Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-6. Linear Transformations

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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- **Ilijas Farah, York University**

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Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\vec{x} \in \mathbb{R}^n$. Since T preserves scalar multiplication,

1. $T(0\vec{x}) = 0T(\vec{x})$ implying $T(\vec{0}) = \vec{0}$, so T preserves the zero vector.
2. $T((-1)\vec{x}) = (-1)T(\vec{x})$, implying $T(-\vec{x}) = -T(\vec{x})$, so T preserves the negative of a vector.

Suppose $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ are vectors in $\mathbb{R}^n$ and for some $a_1, a_2, \dots, a_k \in \mathbb{R}$.

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k.$$

$$\Downarrow$$

3.

$$\begin{aligned} T(\vec{y}) &= T(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k) \\ &= a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \cdots + a_kT(\vec{x}_k), \end{aligned}$$

i.e., T preserves linear combinations.

Problem

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}. \text{ Find } T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}.$$

Solution

The only way it is possible to solve this problem is if

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} \text{ is a linear combination of } \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix},$$

i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Solution (continued)

To find a and b, solve the system of three equations in two variables:

$$\left[\begin{array}{rr|r} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus a $= 1$, b $= -2$, and

$$\left[\begin{array}{r} -7 \\ 3 \\ -9 \end{array}\right] = \left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right] - 2\left[\begin{array}{r} 4 \\ 0 \\ 5 \end{array}\right].$$

Solution (continued)

We now use that fact that linear transformations preserve linear combinations, implying that

$$\begin{aligned}
T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}\right) \\
&= T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}
\end{aligned}$$

Therefore, $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$. ■

Problem

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}. \text{ Find } T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix}.$$

Solution (Final Answer)

$$T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -3 \end{bmatrix}.$$

■

Theorem

Every matrix transformation is a linear transformation.

Proof.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation induced by the $m \times n$ matrix A, i.e., $T(\vec{x}) = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. Then

$$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T(\vec{x}) + T(\vec{y}),$$

proving that T preserves addition. Also,

$$T(a\vec{x}) = A(a\vec{x}) = a(A\vec{x}) = aT(\vec{x}),$$

proving that T preserves scalar multiplication.

Since T preserves addition and scalar multiplication T is a linear transformation. ∎

Example (The Zero Transformation)

If A is the $m \times n$ matrix of zeros, then the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ induced by A is called the zero transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$

$$T(\vec{x}) = A\vec{x} = O\vec{x} = \vec{0}.$$

The zero transformation is usually written as $T = 0$.

Example (The Identity Transformation)

The transformation of $\mathbb{R}^n$ induced by I_n, the $n \times n$ identity matrix, is called the identity transformation because for every vector $\vec{x}$ in $\mathbb{R}^n$,

$$T(\vec{x}) = I_n\vec{x} = \vec{x}.$$

The identity transformation on $\mathbb{R}^n$ is usually written as $1_{\mathbb{R}^n}$.

Problem (Revisited)

Is the following $T : \mathbb{R}^3 \to \mathbb{R}^4$ a matrix transformation?

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

Solution

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix} = T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Yes, T is a matrix transformation. $\blacksquare$

Problem (Not all transformations are matrix transformations)

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Show that T NOT a matrix transformation.

Solution

We have $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Since every matrix transformation is a linear transformation, we consider $T(0)$, where 0 is the zero vector of $\mathbb{R}^2$.

$$T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

violating one of the properties of a linear transformation.

Therefore, T is not a linear transformation, and hence is not a matrix transformation. ■

Remark

Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Theorem (Every Linear Transformation is a Matrix Transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then we can find an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

In this case, we say that T is induced, or determined, by A and we write

$$T_A(\vec{x}) = A\vec{x}$$

"linear" = "matrix"

Problem

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^3$ is another matrix transformation, that is, $T(\vec{x}) = A\vec{x}$ for some matrix A. Can you find a matrix A that works?

Solution

First, since $T : \mathbb{R}^3 \to \mathbb{R}^4$, we know that A must have size 4×3. Now consider the product

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix},$$

and try to fill in the values of the matrix.

$$\vdots$$

We can deduce from the product that T is induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

$\blacksquare$

Finding the Matrix of a Linear Transformation

Is there an easier way to find the matrix of T? For some transformations guess and check will work, but this is not an efficient method. The next theorem gives a method for finding the matrix of T.

Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of I_n is called the standard basis of $\mathbb{R}^n$.

Theorem (Matrix of a Linear Transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. Furthermore, T is induced by the unique matrix

$$A = \left[\begin{array}{cccc} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array} \right],$$

where $\vec{e}_j$ is the j-th column of I_n, and $T(\vec{e}_j)$ is the j-th column of A.

Corollary

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

“linear” = “matrix”

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A, of T.

Solution

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2(0) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 2(1) \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Downarrow$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

■

Sometimes, T is defined through its actions several concrete vectors.

Problem

Find the matrix A of T where T is given as

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution (continued)

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. First, find x and y such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Once we find x and y we can compute

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = xT \begin{bmatrix} 1 \\ 1 \end{bmatrix} + yT \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution (continued)

Finding x and y involves solving the following system of equations.

$$\begin{aligned} x &= 1 \\ x - y &= 0 \end{aligned}$$

The solution is $x = 1, y = 1$. Hence, we can find $T(\vec{e}_1)$ as follows.

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

As for $T(\vec{e}_2)$,

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -T\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

$$\Downarrow$$

$$A = \begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$$

■

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix}$.

Is T a linear transformation?

Solution

If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

It remains to verify the matrix transform induced by A indeed coincides with T:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, T is a matrix transformation induced by A above. ■

Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x+y \end{bmatrix}$. Is T a linear transformation?

Solution

If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

However, the matrix transform induced by A doesn't pass the verification:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x+y \end{bmatrix} \neq \begin{bmatrix} xy \\ x+y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, T in NOT a linear transformation. $\blacksquare$

Composition of Linear Transformations

Definition

Suppose $T : \mathbb{R}^k \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations. The composite (or composition) of S and T is

$$S \circ T : \mathbb{R}^k \to \mathbb{R}^m,$$

is defined by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k.$$

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, bend left, "T"] \arrow[rr, bend right, "S \circ T"'] & \mathbb{R}^n \arrow[r, bend left, "S"] & \mathbb{R}^m
\end{tikzcd}$$

Remark (Convention on the order)

$S \circ T$ means that the transformation T is applied first, followed by the transformation S.

Theorem

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and suppose that S is induced by matrix A, and T is induced by matrix B. Then $S \circ T$ is a linear transformation, and $S \circ T$ is induced by the matrix AB.

Problem

Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations defined by

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{and} \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Find $S \circ T$.

Solution

Then S and T are induced by matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively. The composite of S and T is the transformation $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(S \circ T) \begin{bmatrix} x \\ y \end{bmatrix} = S\left(T \begin{bmatrix} x \\ y \end{bmatrix}\right),$$

and has matrix (or is induced by the matrix)

$$AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Example (continued)

Therefore the composite of S and T is the linear transformation

$$(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = AB\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix},$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. ■

Remark

Compare this with the composite of T and S which is the linear transformation

$$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix},$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Rotations in $\mathbb{R}^2$

The rest part is an application of the linear transform to the study of the rotations in $\mathbb{R}^2$. This is left your motivated students to study by themselves.

Definition

The transformation

$$\mathrm{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $\mathrm{R}_\theta(\vec{\mathrm{e}}_1)$ and $\mathrm{R}_\theta(\vec{\mathrm{e}}_2)$, where

$$\vec{\mathrm{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\mathrm{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\mathrm{R}_\theta(\vec{\mathrm{e}}_1) = \mathrm{R}_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

and

$$\mathrm{R}_\theta(\vec{\mathrm{e}}_2) = \mathrm{R}_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

The Matrix for R_θ

The rotation $\mathrm{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

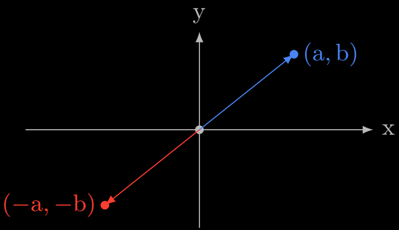
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π.



We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

Solution

First,

$$R_{\frac{\pi}{2}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

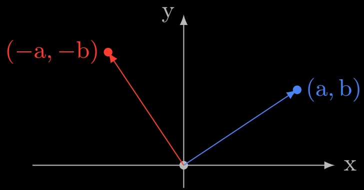
$\blacksquare$

Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Reflection in $\mathbb{R}^2$

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

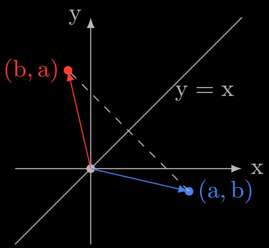
Example

In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



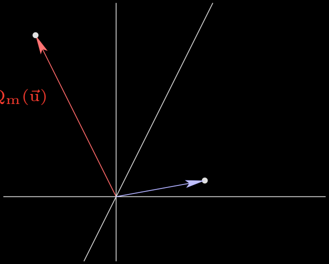
This is a matrix transformation because

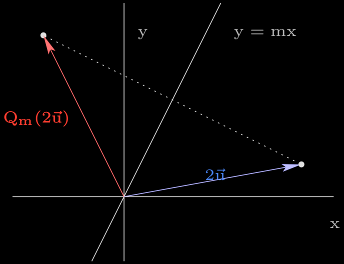
$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Reflection in the line

Example (Reflection in $y = mx$ preserves scalar multiplication)

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.





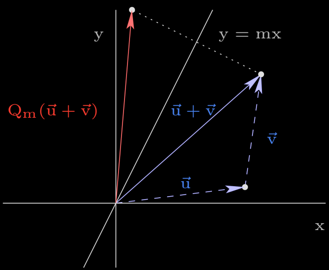
The figure indicates that $Q_m(2\vec{u}) = 2Q_m(\vec{u})$. In general, for any scalar k,

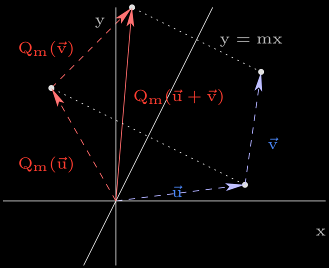
$$Q_m(kX) = kQ_m(X),$$

i.e., Q_m preserves scalar multiplication.

Example (Reflection in $y = mx$ preserves vector addition)

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$.





The figure indicates that

$$Q_m(\vec{u}) + Q_m(\vec{v}) = Q_m(\vec{u} + \vec{v}),$$

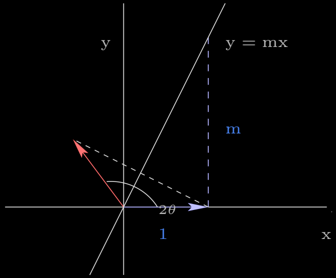
i.e., Q_m preserves vector addition.

Since Q_m preserves addition and scalar multiplication, Q_m is a linear transformation, and hence a matrix transformation.

The matrix that induces Q_m can be found by computing $Q_m(\vec{e}_1)$ and $Q_m(\vec{e}_2)$, where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

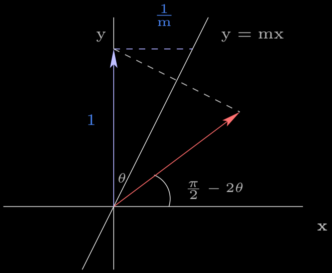
$Q_m(\vec{e}_1)$



$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

$$Q_m(\vec{e}_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta \\ 2\sin\theta\cos\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 \\ 2m \end{bmatrix}$$

$Q_m(\vec{e}_2)$



$$\cos\theta = \frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{1+m^2}}$$

$$\begin{aligned} Q_m(\vec{e}_2) &= \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2}\cos(2\theta) + \sin\frac{\pi}{2}\sin(2\theta) \\ \sin\frac{\pi}{2}\cos(2\theta) - \cos\frac{\pi}{2}\sin(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix} = \begin{bmatrix} 2\sin\theta\cos\theta \\ \cos^2\theta - \sin^2\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 2m \\ m^2 - 1 \end{bmatrix} \end{aligned}$$

Alternatively, we can use the following relation to find $\mathrm{Q_m}$:

$$\mathrm{Q_m} = \mathrm{R}_\theta \circ \mathrm{Q}_0 \circ \mathrm{R}_{-\theta}$$

$$\mathrm{R}_\theta \sim \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad \mathrm{Q}_0 \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathrm{R}_{-\theta} \sim \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

Then multiply these three matrices ...

The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Problem (Multiple Actions)

Find the rotation or reflection that equals reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

Solution

Let Q_0 denote the reflection in the x-axis, and $R_{\frac{\pi}{2}}$ denote the rotation through an angle of $\frac{\pi}{2}$. We want to find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right]$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \left[\begin{array}{rr} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$$

Solution

Hence $R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array}\right] = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right].$$

Notice that $BA = \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array}\right]$ is a reflection matrix.

How do we know this?

Solution (continued)

Compare BA to

$$Q_m = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Now, since $1-m^2=0$, we know that $m=1$ or $m=-1$. But $\frac{2m}{1+m^2}=1>0$, so $m>0$, implying $m=1$.

Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y=x$. ∎

Problem (Reflection followed by Reflection)

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y-axis.

Solution

We must find the matrix for the transformation $Q_Y \circ Q_{-1}$.

Q_{-1} is induced by

$$A = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and Q_Y is induced by

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $Q_Y \circ Q_{-1}$ is induced by BA.

Solution (continued)

$$BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

What transformation does BA induce?

Rotation through an angle θ such that

$$\cos\theta = 0 \quad \text{and} \quad \sin\theta = -1.$$

Therefore, $Q_Y \circ Q_{-1} = R_{-\frac{\pi}{2}} = R_{\frac{3\pi}{2}}$. ■

Remark (Summary)

In general,

- The composite of two rotations is a rotation

$$\mathrm{R}_\theta \circ \mathrm{R}_\eta = \mathrm{R}_{\theta+\eta}$$

- The composite of two reflections is a rotation.

$$\mathrm{Q}_\mathrm{m} \circ \mathrm{Q}_\mathrm{n} = \mathrm{R}_\theta$$

  where θ is $2\times$ the angle between lines $\mathrm{y} = \mathrm{mx}$ and $\mathrm{y} = \mathrm{nx}$.

- The composite of a reflection and a rotation is a reflection.

$$\mathrm{R}_\theta \circ \mathrm{Q}_\mathrm{n} = \mathrm{Q}_\mathrm{m} \circ \mathrm{Q}_\mathrm{n} \circ \mathrm{Q}_\mathrm{n} = \mathrm{Q}_\mathrm{m}$$