Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces
§6-3. Linear Independence and Dimension

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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Linear Independence

Definition

Let V be a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ a subset of V. The set S is linearly independent or simply independent if the following condition holds:

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k = \mathbf{0} \quad \Rightarrow \quad s_1 = s_2 = \cdots = s_k = 0$$

i.e., the only linear combination that vanishes is the trivial one. If S is not linearly independent, then S is said to be dependent.

Example

The set $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ is a dependent subset of $\mathbb{R}^3$

because

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solutions, for example a = 2, b = 3 and c = −1.

Is the set $T = \{3x^2 - x + 2, x^2 + x - 1, x^2 - 3x + 4\}$ an independent subset of $\mathcal{P}_2$?

Suppose $a(3x^2 - x + 2) + b(x^2 + x - 1) + c(x^2 - 3x + 4) = 0$, for some $a, b, c \in \mathbb{R}$. Then

$$x^2(3a + b + c) + x(-a + b - 3c) + (2a - b + 4c) = 0,$$

implying that

$$
\begin{aligned}
3a + b + c &= 0 \\
-a + b - 3c &= 0 \\
2a - b + 4c &= 0
\end{aligned}
$$

Solving this linear system of three equations in three variables

$$
\left[
\begin{array}{ccc|c}
3 & 1 & 1 & 0 \\
-1 & 1 & -3 & 0 \\
2 & -1 & 4 & 0
\end{array}
\right]
\rightarrow
\left[
\begin{array}{ccc|c}
1 & 0 & 1 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0
\end{array}
\right].
$$

Since there is nontrivial solution, T is a dependent subset of $\mathcal{P}_2$. ■

Problem

Is $U = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ an independent subset of $\mathbf{M}_{22}$?

Solution

Suppose $a \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$.

$$\Downarrow$$

$$
\begin{array}{rclrcl}
a + c & = & 0 \,, & a + b & = & 0 \,, \\
b + c & = & 0 \,, & a + c & = & 0 \,.
\end{array}
$$

This system of four equations in three variables has unique solution $a = b = c = 0$,

$$\Downarrow$$

U is an independent subset of $\mathbf{M}_{22}$.

■

Example (An independent subset of $\mathcal{P}_n$)

Consider $\{1, x, x^2, \ldots, x^n\}$, and suppose that

$$a_0 \cdot 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

for some $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then $a_0 = a_1 = \cdots = a_n = 0$, and thus $\{1, x, x^2, \ldots, x^n\}$ is an independent subset of $\mathcal{P}_n$.

Example (Polynomials with distinct degrees)

Any set of polynomials with DISTINCT degrees is independent.

For example,

$$\left\{ 2x^4 - x^3 + 5, \quad -3x^3 + 2x^2 + 2, \quad 4x^2 + x - 3, \quad 2x - 1, \quad 3 \right\}$$

is an independent subset of $\mathcal{P}_4$.

Example

As we saw earlier, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ (the standard basis of $\mathbb{R}^n$) is an independent subset of $\mathbb{R}^n$.

Example

$$U = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is an independent subset of $\mathbf{M}_{32}$.

Example (An independent subset of $\mathbf{M}_{mn}$)

In general, the set of mn $m \times n$ matrices that have a '1' in position (i, j) and zeros elsewhere, $1 \leq i \leq m$, $1 \leq j \leq n$, constitutes an independent subset of $\mathbf{M}_{mn}$.

Let V be a vector space.

1. If $\mathbf{v}$ is a nonzero vector of V, then $\{\mathbf{v}\}$ is an independent subset of V.

   Proof. Suppose that $k\mathbf{v} = \mathbf{0}$ for some $k \in \mathbb{R}$. Since $\mathbf{v} \neq \mathbf{0}$, it must be that $k = 0$, and therefore $\{\mathbf{v}\}$ is an independent set. ∎

2. The zero vector of V, $\mathbf{0}$ is never an element of an independent subset of V.

   Proof. Suppose $S = \{\mathbf{0}, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\}$ is a subset of V. Then

   $$1(\mathbf{0}) + 0(\mathbf{v}_2) + 0(\mathbf{v}_3) + \cdots + 0(\mathbf{v}_k) = \mathbf{0}.$$

   Since the coefficient of $\mathbf{0}$ (on the left-hand side) is '1', we have a nontrivial vanishing linear combination of the vectors of S. Therefore, S is dependent. ∎

Problem

Let V be a vector space and let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an independent subset of V. Is

$$S = \{\mathbf{u} + \mathbf{v}, 2\mathbf{u} + \mathbf{w}, \mathbf{v} - 5\mathbf{w}\}$$

an independent subset of V? Justify your answer.

Solution

Suppose that a linear combination of the vectors of S is equal to zero, i.e.,

$$a(\mathbf{u} + \mathbf{v}) + b(2\mathbf{u} + \mathbf{w}) + c(\mathbf{v} - 5\mathbf{w}) = \mathbf{0}$$

for some $a, b, c \in \mathbb{R}$. Then $(a + 2b)\mathbf{u} + (a + c)\mathbf{v} + (b - 5c)\mathbf{w} = \mathbf{0}$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent,

$$
\begin{aligned}
a + 2b &= 0 \\
a + c &= 0 \\
b - 5c &= 0.
\end{aligned}
$$

Solving for a, b and c, we find that the system has unique solution $a = b = c = 0$. Therefore, S is linearly independent. ∎

Problem

Suppose that A is an n × n matrix with the property that $A^k = \mathbf{0}$ but $A^{k-1} \neq \mathbf{0}$. Prove that

$$B = \{I, A, A^2, \ldots, A^{k-1}\}$$

is an independent subset of $\mathbf{M}_{nn}$.

Solution

We need to show that

$$r_0 I + r_1 A + r_2 A^2 + \cdots + r_{k-1} A^{k-1} = \mathbf{0} \quad \overset{?}{\Longrightarrow} \quad r_0 = r_1 = \cdots = r_{k-1} = 0.$$

Multiply A^{k-1} on both sides:

$$r_0 A^{k-1} + r_1 A^k + r_2 A^{k+1} + \cdots + r_{k-1} A^{2k-2} = \mathbf{0}$$

$$\Downarrow$$

$$r_0 A^{k-1} = \mathbf{0}.$$

Since $A^{k-1} \neq \mathbf{0}$, we see that $r_0 = 0$. Repeat the above processes to show that all $r_i = 0$ for $i = 0, 1, \cdots, k-1$. ∎

Theorem (Unique Representation Theorem)

Let V be a vector space and let $U = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$ be an independent set. If $\mathbf{v}$ is in span(U), then $\mathbf{v}$ has a unique representation as a linear combination of elements of U.

Proof.

If a vector $\mathbf{v}$ has two (ostensibly different) representations

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n$$
$$\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_n\mathbf{v}_n$$

$$\Downarrow$$

$$(s_1 - t_1)\mathbf{v}_1 + (s_2 - t_2)\mathbf{v}_2 + \cdots + (s_n - t_n)\mathbf{v}_n = \mathbf{0}$$

$$\Downarrow$$

$$s_1 - t_1 = 0, \quad s_2 - t_2 = 0, \quad \cdots, \quad s_n - t_n = 0.$$

$$\Downarrow$$

The two representations are the same one.

■

The Fundamental Theorem

The Fundamental Theorem for $\mathbb{R}^n$ generalizes to an arbitrary vector space.

Theorem (Fundamental Theorem)

Let V be a vector space that can be spanned by a set of n vectors, and suppose that V contains an independent subset of m vectors. Then $m \leq n$.

Proof.

Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and let $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m\}$. Suppose $V = \text{span}(X)$ and that Y is an independent subset of V. Each vector in Y can be written as a linear combination of vectors of X: for some $a_{ij} \in \mathbb{R}$, $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$
\begin{aligned}
\mathbf{y}_1 &= a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + \cdots + a_{1n}\mathbf{x}_n \\
\mathbf{y}_2 &= a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{2n}\mathbf{x}_n \\
&\ \ \vdots \\
\mathbf{y}_m &= a_{m1}\mathbf{x}_1 + a_{m2}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_n.
\end{aligned}
$$

Proof. (continued)

Let $A = [\ a_{ij}\ ]$, and suppose that $m > n$. Since rank $(A) = \dim(\text{row}(A)) \leq n$, it follows that the rows of A form a dependent subset of $\mathbb{R}^n$, and hence there is a nontrivial linear combination of the rows of A that is equal to the $1 \times n$ vector of all zeros, i.e., there exist $s_1, s_2, \ldots, s_m \in \mathbb{R}$, not all equal to zero, such that

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_m \end{bmatrix} A = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} = \mathbf{0}_{1n}.$$

It follows that for each j, $1 \leq j \leq n$,

$$s_1 a_{1j} + s_2 a_{2j} + \ldots + s_m a_{mj} = 0. \tag{1}$$

Consider the (nontrivial) linear combination of vectors of Y:

$$s_1 \mathbf{y}_1 + s_2 \mathbf{y}_2 + \cdots + s_m \mathbf{y}_m.$$

Proof. (continued)

$$
\begin{aligned}
s_1\mathbf{y}_1 + s_2\mathbf{y}_2 + \cdots + s_m\mathbf{y}_m \;=\;& s_1(a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + \cdots + a_{1n}\mathbf{x}_n) + \\
& s_2(a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{2n}\mathbf{x}_n) + \\
& \vdots \\
& s_m(a_{m1}\mathbf{x}_1 + a_{m2}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_n) \\
=\;& (s_1a_{11} + s_2a_{21} + \ldots + s_ma_{m1})\mathbf{x}_1 + \\
& (s_1a_{12} + s_2a_{22} + \ldots + s_ma_{m2})\mathbf{x}_2 + \\
& \vdots \\
& (s_1a_{1n} + s_2a_{2n} + \ldots + s_ma_{mn})\mathbf{x}_n.
\end{aligned}
$$

By Equation (1), it follows that

$$
s_1\mathbf{y}_1 + s_2\mathbf{y}_2 + \cdots + s_m\mathbf{y}_m = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_n = \mathbf{0}.
$$

Therefore, $s_1\mathbf{y}_1 + s_2\mathbf{y}_2 + \cdots + s_m\mathbf{y}_m = \mathbf{0}$ is a nontrivial vanishing linear combination of the vectors of Y. This contradicts the fact that Y is independent, and therefore $m \leq n$. ■

Definition

Let V be a vector space and let B = $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq$ V. We say B is a basis of V if
(i) B is an independent subset of V and
(ii) span(B) = V.

Remark (Unique Representation Theorem)

Recall that if V is a vector space and B is a basis of V, then as seen earlier, any vector $\mathbf{u} \in$ V can be expressed uniquely as a linear combination of vectors of B.

Example

As we saw earlier, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, called the standard basis of $\mathbb{R}^n$.

Example (A basis of $\mathcal{P}_n$)

We've already seen that
$$\{1, x, x^2, \ldots, x^n\}$$
spans $\mathcal{P}_n$ and is an independent subset of $\mathcal{P}_n$, and is thus a basis of $\mathcal{P}_n$.

$$\{1, x, x^2, \ldots, x^n\}$$

is called the standard basis of $\mathcal{P}_n$.

Example (A basis of $\mathbf{M}_{mn}$)

The set of mn $m \times n$ matrices that have a '1' in position (i, j) and zeros elsewhere, $1 \leq i \leq m$, $1 \leq j \leq n$, spans $\mathbf{M}_{mn}$ and is an independent subset of $\mathbf{M}_{mn}$. Therefore, this set constitutes a basis of $\mathbf{M}_{mn}$ and is called the standard basis of $\mathbf{M}_{mn}$.

The Invariance Theorem generalizes from $\mathbb{R}^n$ to an arbitrary vector space V. The proof is identical, and involves two applications of the Fundamental Theorem.

Theorem (Invariance Theorem)

If V is a vector space with bases $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$, then m = n.

Definition (Dimension of a vector space)

Let V be a vector space and suppose $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of V. The dimension of V is the number of vectors in B, and we write $\dim(V) = n$. By convention, $\dim(\{\mathbf{0}\}) := 0$.

Example

Let V be a vector space and **u** a NONZERO vector of V. Then $U = \text{span}\{\mathbf{u}\}$ is spanned by $\{\mathbf{u}\}$. Since $\{\mathbf{u}\}$ is independent, $\{\mathbf{u}\}$ is a basis of U, and thus $\dim(U) = 1$.

Example

Since $\{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathcal{P}_n$, $\dim(\mathcal{P}_n) = n + 1$.

Example

$\dim(\mathbf{M}_{mn}) = mn$ since the standard basis of $\mathbf{M}_{mn}$ consists of mn matrices.

Problem

Let $U = \left\{ A \in \mathbf{M}_{22} \;\middle|\; A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A \right\}$. Then U is a subspace of $\mathbf{M}_{22}$. Find a basis of U, and hence $\dim(U)$.

Solution

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}$. Then

$$A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix}.$$

If $A \in U$, then $\begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix}.$

Solution (continued)

Equating entries leads to a system of four equations in the four variables a, b, c and d.

$$
\begin{aligned}
a + b &= a + c \\
-b &= b + d \\
c + d &= -c \\
-d &= -d
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
b - c &= 0 \\
-2b - d &= 0 \\
2c + d &= 0
\end{aligned} \ .
$$

The solution to this system is $a = s$, $b = -\frac{1}{2}t$, $c = -\frac{1}{2}t$, $d = t$ for any $s, t \in \mathbb{R}$, and thus $A = \begin{bmatrix} s & \frac{t}{2} \\ -\frac{t}{2} & t \end{bmatrix}$, $s, t \in \mathbb{R}$. Since $A \in U$ is arbitrary,

$$
\begin{aligned}
U &= \left\{ \begin{bmatrix} s & \frac{t}{2} \\ -\frac{t}{2} & t \end{bmatrix} \ \middle| \ s, t \in \mathbb{R} \right\} \\
&= \left\{ s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \ \middle| \ s, t \in \mathbb{R} \right\} \\
&= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \right\} .
\end{aligned}
$$

Solution (continued)

Let

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \right\}.$$

Then $\operatorname{span}(B) = U$, and it is routine to verify that B is an independent subset of $\mathbf{M}_{22}$. Therefore, B is a basis of U, and $\dim(U) = 2$. ∎

Let U = {p(x) $\in \mathcal{P}_2$ | p(1) = 0}. Then U is a subspace of $\mathcal{P}_2$. Find a basis of U, and hence dim(U).

Solution

Final Answer B = $\{x - x^2, 1 - x^2\}$ is a basis of U and thus dim(U) = 2. ■