

# Math 221: LINEAR ALGEBRA

## Chapter 8. Orthogonality

### §8-3. Positive Definite Matrices

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.



# Linear Algebra with Applications

## Lecture Notes

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Positive Definite Matrices

Cholesky factorization – Square Root of a Matrix

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## Positive Definite Matrices

Cholesky factorization – Square Root of a Matrix

# Positive Definite Matrices

## Definition

An  $n \times n$  matrix  $A$  is **positive definite** if it is **symmetric** and has **positive** eigenvalues, i.e., if  $\lambda$  is a eigenvalue of  $A$ , then  $\lambda > 0$ .


## Theorem

If  $A$  is a positive definite matrix, then  $\det(A) > 0$  and  $A$  is invertible.

## Proof.

Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  denote the (not necessarily distinct) eigenvalues of  $A$ . Since  $A$  is symmetric,  $A$  is orthogonally diagonalizable. In particular,  $A \sim D$ , where  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Similar matrices have the same determinant, so

$$\det(A) = \det(D) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Since  $A$  is positive definite,  $\lambda_i > 0$  for all  $i$ ,  $1 \leq i \leq n$ ; it follows that  $\det(A) > 0$ , and therefore  $A$  is invertible. 

## Theorem

A symmetric matrix  $A$  is positive definite if and only if  $\vec{x}^T A \vec{x} > 0$  for all  $\vec{x} \in \mathbb{R}^n$ ,  $\vec{x} \neq \vec{0}$ .

## Proof.

Since  $A$  is symmetric, there exists an orthogonal matrix  $P$  so that

$$P^T A P = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = D,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the (not necessarily distinct) eigenvalues of  $A$ . Let  $\vec{x} \in \mathbb{R}^n$ ,  $\vec{x} \neq \vec{0}$ , and define  $\vec{y} = P^T \vec{x}$ . Then

$$\vec{x}^T A \vec{x} = \vec{x}^T (P D P^T) \vec{x} = (\vec{x}^T P) D (P^T \vec{x}) = (P^T \vec{x})^T D (P^T \vec{x}) = \vec{y}^T D \vec{y}.$$

Writing  $\vec{y}^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$ ,

$$\begin{aligned} \vec{x}^T A \vec{x} &= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2. \end{aligned}$$

### Proof. (continued)

( $\Rightarrow$ ) Suppose  $A$  is positive definite, and  $\vec{x} \in \mathbb{R}^n$ ,  $\vec{x} \neq \vec{0}$ . Since  $P^T$  is invertible,  $\vec{y} = P^T \vec{x} \neq \vec{0}$ , and thus  $y_j \neq 0$  for some  $j$ , implying  $y_j^2 > 0$  for some  $j$ . Furthermore, since all eigenvalues of  $A$  are positive,  $\lambda_i y_i^2 \geq 0$  for all  $i$ ; in particular  $\lambda_j y_j^2 > 0$ . Therefore,  $\vec{x}^T A \vec{x} > 0$ .

( $\Leftarrow$ ) Conversely, if  $\vec{x}^T A \vec{x} > 0$  whenever  $\vec{x} \neq \vec{0}$ , choose  $\vec{x} = P \vec{e}_j$ , where  $\vec{e}_j$  is the  $j^{\text{th}}$  column of  $I_n$ . Since  $P$  is invertible,  $\vec{x} \neq \vec{0}$ , and thus

$$\vec{y} = P^T \vec{x} = P^T (P \vec{e}_j) = \vec{e}_j.$$

Thus  $y_j = 1$  and  $y_i = 0$  when  $i \neq j$ , so

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 = \lambda_j,$$

i.e.,  $\lambda_j = \vec{x}^T A \vec{x} > 0$ . Therefore,  $A$  is positive definite. ■




## Theorem (Constructing Positive Definite Matrices)

Let  $U$  be an  $n \times n$  invertible matrix, and let  $A = U^T U$ . Then  $A$  is positive definite.

Proof.

Let  $\vec{x} \in \mathbb{R}^n$ ,  $\vec{x} \neq \vec{0}$ . Then

$$\begin{aligned}\vec{x}^T A \vec{x} &= \vec{x}^T (U^T U) \vec{x} \\ &= (\vec{x}^T U^T)(U \vec{x}) \\ &= (U \vec{x})^T (U \vec{x}) \\ &= \|U \vec{x}\|^2.\end{aligned}$$

Since  $U$  is invertible and  $\vec{x} \neq \vec{0}$ ,  $U \vec{x} \neq \vec{0}$ , and hence  $\|U \vec{x}\|^2 > 0$ , i.e.,  $\vec{x}^T A \vec{x} = \|U \vec{x}\|^2 > 0$ . Therefore,  $A$  is positive definite. 

## Definition

Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. For  $1 \leq r \leq n$ ,  $^{(r)}A$  denotes the  $r \times r$  submatrix in the upper left corner of  $A$ , i.e.,

$$^{(r)}A = [a_{ij}], \quad 1 \leq i, j \leq r.$$

$^{(1)}A, ^{(2)}A, \dots, ^{(n)}A$  are called the **principal submatrices** of  $A$ .

## Lemma

If  $A$  is an  $n \times n$  positive definite matrix, then each principal submatrix of  $A$  is positive definite.

Proof.

Suppose  $A$  is an  $n \times n$  positive definite matrix. For any integer  $r$ ,  $1 \leq r \leq n$ , write  $A$  in block form as

$$A = \begin{bmatrix} {}^{(r)}A & B \\ C & D \end{bmatrix},$$

where  $B$  is an  $r \times (n - r)$  matrix,  $C$  is an  $(n - r) \times r$  matrix, and  $D$  is an


$$(n - r) \times (n - r) \text{ matrix. Let } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} \neq \vec{0} \text{ and let } \vec{x} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \text{ Then}$$

$\vec{x} \neq \vec{0}$ , and by the previous theorem,  $\vec{x}^T A \vec{x} > 0$ .

Proof. (continued)

But

$$\vec{x}^T A \vec{x} = \begin{bmatrix} y_1 & \cdots & y_r & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} {}^{(r)}A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{y}^T \left( {}^{(r)}A \right) \vec{y},$$

and therefore  $\vec{y}^T \left( {}^{(r)}A \right) \vec{y} > 0$ . Then  ${}^{(r)}A$  is positive definite again by the previous theorem. 

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Positive Definite Matrices

Cholesky factorization – Square Root of a Matrix

## Cholesky factorization – Square Root of a Matrix

$$4 = 2 \times 2^T$$

$$\begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

### Theorem

Let  $A$  be an  $n \times n$  symmetric matrix. Then the following conditions are equivalent.

1.  $A$  is positive definite.
2.  $\det({}^{(r)}A) > 0$  for  $r = 1, 2, \dots, n$ .
3.  $A = U^T U$  where  $U$  is upper triangular and has positive entries on its main diagonal. Furthermore,  $U$  is unique. The expression  $A = U^T U$  is called the **Cholesky factorization** of  $A$ .

## Algorithm for Cholesky Factorization

Let  $A$  be a positive definite matrix. The Cholesky factorization  $A = U^T U$  can be obtained as follows.

1. Using only type 3 elementary row operations, with multiples of rows added to lower rows, put  $A$  in upper triangular form. Call this matrix  $\hat{U}$ ; then  $\hat{U}$  has positive entries on its main diagonal (this can be proved by induction on  $n$ ).
2. Obtain  $U$  from  $\hat{U}$  by dividing each row of  $\hat{U}$  by the square root of the diagonal entry in that row.

## Problem

Show that  $A = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix}$  is positive definite, and find the Cholesky factorization of  $A$ .

## Solution

$$^{(1)}A = \begin{bmatrix} 9 \end{bmatrix} \quad \text{and} \quad ^{(2)}A = \begin{bmatrix} 9 & -6 \\ -6 & 5 \end{bmatrix},$$

so  $\det(^{(1)}A) = 9$  and  $\det(^{(2)}A) = 9$ . Since  $\det(A) = 36$ , it follows that  $A$  is positive definite.

$$\begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row, to give

$$U = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad U^T U = A.$$





### Problem

Verify that

$$A = \begin{bmatrix} 12 & 4 & 3 \\ 4 & 2 & -1 \\ 3 & -1 & 7 \end{bmatrix}$$

is positive definite, and find the Cholesky factorization of  $A$ .

### Solution ( Final Answer )

$\det \left( {}^{(1)}A \right) = 12$ ,  $\det \left( {}^{(2)}A \right) = 8$ ,  $\det (A) = 2$ ; by the previous theorem,  $A$  is positive definite.

$$U = \begin{bmatrix} 2\sqrt{3} & 2\sqrt{3}/3 & \sqrt{3}/2 \\ 0 & \sqrt{6}/3 & -\sqrt{6} \\ 0 & 0 & 1/2 \end{bmatrix}$$

and  $U^T U = A$ .