Math 221: LINEAR ALGEBRA

§AppendixA. Complex Numbers

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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Complex Numbers

Why complex numbers?

- Counting numbers: $1, 2, 3, 4, 5, \ldots$
- Integers: $0, 1, 2, 3, 4 \ldots$ but also $-1, -2, -3 \ldots$
- To solve $3x + 2 = 0$, integers aren't enough, so we have rational numbers (fractions), i.e.,

$$\text{if } 3x + 2 = 0, \text{ then } x = -\frac{2}{3}.$$

- We still can't solve $x^2 - 2 = 0$ because there are no rational numbers x with the property that $x^2 - 2 = 0$, so we have irrational numbers, i.e.,

$$\text{if } x^2 - 2 = 0, \text{ then } x = \pm\sqrt{2}.$$

- The set of real numbers, $\mathbb{R}$, consists of all rational and irrational numbers (note that integers are rational numbers). However, we still can't solve

$$x^2 + 1 = 0$$

  because this requires $x^2 = -1$, but any real number x has the property that $x^2 \geq 0$.

Definitions

- The imaginary unit, denoted i, is defined to be a number with the property that $i^2 = -1$.
- A pure imaginary number has the form bi where $b \in \mathbb{R}$, $b \neq 0$, and i is the imaginary unit.
- A complex number is any number z of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit.

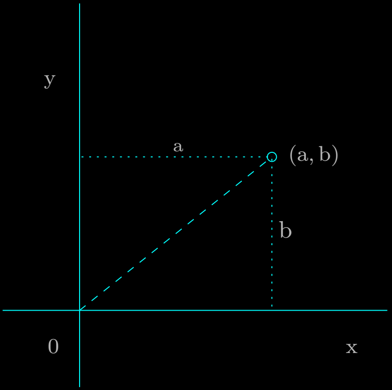
(1) a is called the real part of z.

(2) b is called the imaginary part of z

(3) If $b = 0$, then z is a real number.

The Complex Plane

A complex number $z = a + bi$ can be represented geometrically by the point (a, b) in the xy-plane, where the x-axis is the real axis and the y-axis is the imaginary axis.



- Real numbers: $a + 0i$ lie on the x-axis.
- Pure imaginary numbers: $0 + bi$ $(b \neq 0)$ lie on the y-axis.

Addition and Subtraction of Complex Numbers

Let $z = a + bi$ and $w = c + di$ be complex numbers.

- Equality $z = w$ if and $a = c$ and $b = d$.
- Addition

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

- Subtraction

$$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

Examples

- $(-3 + 6i) + (5 - i) = 2 + 5i.$
- $(4 - 7i) + (6 - 2i) = 10 - 9i.$
- $(-3 + 6i) - (5 - i) = -8 + 7i.$
- $(4 - 7i) - (6 - 2i) = -2 - 5i.$

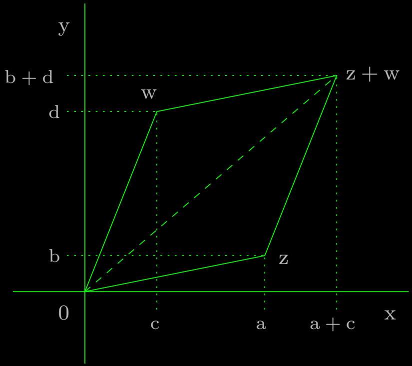
Properties of Addition

Let z, w, and v be complex numbers.

1. $z + w = w + z$. (addition is commutative)
2. $(z + w) + v = z + (w + v)$. (addition is associative)
3. $z + 0 = z$. (existence of an additive identity)
4. For every $z = a + bi$ there exists a complex number $-z = -a - bi$ such that $z + (-z) = 0$. (existence of an additive inverse)

Addition in the Complex Plane

If $z = a + bi$ and $w = c + di$, then $z + w = (a + c) + (b + d)i$. Geometrically, we have:



0, z, w, and $z + w$ are the vertices of a parallelogram.

Multiplication of Complex Numbers

Let $z = a + bi$ and $w = c + di$ be complex numbers. Then the product of z and w is

$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

The multiplication is done essentially as the product of two linear polynomials, with i^2 replaced by -1.

Example

$$\begin{aligned} (2-3i)(-3+4i) = \quad &= ((2)(-3) - (-3)(4)) + ((2)(4) + (-3)(-3))i \\ &= (-6+12) + (8+9)i \\ &= 6 + 17i \end{aligned}$$

Properties of Multiplication

Let z, w and v be complex numbers.

- $zw = wz.$ (multiplication is commutative)
- $(zw)v = z(wv).$ (multiplication is associative)
- $z(w + v) = zw + zv.$ (multiplication distributes over addition)
- $1z = z.$ ('1' is the multiplicative identity)
- For each $z \neq 0$, there exists z^{-1} such that $zz^{-1} = 1.$ (existence of a multiplicative inverse)

Problem

Find all complex numbers z so that $z^2 = -3 + 4i$.

Solution

Let $z = a + bi$. Then

$$z^2 = (a + bi)^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so

$$a^2 - b^2 = -3 \quad \text{and} \quad 2ab = 4.$$

Since $2ab = 4$, $a = \frac{2}{b}$. Substituting this into the first equation gives us

$$\begin{aligned} a^2 - b^2 &= -3 \\ \left(\frac{2}{b}\right)^2 - b^2 &= -3 \\ \frac{4}{b^2} - b^2 &= -3 \\ 4 - b^4 &= -3b^2 \\ b^4 - 3b^2 - 4 &= 0. \end{aligned}$$

Solution (continued)

Now, $b^4 - 3b^2 - 4 = 0$ can be factored into

$$\begin{aligned}(b^2-4)(b^2+1) &= 0\\ (b-2)(b+2)(b^2+1) &= 0.\end{aligned}$$

Since $b \in \mathbb{R}$ and b^2+1 has no real roots, $b = 2$ or $b = -2$.

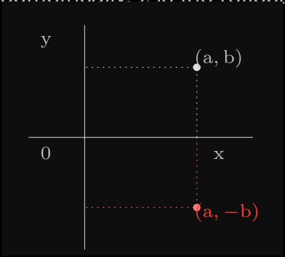
Since $a = \frac{2}{b}$, it follows that

- when $b = 2$, $a = 1$, and $z = a + bi = 1 + 2i$;
- when $b = -2$, $a = -1$, and $z = a + bi = -1 - 2i$.

Therefore, if $z^2 = -3 + 4i$, then $z = 1 + 2i$ or $z = -1 - 2i$.

The Conjugate of a Complex Number

Let $z = a + bi$ be a complex number. The conjugate of z is the complex number $\overline{z} = a - bi$. Geometrically, $\overline{z}$ is the reflection of z in the x-axis.



Examples

- If $z = 3 + 4i$, then $\overline{z} = 3 - 4i$, i.e., $\overline{3 + 4i} = 3 - 4i$.
- $\overline{-2 + 5i} = -2 - 5i$.
- $\overline{i} = -i$.
- $\overline{7} = 7$.

Properties of the Conjugate

Let z and w be complex numbers.

- $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
- $\overline{(zw)} = \overline{z}\,\overline{w}$.
- $\overline{(\overline{z})} = z$.
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.
- z is real if and only if $\overline{z} = z$.

Remark

If $z = a + bi$, then

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Division of Complex Numbers

Let $z = a + bi$ and $w = c + di$ be complex numbers. Suppose that c, d are not both zero. Then the quotient z divided by w is

$$
\begin{aligned}
\frac{z}{w} = \frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di} \\
&= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\
&= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.
\end{aligned}
$$

The quotient $\frac{z}{w}$ is obtained by multiplying both top and bottom of $\frac{z}{w}$ by $\overline{w}$ and then simplifying the expression.

Examples

- $$\frac{1}{\mathrm{i}} = \frac{1}{\mathrm{i}} \times \frac{-\mathrm{i}}{-\mathrm{i}} = \frac{-\mathrm{i}}{-\mathrm{i}^2} = -\mathrm{i}.$$

- $$\frac{2-\mathrm{i}}{3+4\mathrm{i}} = \frac{2-\mathrm{i}}{3+4\mathrm{i}} \times \frac{3-4\mathrm{i}}{3-4\mathrm{i}} = \frac{(6-4)+(-3-8)\mathrm{i}}{3^2+4^2} = \frac{2-11\mathrm{i}}{25} = \frac{2}{25} - \frac{11}{25}\mathrm{i}.$$

- $$\frac{1-2\mathrm{i}}{-2+5\mathrm{i}} = \frac{1-2\mathrm{i}}{-2+5\mathrm{i}} \times \frac{-2-5\mathrm{i}}{-2-5\mathrm{i}} = \frac{(-2-10)+(4-5)\mathrm{i}}{2^2+5^2} = -\frac{12}{29} - \frac{1}{29}\mathrm{i}.$$

The Multiplicative Inverse

Every nonzero complex number $z = a + bi$ has a unique multiplicative inverse $z^{-1} = \frac{1}{z}$ such that $zz^{-1} = 1$, and

$$\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

Since z is nonzero, $a^2 + b^2 \neq 0$, so the inverse is defined.

Example

When $z = 2 + 6i$, z^{-1} is defined, and

$$\frac{1}{z} = \frac{1}{2 + 6i} = \frac{1}{2 + 6i} \times \frac{2 - 6i}{2 - 6i} = \frac{2 - 6i}{2^2 + 6^2} = \frac{2 - 6i}{40} = \frac{1}{20} - \frac{3}{20} i.$$

You can always check that $zz^{-1} = 1$.

Modulus

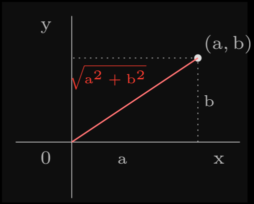
Definition

The absolute value or modulus of a complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$

Note that this is consistent with the definition of the absolute value of a real number.

Geometrically, $|z| = \sqrt{a^2 + b^2}$ is the distance from z to the origin.



Examples

1. $|-3+4\mathrm{i}| = \sqrt{3^2+4^2} = \sqrt{25} = 5.$
2. $|3-2\mathrm{i}| = \sqrt{3^2+2^2} = \sqrt{13}.$
3. $|\mathrm{i}| = \sqrt{1^2} = 1.$

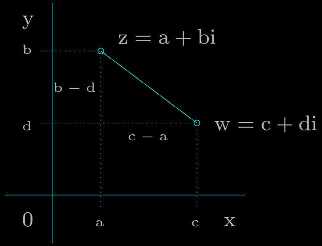
Theorem (Properties of the Modulus)

Let z and w be complex numbers.

1. $z \cdot \overline{z} = |z|^2$.
2. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.
3. $|z| \geq 0$ for all z.
4. $|z| = 0$ if and only if $z = 0$.
5. $|zw| = |z|\;|w|$.
6. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.
7. The Triangle Inequality
   $|z + w| \leq |z| + |w|$.

Example (The Triangle Inequality: Geometrically)

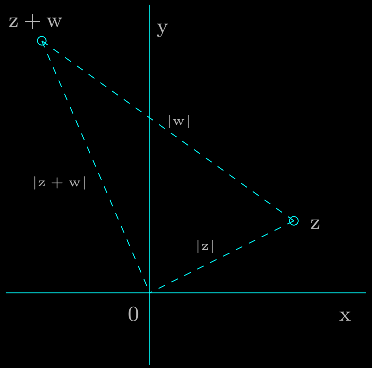
If $z = a + bi$ and $w = c + di$, then $|z - w| = \sqrt{(a-c)^2 + (b-d)^2}$.



This shows that the distance between z and w in the complex plane is just the absolute value of their difference.

Example (continued)

Now consider the points z, $z + w$, and the origin 0 in the complex plane.

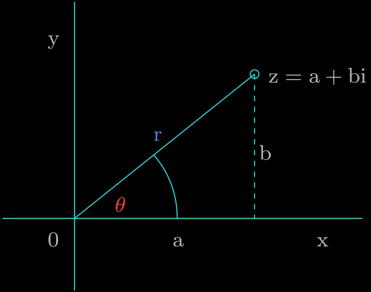


The triangle formed by these points has sides of length $|z|$, and $|z + w|$ and $|w|$ (the absolute value of the difference between $z + w$ and z).

Since the length of any side of a triangle is at most the sum of the lengths of the other two sides, we get $|z + w| \leq |z| + |w|$.

Complex Numbers in Polar Form

Suppose $z = a + bi$, and let $r = |z| = \sqrt{a^2 + b^2}$. Then r is the distance from z to the origin. Denote by θ the angle that the line through 0 and z makes with the positive x-axis (measured clockwise).



Then θ is an angle defined by $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$, so

$$z = r\cos\theta + r\sin\theta i = r(\cos\theta + i\sin\theta).$$

θ is called an argument of z, and is denoted $\arg z$.

Definition (Polar Form of a Complex Number)

Let z be a complex number with $|z| = r$ and $\arg z = \theta$. Then

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

is called a polar form of z.

Remark

Since $\arg z$ is not unique, we do not write the polar form of z.

Definition

Let z be a complex number with $|z| = r$. The principal argument of z is the unique angle $\theta = \arg z$ (measured in radians) such that

$$-\pi < \theta \le \pi.$$

Example

Find the polar form for the number $z = 1$.

Solution

To convert z to polar form, we need to find r and θ so that $1 = re^{i\theta}$. Now $r = |z| = \sqrt{1^2} = 1$, and $\theta = 0$ is an argument for $z = 1$. However, we may also write

$$1 = e^{2\pi i}, 1 = e^{-2\pi i}, e^{4\pi i}, e^{6\pi i}, \ldots$$

Since sine and cosine have periodicity 2π, we may add (or subtract) multiples of 2π to any argument.

Example (Converting to Polar Form)

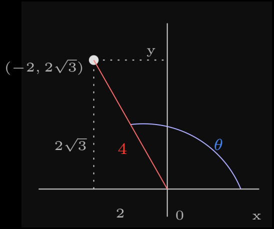
Convert the number $z = -2 + 2\sqrt{3}i$ to polar form.

Solution

To convert z to polar form, we need to find r and θ so that $-2 + 2\sqrt{3}i = re^{i\theta}$. Since $r = |z|$,

$$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 4(3)} = \sqrt{16} = 4.$$

There are two approaches to finding an argument, θ. One is to graph $-2 + 2\sqrt{3}$ in the complex plane.



Solution (continued)

The triangle sitting on the negative half of the real axis has sides of length 2, $2\sqrt{3}$, and 4; you should recognize this as a right triangle whose other two angles measure $\frac{\pi}{3}$ and $\frac{\pi}{6}$. From this, we see that $\theta = \frac{2\pi}{3}$ is an argument of z. Therefore, z can be written in polar form as $z = 4e^{i(2\pi/3)}$.

The other approach to finding an argument, θ, for $z = -2 + 2\sqrt{3}i$ is as follows. We've already calculated $|z| = r = 4$. By definition, θ is an angle satisfying

$$\cos\theta = \frac{-2}{4} = -\frac{1}{2} \quad \text{and} \quad \sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}.$$

By graphing the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, we again determine that $\theta = \frac{2\pi}{3}$, and thus z can be written in polar form as $z = 4e^{i(2\pi/3)}$. ■

Problem

Convert each of the following complex numbers to polar form.

1. $3\mathrm{i} = 3\mathrm{e}^{(\pi/2)\mathrm{i}}$.
2. $-1 - \mathrm{i} = \sqrt{2}\mathrm{e}^{-(3\pi/4)\mathrm{i}} = \sqrt{2}\mathrm{e}^{(5\pi/4)\mathrm{i}}$.
3. $\sqrt{3} - \mathrm{i} = 2\mathrm{e}^{-(\pi/6)\mathrm{i}}$.
4. $\sqrt{3} + 3\mathrm{i} = 2\sqrt{3}\mathrm{e}^{(\pi/3)\mathrm{i}}$.

Problem (Converting from Polar Form to Cartesian form)

Let $z = 2e^{2\pi i/3}$. Write z in the form $z = a + bi$ (this is called Cartesian form or Standard form).

Solution

First, remember that $e^{i\theta} = \cos\theta + i\sin\theta$, and thus

$$\begin{aligned} e^{2\pi i/3} &= \cos(2\pi/3) + i\sin(2\pi/3) \\ &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \end{aligned}$$

Therefore

$$\begin{aligned} z = 2e^{2\pi i/3} &= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{3}\right) \\ &= -1 + \sqrt{3}i. \end{aligned}$$

Problem

Express each of the following complex numbers in Cartesian form.

1. $3e^{-\pi i} = -3$
2. $2e^{3\pi i/4} = -\sqrt{2} + i\sqrt{2}$
3. $2\sqrt{3}e^{-2\pi i/6} = \sqrt{3} - 3i$

Problems involving multiplication of complex numbers can often be solved more easily by using polar forms of the complex numbers.

Theorem

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are complex numbers, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Theorem (De Moivre's Theorem)

If θ is any angle and n is a positive integer,

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

As an immediate consequence of De Moivre's Theorem, we have that for any real number $r > 0$ and any positive integer n,

$$\begin{aligned} (re^{i\theta})^n &= r^n e^{in\theta} \\ (r(\cos\theta + i\sin\theta))^n &= r^n(\cos n\theta + i\sin n\theta) \end{aligned}$$

Problem

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

Solution

Let $z = 1 - i = \sqrt{2}e^{-\pi i/4}$ and $w = \sqrt{3} + i = 2e^{\pi i/6}$. We want to compute $z^6 w^3$.

$$\begin{aligned}
z^6 w^3 &= (\sqrt{2}e^{-\pi i/4})^6 (2e^{\pi i/6})^3 \\
&= (2^3 e^{-6\pi i/4})(2^3 e^{3\pi i/6}) \\
&= (8e^{-3\pi i/2})(8e^{\pi i/2}) \\
&= 64e^{-\pi i} \\
&= 64e^{\pi i} \\
&= 64(\cos\pi + i\sin\pi) \\
&= -64.
\end{aligned}$$

Problem

Express $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)^{17}$ in the form $\mathrm{a} + \mathrm{bi}$.

Solution

Let $\mathrm{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} = \mathrm{e}^{-\pi\mathrm{i}/3}$. Then

$$
\begin{aligned}
\mathrm{z}^{17} &= \left(\mathrm{e}^{-\pi\mathrm{i}/3}\right)^{17} \\
&= \mathrm{e}^{-17\pi\mathrm{i}/3} \\
&= \mathrm{e}^{\pi\mathrm{i}/3} \\
&= \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3} \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}.
\end{aligned}
$$

Roots of Complex Numbers

Definition

Let z and q be complex numbers, and let n be a positive integer. Then z is called an n^{th} root of q if $z^n = q$.

De Moivre's Theorem and its implication

If θ is any angle and n is a positive integer, $\left(e^{i\theta}\right)^n = e^{in\theta}$. This implies that for any real number $r > 0$ and any positive integer n,

$$(re^{i\theta})^n = r^n e^{in\theta}.$$

This leads to the following result.

Corollary

Let q be a nonzero complex number and n a positive integer. Then $z^n = q$ has exactly n complex solutions, i.e., q has exactly n complex n^{th} roots.

Example

For any positive real number a, $z^2 = a$ has two complex (in this case, real) solution, $z = \sqrt{a}$ and $z = -\sqrt{a}$. This is equivalent to the statement that a has two complex (in this case, real) square roots.

- One particular example: 25 has two square roots, 5 and -5, and these are the two solutions to $z^2 = 25$.
- But we all knew that. A more interesting example is that -1 has no real square roots, but suddenly it has two (complex) square roots, i and $-i$. These are the two (complex) solutions to $z^2 = 1$.

Example (Cube Roots)

To find the (three) cube roots of i, we solve the equation $z^3 = i$. To do so, we express both z and i in polar form: convert i to polar form, and write $z = re^{i\theta}$, giving us

$$(re^{i\theta})^3 = e^{\pi i/2}.$$

Thus $r^3 e^{3i\theta} = 1e^{\pi i/2}$, implying that $r^3 = 1$ and $3\theta = \frac{\pi}{2}$.

- Since r is a real number, $r^3 = 1$ implies that $r = 1$.
- The statement $3\theta = \frac{\pi}{2}$ is not completely correct. The problem that arises is that the argument for i, $\frac{\pi}{2}$ is not unique. Instead, we could have written

  $$i = e^{5\pi i/2} \text{ or } i = e^{9\pi i/2} \text{ or } i = e^{-3\pi i/2}.$$

  In fact, there are infinitely many choices for the argument of i. The important thing to notice is that any two different arguments differ by a multiple of 2π, and thus we may write

  $$3\theta = \frac{\pi}{2} + 2\pi\ell, \ \ell \in \mathbb{Z}.$$

  ($\mathbb{Z}$ denotes the set of integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$).

Example (continued)

Dividing both sides of $3\theta = \frac{\pi}{2} + 2\pi\ell$ by 3:

$$\theta = \frac{\pi}{6} + \frac{2}{3}\pi\ell,$$

where ℓ is any integer. The Corollary to De Moivre's Theorem tells us that there are only three different cube roots. These are obtained by using $\ell = 0$, $\ell = 1$, and $\ell = 2$, resulting in three values of θ:

$$\frac{\pi}{6}, \frac{5\pi}{6}, \quad \text{and} \quad \frac{9\pi}{6} = \frac{3\pi}{2}.$$

Thus the cube roots of i are

$$\mathrm{e}^{\pi \mathrm{i}/6}, \mathrm{e}^{5\pi \mathrm{i}/6}, \quad \text{and} \quad \mathrm{e}^{3\pi \mathrm{i}/2}.$$

We now convert these to Cartesian form.

Example (continued)

$$\begin{aligned} \mathrm{e}^{\pi \mathrm{i}/6} &= \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, \\ \mathrm{e}^{\pi \mathrm{i}/6} &= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, \\ \mathrm{e}^{3\pi \mathrm{i}/2} &= -\mathrm{i}. \end{aligned}$$

You can check your work by computing the cube of each of these.

This process is summarized in the following procedure.

Finding Roots of a Complex Number

Let w be a complex number. We wish to find the n^{th} roots of w, that is all z such that $z^n = w$.
There are n distinct n^{th} roots and they can be found as follows:.

1. Express both z and w in polar form $z = re^{i\theta}, w = se^{i\phi}$. Then $z^n = w$ becomes:

$$(re^{i\theta})^n = r^n e^{in\theta} = se^{i\phi}$$

   We need to solve for r and θ.

2. Solve the following two equations:

$$\begin{aligned} r^n &= s \\ e^{in\theta} &= e^{i\phi} \end{aligned} \tag{1}$$

Continued

3. The solutions to $r^n = s$ are given by $r = \sqrt[n]{s}$.
4. The solutions to $e^{in\theta} = e^{i\phi}$ are given by:

$$n\theta = \phi + 2\pi\ell, \text{ for } \ell = 0, 1, 2, \cdots, n-1$$

or

$$\theta = \frac{\phi}{n} + \frac{2}{n}\pi\ell, \text{ for } \ell = 0, 1, 2, \cdots, n-1$$

5. Using the solutions r, θ to the equations given in (1) construct the n^{th} roots of the form $z = re^{i\theta}$.

Problem

Find all complex numbers z such that $z^4 = 2(\sqrt{3}i - 1)$, and express each in the form $a + bi$.

Solution

1. Convert $2(\sqrt{3}i - 1) = -2 + 2\sqrt{3}i$ to polar form:

$$|z^4| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4.$$

   If ϕ is an argument for $-2 + 2\sqrt{3}i$, then

$$\cos\phi = \frac{-2}{4} = -\frac{1}{2} \quad \text{and} \quad \sin\phi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}, \text{ so } \phi = \frac{2\pi}{3}.$$

   Thus $z^4 = 4e^{2\pi i/3}$. Let $z = re^{i\theta}$.

2. The equation becomes $r^4 e^{i4\theta} = 4e^{2\pi i/3}$, so we need to solve

$$\begin{aligned} r^4 &= 4 \\ e^{i4\theta} &= e^{2\pi i/3} \end{aligned}$$

Solution (continued)

3. Since $r^4 = 4$, $r^2 = \pm 2$. But r is real, and so $r^2 = 2$, implying $r = \pm\sqrt{2}$. However $r \geq 0$, and therefore $r = \sqrt{2}$.
4. The solutions to $e^{i4\theta} = e^{2\pi i/3}$ are given by

$$4\theta = \frac{2}{3}\pi + 2\pi\ell, \ell = 0, 1, 2, 3.$$

Therefore,

$$\theta = \frac{2\pi}{12} + \frac{2\pi\ell}{4} = \frac{\pi}{6} + \frac{\pi\ell}{2} = \frac{\pi(3\ell + 1)}{6}, \text{ for } \ell = 0, 1, 2, 3.$$

Solution (continued)

5. Thus $r = \sqrt{2}$ and $\theta = \left(\frac{3\ell+1}{6}\right)\pi$, $\ell = 0, 1, 2, 3$. Converting to Cartesian form:

$$\begin{array}{llll}
\ell = 0: & z = \sqrt{2}e^{\pi i/6} & = \sqrt{2}(\frac{(\sqrt{3}}{2} + \frac{1}{2}i) & = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i \\
\ell = 1: & z = \sqrt{2}e^{2\pi i/3} & = \sqrt{2}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) & = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i \\
\ell = 2: & z = \sqrt{2}e^{7\pi i/6} & = \sqrt{2}(-\frac{\sqrt{3}}{2} - \frac{1}{2}i) & = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i \\
\ell = 3: & z = \sqrt{2}e^{5\pi i/3} & = \sqrt{2}(\frac{1}{2} - \frac{\sqrt{3}}{2}i) & = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i
\end{array}$$

Therefore, the fourth roots of $2(\sqrt{3}i - 1)$ are:

$$\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i, -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i.$$

■

Roots of Unity

Definition

A complex number z is a root of unity if there exists a positive integer n so that $z^n = 1$.

Problem

Find the sixth roots of unity, i.e., all solutions to $z^6 = 1$.

Solution

Write $z = re^{i\theta}$ and convert 1 to polar form to get

$$(re^{i\theta})^6 = e^{i0}, \text{ and so } r^6 e^{6\theta i} = e^{i0}.$$

Equating the absolute values and arguments,

$$r^6 = 1 \quad \text{and} \quad 6\theta = 0 + 2\pi\ell, \ \ell = 0, 1, 2, 3, 4, 5.$$

Since r is real, $r = 1$. The six arguments for the solutions are

$$\theta = \frac{2\pi\ell}{6} = \frac{\pi\ell}{3}, \ \ell = 0, 1, 2, 3, 4, 5.$$

Solution (continued)

The six arguments for the solutions are

$$\theta = \frac{2\pi\ell}{6} = \frac{\pi\ell}{3}, \ \ell = 0, 1, 2, 3, 4, 5.$$

Converting these to Cartesian form:

ℓ	θ	z
0	0	$e^{0i} = 1$
1	$\frac{\pi}{3}$	$e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
2	$\frac{2\pi}{3}$	$e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
3	π	$e^{\pi i} = -1$
4	$\frac{4\pi}{3}$	$e^{4\pi i/3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
5	$\frac{5\pi}{3}$	$e^{5\pi i/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

If you graph these six point in the complex plane, you'll see that they result in six equally spaced points on the unit circle, one of them being $(1, 0)$. ■

Definition

Roots of Unity For any integer $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{2\pi \ell i/n} \text{ for } \ell = 0, 1, 2, \ldots, n-1.$$

Furthermore, the n^{th} roots of unity correspond to n equally spaced points on the unit circle, one of them being $(1, 0)$.

The Quadratic Formula

Definition

A real quadratic is an expression of the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

To find the roots of a real quadratic, we can either factor by inspection, or use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ is called the discriminant, and

- if $b^2 - 4ac \geq 0$, then the roots of the quadratic are real;
- if $b^2 - 4ac < 0$, then the quadratic has no real roots.

Definition

A real quadratic $ax^2 + bx + c$ is called irreducible if the discriminant is less than zero, i.e., $b^2 - 4ac < 0$.

Notice that if $b^2 - 4ac < 0$, then

$$\sqrt{b^2 - 4ac} = \sqrt{(-1)(4ac - b^2)} = (\pm)i\sqrt{4ac - b^2}.$$

It follows that the roots of an irreducible quadratic are

$$\frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \begin{cases} -\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}i \\ -\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}i \end{cases},$$

and we see that the two roots are complex conjugates of each other.
We denote the two roots by

$$u = -\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}i \quad \text{and} \quad \overline{u} = -\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}i.$$

Example (Real Quadratics with Complex Roots)

The quadratic $x^2 - 14x + 58$ has roots

$$\begin{aligned} x &= \frac{14 \pm \sqrt{196 - 4 \times 58}}{2} \\ &= \frac{14 \pm \sqrt{196 - 232}}{2} \\ &= \frac{14 \pm \sqrt{-36}}{2} \\ &= \frac{14 \pm 6i}{2} \\ &= 7 \pm 3i, \end{aligned}$$

so the roots are $7 + 3i$ and $7 - 3i$.

Conversely, given $u = a + bi$ with $b \neq 0$, there is an irreducible quadratic having roots u and $\overline{u}$.

Problem

Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?

Solution

$$\begin{aligned} (x-u)(x-\overline{u}) &= (x-(5-2i))(x-(5+2i)) \\ &= x^2 - (5-2i)x - (5+2i)x + (5-2i)(5+2i) \\ &= x^2 - 10x + 29. \end{aligned}$$

Therefore, $x^2 - 10x + 29$ is an irreducible quadratic with roots $5 - 2i$ and $5 + 2i$.
Notice that $-10 = -(u + \overline{u})$ and $29 = u\overline{u} = |u|^2$. ■

Problem

Find an irreducible quadratic with root $u = -3+4i$, and find the other root.

Solution (answer)

$x^2 + 6x + 25$ has roots $u = -3 + 4i$ and $\overline{u} = -3 - 4i$.

Problem (Quadratics with Complex Coefficients)

Find the roots of the quadratic $x^2 - (3-2i)x + (5-i) = 0$.

Solution

Using the quadratic formula

$$x = \frac{3 - 2i \pm \sqrt{(-(3-2i))^2 - 4(5-i)}}{2}.$$

Now,

$$(-(3-2i))^2 - 4(5-i) = 5 - 12i - 20 + 4i = -15 - 8i,$$

so

$$x = \frac{3 - 2i \pm \sqrt{-15-8i}}{2}.$$

To find $\pm\sqrt{-15-8i}$, solve $z^2 = -15 - 8i$ for z.

Solution (continued)

Let $z = a + bi$ and $z^2 = -15 - 8i$. Then

$$(a^2 - b^2) + 2abi = -15 - 8i,$$

so $a^2 - b^2 = -15$ and $2ab = -8$.

Solving for a and b gives us $z = 1 - 4i, -1 + 4i$, i.e., $z = \pm(1 - 4i)$. Therefore,

$$x = \frac{3 - 2i \pm (1 - 4i)}{2},$$

and

$$\tfrac{3-2i+(1-4i)}{2} = \tfrac{4-6i}{2} = 2 - 3i,$$

$$\tfrac{3-2i-(1-4i)}{2} = \tfrac{2+2i}{2} = 1 + i.$$

Thus the roots of $x^2 - (3 - 2i)x + (5 - i)$ are $2 - 3i$ and $1 + i$. ■

Problem

Find the roots of $x^2 - 3ix + (-3 + i)$.

Solution (answer)

$1 + i$ and $-1 + 2i$.

Problem

Verify that $u_1 = (4 - i)$ is a root of

$$x^2 - (2 - 3i)x - (10 + 6i)$$

and find the other root, u_2.

Solution

First,

$$\begin{aligned} u_1^2 - (2-3i)u_1 - (10+6i) &= (4-i)^2 - (2-3i)(4-i) - (10+6i) \\ &= (15-8i) - (5-14i) - (10+6i) \\ &= 0, \end{aligned}$$

so $u_1 = (4 - i)$ is a root.

Solution (continued)

Recall that if u_1 and u_2 are the roots of the quadratic, then

$$u_1 + u_2 = (2 - 3i) \quad \text{and} \quad u_1 u_2 = -(10 + 6i).$$

Solve for u_2 using either one of these equations.
Since $u_1 = 4 - i$ and $u_1 + u_2 = 2 - 3i$,

$$u_2 = 2 - 3i - u_1 = 2 - 3i - (4 - i) = -2 - 2i.$$

Therefore, the other root is $u_2 = -2 - 2i$.
You can easily verify your answer by computing $u_1 u_2$:

$$u_1 u_2 = (4 - i)(-2 - 2i) = -10 - 6i = -(10 + 6i).$$

$\blacksquare$