

Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra §2-5. Elementary Matrices

Le Chen¹

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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Elementary Matrices

Inverses of elementary matrices

Smith Normal Form

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Smith Normal Form

Elementary Matrices

Definition

An **elementary matrix** is a matrix obtained from an identity matrix by performing a **single** elementary row operation.

Remark (Three Types of Elementary Row Operations)

(\sim bases for genomic sequences)

- ▶ Type I: Interchange two rows.
- ▶ Type II: Multiply a row by a nonzero number.
- ▶ Type III: Add a (nonzero) multiple of one row to a different row.

Example

Switch the 2nd row
and the 4th row

Multiply -2 to the
3rd row

Add -3 multiple of
1st row to the 3rd row

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

are examples of elementary matrices of types I, II and III, respectively.

Example (continued)

Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$$

We are interested in the effect that (left) multiplication of A by E, F and G has on the matrix A. Computing EA, FA, and GA ...

Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

Switch the 2nd row
and the 4th row

$$FA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -6 & -6 \\ 4 & 4 \end{bmatrix}$$

Multiply -2 to the
3rd row

$$GA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

Add -3 multiple of
1st row to the 3rd row



Remark

The elementary matrices are the programmed receipts for your cooking!

Theorem (Multiplication by an Elementary Matrix)

Let A be an $m \times n$ matrix.

If B is obtained from A by performing **one single elementary** row operation,

then $B = EA$

where E is the elementary matrix obtained from I_m by performing the same elementary operation on I_m as was performed on A .

$$A \longrightarrow B$$

$$\text{El. Op.} \quad \implies \quad A = EB$$

$$I \longrightarrow E$$

Problem

Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

Find elementary matrices E and F so that $C = FEA$.

Solution

Note. The statement of the problem implies that **C can be obtained from A by a sequence of two elementary row operations**, represented by elementary matrices E and F.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \xrightarrow{F} \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = C$$

where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. Thus we have the sequence $A \rightarrow EA \rightarrow F(EA) = C$, so $C = FEA$, i.e.,

$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}.$$



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Elementary Matrices

Inverses of elementary matrices

Smith Normal Form

Inverses of Elementary Matrices

Lemma

Every elementary matrix E is invertible, and E^{-1} is also an elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E .

The following table gives the inverse of each type of elementary row operation:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q

Note that elementary matrices of type I are self-inverse.

Inverses of Elementary Matrices

Example

Without using the matrix inversion algorithm, find the inverse of the elementary matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hint. What row operation can be applied to G to transform it to I_4 ? The row operation $G \rightarrow I_4$ is to **add** three times row one to row three, and thus

$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check by computing $G^{-1}G$.

Example (continued)

Similarly,

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose A is an $m \times n$ matrix and that B can be obtained from A by a sequence of k elementary row operations. Then there exist elementary matrices E_1, E_2, \dots, E_k such that

$$B = E_k(E_{k-1}(\cdots(E_2(E_1A))\cdots))$$

Since matrix multiplication is associative, we have

$$B = (E_k E_{k-1} \cdots E_2 E_1)A$$

or, more concisely, $B = UA$ where $U = E_k E_{k-1} \cdots E_2 E_1$.

To find U so that $B = UA$, we **could** find E_1, E_2, \dots, E_k and multiply these together (in the correct order), but there is an easier method for finding U .

Definition

Let A be an $m \times n$ matrix. We write

$$A \rightarrow B$$

if B can be obtained from A by a sequence of elementary row operations. In this case, we call A and B are **row-equivalent**.

Theorem

Suppose A is an $m \times n$ matrix and that $A \rightarrow B$. Then

1. there exists an **invertible** $m \times m$ matrix U such that $B = UA$;
2. U can be computed by performing elementary row operations on $\left[\begin{array}{c|c} A & I_m \end{array} \right]$ to transform it into $\left[\begin{array}{c|c} B & U \end{array} \right]$;
3. $U = E_k E_{k-1} \cdots E_2 E_1$, where E_1, E_2, \dots, E_k are elementary matrices corresponding, in order, to the elementary row operations used to obtain B from A .

Problem

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, and let R be the reduced row-echelon form of A . Find a matrix U so that $R = UA$.

Solution

$$\begin{aligned} \left[\begin{array}{ccc|cc} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & -1 \\ 0 & -3 & -2 & -2 & 3 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array} \right] \end{aligned}$$

Starting with $[A \mid I]$, we've obtained $[R \mid U]$.

Therefore $R = UA$, where

$$U = \begin{bmatrix} 1/3 & 0 \\ 2/3 & -1 \end{bmatrix}.$$



Example (A Matrix as a product of elementary matrices)

Let

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}.$$

Suppose we do row operations to put A in reduced row-echelon form, and write down the corresponding elementary matrices.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix} &\xrightarrow{E_1} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3} \\ &\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Notice that the reduced row-echelon form of A equals I_3 . Now find the matrices E_1, E_2, E_3, E_4 and E_5 .

Example (continued)

$$\begin{aligned} E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ E_4 &= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

It follows that

$$\begin{aligned} (E_5(E_4(E_3(E_2(E_1A)))))) &= I \\ (E_5E_4E_3E_2E_1)A &= I \end{aligned}$$

and therefore

$$A^{-1} = E_5E_4E_3E_2E_1$$

Example (continued)

Since $A^{-1} = E_5 E_4 E_3 E_2 E_1$,

$$\begin{aligned} A^{-1} &= E_5 E_4 E_3 E_2 E_1 \\ (A^{-1})^{-1} &= (E_5 E_4 E_3 E_2 E_1)^{-1} \\ A &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \end{aligned}$$

This example illustrates the following result.

Theorem

Let A be an $n \times n$ matrix. Then, A^{-1} exists if and only if A can be written as the product of elementary matrices.

Example (revisited – Matrix inversion algorithm)

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & I \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 2 & -4 & \\ -3 & -6 & 13 & \\ 0 & -1 & 2 & \end{array} \right]$$

$$E_1 \left[\begin{array}{ccc|c} 1 & 2 & -4 & \\ 0 & 0 & 1 & E_1 \\ 0 & -1 & 2 & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$E_2 E_1 \left[\begin{array}{ccc|c} 1 & 2 & -4 & \\ 0 & -1 & 2 & E_2 E_1 \\ 0 & 0 & 1 & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array} \right]$$

Example (continued)

$$E_3E_2E_1[A \mid I] = \left[\begin{array}{ccc|c} 1 & 2 & -4 & E_3E_2E_1 \\ 0 & 1 & -2 & \\ 0 & 0 & 1 & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array} \right]$$

$$E_4E_3E_2E_1[A \mid I] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & E_4E_3E_2E_1 \\ 0 & 1 & -2 & \\ 0 & 0 & 1 & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array} \right]$$

$$E_5E_4E_3E_2E_1[A \mid I] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & E_5E_4E_3E_2E_1 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 6 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array} \right]$$

$$A^{-1} = E_5E_4E_3E_2E_1 = \left[\begin{array}{ccc} 1 & 0 & 2 \\ 6 & 2 & -1 \\ 3 & 1 & 0 \end{array} \right]$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Since $E_4 E_3 E_2 E_1 A = I$, $A^{-1} = E_4 E_3 E_2 E_1$, and hence

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

Solution (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/11 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1}$$

i.e.,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$



One result that we have assumed in all our work involving reduced row-echelon matrices is the following.

Theorem (Uniqueness of the Reduced Echelon Form)

If A is an $m \times n$ matrix and R and S are reduced row-echelon forms of A , then $R = S$.

Remark

This theorem ensures that the reduced row-echelon form of a matrix is **unique**, and its proof follows from the results about elementary matrices.

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Elementary Matrices

Inverses of elementary matrices

Smith Normal Form

Smith Normal Form

Definition

If A is an $m \times n$ matrix of rank r , then the matrix $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$ is called the **Smith normal form** of A .

Theorem

If A is an $m \times n$ matrix of rank r , then there exist invertible matrices U and V of size $m \times m$ and $n \times n$, respectively, such that

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$$

Proof.

1. Apply the elementary row operations:

$$[A|I_m] \xrightarrow{\text{e.r.o.}} [\text{rref}(A)|U]$$

2. Apply the elementary column operations:

$$\begin{pmatrix} \text{rref}(A) \\ I_n \end{pmatrix} \xrightarrow{\text{e.c.o.}} \begin{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n} \\ V \end{pmatrix}_{2m \times n}$$



Remark

The elementary column operations above are equivalent to the elementary row operations on the transpose:

$$[\text{rref}(A)^T | I_n] \xrightarrow{\text{e.r.o.}} \left[\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n \times m} \middle| V^T \right]_{n \times 2m}$$

Problem

Find the decomposition of $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ into the Smith normal form:

$A = \tilde{U}N\tilde{V}$, where N is the Smith normal form of A and \tilde{U}, \tilde{V} are some invertible matrices.

Solution

We have seen that

$$[A|I_2] = \left[\begin{array}{ccc|cc} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array} \right] = [\text{rref}(A)|U]$$

Now,

$$\left(\text{rref}(A)^T \mid I_3 \right) = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{array} \right] = [N^T|V^T]$$

Solution (Continued)

Hence, we find $N = UAV$, namely,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, since U and V are invertible, we see that

$$A = U^{-1}NV^{-1},$$

namely,

$$\begin{aligned} A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} &= \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \tilde{U}N\tilde{V}. \end{aligned}$$

