Math 221: LINEAR ALGEBRA

§AppendixA. Complex Numbers

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

In addition we recognize the following contributors. All new content contributed is released under the same license as noted below.

- **Ilijas Farah, York University**

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- The set of real numbers, $\mathbb{R}$, consists of all rational and irrational numbers (note that integers are rational numbers). However, we still can't solve

$$x^2 + 1 = 0$$

  because this requires $x^2 = -1$, but any real number x has the property that $x^2 \geq 0$.

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(3) If $b = 0$, then z is a real number.

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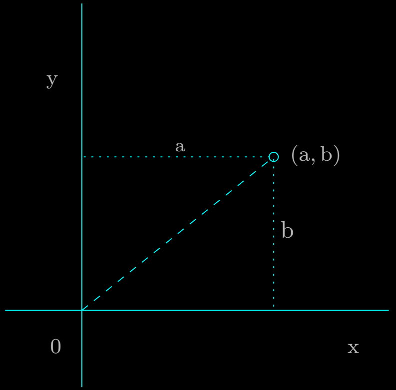

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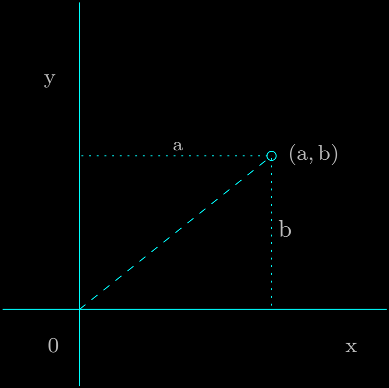

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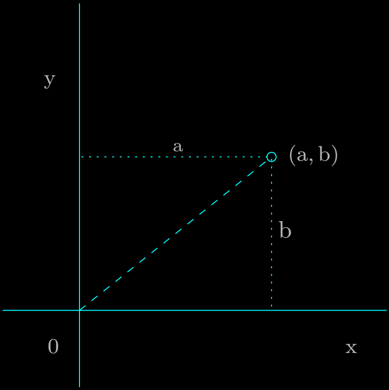

- Real numbers: $a + 0i$ lie on the x-axis.
- Pure imaginary numbers: $0 + bi$ $(b \neq 0)$ lie on the y-axis.

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4. For every $z = a + bi$ there exists a complex number $-z = -a - bi$ such that $z + (-z) = 0$. (existence of an additive inverse)

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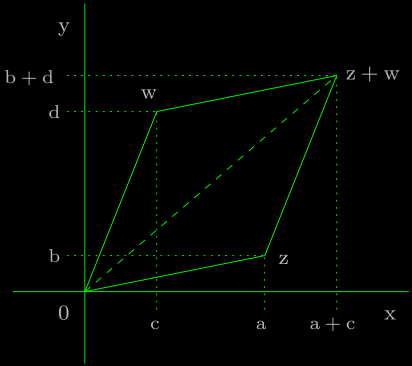

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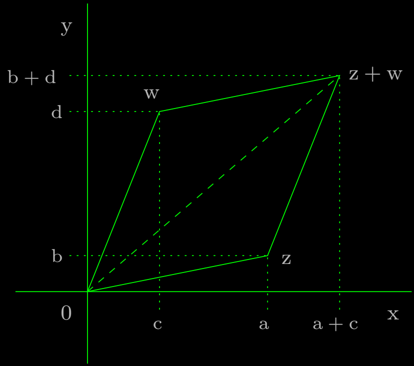

0, z, w, and $z + w$ are the vertices of a parallelogram.

Multiplication of Complex Numbers

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Example

$$\begin{aligned}(2-3i)(-3+4i) = \quad &= ((2)(-3) - (-3)(4)) + ((2)(4) + (-3)(-3))i \\ &= (-6 + 12) + (8 + 9)i \\ &= 6 + 17i\end{aligned}$$

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- For each $z \neq 0$, there exists z^{-1} such that $zz^{-1} = 1.$ (existence of a multiplicative inverse)

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Therefore, if $z^2 = -3 + 4i$, then $z = 1 + 2i$ or $z = -1 - 2i$.

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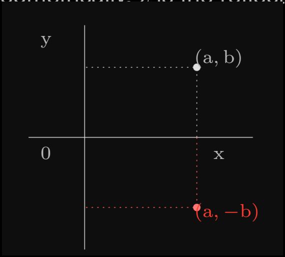

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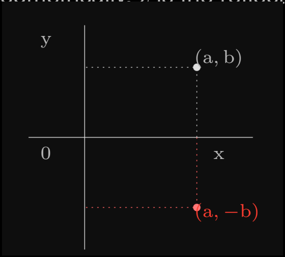

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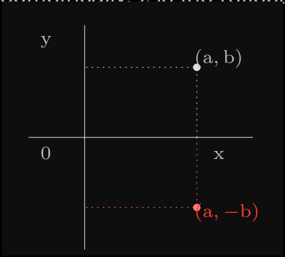

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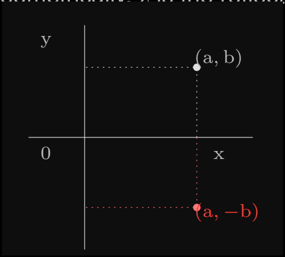

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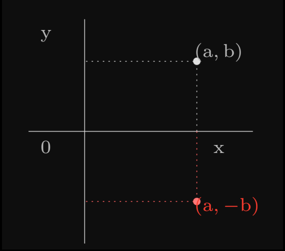

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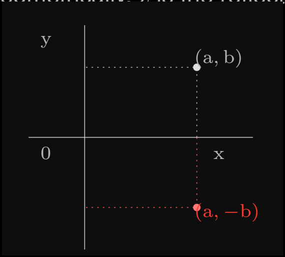

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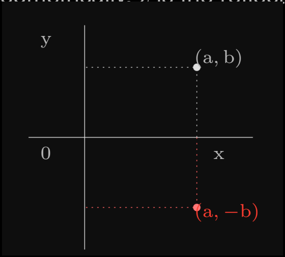

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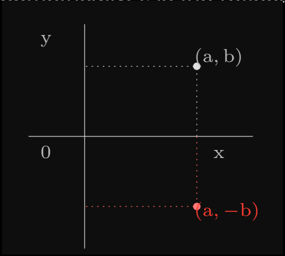

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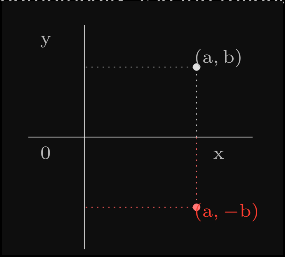

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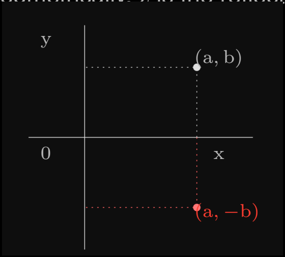

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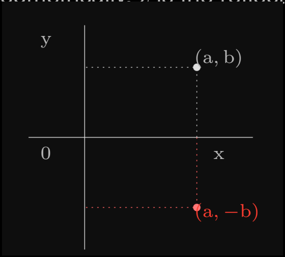

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The quotient $\frac{z}{w}$ is obtained by multiplying both top and bottom of $\frac{z}{w}$ by $\overline{w}$ and then simplifying the expression.

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You can always check that $zz^{-1} = 1$.

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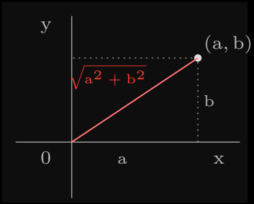
Geometrically, $|z| = \sqrt{a^2 + b^2}$ is the distance from z to the origin.



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7. The Triangle Inequality
   $|z + w| \leq |z| + |w|$.

Example (The Triangle Inequality: Geometrically)

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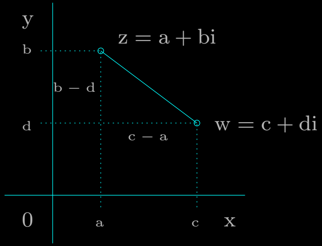

This shows that the distance between z and w in the complex plane is just the absolute value of their difference.

Example (continued)

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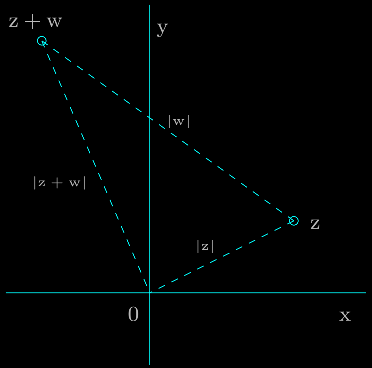

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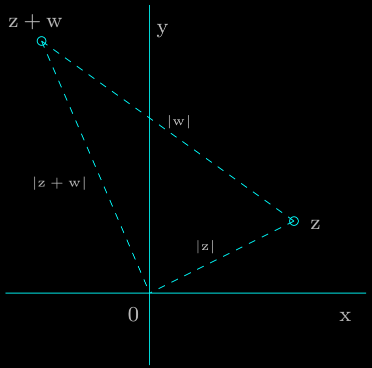

The triangle formed by these points has sides of length $|z|$, and $|z + w|$ and $|w|$ (the absolute value of the difference between $z + w$ and z).

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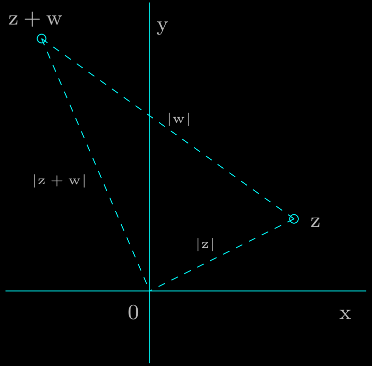

The triangle formed by these points has sides of length $|z|$, and $|z + w|$ and $|w|$ (the absolute value of the difference between $z + w$ and z).
Since the length of any side of a triangle is at most the sum of the lengths of the other two sides, we get $|z + w| \leq |z| + |w|$.

Complex Numbers in Polar Form

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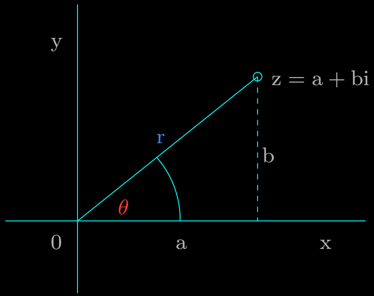
Suppose $z = a + bi$, and let $r = |z| = \sqrt{a^2 + b^2}$. Then r is the distance from z to the origin. Denote by θ the angle that the line through 0 and z makes with the positive x-axis (measured clockwise).



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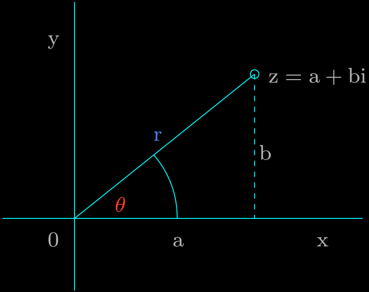

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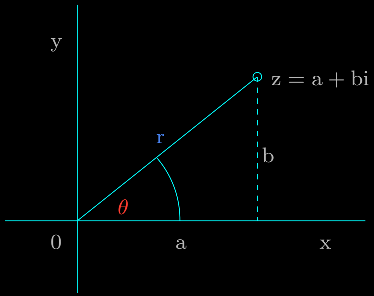

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θ is called an argument of z, and is denoted $\arg z$.

Definition (Polar Form of a Complex Number)

Let z be a complex number with $|z| = r$ and $\arg z = \theta$. Then

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Since $\arg \mathrm{z}$ is not unique, we do not write the polar form of z.

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Definition

Let z be a complex number with $|z| = r$. The principal argument of z is the unique angle $\theta = \arg z$ (measured in radians) such that

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Since sine and cosine have periodicity 2π, we may add (or subtract) multiples of 2π to any argument.

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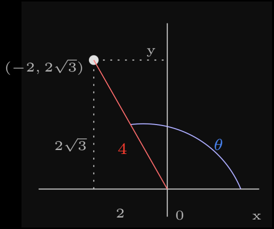

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By graphing the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, we again determine that $\theta = \frac{2\pi}{3}$, and thus z can be written in polar form as $z = 4e^{i(2\pi/3)}$. ■

Problem

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As an immediate consequence of De Moivre's Theorem, we have that for any real number $r > 0$ and any positive integer n,

$$\begin{aligned}(re^{i\theta})^n &= r^n e^{in\theta} \\ (r(\cos\theta + i\sin\theta))^n &= r^n(\cos n\theta + i\sin n\theta)\end{aligned}$$

Problem

Express $(1 - \mathrm{i})^6(\sqrt{3} + \mathrm{i})^3$ in the form $\mathrm{a} + \mathrm{bi}$.

Problem

Express $(1-\mathrm{i})^6(\sqrt{3}+\mathrm{i})^3$ in the form $a+bi$.

Solution

Let $z = 1-\mathrm{i} = \sqrt{2}e^{-\pi\mathrm{i}/4}$ and $w = \sqrt{3}+\mathrm{i} = 2e^{\pi\mathrm{i}/6}$.

Problem

Express $(1-\mathrm{i})^6(\sqrt{3}+\mathrm{i})^3$ in the form $\mathrm{a}+\mathrm{bi}$.

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\mathrm{z}^6\mathrm{w}^3 &= (\sqrt{2}\mathrm{e}^{-\pi\mathrm{i}/4})^6(2\mathrm{e}^{\pi\mathrm{i}/6})^3 \\
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&= 64(\cos\pi+\mathrm{i}\sin\pi) \\
&= -64.
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Problem

Express $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)^{17}$ in the form $\mathrm{a}+\mathrm{bi}$.

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Solution

Let $\mathrm{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = \mathrm{e}^{-\pi\mathrm{i}/3}$.

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Solution

Let $\mathrm{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = \mathrm{e}^{-\pi\mathrm{i}/3}$. Then

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Let $z = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = e^{-\pi i/3}$. Then

$$\begin{aligned} z^{17} &= \left(e^{-\pi i/3}\right)^{17} \\ &= e^{-17\pi i/3} \\ &= e^{\pi i/3} \\ &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \end{aligned}$$

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If θ is any angle and n is a positive integer, $\left(e^{i\theta}\right)^n = e^{in\theta}$. This implies that for any real number $r > 0$ and any positive integer n,

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This leads to the following result.

Corollary

Let q be a nonzero complex number and n a positive integer. Then $z^n = q$ has exactly n complex solutions, i.e., q has exactly n complex n^{th} roots.

Example

For any positive real number a, $z^2 = a$ has two complex (in this case, real) solution, $z = \sqrt{a}$ and $z = -\sqrt{a}$. This is equivalent to the statement that a has two complex (in this case, real) square roots.

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- One particular example: 25 has two square roots, 5 and -5, and these are the two solutions to $z^2 = 25$.
- But we all knew that. A more interesting example is that -1 has no real square roots, but suddenly it has two (complex) square roots, i and $-i$. These are the two (complex) solutions to $z^2 = 1$.

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  ($\mathbb{Z}$ denotes the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$).

Example (continued)

Dividing both sides of $3\theta = \frac{\pi}{2} + 2\pi\ell$ by 3:

$$\theta = \frac{\pi}{6} + \frac{2}{3}\pi\ell,$$

where ℓ is any integer.

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We now convert these to Cartesian form.

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You can check your work by computing the cube of each of these.

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This process is summarized in the following procedure.

Finding Roots of a Complex Number

Let w be a complex number. We wish to find the n^{th} roots of w, that is all z such that $\mathrm{z}^{\mathrm{n}} = \mathrm{w}$.
There are n distinct n^{th} roots and they can be found as follows:.

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1. Express both z and w in polar form $\mathrm{z} = \mathrm{r}e^{i\theta}, \mathrm{w} = \mathrm{s}e^{i\phi}$. Then $\mathrm{z}^{\mathrm{n}} = \mathrm{w}$ becomes:
$$(\mathrm{r}e^{i\theta})^{\mathrm{n}} = \mathrm{r}^{\mathrm{n}}e^{\mathrm{i}\mathrm{n}\theta} = \mathrm{s}e^{i\phi}$$
   We need to solve for r and θ.

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$$(re^{i\theta})^n = r^n e^{in\theta} = se^{i\phi}$$

   We need to solve for r and θ.

2. Solve the following two equations:

$$\begin{aligned} r^n &= s \\ e^{in\theta} &= e^{i\phi} \end{aligned} \tag{1}$$

Continued

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$$\mathrm{n}\theta = \phi + 2\pi\ell, \text{ for } \ell = 0, 1, 2, \cdots, \mathrm{n}-1$$

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$$\theta = \frac{\phi}{\mathrm{n}} + \frac{2}{\mathrm{n}}\pi\ell, \text{ for } \ell = 0, 1, 2, \cdots, \mathrm{n}-1$$

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5. Using the solutions r, θ to the equations given in (1) construct the n^{th} roots of the form $\mathrm{z} = \mathrm{r}\mathrm{e}^{\mathrm{i}\theta}$.

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2. The equation becomes $r^4 e^{i4\theta} = 4e^{2\pi i/3}$, so we need to solve

$$\begin{aligned} r^4 &= 4 \\ e^{i4\theta} &= e^{2\pi i/3} \end{aligned}$$

Solution (continued)

3. Since $r^4 = 4$, $r^2 = \pm 2$. But r is real, and so $r^2 = 2$, implying $r = \pm\sqrt{2}$. However $r \geq 0$, and therefore $r = \sqrt{2}$.

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$$4\theta = \frac{2}{3}\pi + 2\pi\ell, \ell = 0, 1, 2, 3.$$

Therefore,

$$\theta = \frac{2\pi}{12} + \frac{2\pi\ell}{4} = \frac{\pi}{6} + \frac{\pi\ell}{2} = \frac{\pi(3\ell+1)}{6}, \text{ for } \ell = 0, 1, 2, 3.$$

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\ell = 2: \quad & \mathrm{z} = \sqrt{2}\mathrm{e}^{7\pi \mathrm{i}/6} && = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) && = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}
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Therefore, the fourth roots of $2(\sqrt{3}i - 1)$ are:

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$\blacksquare$

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If you graph these six point in the complex plane, you'll see that they result in six equally spaced points on the unit circle, one of them being $(1, 0)$. ■

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Roots of Unity For any integer $n \geq 1$, the (complex) solutions to $z^n = 1$ are

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To find the roots of a real quadratic, we can either factor by inspection, or use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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The Quadratic Formula

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- if $b^2 - 4ac \geq 0$, then the roots of the quadratic are real;
- if $b^2 - 4ac < 0$, then the quadratic has no real roots.

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It follows that the roots of an irreducible quadratic are

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and we see that the two roots are complex conjugates of each other. We denote the two roots by

$$u = -\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}i \quad \text{and} \quad \overline{u} = -\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}i.$$

Example (Real Quadratics with Complex Roots)

The quadratic $x^2 - 14x + 58$ has roots

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so the roots are $7 + 3i$ and $7 - 3i$.

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Therefore, $x^2 - 10x + 29$ is an irreducible quadratic with roots $5 - 2i$ and $5 + 2i$.
Notice that $-10 = -(u + \overline{u})$ and $29 = u\overline{u} = |u|^2$. ■

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Solution (answer)

$x^2 + 6x + 25$ has roots $u = -3+4i$ and $\overline{u} = -3-4i$.

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To find $\pm\sqrt{-15-8i}$, solve $z^2 = -15 - 8i$ for z.

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Thus the roots of $x^2 - (3 - 2i)x + (5 - i)$ are $2 - 3i$ and $1 + i$. ■

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Solution (answer)

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First,

$$\begin{aligned} u_1^2 - (2 - 3i)u_1 - (10 + 6i) &= (4 - i)^2 - (2 - 3i)(4 - i) - (10 + 6i) \\ &= (15 - 8i) - (5 - 14i) - (10 + 6i) \\ &= 0, \end{aligned}$$

so $u_1 = (4 - i)$ is a root.

Solution (continued)

Recall that if u_1 and u_2 are the roots of the quadratic, then

$$u_1 + u_2 = (2 - 3i) \quad \text{and} \quad u_1 u_2 = -(10 + 6i).$$

Solve for u_2 using either one of these equations.

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Therefore, the other root is $u_2 = -2 - 2i$.
You can easily verify your answer by computing $u_1 u_2$:

$$u_1 u_2 = (4 - i)(-2 - 2i) = -10 - 6i = -(10 + 6i).$$

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