Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry
§4-3. More on the Cross Product

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Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

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More on the Cross Product

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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A new topic
A new example or problem
A new or better proof to an existing theorem
Any other suggestions to improve the material

Contact Lyryx at info@lyryx.com with your ideas.

More on the Cross Product

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

Theorem

Given three vectors $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, it holds that

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}.$$

Proof.

Let $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

$\blacksquare$

Proof.

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$$\begin{aligned}
\vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \cdot \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix} \\
&= x_0(y_1 z_2 - z_1 y_2) - y_0(x_1 z_2 - z_1 x_2) + z_0(x_1 y_2 - y_1 x_2) \\
&= x_0 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} - y_0 \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} + z_0 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \\
&= \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}.
\end{aligned}$$

$\blacksquare$

Theorem (Properties of the Cross Product)

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

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5. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ for any scalar k.

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6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ for any scalar k.
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

Theorem (Properties of the Cross Product)

Let $\vec{\mathrm{u}}, \vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$ be in $\mathbb{R}^3$.

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7. $\vec{\mathrm{u}} \times (\vec{\mathrm{v}} + \vec{\mathrm{w}}) = \vec{\mathrm{u}} \times \vec{\mathrm{v}} + \vec{\mathrm{u}} \times \vec{\mathrm{w}}$.
8. $(\vec{\mathrm{v}} + \vec{\mathrm{w}}) \times \vec{\mathrm{u}} = \vec{\mathrm{v}} \times \vec{\mathrm{u}} + \vec{\mathrm{w}} \times \vec{\mathrm{u}}$.

Theorem (The Lagrange Identity)

If $\vec{u}, \vec{v} \in \mathbb{R}^3$, then

$$||\vec{u} \times \vec{v}||^2 = ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2.$$

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Proof.

Write $\vec{u} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then both sides are equal to

$$(a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_2 b_3 - a_3 b_2)^2 .$$

Work out these by yourself! ∎

As a consequence of the Lagrange Identity and the fact that

$$\vec{\mathrm{u}} \cdot \vec{\mathrm{v}} = ||\vec{\mathrm{u}}||\,||\vec{\mathrm{v}}|| \cos\theta,$$

we have

$$\begin{aligned}
||\vec{\mathrm{u}} \times \vec{\mathrm{v}}||^2 &= ||\vec{\mathrm{u}}||^2 ||\vec{\mathrm{v}}||^2 - (\vec{\mathrm{u}} \cdot \vec{\mathrm{v}})^2 \\
&= ||\vec{\mathrm{u}}||^2 ||\vec{\mathrm{v}}||^2 - ||\vec{\mathrm{u}}||^2 ||\vec{\mathrm{v}}||^2 \cos^2\theta \\
&= ||\vec{\mathrm{u}}||^2 ||\vec{\mathrm{v}}||^2 (1 - \cos^2\theta) \\
&= ||\vec{\mathrm{u}}||^2 ||\vec{\mathrm{v}}||^2 \sin^2\theta.
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Taking square roots on both sides yields,

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Note that since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$.

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If $\theta = 0$ or $\theta = \pi$, then $\sin\theta = 0$, and $||\vec{\mathrm{u}} \times \vec{\mathrm{v}}|| = 0$. This is consistent with our earlier observation that if $\vec{\mathrm{u}}$ and $\vec{\mathrm{v}}$ are parallel, then $\vec{\mathrm{u}} \times \vec{\mathrm{v}} = \vec{0}$.

Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$, and let θ denote the angle between $\vec{u}$ and $\vec{v}$.

1. $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin\theta$, and is the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.
2. $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

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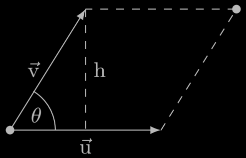
Proof. (area of parallelogram)

The area of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is $||\vec{u}||h$, where h is the height of the parallelogram.



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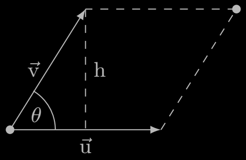

Since $\sin\theta = \frac{h}{||\vec{v}||}$, we see that $h = ||\vec{v}|| \sin\theta$.

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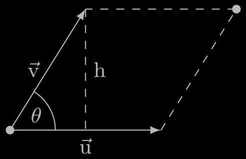


Since $\sin\theta = \frac{h}{||\vec{v}||}$, we see that $h = ||\vec{v}|| \sin\theta$. Therefore, the area is

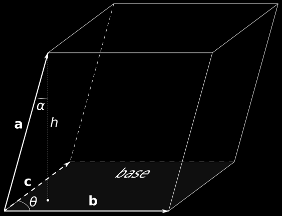
$$||\vec{u}||\ ||\vec{v}|| \sin\theta.$$

■

Theorem

The volume of the parallelepiped determined by the three vectors $\vec{b}$, $\vec{c}$, and $\vec{a}$ in $\mathbb{R}^3$ is

$$|\vec{a} \cdot (\vec{b} \times \vec{c})|.$$



Proof.

Volume $=$ base area $\times h$, where base area $= |\vec{b} \times \vec{c}|$ and the height $h = |\vec{a}||\cos(\alpha)|$. Hence,

$$\text{Vol} = |\vec{b} \times \vec{c}|\,|\vec{a}||\cos(\alpha)| = |(\vec{b} \times \vec{c}) \cdot \vec{a}|.$$

∎

Problem

Find the area of the triangle having vertices $A(3, -1, 2)$, $B(1, 1, 0)$ and $C(1, 2, -1)$.

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Solution

The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Problem

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Solution

The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. $\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$.

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$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

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The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. $\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$. Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

so the area of the triangle is $\frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \sqrt{2}$. ∎

Problem

Find the volume of the parallelepiped determined by the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Problem

Find the volume of the parallelepiped determined by the vectors
$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Solution

The volume of the parallelepiped is

$$|\vec{w} \cdot (\vec{u} \times \vec{v})| = \left| \det \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \right| = 2.$$