Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations
§7-2. Kernel and Image

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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What are the Kernel and the Image?

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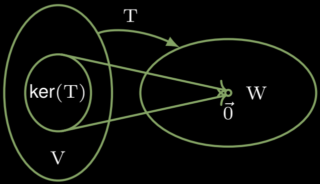
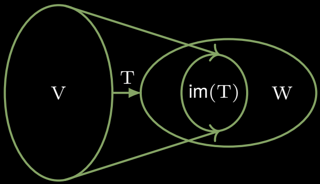
Definition

Let V and W be vector spaces, and $T : V \to W$ a linear transformation.

1. The kernel of T (sometimes called the null space of T) is defined to be the set
$$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$
2. The image of T is defined to be the set
$$\mathrm{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$





Remark

If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation induced by A, then

- $\ker(T_A) = \text{null}(A)$;
- $\text{im}(T_A) = \text{im}(A)$.

Problem

Let $T : \mathcal{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1.$$

Find $\ker(T)$ and $\operatorname{im}(T)$.

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Find $\ker(T)$ and $\operatorname{im}(T)$.

Solution

$$\begin{aligned}
\ker(T) &= \{p(x) \in \mathcal{P}_1 \mid p(1) = 0\} \\
&= \{ax + b \mid \forall a, b \in \mathbb{R} \quad \text{and} \quad a + b = 0\} \\
&= \{ax - a \mid \forall a \in \mathbb{R}\}.
\end{aligned}$$

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\end{aligned}$$

$$\begin{aligned}
\mathrm{im}(T) &= \{p(1) \mid p(x) \in \mathcal{P}_1\} \\
&= \{a + b \mid ax + b \in \mathcal{P}_1\} \\
&= \{a + b \mid \forall a, b \in \mathbb{R}\} \\
&= \mathbb{R}.
\end{aligned}$$

■

Theorem

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then $\ker(T)$ is a subspace of V and $\operatorname{im}(T)$ is a subspace of W.

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Proof. (that $\ker(T)$ is a subspace of V)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. T is a linear transformation $\Rightarrow T(\vec{0}_V) = \vec{0}_W \Rightarrow \vec{0}_V \in \ker(T)$.

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2. Let $\vec{v}_1, \vec{v}_2 \in \ker(T)$. Then $T(\vec{v}_1) = \vec{0}$, $T(\vec{v}_2) = \vec{0}$, and

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}.$$

   Thus $\vec{v}_1 + \vec{v}_2 \in \ker(T)$.

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Thus $\vec{v}_1 + \vec{v}_2 \in \ker(T)$.
3. Let $\vec{v}_1 \in \ker(T)$ and let $k \in \mathbb{R}$. Then $T(\vec{v}_1) = \vec{0}$, and
$$T(k\vec{v}_1) = kT(\vec{v}_1) = k(\vec{0}) = \vec{0}.$$
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$$T(k\vec{v}_1) = kT(\vec{v}_1) = k(\vec{0}) = \vec{0}.$$
Thus $k\vec{v}_1 \in \ker(T)$.

By the Subspace Test, $\ker(T)$ is a subspace of V. ■

Proof. (that $\mathrm{im}(T)$ is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. T is a linear transformation $\Rightarrow T(\vec{0}_V) = \vec{0}_W \Rightarrow \vec{0}_W \in \mathrm{im}(T)$.

Proof. (that $\mathrm{im}(T)$ is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. T is a linear transformation $\Rightarrow T(\vec{0}_V) = \vec{0}_W \Rightarrow \vec{0}_W \in \mathrm{im}(T)$.
2. Let $\vec{w}_1, \vec{w}_2 \in \mathrm{im}(T)$. Then there exist $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$, and thus

$$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2).$$

Since $\vec{v}_1 + \vec{v}_2 \in V$, $\vec{w}_1 + \vec{w}_2 \in \mathrm{im}(T)$.

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Since $\vec{v}_1 + \vec{v}_2 \in V$, $\vec{w}_1 + \vec{w}_2 \in \mathrm{im}(T)$.
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$$k\vec{w}_1 = kT(\vec{v}_1) = T(k\vec{v}_1).$$
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Proof. (that im(T) is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. T is a linear transformation $\Rightarrow T(\vec{0}_V) = \vec{0}_W \Rightarrow \vec{0}_W \in \mathrm{im}(T)$.
2. Let $\vec{w}_1, \vec{w}_2 \in \mathrm{im}(T)$. Then there exist $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$, and thus
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3. Let $\vec{w}_1 \in \mathrm{im}(V)$ and let $k \in \mathbb{R}$. Then there exists $\vec{v}_1 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, and
$$k\vec{w}_1 = kT(\vec{v}_1) = T(k\vec{v}_1).$$
Since $k\vec{v}_1 \in V$, $k\vec{w}_1 \in \mathrm{im}(T)$.

By the Subspace Test, im(T) is a subspace of W. ■

Definition

Let V and W be vector spaces and $T : V \to W$ a linear transformation.

1. The dimension of $\ker(T)$, $\dim(\ker(T))$ is called the nullity of T and is denoted $\text{nullity}(T)$, i.e.,

$$\text{nullity}(T) = \dim(\ker(T)).$$

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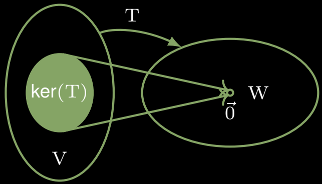
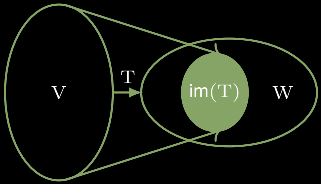
2. The dimension of $\mathrm{im}(T)$, $\dim(\mathrm{im}(T))$ is called the rank of T and is denoted $\mathrm{rank}\ (T)$, i.e.,

$$\mathrm{rank}(T) = \dim(\mathrm{im}(T)).$$

Nullity of T



Rank of T



Example

If A is an $m \times n$ matrix, then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and

$$\mathrm{im}(T_A) = \mathrm{im}(A) = \mathrm{col}(A)$$

$$\Downarrow$$

$$\begin{aligned} \mathrm{rank}\ (T_A) &= \dim(\mathrm{im}(T_A)) \\ &= \dim(\mathrm{col}(A)) \\ &= \mathrm{rank}\ (A) \\ &= \dim(\mathrm{row}(A)) \end{aligned}$$

$$\ker(T_A) = \mathrm{null}(A)$$

$$\Downarrow$$

$$\begin{aligned} &\mathrm{nullity}(T_A) = \dim(\mathrm{null}(A)) \\ &= \text{“\# of free parameters in } Ax = 0\text{"} \\ &= n - \mathrm{rank}\ (A) \end{aligned}$$

$$\Updownarrow$$

$$\boxed{\mathrm{rank}\ (A) + \mathrm{nullity}(T_A) = \dim(\mathbb{R}^n)}$$

Finding bases of the kernel and the image

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Example (continued)

For the linear transformation T defined by $T : \mathcal{P}_1 \to \mathbb{R}$

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1,$$

we found that

$$\ker(T) = \{ax - a \mid a \in \mathbb{R}\} \quad \text{and} \quad \operatorname{im}(T) = \mathbb{R}.$$

Finding bases of the kernel and the image

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For the linear transformation T defined by $T : \mathcal{P}_1 \to \mathbb{R}$

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we found that

$$\ker(T) = \{ax - a \mid a \in \mathbb{R}\} \quad \text{and} \quad \mathrm{im}(T) = \mathbb{R}.$$

- $\ker(T) = \mathrm{span}\{(x-1)\}$ and $\dim(\ker(T)) = 1 = \mathrm{nullity}(T)$.
- $\mathrm{im}(T) = \mathrm{span}\{1\}$ and $\dim(\mathrm{im}(T)) = 1 = \mathrm{rank}\ (T)$
- Hence,
$$\mathrm{nullity}(T) + \mathrm{rank}\ (T) = \dim(\mathcal{P}_1) = 2.$$

∎

Problem

Let $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Then T is a linear transformation (you should be able to prove this). Find a basis of $\ker(T)$ and a basis of $\mathrm{im}(T)$.

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Then T is a linear transformation (you should be able to prove this). Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

Solution

Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T)$. Then

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

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Solution

Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T)$. Then

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives us a system of four equations in the four variables a, b, c, d:

$$\begin{cases} a+b=0 \\ b+c=0 \\ c+d=0 \\ d+a=0 \end{cases}$$

Solution (continued)

This system has solution $a = -t, b = t, c = -t, d = t$ for any $t \in \mathbb{R}$, and thus

$$\ker(T) = \left\{ \begin{bmatrix} -t & t \\ -t & t \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$$

Solution (continued)

This system has solution $a = -t, b = t, c = -t, d = t$ for any $t \in \mathbb{R}$, and thus

$$\ker(T) = \left\{ \left[\begin{array}{cc} -t & t \\ -t & t \end{array} \right] \;\middle|\; t \in \mathbb{R} \right\} = \operatorname{span} \left\{ \left[\begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right] \right\}.$$

Let

$$B = \left\{ \left[\begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right] \right\}.$$

Since B is an independent subset of $\mathbf{M}_{22}$ and $\operatorname{span}(B) = \ker(T)$, B is a basis of $\ker(T)$.

Solution (continued)

As for im(T), notice that

$$\begin{aligned} \mathrm{im}(T) &= \left\{ \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\} \\ &= \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Solution (continued)

As for $\mathrm{im}(T)$, notice that

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Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Solution (continued)

As for $\operatorname{im}(T)$, notice that

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\operatorname{im}(T) &= \left\{ \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\} \\
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\end{aligned}$$

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S is a dependent subset of $\mathbf{M}_{22}$, but (check this yourselves)

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is an independent subset of S. Since $\operatorname{span}(C) = \operatorname{span}(S) = \operatorname{im}(T)$ and C is independent, C is a basis of $\operatorname{im}(T)$. $\blacksquare$

Remark

$$\dim(\mathbf{M}_{22}) = 4$$

$$\text{nullity}(\mathrm{T}) = \dim(\ker(\mathrm{T})) = 1$$

$$\text{rank }(\mathrm{T}) = \dim(\text{im}(\mathrm{T})) = 3$$

$$\Downarrow$$

$$\text{nullity}(\mathrm{T}) + \text{rank }(\mathrm{T}) = \dim(\mathbf{M}_{22})$$

Surjections and Injections

Surjections and Injections

Definition

Let V and W be vector spaces and $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ a linear transformation.

1. T is onto (or surjective) if $\mathrm{im}(\mathrm{T}) = \mathrm{W}$.
2. T is one-to-one (or injective) if,

$$\mathrm{T}(\vec{\mathrm{v}}) = \mathrm{T}(\vec{\mathrm{w}}) \quad \forall \vec{\mathrm{v}}, \vec{\mathrm{w}} \in \mathrm{V} \qquad \Rightarrow \qquad \vec{\mathrm{v}} = \vec{\mathrm{w}}.$$

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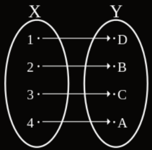
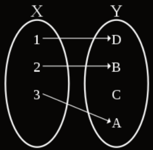
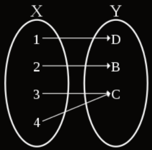
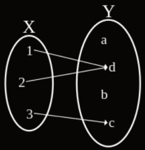
$$T(\vec{v}) = T(\vec{w}) \quad \forall \vec{v}, \vec{w} \in V \qquad \Rightarrow \qquad \vec{v} = \vec{w}.$$

Example

Let V be a vector space. Then the identity operator on V, $1_V : V \to V$, is one-to-one and onto.

	surjective	non-surjective
injective	 bijective	 injective-only
non-injective	 surjective-only	 general

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Proof.

$(\Rightarrow)$ Let $\vec{v} \in \ker(T)$. Then

$$T(\vec{v}) = \vec{0} = T(\vec{0}).$$

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$$\text{T is one-to-one} \quad \Rightarrow \quad \vec{v} = \vec{0} \quad \Rightarrow \quad \ker T = \{\vec{0}\}$$

($\Leftarrow$) Conversely, suppose that $\ker(T) = \{\vec{0}\}$, and let $\vec{v}, \vec{w} \in V$ be such that

$$T(\vec{v}) = T(\vec{w}).$$

Then $T(\vec{v}) - T(\vec{w}) = \vec{0}$, and since T is a linear transformation

$$T(\vec{v} - \vec{w}) = \vec{0}.$$

By definition, $\vec{v} - \vec{w} \in \ker(T)$, implying that $\vec{v} - \vec{w} = \vec{0}$. Therefore $\vec{v} = \vec{w}$, and hence T is one-to-one. ■

Problem

Let $T : \mathbf{M}_{22} \to \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + d \\ b + c \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Prove that T is onto but not one-to-one.

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Prove that T is onto but not one-to-one.

Proof.

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Since $\mathrm{T}\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, T is onto.

Observe that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \ker(\mathrm{T})$, so $\ker(\mathrm{T}) \neq \{\vec{0}_{22}\}$. By the previous Theorem, T is not one-to-one. ■

Problem

Suppose U is an invertible $m \times m$ matrix and let $T : \mathbf{M}_{mn} \to \mathbf{M}_{mn}$ be defined by

$$T(A) = UA \text{ for all } A \in \mathbf{M}_{mn}.$$

Then T is a linear transformation (this is left to you to verify). Prove that T is one-to-one and onto.

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Then T is a linear transformation (this is left to you to verify). Prove that T is one-to-one and onto.

Proof.

Suppose $A, B \in \mathbf{M}_{mn}$ and that $T(A) = T(B)$. Then $UA = UB$; since U is invertible

$$\begin{aligned} U^{-1}(UA) &= U^{-1}(UB) \\ (U^{-1}U)A &= (U^{-1}U)B \\ I_{mm}A &= I_{mm}B \\ A &= B. \end{aligned}$$

Therefore, T is one-to-one.

Proof. (continued)

To prove that T is onto, let $B \in \mathbf{M}_{mn}$ and let $A = U^{-1}B$. Then

$$T(A) = UA = U(U^{-1}B) = (UU^{-1})B = I_{mm}B = B,$$

and therefore T is onto. ∎

Problem

Let $S : \mathcal{P}_2 \to \mathbf{M}_{22}$ be a linear transformation defined by

$$S(ax^2 + bx + c) = \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \text{ for all } ax^2 + bx + c \in \mathcal{P}_2.$$

Prove that S is one-to-one but not onto.

Problem

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Prove that S is one-to-one but not onto.

Proof.

By definition,

$$\ker(S) = \{ax^2 + bx + c \in \mathcal{P}_2 \mid a+b=0, a+c=0, b-c=0, b+c=0\}.$$

Suppose $p(x) = ax^2 + bx + c \in \ker(S)$. This leads to a homogeneous system of four equations in three variables:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the unique solution is $a = b = c = 0$, $\ker(S) = \{\vec{0}\}$, and thus S is one-to-one.

Proof. (continued)

To show that S is not onto, show that $\mathrm{im}(S) \neq \mathcal{P}_2$; i.e., find a matrix $A \in \mathbf{M}_{22}$ such that for every $p(x) \in \mathcal{P}_2$, $S(p(x)) \neq A$. Let

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 2 \end{array}\right],$$

and suppose $p(x) = ax^2 + bx + c \in \mathcal{P}_2$ is such that $S(p(x)) = A$. Then

$$\begin{array}{ll} a + b = 0 & a + c = 1 \\ b - c = 0 & b + c = 2 \end{array}$$

Solving this system

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

Since the system is inconsistent, there is no $p(x) \in \mathcal{P}_2$ so that $S(p(x)) = A$, and therefore S is not onto. ∎

Problem (One-to-one linear transformations preserve independent sets)

Let V and W be vector spaces and $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ a linear transformation. Prove that if T is one-to-one and $\{\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2, \ldots, \vec{\mathrm{v}}_\mathrm{k}\}$ is an independent subset of V, then $\{\mathrm{T}(\vec{\mathrm{v}}_1), \mathrm{T}(\vec{\mathrm{v}}_2), \ldots, \mathrm{T}(\vec{\mathrm{v}}_\mathrm{k})\}$ is an independent subset of W.

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Proof.

Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Suppose that

$$a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \cdots + a_k T(\vec{v}_k) = \vec{0}_W$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$. Since linear transformations preserve linear combinations (addition and scalar multiplication),

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W.$$

Now, since T is one-to-one, $\ker(T) = \{\vec{0}_V\}$, and thus

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V.$$

However, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is independent, and hence $a_1 = a_2 = \cdots = a_k = 0$. Therefore, $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is independent. ∎

Problem (Onto linear transformations preserve spanning sets)

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Prove that if T is onto and $V = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, then

$$W = \mathrm{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

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Proof.

Suppose that T is onto and let $\vec{w} \in W$. Then there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $V = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, there exist $a_1, a_2, \ldots a_k \in \mathbb{R}$ such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$. Since T is a linear transformation,

$$\begin{aligned} \vec{w} = T(\vec{v}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) \\ &= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + \cdots + a_kT(\vec{v}_k), \end{aligned}$$

i.e., $\vec{w} \in \mathrm{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$, and thus

$$W \subseteq \mathrm{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

On the other hand,

$$T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k) \in W \quad \Longrightarrow \quad \mathrm{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq W.$$

Therefore, $W = \mathrm{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$. $\blacksquare$

Suppose A is an $m \times n$ matrix. How do we determine if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is onto? How do we determine if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one?

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Theorem

Let A be an $m \times n$ matrix, and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear transformation induced by A.

1. T_A is onto if and only if $\text{rank}(A) = m$.
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Proof. (sketch)

1. T_A is onto if and only if $\operatorname{im}(T_A) = \mathbb{R}^m$. This is equivalent to $\operatorname{col}(A) = \mathbb{R}^m$, which occurs if and only if $\dim(\operatorname{col}(A)) = m$, i.e., $\operatorname{rank}(A) = m$.
2. $\ker(T_A) = \operatorname{null}(A)$, and $\operatorname{null}(A) = \{\vec{0}\}$ if and only if $A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$. Thus and row echelon form of A has a leading one in every column, which occurs if and only if $\operatorname{rank}(A) = n$. $\blacksquare$

The Dimension Theorem (Rank-Nullity Theorem)

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Suppose A is an $m \times n$ matrix with rank r. Since $\mathrm{im}(T_A) = \mathrm{col}(A)$,

$$\dim(\mathrm{im}(T_A)) = \mathrm{rank}\ (A) = r.$$

We also know that $\ker(T_A) = \mathrm{null}(A)$, and that $\dim(\mathrm{null}(A)) = n - r$. Thus,

$$\dim(\mathrm{im}(T_A)) + \dim(\ker(T_A)) = n = \dim\ \mathbb{R}^n.$$

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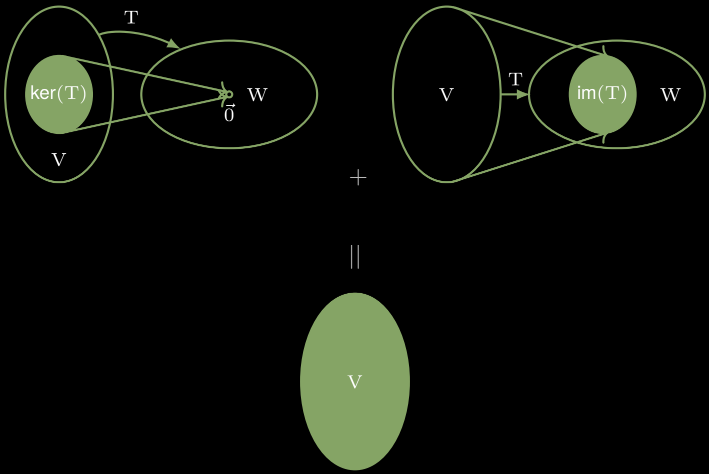
Theorem (Dimension Theorem (Rank-Nullity Theorem))

Let V and W be vector spaces and $T : V \to W$ a linear transformation. If $\ker(T)$ and $\mathrm{im}(T)$ are both finite dimensional, then V is finite dimensional, and

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)).$$

Equivalently, $\dim(V) = \mathrm{nullity}(T) + \mathrm{rank}\,(T)$.

$$
\begin{tikzcd}
 & W \arrow[dd, dash] \\
V \arrow[dd, dash] \arrow[dr] \arrow[ddd, dashed, dash, bend right=60, "\dim V"'] & \\
 & \operatorname{im} T \arrow[dd, dash] \arrow[dd, dashed, dash, bend left=60, "\dim \operatorname{im} T"] \\
\ker T \arrow[d, dash] \arrow[dr] \arrow[d, dashed, dash, bend right=60, "\dim \ker T"'] & \\
0 \arrow[r] & 0
\end{tikzcd}
$$


T
ker(T)
V
W
$\vec{0}$
V
T
im(T)
W
+
||
V

Proof. (Outline)

Let $\vec{w} \in \mathrm{im}(T)$; then $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$. Suppose

$$\left\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_r)\right\}$$

is a basis of $\mathrm{im}(T)$, and that

$$\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\}$$

is a basis of $\ker(T)$. We define

$$B = \left\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_r, \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\}.$$

To prove that B is a basis of V, it remains to prove that B spans V and that B is linearly independent.

Since B is independent and spans V, B is a basis of V, implying V is finite dimensional (V is spanned by a finite set of vectors). Furthermore, $|B| = r + k$, so

$$\dim(V) = \dim(\mathrm{im}(T)) + \dim(\ker(T)).$$

■

Remark

1. It is not an assumption of the theorem that V is finite dimensional. Rather, it is a consequence of the assumption that both im(T) and ker(T) are finite dimensional.

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Remark

1. It is not an assumption of the theorem that V is finite dimensional. Rather, it is a consequence of the assumption that both $\mathrm{im}(T)$ and $\ker(T)$ are finite dimensional.
2. As a consequence of the Dimension Theorem, if V is a finite dimensional vector space and either $\dim(\ker(T))$ or $\dim(\mathrm{im}(T))$ is known, then the other can be easily found.

Example

Let V and W be vector spaces and $T : V \to W$ a linear transformation. If V is finite dimensional, then it follows that

$$\dim(\ker(T)) \leq \dim(V) \quad \text{and} \quad \dim(\mathrm{im}(T)) \leq \dim(V).$$

Problem

For $a \in \mathbb{R}$, recall that the linear transformation $E_a : \mathcal{P}_n \to \mathbb{R}$, the evaluation map at a, is defined as

$$E_a(p(x)) = p(a) \text{ for all } p(x) \in \mathcal{P}_n.$$

Prove that E_a is onto, and that

$$B = \{(x - a), (x - a)^2, (x - a)^3, \dots, (x - a)^n\}$$

is a basis of $\ker(E_a)$.

Proof.

Let $t \in \mathbb{R}$, and choose $p(x) = t \in \mathcal{P}_n$. Then $p(a) = t$, so $E_a(p(x)) = t$, i.e., E_a is onto.

Proof.

Let $t \in \mathbb{R}$, and choose $p(x) = t \in \mathcal{P}_n$. Then $p(a) = t$, so $E_a(p(x)) = t$, i.e., E_a is onto.

By the Dimension Theorem,

$$n + 1 = \dim(\mathcal{P}_n) = \dim(\ker(E_a)) + \dim(\mathrm{im}(E_a)).$$

Since E_a is onto, $\dim(\mathrm{im}(E_a)) = \dim(\mathbb{R}) = 1$, and thus $\dim(\ker(E_a)) = n$.

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Note that $(x - a)^j \in \ker(E_a)$ for $j = 1, 2, \ldots, n$, so $B \subseteq \ker(E_a)$.

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Since $|B| = n = \dim(\ker(E_a))$, B spans $\ker(E_a)$.

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Therefore, B is a basis of $\ker(E_a)$. $\blacksquare$

Theorem

Let V and W be vector spaces, $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ a linear transformation, and

$$\mathrm{B} = \left\{\vec{\mathrm{b}}_1, \vec{\mathrm{b}}_2, \ldots, \vec{\mathrm{b}}_\mathrm{r}, \vec{\mathrm{b}}_{\mathrm{r}+1}, \vec{\mathrm{b}}_{\mathrm{r}+2}, \ldots, \vec{\mathrm{b}}_\mathrm{n}\right\}$$

a basis of V with the property that $\left\{\vec{\mathrm{b}}_{\mathrm{r}+1}, \vec{\mathrm{b}}_{\mathrm{r}+2}, \ldots, \vec{\mathrm{b}}_\mathrm{n}\right\}$ is a basis of $\ker(\mathrm{T})$. Then

$$\left\{\mathrm{T}(\vec{\mathrm{b}}_1), \mathrm{T}(\vec{\mathrm{b}}_2), \ldots, \mathrm{T}(\vec{\mathrm{b}}_\mathrm{r})\right\}$$

is a basis of $\mathrm{im}(\mathrm{T})$, and therefore $\mathrm{r} = \mathrm{rank}\ (\mathrm{T})$.

Theorem

Let V and W be vector spaces, $T : V \to W$ a linear transformation, and

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a basis of V with the property that $\left\{\vec{b}_{r+1}, \vec{b}_{r+2}, \ldots, \vec{b}_n\right\}$ is a basis of $\ker(T)$. Then

$$\left\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_r)\right\}$$

is a basis of $\mathrm{im}(T)$, and therefore $r = \mathrm{rank}\,(T)$.

Remark (How is this useful?)

Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation. If you find a basis of $\ker(T)$, then this may be used to find a basis of $\mathrm{im}(T)$: extend the basis of $\ker(T)$ to a basis of V; applying the transformation T to each of the vectors that was added to the basis of $\ker(T)$ produces a set of vectors that is a basis of $\mathrm{im}(T)$.

Problem

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and let $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be a linear transformation defined by

$$T(X) = XA - AX \text{ for all } X \in \mathbf{M}_{22}.$$

Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

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Find a basis of $\ker(T)$ and a basis of $\mathrm{im}(T)$.

Solution

First note that by the Dimension Theorem,

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(\mathbf{M}_{22}) = 4.$$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned} T(X) &= AX - XA \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c & d \\ a & b \end{bmatrix} - \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} \end{aligned}$$

Solution (continued)

If $X \in \ker(T)$, then $T(X) = \vec{0}_{22}$ so

$$\begin{cases} c - b = 0 \\ d - a = 0 \\ a - d = 0 \\ b - c = 0 \end{cases} \implies \begin{cases} a = s \\ b = t \\ c = t \\ d = s \end{cases} \quad \text{for } s, t \in \mathbb{R}.$$

Therefore,

$$\ker(T) = \left\{ \left[\begin{array}{cc} s & t \\ t & s \end{array} \right] \middle| \; s, t \in \mathbb{R} \right\} = \operatorname{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}.$$

Let

$$B = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}$$

Since B is independent and spans $\ker(T)$, B_k is a basis of $\ker(T)$.

Solution (continued)

To find a basis of $\mathrm{im}(T)$, extend the basis of $\ker(T)$ to a basis of $\mathbf{M}_{22}$: here is one such basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

Therefore,

$$C = \left\{ T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of $\mathrm{im}(T)$. ■