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The divergence of fluctuations for shape in first passage percolation

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Abstract. We consider the first passage percolation model on $\mathbf{Z}^d$ for $d \geq 2$. In this model, we assign independently to each edge the value zero with probability p and the value one with probability $1-p$. We denote by $T(\mathbf{0}, v)$ the passage time from the origin to v for $v \in \mathbf{R}^d$ and

$$B(t) = \{v \in \mathbf{R}^d : T(\mathbf{0}, v) \leq t\} \text{ and } G(t) = \{v \in \mathbf{R}^d : ET(\mathbf{0}, v) \leq t\}.$$

It is well known that if $p < p_c$, there exists a compact shape $B_d \subset \mathbf{R}^d$ such that for all $\epsilon > 0$,

$$tB_d(1-\epsilon) \subset B(t) \subset tB_d(1+\epsilon) \text{ and } G(t)(1-\epsilon) \subset B(t) \subset G(t)(1+\epsilon) \text{ eventually w.p.1.}$$

We denote the fluctuations of $B(t)$ from tB_d and $G(t)$ by

$$F(B(t), tB_d) = \inf\left\{l : tB_d\left(1 - \frac{l}{t}\right) \subset B(t) \subset tB_d\left(1 + \frac{l}{t}\right)\right\},$$
$$F(B(t), G(t)) = \inf\left\{l : G(t)\left(1 - \frac{l}{t}\right) \subset B(t) \subset G(t)\left(1 + \frac{l}{t}\right)\right\}.$$

In this paper, we show that for all $d \geq 2$ with a high probability, the fluctuations $F(B(t), G(t))$ and $F(B(t), tB_d)$ diverge with a rate of at least $C \log t$ for some constant C. The proof of this argument depends on the linearity between the number of pivotal edges of all minimizing paths and the paths themselves. This linearity is also independently interesting.

1. Introduction of the model and results

In this model, we consider the $\mathbf{Z}^d$ lattice for $d \geq 2$ as a graph with edges connecting each pair of vertices $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ with $d(x, y) = 1$, where $d(x, y)$ is the distance between x and y. For any two vertex sets $A, B \subset \mathbf{Z}^d$, the distance between A and B is also defined by

$$d(A, B) = \min\{d(u, v) : u \in A \text{ and } v \in B\}. \tag{1.0}$$

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We assign independently to each edge the value $t(e) = 0$ with a probability p or $t(e) = 1$ with a probability $1 - p$. More formally, we consider the following probability space. As the sample space, we take $\Omega = \prod_{e \in \mathbf{Z}^d} \{0, 1\}$, points of which are represented as *configurations*. Let $P = P_p$ be the corresponding product measure on Ω. The expectation and variance with respect to P are denoted by $E(\cdot) = E_p(\cdot)$ and $\sigma^2(\cdot) = \sigma_p^2(\cdot)$. For any two vertices u and v, a path γ from u to v is an alternating sequence $(v_0, e_1, v_1, \ldots, e_n, v_n)$ of vertices v_i and edges e_i in $\mathbf{Z}^d$ with $v_0 = u$ and $v_n = v$. Given a path γ, we define the *passage time* of γ as

$$T(\gamma) = \sum_{i=1}^{n} t(e_i). \tag{1.1}$$

For any two sets A and B, we define the passage time from A to B as

$$T(A, B) = \inf\{T(\gamma) : \gamma \text{ is a path from some vertex of } A \text{ to some vertex in } B\},$$

where the infimum takes over all possible finite paths. A path γ from A to B with $t(\gamma) = T(A, B)$ is called the *route* of $T(A, B)$. If we focus on a special configuration ω, we may write $T(A, B)(\omega)$ instead of $T(A, B)$. When $A = \{u\}$ and $B = \{v\}$ are single vertex sets, $T(u, v)$ is the passage time from u to v. We may extend the passage time over $\mathbf{R}^d$. If x and y are in $\mathbf{R}^d$, we define $T(x, y) = T(x', y')$, where x' (resp., y') is the nearest neighbor of x (resp., y) in $\mathbf{Z}^d$. Possible indetermination can be eliminated by choosing an order on the vertices of $\mathbf{Z}^d$ and taking the smallest nearest neighbor for this order.

[7] introduced the following point-point passage time:

$$a_{m,n} = \inf\{T(\gamma) : \gamma \text{ is a path from } (m, \ldots, 0) \text{ to } (n, \ldots, 0)\}.$$

By Kingman's subadditive argument, we have

$$\lim_{n \to \infty} \frac{1}{n} a_{0n} = \mu_p \text{ a.s. and in } L_1, \tag{1.2}$$

and (see Theorem 6.1 in [11])

$$\mu_p = 0 \text{ iff } p \geq p_c, \tag{1.3}$$

where $p_c = p_c(d)$ is the critical probability for Bernoulli (bond) percolation on $\mathbf{Z}^d$ and the non-random constant μ_p is called the *time constant*.

Given a unit vector $x \in \mathbf{R}^d$, by the same arguments in (1.2) and (1.3),

$$\lim_{n \to \infty} \frac{1}{n} T(\mathbf{0}, nx) = \liminf_{n} \frac{1}{n} ET(\mathbf{0}, nx) = \mu_p(x) \text{ a.s. and in } L_1, \tag{1.4}$$

and

$$\mu_p(x) = 0 \text{ iff } p \geq p_c.$$

The map $x \to \mu_p(x)$ induces a norm on $\mathbf{R}^d$. The unit radius ball for this norm is denoted by $B_d := B_d(p)$ and is called the *asymptotic shape*. The boundary of B_d is

$$\partial B_d := \{x \in \mathbf{R}^d : \mu_p(x) = 1\}.$$

If $p < p_c$, B_d is a compact convex deterministic set and ∂B_d is a continuous convex closed curve [11]. Define for all $t > 0$,

$$B(t) := \{v \in \mathbf{R}^d,\ T(\mathbf{0}, v) \leq t\}.$$

The shape theorem (see Theorem 1.7 of [11]) is the well-known result stating that for any $\epsilon > 0$,

$$tB_d(1-\epsilon) \subset B(t) \subset tB_d(1+\epsilon) \text{ eventually w.p.1.} \tag{1.5}$$

In addition to tB_d, we can consider the mean of $B(t)$ to be

$$G(t) = \{v \in \mathbf{R}^d : ET(\mathbf{0}, v) \leq t\}.$$

By (1.4), we have

$$tB_d \subset G(t) \tag{1.6}$$

and

$$G(t)(1-\epsilon) \subset B(t) \subset G(t)(1+\epsilon) \text{ eventually w.p.1.} \tag{1.7}$$

The natural, or perhaps the most challenging, question in this field (see [11] and [20]), is to ask how "fast" or how "rough" the boundary or interface is of the set $B(t)$ from the deterministic boundaries tB_d and $G(t)$. It is widely conjectured (see (1.8) in [12]) that if $p < p_c$ for a lower d, there exists $l = l(t)$ such that the following probabilities

$$P\left(tB_d\left(1-\frac{xl}{t}\right) \subset B(t) \subset tB_d\left(1+\frac{xl}{t}\right)\right) \tag{1.8}$$

are close to one for large $x > 0$, in particular, when $d = 2$, $l \approx t^{1/3}$.

This problem has also received a great amount of attention from statistical physicists because of its equivalence with one version of the Eden growth model. They believe that there is a scaling relation for the shape fluctuations in growth models. For each unit vector x, we may denote by $h_t(x)$ the *height* of the interface (see page 490 in [16]). The initial condition is $h_0(x) = 0$. Being interested in fluctuation, we consider the height fluctuation function

$$\bar{h}_t(x) = h_t(x) - Eh_t(x).$$

Statistical physicists believe that $\bar{h}_t(x)$ should be satisfied (see (3.1) in [16]) by the statistical properties of the rescaled process

$$\bar{h}_t(x) = b^{\zeta}\bar{h}_{b^z t}(bx)$$

with the scaling exponents ζ and z for an arbitrary rescaling factor b. With this rescaling equation, we should have (see (7.9) in [16]), for a lower d and for all vectors x,

$$\bar{h}_t(x) \approx t^{\zeta/z} \text{ pointwisely or } \sigma(h_t(x)) \approx t^{\zeta/z}. \tag{1.9}$$

In particular, (1.9) should hold for $\zeta = 1/2$ and $z = 2/3$ when $d = 2$. There have been varying discussions about the nature of the fluctuations of $\bar{h}_t(x)$ for a large d, including the possible independence of $\chi(d) = \zeta/z$ on d [10]; the picture of $\chi(d)$ decreasing with d but always remaining strictly positive (see [21] and [15]); the possibility that for d above some d_c, $\chi(d) = 0$; and the idea that the fluctuations do not even diverge (see [17, 6], and [4]). Simulations for a large d seem to be difficult to pursue, as mentioned by [15], because the computation time would be prohibitive.

Mathematicians have also made significant efforts in this direction. Before introducing mathematical estimates, we would like to give a precise definition of the fluctuations of $B(t)$ from tB_d or $G(t)$ by using (1.8) and (1.9). In fact, if we ask how fast or how rough the boundary of an interface from tB_d or from $G(t)$ is, we should consider two versions:

(a) (*directional fluctuation*) for a fixed unit vector x, consider the mean of the distance between the boundaries of $B(t)$ and $G(t)$ or of $B(t)$ and tB_d along the direction x.
(b) (*maximum fluctuation*) Consider the mean of the maximum distance between the boundaries of $B(t)$ and $G(t)$ or of $B(t)$ and tB_d among all vectors.

Note that large directional fluctuations imply large maximum fluctuations. In this paper, we primarily focus on maximum fluctuations by using the following definition, which is more general than the definition in (b). For a connected set Γ of $\mathbf{R}^d$ containing the origin, let

$$\Gamma_l^+ = \{v \in \mathbf{R}^d : d(v, \Gamma) \leq l\} \text{ and } \Gamma_l^- = \{v \in \Gamma : d(v, \partial\Gamma) \geq l\}.$$

In words, we enlarge or shrink Γ with l units to have Γ_l^+ or Γ_l^-. Note that $\Gamma_l^- \subset \Gamma$ and $\Gamma \subset \Gamma_l^+$. Note also that Γ_l^- might be empty even though Γ is non-empty. For a fixed Γ, we define the following random variable as

$$F(B(t), \Gamma) = F(B(t)(\omega), \Gamma) = \inf\{l : \Gamma_l^- \subset B(t)(\omega) \subset \Gamma_l^+\}. \tag{1.10}$$

If we set $\Gamma = tB_d$, we may view $F(B(t), \Gamma)$ as the maximum fluctuation of $B(t)$ from Γ. The definition of (1.10) was used by [18]. With this definition, the conjecture in (1.8) is equivalent to the question

$$F(B(t), tB_2) = O(t^{1/3}) \text{ with a high probability.}$$

When $p \geq p_c$, B_d is unbounded, and so is $B(t)$. Furthermore, when $p = 0$, there are no fluctuations, so we require in this paper that

$$0 < P(t(e) = 0) = p < p_c. \tag{1.11}$$

The mathematical estimates for the upper bound of the fluctuation $F(B(t), \Gamma)$, when $\Gamma = tB_d$ and $\Gamma = G(t)$, are quite promising. [12] and [1, 2] showed that for $p < p_c(d)$ and all $d \geq 2$, there is a constant C_1 such that

$$F(B(t), tB_d) \leq C_1 \sqrt{t} \log t \text{ eventually w.p.1} \tag{1.12}$$

and

$$tB_d \subset G(t) \subset (t + C_2 \sqrt{t} \log t) B_d,$$

where log denotes the natural logarithm. In this paper, C and C_i are always positive constants that may depend on p or d, but not on t, and their values are not significant and change from appearance to appearance. [3] also showed that when $t(e)$ only takes two values $0 < a < b$ with a half probability for each one,

$$F(B(t), tB_d) \leq C\sqrt{t}/\log t \text{ eventually w.p.1.} \tag{1.13}$$

On the other hand, the estimates for the lower bound of the fluctuations are quite unsatisfactory. Under (1.11), it seems that the only result for all $d \geq 2$ (see [12]) is

$$F(B(t), tB_d) \geq \text{ a non-zero constant eventually w.p.1.} \tag{1.14}$$

For $d = 2$, [18] showed that

$$F(B(t), tB_d) \geq t^{1/8} \text{ and } F(B(t), G(t)) \geq t^{1/8} \text{ eventually w.p.1.} \tag{1.15}$$

If we focus on the directional fluctuation, we see it has been proved (see [18] and [19]) that for $d = 2$,

$$\sigma^2(T(\mathbf{0}, xt)) \geq C \log t.$$

Clearly, one of most intriguing questions in this field is to ask if the fluctuations of $B(t)$ diverge, as some statistical physicists believe to be true while others do not. More precisely, we may ask whether there is a vector x such that for all $d > 2$,

$$\sigma^2(T(\mathbf{0}, xt)) \to \infty \text{ as } t \to \infty \tag{1.16}$$

or at least whether the maximum fluctuation

$$EF(B(t), tB_d) \to \infty \text{ and } EF(B(t), G(t)) \to \infty \text{ as } t \to \infty.$$

In this paper, we answer the second conjecture affirmatively to show that the maximum fluctuations of $B(t)$ always diverge for all $d \geq 2$. We can even tell that the divergence rate is at least $C \log t$.

Theorem 1. *For $0 < p < p_c$ and $3 \leq d$, and for any choice of $0 < \delta < 1/(-2\log p)$, there exists a positive constant $C = C(p, d, \delta)$ such that*

$$P(F(B(t), \Gamma) \leq \delta \log t) \leq Ct^{-d+2-2\delta \log p}$$

for all $1 \leq t$ and for any deterministic set Γ.

Remark 1. If we set $\Gamma = tB_d$ or $\Gamma = G(t)$, together with (1.15), the fluctuations $F(B(t), tB_d)$ and $F(B(t), G(t))$ are at least $\delta \log t$ with a large probability. Also, it follows from this probability estimate that

$$E(F(B(t), \Gamma)) \geq C \log t \tag{1.17}$$

for a constant $C = C(p, d) > 0$.

Remark 2. We are unable to estimate whether $F(tB_d, G(t))$ diverges, even though we believe it does.

Remark 3. Theorem 1 only shows that the maximum fluctuations diverge; we do not know whether the directional fluctuations diverge.

The proof of Theorem 1 is constructive. In fact, if $F(B(t), \Gamma) \leq \delta \log t$, then we can construct $t^{d-1+2\delta \log p}$ zero-paths from $\partial B(t)$ to $\Gamma^+_{\delta \log t}$. For each such path, we can use the geometric property introduced in Section 2 to show that the path contains a *pivotal* edge defined in Section 3. Therefore, we can construct about $t^{d-1+2\delta \log p}$ pivotal edges. However, in Section 3, we can also show that the number of pivotal edges is of order t. Therefore, for a suitable δ, we cannot have as many pivotal edges as we constructed. The contradiction tells us that $F(B(t), \Gamma) \geq \delta \log t$.

With these estimates for pivotal edges in Section 3, we can also estimate the number of the total vertices in all routes. This estimate is independently interesting. For a connected set $\Gamma \subset \mathbf{R}^d$ with $\alpha_1 tB_d \subset \Gamma \subset \alpha_2 tB_d$ for some constants $0 < \alpha_1 < 1 < \alpha_2$, let

$$R_\Gamma = \bigcup \gamma_t, \text{where} \gamma_t \text{is a route for } T(\mathbf{0}, \partial\Gamma).$$

Theorem 2. *For $0 < p < p_c$ and $2 \leq d$, and for any choice of $0 < \alpha_1 < 1 < \alpha_2$, there exists $C = C(p, d, \alpha_1, \alpha_2)$ such that*

$$E(|R_\Gamma|) \leq Ct$$

for all $1 \leq t$ and for any deterministic set Γ with $\alpha_1 tB_d \subset \Gamma \subset \alpha_2 tB_d$, where $|A|$ denotes the cardinality of A for some set A.

Remark 4. [11] showed that there exists a route for $T(\mathbf{0}, \Gamma)$ with length Ct for some positive constant C. Theorem 2 gives a stronger result, with the number of vertices in all routes for $T(\mathbf{0}, \partial\Gamma)$ also in order t. For quite some time, the author believed that the routes of $T(\mathbf{0}, \partial\Gamma)$ resembled a spiderweb centered at the origin and thus the number of vertices in the routes should be of order t^d. However, Theorem 2 negates this assumption.

Remark 5. Clearly, there might be many routes for the passage time $T(\mathbf{0}, \partial\Gamma)$. As a consequence of Theorem 2, each route contains at most Ct vertices. Specifically, let

$$M_{t,\Gamma} = \sup\{k : \text{there exists a route of } T_\Gamma(\mathbf{0}, \partial\Gamma) \text{ containing k edges}\}.$$

If $0 < p < p_c$, for all $d \geq 2$ and $t > 0$, there exists a positive constant $C = C(p, d)$ such that

$$E(M_{t,\Gamma}) \leq Ct. \tag{1.18}$$

Remark 6. We may also consider Theorem 2 for a point-point passage time. Let

$$R_{x,t} = \bigcup \gamma_t, \text{ where} \gamma_t \text{is a route for } T(\mathbf{0}, xt)$$

for a unit vector x. We may use the same argument of Theorem 2 to show if $0 < p < p_c$, for all $d \geq 2$ and $t > 0$, there exists a positive constant $C = C(p, d)$ such that

$$E(|R_{x,t}|) \leq Ct. \tag{1.19}$$

Remark 7. The condition $p > 0$ in Theorem 2 is crucial. As $p \downarrow 0$, the constant C in Theorem 2, (1.18) and (1.19) may go to infinity. When $p = 0$, all edges have to take value one. If we take Γ as a diamond shape with a diagonal length of $2t$ in both the vertical and the horizontal directions, it is easy to say that all edges inside the diamond belong to R_Γ. Therefore,

$$|R_\Gamma| = \text{ the number of edges in } \Gamma = O(t^d). \tag{1.20}$$

This tells us that Theorem 2 will not work when $p = 0$.

2. Geometric properties of $B(t)$

In this section, we would like to introduce a few geometric properties for $B(t)$. Given a set $\Gamma \subset \mathbf{R}^d$, we let Γ' be all vertices contained in $\Gamma \cap \mathbf{Z}^d$. It is easy to see that

$$\Gamma' \subset \Gamma \subset \{v + (-1, 1)^d : v \in \Gamma'\}. \tag{2.0}$$

As we mentioned in the last section, both $B(t)$ and $G(t)$ are finite, as are $B'(t)$ and $G'(t)$. We now show that $B'(t)$ and $G'(t)$ are also connected. Here a set A is said to be connected in $\mathbf{Z}^d$ if any two vertices of A are connected by a path in A.

Proposition 1. *$B'(t)$ and $G'(t)$ are connected.*

Proof. Since $T(\mathbf{0}, \mathbf{0}) = 0 \leq t$, $\mathbf{0} \in B'(t)$. We pick a vertex $v \in B'(t)$, so $T(\mathbf{0}, v) \leq t$. This tells us that there exists a path γ such that

$$T(\gamma) \leq t.$$

Therefore, for any $u \in \gamma$,

$$T(\mathbf{0}, u) \leq t, \text{ and thus } u \in B'(t). \tag{2.1}$$

This implies that $\gamma \subset B'(t)$, so we know $B'(t)$ is connected. The same argument shows that $G'(t)$ is connected. □

Given a finite set Γ of $\mathbf{Z}^d$, we define its vertex boundary as follows. For each $v \in \Gamma$, $v \in \Gamma$ is said to be a boundary vertex of Γ if there exists $u \not\in \Gamma$ but u is adjacent to v. We denote by $\partial\Gamma$ all boundary vertices of Γ. We also let $\partial_o\Gamma$ be all vertices not in Γ, but adjacent to $\partial\Gamma$. With these definitions, we have the following proposition.

Proposition 2. *For all $v \in \partial B'(t)$, $T(\mathbf{0}, v) = t$, and for all $u \in \partial_o B'(t)$, $T(\mathbf{0}, u) = t + 1$.*

Proof. We pick $v \in \partial B'(t)$. By the definition of the boundary, $v \in B'(t)$, and thus $T(\mathbf{0}, v) \leq t$. Now we show $T(\mathbf{0}, v) \geq t$ for all $v \in \partial B'(t)$. If we suppose that $T(\mathbf{0}, v) < t$ for some $v \in \partial B'(t)$, then $T(\mathbf{0}, v) \leq t - 1$, since $T(\mathbf{0}, v)$ is an integer. Note that $t(e)$ only takes zero or one, so there exists $u \in \partial_o B(t)$ and u is adjacent to v such that $T(\mathbf{0}, u) \leq t$. This tells us that $u \in B'(t)$. But we know as we defined earlier that

$$\partial_o B'(t) \cap B(t) = \emptyset. \tag{2.2}$$

This contradiction tells us that $T(\mathbf{0}, v) \geq t$ for all $v \in \partial B'(t)$.

Now we will prove the second part of Proposition 2. We pick a vertex $u \in \partial_o B'(t)$. Since u is adjacent to $v \in B'(t)$,

$$T(\mathbf{0}, u) \leq 1 + T(\mathbf{0}, v) \leq 1 + t. \tag{2.3}$$

On the other hand, any path from $\mathbf{0}$ to u must pass through a vertex of $\partial B'(t)$ before reaching $\partial_o B'(t)$. We denote the vertex by v. As we proved, $T(\mathbf{0}, v) = t$. The passage time of the rest of the path from v to u has to be greater than or equal to one, otherwise $u \in B'(t)$. Therefore, any path from $\mathbf{0}$ to u has a passage time greater than or equal to $t + 1$, that is,

$$T(\mathbf{0}, u) \geq t + 1.$$

Therefore, $T(\mathbf{0}, \partial_o B'(t)) = t + 1$. □

Given a fixed connected set $\kappa = \kappa_t$ containing the origin, define the event as

$$\{B'(t) = \kappa\} = \{\omega : B'(t)(\omega) = \kappa\}.$$

Proposition 3. *The event of* $\{B'(t) = \kappa\}$ *only depends on the zeros and ones of the edges on* $\kappa \cup \partial_o \kappa$.

Proposition 3 for $d = 2$ was proven by [14]. In fact, they gave a precise structure of $B'(t)$. We may adapt their idea to prove Proposition 3 for $d \geq 3$ by using the plaquette surface (see the definition in Section 12.4 of [5]. To avoid the complicated definition of the plaquette surface, we would rather give the following direct proof.

Proof. Let κ^C denote the vertices of $\mathbf{Z}^d \setminus \kappa$ and

$$\{\omega(\kappa)\} = \prod_{\text{edge in } \kappa} \{0, 1\} \text{ and } \{\omega(\kappa^C)\} = \prod_{\text{edge in } \kappa^C} \{0, 1\},$$

where the edges in κ are the edges whose two vertices belong to $\kappa \cup \partial_o \kappa$ and the edges in κ^C are the other edges. For each $\omega \in \Omega$, we may rewrite ω as

$$\omega = (\omega(\kappa), \omega(\kappa^C)).$$

Suppose that Proposition 3 is not true, so the zeros and ones in $\omega(\kappa^C)$ can affect the event $\{B'(t) = \kappa\}$. In other words, there exist two different $\omega_1, \omega_2 \in \Omega$ with

$$\omega_1 = (\omega(\kappa), \omega_1(\kappa^C)) \text{ and } \omega_2 = (\omega(\kappa), \omega_2(\kappa^C))$$

such that

$$B'(t)(\omega_1) = \kappa \text{ but } B'(t)(\omega_2) \neq \kappa. \tag{2.4}$$

From (2.4) there are two cases:

(a) there exists u such that $u \in B'(t)(\omega_2)$, but $u \notin \kappa$.
(b) there exists u such that $u \in \kappa$, but $u \notin B'(t)(\omega_2)$.

If (a) holds,

$$T(\mathbf{0}, u)(\omega_2) \leq t. \tag{2.5}$$

There exists a path γ from $\mathbf{0}$ to u such that

$$T(\gamma)(\omega_2) \leq t. \tag{2.6}$$

Since $u \notin \kappa$, any path from $\mathbf{0}$ to u has to pass through $\partial_o \kappa = \partial_o B'(t)(\omega_1)$. Let γ' be the subpath of γ from $\mathbf{0}$ to $\partial_o \kappa = \partial_o B'(t)(\omega_1)$. Then by Proposition 2,

$$T(\gamma')(\omega_1) \geq t + 1.$$

Note that the zeros and ones on the edges of κ for both $\omega_2 = (\omega(\kappa), \omega_2(\kappa^C))$ and $\omega_1 = (\omega(\kappa), \omega_1(\kappa^C))$ are the same, so

$$t + 1 \leq T(\gamma')(\omega_1) = T(\gamma')(\omega_2) \leq T(\gamma)(\omega_2). \tag{2.7}$$

By (2.6) and (2.7), (a) cannot hold. Now we assume that (b) holds. Since any path from $\mathbf{0}$ to u has to pass through $\partial_o B'(t)(\omega_2)$, by Proposition 2,

$$T(\mathbf{0}, u)(\omega_2) \geq t + 1. \tag{2.8}$$

But since $u \in \kappa$ and $B'(t)(\omega_1) = \kappa$, there exists a path γ inside $B'(t)(\omega_1)$ from $\mathbf{0}$ to u such that

$$T(\gamma)(\omega_1) \leq t.$$

Therefore,

$$T(\mathbf{0}, u)(\omega_2) \leq T(\gamma)(\omega_2) \leq t, \tag{2.9}$$

since $\gamma \subset \kappa$ and the zeros and ones on the edges of κ for both $\omega_2 = (\omega(\kappa), \omega_2(\kappa^C))$ and $\omega_1 = (\omega(\kappa), \omega_1(\kappa^C))$ are the same. The contradiction of (2.8) and (2.9) tells us that (b) cannot hold. Proposition 3 follows since (2.4) cannot hold. □

3. The linearity of a number of pivotal edges

In this section, we discuss a fixed value $0 < p < p_c$ and a fixed open interval $I_p \subset (0, p_c)$ centered at p. First, we show that the length of a route from the origin to $\partial B'(t)$ is of order t.

Lemma 1. *For a small interval $I_p \subset (0, p_c)$ centered at p, there exist positive constants $\alpha = \alpha(I_p, d)$, $C_1 = C_1(I_p, d)$, and $C_2 = C_2(I_p, d)$ such that for all t and all $p' \in I_p$,*

$$P(\exists \text{ a route} \gamma \text{from the origin to} \partial B'(t) \text{with } |\gamma| \geq \alpha t) \leq C_1 \exp(-C_2 t).$$

Proof. By Theorem 5.2 and 5.8 in [11], for all $p' \in I_p$ and for all t, there exist $C_3 = C_3(I_p, d)$ and $C_4 = C_4(I_p, d)$ such that

$$\begin{aligned} P\left(\frac{2t}{3} B_d \not\subset B(t)\right) &\leq C_3 \exp(-C_4 t) \text{ and} \\ P\left(B(t) \not\subset \frac{3t}{2} B_d\right) &\leq C_3 \exp(-C_4 t). \end{aligned} \tag{3.0}$$

If we put these two inequalities from (3.0) together, we have for all $p' \in I_p$ and all t,

$$P\left(\frac{t}{2} B_d \subset B'(t) \subset 2t B_d \text{ for all large } t\right) \geq 1 - C_3 \exp(-C_4 t). \tag{3.1}$$

On the event of

$$\{\exists \text{ a route} \gamma \text{from the origin to} \partial B(t) \text{with} |\gamma| \geq \alpha t\} \cap \left\{\frac{t}{2} B_d \subset B(t) \subset 2t B_d\right\},$$

we can assume that there exists a route γ from the origin to some vertex $u \in \partial B'(t)$ with

$$\frac{t}{2} \leq d(\mathbf{0}, u) \leq 2t \text{ and } |\gamma| \geq \alpha t$$

such that

$$t = T(\mathbf{0}, \partial B'(t)) = T(\gamma) = T(\mathbf{0}, u).$$

Therefore, by (3.1),

$$\begin{aligned} & P(\exists \text{ a route} \gamma \text{from the origin to} \partial B'(t) \text{with } T(\gamma) \leq t, |\gamma| \geq \alpha t) \\ \leq & \sum_{t/2 \leq d(\mathbf{0},u) \leq 2t} P(\exists \text{ a route} \gamma \text{from the origin to u with } T(\gamma) \\ & \leq t, |\gamma| \geq \alpha t) + C_3 \exp(-C_4 t). \end{aligned}$$

Proposition 5.8 in [11] tells us that there exist positive constants $\beta(I_p, d)$, $C_5 = C_5(I_p, d)$, and $C_6 = C_6(I_p, d)$ such that for all $p' \in I_p$ and t,

$$\begin{aligned} & P(\exists \text{a self-avoiding path} \gamma \text{from } (0,0) \text{to y that contains n edges, but with} \\ & T(\gamma) \leq \beta n) \leq C_5 \exp(-C_6 n), \end{aligned} \tag{3.2}$$

where n is the largest integer less than t. If we take a suitable $\alpha = \alpha(I_p, d)$ together with these two observations, we have for all $p' \in I_p$,

$$P(\exists \text{ a route} \gamma \text{ from the origin to} \partial B'(t) \text{ with } |\gamma| \geq \alpha t) \leq (2t)^d C_5 \exp(-C_6 t).$$

Lemma 1 follows. □

To show the theorems, we may concentrate on a "regular" set satisfying (3.1). Here we give the following precise definition. Given a deterministic connected finite set $\Gamma = \Gamma_t \subset \mathbf{R}^d$, Γ is said to be *regular* if there exists t such that

$$\frac{t}{2} B_d \subset \Gamma \subset 2t B_d. \tag{3.3}$$

For a regular set Γ, we denote by

$$T_\Gamma(\mathbf{0}, \partial\Gamma) = \inf\{T(\gamma) : \gamma \subset \Gamma' \text{ is a path from the origin to some vertex of} \partial\Gamma'\}.$$

Now we try to compute the derivative of $ET_\Gamma(\mathbf{0}, \partial\Gamma)$ in p for a regular set Γ. As a result, we have

$$ET_\Gamma(\mathbf{0}, \partial\Gamma) = \sum_{i \geq 1} P(T_\Gamma((0,0), \partial\Gamma) \geq i).$$

An event $\mathcal{A}$ is said to be increasing if

$$1 - I_{\mathcal{A}}(\omega) \leq 1 - I_{\mathcal{A}}(\omega') \text{ whenever } \omega \leq \omega',$$

where $I_{\mathcal{A}}$ is the indicator of $\mathcal{A}$. Note that Γ is a finite set, so $\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp}$ exists. We have

$$\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp} = \sum_{i\geq 1} \frac{dPT_\Gamma(\mathbf{0},\partial\Gamma)\geq i}{dp}. \tag{3.5}$$

Note that

$$\{T_\Gamma(\mathbf{0},\partial\Gamma)\geq i\}$$

is decreasing, so by Russo's formula

$$\frac{dE_pT_\Gamma(\mathbf{0},\partial\Gamma)}{dp} = -\sum_{i\geq 1}\sum_{e\in\Gamma} P(\{T_\Gamma(\mathbf{0},\partial\Gamma)\geq i\}(e)), \tag{3.6}$$

where $\{T_\Gamma(\mathbf{0},\partial\Gamma)\geq i\}(e)$ is the event that e is a pivotal edge for $\{T_\Gamma(\mathbf{0},\partial\Gamma)\geq i\}$. In fact, given a configuration ω, e is said to be a pivotal edge for $\{T_\Gamma(\mathbf{0},\partial\Gamma)(\omega)\geq i\}$ if $t(e)(\omega)=1$ and

$$T_\Gamma(\mathbf{0},\partial\Gamma)(\omega')=i-1,$$

where ω' is the configuration that $t(b)(\omega)=t(b)(\omega')$ for all edges $b\in\Gamma$ except e and $t(e)(\omega')=0$. The event $\{T_\Gamma(\mathbf{0},\partial\Gamma)\geq i\}(e)$ is equivalent to the event that there exists a route of $T_\Gamma(\mathbf{0},\partial\Gamma)$ with passage time i passing through e and $t(e)=1$. With this observation,

$$\begin{aligned}
&\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp}\\
&= -\sum_{i\geq 1}\sum_{e\in\Gamma} P(\exists \text{ a route of} T_\Gamma(\mathbf{0},\partial\Gamma)\text{passing through} e \text{with} T_\Gamma(\mathbf{0},\partial\Gamma)\\
&= i \text{ and } t(e)=1)\\
&= -\sum_{e\in\Gamma} P(\exists \text{ a route of} T_\Gamma(\mathbf{0},\partial\Gamma)\text{passing through e and} t(e)=1).
\end{aligned}$$

Let K_Γ be the number of edges $\{e\}\subset\Gamma'$ such that a route from the origin to $\partial\Gamma'$ passes through e and $t(e)=1$. We have

$$-\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp} = E(K_\Gamma). \tag{3.7}$$

Now we can give an upper bound for $E(K_\Gamma)$ by giving an upper bound for $-\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp}$. But before doing that, we shall define the route length for $T_\Gamma(\mathbf{0},\partial\Gamma)$ by

$$N_\Gamma(\omega)=\min\{k : \text{there exists a route of} T_\Gamma(\mathbf{0},\partial\Gamma)(\omega)\text{containing k edges}\}.$$

We show that the size of N_Γ cannot be more than Ct for some constant C.

Lemma 2. *For a regular set* Γ *and the interval* I_p*, there exist positive constants* $C_i = C_i(I_p, d)$ $(i = 1, 2, 3)$ *such that for all* $p' \in I_p$ *and* t*,*

$$P(N_\Gamma \geq C_1 t) \leq C_2 \exp(-C_3 t).$$

Proof. We follow the proof of Theorem 8.2 in [20] to control N_Γ by $T_\Gamma(\mathbf{0}, \partial\Gamma)$. Let $\omega + r$ denote the time state of the lattice obtained by adding the r to $t(e)$ for each edge e. It follows from the definitions of the passage time and N_Γ that

$$T_\Gamma(\mathbf{0}, \partial\Gamma)(\omega + r) \leq T_\Gamma(\mathbf{0}, \partial\Gamma)(\omega) + r N_\Gamma(\omega). \tag{3.8}$$

If we take a negative r in (3.8), we have

$$N_\Gamma(\omega) \leq \frac{T_\Gamma(\mathbf{0}, \partial\Gamma)(\omega + r) - T_\Gamma(\mathbf{0}, \partial\Gamma)}{r}. \tag{3.9}$$

Note that Γ is regular, so $\Gamma \subset 2tB_d$. If we denote by L the segment from the origin to $\partial(2tB_d)$ along the X-axis, then L has to go through $\partial\Gamma$ somewhere since $\Gamma \subset 2tB_d$. Therefore,

$$\frac{-T_\Gamma(\mathbf{0}, \Gamma)}{r} \leq \frac{-T(L)}{r} \leq \frac{-2t}{\mu r}. \tag{3.10}$$

If we can show that for some $r < 0$ there exist constants $C_4 = C_4(I_p, d)$ and $C_5 = C_5(I_p, d)$ such that for all $p' \in I_p$ and all t,

$$P(T_\Gamma(\mathbf{0}, \partial\Gamma)(\omega + r) \leq 0) \leq C_4 \exp(-C_5 t), \tag{3.11}$$

then by (3.9) and (3.10), Lemma 2 holds. Therefore, to show Lemma 2, we need to show (3.11). Note that Γ is a finite connected set, so for each ω, there exists $x = x(\omega) \in \partial\Gamma$ such that

$$T_\Gamma(\mathbf{0}, \partial\Gamma)(\omega + r) = T_\Gamma(\mathbf{0}, x)(\omega + r) \geq T(\mathbf{0}, x)(\omega + r).$$

Since $x \in \partial\Gamma$ and Γ is regular, then

$$t/2 \leq d(\mathbf{0}, x) \leq 2t.$$

We have

$$\begin{aligned} P(T_\Gamma(\mathbf{0}, \partial\Gamma)(\omega + r) \leq 0) &\leq P(T(\mathbf{0}, x(\omega))(\omega + r) \leq 0) \\ &\leq \sum_{t/2 \leq d(\mathbf{0}, y) \leq 2t} P(T(\mathbf{0}, y)(\omega + r) \leq 0). \end{aligned} \tag{3.12}$$

Therefore, by (3.2) and (3.12), we take β in (3.2) and $|r|$ small with $r < 0$ and $\beta > |r| > 0$ to obtain for all $p' \in I_p$,

$$\begin{aligned} &\sum_{t/2 \leq d(\mathbf{0}, y) \leq 2t} P(T(\mathbf{0}, y)(\omega + r) \leq 0) \\ &\leq \sum_{t/2 \leq d(\mathbf{0}, y) \leq 2t} P(\exists \text{ a self-avoiding path} \gamma \text{from } \mathbf{0} \text{ to } y \text{ that contains n edges,} \\ &\qquad \text{but with } T(\gamma)(\omega) \leq 2\beta n) \\ &\leq C_6 t^d \exp(-C_7 t), \end{aligned} \tag{3.13}$$

where n is the largest integer less than t. Therefore, (3.11) follows from (3.13). □

With Lemma 2, we are ready to give an upper bound for $-\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp}$.

Lemma 3. *For a regular set* Γ*, there exists a constant* $C(I_p, d)$ *such that for all* $p' \in I_p$ *and* t*,*

$$-\frac{dET_\Gamma(\mathbf{0},\partial\Gamma)}{dp} \leq Ct.$$

Proof. We assign $s(e) \geq t(e)$ either zero or one independently from edge to edge with probabilities $p-h$ or $1-(p-h)$ for a small number $h>0$, respectively. With this definition,

$$\begin{aligned} P(s(e)=1, t(e)=0) &= P(s(e)=1) - P(s(e)=1, t(e)=1) \\ &= P(s(e)=1) - P(t(e)=1) = 1-(p-h)-(1-p) = h. \end{aligned} \tag{3.14}$$

Let γ^t be a route for $T_\Gamma^t(\mathbf{0},\partial\Gamma)$ with time state $t(e)$ and let γ^s be a route for $T_\Gamma^s(\mathbf{0},\partial\Gamma)$ with time state $s(e)$. Here we pick γ^t such that

$$|\gamma^t| = N_\Gamma.$$

For each edge $e \in \gamma^t$, if $t(e)=1$, then $s(e)=1$. If $t(e)=0$ but $s(e)=1$, we just add one for this edge. Therefore,

$$T_\Gamma^s(\mathbf{0},\partial\Gamma) \leq T(\gamma^t) + \sum_{e\in\gamma^t} I_{(t(e)=0,s(e)=1)}. \tag{3.15}$$

Clearly, γ^t may not be unique, so we select a unique route from these γ^t. We still write γ^t for the unique route without loss of generality. By (3.15) and this selection,

$$ET_\Gamma^s(\mathbf{0},\partial\Gamma) \leq ET(\gamma^t) + \sum_\beta \sum_{e\in\beta} P(t(e)=0, s(e)=1, \gamma^t=\beta), \tag{3.16}$$

where the first sum in (3.16) takes over all possible paths β from $\mathbf{0}$ to $\partial\Gamma'$. Let us estimate

$$\sum_\beta \sum_{e\in\beta} P(t(e)=0, s(e)=1, \gamma^t=\beta).$$

Since Γ is regular, the longest path from $(0,0)$ to $\partial\Gamma'$ is less than $(2t)^d$. By Lemma 2, there exist $C_1 = C_1(I_p,d)$, $C_2(I_p,d)$, and $C_3(I_p,d)$ such that

$$\begin{aligned} &\sum_\beta \sum_{e\in\beta} P(t(e)=0, s(e)=1, \gamma^t=\beta) \\ &\quad \leq \sum_{|\beta|\leq C_1 t} \sum_{e\in\beta} P(t(e)=0, s(e)=1, \gamma^t=\beta) + C_2 t^d \exp(-C_3 t). \end{aligned} \tag{3.17}$$

Note that the value of $s(e)$ may depend on the value of $t(e)$, but not on the other values of $t(b)$ for $b \neq e$, so by (3.14),

$$\begin{aligned} & P(t(e)=0, s(e)=1, \gamma^t=\beta) = P(s(e)=1 \mid t(e)=0, \gamma^t=\beta) \\ & P(t(e)=0, \gamma^t=\beta) \\ & \qquad \leq P(s(e)=1 \mid t(e)=0) P(\gamma^t=\beta) = hp^{-1} P(\gamma^t=\beta). \end{aligned} \tag{3.18}$$

By (3.18), we have

$$\begin{aligned} & \sum_{|\beta| \leq C_1} \sum_{e \in \beta} P(t(e)=0, s(e)=1, \gamma^t=\beta) \\ & \qquad \leq \sum_{|\beta| \leq C_1 t} \sum_{e \in \beta} hp^{-1} P(\gamma^t=\beta) \leq C_1 htp^{-1}. \end{aligned} \tag{3.19}$$

By (3.17) and (3.19), there exists $C_4 = C_4(I_p, d)$ such that

$$E(T_\Gamma^s(\mathbf{0}, \partial\Gamma)) \leq E(T_\Gamma^t(\mathbf{0}, \partial\Gamma)) + C_4 th. \tag{3.20}$$

If we set

$$f(p) = E(T_\Gamma(\mathbf{0}, \partial\Gamma)) \text{for time state} t(e) \text{with} P(t(e)=0) = p,$$

then by (3.20),

$$-\frac{df(p)}{dp} = \lim_{h \to 0} -\frac{f(p-h) - f(p)}{-h} \leq C_4 t. \tag{3.21}$$

Therefore, we have

$$-\frac{dET_\Gamma(\mathbf{0}, \partial\Gamma)}{dp} = -\frac{df(p)}{dp} \leq C_4 t, \tag{3.22}$$

so Lemma 3 follows from (3.22). □

Together with (3.7) and Lemma 3, we have the following proposition.

Proposition 4. *If $0 < p < p_c$, then for a regular set Γ there exists a constant $C = C(p)$ such that*

$$EK_\Gamma \leq Ct.$$

4. Proof of Theorem 1

In this section, we only show Theorem 1 for $d = 3$. The same proof for $d > 3$ can be adapted directly. Given a fixed set $\Gamma \subset \mathbf{R}^3$ defined in Section 2, $\Gamma' \subset \mathbf{Z}^3$ is the set of all vertices contained in Γ, where

$$\Gamma' \subset \Gamma \subset \{v + (-1, 1)^3 : v \in \Gamma'\}. \tag{4.0}$$

Suppose that there exists a deterministic set Γ such that

$$F(B(t), \Gamma) \leq \delta \log t. \tag{4.1}$$

Then (4.1) means that

$$\Gamma^-_{\delta \log t} \subset B(t) \subset \Gamma^+_{\delta \log t},$$

where

$$\Gamma^+_l = \{v \in \mathbf{R}^3 : d(v, \Gamma) \leq l\} \text{ and } \Gamma^-_l = \{v \in \Gamma : d(v, \partial\Gamma) \geq l\}.$$

We first show that if $\Gamma^+_{\delta \log t}$ does not satisfy the regularity condition in (3.3), then the probability of the event in (4.1) is exponentially small. We assume that

$$\Gamma^+_{\delta \log t} \not\subset 2t B_d. \tag{4.2}$$

If

$$F(B(t), \Gamma) \leq \delta \log t \text{ with } \delta \log t < \frac{t}{3}, \tag{4.3}$$

then we claim that

$$B(t) \not\subset \frac{3t}{2} B_d. \tag{4.4}$$

To see (4.4), note that

$$B(t) \subset \frac{3t}{2} B_d \text{ implies that } \Gamma^+_{\delta \log t} \subset 2t B_d. \tag{4.5}$$

Therefore, (4.4) follows from (4.2). Under (4.2), by (3.0) there exist $C_1(p, d)$ and $C_2(p, d)$ such that

$$P(F(B(t), \Gamma) \leq \delta \log t) \leq C_1 \exp(-C_2 t). \tag{4.6}$$

Similarly, if we assume that $(t/2)B_d \not\subset \Gamma^+_{\delta \log t}$ for a set Γ, we have

$$P(F(B(t), \Gamma) \leq \delta \log t) \leq C_1 \exp(-C_2 t). \tag{4.7}$$

With (4.6) and (4.7), and with $\Gamma^+_{\delta \log t}$ not satisfying the regularity condition in (3.3), we have

$$P((F(B(t), \Gamma) \leq \delta \log t) \leq C_1 \exp(-C_2 t). \tag{4.8}$$

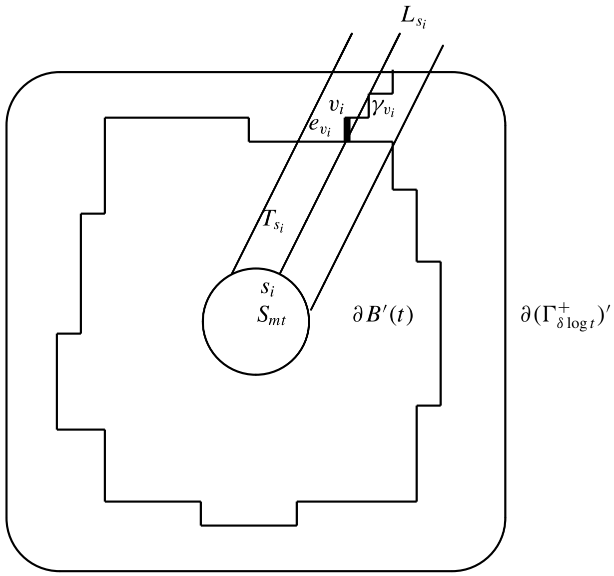

Fig. 1. The graphs show S_{mt}, $\partial B'(t)$, $\partial(\Gamma^+_{\delta \log t})'$, the cylinder T_{s_i} with the center at L_{s_i}, the pivotal edge e_{v_i}, and the zigzag path γ_{v_i} from v_i to $\partial(\Gamma^+_{\delta \log t})'$

Now we focus on $\Gamma^+_{\delta \log t}$ satisfying (3.3). We need to show that under (4.1) there are of order t^2 disjoint zero-paths from $\partial B(t)$ to $\Gamma^+_{\delta \log t}$. To accomplish this, let S_{mt} denote a sphere with the center at the origin and a radius tm for a small but positive number m. Then by (3.1), for a suitable $m > 0$,

$$P(S_{mt} \subset B(t) \subset 2tB_d) \geq 1 - C_1 \exp(-C_2 t). \tag{4.9}$$

Here we select the sphere S_{mt} without a special purpose since the sphere is easy to describe. For each $s \in \partial S_{mt}$, let L_s be the normal line passing though s, that is, the line orthogonal to the tangent plane of S_{mt} at s. We denote the cylinder with the center at L_s (see Fig. 1) by

$$T_s(M) = \{(x, y, z) \in \mathbf{R}^3 : d((x, y, z), L_s) \leq M\} \text{ for some constant } M > 0.$$

Now we work on the regular polyhedron with ct^2 faces embedded on S_{mt}, where ct^2 is an integer and $c = c(m, M)$ is a small number such that the radius of each face of the regular polyhedron is larger than M. We denote the center of each face in the regular polyhedron by $\{s_i\}_{i=1}^{ct^2}$. By this construction, we have at least ct^2 disjoint cylinders $\{T_s(M)\}$. We denote them by $\{T_{s_i}(M)\}_{i=1}^{ct^2}$.

For each s_i, we may take M large such that there exists a path $\gamma_{s_i} \subset Z^d \cap T_{s_i}(M)$ from some vertex of $\mathbf{Z}^3$ in S_{mt} to ∞. To see the existence of such a path, if L_{s_i} is the ray going along the coordinate axis, we simply use L_{s_i} as the path. If it is not, we can construct a zigzag path in $\mathbf{Z}^d$ next to L_{s_i} from s_i to ∞ (see Fig. 1).

In fact, we may take $M = 2$ to keep our zigzag path inside $T_{s_i}(M)$. For simplicity, we use T_{s_i} to denote $T_{s_i}(2)$. There might be many such zigzag paths, so we just select one in a unique manner. We denote by $u_i \in \mathbf{Z}^d$, with $d(u_i, s_i) \leq 2$, the initial vertex in γ_{s_i}. Since γ_{s_i} is next to L_{s_i}, for any point x on the ray L_{s_i}, there is v in γ_{s_i} with $d(v, x) \leq 2$. Furthermore, by a simple induction we conclude that

$$\text{the number of vertices from } u_i \text{ to v along } \gamma_{s_i} \text{is less than } 2d(s_i, x). \quad (4.10)$$

Thus, (4.10) tells us that the length of γ_{s_i} is linear to the length of L_{s_i}. Since γ_{s_i} is from S_{mt} to $\partial(\Gamma^+_{\delta \log t})'$, it must reach outside $B'(t)$ from its inside. Let v_i be the last vertex from these intersections of $\partial_o B'(t) \cap \gamma_{s_i}$ and let γ_{v_i} be the piece of γ_{s_i} outside $B'(t)$ from v_i to $\partial(\Gamma^+_{\delta \log t})'$ (see Fig. 1). On $F(B(t), \Gamma) \leq \delta \log t$ for a regular Γ, we know that

$$B'(t) \subset (\Gamma^+_{\delta \log t})'.$$

Therefore, by our construction (see Fig. 1),

$$\gamma_{v_i} \subset (\Gamma^+_{\delta \log t})'. \quad (4.11)$$

Also, by our special construction in (4.10), we have

$$|\gamma_{v_i}| \leq 2\delta \log t. \quad (4.12)$$

When $B'(t) = \kappa$ for a fixed vertex set κ, then γ_{v_i} is a fixed path from $\partial_o \kappa$ to $\partial(\Gamma^+_{\delta \log t})'$ with a length less than $2\delta \log t$. Therefore, on $B'(t) = \kappa$,

$$P(\gamma_{v_i} \text{ is a zero-path}) \geq p^{2\delta \log t}. \quad (4.13)$$

We say T_{s_i} is *good* if there exists such a zero-path γ_{v_i}. On $B'(t) = \kappa$,

let $M(\Gamma, \kappa)$ be the number of such good cylinders T_{s_i}. By (4.13), we have

$$EM(\Gamma, \kappa) \geq (ct^2) p^{2\delta \log t} = ct^{2+2\delta \log p}. \quad (4.14)$$

On $B'(t) = \kappa$, note that the event that T_{s_i} is good depends on the zeros and ones of the edges being inside T_{s_i} but outside $\partial_o \kappa$. Note also that T_{s_i} and T_{s_j} are disjoint for $i \neq j$. With these observations, by the Azuma-Hoeffding inequality, there exist $C_i = C_i(p, d)$ for $i = 1, 2$ such that

$$P(M(\Gamma, \kappa) \leq ct^{(2+2\delta \log p)}/2) \leq C_1 \exp\left(-C_2 t^{(2+4\delta \log p)}\right). \quad (4.15)$$

We denote by

$$\mathcal{D}(\Gamma, \kappa) = \{M(\Gamma, \kappa) \geq ct^{(2+2\delta \log p)}/2\}.$$

Note that

$$\mathcal{D}(\Gamma, \kappa) \text{ only depends on the zeros and ones outside } \partial_o \kappa. \quad (4.16)$$

By Proposition 3 and (4.16),

$$\{B'(t) = \kappa\} \text{ and } \mathcal{D}(\Gamma, \kappa) \text{ are independent.} \quad (4.17)$$

By Proposition 2, any route from $(0, 0, 0)$ to v_i in $B'(t) \cup \partial_o B'(t)$ has a passage time $t + 1$. We just pick one from these routes and denote it by $\gamma(\mathbf{0}, v_i)$. On $F(B(t), \Gamma) \leq \delta \log t$, if T_{s_i} is good, there exists a zero-path γ_{v_i} from v_i to $\partial(\Gamma^+_{\delta \log t})'$. This implies that there exists a path

$$\gamma(\mathbf{0}, \partial\Gamma^+_{\delta \log t}) = \gamma(\mathbf{0}, v_i) \cup \gamma_{v_i}$$

from $(0, 0, 0)$ to $\partial(\Gamma^+_{\delta \log t})'$ with a passage time $t + 1$ and that the path passes through the edge adjacent to v_i between $\partial B(t)$ and $\partial_o B(t)$. On the other hand, note that any path from the origin to $\partial(\Gamma^+_{\delta \log t})'$ has to pass through $\partial_o B'(t)$ first, so by Proposition 2 it has to spend at least passage time $t + 1$. Therefore, if we denote by e_{v_i} the edge adjacent to v_i from $\partial B(t)$ to $\partial_o B(t)$, then the path $\gamma(\mathbf{0}, \partial\Gamma^+_{\delta \log t})$ with passage time $T((0, 0, 0), \partial\Gamma^+_{\delta \log t})$ passes through e_{v_i} and $t(e_{v_i}) = 1$. By (4.11) and

$$B'(t) \subset (\Gamma^+_{\delta \log t})',$$

the path $\gamma(\mathbf{0}, \partial\Gamma^+_{\delta \log t})$ has to stay inside $(\Gamma^+_{\delta \log t})'$. These observations tell us that e_{v_i} is a pivotal edge for $T_{\Gamma^+_{\delta \log t}}((0, 0), \Gamma^+_{\delta \log t})$. Therefore, on $F(B(t), \Gamma) \leq \delta \log t$, if T_{s_i} is good,

$$T_{s_i} \text{ contains at least one pivotal edge for } T_{\Gamma^+_{\delta \log t}}((0, 0), \Gamma^+_{\delta \log t}). \tag{4.18}$$

With these preparations, we now show Theorem 1.

Proof of Theorem 1. If $\Gamma^+_{\delta \log t}$ is not regular,

$$1/2tB_d \not\subset \Gamma^+_{\delta \log t} \text{ or } \Gamma^+_{\delta \log t} \not\subset 2tB_d,$$

by (4.8) there are $C_1 = C_1(p, d)$ and $C_2(p, d)$ such that

$$P_p(F(B(t), \Gamma) \leq \delta \log t) \leq C_1 \exp(-C_2 t). \tag{4.19}$$

Now we only need to focus on a regular $\Gamma^+_{\delta \log t}$.

$$P_p(F(B(t), \Gamma) \leq \delta \log t) = \sum_\kappa P(F(B(t), \Gamma) \leq \delta \log t, B'(t) = \kappa), \tag{4.20}$$

where the sum takes over all possible sets κ. For each fixed κ, by (4.17) and (4.15), there exist $C_3 = C_3(p, d)$ and $C_4 = C_4(p, d)$ such that

$$\begin{aligned}
&\sum_\kappa P(F(B(t), \Gamma) \leq \delta \log t, B'(t) = \kappa) \\
&\quad \leq \sum_\kappa P(F(B(t), \Gamma) \leq \delta \log t, B'(t) = \kappa, \mathcal{D}(\Gamma, \kappa)) + C_3 \exp\left(-C_4 t^{2+4\delta \log p}\right).
\end{aligned}$$

By (4.18),

$$\sum_{\kappa} P\left(F(B(t), \Gamma) \leq \delta \log t,\, B'(t) = \kappa,\, \mathcal{D}(\Gamma, \kappa)\right)$$
$$\leq \sum_{\kappa} P\left(K_{\Gamma^+_{\delta \log t}} \geq \frac{ct^{2+2\delta \log p}}{2},\, B'(t) = \kappa\right). \tag{4.21}$$

We combine (4.20) and (4.21) together to have

$$P(F(B(t), \Gamma) \leq \delta \log t) \leq P\left(K_{\Gamma^+_{\delta \log t}} \geq \frac{ct^{2+2\delta \log p}}{2}\right)$$
$$+C_5 \exp\left(-C_6 t^{2+4\delta \log p}\right) \tag{4.22}$$

for $C_5 = C_5(p, d)$ and $C_6 = C_6(p, d)$. By Markov's inequality and Proposition 4, if we select a suitable $0 < \delta < 1/(-2 \log p)$, for a regular Γ there exists $C_7 = C_7(p, d, \delta)$ such that

$$P(F(B(t), \Gamma) \leq \delta \log t) \leq C_7 t^{-1-2\delta \log p}. \tag{4.23}$$

Theorem 1 follows from (4.19) and (4.23).

5. Proof of Theorem 2.

Since Γ is regular, by Proposition 4,

$$EK_\Gamma \leq Ct. \tag{5.1}$$

By (5.1) for a large positive number M,

$$E|R_\Gamma| \leq E(|R_\Gamma|; |R_\Gamma| \geq MK_\Gamma) + MCt. \tag{5.2}$$

Now we estimate $E(|R_\Gamma|; |R_\Gamma| \geq MK_\Gamma)$ by using the method of renormalization in [13]. We define, for integer $k \geq 1$ and $u \in \mathbf{Z}^d$, the cube

$$B_k(u) = \prod_{i=1}^{d} [ku_i, ku_i + k)$$

with the lower left-hand corner at ku and the fattened R_Γ by

$$\hat{R}_\Gamma(k) = \{u \in \mathbf{Z}^d : B_k(u) \cap R_\Gamma \neq \emptyset\}.$$

By our definition,

$$|\hat{R}_\Gamma(k)| \geq \frac{|R_\Gamma|}{k^d}. \tag{5.3}$$

For each cube $B_k(u)$, there are at most 4^d neighbor cubes, where we count diagonal neighbor cubes. We say that these neighbors are connected to $B_k(u)$ and denote by $\bar{B}_k(u)$ the vertex set of $B_k(u)$ and all its neighbor cubes. Note that, by the definition, R_Γ is a connected set that contains the origin, so $\hat{R}_\Gamma(k)$ is also connected in the sense of the connection of two of its diagonal vertices. If Γ is regular, then $|R_\Gamma| \geq t/2$. By (5.3), we have

$$P(|R_\Gamma| \geq MK_\Gamma) = \sum_{m \geq Mt/(2k^d)} P(|R_\Gamma| \geq MK_\Gamma, |\hat{R}_\Gamma(k)| = m). \tag{5.4}$$

We say a cube $B_k(u)$ for $u \in \hat{R}_\Gamma(k)$ is *bad* if there does not exist an edge $e \in \bar{B}_k(u) \cap R_\Gamma$ such that $t(e) = 1$. Otherwise, we say the cube is *good*. Let $\mathcal{B}_k(u)$ be the event that $B_k(u)$ is bad and let D_Γ be the number of bad cubes $B_k(u)$ for $u \in \hat{R}_\Gamma$.

If $\mathcal{B}_k(u)$ occurs, there is a zero-path $\gamma_k \subset R_\Gamma$ from $\partial B_k(u)$ to $\partial \bar{B}_k(u)$. Note that $p < p_c$, by Theorem 5.4 in [5], there exist $C_1 = C_1(p,d)$ and $C_2 = C_2(p,d)$ such that for fixed $B_k(u)$,

$$P(\mathcal{B}_k(u)) \leq C_1 \exp(-C_2 k). \tag{5.5}$$

On $\{|\hat{R}_\Gamma(k)| = m, |R_\Gamma| \geq MK_\Gamma\}$, if $2(4k)^d < M$, we claim

$$D_\Gamma \geq \frac{m}{2}. \tag{5.6}$$

To see this, suppose that there are $m/2$ good cubes. For each good cube $B_k(u)$, $\bar{B}_k(u)$ contains an edge $e \in R_\Gamma$ with $t(e) = 1$, so e is a pivotal edge. Note that each $B_k(u)$ has at most 4^d neighbor cubes adjacent to $B_k(u)$, so there are at least $\frac{m}{4^d 2}$ pivotal edges. Therefore, $K_\Gamma > \frac{m}{4^d 2}$. By (5.3) on $\{|R_\Gamma| \geq MK_\Gamma, |\hat{R}_\Gamma(k)| = m\}$,

$$|R_\Gamma| \geq MK_\Gamma \geq \frac{Mm}{4^d 2} \geq \frac{M|\hat{R}_\Gamma(k)|}{4^d 2} > |\hat{R}_\Gamma(k)| k^d. \tag{5.7}$$

The contradiction of (5.3) and (5.7) tells us that (5.6) holds.

By this observation and (5.4), we take $2(4k)^d < M$ to obtain

$$P(|R_\Gamma| \geq MK_\Gamma) = \sum_{m \geq Mt/(2k^d)} P(|R_\Gamma| \geq Mt, |\hat{R}_\Gamma(k)| = m, D_\Gamma \geq m/2). \tag{5.8}$$

Now we fix $\hat{R}_\Gamma$ to have

$$P(|R_\Gamma| \geq MK_\Gamma) = \sum_{m \geq Mt/(2k^d)} \sum_{\kappa_m} P(|R_\Gamma| \geq MK_\Gamma, \hat{R}_\Gamma(k) = \kappa_m, D_\Gamma \geq m/2), \tag{5.9}$$

where κ_m is a fixed connected vertex set with m vertices, and the second sum in (5.9) takes over all possible such κ_m. For each fixed $\hat{R}_\Gamma(k) = \kappa_m$, there are at most $\binom{m}{i}$ choices for these $i, i = m/2, \cdots, m$, bad cubes, so by (5.5),

$$P(|R_\Gamma| \geq MK_\Gamma, \hat{R}_\Gamma(k) = \kappa_m, D_\Gamma \geq m/2) \leq C_1 m \binom{m}{m/2} \exp(-C_2 km/2). \tag{5.10}$$

Substitute the upper bound of (5.10) for each term of the sums in (5.9) to obtain

$$P(|R_\Gamma| \geq MK_\Gamma) \leq \sum_{m \geq Mt/(2k^d)} \sum_{\kappa_m} C_1 m \binom{m}{m/2} \exp(-C_2 km/2). \quad (5.11)$$

As we mentioned, $\hat{R}_\Gamma$ is connected, so there are at most $(4)^{dm}$ choices for all possible κ_m. With this observation and (5.11), we have

$$\begin{aligned} P(|R_\Gamma| \geq MK_\Gamma) &= \sum_{m \geq Mt/(2k^d)} (4)^{dm} m \binom{m}{m/2} \exp(-C_2 km/2) \\ &\leq C_1 \sum_{m \geq Mt/k^d} m[4^d 2 \exp(-C_2 k/2)]^m. \end{aligned} \quad (5.12)$$

We choose k large to make

$$4^d 2 \exp(-C_2 k/2) < 1/2.$$

By (5.12), there are $C_3 = C_3(p, d)$ and $C_4 = C_4(p, d)$ such that

$$P(|R_\Gamma| \geq MK_\Gamma) \leq C_3 \exp(-C_4 t). \quad (5.13)$$

Therefore, by (5.2), note that there are at most t^{2d} vertices on Γ, so there exists $C_5 = C_5(p, d)$ such that

$$E|R_\Gamma| = C_3 t^{2d} \exp(-C_4 t) + MCt \leq C_5 t. \quad (5.14)$$

Theorem 2 follows from (5.14).

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