Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations
§7-3. Isomorphisms and Composition

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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What is isomorphism?

Proving vector spaces are isomorphic

Characterizing isomorphisms

Composition of transformations

Inverses

What is an isomorphism?

Example

$\mathcal{P}_1 = \{ax + b \mid a, b \in \mathbb{R}\}$, has addition and scalar multiplication defined as follows:

$$\begin{aligned} (a_1x + b_1) + (a_2x + b_2) &= (a_1 + a_2)x + (b_1 + b_2), \\ k(a_1x + b_1) &= (ka_1)x + (kb_1), \end{aligned}$$

for all $(a_1x + b_1), (a_2x + b_2) \in \mathcal{P}_1$ and $k \in \mathbb{R}$.

The role of the variable x is to distinguish a_1 from b_1, a_2 from b_2, $(a_1 + a_2)$ from $(b_1 + b_2)$, and (ka_1) from (kb_1).

Example (continued)

This can be accomplished equally well by using vectors in $\mathbb{R}^2$.

$$\mathbb{R}^2 = \left\{ \left[\begin{array}{c} a \\ b \end{array} \right] \;\middle|\; a, b \in \mathbb{R} \right\}$$

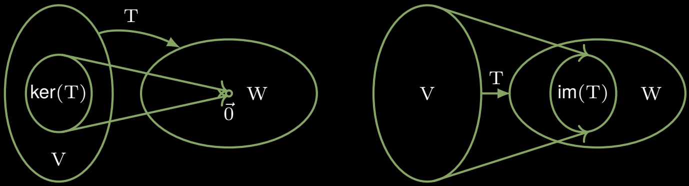
where addition and scalar multiplication are defined as follows:

$$\left[\begin{array}{c} a_1 \\ b_1 \end{array} \right] + \left[\begin{array}{c} a_2 \\ b_2 \end{array} \right] = \left[\begin{array}{c} a_1 + a_2 \\ b_1 + b_2 \end{array} \right], \; k \left[\begin{array}{c} a_1 \\ b_1 \end{array} \right] = \left[\begin{array}{c} ka_1 \\ kb_1 \end{array} \right]$$

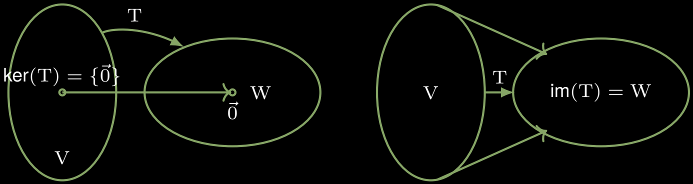
for all $\left[\begin{array}{c} a_1 \\ b_1 \end{array} \right], \left[\begin{array}{c} a_2 \\ b_2 \end{array} \right] \in \mathbb{R}^2$ and $k \in \mathbb{R}$.

Definition

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. T is an isomorphism if and only if T is both one-to-one and onto (i.e., $\ker(T) = \{\mathbf{0}\}$ and $\operatorname{im}(T) = W$). If $T : V \to W$ is an isomorphism, then the vector spaces V and W are said to be isomorphic, and we write $V \cong W$.



General linear transformation T



Isomorphism T

Example

The identity operator on any vector space is an isomorphism.

Example

$T : \mathcal{P}_n \to \mathbb{R}^{n+1}$ defined by

$$T(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

for all $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathcal{P}_n$ is an isomorphism. To verify this, prove that T is a linear transformation that is one-to-one and onto.

Proving isomorphism of vector spaces

Problem

Prove that $\mathbf{M}_{22}$ and $\mathbb{R}^4$ are isomorphic.

Proof.

Let $\mathrm{T} : \mathbf{M}_{22} \to \mathbb{R}^4$ be defined by

$$\mathrm{T}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

It remains to prove that

1. T is a linear transformation;
2. T is one-to-one;
3. T is onto.

Solution (continued – 1. linear transformation)

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in \mathbf{M}_{22}$ and let $k \in \mathbb{R}$. Then

$$T(A) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad \text{and} \quad T(B) = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

$$\Downarrow$$

$$T(A+B) = T\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \\ a_4+b_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = T(A)+T(B)$$

$$\Downarrow$$

T preserves addition.

Solution (continued – 1. linear transformation)

Also

$$T(kA) = T \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \\ ka_4 \end{bmatrix} = k \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = kT(A)$$

$$\Downarrow$$

T preserves scalar multiplication.

Since T preserves addition and scalar multiplication, T is a linear transformation.

Solution (continued – 2. One-to-one)

By definition,

$$\begin{aligned}\ker(T) &= \{A \in \mathbf{M}_{22} \mid T(A) = \mathbf{0}\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.\end{aligned}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker T$, then $a = b = c = d = 0$, and thus $\ker(T) = \{\mathbf{0}_{22}\}$.

$$\Downarrow$$

T is one-to-one.

Solution (continued – 3. Onto)

Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4,$$

and define matrix $A \in \mathbf{M}_{22}$ as follows:

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Then $T(A) = X$, and therefore T is onto.

Finally, since T is a linear transformation that is one-to-one and onto, T is an isomorphism. Therefore, $\mathbf{M}_{22}$ and $\mathbb{R}^4$ are isomorphic vector spaces. ■

Example (Other isomorphic vector spaces)

1. For all integers $n \geq 0$, $\mathcal{P}_n \cong \mathbb{R}^{n+1}$.
2. For all integers m and n, $m, n \geq 1$, $\mathbf{M}_{mn} \cong \mathbb{R}^{m \times n}$.
3. For all integers m and n, $m, n \geq 1$, $\mathbf{M}_{mn} \cong \mathcal{P}_{mn-1}$.

You should be able to define appropriate linear transformations and prove each of these statements.

Characterizing isomorphisms

Theorem

Let V and W be finite dimensional vector spaces and $T : V \to W$ a linear transformation. The following are equivalent.

1. T is an isomorphism.
2. If $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is any basis of V, then $\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)\}$ is a basis of W.
3. There exists a basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of V such that $\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)\}$ is a basis of W.

Proof.

$(1) \Rightarrow (2)$: This is because

- One-to-one linear transformations preserve independent sets.
- Onto linear transformations preserve spanning sets.

$(2) \Rightarrow (3)$ is trivial.

Proof. (Continued)

(3) $\Rightarrow$ (1). We need to prove that T is both onto and one-to-one.

If $T(\vec{v}) = \vec{0}$, write $\vec{v} = v_1\vec{b}_1 + \cdots + v_n\vec{b}_n$ where each v_i is in $\mathbb{R}$. Then

$$\vec{0} = T(\vec{v}) = v_1 T(\vec{b}_1) + \cdots + v_n T(\vec{b}_n)$$

so $v_1 = \cdots = v_n = 0$ by (3). Hence $\vec{v} = \vec{0}$, so $\ker T = \{\vec{0}\}$ and T is one-to-one.

To show that T is onto, let $\vec{w}$ be any vecor in W. By (3) there exist $w_1, \ldots, w_n$ in $\mathbb{R}$ such that

$$\vec{w} = w_1 T(\vec{b}_1) + \cdots + w_n T(\vec{b}_n) = T(w_1\vec{b}_1 + \cdots + w_n\vec{b}_n)$$

Thus T is onto. ■

Suppose V and W are finite dimensional vector spaces with $\dim(V) = \dim(W)$, and let

$$\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\} \quad \text{and} \quad \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\}$$

be bases of V and W respectively. Then $T : V \to W$ defined by

$$T(\vec{b}_i) = \vec{f}_i \text{ for } 1 \leq k \leq n$$

is a linear transformation that maps a basis of V to a basis of W. By the previous Theorem, T is an isomorphism.

Conversely, if V and W are isomorphic and $T : V \to W$ is an isomorphism, then (by the previous Theorem) for any basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of V, $\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)\}$ is a basis of W, implying that $\dim(V) = \dim(W)$.

This proves the next theorem.

Theorem

Finite dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Corollary

If V is a vector space with $\dim(V) = n$, then V is isomorphic to $\mathbb{R}^n$.

Problem

Let V denote the set of 2×2 real symmetric matrices. Then V is a vector space with dimension three. Find an isomorphism $T : \mathcal{P}_2 \to V$ with the property that $T(1) = I_2$ (the 2×2 identity matrix).

Solution

$$V = \left\{ \left[\begin{array}{cc} a & b \\ b & c \end{array} \right] \;\middle|\; a, b, c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Let

$$B = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}.$$

Then B is independent, and $\operatorname{span}(B) = V$, so B is a basis of V. Also, $\dim(V) = 3 = \dim(\mathcal{P}_2)$. However, we want a basis of V that contains I_2.

Solution (continued)

Let

$$B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Since B' consists of $\dim(V)$ symmetric independent matrices, B' is a basis of V. Note that $I_2 \in B'$. Define

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T(x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then for all $ax^2 + bx + c \in \mathcal{P}_2$,

$$T(ax^2 + bx + c) = \begin{bmatrix} c & b \\ b & a + c \end{bmatrix},$$

and $T(1) = I_2$.

By the previous Theorem, $T : \mathcal{P}_2 \to V$ is an isomorphism. ■

Theorem

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. If $\dim(V) = \dim(W) = n$, then T is an isomorphism if and only if T is either one-to-one or onto.

Proof.

($\Rightarrow$) By definition, an isomorphism is both one-to-one and onto.

($\Leftarrow$) Suppose that T is one-to-one. Then $\ker(T) = \{\vec{0}\}$, so $\dim(\ker(T)) = 0$. By the Dimension Theorem,

$$\begin{aligned} \dim(V) &= \dim(\mathrm{im}(T)) + \dim(\ker(T)) \\ n &= \dim(\mathrm{im}(T)) + 0 \end{aligned}$$

so $\dim(\mathrm{im}(T)) = n = \dim(W)$. Furthermore $\mathrm{im}(T) \subseteq W$, so it follows that $\mathrm{im}(T) = W$. Therefore, T is onto, and hence is an isomorphism.

Proof. (continued)

($\Leftarrow$) Suppose that T is onto. Then $\mathrm{im}(T) = W$, so $\dim(\mathrm{im}(T)) = \dim(W) = n$. By the Dimension Theorem,

$$\begin{aligned} \dim(V) &= \dim(\mathrm{im}(T)) + \dim(\ker(T)) \\ n &= n + \dim(\ker(T)) \end{aligned}$$

so $\dim(\ker(T)) = 0$. The only vector space with dimension zero is the zero vector space, and thus $\ker(T) = \{\vec{0}\}$. Therefore, T is one-to-one, and hence is an isomorphism. ∎

Copyright

What is isomorphism?

Proving vector spaces are isomorphic

Characterizing isomorphisms

Composition of transformations

Inverses

Composition of transformations

Definition

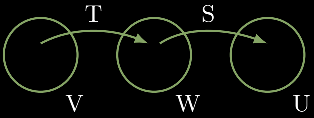
Let V, W and U be vector spaces, and let

$$\mathrm{T} : \mathrm{V} \to \mathrm{W} \quad \text{and} \quad \mathrm{S} : \mathrm{W} \to \mathrm{U}$$

be linear transformations. The composite of T and S is

$$\mathrm{ST} : \mathrm{V} \to \mathrm{U}$$

where $(\mathrm{ST})(\vec{\mathrm{v}}) = \mathrm{S}(\mathrm{T}(\vec{\mathrm{v}}))$ for all $\vec{\mathrm{v}} \in \mathrm{V}$. The process of obtaining ST from S and T is called composition.



Example

Let $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ and $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be linear transformations such that

$$S(A) = -A^T \quad \text{and} \quad T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix} \text{ for all } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Then

$$(ST) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = S \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} -b & -d \\ -a & -c \end{bmatrix},$$

and

$$(TS) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = T \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} = \begin{bmatrix} -c & -a \\ -d & -b \end{bmatrix}.$$

If a, b, c and d are distinct, then $(ST)(A) \neq (TS)(A)$.

This illustrates that, in general, $ST \neq TS$.

Theorem

Let V, W, U and Z be vector spaces and

$$V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$$

be linear transformations. Then

1. ST is a linear transformation.
2. $T1_V = T$ and $1_W T = T$.
3. $(RS)T = R(ST)$.

Problem (The composition of onto transformations is onto)

Let V, W and U be vector spaces, and let

$$\mathrm{V} \xrightarrow{\mathrm{T}} \mathrm{W} \xrightarrow{\mathrm{S}} \mathrm{U}$$

be linear transformations. Prove that if T and S are onto, then ST is onto.

Proof.

Let $\mathbf{z} \in \mathrm{U}$. Since S is onto, there exists a vector $\mathbf{y} \in \mathrm{W}$ such that $\mathrm{S}(\mathbf{y}) = \mathbf{z}$. Furthermore, since T is onto, there exists a vector $\mathbf{x} \in \mathrm{V}$ such that $\mathrm{T}(\mathbf{x}) = \mathbf{y}$. Thus

$$\mathbf{z} = \mathrm{S}(\mathbf{y}) = \mathrm{S}(\mathrm{T}(\mathbf{x})) = (\mathrm{ST})(\mathbf{x}),$$

showing that for each $\mathbf{z} \in \mathrm{U}$ there exists and $\mathbf{x} \in \mathrm{V}$ such that $(\mathrm{ST})(\mathbf{x}) = \mathbf{z}$. Therefore, ST is onto. ∎

Problem (The composition of one-to-one transformations is one-to-one)

Let V, W and U be vector spaces, and let

$$V \xrightarrow{T} W \xrightarrow{S} U$$

be linear transformations. Prove that if T and S are one-to-one, then ST is one-to-one.

The proof of this is left as an exercise.

Inverses

Theorem

Let V and W be finite dimensional vector spaces, and $T : V \to W$ a linear transformation. Then the following statements are equivalent.

1. T is an isomorphism.
2. There exists a linear transformation $S : W \to V$ so that
$$ST = 1_V \quad \text{and} \quad TS = 1_W.$$

In this case, the isomorphism S is uniquely determined by T:
$$\text{if } \vec{w} \in W \quad \text{and} \quad \vec{w} = T(\vec{v}), \text{ then } S(\vec{w}) = \vec{v}.$$

Given an isomorphism $T : V \to W$, the unique isomorphism satisfying the second condition of the theorem is the inverse of T, and is written T^{-1}.

Remark (Fundamental Identities (relating T and T^{-1}))

If V and W are vector spaces and $T : V \to W$ is an isomorphism, then $T^{-1} : W \to V$ is a linear transformation such that

$$(T^{-1}T)(\vec{v}) = \vec{v} \quad \text{and} \quad (TT^{-1})(\vec{w}) = \vec{w}$$

for each $\vec{v} \in V$, $\vec{w} \in W$. Equivalently,

$$T^{-1}T = 1_V \quad \text{and} \quad TT^{-1} = 1_W.$$

Problem

The function $T : \mathcal{P}_2 \to \mathbb{R}^3$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - c \\ 2b \\ a + c \end{bmatrix} \text{ for all } a + bx + cx^2 \in \mathcal{P}_2$$

is a linear transformation (this is left for you to verify). Does T have an inverse? If so, find T^{-1}.

Solution

Since $\dim(\mathcal{P}_2) = 3 = \dim(\mathbb{R}^3)$, it suffices to prove that T is either one-to-one or onto.

Suppose $a + bx + cx^2 \in \ker(T)$. Then

$$\begin{cases} a - c = 0 \\ 2b = 0 \\ a + c = 0 \end{cases} \implies \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

Therefore, $\ker(T) = \{\mathbf{0}\}$, and hence T is one-to-one. By our earlier observation, it follows that T is onto, and thus is an isomorphism.

Solution (continued)

To find T^{-1}, we need to specify $T^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ for any $\begin{bmatrix} p \\ q \\ r \end{bmatrix} \in \mathbb{R}^3$.

Let $a + bx + cx^2 \in \mathcal{P}_2$, and suppose

$$T(a + bx + cx^2) = \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

By the definition of T, $p = a - c$, $q = 2b$ and $r = a + c$. We now solve for a, b and c in terms of p, q and r.

$$\left[\begin{array}{rrr|c} 1 & 0 & -1 & p \\ 0 & 2 & 0 & q \\ 1 & 0 & 1 & r \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & (r+p)/2 \\ 0 & 1 & 0 & q/2 \\ 0 & 0 & 1 & (r-p)/2 \end{array}\right].$$

Solution (continued)

We now have $a = \frac{r+p}{2}$, $b = \frac{q}{2}$ and $c = \frac{r-p}{2}$, and thus

$$T(a + bx + cx^2) = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = T\left(\frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2\right)$$

Therefore,

$$\begin{aligned} T^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix} &= T^{-1}\left(T\left(\frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2\right)\right) \\ &= (T^{-1}T)\left(\frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2\right) \\ &= \frac{r+p}{2} + \frac{q}{2}x + \frac{r-p}{2}x^2. \end{aligned}$$

■

Definition

Let V be a vector space with $\dim(V) = n$, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a fixed basis of V, and let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denote the standard basis of $\mathbb{R}^n$. We define a transformation $C_B : V \to \mathbb{R}^n$ by

$$C_B(a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n) = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Then C_B is a linear transformation such that $C_B(\vec{b}_i) = \vec{e}_i$, $1 \leq i \leq n$, and thus C_B is an isomorphism, called the coordinate isomorphism corresponding to B.

Example

Let V be a vector space and let $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a fixed basis of V. Then $C_B : V \to \mathbb{R}^n$ is invertible, and it is clear that $C_B^{-1} : \mathbb{R}^n \to V$ is defined by

$$C_B^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n \text{ for each } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n.$$