Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra

§2-2. Equations, Matrices, and Transformations

Le Chen[1]

Emory University, 2021 Spring

(last updated on 01/12/2023)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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A new topic
A new example or problem
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Any other suggestions to improve the material

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Vectors

Vectors

Definitions

A row matrix or column matrix is often called a vector, and such matrices are referred to as row vectors and column vectors, respectively. If $\vec{x}$ is a row vector of size $1 \times n$, and $\vec{y}$ is a column vector of size $m \times 1$, then we write

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Definition (Vector form of a system of linear equations)

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

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Such a system can be expressed in vector form or as a vector equation by using linear combinations of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Problem

Express the following system of linear equations in vector form:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

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Solution

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Matrix vector multiplication

Matrix vector multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, written $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$, and let $\vec{x}$ be an $n \times 1$ column vector,

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$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product of matrix A and (column) vector $\vec{x}$ is the $m \times 1$ column vector given by

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \sum_{j=1}^{n} x_j\vec{a}_j$$

that is, $A\vec{x}$ is a linear combination of the columns of A.

Problem

Compute the product $A\vec{x}$ for

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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Solution

$$A\vec{x} = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

Problem

Compute $A\vec{y}$ for

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

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Solution

$$A\vec{y} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

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Such a system can be expressed in matrix form using matrix vector multiplication,

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Thus a system of linear equations can be expressed as a matrix equation

$$A\vec{x} = \vec{b},$$

where A is the coefficient matrix, $\vec{b}$ is the constant matrix, and $\vec{x}$ is the matrix of variables.

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Solution

$$\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Theorem

1. Every system of m linear equations in n variables can be written in the form $A\vec{x} = \vec{b}$ where A is the coefficient matrix, $\vec{x}$ is the matrix of variables, and $\vec{b}$ is the constant matrix.

Theorem (continued)

2. The system $A\vec{x} = \vec{b}$ is consistent (i.e., has at least one solution) if and only if $\vec{b}$ is a linear combination of the columns of A.

Theorem (continued)

3. The vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the system $A\vec{x} = \vec{b}$ if and only if $x_1, x_2, \ldots, x_n$ are a solution to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n = \vec{b}$$

where $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the columns of A.

Problem

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express $\vec{b}$ as a linear combination of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ of A, or show that this is impossible.

Solution

Solve the system $A\vec{x} = \vec{b}$ where $\vec{x}$ is a column vector with four entries.

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$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 1/7 \\ 0 & 1 & 0 & 1 & -5/7 \\ 0 & 0 & 1 & -1 & 3/7 \end{array}\right]$$

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Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4.

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Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4. Choosing $x_4 = 0$ (which is the simplest thing to do) gives us

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{7}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{7}\vec{a}_1 - \frac{5}{7}\vec{a}_2 + \frac{3}{7}\vec{a}_3 + 0\vec{a}_4.$$

■

Remark

The problem may ask to to find all possible linear combinations of the columns $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$ of A.

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This is equivalent to find all solutions to the corresponding system of linear equations:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} - s \\ -\frac{5}{7} - s \\ \frac{3}{7} + s \\ s \end{bmatrix}$$

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Hence, all possible linear combinations are:

$$\vec{b} = \left(\frac{1}{7} - s\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{5}{7} + s\right) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{7} + s\right) \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Theorem

Let A and B be $m \times n$ matrices, and let $\vec{x}$ and $\vec{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.
2. $A(a\vec{x}) = a(A\vec{x}) = (aA)\vec{x}$ for all scalars a.
3. $(A + B)\vec{x} = A\vec{x} + B\vec{x}$.

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3. $(A+B)\vec{x} = A\vec{x} + B\vec{x}$.

This provides a useful way to describe the solutions to a system $A\vec{x} = \vec{b}$.

Structure of solutions:

$$\text{General solution} = \text{Sol. to the Homog. Eq.} + \text{A Particular Solution}.$$

$$A\vec{x} = A\left(\vec{x}_0 + \vec{x}_1\right) = \underbrace{A\vec{x}_0}_{\vec{x}_0:\ \text{homogeneous sol.}} + \underbrace{A\vec{x}_1}_{\vec{x}_1:\ \text{particular sol.}} = \vec{0} + \vec{b} = \vec{b}.$$

Copyright

Vectors

Matrix Vector Multiplication

The Dot Product

Transformations

Rotations in $\mathbb{R}^2$

The Dot Product

The Dot Product

Definition

If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two ordered n-tuples, their dot product is defined to be the number

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

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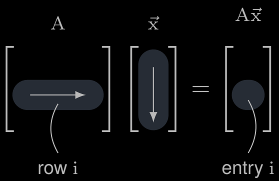
This give an alternative way to carry out the matrix-vector product $A\vec{x}$.



$$A\vec{x}$$

$$\|$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\|$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + x_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \qquad \text{(Def.)}$$

$$\|$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \end{bmatrix} \qquad \text{(Alternative)}$$

Problem

If $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$, compute $A\vec{x}$.

Problem

If $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$, compute $A\vec{x}$.

Solution

The entries of $A\vec{x}$ are the dot products of the rows of A with $\vec{x}$:

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 & + & 0(-1) & + & 2 \cdot 1 & + & (-1)4 \\ 2 \cdot 2 & + & (-1)(-1) & + & 0 \cdot 1 & + & 1 \cdot 4 \\ 3 \cdot 2 & + & 1(-1) & + & 3 \cdot 1 & + & 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}. \end{aligned}$$

Of course, this agrees with the outcome of the previous example. ■

Definition (Identity Matrix)

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Example

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \ldots$$

Problem

Show that $I_n\vec{x} = \vec{x}$ for each n-vector $\vec{x}$ in $\mathbb{R}^n$, $n \geq 1$.

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Show that $I_n\vec{x} = \vec{x}$ for each n-vector $\vec{x}$ in $\mathbb{R}^n$, $n \geq 1$.

Solution

We verify the case $n = 4$. Given the 4-vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + 0 + 0 \\ 0 + x_2 + 0 + 0 \\ 0 + 0 + x_3 + 0 \\ 0 + 0 + 0 + x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x}.$$

In general, $I_n\vec{x} = \vec{x}$ because entry k of $I_n\vec{x}$ is the dot product of row k of I_n with $\vec{x}$, and row k of I_n has 1 in position k and zeros elsewhere. ∎

Transformations

Notation and Terminology

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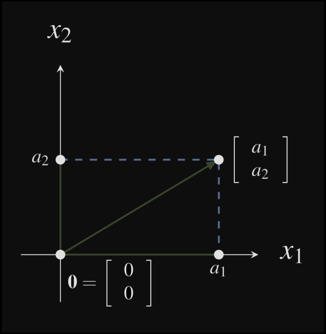
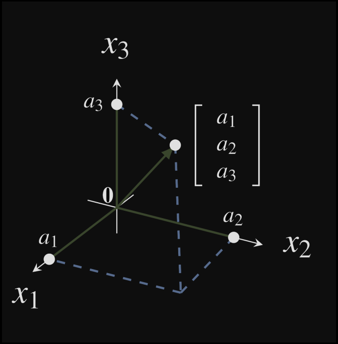
$\mathbb{R}^2$ and $\mathbb{R}^3$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric interpretations as position vectors of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.



x_2
a_2
$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$
x_1
a_1
$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



x_3
a_3
$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$
$\mathbf{0}$
a_2
a_1
x_2
x_1

Definition (Transformations)

A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, and is called a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

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What do we mean by a function?

Definition (Transformations)

A transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, and is called a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $m = n$, then we say T is a transformation of $\mathbb{R}^n$.

What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that, for each vector in $\mathbb{R}^n$, assigns exactly one vector of $\mathbb{R}^m$

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Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a matrix transformation, and write $T_A(\vec{x}) = A\vec{x}$.

Definition (Equality of Transformations)

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Example (Specifying the action of a transformation)

$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

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$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a transformation that transforms the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ into the vector

$$T\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 4+7 \\ 1-7 \\ 7-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

Example (Transformation by matrix multiplication)

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

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Transforming this vector by A looks like:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}.$$

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For example:

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Rotations in $\mathbb{R}^2$

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

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Definition

The transformation

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

Example (Rotation through π)

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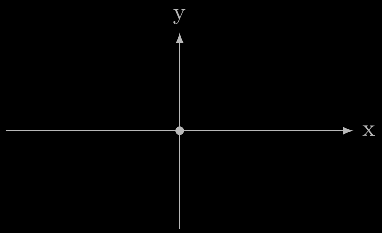

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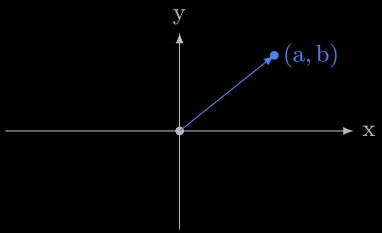

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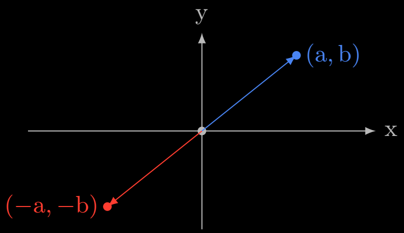

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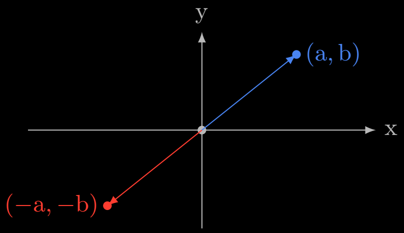

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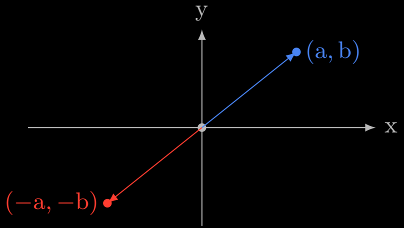

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Example (Rotation through $\pi/2$)

We denote by

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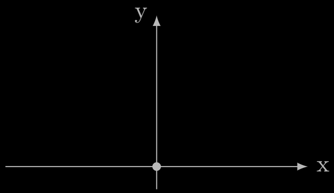

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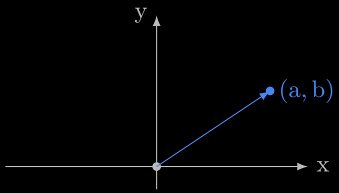

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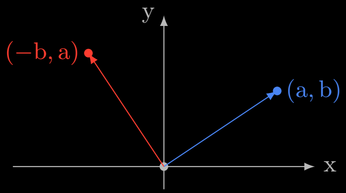

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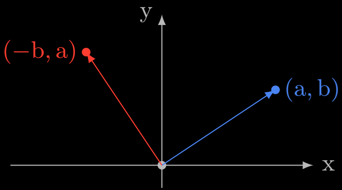

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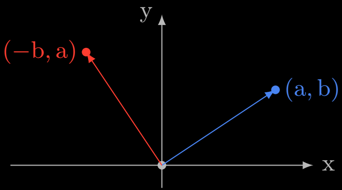

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Remark

In general, the rotation (counterclockwise) about the origin for an angle θ is

$$\mathrm{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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