Math 221: LINEAR ALGEBRA

Chapter 3. Determinants and Diagonalization
§3-2. Determinants and Matrix Inverses

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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- Determinants and Matrix Inverses
- Adjugates
- Cramer's Rule
- Polynomial Interpolation and Vandermonde Determinant

Determinants and Matrix Inverses

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Theorem (Product Theorem)

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and

$$A = E_1 E_2 \cdots E_p \quad \text{and} \quad B = F_1 F_2 \cdots F_q.$$

where E_i and F_j are elementary matrices.

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$$A = E_1 E_2 \cdots E_p \quad \text{and} \quad B = F_1 F_2 \cdots F_q.$$

where E_i and F_j are elementary matrices. Then by the relation of elementary row operations with determinants (Theorem 3.1.2), we see that

$$\begin{aligned} |AB| &= |E_1 \cdots E_p F_1 \cdots F_q| \\ &= |E_1| \cdots |E_p||F_1| \cdots |F_q| \\ &= |E_1 \cdots E_p||F_1 \cdots F_q| \\ &= |A||B|. \end{aligned}$$

$\blacksquare$

Theorem (Determinant of Matrix Inverse)

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det A}.$$

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"$\Rightarrow$":

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"$\Leftarrow$": If $|A| \neq 0$, then $\mathrm{rref}(A) = I$ because otherwise one obtains contradiction by Theorem 3.1.2. This is another way to say that A is invertible: (recall the matrix inverse algorithm)

$$[A|I] \to \left[\underbrace{\mathrm{rref}(A)}_{=I} \middle| A^{-1}\right].$$

$\blacksquare$

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

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$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

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$$= c(10 - c^2) - c = c(9 - c^2) = c(3 - c)(3 + c).$$

Therefore, A is invertible for all $c \neq 0, 3, -3$.

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2. If A is not invertible, then neither is A^T. Hence, $\det A = 0 = \det A^T$.
3. If A is invertible, then $A = E_k E_{k-1} \cdots E_2 E_1$. Hence, by Case 1,

$$\begin{aligned}
\left|A^T\right| &= \left|(E_k E_{k-1} \cdots E_2 E_1)^T\right| \\
&= \left|E_1^T E_2^T \cdots E_{k-1}^T E_k^T\right| \\
&= \left|E_1^T\right| \left|E_2^T\right| \cdots \left|E_{k-1}^T\right| \left|E_k^T\right| \\
&= |E_1| \, |E_2| \cdots |E_{k-1}| \, |E_k| \\
&= |E_k| \, |E_{k-1}| \cdots |E_2| \, |E_1| \\
&= |E_k E_{k-1} \cdots E_2 E_1| \\
&= |A| \, .
\end{aligned}$$

∎

Problem

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Therefore, $\det A = -2$.

Solution (continued)

Now,

$$\begin{aligned}
\det(A^3(B^{-1})^T) &= -4 \\
(\det A)^3 \det(B^{-1}) &= -4 \\
(-2)^3 \det(B^{-1}) &= -4 \\
(-8) \det(B^{-1}) &= -4 \\
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Therefore, $\det B = 2$. ∎

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Suppose A, B and C are 4×4 matrices with

$$\det \mathrm{A} = -1, \det \mathrm{B} = 2, \quad \text{and} \quad \det \mathrm{C} = 1.$$

Find $\det(2\mathrm{A}^2(\mathrm{B}^{-1})(\mathrm{C}^{\mathrm{T}})^3\mathrm{B}(\mathrm{A}^{-1}))$.

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Solution

$$\begin{aligned}
\det(2A^2(B^{-1})(C^T)^3 B(A^{-1})) &= 2^4(\det A)^2 \frac{1}{\det B}(\det C)^3(\det B)\frac{1}{\det A} \\
&= 16(\det A)(\det C)^3 \\
&= 16 \times (-1) \times 1^3 \\
&= -16.
\end{aligned}$$

$\blacksquare$

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Since $A^T = A^{-1}$,

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Assuming A is a real matrix, this implies that $\det A = \pm 1$, i.e., $\det A = 1$ or $\det A = -1$. ■

Adjugates

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For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen the adjugate of A defined as

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and observed that

$$\begin{aligned} A \operatorname{adj}(A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (\det A) I_2 \end{aligned}$$

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Furthermore, if $\det A \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

Definition (Adjugate Matrix)

If A is an $n \times n$ matrix, then the adjugate matrix of A is defined to be

$$\mathrm{adj}(A) \stackrel{\mathrm{def}}{=} \left[\ c_{ij}(A)\ \right]^T = \left[\ (-1)^{i+j} \det(A_{ij})\ \right]^T,$$

where $c_{ij}(A)$ is the (i, j)-cofactor of A, i.e., $\mathrm{adj}(A)$ is the transpose of the cofactor matrix (matrix of cofactors).

Problem

Find adj(A) when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$ and compute $A\,\mathrm{adj}(A)$.

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Notice that

$$A\,\mathrm{adj}(A) = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix} = \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix}$$

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$$\det A = \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 19 & 0 & 22 \\ 3 & 0 & -6 \end{vmatrix} = (-1)\begin{vmatrix} 19 & 22 \\ 3 & -6 \end{vmatrix} = 180,$$

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Therefore,

$$A \operatorname{adj}(A) = (\det A)I. \quad \blacksquare$$

Theorem (The Adjugate Formula)

If A is an $n \times n$ matrix, then

$$A\,\mathrm{adj}(A) = (\det A)I = \mathrm{adj}(A)A.$$

Furthermore, if $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}(A).$$

Proof.

We only prove the case when $n = 3$.

$$A\ \mathrm{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

where, for example,

$$\text{(3,2)-th entry} = a_{31}c_{21} + a_{32}c_{22} + a_{33}c_{23} = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0.$$

$\blacksquare$

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Using the adjugate formula,

$$\begin{aligned}
A \operatorname{adj}(A) &= (\det A)I \\
\det(A \operatorname{adj}(A)) &= \det((\det A)I) \\
(\det A) \times \det \operatorname{adj}(A) &= (\det A)^n (\det I) \\
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If $\det A \neq 0$, then divide both sides of the last equation by $\det A$:

$$\det\ \mathrm{adj}(A) = (\det A)^{n-1}.$$

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det\ \mathrm{adj}(A) = 0, \tag{$\star$}$$

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$$\det\ \mathrm{adj}(A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

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We will prove ($\star$) by contradiction.

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We will prove $(\star)$ by contradiction. Indeed, if $\det A = 0$, then

$$A\,\mathrm{adj}(A) = (\det A)I = (0)I = O,$$

i.e., $A\,\mathrm{adj}(A)$ is the zero matrix.

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i.e., $A \operatorname{adj}(A)$ is the zero matrix. If $\det \operatorname{adj}(A) \neq 0$, then $\operatorname{adj}(A)$ would be invertible, and $A \operatorname{adj}(A) = O$ would imply $A = O$.

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Problem

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Solution

Notice that

$$(A + B^T)^T = A^T + (B^T)^T = A^T + B.$$

Since a matrix and it's transpose have the same determinant

$$\det(A + B^T) = \det((A + B^T)^T) = \det(A^T + B).$$

$\blacksquare$

Problem

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

1. If $\mathrm{adj}(A)$ exists, then A is invertible.
2. If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.

Problem

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1. If $\mathrm{adj}(A)$ exists, then A is invertible.
2. If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.

Problem

Prove or give a counterexample to the following statement:

$$\text{If } \det A = 1, \text{ then } \mathrm{adj}(A) = A.$$

Cramer's Rule

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Theorem (Cramer's Rule)

Let A be an $n \times n$ invertible matrix, the solution to the system $A\vec{x} = \vec{b}$ of n equations in teh variables $x_1, x_2 \cdots x_n$ is given by

$$x_1 = \frac{\det\left(A_1(\vec{b})\right)}{\det A}, \quad x_2 = \frac{\det\left(A_2(\vec{b})\right)}{\det A}, \quad \cdots, \quad x_n = \frac{\det\left(A_n(\vec{b})\right)}{\det A}$$

where, for each j, the matrix $A_j(\vec{b})$ is obtained from A by replacing column j with $\vec{b}$:

$$A_j(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix}$$

Proof.

- Notice that

$$
\begin{aligned}
A_j(\vec{b}) &= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix} \\
&= \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{e}_{j-1} & A\vec{x} & A\vec{e}_{j+1} & \cdots & A\vec{e}_n \end{bmatrix} \\
&= A \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix} \\
&= A\, I_j(\vec{x})
\end{aligned}
$$

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- Notice that

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A_j(\vec{b}) &= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix} \\
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&= A \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix} \\
&= A\, I_j(\vec{x})
\end{aligned}
$$

where

$$
I_j(\vec{x}) = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & & & x_1 & & & \\
& \ddots & & \vdots & & & \\
& & 1 & x_{j-1} & & & \\
& & & x_j & & & \\
& & & x_{j+1} & 1 & & \\
& & & \vdots & & \ddots & \\
& & & x_n & & & 1
\end{bmatrix}
$$

Proof. (continued)

- Hence, by taking the determinants on both sides, we have

$$\begin{aligned}\det(A_j(\vec{b})) &= \det(A\, I_j(\vec{x}))\\ &= \det(A)\det(I_j(\vec{x}))\end{aligned}$$

- And because $\det(A) \neq 0$, we can then write:

$$\det(I_j(\vec{x})) = \frac{\det(A_j(\vec{b}))}{\det(A)}$$

- Finally, notice that $\quad \det(I_j(\vec{x})) = \cdots$

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$\blacksquare$

Problem

Find x_3 such that

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

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Solution

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

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Therefore, $x_3 = \frac{-6}{-4} = \frac{3}{2}$. ■

Remark

For practice, you should compute $\det \mathrm{A}_1$ and $\det \mathrm{A}_2$, where

$$\mathrm{A}_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathrm{A}_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

and then solve for x_1 and x_2.

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and then solve for x_1 and x_2.

Solution. $x_1 = -1$, $x_2 = 7/2$. ∎

Polynomial Interpolation and Vandermonde Determinant

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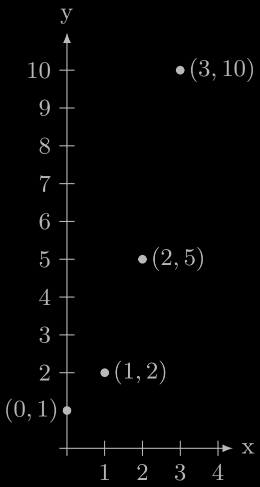
Problem

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial p(x) of degree at most three, and then estimate the value of y corresponding to $x = 3/2$.



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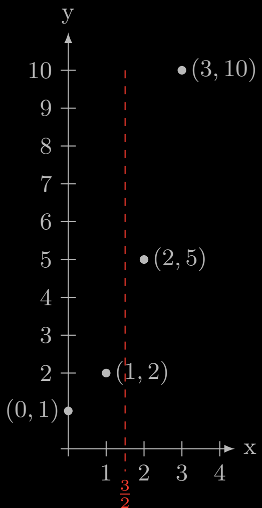

Solution

We want to find the coefficients r_0, r_1, r_2 and r_3 of

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$\begin{aligned}
p(0) &= r_0 = 1 \\
p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\
p(2) &= r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\
p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 10
\end{aligned}$$

Solution (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

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$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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Therefore, $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

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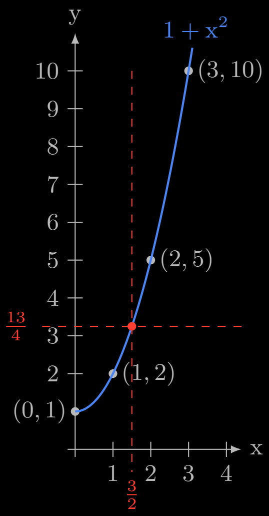
Finally, the estimate is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4}.$$

■


y
$1 + x^2$
10
9
8
7
6
5
4
3
2
$\frac{13}{4}$
(3, 10)
(2, 5)
(1, 2)
(0, 1)
x
1
2
3
4
$\frac{3}{2}$

Theorem (Polynomial Interpolation)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

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The polynomial $p(x)$ is called the interpolating polynomial for the data.

To find $p(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{n-1}x^{n-1}$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$.

$$\begin{aligned}
r_0 + r_1x_1 + r_2x_1^2 + \cdots + r_{n-1}x_1^{n-1} &= y_1 \\
r_0 + r_1x_2 + r_2x_2^2 + \cdots + r_{n-1}x_2^{n-1} &= y_2 \\
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\vdots \qquad\qquad & \vdots \quad \vdots \\
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The coefficient matrix for this system is

$$\begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
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\vdots & \vdots & \vdots & \vdots & \vdots \\
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$$\begin{array}{rcl} r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} & = & y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} & = & y_2 \\ r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} & = & y_3 \\ \vdots & \vdots & \vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} & = & y_n \end{array}$$

The coefficient matrix for this system is

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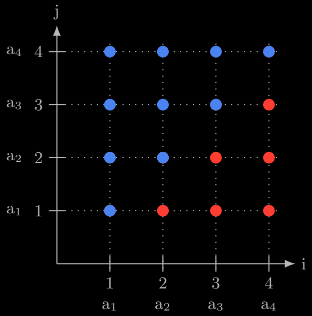
- Such matrix is called Vandermonde matrix.
- Its determinant is called Vandermonde determinant.

Theorem (Vandermonde Determinant)

Let $a_1, a_2, \ldots, a_n$ be real numbers, $n \geq 2$. The corresponding Vandermonde determinant is

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1 \leq j < i \leq n} (a_i - a_j).$$



Proof.

We will prove this by induction. It is clear that when $n = 2$,

$$\det \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1 = \prod_{1 \le j < i \le 2} (a_i - a_j).$$

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Assume that it is true for $n-1$. Now let's consider the case n. Denote

$$p(x) := \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}.$$

Proof. (continued)

Because $p(a_1) = \cdots = p(a_{n-1}) = 0$ (why?), $p(x)$ has to take the following form:

$$p(x) = c(x - a_1)(x - a_2)\cdots(x - a_{n-1}).$$

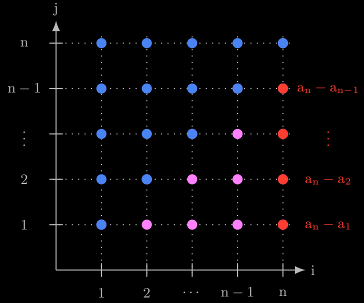
To identify the constant c, notice that c is the coefficient for x^{n-1}. By cofactor expansion of the determinant along the last row,

$$\begin{aligned} c &= (-1)^{n+n} \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{bmatrix} \\ &= \prod_{1 \le j < i \le n-1} (a_i - a_j). \end{aligned}$$

Proof. (continued)

Hence,

$$p(a_n) = \left(\prod_{1 \le j < i \le n-1} (a_i - a_j) \right) \times (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})$$



$$p(a_n) = \prod_{1 \le j < i \le n} (a_i - a_j).$$

∎

Example

In our earlier example with the data points $(0,1)$, $(1,2)$, $(2,5)$ and $(3,10)$, we have

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

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According to the previous theorem, this determinant is equal to

$$\begin{aligned} & (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ =& (1 - 0)(2 - 0)(2 - 1)(3 - 0)(3 - 1)(3 - 2) \\ =& 2 \times 3 \times 2 \\ =& 12. \end{aligned}$$

Corollary

The Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

Corollary

The Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with distinct x_i, then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}.$$