Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces

§6-1. Examples and Basic Properties

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Algebra with Applications

Lecture Notes

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These lecture notes were originally developed by Karen Seyffarth of the University of Calgary. Edits, additions, and revisions have been made to these notes by the editorial team at Lyryx Learning to accompany their text Linear Algebra with Applications based on W. K. Nicholson's original text.

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What is a vector space?

1. $\mathbb{R}^n$

2. Polynomials of order at most n:

$$\{a_0 + a_1 x + \cdots + a_n x^n | a_i \in \mathbb{R},\ i = 1, \cdots, n\}$$

3. The set of $m \times n$ matrices.

4. The set of continuous functions on $[0, 1]$, i.e., $C([0, 1])$.

5. The set of functions on $[0, 1]$ having nth continuous derivatives, i.e., $C^n([0, 1])$.

$\vdots \qquad \vdots$

Definition (Vector Space)

Let V be a nonempty set of objects with two operations:

vector addition and scalar multiplication.

Then V is called a vector space if it satisfies the following

- Axioms of Addition

  and

- Axioms of Scalar Multiplication.

The elements of V are called vectors.

Definition (continued – Axioms of ADDITION)

A1. V is closed under addition.
$\mathbf{v}, \mathbf{w} \in \mathrm{V} \quad \Longrightarrow \quad \mathbf{u} + \mathbf{v} \in \mathrm{V}$

A2. Addition is commutative.
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$.

A3. Addition is associative.
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$.

A4. Existence of an additive identity.
There exists an element $\mathbf{0}$ in V so that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathrm{V}$.

A5. Existence of an additive inverse.
For each $\mathbf{u} \in \mathrm{V}$ there exists an element $-\mathbf{u} \in \mathrm{V}$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Definition (continued – Axioms of SCALAR MULTIPLICATION)

S1. V is closed under scalar multiplication.
$\mathbf{v} \in \mathrm{V}$ and $\mathrm{k} \in \mathbb{R}, \implies \mathrm{k}\mathbf{v} \in \mathrm{V}$.

S2. Scalar multiplication distributes over vector addition.
$\mathrm{a}(\mathbf{u} + \mathbf{v}) = \mathrm{a}\mathbf{u} + \mathrm{a}\mathbf{v}$ for all $\mathrm{a} \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathrm{V}$.

S3. Scalar multiplication distributes over scalar addition.
$(\mathrm{a} + \mathrm{b})\mathbf{u} = \mathrm{a}\mathbf{u} + \mathrm{b}\mathbf{u}$ for all $\mathrm{a}, \mathrm{b} \in \mathbb{R}$ and $\mathbf{u} \in \mathrm{V}$.

S4. Scalar multiplication is associative.
$\mathrm{a}(\mathrm{b}\mathbf{u}) = (\mathrm{ab})\mathbf{u}$ for all $\mathrm{a}, \mathrm{b} \in \mathbb{R}$ and $\mathbf{u} \in \mathrm{V}$.

S5. Existence of a multiplicative identity for scalar multiplication.
$1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathrm{V}$.

Definition (Vector Difference)

Let V be a vector space and $\mathbf{u}, \mathbf{v} \in \mathrm{V}$. The difference of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

(where $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$).

Theorem

Let V be a vector space, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{V}$, and $\mathrm{a} \in \mathbb{R}$.

1. If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
2. The equation $\mathbf{x} + \mathbf{v} = \mathbf{u}$, has a unique solution $\mathbf{x} \in \mathrm{V}$ given by $\mathbf{x} = \mathbf{u} - \mathbf{v}$.
3. $\mathrm{a}\mathbf{v} = \mathbf{0}$ if and only if $\mathrm{a} = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.
5. $(-\mathrm{a})\mathbf{v} = -(\mathrm{a}\mathbf{v}) = \mathrm{a}(-\mathbf{v})$.

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What is a vector space?

Example One – Matrices

Example Two – Polynomials

More Examples

Example One – Matrices

Example

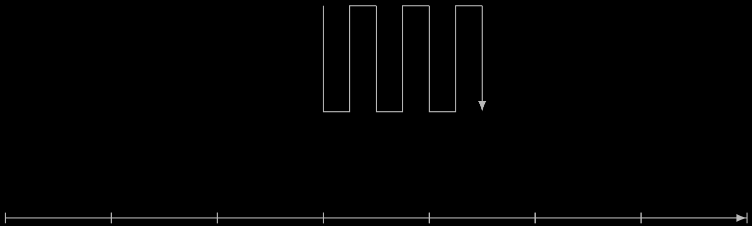
$\mathbb{R}^n$ with matrix addition and scalar multiplication is a vector space.

Example

$\mathbf{M}_{mn}$, the set of all $m \times n$ matrices (of real numbers) with matrix addition and scalar multiplication is a vector space. It is left as an exercise to verify the ten vector space axioms.

Remark

1. Notation: the $m \times n$ matrix of all zeros is written $\mathbf{0}$ or, when the size of the matrix needs to be emphasized, $\mathbf{0}_{mn}$.
2. The vector space $\mathbf{M}_{mn}$ "is the same as" the vector space $\mathbb{R}^{mn}$.



Problem

Let V be the set of all 2×2 matrices of real numbers whose entries sum to zero. We use the usual addition and scalar multiplication of $\mathbf{M}_{22}$. Show that V is a vector space.

Solution

The matrices in V may be described as follows:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22} \;\middle|\; a + b + c + d = 0 \right\}.$$

Since we are using the matrix addition and scalar multiplication of $\mathbf{M}_{22}$, it is automatic that addition is commutative and associative, and that scalar multiplication satisfies the two distributive properties, the associative property, and has 1 as an identity element.

What needs to be shown is closure under addition (for all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} + \mathbf{w} \in V$), and closure under scalar multiplication (for all $\mathbf{v} \in V$ and $k \in \mathbb{R}$, $k\mathbf{v} \in V$), as well as showing the existence of an additive identity and additive inverses in the set V.

Solution (continued)

- **Closure under addition:** Suppose

$$A = \begin{bmatrix} w_1 & x_1 \\ y_1 & z_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} w_2 & x_2 \\ y_2 & x_2 \end{bmatrix}$$

are in V. Then $w_1 + x_1 + y_1 + z_1 = 0, w_2 + x_2 + y_2 + z_2 = 0$, and

$$A + B = \begin{bmatrix} w_1 & x_1 \\ y_1 & z_1 \end{bmatrix} + \begin{bmatrix} w_2 & x_2 \\ y_2 & z_2 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 & x_1 + x_2 \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix}.$$

Since

$$\begin{aligned} &(w_1 + w_2) + (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ &= (w_1 + x_1 + y_1 + z_1) + (w_2 + x_2 + y_2 + z_2) \\ &= 0 + 0 = 0, \end{aligned}$$

A + B is in V, so V is closed under addition.

Solution (continued)

- **Closure under scalar multiplication:** Suppose $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is in V and $k \in \mathbb{R}$. Then $w + x + y + z = 0$, and

$$kA = k\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} kw & kx \\ ky & kz \end{bmatrix}.$$

  Since

$$kw + kx + ky + kz = k(w + x + y + z) = k(0) = 0,$$

  kA is in V, so V is closed under scalar multiplication.

Solution (continued)

- **Existence of an additive identity:** The additive identity of $\mathbf{M}_{22}$ is the 2×2 matrix of zeros,
$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$
Since $0 + 0 + 0 + 0 = 0$, $\mathbf{0}$ is in V, and has the required property (as it does in $\mathbf{M}_{22}$).

Solution (continued)

- **Existence of an additive inverse:** Let $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be in V. Then $w + x + y + z = 0$, and its additive inverse in $\mathbf{M}_{22}$ is

$$-A = \begin{bmatrix} -w & -x \\ -y & -z \end{bmatrix}.$$

Since

$$(-w) + (-x) + (-y) + (-z) = -(w + x + y + x) = -0 = 0,$$

$-A$ is in V and has the required property (as it does in $\mathbf{M}_{22}$). ■

Problem

Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \quad \text{and} \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0. \right\}.$$

We use the usual addition and scalar multiplication of $\mathbf{M}_{22}$. Show that V is NOT a vector space.

Solution

We need to find a counter example that violates some axioms. Indeed, if

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

then $\det(A) = 0$ and $\det(B) = 0$, so $A, B \in V$. However,

$$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

and $\det(A + B) = -1$, so $A + B \notin V$, i.e., V is not closed under addition. ■

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What is a vector space?

Example One – Matrices

Example Two – Polynomials

More Examples

Example Two – Polynomials

Definition

Let $\mathcal{P}$ be the set of all polynomials in x, with real coefficients, and let $p \in \mathcal{P}$. Then

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

for some integer n.

The degree of p is the highest power of x with a nonzero coefficient.

Definition (continued)

- Addition. Suppose $p, q \in \mathcal{P}$. Then

$$p(x) = \sum_{i=0}^{n} a_i x^i \quad \text{and} \quad q(x) = \sum_{i=0}^{m} b_i x^i.$$

We may assume, without loss of generality, that $n \geq m$; for $j = m+1, m+2, \ldots, n-1, n$, we define $b_j = 0$. Then

$$(p+q)(x) = p(x) + q(x) = \sum_{i=0}^{n} (a_i x^i + b_i x^i) = \sum_{i=0}^{n} (a_i + b_i) x^i.$$

Remark

Note that this definition ensures that $\mathcal{P}$ is closed under addition.

Definition (continued)

- Scalar Multiplication. Suppose $p \in \mathcal{P}$ and $k \in \mathbb{R}$. Then

$$p(x) = \sum_{i=0}^{n} a_i x^i,$$

and

$$(kp)(x) = k(p(x)) = \sum_{i=0}^{n} k(a_i x^i) = \sum_{i=0}^{n} (ka_i)x^i.$$

- The zero polynomial is denoted $\mathbf{0}$. Note that $\mathbf{0} = 0$, but we use $\mathbf{0}$ to emphasize that it is the zero vector of $\mathcal{P}$.

Remark

Note that this definition ensures that $\mathcal{P}$ is closed under scalar multiplication.

Example

The set of polynomials $\mathcal{P}$, with addition and scalar multiplication as defined, is a vector space. It is left as an exercise to verify the ten vector space axioms.

Example

For $n \geq 1$, let $\mathcal{P}_n$ denote the set of all polynomials of degree at most n, along with the zero polynomial, with addition and scalar multiplication as in $\mathcal{P}$, i.e.,

$$\mathcal{P}_n = \left\{a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n \mid a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in \mathbb{R}\right\}.$$

Then $\mathcal{P}_n$ is a vector space, and it is left as an exercise to verify the $\mathcal{P}_n$ is closed under addition and scalar multiplication, and satisfies the ten vector space axioms.

More Examples

Problem

Let $V = \{(x, y) \mid x, y \in \mathbb{R}\}$, with addition $\oplus$ and scalar multiplication $\odot$ defined as follows:

For $(x_1, y_1), (x_2, y_2) \in V$, and $a, b \in \mathbb{R}$:

1. Addition. $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + 1)$.
2. Scalar Multiplication. $a \odot (x_1, y_1) = (ax_1, ay_1 + a - 1)$.

Show that V, with addition and scalar multiplication as defined, is a vector space.

Proof.

1. It is clear that V is closed under $\oplus$ and $\odot$, since both operations produce ordered pairs of real numbers.
2. It is routine to verify that $\oplus$ is commutative and associative.
3. What is the additive identity?
4. What is the additive inverse of $(x, y) \in V$?
5. Verify that $(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$.
6. Verify that $a \odot ((x_1, y_1) \oplus (x_2, y_2)) = (a \odot (x_1, y_1)) \oplus (a \odot (x_2, y_2))$.
7. Verify that $a \odot (b \odot (x_1, y_1)) = (ab) \odot (x_1, y_1)$.
8. Verify that $1 \odot (x, y) = (x, y)$. $\blacksquare$

Problem

Let $\mathbb{R}_+$ be the set of positive reals. Let the addition $\oplus$ and the scalar multiplication $\odot$ defined as follows:

For $x, y \in \mathbb{R}_+$, and $a \in \mathbb{R}$:

1. Addition. $x \oplus y = xy$.
2. Scalar Multiplication. $a \odot x = x^a$.

Prove that $\mathbb{R}_+$ equipped with $\oplus$ and $\odot$ is a vector space.

Proof.

Verify ten properties in the Axioms! ∎

Problem

1. Let $\mathrm{C}([0,1])$ be the set of continuous functions defined on $[0,1]$ equipped with usual addition and scalar multiplication. Prove that $\mathrm{C}([0,1])$ is a vector space.
2. Let $\mathrm{C}^{\mathrm{n}}([0,1])$ be the set of functions that have continuous nth derivatives $(\mathrm{n} \geq 0)$ defined on $[0,1]$, equipped with usual addition and scalar multiplication. Prove that $\mathrm{C}^{\mathrm{n}}([0,1])$ is a vector space.

Proof.

Verify ten properties in the Axioms! ■