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Self-excitation of a nonlinear scalar field in a random medium

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ABSTRACT We discuss the evolution in time of a scalar field under the influence of a random potential and diffusion. The cases of a short-correlation in time and of stationary potentials are considered. In a linear approximation and for sufficiently weak diffusion, the statistical moments of the field grow exponentially in time at growth rates that progressively increase with the order of the moment; this indicates the intermittent nature of the field. Nonlinearity halts this growth and in some cases can destroy the intermittency. However, in many nonlinear situations the intermittency is preserved: high, persistent peaks of the field exist against the background of a smooth field distribution. These widely spaced peaks may make a major contribution to the average characteristics of the field.

A model of evolution of a scalar field in a prescribed random medium can be formulated in terms of a parabolic equation with a random rate of reproduction or extinction, U (the potential, in short):

$$\frac{\partial\varphi}{\partial t} = \mathrm{U}(t, \mathbf{x}, \omega)\varphi + æ\Delta\varphi,$$
$$\varphi(0, \mathbf{x}) = \varphi_0(\mathbf{x}) \geqslant 0, \qquad [1]$$

where æ is the diffusion coefficient and ω is the random parameter, the potential U is a certain function of t and x for every value of ω. This problem has important applications in biology and chemical kinetics, where the scalar field is usually understood as a concentration (1). Some problems of transport of impurities and equations for vector fields (2) may be reduced, under certain approximations, to equations of type **1**. For example, the transport of an impurity by a prescribed potential flow is described by the equation

$$\frac{\partial n}{\partial t} + \mathrm{div}(nv) = æ\Delta n, \qquad \mathbf{v} = \nabla\psi,$$

which is equivalent to Eq. **1** when the term $\nabla r\cdot\nabla\psi$ can be neglected in comparison with $n\nabla\psi$—e.g., at the early stages of evolution of an initially smooth distribution $n_0(\mathbf{x})$.

In the linear problem (Eq. **1**), near those positions where the potential is positive, there occurs self-excitation (exponential growth) of the field that propagates through diffusion. Random behavior of the potential results in intermittency of the field—i.e., in the appearance of concentrated structures of $\varphi(t, \mathbf{x})$ that determine the statistical averages of the field (3, 4). These structures should not be confused with structures of the caustic type (see, e.g., ref. 5) or with those studied by synergetics, whose existence is associated with nonlinear effects.

Here we show how the processes of self-excitation and intermittency are affected by nonlinearity, which models the back-action of the field on the medium. We emphasize that we consider the simplest nonlinearities, which cannot produce structures alone, without random effects. It is natural to expect that nonlinear back-action will result primarily in suppression of the growth of high maxima in the solution, thus smoothing the intermittency. However, the result proves to be less trivial and depends radically on both the time behavior of the potential and the form of nonlinearity.

For the sake of simplicity, put æ = 0; i.e., neglect the spatial diffusion of the field. We shall indicate the influence of a weak diffusion on the results separately. We consider two types of potentials. The first is characterized by a short correlation time

$$\mathrm{U}(t, \mathbf{x}, \omega) = u + \tau^{-1/2}\frac{d\mathrm{w}_t}{dt}, \qquad [2]$$

where u is the stationary value of the potential, τ is a certain characteristic time similar to the characteristic turbulent diffusion time in a random flow, w_t is the Wiener random process, $\langle \mathrm{w}_t\rangle = 0$, $\langle \mathrm{w}_t^2\rangle = t$, and $\langle\ \rangle$ denotes the ensemble average.

The time is conveniently measured in units of τ; the dimensionless Wiener process w_t is then obtained $\tau^{-1/2}\mathrm{w}_t$. The value of $\mathrm{U} - u$ obviously has the property of white noise.

Another type of potential considered here is that of a stationary potential $\mathrm{U} = \mathrm{U}(\mathbf{x}, \mathrm{w})$ that does not include time dependence. All potentials considered are statistically homogeneous in time and statistically homogeneous and isotropic in space.

(*i*) Consider first a nonlinearity that is not associated with the potential

$$\frac{\partial\varphi}{\partial t} = \mathrm{U}\varphi - \varepsilon f(\varphi), \qquad [3]$$

where ε is a certain parameter and $f(\varphi)$ is a given nonlinear function that grows sufficiently fast with φ.

The random process described by Eq. **3** with potential **2** can be considered as a one-dimensional diffusion process along a line, φ being the coordinate along that line

$$d\varphi = \varphi dw_t + (u\varphi - \varepsilon f)dt.$$

The generating operator of this diffusion process is given by

$$\hat{L} = \frac{\varphi^2}{2}\frac{d^2}{d\varphi^2} + (u\varphi - \varepsilon f)\frac{d}{d\varphi}. \qquad [4]$$

Thus $\varphi^2/2$ plays the role of the diffusion coefficient, while $-(u\varphi - \varepsilon f)$ represents the convective velocity. Let us denote by $p(t, \bar{\varphi}, \varphi)$ the fundamental solution (transition density) of the evolutionary problem posed for operator **4**, so that $p(0, \bar{\varphi}, \varphi) = \delta(\bar{\varphi} - \varphi)$ and

$$\frac{\partial p}{\partial t} = \hat{L}(\bar{\varphi})p = \hat{L}^*(\varphi)p,$$

where the asterisk denotes the adjoint operator.

Suppose that a limiting stationary density distribution exists, $\pi(\varphi) \equiv \lim_{t\to\infty} p(t, \tilde{\varphi}, \varphi)$. It obeys the following equation:

$$\hat{L}^* \pi(\varphi) = 0 \qquad [5]$$

with the natural normalization condition $\int_0^\infty \pi d\varphi = 1$. The problem **5** with the operator, adjoint to **4**, has solutions. Indeed, the equation

$$\frac{1}{2}\frac{d^2}{d\varphi^2}(\pi\varphi^2) - \frac{d}{d\varphi}\pi(u\varphi - \varepsilon f) = 0$$

has the first integral

$$\frac{1}{2}\frac{d}{d\varphi}\pi\varphi^2 = \frac{u\varphi - \varepsilon f}{\varphi^2} + \mathrm{C}_1.$$

Solutions with $\mathrm{C}_1 \neq 0$ are not integrable at $\varphi = 0$ and must be rejected. As a result, we obtain that when the function $f(\varphi)$ grows at least as φ^2, the limiting distribution $\pi(\varphi)$ is given by

$$\pi(\varphi) = \frac{\mathrm{C}}{\varphi^2}\exp\left(2\int\frac{u\varphi - \varepsilon f}{\varphi^2}\,d\varphi\right),$$

where C is the normalization constant.

Consider $f = \varphi^3$ as an example. Then

$$\pi(\varphi) = \frac{(2\varepsilon)^{u-1/2}}{\Gamma(2n-1)}\,\varphi^{2(u-1)}\exp(-\,\varepsilon\varphi^2), \qquad [6]$$

where Γ is Euler's γ function. An important condition of existence of the density $\pi(\varphi)$ follows from this as $u > \frac{1}{2}$. When $u < \frac{1}{2}$, the density π concentrates at $\varphi = 0$ and the solution decays. When $\frac{1}{2} < u < 1$, the function $\pi(\varphi)$ decreases monotonically with φ. In the limit $\varepsilon \to 0$, which corresponds to the transition to the linear problem, we obtain $\pi \to 0$. In this limit and for $u > 1$, this maximum value of the density decreases as $\varepsilon^{1/2}$. It can be shown that $\pi(\varphi)$ and all its derivatives vanish at $\varphi = 0$ when the spatial diffusion is taken into account, æ $\neq 0$. In a statistically space-homogeneous medium without spatial diffusion at large times, the solution is distributed over all positions with equal probability and density $\pi(\varphi)$; the values of φ at different positions are statistically independent. The spatial diffusion results in the establishment of spatial correlations that decay with distance at the greater rate the smaller is æ.

Statistical moments of the scalar field are

$$\langle\varphi^q\rangle = \frac{\Gamma(q + 2u - 1)}{2^{q/2}\Gamma(2u-1)}\cdot\varepsilon^{-q/2}.$$

Note that the distribution **6** for $u = 1$ coincides with the "half" of the Gaussian distribution since $\varphi \geqslant 0$ in our problem.

Although the derived distribution contains maxima of arbitrary height, the probability of high maxima is low and in general the situation is close to Gaussian statistics; intermittency is absent.

Consider now a random potential that does not depend on time—e.g., the Gaussian one with the average value u and the rms deviation σ. At those positions where the potential is positive the solution for $f = \varphi^3$ is obviously distributed as $(\mathrm{U}/\varepsilon)^{1/2}$. At those positions where $\mathrm{U} < 0$ the solution tends to zero. For the Gaussian potential the limiting distribution has the form

$$\pi(\varphi) = \frac{1}{2}\delta(\varphi) + \frac{\varepsilon\varphi}{\sqrt{\pi\sigma}}\exp\left(-\frac{\varepsilon^2\varphi^4}{2\sigma}\right), \qquad [7]$$

where diffusion is neglected and the simplification $u = 0$ is adopted. This distribution has two characteristic maxima, at the origin and at $\varphi_* = (\sigma/2\varepsilon^2)^{1/4}$. Thus, we see the evidence of an island intermittency: one part of the space is occupied by the field distributed around a high maximum φ_* while the remaining half is empty. When the average value of U is nonvanishing, the relation between the field and empty regions is different and the problem arises of the percolation structure of this distribution, similar to the problem considered in ref. 6. A weak diffusion smoothes the δ-function-like maximum of distribution **7** but does not affect its intermittent character.

(*ii*) Now let us introduce nonlinearity into the potential term

$$\frac{\partial\varphi}{\partial t} = g(\varphi)\mathrm{U}\varphi.$$

In the simplest case, nonlinearity is described by a step function: $g = 1$ for $\varphi < \varphi_m$ and $g = 0$ for $\varphi \geqslant \varphi$. Let us first consider the short-correlation potential. For $u > \frac{1}{2}$, the solution increases exponentially until $\varphi = \varphi_m$ and remains constant afterward. The case with $u < \frac{1}{2}$ is less trivial. In this case, a typical field realization decays exponentially at the linear stage but the higher moments increase; i.e., sparse peaks persist. Those peaks that rise above φ_m are preserved at this level. As a result, the following stationary picture arises: at randomly scattered points there are maxima of height φ_m while the field between them decays exponentially. In the simple example considered, all peaks are as identical as randomly scattered telegraph poles whose positions are fixed in time. To remove this restriction and obtain some distribution of maxima in height and its evolution in time, it is sufficient to consider the function decreasing at infinity. Consider, for instance, $g = (1 + \varphi/\varphi_m)^{-1}$. To simplify the exposition, put $u = 0$. Then

$$\varphi(t, \mathbf{x}, \omega) = \begin{cases} \varphi_0\exp(w_t - t/2), & \varphi \leqslant \varphi_m \\ \varphi_m w_t, & \varphi \geqslant \varphi_m. \end{cases}$$

It is clear that at those positions where the solution has not reached the level of φ_m, it decays exponentially in time almost everywhere except at sparse peaks. Meanwhile, at those positions where the solution has exceeded the level φ_m at the linear stage, it can still exceed this level for a prolonged period but, sooner or later, it decreases to φ_m, again because of recovery of the one-dimensional Wiener process, and begins to decay. Of course, oscillations are possible around the level but the larger the number of such oscillations, the lower their probability. As a result, a typical realization decays in time. However, it contains sparse peaks against a smooth background and these peaks determine the values of averaged quantities; i.e., the intermittency is present.

Indeed, the average value $\langle\varphi\rangle$ remains constant in time notwithstanding the fact that a typical realization decays. This can be seen in the following way. Let the solution reach and exceed the level φ_m at moment t_m—i.e., to exceed unity and, say, become equal to two. Since w_t is a random process with independent increments, we can choose "two" as its starting point. Then the probability P that the trajectory does not reach "unity" up to moment t is proportional to $(t - t_m)^{1/2}$ since, with overwhelming probability, the random process concentrates somewhere in a length interval of order $(t - t_m)^{1/2}$ in this time. Since the typical realization of the process w_t is also of order $t^{1/2}$, the mean value is $\langle\varphi\rangle \sim P\varphi_m w_{t-t_m} \simeq$ constant. Actually, this fact is true for any dependence of $g(\varphi)$ because it follows from the fact that the average value of the Ito stochastic integral $\int g(\varphi)\varphi(w_t)dw_t$ vanishes (7).

Other statistical moments increase as $\langle\varphi^q\rangle \sim (t^{-1/2})^q t^{1/2} \sim t^{(q-1)/2}$. One can expect that this weak, power-law growth can be stabilized by the weak diffusion æ. The resulting distribution of the peaks over heights is of the Gaussian type. However, the distribution is far from being Gaussian in general, since between the sparse peaks, whose height is inversely proportional to a power of æ, the solution is small, being determined by diffusion from widely separated peaks. Meanwhile, in the Gaussian distribution a typical realization behaves as the average plus or minus the rms deviation. Of course, for sufficiently large æ the diffusion smoothes out the peaks and the intermittency disappears.

For the case of a stationary potential and the step-function $g(\varphi)$, we again consider regions with positive and negative values of the potential to obtain a limiting distribution of type **7**:

$$\pi(\varphi) = p\delta(\varphi) + (1-p)\delta(\varphi - \varphi_m),$$

where p is the probability that the potential has a negative value. For more complicated forms of $g(\varphi)$, the limiting distribution arises only when at least weak diffusion is taken into account. It also has two maxima that are not δ-function-like now while the quantity analogous to φ_m is determined by the diffusion coefficient æ.

Thus, for a stationary potential, the island intermittency arises independently of the form of the local nonlinear influence.

(*iii*) The intermittency is still more pronounced when the nonlinear influence has a nonlocal nature—e.g., when the nonlinear function in **3** depends on $\langle\varphi\rangle$ or $\langle\varphi^q\rangle$ rather than on φ. Obviously, this leads to stabilization of $\langle\varphi\rangle$ or $\langle\varphi\rangle^q$, respectively. All moments with $q < q_0$ and a typical field realization decay exponentially while moments with $q > q_0$ continue their intermittent growth typical of the linear problem (q_0 is a certain fixed number).

Thus, the behavior of nonlinear solutions depends radically on the time behavior of the potential and on the form of nonlinearity. In realistic problems, there commonly appears a condition of conservation of some moment—e.g., of the average value in the problem of the breeding and death of bacteria that struggle for food (1) or of the average square (the energy) in the problem of radiowave propagation in a random medium (8). In the vectorial problem of the hydromagnetic dynamo, the nonlinear influence of the magnetic field on the flow is equivalent to the influence on the potential in a scalar problem (9), which makes it reasonable to expect that the intermittency of a magnetic field would be preserved in a nonlinear regime. Nonlinearity can be of a combined nature, combining, say, deterministic nonlinear damping of the solution with a nonlinear influence on the potential. In this case, the properties of nonlinear solutions are determined by a competition of the two nonlinear mechanisms, the first of which tends to suppress the intermittency (for a short-correlation potential) while the second of which tends to preserve it. A quantitative detailed analysis of this competition is a promising field for future studies.

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