so U is closed under addition.
- Suppose $X \in U$ and $k \in \mathbb{R}$. Then $AX = XA$, implying that

  $$A(kX) = k(AX) = k(XA) = (kX)A;$$

  thus $kX \in U$, so U is closed under scalar multiplication.

By the subspace test, U is a subspace of $\mathbf{M}_{nn}$. ■

Problem

Let $t \in \mathbb{R}$, and let

$$U = \{p \in \mathcal{P} \mid p(t) = 0\},$$

i.e., U is the subset of polynomials that have t as a root. Prove that U is a vector space.

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Proof.

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Proof.

- Let $\mathbf{0}$ denote the zero polynomial. Then $\mathbf{0}(t) = 0$, and thus $\mathbf{0} \in U$.
- Let $q, r \in U$. Then $q(t) = 0$, $r(t) = 0$, and

  $$(q + r)(t) = q(t) + r(t) = 0 + 0 = 0.$$

  Therefore, $q + r \in U$, so U is closed under addition.

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- Let $q \in U$ and $k \in \mathbb{R}$. Then $q(t) = 0$ and

  $$(kq)(t) = k(q(t)) = k \cdot 0 = 0.$$

  Therefore, $kq \in U$, so U is closed under scalar multiplication.

By the subspace test, U is a subspace of $\mathcal{P}$, and thus is a vector space. $\blacksquare$

Examples (more...)

1. It is routine to verify that $\mathcal{P}_{\mathrm{n}}$ is a subspace of $\mathcal{P}$ for all $\mathrm{n} \geq 0$.

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1. It is routine to verify that $\mathcal{P}_n$ is a subspace of $\mathcal{P}$ for all $n \geq 0$.

2. $U = \{A \in \mathbf{M}_{22} \mid A^2 = A\}$ is NOT a subspace of $\mathbf{M}_{22}$.

   To prove this, notice that I_2, the two by two identity matrix, is in U, but $2I_2 \notin U$ since $(2I_2)^2 = 4I_2 \neq 2I_2$, so U is not closed under scalar multiplication.

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3. $U = \{p \in \mathcal{P}_2 \mid p(1) = 1\}$ is NOT a subspace of $\mathcal{P}_2$.

   Because the zero polynomial is not in U: $\mathbf{0}(1) = 0$.

4. $C^n([0,1])$, $n \geq 1$, is a subspace of $C([0,1])$.

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Definitions (Linear Combinations and Spanning)

Let V be a vector space and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a subset of V.

1. A vector $\mathbf{u} \in V$ is called a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ if there exist scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n.$$

2. The set of all linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called the span of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$, and is defined as

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} = \{a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n \mid a_1, a_2, \ldots, a_n \in \mathbb{R}\}.$$

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3. If $U = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is called a spanning set of U.

Problem

Is it possible to express $x^2 + 1$ as a linear combination of

$$x + 1, \quad x^2 + x, \quad \text{and} \quad x^2 + 2 \ ?$$

Equivalently, is $x^2 + 1 \in \operatorname{span}\{x + 1, x^2 + x, x^2 + 2\}$?

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Solution

Suppose that there exist $a, b, c \in \mathbb{R}$ such that

$$x^2 + 1 = a(x + 1) + b(x^2 + x) + c(x^2 + 2).$$

Then

$$x^2 + 1 = (b + c)x^2 + (a + b)x + (a + 2c),$$

implying that $b + c = 1$, $a + b = 0$, and $a + 2c = 1$.

Solution (continued)

Hence,

1. If this system is consistent, then we have found a way to express $x^2 + 1$ as a linear combination of the other vectors; otherwise,
2. if the system is inconsistent and it is impossible to express $x^2 + 1$ as a linear combination of the other vectors.

Because

$$\det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\right) = -3 \neq 0,$$

Answer: Yes, i.e., $x^2 + 1 \in \operatorname{span}\{x + 1, x^2 + x, x^2 + 2\}$. ■

Remark

By solving the linear equation

$$\begin{array}{ccccccc} & & b & + & c & = & 1 \\ a & + & b & + & & = & 0 \\ a & + & & & 2c & = & 1 \end{array}$$

we find that

$$a = -\frac{1}{3}, \quad b = \frac{1}{3}, \quad c = \frac{2}{3}.$$

Hence,

$$x^2 + 1 = -\frac{1}{3}(x+1) + \frac{1}{3}(x^2+x) + \frac{2}{3}(x^2+2)$$

Problem

Let

$$\mathbf{u} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}.$$

Is $\mathbf{w} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$? Prove your answer.

Problem

Let

$$\mathbf{u} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}.$$

Is $\mathbf{w} \in \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$? Prove your answer.

Solution (partial)

Suppose there exist $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = a \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} + b \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} a + 2b &= 1 \\ -a + b &= 3 \\ 2a + b &= -1 \\ a + 0b &= 1. \end{aligned}$$

What remains is to determine whether or not this system is consistent.
Answer: No. $\blacksquare$

Example

The set of 3×2 real matrices,

$$\mathbf{M}_{32} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

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Remark (A Spanning Set of $\mathbf{M}_{mn}$)

In general, the set of mn $m \times n$ matrices that have a '1' in position (i, j) and zeros elsewhere, $1 \leq i \leq m$, $1 \leq j \leq n$, constitutes a spanning set of $\mathbf{M}_{mn}$.

Example

Let $p(x) \in \mathcal{P}_3$. Then $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Therefore,

$$\mathcal{P}_3 = \mathrm{span}\{1, x, x^2, x^3\}.$$

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$$\mathcal{P}_3 = \text{span}\{1, x, x^2, x^3\}.$$

Remark (A Spanning Set of $\mathcal{P}_n$)

For all $n \geq 0$,

$$\mathcal{P}_n = \text{span}\{x^0, x^1, x^2, \ldots, x^n\} = \text{span}\{1, x, x^2, \ldots, x^n\}.$$

span$\{\cdots\}$ is a subspace and the smallest one.

Theorem

Let V be a vector space, let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in V$, and let

$$U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}.$$

Then

1. U is a subspace of V containing $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.
2. If W is a subspace of V and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in W$, then $U \subseteq W$. In other words, U is the "smallest" subspace of V that contains $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

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Remark

This theorem should be familiar as it was covered in the particular case $V = \mathbb{R}^n$. The proof of the result in $\mathbb{R}^n$ immediately generalizes to an arbitrary vector space V.

Problem

Let

$$A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Show that $\mathbf{M}_{22} = \text{span}\{A_1, A_2, A_3, A_4\}$.

Problem

Let

$$A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Show that $\mathbf{M}_{22} = \operatorname{span}\{A_1, A_2, A_3, A_4\}$.

Remark

We need to prove two inclusions

$$\mathbf{M}_{22} \subseteq \operatorname{span}\{A_1, A_2, A_3, A_4\}$$

and

$$\operatorname{span}\{A_1, A_2, A_3, A_4\} \subseteq \mathbf{M}_{22}$$

Proof. (First proof)

Let

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\mathbf{M}_{22} = \operatorname{span}\{E_1, E_2, E_3, E_4\}$ and $A_1, A_2, A_3, A_4 \in \mathbf{M}_{22}$, it follows from the previous Theorem that

$$\operatorname{span}\{A_1, A_2, A_3, A_4\} \subseteq \mathbf{M}_{22}.$$

Proof. (First proof)

Let

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\mathbf{M}_{22} = \mathrm{span}\{E_1, E_2, E_3, E_4\}$ and $A_1, A_2, A_3, A_4 \in \mathbf{M}_{22}$, it follows from the previous Theorem that

$$\mathrm{span}\{A_1, A_2, A_3, A_4\} \subseteq \mathbf{M}_{22}.$$

Now show that E_i, $1 \leq i \leq 4$, can be written as a linear combination of A_1, A_2, A_3, A_4, i.e., $E_i \in \mathrm{span}\{A_1, A_2, A_3, A_4\}$ (lots of work to be done here!), and apply the previous Theorem again to show that

$$\mathbf{M}_{22} \subseteq \mathrm{span}\{A_1, A_2, A_3, A_4\}.$$

$\blacksquare$

Proof. (Second proof)

(1) Since $A_1, A_2, A_3, A_4 \in \mathbf{M}_{22}$ and $\mathbf{M}_{22}$ is a vector space,

$$\mathrm{span}\{A_1, A_2, A_3, A_4\} \subseteq \mathbf{M}_{22}.$$

Proof. (Second proof)

(1) Since $A_1, A_2, A_3, A_4 \in \mathbf{M}_{22}$ and $\mathbf{M}_{22}$ is a vector space,

$$\operatorname{span}\{A_1, A_2, A_3, A_4\} \subseteq \mathbf{M}_{22}.$$

(2) For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}$, we need to find $x_1, \cdots, x_4$, such that

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{array}{rcrcrcrcl} x_1 & & & + & x_3 & + & x_4 & = & a \\ -x_1 & + & x_2 & - & x_3 & & & = & b \\ -x_1 & + & x_2 & - & x_3 & + & x_4 & = & c \\ x_1 & - & x_2 & & & + & x_4 & = & d \end{array}$$

Since the coefficient matrix is invertible one can find unique solution and so

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \operatorname{span}\{A_1, A_2, A_3, A_4\}.$$

Therefore, $\mathbf{M}_{22} \subseteq \operatorname{span}\{A_1, A_2, A_3, A_4\}$. ■

Problem

Let $p(x) = x^2 + 1$, $q(x) = x^2 + x$, and $r(x) = x + 1$. Prove that $\mathcal{P}_2 = \mathrm{span}\{p(x), q(x), r(x)\}$.

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Solution (sketch)

(1) Since $p(x), q(x), r(x) \in \mathcal{P}_2$ and $\mathcal{P}_2$ is a vector space,

$$\text{span}\{p(x), q(x), r(x)\} \subseteq \mathcal{P}_2.$$

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Solution (sketch)

(1) Since $p(x), q(x), r(x) \in \mathcal{P}_2$ and $\mathcal{P}_2$ is a vector space,

$$\mathrm{span}\{p(x), q(x), r(x)\} \subseteq \mathcal{P}_2.$$

(2) As we've already observed, $\mathcal{P}_2 = \mathrm{span}\{1, x, x^2\}$. To complete the proof, show that each of 1, x and x^2 can be written as a linear combination of $p(x), q(x)$ and $r(x)$, i.e., show that

$$1, x, x^2 \in \mathrm{span}\{p(x), q(x), r(x)\}.$$

Then apply the previous Theorem. ■