

Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry

§4-4. Linear Operators on \mathbb{R}^3

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Emory University, 2021 Spring

(last updated on 01/12/2023)



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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

Current Lecture Notes Revision: Version 2018 — Revision B

These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text [Linear Algebra with Applications](#) based on W. K. Nicholson's original text.

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- Ilijas Farah, York University

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Rotations

Reflections

Multiple Actions

Summary

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

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Summary

Rotations

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the **matrix transformation induced by A** .

Definition (Rotations in \mathbb{R}^2)

The transformation

$$R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ .

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$R_\theta(E_1) = R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix},$$

and

$$R_\theta(E_2) = R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

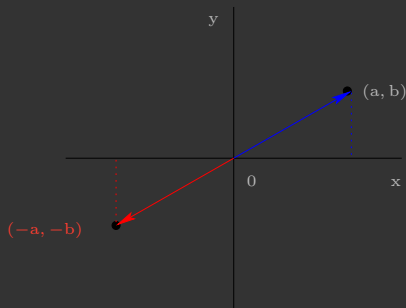
$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π .



We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denotes a **counterclockwise** rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

Solution

First,

$$R_{\frac{\pi}{2}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

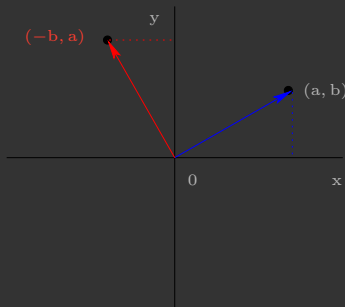
$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

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Example

In \mathbb{R}^2 , reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

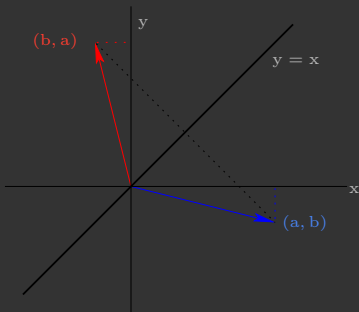
Example

In \mathbb{R}^2 , reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.

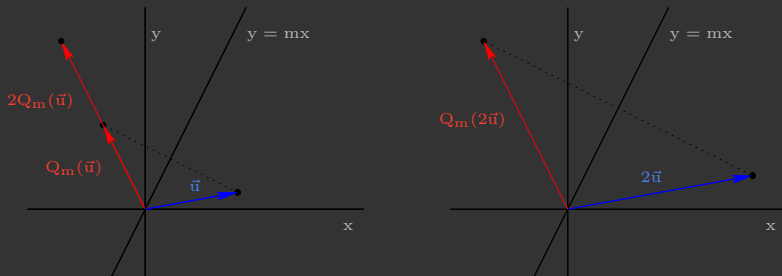


This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example (Reflection in $y = mx$ preserves scalar multiplication)

Let $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.



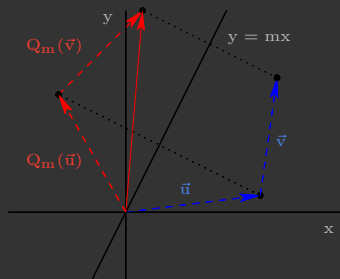
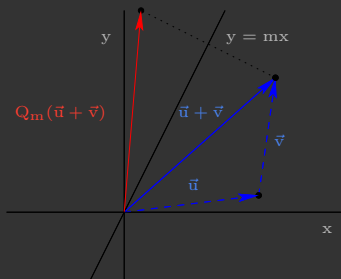
The figure indicates that $Q_m(2\vec{u}) = 2Q_m(\vec{u})$. In general, for any scalar k ,

$$Q_m(k\vec{x}) = kQ_m(\vec{x}),$$

i.e., Q_m preserves scalar multiplication.

Example (Reflection in $y = mx$ preserves vector addition)

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$.



The figure indicates that

$$Q_m(\vec{u}) + Q_m(\vec{v}) = Q_m(\vec{u} + \vec{v}),$$

i.e., Q_m preserves vector addition.

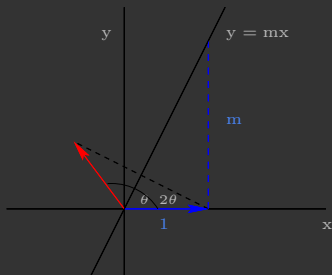
Q_m is a linear transformation

Since Q_m preserves addition and scalar multiplication, Q_m is a linear transformation, and hence a matrix transformation.

The matrix that induces Q_m can be found by computing $Q_m(E_1)$ and $Q_m(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

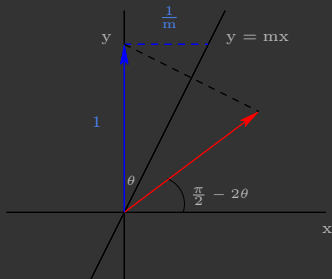
Example $(Q_m(E_1))$



$$\cos \theta = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin \theta = \frac{m}{\sqrt{1+m^2}}$$

$$Q_m(E_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta \\ 2 \sin \theta \cos \theta \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 \\ 2m \end{bmatrix}$$

Example ($Q_m(E_2)$)



$$\cos \theta = \frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \sin \theta = \frac{1}{\sqrt{1+m^2}}$$

$$\begin{aligned} Q_m(E_2) &= \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{2} \cos(2\theta) + \sin \frac{\pi}{2} \sin(2\theta) \\ \sin \frac{\pi}{2} \cos(2\theta) - \cos \frac{\pi}{2} \sin(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix} = \begin{bmatrix} 2 \sin \theta \cos \theta \\ \cos^2 \theta - \sin^2 \theta \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 2m \\ m^2 - 1 \end{bmatrix} \end{aligned}$$

The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

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Problem

Find the rotation or reflection that equals reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

Solution

Let Q_0 denote the reflection in the x-axis, and $R_{\frac{\pi}{2}}$ denote the rotation through an angle of $\frac{\pi}{2}$. We want to find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution

Hence $R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a **reflection** matrix.

How do we know this?

Solution (continued)


Compare BA to

$$Q_m = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Now, since $1-m^2=0$, we know that $m=1$ or $m=-1$. But $\frac{2m}{1+m^2}=1>0$, so $m>0$, implying $m=1$.

Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y=x$. 

Problem (Relection followed by reflection)

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y -axis.

Solution

We must find the matrix for the transformation $Q_Y \circ Q_{-1}$.

Q_{-1} is induced by

$$A = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and Q_Y is induced by

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $Q_Y \circ Q_{-1}$ is induced by BA .

Solution (continued)

$$\mathbf{BA} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

What transformation does \mathbf{BA} induce?

Rotation through an angle θ such that

$$\cos \theta = 0 \quad \text{and} \quad \sin \theta = -1.$$

Therefore, $\mathbf{Q}_Y \circ \mathbf{Q}_{-1} = \mathbf{R}_{-\frac{\pi}{2}} = \mathbf{R}_{\frac{3\pi}{2}}$.

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In general,

- The composite of two rotations is a rotation

$$R_\theta \circ R_\eta = R_{\theta+\eta}.$$

- The composite of two reflections is a rotation.

$$Q_m \circ Q_n = R_\theta$$

where θ is $2\times$ the angle between lines $y = mx$ and $y = nx$.

- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$