Math 221: LINEAR ALGEBRA

Chapter 3. Determinants and Diagonalization
§3-1. The Cofactor Expansion

Le Chen[1]

Emory University, 2021 Spring

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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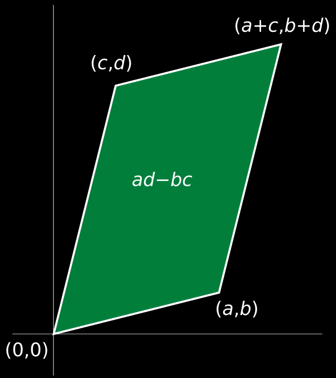
Determinant of Small Matrices

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A is defined as

$$\det A = ad - bc,$$

and that A is invertible if and only if $\det A \neq 0$.

Notation: For $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we often write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, i.e., use vertical bars instead of square brackets.



$$\det \begin{pmatrix} \mathrm{a} & \mathrm{c} \\ \mathrm{b} & \mathrm{d} \end{pmatrix} = \text{signed area of parallelogram}$$

Problem

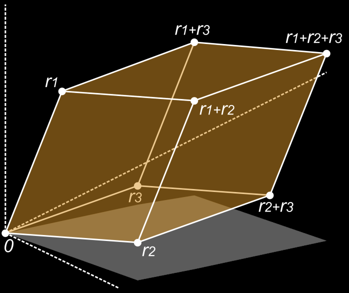
How to define determinant for a general $n \times n$ matrix?

$$2 \times 2$$

$$\begin{array}{c r} & - \\ \begin{vmatrix} a_{12} & a_{22} \\ a_{11} & a_{21} \end{vmatrix} & \\ & + \end{array}$$

3×3

$$\begin{array}{c|ccc|cc} & & & - & - & - \\ & a_{13} & a_{23} & a_{33} & a_{13} & a_{23} \\ & a_{12} & a_{22} & a_{32} & a_{12} & a_{22} \\ & a_{11} & a_{21} & a_{31} & a_{11} & a_{21} \\ & & & + & + & + \end{array}$$



$\det\begin{pmatrix}\vec{r}_1 & \vec{r}_2 & \vec{r}_3\end{pmatrix} =$ signed volume of the parallelepipe

4×4

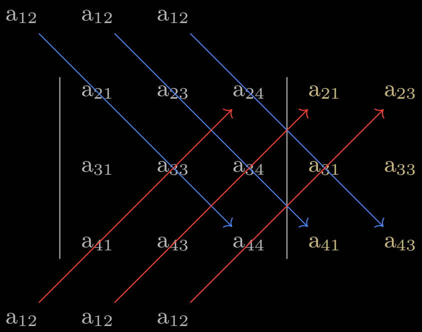
$$\left|\begin{array}{cccc|ccc}
a_{14} & a_{24} & a_{34} & a_{44} & a_{14} & a_{24} & a_{34} \\
a_{13} & a_{23} & a_{33} & a_{43} & a_{13} & a_{23} & a_{33} \\
a_{12} & a_{22} & a_{32} & a_{42} & a_{12} & a_{22} & a_{32} \\
a_{11} & a_{21} & a_{31} & a_{41} & a_{11} & a_{21} & a_{31}
\end{array}\right.$$

Only partially right... still missing many terms...

4×4 part I:

$$
\begin{array}{c|ccc|cc}
a_{11} & a_{11} & a_{11} & & & \\
& a_{22} & a_{23} & a_{24} & a_{22} & a_{23} \\
& a_{32} & a_{33} & a_{34} & a_{32} & a_{33} \\
& a_{42} & a_{43} & a_{44} & a_{42} & a_{43} \\
a_{11} & a_{11} & a_{11} & & &
\end{array}
$$

4×4 part II:



4×4 part III:

$$
\begin{array}{c|ccc|cc}
a_{13} & a_{13} & a_{13} & & & \\
 & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} \\
 & a_{31} & a_{32} & a_{34} & a_{31} & a_{32} \\
 & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} \\
a_{13} & a_{13} & a_{13} & & &
\end{array}
$$

4×4 part IV:

$$
\begin{array}{c|ccc|cc}
a_{14} & a_{14} & a_{14} & & & \\
 & a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
 & a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\
 & a_{41} & a_{42} & a_{43} & a_{41} & a_{42} \\
a_{14} & a_{14} & a_{14} & & &
\end{array}
$$

5×5? ...

The determinant of an $n \times n$ matrix is more effectively defined through recursion...

Cofactor and cofactor expansion

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- The sign of the (i, j) position is $(-1)^{i+j}$. (Thus the sign is 1 if $(i + j)$ is even, and -1 if $(i + j)$ is odd.)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \Rightarrow \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Definition (continued)

- Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j. The (i,j)-cofactor of A is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

- The determinant of A is defined as

$$\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$$

  and is called the cofactor expansion of $\det A$ along row 1.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find $\det A$.

Using cofactor expansion along row 1,

$$\begin{aligned}
\det A &= 1c_{11}(A) + 2c_{12}(A) + 3c_{13}(A) \\
&= 1(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= (45 - 48) - 2(36 - 42) + 3(32 - 35) \\
&= -3 - 2(-6) + 3(-3) \\
&= -3 + 12 - 9 \\
&= 0
\end{aligned}$$

Example (continued)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now try cofactor expansion along column 2.

$$\begin{aligned}
\det A &= 2c_{12}(A) + 5c_{22}(A) + 8c_{32}(A) \\
&= 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\
&= -2(36 - 42) + 5(9 - 21) - 8(6 - 12) \\
&= -2(-6) + 5(-12) - 8(-6) \\
&= 12 - 60 + 48 \\
&= 0.
\end{aligned}$$

We get the same answer!

Theorem (Cofactor Expansion Theorem)

The determinant of an $n \times n$ matrix A can be computed using the cofactor expansion along any row or column of A.

Example

Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0c_{11}(A) + 1c_{12}(A) + 2c_{13}(A) + 1c_{14}(A) \\ &= 1c_{12}(A) + 2c_{13}(A) + c_{14}(A), \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\begin{aligned} \det A &= 0c_{31}(A) + 1c_{32}(A) + (-1)c_{33}(A) + 0c_{34}(A) \\ &= 1c_{32}(A) + (-1)c_{33}(A), \end{aligned}$$

i.e., in the first case we have to compute three cofactors, but in the second we only have to compute two.

Example (continued)

We can save ourselves some work by using cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned}
\det A &= 1c_{32}(A) + (-1)c_{33}(A) \\
&= 1(-1)^5 \begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1)(-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\
&= (-1)2(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} + (-1)1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\
&= 2(10-21) + 1(10-21) \\
&= 2(-11) + (-11) \\
&= -33.
\end{aligned}$$

Example (continued)

Try computing $\det \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$ using cofactor expansion along other rows and columns, for instance column 2 or row 4. You will still get $\det A = -33$. ■

Problem

Find $\det A$ for $A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}$.

Solution

Using cofactor expansion along column 3, $\det A = 0$. ■

Remark

If A is an $n \times n$ matrix with a row or column of zeros, then $\det A = 0$.

Elementary Row Operations and Determinants

Example

Let $A = \left[\begin{array}{ccc} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 1 & 0 & -2 \end{array}\right]$. Then

$$\det A = 4(-1)^4 \left|\begin{array}{cc} 2 & -3 \\ 1 & -2 \end{array}\right| = 4(-1) = -4.$$

Let B_1, B_2, and B_3 be obtained from A by performing a type 1, 2 and 3 elementary row operation, respectively, i.e.,

$r_2 \leftrightarrow r_3$	$-3 \times r_3 \rightarrow r_3$	$2 \times r_1 + r_3 \rightarrow r_3$
$B_1 = \left[\begin{array}{ccc} 2 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 4 & 0 \end{array}\right]$	$B_2 = \left[\begin{array}{ccc} 2 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{array}\right]$	$B_3 = \left[\begin{array}{ccc} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 5 & 0 & -8 \end{array}\right]$.

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

$$\det B_2 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ -3 & 6 \end{vmatrix} = 4(12-9) = 4 \times 3 = 12 = -3\det A.$$

$$\det B_3 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = 4(-16+15) = 4(-1) = -4 = \det A.$$

Theorem (Determinant and Elementary Row Operations)

Let A be an $n \times n$ matrix.

1. If B is obtained from A by exchanging two different rows (or columns) of A, then $\det B = -\det A$.
2. If B is obtained from A by multiplying a row (or column) of A by a scalar $k \in \mathbb{R}$, then $\det B = k \det A$.
3. If B is obtained from A by adding k times one row of A to a different row of A (or adding k times one column of A to a different column of A) then $\det B = \det A$.
4. If A has a row or column of zeros, then $\det A = 0$.
5. If two different rows (or columns) of A are identical, then $\det A = 0$.

Example

$$\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix} = 36 - 36 = 0.$$

$\blacksquare$

Example

$$
\begin{aligned}
\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} &= \begin{vmatrix} 0 & -8 & 26 & 4 \\ -1 & -3 & 8 & 0 \\ 0 & -4 & 13 & 5 \\ 0 & -2 & 10 & -1 \end{vmatrix} \\
&= (-1)(-1)^3 \begin{vmatrix} -8 & 26 & 4 \\ -4 & 13 & 5 \\ -2 & 10 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -14 & 8 \\ 0 & -7 & 7 \\ -2 & 10 & -1 \end{vmatrix} \\
&= (-2)(-1)^4 \begin{vmatrix} -14 & 8 \\ -7 & 7 \end{vmatrix} \\
&= -2 \begin{vmatrix} 0 & -6 \\ -7 & 7 \end{vmatrix} \\
&= (-2)(-42) = 84.
\end{aligned}
$$

$\blacksquare$

Problem

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1$, find $\det \begin{bmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{bmatrix}$.

Solution

$$\begin{vmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{vmatrix} = (-1)(2) \begin{vmatrix} x & y & z \\ 3p+a & 3q+b & 3r+c \\ p & q & r \end{vmatrix}$$

$$= (-2) \begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = (-2)(-1) \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = 2(-1) \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$= (-2)(-1) = 2.$$ ■

Example

$$\begin{aligned}
\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1\det\begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= (1)(5)\det\begin{bmatrix} 9 \end{bmatrix} \\
&= (1)(5)(9) \\
&= 45.
\end{aligned}$$

$\blacksquare$

Determinant and Scalar Multiple

Problem

Suppose A is a 3×3 matrix with $\det A = 7$. What is $\det(-3A)$?

Solution

Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then $-3A = \begin{bmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{bmatrix}$.

$$\det(-3A) = \begin{vmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix} = (-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix}$$

$$= (-3)(-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix} = (-3)(-3)(-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-3)^3 \det A = (-27) \times 7 = -189. \quad \blacksquare$$

Theorem (Determinant of Scalar Multiple of Matrices)

If A is an $n \times n$ matrix and $k \in \mathbb{R}$ is a scalar, then

$$\det(kA) = k^n \det A.$$

Problem

Let

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix}$$

Show that $\det B = 9 \det A$.

Solution

$$\det B = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = \begin{vmatrix} p-4x & q-4y & r-4z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$= \begin{vmatrix} p-4x & q-4y & r-4z \\ 9x & 9y & 9z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = 9\begin{vmatrix} p-4x & q-4y & r-4z \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$= 9\begin{vmatrix} p & q & r \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = 9\begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix} = -9\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$= 9\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \det A. \qquad \blacksquare$$

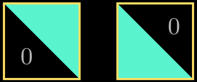
Determinant of Triangular Matrices

Theorem

If $A = [a_{ij}]$ is an $n \times n$ (square, upper or lower) triangular matrix, then

$$\det A = a_{11}a_{22}a_{33}\cdots a_{nn},$$

i.e., $\det A$ is the product of the entries of the main diagonal of A.



Determinants of Upper Triangular Matrices

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

$$\Downarrow$$

$$\det(U) = u_{11} u_{22} \cdots u_{nn}$$

Determinants of lower Triangular Matrices

$$\mathrm{L} = \begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix}$$

$$\Downarrow$$

$$\det(\mathrm{L}) = \ell_{11}\ell_{22}\cdots\ell_{nn}$$

Determinants of diagonal Matrices

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

$$\Downarrow$$

$$\text{det}(D) = d_{11}d_{22}\cdots d_{nn}$$

Determinant of Block Matrices

Theorem

Consider the matrices

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$$

where A and B are square matrices. Then

$$\det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \det A \det B \quad \text{and} \quad \det \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix} = \det A \det B.$$

Example

$$\det\begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \det\left[\begin{array}{ccc|cc} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$= \det\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix} \det\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \det\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= 3 \times (-2) = -6.$$

Some More Exercises

Example (From Exercise)

Evaluate by inspection.

$$\det \begin{bmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{bmatrix} = \ ?$$

$$\text{row2} + \text{row3} - 2(\text{row1}) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Example (From Exercise)

(a) Find $\det A$ if A is 3×3 and $\det(2A) = 6$.

(b) Let A be an $n \times n$ matrix. Under what conditions is $\det(-A) = \det A$?

Example (From Exercise)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

(a) $\det(A + B) = \det A + \det B$.

(c) If A is 2×2, then $\det(A^T) = \det A$.

(e) If A is 2×2, then $\det(7A) = 49 \det A$.

(g) $\det(-A) = -\det A$.