Math 221: LINEAR ALGEBRA

§Review session for test III

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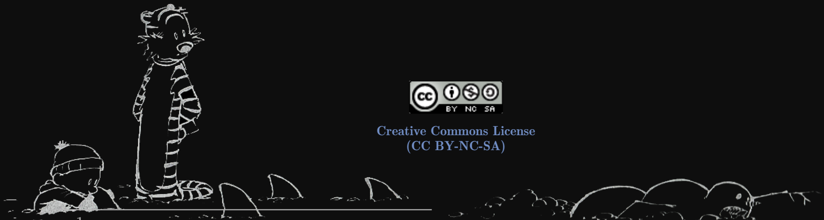

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Exercise 5.1.7 (L)

Problem

If $U = \mathrm{span}\{\vec{x}, \vec{y}, \vec{z}\}$ in $\mathbb{R}^n$, show that $U = \mathrm{span}\{\vec{x} + t\vec{z}, \vec{y}, \vec{z}\}$ for every $t \in \mathbb{R}$.

Exercise 5.1.8 (H)

Problem

If $U = \text{span}\{\vec{x}, \vec{y}, \vec{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\vec{x} + \vec{y}, \vec{y} + \vec{z}, \vec{z} + \vec{x}\}$.

Exercise 5.1.13 (L)

Problem

If A is an $m \times n$ matrix, show that, for each invertible $m \times m$ matrix U, $\text{null}(A) = \text{null}(UA)$.

Exercise 5.1.14 (H)

Problem

If A is an $m \times n$ matrix, show that, for each invertible $n \times n$ matrix V, $\mathrm{im}(A) = \mathrm{im}(AV)$.

Exercise 5.1.18 (P)

Problem

Suppose that $\vec{x}_1$, $\vec{x}_2, \cdots, \vec{x}_k$ are vectors in $\mathbb{R}^n$. If $\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k$ where $a_1 \neq 0$, show that

$$\mathrm{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\} = \mathrm{span}\{\vec{y}_1, \vec{x}_2, \cdots, \vec{x}_k\}.$$

Exercise 5.1.23 (P)

Problem

Let P denote an invertible $n \times n$ matrix. If λ is a number, show that

$$E_\lambda(PAP^{-1}) = \{P\vec{x} \mid \vec{x} \text{ is in } E_\lambda(A)\}$$

for each $n \times n$ matrix A.

Exercise 5.2.5 (P)

Problem

Suppose that $\{\vec{x}, \vec{y}, \vec{z}, \vec{w}\}$ is a basis of $\mathbb{R}^4$. Show that

1. $\{\vec{x} + a\vec{w}, \vec{y}, \vec{z}, \vec{w}\}$ is also a basis of $\mathbb{R}^4$ for any choice of scalar a.
2. $\{\vec{x} + \vec{w}, \vec{y} + \vec{w}, \vec{z} + \vec{w}, \vec{w}\}$ is also a basis of $\mathbb{R}^4$.
3. $\{\vec{x}, \vec{x} + \vec{y}, \vec{x} + \vec{y} + \vec{z}, \vec{x} + \vec{y} + \vec{z} + \vec{w}\}$ is also a basis of $\mathbb{R}^4$.

Exercise 5.2.8 (P)

Problem

If A is an $n \times n$ matrix, show that $\det(A) = 0$ if and only if some column of A is a linear combinations of the other columns.

Exercise 5.2.12 (L)

Problem

If $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$ is independent, show that $\{\vec{x}_1, \vec{x}_1 + \vec{x}_2, \cdots, \vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k\}$ is independent too.

Exercise 5.2.13 (H)

Problem

If $\{\vec{y}, \vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$ is independent, show that $\{\vec{y}+\vec{x}_1, \vec{y}+\vec{x}_2, \cdots, \vec{y}+\vec{x}_k\}$ is independent too.

Exercise 5.2.14 (P)

Problem

If $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$ is independent, and if $\vec{y}$ is not in $\mathrm{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$, show that $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k, \vec{y}\}$ is independent.

Exercise 5.2.15 (P)

Problem

If A and B are matrices and the columns of AB are independent, show that the columns of B are independent.

Exercise 5.2.16 (P)

Problem

Suppose that $\{\vec{x}, \vec{y}\}$ is a basis of $\mathbb{R}^2$, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that A is invertible if and only if $\{a\vec{x} + b\vec{y}, c\vec{x} + d\vec{y}\}$ is a basis of $\mathbb{R}^2$.

Exercise 5.2.17 (P)

Problem

Let A denote an $m \times n$ matrix.

1. Show that $\operatorname{null}(A) = \operatorname{null}(UA)$ for every invertible $m \times m$ matrix U.
2. Show that $\dim(\operatorname{null}(A)) = \dim(\operatorname{null}(AV))$ for every invertible $n \times n$ matrix V.

Exercise 5.2.18 (P)

Problem

Let A denote an $m \times n$ matrix.

1. Show that $\mathrm{im}(A) = \mathrm{im}(AV)$ for every invertible $n \times n$ matrix V.
2. Show that $\dim(\mathrm{im}(A)) = \dim(\mathrm{im}(UA))$ for every invertible $m \times m$ matrix U.

Exercise 5.3.2 (P)

Problem

In each case, show that the set of vectors is orthogonal in $\mathbb{R}^4$.

1. $\{(1,-1,3,5),(4,1,1,-1),(-7,28,5,5)\}$
2. $\{(2,-1,4,5),(0,-1,1,-1),(0,3,2,-1)\}$

Exercise 5.3.9 (P)

Problem

If A is an $m \times n$ matrix with orthonormal columns, show that $A^T A = I_n$.

Exercise 5.3.12 (H)

Problem

1. Show that $\vec{x}$ and $\vec{y}$ are orthogonal in $\mathbb{R}^n$ if and only if $||\vec{x}+\vec{y}|| = ||\vec{x}-\vec{y}||$.
2. Show that $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$ are orthogonal in $\mathbb{R}^n$ if and only if $||\vec{x}|| = ||\vec{y}||$.

Exercise 5.3.16 (H)

Problem

If $\mathrm{R}^{\mathrm{n}} = \mathrm{span}\{\vec{\mathrm{x}}_1, \cdots, \vec{\mathrm{x}}_{\mathrm{m}}\}$ and $\vec{\mathrm{x}} \cdot \vec{\mathrm{x}}_{\mathrm{i}} = 0$ for each i, show that $\vec{\mathrm{x}} = \vec{0}$.

Exercise 5.3.17 (P)

Problem

If $\mathrm{R}^{\mathrm{n}} = \mathrm{span}\{\vec{\mathrm{x}}_1, \cdots, \vec{\mathrm{x}}_{\mathrm{m}}\}$ and $\vec{\mathrm{x}} \cdot \vec{\mathrm{x}}_{\mathrm{i}} = \vec{\mathrm{y}} \cdot \vec{\mathrm{x}}_{\mathrm{i}}$ for each i, show that $\vec{\mathrm{x}} = \vec{\mathrm{y}}$.

Exercise 5.3.18 (L)

Problem

Let $\{\vec{e}_1, \cdots, \vec{e}_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Given $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, show that

$$\vec{x} \cdot \vec{y} = \frac{(\vec{x} \cdot \vec{e}_1)(\vec{y} \cdot \vec{e}_1)}{||\vec{e}_1||^2} + \cdots + \frac{(\vec{x} \cdot \vec{e}_n)(\vec{y} \cdot \vec{e}_n)}{||\vec{e}_n||^2}.$$

Exercise 5.4.3

Problem

1. Can 3×4 matrix have independent columns? Independent rows? Explain.
2. If A is 4×3 and rank $(A) = 2$, can A have independent columns? Independent rows? Explain.
3. If A is an $m \times n$ matrix and rank $(A) = m$, show that $m \leq n$.
4. Can a non-square matrix have its rows independent and its columns independent too? Explain.
5. Can the null space of a 3×6 matrix have dimension 2? Explain.
6. Suppose that A is 5×4 and $\mathrm{null}(A) = \{c\vec{x} | c \in \mathbb{R}\}$ for some $\vec{x} \neq \vec{0}$. Can $\dim(\mathrm{im}(A)) = 2$?

Exercise 5.4.5 (H)

Problem

If A is $m \times n$ and B is $n \times m$, show that $AB = 0$ if and only if $\mathrm{col}(B) \subseteq \mathrm{null}(A)$.

Exercise 5.4.8 (L)

Problem

Let $A = \vec{c}\vec{r}$ where $\vec{c} \neq \vec{0}$ is a column vector in $\mathbb{R}^m$ and $\vec{r} \neq \vec{0}$ is a row vector in $\mathbb{R}^n$.

1. Show that $\mathrm{col}\,(A) = \mathrm{span}\{\vec{c}\}$ and $\mathrm{row}(A) = \mathrm{span}\{\vec{r}\}$.
2. Find $\dim(\mathrm{null}(A))$
3. Show that $\mathrm{null}(A) = \mathrm{null}(\vec{r})$

Exercise 5.4.10 (L)

Problem

Let A be an $n \times n$ matrix.

1. Show that $A^2 = 0$ if and only if $\mathrm{col}(A) \subseteq \mathrm{null}(A)$
2. Conclude that if $A^2 = 0$, then $\mathrm{rank}\,(A) \leq \frac{n}{2}$
3. Find a matrix A for which $\mathrm{col}(A) = \mathrm{null}(A)$

Exercise 5.4.11

Problem

Let B be an $m \times n$ matrix and let AB be $k \times n$ matrix. If rank (B) = rank (AB), show that null(B) = null(AB).

Exercise 5.4.13 (H)

Problem

Let A be an $m \times n$ matrix with columns $\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_n$. If $\operatorname{rank}(A) = n$, show that $\{A^T\vec{c}_1, A^T\vec{c}_2, \cdots, A^T\vec{c}_n\}$ is a basis of $\mathbb{R}^n$.

Exercise 5.4.18 (P)

Problem

1. Show that if A and B have independent columns, so does AB.
2. Show that if A and B have independent rows, so does AB.

Exercise 5.5.3 (P)

Problem

If $A \sim B$, show that

1. $A^T \sim B^T$
2. $A^{-1} \sim B^{-1}$
3. $rA \sim rB$ for $r \in \mathbb{R}$
4. $A^n \sim B^n$ for $n \geq 1$

Exercise 5.5.7 (P)

Problem

Let λ be an eigenvalue of A with corresponding eigenvector $\vec{x}$. If $B = P^{-1}AP$ is similar to A, show that $P^{-1}\vec{x}$ is an eigenvector of B corresponding to λ.

Exercise 5.5.10 (P)

Problem

Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ (including multiplicity). Show that:

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n \quad \text{and} \quad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Exercise 5.5.12 (P)

Problem

Let P be an invertible $n \times n$ matrix. If A is any $n \times n$ matrix, write $T_P(A) = P^{-1}AP$. Verify that

1. $T_P(I) = I$
2. $T_P(AB) = T_P(A)T_P(B)$
3. $T_P(A + B) = T_P(A) + T_P(B)$
4. $T_P(rA) = rT_P(A)$
5. $T_P(A^k) = [T_P(A)]^k$
6. If A is invertible, $T_P(A^{-1}) = [T_P(A)]^{-1}$
7. If Q is invertible, $T_Q(T_P(A)) = T_{PQ}(A)$

Exercise 5.5.17 (P)

Problem

Let $A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$.

1. Show that $x^3 - (a^2 + b^2 + c^2)x - 2abc$ has real roots by considering A
2. Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$ by considering B

Exercise 5.5.18 (P)

Problem

Assume the 2×2 matrix A is similar to an upper triangular matrix. If $\mathrm{tr}(\mathrm{A}) = 0 = \mathrm{tr}(\mathrm{A}^2)$, show that $\mathrm{A}^2 = \mathrm{O}_{2\times 2}$, where $\mathrm{O}_{2\times 2}$ is a 2×2 zero matrix.

Bank 5.44

Problem

Find a basis for the solution space of $A\vec{x} = 0$ if

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 3 & -5 & 7 & 8 \end{bmatrix}$$