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Stationary solutions of SPDEs and infinite horizon BDSDEs

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Abstract

In this paper we study the existence of stationary solutions for stochastic partial differential equations. We establish a new connection between $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued solutions of backward doubly stochastic differential equations (BDSDEs) on infinite horizon and the stationary solutions of the SPDEs. Moreover, we prove the existence and uniqueness of the solutions of BDSDEs on both finite and infinite horizons, so obtain the solutions of initial value problems and the stationary solutions (independent of any initial value) of SPDEs. The connection of the weak solutions of SPDEs and BDSDEs has independent interests in the areas of both SPDEs and BSDEs.

Keywords: Backward doubly stochastic differential equations; Weak solutions; Stochastic partial differential equations; Stationary solution; Random dynamical systems

1. Introduction

Let $u:[0,\infty)\times U\times \Omega\to U$ be a measurable random dynamical system on a measurable space $(U,\mathcal{B})$ over a metric dynamical system $(\Omega,\mathscr{F},P,(\theta_t)_{t\geqslant 0})$, then a stationary solution is a $\mathscr{F}$ measurable random variable $Y:\Omega\to U$ such that (Arnold [1])

$$u\big(t,Y(\omega),\omega\big)=Y(\theta_t\omega)\quad \text{for all } t\geqslant 0 \text{ a.s.} \tag{1.1}$$

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This "one-force, one-solution" setting is a natural extension of equilibria or fixed points in deterministic systems to stochastic counterparts. The simplest nontrivial example is the Ornstein–Uhlenbeck process defined by the stochastic differential equation $du(t) = -u(t)\,dt + dB_t$. It defines a random dynamical system $u(t,u_0) = u_0 e^{-t} + \int_0^t e^{-(t-s)}\,dB_s$ and its stationary point is given by $Y(\omega) = \int_{-\infty}^0 e^s\,dB_s$. Moreover, for any u_0, $u(t,u_0,\theta_{-t}\omega) \to Y(\omega)$ as $t\to\infty$, where θ_t is the shift operator of the Brownian path: $(\theta_t B)(s) = B(t+s) - B(s)$ for any $s \in (-\infty,+\infty)$. A pathwise stationary solution describes the pathwise invariance of the stationary solution over time along the measurable and P-preserving transformation $\theta_t \colon \Omega \to \Omega$, and the pathwise limit of the solutions of random dynamical systems. Needless to say, it is one of the fundamental questions of basic importance [1,7,14,21,29,30]. For random dynamical systems generated by stochastic partial differential equations (SPDEs), such random fixed points consist of infinitely many random moving invariant surfaces on the configuration space due to the random external force pumped to the system constantly. They are more realistic models than many deterministic models as it demonstrates some complicated phenomena such as turbulence. Their existence and stability are of great interests in both mathematics and physics. However, in contrast to the deterministic dynamical systems, also due to the fact that the external random force exists at all time, the existence of stationary solutions of stochastic dynamical systems generated e.g. by stochastic differential equations (SDEs) or SPDEs, is a difficult and subtle problem. We would like to point out that there have been extensive works on stability and invariant manifolds of random dynamical systems, and researchers usually assume there is an invariant set (or a single point: a stationary solution or a fixed point, often assumed to be 0), then prove invariant manifolds and stability results at a point of the invariant set (Arnold [1] and references therein, Ruelle [28], Duan, Lu and Schaumulfuss [10,11], Li and Lu [19], Mohammed, Zhang and Zhao [21] to name but a few). But the invariant manifolds theory gives neither the existence results of the invariant set and the stationary solution nor a way to find them. In particular, for the existence of stationary solutions for SPDEs, results are only known in very few cases [7,14,21,29,30]. In [29,30], the stationary strong solution of the stochastic Burgers' equations with periodic or random forcing (C^3 in the space variable) was established by Sinai using the Hopf–Cole transformation. In [21], the stationary solution of the stochastic evolution equations was identified as a solution of the corresponding integral equation up to time $+\infty$ and the existence was established for certain SPDEs by Mohammed, Zhang and Zhao. But the existence of solutions of such a stochastic integral equations in general is far from clear.

The main purpose of this paper is to find the pathwise stationary solution of the following SPDE

$$\begin{aligned} dv(t,x) = {}& \big[\mathscr{L}v(t,x) + f\big(x, v(t,x), \sigma^*(x)Dv(t,x)\big)\big]\,dt \\ & + g\big(x, v(t,x), \sigma^*(x)Dv(t,x)\big)\,dB_t, \end{aligned} \tag{1.2}$$

without assumption that there is an invariant set. Here B is a two-sided cylindrical Brownian motion on a separable Hilbert space U_0; $\mathscr{L}$ is the infinitesimal generator of a diffusion process $X_s^{t,x}$ (solution of Eq. (2.11)) given by

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} \tag{1.3}$$

with $(a_{ij}(x)) = \sigma\sigma^*(x)$. Equation (1.2) is very general, especially the nonlinear functions f and g can include ∇u and the second order differential operator $\mathscr{L}$ is allowed to be degenerate, while in most literature, g is not allowed to depend on ∇u or g only depends on ∇u linearly (Da Prato and Zabczyk [8], Krylov [16], Pardoux [23]). As an intermediate step, the result of existence and uniqueness of the weak solutions of (1.2), obtained by solving the corresponding backward doubly stochastic differential equations (BDSDEs), appears also new. The existence and uniqueness of such equations when g is independent of ∇u or linearly dependent of ∇u were studied by Da Prato and Zabczyk [8], Krylov [16]. But we do not claim here our results on the existence and uniqueness for the types of SPDEs studied in [8,16] have superseded their previous results.

Note that from the pathwise stationary solution obtained in this paper, we can construct an invariant measure for the skew product of the metric dynamical system and the random dynamical system. In this connection, we mention that in recent years, substantial results on the existence and uniqueness of invariant measures for SPDEs and weak convergence of the law of the solutions as time tends to infinity have been proved for many important SPDEs [5,6,9,12,13] to name but a few). The invariant measure describes the invariance of a certain solution in law when time changes, therefore it is a stationary measure of the Markov transition probability. It is well known that an invariant measure gives a stationary solution when it is a random Dirac measure. Although an invariant measure of a random dynamical system on $\mathbb{R}^1$ gives a stationary solution, in general, this is not true unless one considers an extended probability space. However, considering the extended probability space, one essentially regards the random dynamical system as noise as well, so the dynamics is different. See [20] for some examples of SDEs on $\mathbb{R}^1$ and a perfect cocycle on $\mathbb{S}^1$ having an invariant measure, but not a stationary solution. In fact, the stationary solution we study in this paper gives the support of the corresponding invariant measure, so reveals more detailed information than an invariant measure.

In this paper, BDSDEs will be used as our tool to study stationary solutions of SPDEs. We will prove that the solutions of the corresponding infinite horizon BDSDEs give the desired stationary solutions of the SPDEs (1.2). Backward stochastic differential equations (BSDEs) have been studied extensively in the last 16 years since the pioneering work of Pardoux and Peng [24]. The connection between BSDEs and quasilinear parabolic partial differential equations (PDEs) was discovered by Pardoux and Peng in [25] and Peng in [27]. The study of the connection of weak solutions of PDEs and BSDEs began in Barles and Lesigne [4]. The BDSDEs and their connections with the SPDEs were studied by Pardoux and Peng in [26] for the strong solutions, and by Bally and Matoussi in [3] for the weak solutions. On the other hand, the infinite horizon BSDE was first studied by Peng in [27] and it was shown that the corresponding PDE is a Poisson equation (elliptic equation). This was studied systematically by Pardoux in [22]. Notice that the solutions of the Poisson equations can be regarded as the stationary solutions of the parabolic PDEs. Deepening this idea, it would not be unreasonable to conjecture that the solutions of infinite horizon BDSDEs (if exists) be the stationary solutions of the corresponding SPDEs. Of course, we cannot write them as solutions of Poisson equations or stochastic Poisson equations like in the deterministic cases. However, it is very natural to describe the stationary solutions of SPDEs by the solutions of infinite horizon BDSDEs. In this sense, BDSDEs (or BSDEs) can be regarded as more general SPDEs (or PDEs).

As far as we know, the connection of the pathwise stationary solutions of the SPDEs and infinite horizon BDSDEs we study in this paper is new (Section 2). We believe this new method can be used to many SPDEs such as those with quadratic or polynomial growth nonlinear terms. We do not intend to include all these results in the present paper, but only study Lipschitz continuous nonlinear term to initiate this intrinsic method to the study of this basic problem in dynamics of SPDEs. We would like to point out that our BDSDE method depends on neither the continuity of the random dynamical system (continuity means $u(t,\cdot,\omega):U\to U$ is a.s. continuous) nor on the method of the random attractors. The continuity problem for the SPDE (1.2) with the nonlinear noise considered in this paper still remains open mainly due to the failure of Kolmogorov's continuity theorem in infinite-dimensional setting as pointed out by some researchers (e.g. [10,21]).

One of the necessary intermediate steps is to study the BDSDEs on finite horizon and establish their connections with the weak solutions of SPDEs (Sections 3 and 4). Our method to study the $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$ valued solutions of BDSDEs on finite horizon was inspired by Bally and Matoussi's approach on the existence and uniqueness of solutions of BDSDEs with finite-dimensional Brownian motions [3]. But our results are stronger and our conditions are weaker. We will solve the BDSDEs driven by the cylindrical Brownian motion and nonlinear terms satisfying Lipschitz conditions in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)\otimes L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$. We obtain a unique solution $(Y^{t,\cdot}_{\cdot},Z^{t,\cdot}_{\cdot})\in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. The result $Y^{t,\cdot}_{\cdot}\in S^{2,0}([t,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$, which plays an important role in solving the nonlinear BDSDEs and proving the connection with the weak solutions of SPDEs (also BSDEs and PDEs), was not obtained in [3]. The generalized equivalence of norm principle (Section 2), which is a simple extension of the equivalence of norm principle obtained by Kunita [17], Barles and Lesigne [4], Bally and Matoussi [3] to random functions, also plays an important role in the proofs of our results. We believe our results for finite time BDSDEs are new even for BSDEs.

In Section 5, we will solve the BDSDEs on infinite horizon and in Section 6, we study continuity of the solution in order to ensure that it gives the perfect stationary solutions of the SPDEs.

2. The stationarity of the solutions of infinite horizon BDSDEs and stationary solutions of SPDEs

On a probability space $(\Omega,\mathscr{F},P)$, let $(\hat{B}_t)_{t\geqslant 0}$ and $(W_t)_{t\geqslant 0}$ be two mutually independent Q-Wiener process valued on U and a standard Brownian motion valued on $\mathbb{R}^d$, respectively. Here U is a separable Hilbert space with countable base $\{e_i\}_{i=1}^\infty$; $Q\in L(U)$ is a symmetric nonnegative trace class operator such that $Qe_i=\lambda_i e_i$ and $\sum_{i=1}^\infty\lambda_i<\infty$. It is well known that $\hat{B}$ has the following expansion [8]: for each t

$$\hat{B}_t=\sum_{j=1}^\infty\sqrt{\lambda_j}\,\hat{\beta}_j(t)e_j, \tag{2.1}$$

where

$$\hat{\beta}_j(t)=\frac{1}{\sqrt{\lambda_j}}\langle\hat{B}_t,e_j\rangle_U,\quad j=1,2,\ldots$$

are mutually independent real-valued Brownian motion on $(\Omega,\mathscr{F},P)$ and the series (2.1) is convergent in $L^2(\Omega,\mathscr{F},P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathscr{F}$. We define

$$\mathscr{F}_{t,T}\triangleq\mathscr{F}^{\hat{B}}_{t,T}\otimes\mathscr{F}^W_t\vee\mathcal{N},\quad\text{for }0\leqslant t\leqslant T;$$

$$\mathscr{F}_t\triangleq\mathscr{F}^{\hat{B}}_{t,\infty}\otimes\mathscr{F}^W_t\vee\mathcal{N},\quad\text{for }t\geqslant 0.$$

Here for any process $(\eta_t)_{t\geqslant 0}$, $\mathscr{F}^{\eta}_{s,t}=\sigma\{\eta_r-\eta_s; 0\leqslant s\leqslant r\leqslant t\}$, $\mathscr{F}^{\eta}_t=\mathscr{F}^{\eta}_{0,t}$, $\mathscr{F}^{\eta}_{t,\infty}=\bigvee_{T\geqslant 0}\mathscr{F}^{\eta}_{t,T}$.

Definition 2.1. Let $\mathbb{S}$ be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ-field $\mathscr{S}$. For $K\in\mathbb{R}^+$, we denote by $M^{2,-K}([0,\infty);\mathbb{S})$ the set of $\mathscr{B}_{\mathbb{R}^+}\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\phi(s)\}_{s\geqslant 0}$ with values on $\mathbb{S}$ satisfying:

(i) $\phi(s):\Omega\to\mathbb{S}$ is $\mathscr{F}_s$ measurable for $s\geqslant 0$;
(ii) $E[\int_0^\infty \mathrm{e}^{-Ks}\|\phi(s)\|^2_{\mathbb{S}}\,ds]<\infty$.

Also we denote by $S^{2,-K}([0,\infty);\mathbb{S})$ the set of $\mathscr{B}_{\mathbb{R}^+}\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\psi(s)\}_{s\geqslant 0}$ with values on $\mathbb{S}$ satisfying:

(i) $\psi(s):\Omega\to\mathbb{S}$ is $\mathscr{F}_s$ measurable for $s\geqslant 0$ and $\psi(\cdot,\omega)$ is continuous P-a.s.;
(ii) $E[\sup_{s\geqslant 0}\mathrm{e}^{-Ks}\|\psi(s)\|^2_{\mathbb{S}}]<\infty$.

Similarly, for $0\leqslant t\leqslant T<\infty$, we define $M^{2,0}([t,T];\mathbb{S})$ and $S^{2,0}([t,T];\mathbb{S})$ on finite time interval.

Definition 2.2. Let $\mathbb{S}$ be a Hilbert space with norm $\|\cdot\|_{\mathbb{S}}$ and Borel σ-field $\mathscr{S}$. We denote by $M^{2,0}([t,T];\mathbb{S})$ the set of $\mathscr{B}_{[t,T]}\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\phi(s)\}_{t\leqslant s\leqslant T}$ with values on $\mathbb{S}$ satisfying:

(i) $\phi(s):\Omega\to\mathbb{S}$ is $\mathscr{F}_{s,T}\vee\mathscr{F}^{\hat{B}}_{T,\infty}$ measurable for $t\leqslant s\leqslant T$;
(ii) $E[\int_t^T\|\phi(s)\|^2_{\mathbb{S}}\,ds]<\infty$.

Also we denote by $S^{2,0}([t,T];\mathbb{S})$ the set of $\mathscr{B}_{[t,T]}\otimes\mathscr{F}/\mathscr{S}$ measurable random processes $\{\psi(s)\}_{t\leqslant s\leqslant T}$ with values on $\mathbb{S}$ satisfying:

(i) $\psi(s):\Omega\to\mathbb{S}$ is $\mathscr{F}_{s,T}\vee\mathscr{F}^{\hat{B}}_{T,\infty}$ measurable for $t\leqslant s\leqslant T$ and $\psi(\cdot,\omega)$ is continuous P-a.s.;
(ii) $E[\sup_{t\leqslant s\leqslant T}\|\psi(s)\|^2_{\mathbb{S}}]<\infty$.

For a positive K, we consider the following infinite horizon BDSDE with the infinite-dimensional Brownian motion $\hat{B}$ as noise and Y_t taking values on a separable Hilbert space H, Z_t taking values on $\mathcal{L}^2_{\mathbb{R}^d}(H)$ (the space of all Hilbert–Schmidt operators from $\mathbb{R}^d$ to H with the Hilbert–Schmidt norm):

$$\begin{aligned}\mathrm{e}^{-Kt}Y_t&=\int_t^\infty \mathrm{e}^{-Kr}f(r,Y_r,Z_r)\,dr+\int_t^\infty K\mathrm{e}^{-Kr}Y_r\,dr\\&\quad-\int_t^\infty \mathrm{e}^{-Kr}g(r,Y_r,Z_r)\,d^\dagger\hat{B}_r-\int_t^\infty \mathrm{e}^{-Kr}Z_r\,dW_r,\quad t\geqslant 0.\end{aligned}\tag{2.2}$$

Assume $f:[0,\infty)\times\Omega\times H\times\mathcal{L}^2_{\mathbb{R}^d}(H)\to H$, $g:[0,\infty)\times\Omega\times H\times\mathcal{L}^2_{\mathbb{R}^d}(H)\to\mathcal{L}^2_{U_0}(H)$ are $\mathscr{B}_{\mathbb{R}^+}\otimes\mathscr{F}\otimes\mathscr{B}_H\otimes\mathscr{B}_{\mathcal{L}^2_{\mathbb{R}^d}(H)}$ measurable such that for any $(t,Y,Z)\in[0,\infty)\times H\times\mathcal{L}^2_{\mathbb{R}^d}(H)$,

$f(t,Y,Z)$, $g(t,Y,Z)$ are $\mathscr{F}_t$ measurable, where $U_0 = Q^{\frac{1}{2}}(U) \subset U$ is a separable Hilbert space with the norm $\langle u,v\rangle_{U_0} = \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle_U$ and the complete orthonormal base $\{\sqrt{\lambda_i}\, e_i\}_{i=1}^{\infty}$, $\mathcal{L}^2_{U_0}(H)$ is the space of all Hilbert–Schmidt operators from U_0 to H with the Hilbert–Schmidt norm. It is noted that the Q-Wiener process $(\hat{B}_t)_{t\geqslant 0}$ is a cylindrical Wiener process on U_0, and both $\mathcal{L}^2_{U_0}(H)$ and $\mathcal{L}^2_{\mathbb{R}^d}(H)$ are Hilbert spaces.

Note that the integral with respect to $\hat{B}$ is a "backward Itô's integral" and the integral with respect to W is a standard forward Itô's integral. The forward integrals in Hilbert space with respect to Q-Wiener processes were defined in Da Prato and Zabczyk [8]. To see the backward one, let $\{h(s)\}_{s\geqslant 0}$ be a stochastic process with values on $\mathcal{L}^2_{U_0}(H)$ such that $h(s)$ is $\mathscr{F}_s$ measurable for any $s\geqslant 0$ and locally square integrable, i.e. for any $0\leqslant a\leqslant b<\infty$, $\int_a^b \|h(s)\|^2_{\mathcal{L}^2_{U_0}(H)}\,ds<\infty$ almost surely. Since $\mathscr{F}_s$ is a backward filtration with respect to $\hat{B}$, so from the one-dimensional backward Itô's integral and relation with forward integral, for $0\leqslant T\leqslant T'$, we have

$$\int_t^T \sqrt{\lambda_j}\,\big\langle h(s)e_j, f_k\big\rangle\, d^\dagger\hat{\beta}_j(s) = -\int_{T'-T}^{T'-t} \sqrt{\lambda_j}\,\big\langle h(T'-s)e_j, f_k\big\rangle\, d\beta_j(s), \quad j,k=1,2,\ldots$$

where $\beta_j(s)=\hat{\beta}_j(T'-s)-\hat{\beta}_j(T')$, $j=1,2,\ldots,$ and so $B_s=\hat{B}_{T'-s}-\hat{B}_{T'}$. Here $\{f_k\}$ is the complete orthonormal basis in H. From approximation theorem of the stochastic integral in Hilbert space [8], we have

$$\int_{T'-T}^{T'-t} h(T'-s)\,dB_s = \sum_{j,k=1}^{\infty}\int_{T'-T}^{T'-t} \sqrt{\lambda_j}\,\big\langle h(T'-s)e_j, f_k\big\rangle\, d\beta_j(s)\, f_k.$$

Similarly we also have

$$\int_t^T h(s)\,d^\dagger\hat{B}_s = \sum_{j,k=1}^{\infty}\int_t^T \sqrt{\lambda_j}\big\langle h(s)e_j, f_k\big\rangle\, d^\dagger\hat{\beta}_j(s)\, f_k.$$

It turns out that

$$\int_t^T h(s)\,d^\dagger\hat{B}_s = -\int_{T'-T}^{T'-t} h(T'-s)\,dB_s \quad \text{a.s.} \tag{2.3}$$

Later we will consider another Hilbert space $\mathcal{L}^p_{U_0}(H)$ $(p>2)$, a subspace of $\mathcal{L}^2_{U_0}(H)$, including all $h\in\mathcal{L}^2_{U_0}(H)$ which satisfy

$$\|h\|^p_{\mathcal{L}^p_{U_0}(H)} \triangleq \sum_{j,k=1}^{\infty} \lambda_j^{\frac{p}{2}}\,\big|\langle he_j, f_k\rangle\big|^p < \infty.$$

Definition 2.3. Let H_0 be a dense subset of H. If $(Y,Z) \in S^{2,-K} \cap M^{2,-K}([0,\infty);H) \otimes M^{2,-K}([0,\infty);\mathcal{L}^2_{\mathbb{R}^d}(H))$, and for any $\varphi \in H_0$,

$$\left\langle e^{-Kt}Y_t,\varphi\right\rangle = \left\langle \int_t^\infty e^{-Kr} f(r,Y_r,Z_r)\,dr,\varphi\right\rangle + \left\langle \int_t^\infty K e^{-Kr} Y_r\,dr,\varphi\right\rangle - \left\langle \int_t^\infty e^{-Kr} g(r,Y_r,Z_r)\,d^\dagger \hat{B}_r,\varphi\right\rangle - \left\langle \int_t^\infty e^{-Kr} Z_r\,dW_r,\varphi\right\rangle, \quad t \geqslant 0\ P\text{-a.s.,} \tag{2.4}$$

or equivalently

$$\begin{cases} \langle Y_t,\varphi\rangle = \langle Y_T,\varphi\rangle + \left\langle \int_t^T f(r,Y_r,Z_r)\,dr,\varphi\right\rangle - \left\langle \int_t^T g(r,Y_r,Z_r)\,d^\dagger \hat{B}_r,\varphi\right\rangle - \left\langle \int_t^T Z_r\,dW_r,\varphi\right\rangle, \\ \lim_{T\to\infty}\left\langle e^{-KT}Y_T,\varphi\right\rangle = 0 \quad \text{a.s.} \end{cases} \tag{2.5}$$

then we call (Y,Z) a solution of Eq. (2.2) in H.

Remark 2.4. (i) Applying Itô's formula in H (see [8]), we have the equivalent form of Eq. (2.2)

$$\begin{cases} Y_t = Y_T + \int_t^T f(r,Y_r,Z_r)\,dr - \int_t^T g(r,Y_r,Z_r)\,d^\dagger \hat{B}_r - \int_t^T Z_r\,dW_r, \\ \lim_{T\to\infty} e^{-KT}Y_T = 0 \quad \text{a.s.} \end{cases} \tag{2.6}$$

(ii) One can easily verify that the above definition does not depend on the choice of H_0 due to the continuity of the inner product.

(iii) The uniqueness of Y in $S^{2,-K}([0,\infty);H)$ implies if (Y',Z') is another solution, then $Y_s = Y'_s$ for all $s \geqslant 0$ a.s. The uniqueness of Z implies $Z_s = Z'_s$ for a.e. $s \in [0,\infty)$ a.s. But we can modify the Z at the measure zero exceptional set of s such that $Z_s = Z'_s$ for all $s \geqslant 0$ a.s.

The first main purpose of this section is to study the stationary property of the solution of BDSDE (2.2) on H if the solution exists and is unique. In order to show the main idea, we first assume that there exists a unique solution of Eq. (2.2). The study of the existence and uniqueness of Eq. (2.2) will be deferred to later sections (Sections 3–5).

We now construct the measurable metric dynamical system through defining a measurable and measure-preserving shift. Let $\hat{\theta}_t : \Omega \to \Omega$, $t \geqslant 0$, be a measurable mapping on $(\Omega,\mathscr{F},P)$, defined by $\hat{\theta}_t \circ \hat{B}_s = \hat{B}_{s+t} - \hat{B}_t$, $\hat{\theta}_t \circ W_s = W_{s+t} - W_t$. Then for any $s,t \geqslant 0$,

(i) $P \cdot \hat{\theta}_t^{-1} = P$;
(ii) $\hat{\theta}_0 = I$, where I is the identity transformation on Ω;
(iii) $\hat{\theta}_s \circ \hat{\theta}_t = \hat{\theta}_{s+t}$.

Also for an arbitrary $\mathscr{F}$ measurable $\phi : \Omega \to H$, set

$$\hat{\theta} \circ \phi(\omega) = \phi\big(\hat{\theta}(\omega)\big).$$

We give the following bounded and stationary conditions for f, g with respect to $\hat{\theta}$.:

(A.1) There exist a constant $M_1 \geqslant 0$, and functions $\tilde{f}(\cdot) \in M^{2,-K}([0,\infty); R^+)$, $\tilde{g}(\cdot) \in M^{2,-K}([0,\infty); R^+)$ such that for any $s \geqslant 0$, $Y \in H$ and $Z \in \mathcal{L}^2_{\mathbb{R}^d}(H)$,

$$\big\| f(s,Y,Z)\big\|_H^2 \leqslant \tilde{f}^2(s) + M_1\|Y\|_H^2 + M_1\|Z\|^2_{\mathcal{L}^2_{\mathbb{R}^d}(H)},$$

$$\big\| g(s,Y,Z)\big\|^2_{\mathcal{L}^2_{U_0}(H)} \leqslant \tilde{g}^2(s) + M_1\|Y\|_H^2 + M_1\|Z\|^2_{\mathcal{L}^2_{\mathbb{R}^d}(H)};$$

(A.2) For any $r,s \geqslant 0$, $Y \in H$ and $Z \in \mathcal{L}^2_{\mathbb{R}^d}(H)$, $\hat{\theta}_r \circ f(s,Y,Z) = f(s+r,Y,Z)$, $\hat{\theta}_r \circ g(s,Y,Z) = g(s+r,Y,Z)$.

We start from the following general result about the stationarity of the solution of infinite horizon BDSDE.

Proposition 2.5. *Assume Eq.* (2.2) *has a unique solution* (Y,Z)*, then under conditions* (A.1) *and* (A.2)*,* $(Y_t,Z_t)_{t\geqslant 0}$ *is a "perfect" stationary solution, i.e.*

$$\hat{\theta}_r \circ Y_t = Y_{t+r}, \qquad \hat{\theta}_r \circ Z_t = Z_{t+r} \quad \text{for all } r,t \geqslant 0 \ \text{a.s.}$$

Proof. Let $B_s = \hat{B}_{T'-s} - \hat{B}_{T'}$ for arbitrary $T' > 0$ and $-\infty < s \leqslant T'$. Then B_s is a Brownian motion with $B_0 = 0$. For any $r \geqslant 0$, applying $\hat{\theta}_r$ on B_s, we have

$$\begin{aligned}\hat{\theta}_r \circ B_s &= \hat{\theta}_r \circ (\hat{B}_{T'-s} - \hat{B}_{T'}) = \hat{B}_{T'-s+r} - \hat{B}_{T'+r}\\ &= (\hat{B}_{T'-s+r} - \hat{B}_{T'}) - (\hat{B}_{T'+r} - \hat{B}'_T) = B_{s-r} - B_{-r}.\end{aligned}$$

So for $0 \leqslant t \leqslant T \leqslant T'$ and a locally square integrable process $\{h(s)\}_{s\geqslant 0}$, by (2.3)

$$\begin{aligned}\hat{\theta}_r \circ \int_t^T h(s)\,d^\dagger\hat{B}_s &= -\hat{\theta}_r \circ \int_{T'-T}^{T'-t} h(T'-s)\,dB_s\\ &= -\int_{T'-T}^{T'-t} \hat{\theta}_r \circ h(T'-s)\,dB_{s-r}\\ &= -\int_{T'-T-r}^{T'-t-r} \hat{\theta}_r \circ h(T'-s-r)\,dB_s\\ &= \int_{t+r}^{T+r} \hat{\theta}_r \circ h(s-r)\,d^\dagger\hat{B}_s.\end{aligned}$$

As T' can be chosen arbitrarily, so we can get for arbitrary $T \geqslant 0$, $0 \leqslant t \leqslant T$, $r \geqslant 0$,

$$\hat{\theta}_r \circ \int_t^T h(s)\, d^\dagger \hat{B}_s = \int_{t+r}^{T+r} \hat{\theta}_r \circ h(s-r)\, d^\dagger \hat{B}_s. \tag{2.7}$$

It is easy to see that $g(\cdot, Y_\cdot, Z_\cdot)$ is locally square integrable from condition (A.1), hence by condition (A.2) and (2.7)

$$\hat{\theta}_r \circ \int_t^T g(s, Y_s, Z_s)\, d^\dagger \hat{B}_s = \int_{t+r}^{T+r} g(s, \hat{\theta}_r \circ Y_{s-r}, \hat{\theta}_r \circ Z_{s-r})\, d^\dagger \hat{B}_s. \tag{2.8}$$

We consider the equivalent form Eq. (2.6) instead of Eq. (2.2). Applying the operator $\hat{\theta}_r$ on both sides of Eq. (2.6) and by (2.8), we know that $\hat{\theta}_r \circ Y_t$ satisfies the following equation:

$$\begin{cases} \hat{\theta}_r \circ Y_t = \hat{\theta}_r \circ Y_T + \int_{t+r}^{T+r} f(s, \hat{\theta}_r \circ Y_{s-r}, \hat{\theta}_r \circ Z_{s-r})\, ds \\ \qquad\qquad - \int_{t+r}^{T+r} g(s, \hat{\theta}_r \circ Y_{s-r}, \hat{\theta}_r \circ Z_{s-r})\, d^\dagger \hat{B}_s - \int_{t+r}^{T+r} \hat{\theta}_r \circ Z_{s-r}\, dW_s, \\ \lim_{T\to\infty} \mathrm{e}^{-K(T+r)} (\hat{\theta}_r \circ Y_T) = 0 \quad \text{a.s.} \end{cases} \tag{2.9}$$

On the other hand, from Eq. (2.6), it follows that

$$\begin{cases} Y_{t+r} = Y_{T+r} + \int_{t+r}^{T+r} f(s, Y_s, Z_s)\, ds - \int_{t+r}^{T+r} g(s, Y_s, Z_s)\, d^\dagger \hat{B}_s - \int_{t+r}^{T+r} Z_s\, dW_s, \\ \lim_{T\to\infty} \mathrm{e}^{-K(T+r)} Y_{T+r} = 0 \quad \text{a.s.} \end{cases} \tag{2.10}$$

Let $\hat{Y}_\cdot = \hat{\theta}_r \circ Y_{\cdot - r}$, $\hat{Z}_\cdot = \hat{\theta}_r \circ Z_{\cdot - r}$. By the uniqueness of solution of Eq. (2.6) and Remark 2.4(iii), it follows from comparing (2.9) with (2.10) that for any $r \geqslant 0$,

$$\hat{\theta}_r \circ Y_t = \hat{Y}_{t+r} = Y_{t+r}, \qquad \hat{\theta}_r \circ Z_t = \hat{Z}_{t+r} = Z_{t+r} \quad \text{for all } t \geqslant 0 \text{ a.s.}$$

Then by perfection procedure [1,2], we can prove above identities are true for all $t, r \geqslant 0$ a.s. We proved the desired result. □

An important application of the BDSDEs is to connect its solution with the solution of the corresponding SPDEs. If some kind of relationship is established, we can transfer stationary solutions from the infinite horizon BDSDEs to SPDEs. In this way, we are in access to stationary solutions of the SPDEs due to the stationary property of solutions of infinite horizon BDSDEs. For this, a specific Hilbert space $H = L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$ defined below is considered. The main aim of rest of this section is to construct the stationary solution of the SPDEs. Some proofs are given in this sections. But many detailed proofs are postponed to later sections.

In the following we consider the case $H = L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$ with the inner product $\langle u_1, u_2 \rangle = \int_{\mathbb{R}^d} u_1(x) u_2(x) \rho^{-1}(x)\, dx$, a ρ-weighted L^2 space. Here $\rho(x) = (1 + |x|)^q$, $q > 3$, is a weight function. It is easy to see that $\rho(x) : \mathbb{R}^d \to \mathbb{R}^1$ is a continuous positive function satisfying $\int_{\mathbb{R}^d} |x|^p \rho^{-1}(x)\, dx < \infty$ for any $p \in (2, q-1)$. Note that we can consider more general ρ which satisfies the above condition and conditions in [3] and all the results of this paper still hold.

We can write down the solution spaces following Definition 2.1: $M^{2,-K}([0,\infty);L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$, $M^{2,-K}([0,\infty);L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and $S^{2,-K}([0,\infty);L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$. Similar to the definition for $M^{2,-K}([0,\infty);L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, we can also define $M^{p,-K}([0,\infty);L^p_\rho(\mathbb{R}^d;\mathbb{R}^d))$.

For $k \geqslant 0$, we denote by $C^k_{l,b}(\mathbb{R}^p,\mathbb{R}^q)$ the set of C^k-functions whose partial derivatives of order less than or equal to k are bounded and by $H^k_\rho(\mathbb{R}^d;\mathbb{R}^1)$ the ρ-weighted Sobolev space (see e.g. [3]). In order to connect BDSDEs with SPDEs, the form of BDSDEs should be a kind of FBDSDEs (forward and backward doubly SDEs). So we first give the following forward SDE.

For $s \geqslant t$, let $X^{t,x}_s$ be a diffusion process given by the solution of

$$X^{t,x}_s = x + \int_t^s b\left(X^{t,x}_u\right)du + \int_t^s \sigma\left(X^{t,x}_u\right)dW_u, \tag{2.11}$$

where $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$, and for $0 \leqslant s < t$, we regulate $X^{t,x}_s = x$.

For any $r \geqslant 0$, $s \geqslant t$, $x \in \mathbb{R}^d$, apply θ_r on SDE (2.11), then

$$\hat{\theta}_r \circ X^{t,x}_s = x + \int_{t+r}^{s+r} b\left(\hat{\theta}_r \circ X^{t,x}_{u-r}\right)du + \int_{t+r}^{s+r} \sigma\left(\hat{\theta}_r \circ X^{t,x}_{u-r}\right)dW_u.$$

So by the uniqueness of the solution and a perfection procedure (cf. [1]), we have

$$\hat{\theta}_r \circ X^{t,x}_s = X^{t+r,x}_{s+r}, \quad \text{for all } r,s,t,x \text{ a.s.} \tag{2.12}$$

Moreover, it is well known that the solution defines a stochastic flow of diffeomorphism $X^{t,\cdot}_s:\mathbb{R}^d \to \mathbb{R}^d$ and denote by $\hat{X}^{t,\cdot}_s$ the inverse flow (see e.g. Kunita [17]). Denote by $J(\hat{X}^{t,x}_s)$ the determinant of the Jacobi matrix of $\hat{X}^{t,x}_s$. For $\varphi \in H^k_\rho(\mathbb{R}^d;\mathbb{R}^1)$, we define a process $\varphi_t:\Omega\times[0,T]\times\mathbb{R}^d \to \mathbb{R}^1$ by $\varphi_t(s,x) = \varphi(\hat{X}^{t,x}_s)J(\hat{X}^{t,x}_s)$. It is proved in [3] that $\varphi_t(s,\cdot) \in H^k_\rho(\mathbb{R}^d;\mathbb{R}^1)$ and for $u \in H^{k*}_\rho(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}\int_{\mathbb{R}^d} u(x)\varphi(x)\,dx &\triangleq \sum_{0\leqslant|\alpha|\leqslant k}\int_{\mathbb{R}^d} u_\alpha(x)D^\alpha\varphi(x)\,dx \\ &\leqslant \sum_{0\leqslant|\alpha|\leqslant k}\sqrt{\int_{\mathbb{R}^d}\left|u_\alpha(x)\right|^2\rho^{-1}(x)\,dx\int_{\mathbb{R}^d}\left|D^\alpha\varphi(x)\right|^2\rho(x)\,dx} < \infty\end{aligned}$$

and $\int_{\mathbb{R}^d} u(y)\varphi_t(s,y)\,dy = \int_{\mathbb{R}^d} u(X^{t,x}_s)\varphi(x)\,dx$.

The following lemma plays an important role in the analysis in this article. It is an extension of equivalence of norm principle given in [3,4,18] to the cases when φ and Ψ are random.

Lemma 2.6 *(Generalized equivalence of norm principle). Let ρ be the weight function defined at the beginning of this section and X be a diffusion process defined above. If $s \in [t,T]$, $\varphi:\Omega\times\mathbb{R}^d \to \mathbb{R}^1$ is independent of $\mathscr{F}^W_{t,s}$ and $\varphi\rho^{-1} \in L^1(\Omega\otimes\mathbb{R}^d)$, then there exist two constants $c > 0$ and $C > 0$ such that*

$$cE\left[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)\,dx\right] \leqslant E\left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)\,dx\right] \leqslant CE\left[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)\,dx\right].$$

Moreover if $\Psi : \Omega \times [t,T] \times \mathbb{R}^d \to \mathbb{R}^1$, $\Psi(s,\cdot)$ *is independent of* $\mathscr{F}_{t,s}^W$ *and* $\Psi\rho^{-1} \in L^1(\Omega \otimes [t,T] \otimes \mathbb{R}^d)$, *then*

$$\begin{aligned} cE\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s,x)|\rho^{-1}(x)\,dx\,ds\right] &\leqslant E\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s,X_s^{t,x})|\rho^{-1}(x)\,dx\,ds\right] \\ &\leqslant CE\left[\int_t^T \int_{\mathbb{R}^d} |\Psi(s,x)|\rho^{-1}(x)\,dx\,ds\right]. \end{aligned}$$

Proof. Using the conditional expectation with respect to $\mathscr{F}_{t,s}^W$ and noting that $\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}$ is $\mathscr{F}_{t,s}^W$ measurable and $|\varphi(y)|\rho^{-1}(y)$ is independent of $\mathscr{F}_{t,s}^W$, we have

$$\begin{aligned} &E\left[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)\,dx\right] \\ &\quad= \int_{\mathbb{R}^d} E\left[E\left[|\varphi(y)|\rho^{-1}(y)\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}\,|\mathscr{F}_{t,s}^W\right]\right]dy \\ &\quad= \int_{\mathbb{R}^d} E\left[|\varphi(y)|\rho^{-1}(y)\right]E\left[\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}\right]dy. \end{aligned}$$

By Lemma 5.1 in [3], $c \leqslant E[\frac{\rho^{-1}(\hat{X}_s^{t,y})J(\hat{X}_s^{t,y})}{\rho^{-1}(y)}] \leqslant C$ for any $y \in \mathbb{R}^d$, $s \in [t,T]$, the first claim follows. The second claim can be proved similarly. □

By Lemma 2.6, it is easy to deduce that $X_\cdot^{t,\cdot} \in M^{p,-K}([0,\infty); L_\rho^p(\mathbb{R}^d;\mathbb{R}^d))$ for $K \in \mathbb{R}^+$. Now we consider the following BDSDE with infinite-dimensional noise on infinite horizon:

$$\begin{aligned} \mathrm{e}^{-Ks}Y_s^{t,x} &= \int_s^\infty \mathrm{e}^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,dr + \int_s^\infty K\mathrm{e}^{-Kr}Y_r^{t,x}\,dr \\ &\quad - \int_s^\infty \mathrm{e}^{-Kr} g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,d^\dagger \hat{B}_r - \int_s^\infty \mathrm{e}^{-Kr}\langle Z_r^{t,x}, dW_r\rangle. \end{aligned} \tag{2.13}$$

Here $\hat{B}_r = \sum_{j=1}^\infty \sqrt{\lambda_j}\,\hat{\beta}_j(r)e_j$, $\{\hat{\beta}_j(r)\}_{j=1,2,\ldots}$ are mutually independent one-dimensional Brownian motions. Note that we will solve Eq. (2.13) for $Y_r^{t,\cdot} \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$ and $Z_r^{t,\cdot} \in \mathcal{L}_{\mathbb{R}^d}^2(L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) = L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)$.

Set $g_j \triangleq g\sqrt{\lambda_j}\, e_j : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^1$, then Eq. (2.13) is equivalent to

$$\begin{aligned}
\mathrm{e}^{-Ks} Y_s^{t,x} = {} & \int_s^\infty \mathrm{e}^{-Kr} f\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\, dr + \int_s^\infty K\mathrm{e}^{-Kr} Y_r^{t,x}\, dr \\
& - \sum_{j=1}^\infty \int_s^\infty \mathrm{e}^{-Kr} g_j\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\, d^\dagger \hat{\beta}_j(r) - \int_s^\infty \mathrm{e}^{-Kr}\big\langle Z_r^{t,x}, dW_r\big\rangle.
\end{aligned}$$

Referring to Definition 2.3 and noting that $C_{\mathrm{c}}^0(\mathbb{R}^d;\mathbb{R}^1)$ is dense in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$ under the norm $(\int_{\mathbb{R}^d} |\cdot|^2 \rho^{-1}(x)\, dx)^{\frac{1}{2}}$, we can define the solution in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$ as follows.

Definition 2.7. A pair of processes $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ is called a solution of Eq. (2.13) if for an arbitrary $\varphi \in C_{\mathrm{c}}^0(\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{e}^{-Ks} Y_s^{t,x} \varphi(x)\, dx = {} & \int_s^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} f\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\, dx\, dr + \int_s^\infty \int_{\mathbb{R}^d} K\mathrm{e}^{-Kr} Y_r^{t,x} \varphi(x)\, dx\, dr \\
& - \sum_{j=1}^\infty \int_s^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\, dx\, d^\dagger \hat{\beta}_j(r) \\
& - \int_s^\infty \bigg\langle \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} Z_r^{t,x} \varphi(x)\, dx, dW_r \bigg\rangle \quad P\text{-a.s.}
\end{aligned} \tag{2.14}$$

Note that in (2.14) we leave out the weight function ρ in the inner product due to the arbitrariness of φ.

If Eq. (2.13) has a unique solution, then for an arbitrary T, $Y_T^{t,x}$ satisfies

$$Y_s^{t,x} = Y_T^{t,x} + \int_s^T f\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\, dr - \int_s^T g\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\, d^\dagger \hat{B}_r - \int_s^T \big\langle Z_r^{t,x}, dW_r\big\rangle. \tag{2.15}$$

In Section 4, we will deduce the following SPDE associated with BDSDE (2.15):

$$\begin{aligned}
u(t,x) = u(T,x) + {} & \int_t^T \big[\mathscr{L}u(s,x) + f\big(x, u(s,x), \big(\sigma^*\nabla u\big)(s,x)\big)\big]\, ds \\
& - \int_t^T g\big(x, u(s,x), \big(\sigma^*\nabla u\big)(s,x)\big)\, d^\dagger \hat{B}_s.
\end{aligned} \tag{2.16}$$

Here $\mathscr{L}$ is given by (1.3), $u(T,x) = Y_T^{T,x}$. But we can normally study general $u(T,x)$ unless we consider the stationary solution.

Now following Definition 2.2 we write down the solution spaces needed in our paper: $M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$, $M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and $S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$.

Definition 2.8. A process u is called a weak solution (solution in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$) of Eq. (2.16) if $(u, \sigma^*\nabla u) \in M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and for an arbitrary $\Psi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$,

$$\begin{aligned}
&\int_t^T\int_{\mathbb{R}^d} u(s,x)\partial_s\Psi(s,x)\,dx\,ds + \int_{\mathbb{R}^d} u(t,x)\Psi(t,x)\,dx - \int_{\mathbb{R}^d} u(T,x)\Psi(T,x)\,dx \\
&\qquad - \frac{1}{2}\int_t^T\int_{\mathbb{R}^d} \big(\sigma^*\nabla u\big)(s,x)\big(\sigma^*\nabla\Psi\big)(s,x)\,dx\,ds - \int_t^T\int_{\mathbb{R}^d} u(s,x)\,\mathrm{div}\big((b-\tilde{A})\Psi\big)(s,x)\,dx\,ds \\
&\quad = \int_t^T\int_{\mathbb{R}^d} f\big(x, u(s,x), \big(\sigma^*\nabla u\big)(s,x)\big)\Psi(s,x)\,dx\,ds \\
&\qquad - \sum_{j=1}^\infty \int_t^T\int_{\mathbb{R}^d} g_j\big(x, u(s,x), \big(\sigma^*\nabla u\big)(s,x)\big)\Psi(s,x)\,dx\,d^\dagger\hat{\beta}_j(s) \quad P\text{-a.s.}
\end{aligned} \tag{2.17}$$

Here $\tilde{A}_j \triangleq \frac{1}{2}\sum_{i=1}^d \frac{\partial a_{ij}(x)}{\partial x_i}$, and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_d)^*$.

This definition can be easily understood if we note the following integration by parts formula: for $\varphi_1, \varphi_2 \in C^2(\mathbb{R}^d)$,

$$-\int_{\mathbb{R}^d} \mathscr{L}\varphi_1(x)\varphi_2(x)\,dx = \frac{1}{2}\int_{\mathbb{R}^d}\big(\sigma^*\nabla\varphi_1\big)(x)\big(\sigma^*\nabla\varphi_2\big)(x)\,dx + \int_{\mathbb{R}^d}\varphi_1(x)\,\mathrm{div}\big((b-\tilde{A})\varphi_2\big)(x)\,dx.$$

The main purpose of this section is to find the stationary solution of SPDE (1.2) via the solution of BDSDE (2.13). We consider the following conditions:

(A.1)′ Functions $f:\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathbb{R}^1$ and $g:\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathcal{L}^2_{U_0}(\mathbb{R}^1)$ are $\mathscr{B}_{\mathbb{R}^d}\otimes\mathscr{B}_{\mathbb{R}^1}\otimes\mathscr{B}_{\mathbb{R}^d}$ measurable, and there exist constants $M_2, M_{2j}, C, C_j, \alpha_j \geqslant 0$ with $\sum_{j=1}^\infty M_{2j} < \infty$, $\sum_{j=1}^\infty C_j < \infty$ and $\sum_{j=1}^\infty \alpha_j < \frac{1}{2}$ such that for any $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X_1, X_2, Z_1, Z_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$, measurable $U:\mathbb{R}^d\to[0,1]$,

$$\int_{\mathbb{R}^d} U(x)\big|f\big(X_1(x), Y_1(x), Z_1(x)\big) - f\big(X_2(x), Y_2(x), Z_2(x)\big)\big|^2\rho^{-1}(x)\,dx$$

$$\leqslant \int_{\mathbb{R}^d} U(x)\big(M_2\big|X_1(x)-X_2(x)\big|^2 + C\big|Y_1(x)-Y_2(x)\big|^2 + C\big|Z_1(x)-Z_2(x)\big|^2\big) \times \rho^{-1}(x)\,dx,$$

$$\int_{\mathbb{R}^d} U(x)\big|g_j\big(X_1(x),Y_1(x),Z_1(x)\big) - g_j\big(X_2(x),Y_2(x),Z_2(x)\big)\big|^2\rho^{-1}(x)\,dx$$

$$\leqslant \int_{\mathbb{R}^d} U(x)\big(M_{2j}\big|X_1(x)-X_2(x)\big|^2 + C_j\big|Y_1(x)-Y_2(x)\big|^2 + \alpha_j\big|Z_1(x)-Z_2(x)\big|^2\big) \times \rho^{-1}(x)\,dx;$$

(A.2)′ For $p \in (2, q-1)$,

$$\int_{\mathbb{R}^d} \big|f(x,0,0)\big|^p \rho^{-1}(x)\,dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \big\|g(x,0,0)\big\|^p_{\mathcal{L}^p_{U_0}(\mathbb{R}^1)} \rho^{-1}(x)\,dx < \infty;$$

(A.3)′ $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^1)$, $\sigma \in C^3_{l,b}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^1)$. Furthermore, for p is given in (A.2)′, if L is the global Lipschitz constant for b and σ, L satisfies $K - pL - \frac{p(p-1)}{2}L^2 > 0$;

(A.4)′ There exists a constant $\mu > 0$ with $2\mu - pK - pC - \frac{p(p-1)}{2}\sum_{j=1}^{\infty} C_j > 0$ such that for any $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X, Z \in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$, measurable $U:\mathbb{R}^d \to [0,1]$,

$$\int_{\mathbb{R}^d} U(x)\big(Y_1(x)-Y_2(x)\big)\big(f\big(X(x),Y_1(x),Z(x)\big) - f\big(X(x),Y_2(x),Z(x)\big)\big)\rho^{-1}(x)\,dx$$

$$\leqslant -\mu \int_{\mathbb{R}^d} U(x)\big|Y_1(x)-Y_2(x)\big|^2\rho^{-1}(x)\,dx.$$

Remark 2.9. We need monotone condition (A.4)′ in order to solve the infinite horizon BDSDEs. But it does not seem obvious to replace the Lipschitz condition for f in (A.1)′ by a weaker condition on f such as f is continuous in y using the infinite horizon BSDE procedure (e.g. [22]). The difficulty is due to the fact that we consider various conditions in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ here rather than pointwise ones, therefore we cannot solve the BDSDEs pointwise in x. However, our conjecture is that the Lipschitz condition can be relaxed if we strengthen some conditions in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ to pointwise ones. We will study this generality in future publications. Here due to the length of the paper, we only consider the Lipschitz continuous function f to initiate this intrinsic method to the study of this basic problem.

We first acknowledge the two theorems below and give their proofs in Section 6.

Theorem 2.10. *Under conditions* (A.1)′–(A.4)′*, Eq.* (2.13) *has a unique solution* $(Y^{t,x}_s, Z^{t,x}_s)$. *Moreover* $E[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} e^{-pKs}|Y^{t,x}_s|^p\rho^{-1}(x)\,dx] < \infty$.

Theorem 2.11. *Under conditions* (A.1)$'$–(A.4)$'$, *let* $u(t,\cdot) \triangleq Y_t^{t,\cdot}$, *where* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ *is the solution of Eq.* (2.13). *Then for* $t \in [0,T]$, $u(t,\cdot)$ *is a weak solution for Eq.* (2.16). *Moreover,* $u(t,\cdot)$ *is* a.s. *continuous with respect to* t *in* $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$.

Then we prove the main theorem in this section.

Theorem 2.12. *Under conditions* (A.1)$'$–(A.4)$'$, *let* $u(t,\cdot) \triangleq Y_t^{t,\cdot}$, *where* $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ *is the solution of Eq.* (2.13). *Then* $u(t,\cdot)$ *has an indistinguishable version which is a "perfect" stationary solution of Eq.* (2.16).

Proof. For $Y \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$, $Z \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)$, let

$$\hat{f}(\mathcal{T},Y,Z) = f\big(X_s^{t,\cdot},Y,Z\big), \qquad \hat{g}(\mathcal{T},Y,Z) = g\big(X_s^{t,\cdot},Y,Z\big).$$

Here we take $\mathcal{T} = (s,t)$ as a dual time variable (t is fixed). By condition (A.1)$'$, we have

$$\begin{aligned}
&\big\|\hat{f}(\mathcal{T},Y,Z)\big\|^2_{L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)} \\
&\quad= \int_{\mathbb{R}^d} \big|f\big(X_s^{t,x},Y(x),Z(x)\big)\big|^2 \rho^{-1}(x)\,dx \\
&\quad\leqslant C_p \int_{\mathbb{R}^d} \big|f\big(X_s^{t,x},0,0\big)\big|^2 \rho^{-1}(x)\,dx + C_p \int_{\mathbb{R}^d} \big|Y(x)\big|^2 \rho^{-1}(x)\,dx + C_p \int_{\mathbb{R}^d} \big|Z(x)\big|^2 \rho^{-1}(x)\,dx.
\end{aligned}$$

Here and in the following, C_p is a generic constant. By Lemma 2.6 and condition (A.2)$'$,

$$\begin{aligned}
E\Bigg[\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Ks}\big|f\big(X_s^{t,x},0,0\big)\big|^2 \rho^{-1}(x)\,dx\,ds\Bigg] &\leqslant C_p \int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Ks}\big|f(x,0,0)\big|^2 \rho^{-1}(x)\,dx\,ds \\
&\leqslant C_p \int_{\mathbb{R}^d} \big|f(x,0,0)\big|^p \rho^{-1}(x)\,dx < \infty.
\end{aligned}$$

We take $\tilde{f}(\mathcal{T}) = (\int_{\mathbb{R}^d} |f(X_s^{t,x},0,0)|^2 \rho^{-1}(x)\,dx)^{\frac{1}{2}}$, then $\hat{f}(\mathcal{T},Y,Z)$ satisfies condition (A.1). Similarly we can also prove $\hat{g}(\mathcal{T},Y,Z)$ satisfies condition (A.1). On the other hand, applying $\hat{\theta}_r$ on $\hat{f}(\mathcal{T},Y,Z)$, by (2.12), we have for any $Y \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$ and $Z \in L_\rho^2(\mathbb{R}^d;\mathbb{R}^d)$,

$$\hat{\theta}_r \circ \hat{f}(\mathcal{T},Y,Z) = f\big(\hat{\theta}_r \circ X_s^{t,\cdot},Y,Z\big) = f\big(X_{s+r}^{t+r,\cdot},Y,Z\big).$$

Verifying $\hat{g}(\mathcal{T},Y,Z)$ in the same way, we know that $\hat{f}(\mathcal{T},Y,Z)$ and $\hat{g}(\mathcal{T},Y,Z)$ satisfy condition (A.2). Since by Theorem 2.10, Eq. (2.13) has a unique solution $(Y_\mathcal{T},Z_\mathcal{T})$, this $(Y_\mathcal{T},Z_\mathcal{T})$ is a stationary solution as a consequence of Proposition 2.5. That is to say for any $t \geqslant 0$

$$\hat{\theta}_r \circ Y_\mathcal{T} = \hat{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot}, \qquad \hat{\theta}_r \circ Z_\mathcal{T} = \hat{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad \text{for all } r \geqslant 0,\ s \geqslant t \ \text{ a.s.}$$

In particular, for any $t \geqslant 0$

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad \text{for all } r \geqslant 0 \text{ a.s.} \tag{2.18}$$

By Theorem 2.11, we know that $u(t,\cdot) \triangleq Y_t^{t,\cdot}$ is the weak solution for Eq. (2.16), so we get from (2.18) that for any $t \geqslant 0$

$$\hat{\theta}_r \circ u(t,\cdot) = u(t+r,\cdot) \quad \text{for all } r \geqslant 0 \text{ a.s.}$$

Until now, we know "crude" stationary property for $u(t,\cdot)$. And by Theorem 2.11, $u(t,\cdot)$ is continuous with respect to t, So we can get an indistinguishable version of $u(t,\cdot)$, still denoted by $u(t,\cdot)$, such that

$$\hat{\theta}_r \circ u(t,\cdot) = u(t+r,\cdot) \quad \text{for all } t, r \geqslant 0 \text{ a.s.}$$

So we proved the desired result. $\square$

By Definition 2.8, conditions (A.1)′ and (A.2)′, one can calculate that $g(\cdot, u(s,\cdot), (\sigma^*\nabla u)(s,\cdot)) \in \mathcal{L}^2_{U_0}(L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ is locally square integrable in $[0,T]$. Now we consider Eq. (1.2) with cylindrical Brownian motion B on U_0. For arbitrary $T > 0$, let Y be the solution of Eq. (2.13) and $u(t,\cdot) = Y_t^{t,\cdot}$ be the stationary solution of Eq. (2.16) with $\hat{B}$ chosen as the time reversal of B from time T, i.e. $\hat{B}_s = B_{T-s} - B_T$ or $\hat{\beta}_j(s) = \beta_j(T-s) - \beta_j(T)$ for $s \geqslant 0$. By (2.3) and integral transformation in Eq. (2.16), we can see that $v(t,x) \triangleq u(T-t,x)$ satisfies (1.2) or its equivalent form

$$\begin{aligned} v(t,x) = v(t,v_0)(x) = v_0(x) &+ \int_0^t \big[\mathscr{L}v(s,x) + f\big(x, v(s,x), \big(\sigma^*\nabla v\big)(s,x)\big)\big]\,ds \\ &+ \sum_{j=1}^{\infty} \int_0^t g_j\big(x, v(s,x), \big(\sigma^*\nabla v\big)(s,x)\big)\,d\beta_j(s), \quad t \geqslant 0. \end{aligned} \tag{2.19}$$

Here $v_0(x) = v(0,x)$.

In fact, we can prove a claim that $v(t,\cdot)(\omega) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ does not depend on the choice of T. For this, we only need to show that for any $T' \geqslant T$, $Y_{T-t}^{T-t,\cdot}(\hat{\omega}) = Y_{T'-t}^{T'-t,\cdot}(\hat{\omega}')$ when $0 \leqslant t \leqslant T$, where $\hat{\omega}(s) = B_{T-s} - B_T$ and $\hat{\omega}'(s) = B_{T'-s} - B_{T'}$. Let $\hat{\theta}_\cdot$ and $\hat{\theta}'_\cdot$ be the shifts of $\hat{\omega}(\cdot)$ and $\hat{\omega}'(\cdot)$, respectively. Since by (2.18), we have

$$\begin{aligned} Y_{T-t}^{T-t,\cdot}(\hat{\omega}) &= \hat{\theta}_{T-t} Y_0^{0,\cdot}(\hat{\omega}) = Y_0^{0,\cdot}(\hat{\theta}_{T-t}\hat{\omega}), \\ Y_{T'-t}^{T'-t,\cdot}(\hat{\omega}') &= \hat{\theta}'_{T'-t} Y_0^{0,\cdot}(\hat{\omega}') = Y_0^{0,\cdot}(\hat{\theta}'_{T'-t}\hat{\omega}'). \end{aligned}$$

So we just need to assert that $\hat{\theta}_{T-t}\hat{\omega} = \hat{\theta}'_{T'-t}\hat{\omega}'$. Indeed we have for any $s \geqslant 0$

$$\begin{aligned}(\hat{\theta}_{T-t}\hat{\omega})(s) &= \hat{\omega}(T-t+s) - \hat{\omega}(T-t)\\ &= (B_{T-(T-t+s)} - B_T) - (B_{T-(T-t)} - B_T)\\ &= B_{t-s} - B_t.\end{aligned}$$

Note that the right-hand side of the above formula does not depend on T, therefore $\hat{\theta}_{T-t}\hat{\omega}(s) = \hat{\theta}'_{T'-t}\hat{\omega}'(s) = B_{t-s} - B_t$.

On probability space $(\Omega, \mathscr{F}, P)$, we define $\theta_t = (\hat{\theta}_t)^{-1}$, $t \geqslant 0$. Actually $\hat{B}$ is a two-sided Brownian motion, so $(\hat{\theta}_t)^{-1} = \hat{\theta}_{-t}$ is well defined (see [1]). It is easy to see that θ_t is a shift with respect to B satisfying:

(i) $P \cdot (\theta_t)^{-1} = P$;
(ii) $\theta_0 = I$;
(iii) $\theta_s \circ \theta_t = \theta_{s+t}$;
(iv) $\theta_t \circ B_s = B_{s+t} - B_t$.

Since $v(t, \cdot)(\omega) = u(T-t, \cdot)(\hat{\omega}) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ a.s., so

$$\theta_r v(t, \cdot)(\omega) = \hat{\theta}_{-r} u(T-t, \cdot)(\hat{\omega}) = u(T-t-r, \cdot)(\hat{\omega}) = v(t+r, \cdot)(\omega),$$

for all $r \geqslant 0$ and $T \geqslant t + r$ a.s. In particular, let $Y(\omega) = v_0(\omega) = Y_T^{T,\cdot}(\hat{\omega})$. Then above formula implies (1.1):

$$\theta_t Y(\omega) = Y(\theta_t\omega) = v(t, \omega) = v\big(t, v_0(\omega), \omega\big) = v\big(t, Y(\omega), \omega\big), \quad \text{for all } t \geqslant 0 \text{ a.s.}$$

That is to say $v(t, \cdot)(\omega) = Y(\theta_t\omega)(\cdot) = Y_{T-t}^{T-t,\cdot}(\hat{\omega})$ is a stationary solution of Eq. (1.2) with respect to θ. Therefore we proved the following theorem

Theorem 2.13. *Under conditions* (A.1)′–(A.4)′, *for arbitrary T and $t \in [0, T]$, let $v(t, \cdot) \triangleq Y_{T-t}^{T-t,\cdot}$, where $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is the solution of Eq.* (2.13) *with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geqslant 0$. Then $v(t, \cdot)$ is a "perfect" stationary solution of Eq.* (1.2).

3. Finite horizon BDSDEs

Before we study the BDSDEs on infinite horizon, we need to study the BDSDEs on finite horizon and establish the connection with SPDEs. For finite-dimensional noise and under Lipschitz condition for a.e. $x \in \mathbb{R}^d$, the problem was studied in Bally and Matoussi [3]. In this section, we consider the following BDSDE with infinite-dimensional noise on finite horizon:

$$\begin{aligned}Y_s^{t,x} &= h\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,dr\\ &\quad - \int_s^T g\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,d^\dagger\hat{B}_r - \int_s^T \big\langle Z_r^{t,x}, dW_r\big\rangle, \quad 0 \leqslant s \leqslant T. \qquad (3.1)\end{aligned}$$

Here $h:\Omega\times\mathbb{R}^d\to\mathbb{R}^1$, $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathbb{R}^1$, $g:[0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathcal{L}^2_{U_0}(\mathbb{R}^1)$. Set $g_j\triangleq g\sqrt{\lambda_j}\,e_j:[0,T]\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathbb{R}^1$, then Eq. (3.1) is equivalent to

$$\begin{aligned}Y_s^{t,x}&=h\big(X_T^{t,x}\big)+\int_s^T f\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,dr\\&\quad-\sum_{j=1}^{\infty}\int_s^T g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,d^\dagger\hat\beta_j(r)-\int_s^T\big\langle Z_r^{t,x},dW_r\big\rangle,\quad 0\leqslant s\leqslant T.\end{aligned}$$

We assume:

(H.1) Function h is $\mathscr{F}^{\hat B}_{T,\infty}\otimes\mathscr{B}_{\mathbb{R}^d}$ measurable and $E[\int_{\mathbb{R}^d}|h(x)|^2\rho^{-1}(x)\,dx]<\infty$;

(H.2) Functions f and g are $\mathscr{B}_{[0,T]}\otimes\mathscr{B}_{\mathbb{R}^d}\otimes\mathscr{B}_{\mathbb{R}^1}\otimes\mathscr{B}_{\mathbb{R}^d}$ measurable and there exist constants $C, C_j,\alpha_j\geqslant 0$ with $\sum_{j=1}^\infty C_j<\infty$ and $\sum_{j=1}^\infty\alpha_j<\frac12$ such that for any $t\in[0,T]$, $Y_1,Y_2\in L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$, $X,Z_1,Z_2\in L^2_\rho(\mathbb{R}^d;\mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{R}^d}\big|f\big(t,X(x),Y_1(x),Z_1(x)\big)-f\big(t,X(x),Y_2(x),Z_2(x)\big)\big|^2\rho^{-1}(x)\,dx\\&\quad\leqslant C\int_{\mathbb{R}^d}\big(\big|Y_1(x)-Y_2(x)\big|^2+\big|Z_1(x)-Z_2(x)\big|^2\big)\rho^{-1}(x)\,dx,\\&\int_{\mathbb{R}^d}\big|g_j\big(t,X(x),Y_1(x),Z_1(x)\big)-g_j\big(t,X(x),Y_2(x),Z_2(x)\big)\big|^2\rho^{-1}(x)\,dx\\&\quad\leqslant\int_{\mathbb{R}^d}\big(C_j\big|Y_1(x)-Y_2(x)\big|^2+\alpha_j\big|Z_1(x)-Z_2(x)\big|^2\big)\rho^{-1}(x)\,dx;\end{aligned}$$

(H.3) $\int_0^T\int_{\mathbb{R}^d}|f(s,x,0,0)|^2\rho^{-1}(x)\,dx\,ds<\infty$ and $\int_0^T\int_{\mathbb{R}^d}\|g(s,x,0,0)\|^2_{\mathcal{L}^2_{U_0}(\mathbb{R}^1)}\rho^{-1}(x)\,dx\,ds<\infty$;

(H.4) $b\in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma\in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d\times\mathbb{R}^d)$.

Needless to say, the conditions (H.1)–(H.4) for the existence and uniqueness of solution of Eq. (3.1) are weaker than what are needed for the case of infinite horizon. We would like to point out that for the finite horizon problem, our conditions are weaker than those in Bally and Matoussi [3]. In (H.1), we allow the terminal function h depending on $\mathscr{F}_{t,T}$ independent sigma field $\mathscr{F}^{\hat B}_{T,\infty}$. One can easily verify that it does not affect the results in [3]. Moreover, here we only need Lipschitz condition in the space $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$ instead of the pathwise Lipschitz condition posed in [3].

Definition 3.1. A pair of processes $(Y_\cdot^{t,\cdot},Z_\cdot^{t,\cdot})\in S^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ is called a solution of Eq. (3.1) if for any $\varphi\in C^0_c(\mathbb{R}^d;\mathbb{R}^1)$,

$$\int_{\mathbb{R}^d} Y_s^{t,x}\varphi(x)\,dx = \int_{\mathbb{R}^d} h\big(X_T^{t,x}\big)\varphi(x)\,dx + \int_s^T\int_{\mathbb{R}^d} f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\,dx\,dr$$
$$- \sum_{j=1}^{\infty}\int_s^T\int_{\mathbb{R}^d} g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)$$
$$- \int_s^T\left\langle \int_{\mathbb{R}^d} Z_r^{t,x}\varphi(x)\,dx, dW_r\right\rangle \quad P\text{-a.s.} \tag{3.2}$$

The main objective of this section is to prove

Theorem 3.2. *Under conditions* (H.1)–(H.4)*, Eq.* (3.1) *has a unique solution.*

This theorem is an extension of Theorem 3.1 in [3]. The idea is to start from Bally and Matoussi's results for finite-dimensional noise and then take limit to obtain the solution for the case of infinite-dimensional noise. But Bally and Matoussi's results cannot apply immediately here as we have a weaker Lipschitz condition and some of the key claims in the proof of Theorem 3.1 [3] are not obvious under their conditions. Moreover, the result $Y_\cdot^{t,\cdot} \in S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ was not obtained in [3]. We study a sequence of BDSDEs

$$Y_s^{t,x,n} = h\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\,dr$$
$$- \sum_{j=1}^{n}\int_s^T g_j\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\,d^\dagger\hat{\beta}_j(r) - \int_s^T\big\langle Z_r^{t,x,n}, dW_r\big\rangle. \tag{3.3}$$

A solution of (3.3) is a pair of processes $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ satisfying the spatial integral form of Eq. (3.3), i.e. (3.2) with a finite number of one-dimensional backward stochastic integrals.

First we do some preparations.

Lemma 3.3. *Under conditions* (H.1)–(H.4)*, if there exists* $(Y_\cdot(\cdot), Z_\cdot(\cdot)) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ *satisfying the spatial integral form of Eq.* (3.3) *for* $t \leqslant s \leqslant T$*, then* $Y_\cdot(\cdot) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ *and therefore* $(Y_s(x), Z_s(x))$ *is a solution of Eq.* (3.3)*.*

Proof. Let us first see $Y_s(\cdot)$ is continuous with respect to s in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$. Since $(Y_s(x), Z_s(x))$ satisfies the form of Eq. (3.3) for $t \leqslant s < T$, a.e. $x \in \mathbb{R}^d$, therefore,

$$\int_{\mathbb{R}^d} \big|Y_{s+\Delta s}(x) - Y_s(x)\big|^2\rho^{-1}(x)\,dx$$
$$\leqslant C_p\int_{\mathbb{R}^d}\int_s^{s+\Delta s} \big|f\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\big|^2\,dr\,\rho^{-1}(x)\,dx$$

$$+C_p\sum_{j=1}^{n}\int_{\mathbb{R}^d}\left|\int_s^{s+\Delta s} g_j\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\, d^\dagger\hat{\beta}_j(r)\right|^2 \rho^{-1}(x)\,dx$$

$$+C_p\int_{\mathbb{R}^d}\left|\int_s^{s+\Delta s}\big\langle Z_r(x), dW_r\big\rangle\right|^2\rho^{-1}(x)\,dx.$$

For the forward stochastic integral part, it is trivial to see that for $0\leqslant \Delta s\leqslant T-s$,

$$\left|\int_s^{s+\Delta s}\big\langle Z_r(x), dW_r\big\rangle\right|^2 \leqslant \sup_{0\leqslant\Delta s\leqslant T-s}\left|\int_s^{s+\Delta s}\big\langle Z_r(x), dW_r\big\rangle\right|^2 \quad \text{a.s.}$$

And we can deduce that $\int_{\mathbb{R}^d}\sup_{0\leqslant\Delta s\leqslant T-s}|\int_s^{s+\Delta s}\langle Z_r(x), dW_r\rangle|^2\rho^{-1}(x)\,dx<\infty$ a.s. by the B–D–G inequality and $Z_\cdot(\cdot)\in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. So by the dominated convergence theorem, $\lim_{\Delta s\to 0^+}\int_{\mathbb{R}^d}|\int_s^{s+\Delta s}\langle Z_r(x), dW_r\rangle|^2\rho^{-1}(x)\,dx=0$. Similarly we can prove $\lim_{\Delta s\to 0^-}\int_{\mathbb{R}^d}|\int_{s+\Delta s}^{s}\langle Z_r(x), dW_r\rangle|^2\rho^{-1}(x)\,dx=0$ for $t<s\leqslant T$. The backward stochastic integral part tends to 0 as $\Delta s\to 0$ can be deduced similarly. So $Y_s(\cdot)$ is continuous with respect to s in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$. From conditions (H.2)–(H.4) and $(Y_\cdot(\cdot), Z_\cdot(\cdot))\in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, it follows that for a.e. $x\in\mathbb{R}^d$, $E[\int_t^T|f(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2\,dr]<\infty$ and $\sum_{j=1}^n E[\int_t^T|g_j(r, X_r^{t,x}, Y_r(x), Z_r(x))|^2\,dr]<\infty$. For a.e. $x\in\mathbb{R}^d$, referring to Lemma 1.4 in [26], we use the generalized Itô's formula (cf. Elworthy, Truman and Zhao [15]) to $\psi_M(Y_r(x))$, where $\psi_M(x)=x^2 I_{\{-M\leqslant x<M\}}+2M(x-M)I_{\{x\geqslant M\}}-2M(x+M)I_{\{x<-M\}}$. Then

$$\begin{aligned}
&\psi_M\big(Y_s(x)\big)+\int_s^T I_{\{-M\leqslant Y_r(x)<M\}}\big|Z_r(x)\big|^2\,dr\\
&\quad=\psi_M\big(h\big(X_T^{t,x}\big)\big)+\int_s^T\psi'_M\big(Y_r(x)\big)f\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\,dr\\
&\qquad+\sum_{j=1}^n\int_s^T I_{\{-M\leqslant Y_r(x)<M\}}\big|g_j\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\big|^2\,dr\\
&\qquad-\sum_{j=1}^n\int_s^T\psi'_M\big(Y_r(x)\big)g_j\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\,d^\dagger\hat{\beta}_j(r)-\int_s^T\big\langle\psi'_M\big(Y_r(x)\big)Z_r(x), dW_r\big\rangle.
\end{aligned}\tag{3.4}$$

We can use the Fubini theorem to perfect (3.4) so that (3.4) is satisfied for a.e. $x\in\mathbb{R}^d$, on a full measure set that is independent of x. Taking integration in $\mathbb{R}^d$ on both sides, applying the stochastic Fubini theorem [8], we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \psi_M\big(Y_s(x)\big)\rho^{-1}(x)\,dx + \int_s^T \int_{\mathbb{R}^d} I_{\{-M \leqslant Y_r(x) < M\}} \big|Z_r(x)\big|^2 \rho^{-1}(x)\,dx\,dr \\
&\quad \leqslant \int_{\mathbb{R}^d} \psi_M\big(h\big(X_T^{t,x}\big)\big)\rho^{-1}(x)\,dx + \int_s^T \int_{\mathbb{R}^d} \psi_M'\big(Y_r(x)\big) f\big(r, X_r^{t,x}, 0, 0\big)\rho^{-1}(x)\,dx\,dr \\
&\qquad + \int_s^T \int_{\mathbb{R}^d} \psi_M'\big(Y_r(x)\big)\big(f\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big) - f\big(r, X_r^{t,x}, 0, 0\big)\big)\rho^{-1}(x)\,dx\,dr \\
&\qquad + C_p \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} \big|g_j\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big) - g_j\big(r, X_r^{t,x}, 0, 0\big)\big|^2 \rho^{-1}(x)\,dx\,dr \\
&\qquad + C_p \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} \big|g_j\big(r, X_r^{t,x}, 0, 0\big)\big|^2 \rho^{-1}(x)\,dx\,dr \\
&\qquad - \int_s^T \bigg\langle \int_{\mathbb{R}^d} \psi_M'\big(Y_r(x)\big) Z_r(x)\rho^{-1}(x)\,dx, dW_r \bigg\rangle \\
&\qquad - \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} \psi_M'\big(Y_r(x)\big) g_j\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\rho^{-1}(x)\,dx\,d^\dagger \hat{\beta}_j(r).
\end{aligned}
$$

Noting that $\psi_M(h(X_T^{t,x})) \leqslant |h(X_T^{t,x})|^2$ and $|\psi_M'(Y_r(x))|^2 \leqslant 4|Y_r(x)|^2$, so by Lemma 2.6, the B–D–G inequality and Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&E\bigg[\sup_{t \leqslant s \leqslant T} \int_{\mathbb{R}^d} \psi_M\big(Y_s(x)\big)\rho^{-1}(x)\,dx\bigg] \\
&\quad \leqslant C_p E\bigg[\int_{\mathbb{R}^d} \big|h(x)\big|^2 \rho^{-1}(x)\,dx\bigg] + C_p E\bigg[\int_t^T \int_{\mathbb{R}^d} \big(\big|Y_r(x)\big|^2 + \big|Z_r(x)\big|^2\big)\rho^{-1}(x)\,dx\,dr\bigg] \\
&\qquad + C_p E\bigg[\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} \big(\big|g_j(r, x, 0, 0)\big|^2 + \big|f(r, x, 0, 0)\big|^2\big)\rho^{-1}(x)\,dx\,dr\bigg] \\
&\qquad + C_p E\bigg[\sqrt{\int_t^T \int_{\mathbb{R}^d} \big|\psi_M'\big(Y_s(x)\big)\big|^2 \rho^{-1}(x)\,dx \int_{\mathbb{R}^d} \sum_{j=1}^n \big|g_j\big(r, X_r^{t,x}, Y_r(x), Z_r(x)\big)\big|^2 \rho^{-1}(x)\,dx\,dr}\,\bigg]
\end{aligned}
$$

$$+ C_p E\left[\sqrt{\int_t^T \int_{\mathbb{R}^d} \left|\psi'_M\left(Y_s(x)\right)\right|^2 \rho^{-1}(x)\,dx \int_{\mathbb{R}^d} \left|Z_r(x)\right|^2 \rho^{-1}(x)\,dx\,dr}\right]$$

$$\leqslant C_p E\left[\int_{\mathbb{R}^d} \left|h(x)\right|^2 \rho^{-1}(x)\,dx\right] + C_p E\left[\int_t^T \int_{\mathbb{R}^d} \left(\left|Y_r(x)\right|^2 + \left|Z_r(x)\right|^2\right) \rho^{-1}(x)\,dx\,dr\right]$$

$$+ C_p E\left[\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} \left(\left|g_j(r,x,0,0)\right|^2 + \left|f(r,x,0,0)\right|^2\right) \rho^{-1}(x)\,dx\,dr\right]$$

$$+ \frac{1}{5} E\left[\sup_{t\leqslant s\leqslant T} \int_{\mathbb{R}^d} \left|\psi'_M\left(Y_s(x)\right)\right|^2 \rho^{-1}(x)\,dx\right]. \tag{3.5}$$

Since $(Y_\cdot(\cdot), Z_\cdot(\cdot)) \in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, taking the limit as $M \to \infty$ and applying the monotone convergence theorem, we have $E[\sup_{t\leqslant s\leqslant T} \int_{\mathbb{R}^d} |Y_s(x)|^2 \times \rho^{-1}(x)\,dx] < \infty$. So $Y_\cdot(\cdot) \in S^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$ follows. That is to say $(Y_s(x), Z_s(x))$ is a solution of Eq. (3.3). □

For the rest of our paper, we will leave out the similar localization argument as in the proof of Lemma 3.3 when applying Itô's formula to save the space of this paper.

Proposition 3.4. *Under conditions* (H.1)–(H.4)*, assume Eq.* (3.3) *has a unique solution* $(Y_r^{t,x,n}, Z_r^{t,x,n})$*, then for any* $t \leqslant s \leqslant T$, $Y_r^{s,X_s^{t,x},n} = Y_r^{t,x,n}$ *and* $Z_r^{s,X_s^{t,x},n} = Z_r^{t,x,n}$ *for any* $r \in [s,T]$ *and a.e.* $x \in \mathbb{R}^d$ *a.s.*

Proof. For $t \leqslant s \leqslant r \leqslant T$, note that $(Y_r^{s,\cdot,n}, Z_r^{s,\cdot,n})$ is $\mathscr{F}^{\hat{B}}_{r,\infty} \otimes \mathscr{F}^W_{s,r}$ measurable, so is independent of $\mathscr{F}^W_{t,s}$. Thus by Lemma 2.6, we have

$$E\left[\int_s^T \int_{\mathbb{R}^d} \left(\left|Y_r^{s,X_s^{t,x},n}\right|^2 + \left|Z_r^{s,X_s^{t,x},n}\right|^2\right) \rho^{-1}(x)\,dx\,dr\right]$$

$$\leqslant C_p E\left[\int_s^T \int_{\mathbb{R}^d} \left(\left|Y_r^{s,x,n}\right|^2 + \left|Z_r^{s,x,n}\right|^2\right) \rho^{-1}(x)\,dx\,dr\right] < \infty.$$

Moreover, it is easy to see that $X_r^{s,X_s^{t,x}} = X_r^{t,x}$ and $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ is $\mathscr{F}^{\hat{B}}_{r,\infty} \otimes \mathscr{F}^W_{t,r}$ measurable, so $(Y_\cdot^{s,X_s^{t,\cdot},n}, Z_\cdot^{s,X_s^{t,\cdot},n}) \in M^{2,0}([s,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([s,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ and $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ satisfies the spatial integral form of Eq. (3.3) for $s \leqslant r \leqslant T$. Define $Y_r^{s,X_s^{t,x},n} = Y_r^{t,x,n}$, $Z_r^{s,X_s^{t,x},n} = Z_r^{t,x,n}$ when $t \leqslant r < s$. Then $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ satisfies the spatial integral form of Eq. (3.3) for $t \leqslant r \leqslant T$ and $(Y_\cdot^{s,X_s^{t,\cdot},n}, Z_\cdot^{s,X_s^{t,\cdot},n}) \in M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. Therefore, by Lemma 3.3, $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n})$ is the solution of

Eq. (3.3). By the uniqueness of the solution of Eq. (3.3), we have for any $s \in [t,T]$, $(Y_r^{s,X_s^{t,x},n}, Z_r^{s,X_s^{t,x},n}) = (Y_r^{t,x,n}, Z_r^{t,x,n})$ for any $r \in [s,T]$ and a.e. $x \in \mathbb{R}^d$ a.s. $\square$

Theorem 3.5. *Under conditions* (H.1)–(H.4)*, Eq.* (3.3) *has a unique solution, i.e. there exists a unique* $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ *such that for an arbitrary* $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$

$$\begin{aligned}\int_{\mathbb{R}^d} Y_s^{t,x,n}\varphi(x)\,dx &= \int_{\mathbb{R}^d} h\big(X_T^{t,x}\big)\varphi(x)\,dx + \int_s^T\int_{\mathbb{R}^d} f\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\varphi(x)\,dx\,dr\\ &\quad - \sum_{j=1}^n \int_s^T\int_{\mathbb{R}^d} g_j\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\\ &\quad - \int_s^T \left\langle \int_{\mathbb{R}^d} Z_r^{t,x,n}\varphi(x)\,dx, dW_r\right\rangle \quad P\text{-a.s.}\end{aligned} \tag{3.6}$$

Proof. *Uniqueness.* Assume there exists another $(\hat{Y}_s^{t,x,n}, \hat{Z}_s^{t,x,n}) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ satisfying (3.6). Define $\bar{Y}_s^{t,x,n} = Y_s^{t,x,n} - \hat{Y}_s^{t,x,n}$ and $\bar{Z}_s^{t,x,n} = Z_s^{t,x,n} - \hat{Z}_s^{t,x,n}$, $t \leqslant s \leqslant T$. From condition (H.2) and $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}), (\hat{Y}_\cdot^{t,\cdot,n}, \hat{Z}_\cdot^{t,\cdot,n}) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$, it follows that for a.e. $x \in \mathbb{R}^d$, $E[\int_t^T |f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - f(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})|^2\,dr] < \infty$ and $\sum_{j=1}^n E[\int_t^T |g_j(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}) - g_j(r, X_r^{t,x}, \hat{Y}_r^{t,x,n}, \hat{Z}_r^{t,x,n})|^2\,dr] < \infty$. For a.e. $x \in \mathbb{R}^d$, similar as in (3.4), we use generalized Itô's formula to $\mathrm{e}^{Kr}\psi_M(\bar{Y}_r^{t,x,n})$ where $K \in \mathbb{R}^1$, then take integration in $\mathbb{R}^d \times \Omega$ on both sides and apply the stochastic Fubini theorem. Note that the stochastic integrals are martingales, so taking the limit as $M \to \infty$, we have

$$\begin{aligned}&E\left[\mathrm{e}^{Ks}\int_{\mathbb{R}^d}\big|\bar{Y}_s^{t,x,n}\big|^2\rho^{-1}(x)\,dx\right] + \left(K - 2C - \sum_{j=1}^\infty C_j - \frac{1}{2}\right)E\left[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\big|\bar{Y}_r^{t,x,n}\big|^2\rho^{-1}(x)\,dx\,dr\right]\\ &\quad + \left(\frac{1}{2} - \sum_{j=1}^\infty \alpha_j\right)E\left[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\big|\bar{Z}_r^{t,x,n}\big|^2\rho^{-1}(x)\,dx\,dr\right] \leqslant 0.\end{aligned} \tag{3.7}$$

All the terms on the left-hand side of (3.7) are positive when taking K sufficiently large, so it is easy to see that for each $s \in [t,T]$, $\bar{Y}_s^{t,x} = 0$ a.e. $x \in \mathbb{R}^d$ a.s. By a "standard" argument taking s in the rational number space and noting $\int_{\mathbb{R}^d}\mathrm{e}^{Ks}|\bar{Y}_s^{t,x,n}|^2\rho^{-1}(x)\,dx$ is continuous with respect to s, we have $\bar{Y}_s^{t,x,n} = 0$ for all $s \in [t,T]$, a.e. $x \in \mathbb{R}^d$ a.s. Also by (3.7), for a.e. $s \in [t,T]$, $\bar{Z}_s^{t,x,n} = 0$ a.e. $x \in \mathbb{R}^d$, a.s. We can modify the values of Z at the measure zero exceptional set of s such that $\bar{Z}_s^{t,x,n} = 0$ for all $s \in [t,T]$, a.e. $x \in \mathbb{R}^d$ a.s.

Existence.

Step 1. We prove for the following equation:

$$\tilde{Y}_s^{t,x,n} = h\big(X_T^{t,x}\big) + \int_s^T \tilde{f}\big(r, X_r^{t,x}\big)\,dr - \sum_{j=1}^n \int_s^T \tilde{g}_j\big(r, X_r^{t,x}\big)\,d^\dagger\hat{\beta}_j(r) - \int_s^T \big\langle \tilde{Z}_r^{t,x,n}, dW_r\big\rangle, \quad (3.8)$$

if (H.1) and (H.4) are satisfied, and $\tilde{f}(\cdot, X_\cdot^{t,\cdot})$, $\tilde{g}_j(\cdot, X_\cdot^{t,\cdot}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, then there exists a unique solution. For this, we can first use a similar method as in the proof of Theorem 2.1 in [3] to prove there exists $(\tilde{Y}_\cdot^{t,\cdot,n}, \tilde{Z}_\cdot^{t,\cdot,n}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ such that for an arbitrary $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$

$$\begin{aligned}
&\int_{\mathbb{R}^d} \tilde{Y}_s^{t,x,n}\varphi(x)\,dx \\
&\quad= \int_{\mathbb{R}^d} h\big(X_T^{t,x}\big)\varphi(x)\,dx + \int_s^T \int_{\mathbb{R}^d} \tilde{f}\big(r, X_r^{t,x}\big)\varphi(x)\,dx\,dr \\
&\qquad- \sum_{j=1}^n \int_s^T \int_{\mathbb{R}^d} \tilde{g}_j\big(r, X_r^{t,x}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r) - \int_s^T \left\langle \int_{\mathbb{R}^d} \tilde{Z}_r^{t,x,n}\varphi(x)\,dx, dW_r\right\rangle \quad P\text{-a.s.}
\end{aligned}$$

By Lemma 3.3, $\tilde{Y}_\cdot^{t,\cdot,n} \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Then Step 1 follows.

Step 2. Given $(Y_s^{t,x,n,N-1}, Z_s^{t,x,n,N-1}) \in S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$, define $(Y_s^{t,x,n,N}, Z_s^{t,x,n,N})$ as follows:

$$\begin{aligned}
Y_s^{t,x,n,N} &= h\big(X_T^{t,x}\big) + \int_s^T f\big(r, X_r^{t,x}, Y_r^{t,x,n,N-1}, Z_r^{t,x,n,N-1}\big)\,dr \\
&\quad- \sum_{j=1}^n \int_s^T g_j\big(r, X_r^{t,x}, Y_r^{t,x,n,N-1}, Z_r^{t,x,n,N-1}\big)\,d^\dagger\hat{\beta}_j(r) - \int_s^T \big\langle Z_r^{t,x,n,N}, dW_r\big\rangle. \quad (3.9)
\end{aligned}$$

Let $(Y_r^{t,x,n,0}, Z_r^{t,x,n,0}) = (0,0)$. By conditions (H.1), (H.3), (H.4) and Lemma 2.6, we know h, $f(r, X_r^{t,x}, 0, 0)$ and $g_j(r, X_r^{t,x}, 0, 0)$ satisfy the conditions in Step 1, so Eq. (3.8) has a unique solution $(Y_\cdot^{t,\cdot,n,1}, Z_\cdot^{t,\cdot,n,1}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ when $\tilde{f}(r, X_r^{t,x}) = f(r, X_r^{t,x}, 0, 0)$ and $\tilde{g}(r, X_r^{t,x}) = g(r, X_r^{t,x}, 0, 0)$. From Proposition 3.4 and the Fubini theorem, we have $Y_r^{t,x,n,1} = Y_r^{r,X_r^{t,x},n,1}$ and $Z_r^{t,x,n,1} = Z_r^{r,X_r^{t,x},n,1}$ for a.e. $r \in [t,T]$, $x \in \mathbb{R}^d$ almost surely. Thus by conditions (H.1)–(H.4) and Lemma 2.6, we have that h,

$$f\big(r, X_r^{t,x}, Y_r^{t,x,n,1}, Z_r^{t,x,n,1}\big) = f\big(r, X_r^{t,x}, Y_r^{r,X_r^{t,x},n,1}, Z_r^{r,X_r^{t,x},n,1}\big) \quad \text{and}$$
$$g_j\big(r, X_r^{t,x}, Y_r^{t,x,n,1}, Z_r^{t,x,n,1}\big) = g_j\big(r, X_r^{t,x}, Y_r^{r,X_r^{t,x},n,1}, Z_r^{r,X_r^{t,x},n,1}\big)$$

satisfy the conditions in Step 1. Following the same procedure, we obtain $(Y_\cdot^{t,\cdot,n,2}, Z_\cdot^{t,\cdot,n,2}) \in M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. In general, we see (3.9) is an iterated mapping from $S^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([t,T]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ to itself and obtain a sequence $\{(Y_r^{t,x,n,i}, Z_r^{t,x,n,i})\}_{i=0,1,2,\ldots}$. We will prove that (3.9) is a contraction mapping. For this, define

$$\begin{aligned}
&\bar{Y}_s^{t,x,n,i} = Y_s^{t,x,n,i} - Y_s^{t,x,n,i-1}, \qquad \bar{Z}_s^{t,x,n,i} = Z_s^{t,x,n,i} - Z_s^{t,x,n,i-1},\\
&\bar{f}^i(s,x) = f\big(s, X_s^{t,x}, Y_s^{t,x,n,i}, Z_s^{t,x,n,i}\big) - f\big(s, X_s^{t,x}, Y_s^{t,x,n,i-1}, Z_s^{t,x,n,i-1}\big),\\
&\bar{g}_j^i(s,x) = g_j\big(s, X_s^{t,x}, Y_s^{t,x,n,i}, Z_s^{t,x,n,i}\big) - g_j\big(s, X_s^{t,x}, Y_s^{t,x,n,i-1}, Z_s^{t,x,n,i-1}\big), \quad i = 1,2,\ldots,
\end{aligned}$$

$t \leqslant s \leqslant T$. Then, for a.e. $x \in \mathbb{R}^d$, $(\bar{Y}_s^{t,x,n,N}, \bar{Z}_s^{t,x,n,N})$ satisfies

$$\bar{Y}_s^{t,x,n,N} = \int_s^T \bar{f}^{N-1}(r,x)\,dr - \sum_{j=1}^n \int_s^T \bar{g}_j^{N-1}(r,x)\,d^\dagger\hat{\beta}_j(r) - \int_s^T \big\langle \bar{Z}_r^{t,x,n,N}, dW_r\big\rangle.$$

Applying generalized Itô's formula to $\mathrm{e}^{Kr}|\bar{Y}_r^{t,x,n,N}|^2$ for a.e. $x \in \mathbb{R}^d$, by the Young inequality and condition (H.2), we can deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{Ks}\big|\bar{Y}_s^{t,x,n,N}\big|^2 \rho^{-1}(x)\,dx + K\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\big|\bar{Y}_r^{t,x,n,N}\big|^2\rho^{-1}(x)\,dx\,dr\\
&\quad + \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\big|\bar{Z}_r^{t,x,n,N}\big|^2\rho^{-1}(x)\,dx\,dr\\
&\leqslant \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\Big(2C\big|\bar{Y}_r^{t,x,n,N}\big|^2 + \frac{1}{2}\big|\bar{Y}_r^{t,x,n,N-1}\big|^2 + \frac{1}{2}\big|\bar{Z}_r^{t,x,n,N-1}\big|^2\Big)\rho^{-1}(x)\,dx\,dr\\
&\quad + \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\left(\sum_{j=1}^\infty C_j\big|\bar{Y}_r^{t,x,n,N-1}\big|^2 + \sum_{j=1}^\infty \alpha_j\big|\bar{Z}_r^{t,x,n,N-1}\big|^2\right)\rho^{-1}(x)\,dx\,dr\\
&\quad - \sum_{j=1}^n \int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}2\bar{Y}_r^{t,x,n,N}\bar{g}_j^{N-1}(r,x)\rho^{-1}(x)\,dx\,d^\dagger\hat{\beta}_j(r)\\
&\quad - \int_s^T\left\langle \int_{\mathbb{R}^d} \mathrm{e}^{Kr}2\bar{Y}_r^{t,x,n,N}\bar{Z}_r^{t,x,n,N}\rho^{-1}(x)\,dx, dW_r\right\rangle.
\end{aligned} \tag{3.10}$$

Then we have

$$(K-2C)E\left[\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\left|\bar{Y}_r^{t,x,n,N}\right|^2\rho^{-1}(x)\,dx\,dr\right]+E\left[\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\left|\bar{Z}_r^{t,x,n,N}\right|^2\rho^{-1}(x)\,dx\,dr\right]$$
$$\leqslant\left(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j\right)E\left[\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\left(\left(1+2\sum_{j=1}^{\infty}C_j\right)\left|\bar{Y}_r^{t,x,n,N-1}\right|^2+\left|\bar{Z}_r^{t,x,n,N-1}\right|^2\right)\right.$$
$$\left.\times\rho^{-1}(x)\,dx\,dr\right].$$

Letting $K=1+2C+2\sum_{j=1}^{\infty}C_j$, we have

$$E\left[\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\left(\left(1+2\sum_{j=1}^{\infty}C_j\right)\left|\bar{Y}_r^{t,x,n,N}\right|^2+\left|\bar{Z}_r^{t,x,n,N}\right|^2\right)\rho^{-1}(x)\,dx\,dr\right]$$
$$\leqslant\left(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j\right)E\left[\int_s^T\int_{\mathbb{R}^d} \mathrm{e}^{Kr}\left(\left(1+2\sum_{j=1}^{\infty}C_j\right)\left|\bar{Y}_r^{t,x,n,N-1}\right|^2+\left|\bar{Z}_r^{t,x,n,N-1}\right|^2\right)\right.$$
$$\left.\times\rho^{-1}(x)\,dx\,dr\right]. \tag{3.11}$$

Note that $E[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}((1+2\sum_{j=1}^{\infty}C_j)|\cdot|^2+|\cdot|^2)\rho^{-1}(x)\,dx\,dr]$ is equivalent to $E[\int_t^T\int_{\mathbb{R}^d}(|\cdot|^2+|\cdot|^2)\rho^{-1}(x)\,dx\,dr]$. From the contraction principle, the mapping (3.9) has a pair of fixed point $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ that is the limit of the Cauchy sequence $\{(Y_\cdot^{t,\cdot,n,N}, Z_\cdot^{t,\cdot,n,N})\}_{N=1}^{\infty}$ in $M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. We then prove $Y_\cdot^{t,\cdot,n}$ is also the limit of $Y_\cdot^{t,\cdot,n,N}$ in $S^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$ as $N\to\infty$. For this, we only need to prove $\{Y_\cdot^{t,\cdot,n,N}\}_{N=1}^{\infty}$ is a Cauchy sequence in $S^{2,0}([t,T];L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$. Similar as in (3.5), by the B–D–G inequality and Cauchy–Schwarz inequality, from (3.10), we have

$$E\left[\sup_{t\leqslant s\leqslant T}\int_{\mathbb{R}^d}\mathrm{e}^{Ks}\left|\bar{Y}_s^{t,x,n,N}\right|^2\rho^{-1}(x)\,dx\right]$$
$$\leqslant M_3E\left[\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\left(\left|\bar{Y}_r^{t,x,n,N-1}\right|^2+\left|\bar{Z}_r^{t,x,n,N-1}\right|^2+\left|\bar{Y}_r^{t,x,n,N}\right|^2+\left|\bar{Z}_r^{t,x,n,N}\right|^2\right)\rho^{-1}(x)\,dx\,dr\right], \tag{3.12}$$

where $M_3>0$ is independent of n and N. Without losing any generality, assume that $M\geqslant N$. We can deduce from (3.11) and (3.12) that

$$\left(E\left[\sup_{t\leqslant s\leqslant T}\int_{\mathbb{R}^d}\left|Y_s^{t,x,n,M}-Y_s^{t,x,n,N}\right|^2\rho^{-1}(x)\,dx\right]\right)^{\frac{1}{2}}$$
$$\leqslant\sum_{i=N+1}^{M}\left(E\left[\sup_{t\leqslant s\leqslant T}\int_{\mathbb{R}^d}\left|\bar{Y}_s^{t,x,n,i}\right|^2\rho^{-1}(x)\,dx\right]\right)^{\frac{1}{2}}$$
$$\leqslant\sum_{i=N+1}^{M}\left(M_3E\left[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\left(\left|\bar{Y}_r^{t,x,n,i-1}\right|^2+\left|\bar{Z}_r^{t,x,n,i-1}\right|^2+\left|\bar{Y}_r^{t,x,n,i}\right|^2+\left|\bar{Z}_r^{t,x,n,i}\right|^2\right)\right.\right.$$
$$\left.\left.\times\,\rho^{-1}(x)\,dx\,dr\right]\right)^{\frac{1}{2}}$$
$$\leqslant\sum_{i=N+1}^{M}\left(2M_3E\left[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\left(\left(1+2\sum_{j=1}^{\infty}C_j\right)\left|\bar{Y}_r^{t,x,n,i-1}\right|^2+\left|\bar{Z}_r^{t,x,n,i-1}\right|^2\right)\rho^{-1}(x)\,dx\,dr\right]\right)^{\frac{1}{2}}$$
$$\leqslant\sum_{i=N+1}^{\infty}\left(\frac{1}{2}+\sum_{j=1}^{\infty}\alpha_j\right)^{\frac{i-2}{2}}\left(2M_3E\left[\int_t^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\left(\left(1+2\sum_{j=1}^{\infty}C_j\right)\left|Y_r^{t,x,n,1}\right|^2+\left|Z_r^{t,x,n,1}\right|^2\right)\right.\right.$$
$$\left.\left.\times\,\rho^{-1}(x)\,dx\,dr\right]\right)^{\frac{1}{2}}\to 0\quad\text{as } M,N\to\infty.$$

The theorem is proved. □

Following a similar procedure as in the proof of Lemma 3.3, and using Itô's formula to $\mathrm{e}^{Kr}|Y_r^{t,x,n}|^2$, by the B–D–G inequality, we have the following estimation for the solution of Eq. (3.3).

Proposition 3.6. *Under the conditions of Theorem* 3.2, $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ *satisfies*

$$\sup_n E\left[\sup_{t\leqslant s\leqslant T}\int_{\mathbb{R}^d}\left|Y_s^{t,x,n}\right|^2\rho^{-1}(x)\,dx\right]+\sup_n E\left[\int_t^T\int_{\mathbb{R}^d}\left|Z_r^{t,x,n}\right|^2\rho^{-1}(x)\,dx\,dr\right]<\infty.$$

Remark 3.7. For $s\in[0,t]$, Eq. (3.3) is equivalent to the following BDSDE:

$$Y_s^{x,n}=Y_t^{t,x,n}+\int_s^t f\left(r,x,Y_r^{x,n},Z_r^{x,n}\right)dr$$
$$-\sum_{j=1}^{n}\int_s^t g_j\left(r,x,Y_r^{x,n},Z_r^{x,n}\right)d^\dagger\hat{\beta}_j(r)-\int_s^t\left\langle Z_r^{x,n},dW_r\right\rangle. \tag{3.13}$$

Note that $Y_t^{t,x,n}$ satisfies condition (H.1). By a similar method as in the proof of Theorem 3.5 and Proposition 3.6, we can obtain a $(Y_\cdot^{\cdot,n}, Z_\cdot^{\cdot,n}) \in S^{2,0}([0,t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0,t]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$, is the unique solution of Eq. (3.13). Moreover,

$$\sup_n E\bigg[\sup_{0\leqslant s\leqslant t}\int_{\mathbb{R}^d}\big|Y_s^{x,n}\big|^2\rho^{-1}(x)\,dx\bigg] + \sup_n E\bigg[\int_0^t\int_{\mathbb{R}^d}\big|Z_r^{x,n}\big|^2\rho^{-1}(x)\,dx\,dr\bigg] < \infty.$$

To unify the notation, we define $(Y_s^{t,x,n}, Z_s^{t,x,n}) = (Y_s^{x,n}, Z_s^{x,n})$ when $s \in [0,t)$. Then $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Furthermore, we have

$$\sup_n E\bigg[\sup_{0\leqslant s\leqslant T}\int_{\mathbb{R}^d}\big|Y_s^{t,x,n}\big|^2\rho^{-1}(x)\,dx\bigg] + \sup_n E\bigg[\int_0^T\int_{\mathbb{R}^d}\big|Z_r^{t,x,n}\big|^2\rho^{-1}(x)\,dx\,dr\bigg] < \infty. \tag{3.14}$$

Proof of Theorem 3.2. The proof of the uniqueness is rather similar to the uniqueness proof in Theorem 3.5.

Existence. By Theorem 3.5 and Remark 3.7, for each n, there exists a unique solution $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}) \in S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$ to Eq. (3.3). We will prove $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ is a Cauchy sequence in $S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. Without losing any generality, assume that $m \geqslant n$, and define

$$\begin{aligned}
&\bar Y_s^{t,x,m,n} = Y_s^{t,x,m} - Y_s^{t,x,n}, \qquad \bar Z_s^{t,x,m,n} = Z_s^{t,x,m} - Z_s^{t,x,n},\\
&\bar f^{m,n}(s,x) = f\big(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}\big) - f\big(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}\big),\\
&\bar g_j^{m,n}(s,x) = g_j\big(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}\big) - g_j\big(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}\big), \quad 0 \leqslant s \leqslant T.
\end{aligned}$$

Then for $0 \leqslant s \leqslant T$ and a.e. $x \in \mathbb{R}^d$,

$$\begin{cases}
d\bar Y_s^{t,x,m,n} = -\bar f^{m,n}(s,x)\,ds + \sum_{j=1}^n \bar g_j^{m,n}(s,x)\,d^\dagger\hat\beta_j(s)\\
\qquad\qquad + \sum_{j=n+1}^m g_j(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m})\,d^\dagger\hat\beta_j(s) + \langle \bar Z_s^{t,x,m,n}, dW_s\rangle,\\
\bar Y_T^{t,x,m,n} = 0 \quad \text{a.s.}
\end{cases}$$

Applying Itô's formula to $\mathrm{e}^{Kr}|\bar Y_r^{t,x,m,n}|^2$ for a.e. $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mathrm{e}^{Ks}\big|\bar Y_s^{t,x,m,n}\big|^2\rho^{-1}(x)\,dx + \bigg(K - 2C - \sum_{j=1}^\infty C_j - \frac{1}{2}\bigg)\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\big|\bar Y_r^{t,x,m,n}\big|^2\rho^{-1}(x)\,dx\,dr\\
&\quad + \bigg(\frac{1}{2} - \sum_{j=1}^\infty \alpha_j\bigg)\int_s^T\int_{\mathbb{R}^d}\mathrm{e}^{Kr}\big|\bar Z_r^{t,x,m,n}\big|^2\rho^{-1}(x)\,dx\,dr
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_p \sum_{j=n+1}^{m} \Bigg\{ (C_j+\alpha_j)\Bigg(\int_s^T \int_{\mathbb{R}^d} \big(\big|Y_r^{t,x,m}\big|^2 + \big|Z_r^{t,x,m}\big|^2\big)\rho^{-1}(x)\,dx\,dr \\
&\quad + \int_s^T \int_{\mathbb{R}^d} \big|g_j\big(r, X_r^{t,x}, 0, 0\big)\big|^2 \rho^{-1}(x)\,dx\,dr \Bigg) \Bigg\} \\
&\quad - \sum_{j=1}^{n} \int_s^T \int_{\mathbb{R}^d} 2\mathrm{e}^{Kr}\, \bar{Y}_r^{t,x,m,n}\, \bar{g}_j^{m,n}(r,x)\rho^{-1}(x)\,dx\,d^\dagger \hat{\beta}_j(r) \\
&\quad - \sum_{j=n+1}^{m} \int_s^T \int_{\mathbb{R}^d} 2\mathrm{e}^{Kr}\, \bar{Y}_r^{t,x,m,n}\, g_j\big(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m}\big)\rho^{-1}(x)\,dx\,d^\dagger \hat{\beta}_j(r) \\
&\quad - \int_s^T \bigg\langle \int_{\mathbb{R}^d} 2\mathrm{e}^{Kr}\, \bar{Y}_r^{t,x,m,n}\, \bar{Z}_r^{t,x,m,n} \rho^{-1}(x)\,dx, dW_r \bigg\rangle.
\end{aligned}
\tag{3.15}
$$

All the terms on the left-hand side of (3.15) are positive when taking K sufficiently large. Take expectation on both sides of (3.15), then by Lemma 2.6 and (3.14), we have

$$
\begin{aligned}
&E\Bigg[\int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big|\bar{Y}_r^{t,x,m,n}\big|^2 \rho^{-1}(x)\,dx\,dr \Bigg] + E\Bigg[\int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big|\bar{Z}_r^{t,x,m,n}\big|^2 \rho^{-1}(x)\,dx\,dr \Bigg] \\
&\quad \leqslant C_p \sum_{j=n+1}^{m} \Bigg\{ (C_j+\alpha_j)\Bigg(\sup_n E\Bigg[\int_0^T \int_{\mathbb{R}^d} \big(\big|Y_r^{t,x,n}\big|^2 + \big|Z_r^{t,x,n}\big|^2\big)\rho^{-1}(x)\,dx\,dr \Bigg] \\
&\qquad + \int_0^T \int_{\mathbb{R}^d} \big|g_j(r,x,0,0)\big|^2 \rho^{-1}(x)\,dx\,dr \Bigg) \Bigg\} \to 0, \quad \text{as } n,m\to\infty.
\end{aligned}
\tag{3.16}
$$

Also by the B–D–G inequality, from (3.15) we have

$$
\begin{aligned}
&E\Bigg[\sup_{0\leqslant s\leqslant T} \int_{\mathbb{R}^d} \mathrm{e}^{Ks} \big|\bar{Y}_s^{t,x,m,n}\big|^2 \rho^{-1}(x)\,dx \Bigg] \\
&\quad \leqslant C_p E\Bigg[\int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{Kr} \big(\big|\bar{Y}_r^{t,x,m,n}\big|^2 + \big|\bar{Z}_r^{t,x,m,n}\big|^2\big)\rho^{-1}(x)\,dx\,dr \Bigg] \\
&\qquad + C_p \sum_{j=n+1}^{m} (C_j+\alpha_j)\Bigg(\sup_n E\Bigg[\int_0^T \int_{\mathbb{R}^d} \big(\big|Y_r^{t,x,n}\big|^2 + \big|Z_r^{t,x,n}\big|^2\big)\rho^{-1}(x)\,dx\,dr \Bigg]\Bigg) \\
&\qquad + C_p \sum_{j=n+1}^{m} \int_0^T \int_{\mathbb{R}^d} \big|g_j(r,x,0,0)\big|^2 \rho^{-1}(x)\,dx\,dr.
\end{aligned}
$$

So by (3.14), (3.16) and condition (H.3), we have

$$E\Big[\sup_{0\leqslant s\leqslant T}\int_{\mathbb{R}^d} \mathrm{e}^{Ks}\big|\bar{Y}_s^{t,x,m,n}\big|^2\rho^{-1}(x)\,dx\Big]\to 0,\quad \text{as } n,m\to\infty.$$

Therefore $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ is a Cauchy sequence in $S^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ with its limit denoted by $(Y_s^{t,x}, Z_s^{t,x})$. We will show that $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is the solution of Eq. (3.1), i.e. $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ satisfies (3.2) for an arbitrary $\varphi\in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$. For this, we will prove that Eq. (3.6) converges to Eq. (3.2) in $L^2(\Omega)$ term by term as $n\to\infty$. Here we only show the convergence of the third term,

$$\begin{aligned}
&E\Bigg[\bigg|\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d} g_j\big(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\\
&\qquad-\sum_{j=1}^\infty\int_s^T\int_{\mathbb{R}^d} g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]\\
&\quad\leqslant 2E\Bigg[\bigg|\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d}\big(g_j\big(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n}\big)-g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]\\
&\qquad+2E\Bigg[\bigg|\sum_{j=n+1}^\infty\int_s^T\int_{\mathbb{R}^d} g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]\\
&\quad\leqslant C_p\sum_{j=1}^\infty(C_j+\alpha_j)E\Bigg[\int_s^T\int_{\mathbb{R}^d}\big(\big|Y_r^{t,x,n}-Y_r^{t,x}\big|^2+\big|Z_r^{t,x,n}-Z_r^{t,x}\big|^2\big)\rho^{-1}(x)\,dx\,dr\Bigg]\\
&\qquad+C_pE\Bigg[\bigg|\sum_{j=n+1}^\infty\int_s^T\int_{\mathbb{R}^d}\big(g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)-g_j\big(r,X_r^{t,x},0,0\big)\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]\\
&\qquad+C_pE\Bigg[\bigg|\sum_{j=n+1}^\infty\int_s^T\int_{\mathbb{R}^d} g_j\big(r,X_r^{t,x},0,0\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg].
\end{aligned}$$

Note

$$\begin{aligned}
&E\Bigg[\bigg|\sum_{j=n+1}^\infty\int_s^T\int_{\mathbb{R}^d}\big(g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)-g_j\big(r,X_r^{t,x},0,0\big)\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]\\
&\quad=E\Bigg[\int_s^T\bigg\|\int_{\mathbb{R}^d}\big(g\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)-g\big(r,X_r^{t,x},0,0\big)\big)\varphi(x)\,dx\bigg(\sum_{j=n+1}^\infty\lambda_je_j\otimes e_j\bigg)^{\frac12}\bigg\|_{L_U}^2dr\Bigg]
\end{aligned}$$

$$= E\left[\int_s^T \sum_{i=1}^{\infty} \left| \int_{\mathbb{R}^d} \big(g\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big) - g\big(r, X_r^{t,x}, 0, 0\big)\big)\varphi(x)\,dx \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \langle e_j, e_i\rangle \right|^2 dr\right]$$

$$= E\left[\sum_{j=n+1}^{\infty} \int_s^T \left| \int_{\mathbb{R}^d} \big(g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big) - g_j\big(r, X_r^{t,x}, 0, 0\big)\big)\varphi(x)\,dx\right|^2 dr\right]$$

$$\leqslant C_p E\left[\sum_{j=n+1}^{\infty} \int_s^T \int_{\mathbb{R}^d} \left|g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big) - g_j\big(r, X_r^{t,x}, 0, 0\big)\right|^2 \rho^{-1}(x)\,dx\,dr\right]$$

$$\leqslant C_p \sum_{j=n+1}^{\infty} (C_j + \alpha_j) E\left[\int_s^T \int_{\mathbb{R}^d} \big(\big|Y_r^{t,x}\big|^2 + \big|Z_r^{t,x}\big|^2\big)\rho^{-1}(x)\,dx\,dr\right] \to 0. \tag{3.17}$$

Here we used $(\sum_{j=n+1}^{\infty} \lambda_j e_j \otimes e_j)^{\frac{1}{2}} = \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \otimes e_j$. This can be verified as follows: for an arbitrary $u \in U$, by definition of tensor operator,

$$\left(\sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \otimes e_j\right)\left(\sum_{i=n+1}^{\infty} \sqrt{\lambda_i} e_i \otimes e_i\right)u = \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \left\langle e_j, \sum_{i=n+1}^{\infty} \sqrt{\lambda_i} e_i \langle e_i, u\rangle\right\rangle$$

$$= \sum_{j=n+1}^{\infty} \sqrt{\lambda_j} e_j \langle \sqrt{\lambda_j} e_j, e_j\rangle \langle e_j, u\rangle$$

$$= \left(\sum_{j=n+1}^{\infty} \lambda_j e_j \otimes e_j\right)u.$$

Similarly we have

$$C_p E\left[\left|\sum_{j=n+1}^{\infty} \int_s^T \int_{\mathbb{R}^d} g_j\big(r, X_r^{t,x}, 0, 0\big)\varphi(x)\,dx\,d^{\dagger}\hat{\beta}_j(r)\right|^2\right]$$

$$\leqslant C_p \int_s^T \int_{\mathbb{R}^d} \sum_{j=n+1}^{\infty} \left|g_j(r, x, 0, 0)\right|^2 \rho^{-1}(x)\,dx\,dr \to 0. \tag{3.18}$$

That is to say $(Y_s^{t,x}, Z_s^{t,x})_{0\leqslant s\leqslant T}$ satisfies Eq. (3.2). The proof of Theorem 3.2 is completed. □

4. Weak solutions of the corresponding SPDEs

In Section 3, we proved the existence and uniqueness of the weak solution of BDSDE (3.1). We obtained the solution $(Y_s^{t,x}, Z_s^{t,x})$ by taking the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ of the solutions of Eq. (3.3) in the space $S^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^1)) \otimes M^{2,0}([0,T]; L_\rho^2(\mathbb{R}^d; \mathbb{R}^d))$. We still start from

Eq. (3.3) in this section. A direct application of Proposition 3.4 and Fubini theorem immediately leads to

Proposition 4.1. *Under conditions* (H.1)–(H.4)*, if we define* $u^n(t,x) = Y_t^{t,x,n}$, $v^n(t,x) = Z_t^{t,x,n}$*, then* $u^n(s, X_s^{t,x}) = Y_s^{t,x,n}$, $v^n(s, X_s^{t,x}) = Z_s^{t,x,n}$ *for a.e.* $s \in [t,T]$, $x \in \mathbb{R}^d$ *a.s.*

We first use the idea of Bally and Matoussi [3] to give the correspondence between the weak solutions of SPDEs and BDSDEs with finite-dimensional noise. Consider the BDSDEs (3.8). Define the mollifier $K^m(x) = mc\exp\{\frac{1}{(mx-1)^2-1}\}$, if $0 < x < \frac{2}{m}$; $K^m(x) = 0$ otherwise, where c is chosen such that $\int_{-\infty}^{+\infty} K^m(x)\,dx = 1$. Define $h^m(x) = \int_{\mathbb{R}^d} h(y)K^m(x-y)\,dy$, $\tilde f^m(r,x) = \int_{\mathbb{R}^d} \tilde f(r,y)K^m(x-y)\,dy$ and $\tilde g_j^m(r,x) = \int_{\mathbb{R}^d} \tilde g_j(r,y)K^m(x-y)\,dy$. It is easy to see from standard results in analysis that $h^m(\cdot) \to h(\cdot)$, $\tilde f^m(r,\cdot) \to \tilde f(r,\cdot)$ and $\tilde g_j^m(r,\cdot) \to \tilde g_j(r,\cdot)$ in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$, respectively. Denote by $(\tilde Y_{s,m}^{t,x,n}, \tilde Z_{s,m}^{t,x,n})$ the solution of the following BDSDEs:

$$\tilde Y_{s,m}^{t,x,n} = h^m\big(X_T^{t,x}\big) + \int_s^T \tilde f^m\big(r, X_r^{t,x}\big)\,dr - \sum_{j=1}^n \int_s^T \tilde g_j^m\big(r, X_r^{t,x}\big)\,d^\dagger\hat\beta_j(r) - \int_s^T \big\langle \tilde Z_{r,m}^{t,x,n}, dW_r\big\rangle.$$

Let $\tilde u_m^n(t,x) = \tilde Y_{t,m}^{t,x,n}$. Then following classical results of Pardoux and Peng [26], we have $\tilde Z_{t,m}^{t,x,n} = \sigma^*\nabla\tilde u_m^n(t,x)$, and $\tilde Y_{s,m}^{t,x,n} = \tilde u_m^n(s, X_s^{t,x}) = \tilde Y_{s,m}^{s,X_s^{t,x},n}$, $\tilde Z_{s,m}^{t,x,n} = \sigma^*\nabla\tilde u_m^n(s,X_s^{t,x}) = \tilde Z_{s,m}^{s,X_s^{t,x},n}$. Moreover $\tilde u_m^n(t,x)$ satisfies the smootherized SPDE. In particular, for any smooth test function $\Psi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$, we still have

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}^d} \tilde u_m^n(s,x)\partial_s\Psi(s,x)\,dx\,ds + \int_{\mathbb{R}^d} \tilde u_m^n(t,x)\Psi(t,x)\,dx - \int_{\mathbb{R}^d} \tilde h^m(x)\Psi(T,x)\,dx \\
&\quad - \frac{1}{2}\int_t^T\int_{\mathbb{R}^d} \big(\sigma^*\nabla\tilde u_m^n\big)(s,x)\big(\sigma^*\nabla\Psi\big)(s,x)\,dx\,ds - \int_t^T\int_{\mathbb{R}^d} \tilde u_m^n(s,x)\nabla\big((b-\tilde A)\Psi\big)(s,x)\,dx\,ds \\
&= \int_t^T\int_{\mathbb{R}^d} \tilde f^m(s,x)\Psi(s,x)\,dx\,ds - \sum_{j=1}^n \int_t^T\int_{\mathbb{R}^d} \tilde g_j^m(s,x)\Psi(s,x)\,dx\,d^\dagger\hat\beta_j(s) \quad P\text{-a.s.}
\end{aligned} \tag{4.1}$$

But by standard estimates

$$E\left[\int_t^T\int_{\mathbb{R}^d} \big(\big|\tilde Y_{s,m}^{t,x,n} - \tilde Y_s^{t,x,n}\big|^2 + \big|\tilde Z_{s,m}^{t,x,n} - \tilde Z_s^{t,x,n}\big|^2\big)\rho^{-1}(x)\,dx\,ds\right] \to 0 \quad \text{as } m \to \infty.$$

And as $m_1, m_2 \to \infty$,

$$E\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|\tilde{u}^n_{m_1}\left(s,X_s^{t,x}\right)-\tilde{u}^n_{m_2}\left(s,X_s^{t,x}\right)\right|^2+\left|\sigma^*\nabla\tilde{u}^n_{m_1}\left(s,X_s^{t,x}\right)-\sigma^*\nabla\tilde{u}^n_{m_2}\left(s,X_s^{t,x}\right)\right|^2\right)\right.$$
$$\left.\times\,\rho^{-1}(x)\,dx\,ds\right]$$
$$=E\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|\tilde{Y}^{t,x,n}_{s,m_1}-\tilde{Y}^{t,x,n}_{s,m_2}\right|^2+\left|\tilde{Z}^{t,x,n}_{s,m_1}-\tilde{Z}^{t,x,n}_{s,m_2}\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]\to 0. \tag{4.2}$$

We define $\mathcal{H}$ to be the set of random fields $\{w(s,x);\ s\in[0,T],\ x\in\mathbb{R}^d\}$ such that $(w,\sigma^*\nabla w)\in M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$ with the norm $(E[\int_0^T\int_{\mathbb{R}^d}(|w(s,x)|^2+|(\sigma^*\nabla)w(s,x)|^2)\rho^{-1}(x)\,dx\,ds)^{\frac{1}{2}}$. Following a standard argument as in the proof of the completeness of the Sobolev spaces, we can prove $\mathcal{H}$ is complete. Now by the generalized equivalence of norm principle and (4.2), we can see that $\tilde{u}^n_m$ is a Cauchy sequence in $\mathcal{H}$. So there exists $\tilde{u}^n\in\mathcal{H}$ such that $(\tilde{u}^n_m,\sigma^*\nabla\tilde{u}^n_m)\to(\tilde{u}^n,\sigma^*\nabla\tilde{u}^n)$ in $M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$. Moreover $\tilde{Y}^{t,x,n}_s=\tilde{u}^n(s,X_s^{t,x})$, $\tilde{Z}^{t,x,n}_s=\sigma^*\nabla\tilde{u}^n(s,X_s^{t,x})$ for a.e. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. Now it is easy to pass the limit as $m\to\infty$ in (4.1) to conclude that $\tilde{u}^n$ is a weak solution of the corresponding SPDEs. For the nonlinear case, we can regard $\tilde{f}(r,x)=f(r,x,\tilde{u}^n(r,x),\sigma^*\nabla\tilde{u}^n(r,x))$, $\tilde{g}_j(r,x)=g_j(r,x,\tilde{u}^n(r,x),\sigma^*\nabla\tilde{u}^n(r,x))$, and $\tilde{f}$, $\tilde{g}_j$ satisfy the conditions in the above argument. If we define $u^n(t,x)=Y_t^{t,x,n}$ and $v^n(t,x)=Z_t^{t,x,n}$, using a similar proof as in the proof of Theorem 3.1 in [3] together with Theorem 3.5 and Proposition 4.1, we have, under conditions (H.1)–(H.4), $v^n(t,x)=(\sigma^*\nabla u^n)(t,x)$. Moreover, $(u^n,\sigma^*\nabla u^n)\in M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))\otimes M^{2,0}([0,T];L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$, $u^n(t,x)$ is the weak solution of the following SPDE:

$$u^n(t,x)=h(x)+\int_t^T\left[\mathscr{L}u^n(s,x)+f\left(s,x,u^n(s,x),\left(\sigma^*\nabla u^n\right)(s,x)\right)\right]ds$$
$$-\sum_{j=1}^n\int_t^T g_j\left(s,x,u^n(s,x),\left(\sigma^*\nabla u^n\right)(s,x)\right)d^\dagger\hat{\beta}_j(s),\quad 0\leqslant t\leqslant s\leqslant T.$$

That is to say, for any $\Psi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R}^1)$, we have

$$\int_t^T\int_{\mathbb{R}^d}u^n(s,x)\partial_s\Psi(s,x)\,dx\,ds+\int_{\mathbb{R}^d}u^n(t,x)\Psi(t,x)\,dx-\int_{\mathbb{R}^d}h(x)\Psi(T,x)\,dx$$
$$-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}\left(\sigma^*\nabla u^n\right)(s,x)\left(\sigma^*\nabla\Psi\right)(s,x)\,dx\,ds-\int_t^T\int_{\mathbb{R}^d}u^n(s,x)\nabla\left((b-\tilde{A})\Psi\right)(s,x)\,dx\,ds$$
$$=\int_t^T\int_{\mathbb{R}^d}f\left(s,x,u^n(s,x),\left(\sigma^*\nabla u^n\right)(s,x)\right)\Psi(s,x)\,dx\,ds$$

$$-\sum_{j=1}^{n}\int_t^T\int_{\mathbb{R}^d} g_j\big(s,x,u^n(s,x),\big(\sigma^*\nabla u^n\big)(s,x)\big)\Psi(s,x)\,dx\,d^\dagger\hat{\beta}_j(s)\quad P\text{-a.s.} \tag{4.3}$$

In this section, we study Eq. (2.16) with f and g allowed to depend on time as discussed in Section 3 and this section. By intuition if we define $u(t,x)=Y_t^{t,x}$, it should be a "weak solution" of the Eq. (2.16) with $u(T,x)=h(x)$. We will prove this result.

First we need some necessary preparations.

Proposition 4.2. *Under conditions* (H.1)–(H.4)*, let* $(Y_s^{t,x},Z_s^{t,x})$ *be the solution of Eq.* (3.1)*. If we define* $u(t,x)=Y_t^{t,x}$*, then* $\sigma^*\nabla u(t,x)$ *exists for a.e.* $t\in[0,T]$*,* $x\in\mathbb{R}^d$ *a.s., and* $u(s,X_s^{t,x})=Y_s^{t,x}$*,* $(\sigma^*\nabla u)(s,X_s^{t,x})=Z_s^{t,x}$ *for a.e.* $s\in[t,T]$*,* $x\in\mathbb{R}^d$ *a.s.*

Proof. First we prove u^n is a Cauchy sequence in $\mathcal{H}$. For this, by Lemma 2.6 and Proposition 4.1, as $m,n\to\infty$, we have

$$\begin{aligned}
&E\bigg[\int_0^T\int_{\mathbb{R}^d}\big(\big|u^m(s,x)-u^n(s,x)\big|^2+\big|\big(\sigma^*\nabla u^m\big)(s,x)-\big(\sigma^*\nabla u^n\big)(s,x)\big|^2\big)\rho^{-1}(x)\,dx\,ds\bigg]\\
&\quad\leqslant C_pE\bigg[\int_0^T\int_{\mathbb{R}^d}\big(\big|u^m\big(s,X_s^{0,x}\big)-u^n\big(s,X_s^{0,x}\big)\big|^2+\big|\big(\sigma^*\nabla u^m\big)\big(s,X_s^{0,x}\big)-\big(\sigma^*\nabla u^n\big)\big(s,X_s^{0,x}\big)\big|^2\big)\\
&\qquad\times\rho^{-1}(x)\,dx\,ds\bigg]\\
&\quad=C_pE\bigg[\int_0^T\int_{\mathbb{R}^d}\big(\big|Y_s^{0,x,m}-Y_s^{0,x,n}\big|^2+\big|Z_s^{0,x,m}-Z_s^{0,x,n}\big|^2\big)\rho^{-1}(x)\,dx\,ds\bigg]\to 0.
\end{aligned}$$

So there exists $\tilde{u}\in\mathcal{H}$ as the limit of u^n such that $\nabla\tilde{u}(s,x)$ exists for a.e. $s\in[0,T]$, $x\in\mathbb{R}^d$ a.s. and $E[\int_0^T\int_{\mathbb{R}^d}(|u^n(s,x)-\tilde{u}(s,x)|^2+|(\sigma^*\nabla u^n)(s,x)-(\sigma^*\nabla\tilde{u})(s,x)|^2)\rho^{-1}(x)\,dx\,ds]\to 0$. We define $u(t,x)=Y_t^{t,x}$, then similar to the proof as in Proposition 4.1, by the uniqueness of solution of Eq. (3.1), we have $u(s,X_s^{t,x})=Y_s^{t,x}$ for a.e. $s\in[t,T]$, $x\in\mathbb{R}^d$ a.s. Since

$$\begin{aligned}
&E\bigg[\int_0^T\int_{\mathbb{R}^d}\big|u(s,x)-\tilde{u}(s,x)\big|^2\rho^{-1}(x)\,dx\,ds\bigg]\\
&\quad\leqslant 2E\bigg[\int_0^T\int_{\mathbb{R}^d}\big(\big|u(s,x)-u^n(s,x)\big|^2+\big|u^n(s,x)-\tilde{u}(s,x)\big|^2\big)\rho^{-1}(x)\,dx\,ds\bigg]\\
&\quad\leqslant C_pE\bigg[\int_0^T\int_{\mathbb{R}^d}\big(\big|Y_s^{0,x}-Y_s^{0,x,n}\big|^2+\big|u^n(s,x)-\tilde{u}(s,x)\big|^2\big)\rho^{-1}(x)\,dx\,ds\bigg]\to 0,
\end{aligned}$$

$u(t,x) = \tilde{u}(t,x)$ for a.e. $t \in [0,T]$, $x \in \mathbb{R}^d$, a.s. So $\sigma^*\nabla u(t,x)$ exists for a.e. $t \in [0,T]$, $x \in \mathbb{R}^d$, almost surely. Using Lemma 2.6 again, we have

$$
\begin{aligned}
&E\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|u\left(s,X_s^{t,x}\right)-Y_s^{t,x}\right|^2+\left|\left(\sigma^*\nabla u\right)\left(s,X_s^{t,x}\right)-Z_s^{t,x}\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]\\
&\quad\leqslant 2E\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|u\left(s,X_s^{t,x}\right)-u^n\left(s,X_s^{t,x}\right)\right|^2+\left|u^n\left(s,X_s^{t,x}\right)-Y_s^{t,x}\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]\\
&\qquad+2E\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|\left(\sigma^*\nabla u\right)\left(s,X_s^{t,x}\right)-\left(\sigma^*\nabla u^n\right)\left(s,X_s^{t,x}\right)\right|^2+\left|\left(\sigma^*\nabla u^n\right)\left(s,X_s^{t,x}\right)-Z_s^{t,x}\right|^2\right)\right.\\
&\qquad\left.\times\rho^{-1}(x)\,dx\,ds\right]\\
&\quad\leqslant C_pE\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|u(s,x)-\tilde{u}(s,x)\right|^2+\left|\tilde{u}(s,x)-u^n(s,x)\right|^2+\left|Y_s^{t,x,n}-Y_s^{t,x}\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]\\
&\qquad+C_pE\left[\int_t^T\int_{\mathbb{R}^d}\left(\left|\left(\sigma^*\nabla u\right)(s,x)-\left(\sigma^*\nabla\tilde{u}\right)(s,x)\right|^2+\left|\left(\sigma^*\nabla\tilde{u}\right)(s,x)-\left(\sigma^*\nabla u^n\right)(s,x)\right|^2\right.\right.\\
&\qquad\left.\left.+\left|Z_s^{t,x,n}-Z_s^{t,x}\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]\to 0.
\end{aligned}
$$

So $u(s,X_s^{t,x}) = Y_s^{t,x}$, $(\sigma^*\nabla u)(s,X_s^{t,x}) = Z_s^{t,x}$ for a.e. $s \in [t,T]$, $x \in \mathbb{R}^d$ a.s. $\square$

From Proposition 4.2 and Lemma 2.6, it is easy to know that

$$
\begin{aligned}
&E\left[\int_t^T\int_{\mathbb{R}^d}\left|u^n(s,x)-u(s,x)\right|^2\rho^{-1}(x)\,dx\,ds\right]\\
&\qquad+E\left[\int_t^T\int_{\mathbb{R}^d}\left|\left(\sigma^*\nabla u^n\right)(s,x)-\left(\sigma^*\nabla u\right)(s,x)\right|^2\rho^{-1}(x)\,dx\,ds\right]\\
&\quad\leqslant C_pE\left[\int_t^T\int_{\mathbb{R}^d}\left|u^n\left(s,X_s^{t,x}\right)-u\left(s,X_s^{t,x}\right)\right|^2\rho^{-1}(x)\,dx\,ds\right]\\
&\qquad+C_pE\left[\int_t^T\int_{\mathbb{R}^d}\left|\left(\sigma^*\nabla u^n\right)\left(s,X_s^{t,x}\right)-\left(\sigma^*\nabla u\right)\left(s,X_s^{t,x}\right)\right|^2\rho^{-1}(x)\,dx\,ds\right]
\end{aligned}
$$

$$= C_p E\left[\int_t^T \int_{\mathbb{R}^d} \left|Y_s^{t,x,n} - Y_s^{t,x}\right|^2 \rho^{-1}(x)\,dx\,ds\right] + C_p E\left[\int_t^T \int_{\mathbb{R}^d} \left|Z_s^{t,x,n} - Z_s^{t,x}\right|^2 \rho^{-1}(x)\,dx\,ds\right]$$
$$\to 0,$$

as $n \to \infty$. This will be used in the following theorem.

Theorem 4.3. *Under conditions* (H.1)–(H.4)*, if we define* $u(t,x) = Y_t^{t,x}$*, where* $(Y_s^{t,x}, Z_s^{t,x})$ *is the solution of Eq.* (3.1)*, then* $u(t,x)$ *is the unique weak solution of Eq.* (2.16) *with* $u(T,x) = h(x)$*. Moreover,* $u(s, X_s^{t,x}) = Y_s^{t,x}$*,* $(\sigma^*\nabla u)(s, X_s^{t,x}) = Z_s^{t,x}$ *for a.e.* $s \in [t,T]$*,* $x \in \mathbb{R}^d$ *a.s.*

Proof. From Proposition 4.2, we only need to verify that this u is the unique weak solution of Eq. (2.16) with $u(T,x) = h(x)$. By Lemma 2.6, it is easy to see that $(\sigma^*\nabla u)(t,x) = Z_t^{t,x}$ for a.e. $t \in [0,T]$, $x \in \mathbb{R}^d$, a.s. Furthermore, by the generalized equivalence of norm principle again we have

$$E\left[\int_0^T \int_{\mathbb{R}^d} \left(\left|u(s,x)\right|^2 + \left|\left(\sigma^*\nabla u\right)(s,x)\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]$$
$$\leqslant C_p E\left[\int_0^T \int_{\mathbb{R}^d} \left(\left|u\left(s, X_s^{0,x}\right)\right|^2 + \left|\left(\sigma^*\nabla u\right)\left(s, X_s^{0,x}\right)\right|^2\right)\rho^{-1}(x)\,dx\,ds\right]$$
$$= C_p E\left[\int_0^T \int_{\mathbb{R}^d} \left(\left|Y_s^{0,x}\right|^2 + \left|Z_s^{0,x}\right|^2\right)\rho^{-1}(x)\,dx\,ds\right] < \infty.$$

Now we verify that $u(t,x)$ satisfies (2.17) with $u(T,x) = h(x)$ by passing the limit in $L^2(\Omega)$ to (4.3). We only show the convergence of the last term. The last term includes infinite-dimensional integral, but

$$E\left[\left|\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} g_j\left(s,x,u^n(s,x),\left(\sigma^*\nabla u^n\right)(s,x)\right)\Psi(s,x)\,dx\,d^\dagger\hat\beta_j(s)\right.\right.$$
$$\left.\left.- \sum_{j=1}^\infty \int_t^T \int_{\mathbb{R}^d} g_j\left(s,x,u(s,x),\left(\sigma^*\nabla u\right)(s,x)\right)\Psi(s,x)\,dx\,d^\dagger\hat\beta_j(s)\right|^2\right]$$
$$\leqslant 2E\left[\left|\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^d} \left(g_j\left(s,x,u^n(s,x),\left(\sigma^*\nabla u^n\right)(s,x)\right)\right.\right.\right.$$
$$\left.\left.\left.- g_j\left(s,x,u(s,x),\left(\sigma^*\nabla u\right)(s,x)\right)\right)\Psi(s,x)\,dx\,d^\dagger\hat\beta_j(s)\right|^2\right]$$

$$+2E\left[\left|\sum_{j=n+1}^{\infty}\int_t^T\int_{\mathbb{R}^d} g_j\big(s,x,u(s,x),\big(\sigma^*\nabla u\big)(s,x)\big)\Psi(s,x)\,dx\,d^\dagger\hat{\beta}_j(s)\right|^2\right]$$

$$\leqslant C_pE\left[\sum_{j=1}^{\infty}(C_j+\alpha_j)\int_t^T\int_{\mathbb{R}^d}\big(\big|u^n(t,x)-u(t,x)\big|^2+\big|\big(\sigma^*\nabla u^n\big)(s,x)-\big(\sigma^*\nabla u\big)(s,x)\big|^2\big)\right.$$
$$\left.\times\rho^{-1}(x)\,dx\,ds\right]$$
$$+C_pE\left[\left|\sum_{j=n+1}^{\infty}\int_t^T\int_{\mathbb{R}^d}\big(g_j\big(s,x,u(s,x),\big(\sigma^*\nabla u\big)(s,x)\big)\right.\right.$$
$$\left.\left.-\,g_j(s,x,0,0)\big)\Psi(s,x)\,dx\,d^\dagger\hat{\beta}_j(s)\right|^2\right]$$
$$+C_pE\left[\left|\sum_{j=n+1}^{\infty}\int_t^T\int_{\mathbb{R}^d} g_j(s,x,0,0)\Psi(s,x)\,dx\,d^\dagger\hat{\beta}_j(s)\right|^2\right].$$

It is obvious that the first term tends to zero as $n\to\infty$. The last two terms can be treated using a similar method as (3.17) and (3.18).

Therefore $u(t,x)$ satisfies (2.17), so is a weak solution of Eq. (2.16) with $u(T,x)=h(x)$. The uniqueness can be proved following a similar argument of Theorem 3.1 in Bally and Matoussi [3]. $\square$

5. Infinite horizon BDSDEs

We consider the following BDSDE with infinite-dimensional noise on infinite horizon,

$$\begin{aligned}\mathrm{e}^{-Ks}Y_s^{t,x}&=\int_s^{\infty}\mathrm{e}^{-Kr}f\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,dr+\int_s^{\infty}K\mathrm{e}^{-Kr}Y_r^{t,x}\,dr\\&\quad-\int_s^{\infty}\mathrm{e}^{-Kr}g\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,d^\dagger\hat{B}_r-\int_s^{\infty}\mathrm{e}^{-Kr}\big\langle Z_r^{t,x},dW_r\big\rangle.\end{aligned}\tag{5.1}$$

Here $f:[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathbb{R}^1$, $g:[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^1\times\mathbb{R}^d\to\mathcal{L}^2_{U_0}(\mathbb{R}^1)$. Eq. (5.1) is equivalent to

$$\begin{aligned}\mathrm{e}^{-Ks}Y_s^{t,x}&=\int_s^{\infty}\mathrm{e}^{-Kr}f\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,dr+\int_s^{\infty}K\mathrm{e}^{-Kr}Y_r^{t,x}\,dr\\&\quad-\sum_{j=1}^{\infty}\int_s^{\infty}\mathrm{e}^{-Kr}g_j\big(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x}\big)\,d^\dagger\hat{\beta}_j(r)-\int_s^{\infty}\mathrm{e}^{-Kr}\big\langle Z_r^{t,x},dW_r\big\rangle.\end{aligned}$$

We assume:

(H.5) Change $\mathscr{B}_{[0,T]}$ to $\mathscr{B}_{\mathbb{R}^+}$ and $t \in [0, T]$ to $t \geqslant 0$ in (H.2);
(H.6) Change $\int_0^T$ to $\int_0^\infty e^{-Ks}$ in (H.3);
(H.7) There exists a constant $\mu > 0$ with $2\mu - K - 2C - \sum_{j=1}^\infty C_j > 0$ such that for any $t \geqslant 0$, $Y_1, Y_2 \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^1)$, $X, Z \in L^2_\rho(\mathbb{R}^d; \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \big(Y_1(x) - Y_2(x)\big)\big(f\big(t, X(x), Y_1(x), Z(x)\big) - f\big(t, X(x), Y_2(x), Z(x)\big)\big)\rho^{-1}(x)\,dx$$
$$\leqslant -\mu \int_{\mathbb{R}^d} \big|Y_1(x) - Y_2(x)\big|^2 \rho^{-1}(x)\,dx.$$

The main objective of this section is to prove

Theorem 5.1. *Under conditions* (H.4)–(H.7)*, Eq.* (5.1) *has a unique solution.*

Proof. *Uniqueness*. Let $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ be two solutions of Eq. (5.1). Define

$$\begin{aligned}
&\bar{Y}_s^{t,x} = \hat{Y}_s^{t,x} - Y_s^{t,x}, \qquad \bar{Z}_s^{t,x} = \hat{Z}_s^{t,x} - Z_s^{t,x},\\
&\bar{f}(s,x) = f\big(s, X_s^{t,x}, \hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}\big) - f\big(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\big),\\
&\bar{g}(s,x) = g\big(s, X_s^{t,x}, \hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}\big) - g\big(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\big), \quad s \geqslant 0.
\end{aligned}$$

Then for $s \geqslant 0$ and a.e. $x \in \mathbb{R}^d$, $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ satisfy

$$\begin{cases} d\bar{Y}_s^{t,x} = -\bar{f}(s,x)\,ds + \sum_{j=1}^\infty \bar{g}_j(s,x)\,d^\dagger\hat{\beta}_j(s) + \langle \bar{Z}_s^{t,x}, dW_s\rangle,\\ \lim_{T\to\infty} e^{-KT}\bar{Y}_T^{t,x} = 0 \quad \text{a.s.} \end{cases}$$

For a.e. $x \in \mathbb{R}^d$, applying Itô's formula for infinite-dimensional noise to $e^{-Ks}|\bar{Y}_s^{t,x}|^2$, and by Young inequality and conditions (H.5), (H.7), we obtain

$$\begin{aligned}
&E\bigg[\int_{\mathbb{R}^d} e^{-Ks}\big|\bar{Y}_s^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg]\\
&\quad + \bigg(2\mu - K - 2C - \sum_{j=1}^\infty C_j\bigg)E\bigg[\int_s^T\int_{\mathbb{R}^d} e^{-Kr}\big|\bar{Y}_r^{t,x}\big|^2\rho^{-1}(x)\,dx\,dr\bigg]\\
&\quad + \bigg(\frac{1}{2} - \sum_{j=1}^\infty \alpha_j\bigg)E\bigg[\int_s^T\int_{\mathbb{R}^d} e^{-Kr}\big|\bar{Z}_r^{t,x}\big|^2\rho^{-1}(x)\,dx\,dr\bigg]\\
&\leqslant E\bigg[\int_{\mathbb{R}^d} e^{-KT}\big|\bar{Y}_T^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg]. \qquad (5.2)
\end{aligned}$$

Taking $K' > K$ such that $2\mu - K' - 2C - \sum_{j=1}^{\infty} C_j > 0$ as well, we can see that (5.2) remains true with K replaced by K'. In particular,

$$E\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-K's}\big|\bar{Y}_s^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg] \leqslant E\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-K'T}\big|\bar{Y}_T^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg].$$

Therefore, we have

$$E\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-K's}\big|\bar{Y}_s^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg] \leqslant \mathrm{e}^{-(K'-K)T} E\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-KT}\big|\bar{Y}_T^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg]. \tag{5.3}$$

Since $\hat{Y}_s^{t,x}, Y_s^{t,x} \in S^{2,-K} \cap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, so

$$\sup_{T\geqslant 0} E\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-KT}\big|\bar{Y}_T^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg] \leqslant E\bigg[\sup_{T\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-KT}\big(2\big|\hat{Y}_T^{t,x}\big|^2 + 2\big|Y_T^{t,x}\big|^2\big)\rho^{-1}(x)\,dx\bigg] < \infty.$$

Therefore, taking the limit as $T \to \infty$ in (5.3), we have

$$E\bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-K's}\big|\bar{Y}_s^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg] = 0.$$

Then the uniqueness is proved.

Existence. For each $n \in \mathbb{N}$, we define a sequence of BDSDEs (3.1) with $h = 0$ and $T = n$ and denote it by Eq. (3.1_n). It is easy to verify that for each n, these BDSDEs satisfy conditions of Theorem 3.2. Therefore, for each n, there exists a $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,0}([0,n]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,0}([0,n]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ which is equivalent to the space $S^{2,-K}([0,n]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,-K}([0,n]; L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is the unique solution of Eq. (3.1_n). That is to say, for an arbitrary $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$, $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{-Ks} Y_s^{t,x,n}\varphi(x)\,dx \\
&\quad = \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} f\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\varphi(x)\,dx\,dr + \int_s^n \int_{\mathbb{R}^d} K\mathrm{e}^{-Kr} Y_r^{t,x,n}\varphi(x)\,dx\,dr \\
&\qquad - \sum_{j=1}^{\infty} \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r) \\
&\qquad - \int_s^n \bigg\langle \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} Z_r^{t,x,n}\varphi(x)\,dx, dW_r\bigg\rangle \quad P\text{-a.s.}
\end{aligned} \tag{5.4}$$

Let $(Y_t^n, Z_t^n)_{t>n} = (0,0)$, then $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,-K} \cap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$. We will prove $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence. For this, let $(Y_s^{t,x,m}, Z_s^{t,x,m})$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ be the solutions of Eqs. (3.1_m) and (3.1_n), respectively. Without losing any generality, assume that $m \geqslant n$, and define

$$\begin{aligned}
&\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m} - Y_s^{t,x,n}, \qquad \bar{Z}_s^{t,x,m,n} = Z_s^{t,x,m} - Z_s^{t,x,n},\\
&\bar{f}^{m,n}(s,x) = f\big(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}\big) - f\big(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}\big),\\
&\bar{g}_j^{m,n}(s,x) = g_j\big(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}\big) - g_j\big(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}\big), \quad s \geqslant 0.
\end{aligned}$$

Consider two cases:

(i) When $n \leqslant s \leqslant m$, $\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m}$. Since $(Y_s^{t,x,m}, Z_s^{t,x,m})$ is the solution of Eq. (3.1_m), we have for any $m \in \mathbb{N}$,

$$\begin{cases}
dY_s^{t,x,m} = -f(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m})\,ds\\
\qquad\qquad + \sum_{j=1}^{\infty} g_j(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m})\,d^\dagger\hat{\beta}_j(s) + \langle Z_s^{t,x,m}, dW_s\rangle,\\
Y_m^{t,x,m} = 0 \quad \text{for } s \in [0,m],\ \text{a.e. } x \in \mathbb{R}^d,\ \text{a.s.}
\end{cases}$$

Noting that $E[\int_0^m \|g(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m})\|^2_{\mathcal{L}^2_{U_0}(\mathbb{R}^1)}\,dr] < \infty$ for a.e. $x \in \mathbb{R}^d$, we can apply Itô's formula to $\mathrm{e}^{-Kr}|Y_r^{t,x,m}|^2$ for a.e. $x \in \mathbb{R}^d$, then taking integration in $\mathbb{R}^d \times \Omega$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}\big|Y_s^{t,x,m}\big|^2 \rho^{-1}(x)\,dx\\
&\quad + \left(2\mu - K - 2C - \sum_{j=1}^{\infty} C_j - \left(1 + \sum_{j=1}^{\infty} C_j\right)\varepsilon\right) \int_s^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big|Y_r^{t,x,m}\big|^2 \rho^{-1}(x)\,dx\,dr\\
&\quad + \left(\frac{1}{2} - \sum_{j=1}^{\infty}\alpha_j - \sum_{j=1}^{\infty}\alpha_j\varepsilon\right) \int_s^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big|Z_r^{t,x,m}\big|^2 \rho^{-1}(x)\,dx\,dr\\
&\leqslant C_p \int_s^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big|f\big(r, X_r^{t,x}, 0, 0\big)\big|^2 \rho^{-1}(x)\,dx\,dr\\
&\quad + C_p \int_s^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \sum_{j=1}^{\infty}\big|g_j\big(r, X_r^{t,x}, 0, 0\big)\big|^2 \rho^{-1}(x)\,dx\,dr\\
&\quad - \sum_{j=1}^{\infty} \int_s^m \int_{\mathbb{R}^d} 2\mathrm{e}^{-Kr} Y_r^{t,x,m} g_j\big(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m}\big)\rho^{-1}(x)\,dx\,d^\dagger\hat{\beta}_j(r)\\
&\quad - \int_s^m \left\langle \int_{\mathbb{R}^d} 2\mathrm{e}^{-Kr} Y_r^{t,x,m} Z_r^{t,x,m} \rho^{-1}(x)\,dx, dW_r\right\rangle.
\end{aligned} \tag{5.5}$$

Note that the constant ε can be chosen to be sufficiently small such that all the terms on the left-hand side of (5.5) are positive. By (5.5), as $n, m \to \infty$ we have

$$E\left[\int_n^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|Y_r^{t,x,m}\right|^2 \rho^{-1}(x)\,dx\,dr\right] + E\left[\int_n^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|Z_r^{t,x,m}\right|^2 \rho^{-1}(x)\,dx\,dr\right]$$
$$\leqslant C_p E\left[\int_n^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left(\left|f\left(r, X_r^{t,x}, 0, 0\right)\right|^2 + \sum_{j=1}^{\infty} \left|g_j\left(r, X_r^{t,x}, 0, 0\right)\right|^2\right) \rho^{-1}(x)\,dx\,dr\right] \to 0. \quad (5.6)$$

Note that the right-hand side of (5.6) converges to 0 follows from the generalized equivalence of norm principle. Also using the B–D–G inequality to deal with (5.5) in the interval $[n, m]$, by (5.6), as $n, m \to \infty$ we have

$$E\left[\sup_{n\leqslant s\leqslant m} \int_{\mathbb{R}^d} \mathrm{e}^{-Ks} \left|Y_s^{t,x,m}\right|^2 \rho^{-1}\,dx\right]$$
$$\leqslant C_p E\left[\int_n^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left(\left|f\left(r, X_r^{t,x}, 0, 0\right)\right|^2 + \sum_{j=1}^{\infty} \left|g_j\left(r, X_r^{t,x}, 0, 0\right)\right|^2\right) \rho^{-1}(x)\,dx\,dr\right]$$
$$+ C_p E\left[\int_n^m \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left(\left|Y_r^{t,x,m}\right|^2 + \left|Z_r^{t,x,m}\right|^2\right) \rho^{-1}(x)\,dx\,dr\right] \to 0. \quad (5.7)$$

(ii) When $0 \leqslant s \leqslant n$,

$$\bar{Y}_s^{t,x,m,n} = Y_n^{t,x,m} + \int_s^n \bar{f}^{m,n}(r,x)\,dr - \sum_{j=1}^{\infty} \int_s^n \bar{g}_j^{m,n}(r,x)\,d^\dagger \hat{\beta}_j(r) - \int_s^n \left\langle \bar{Z}_r^{t,x,m,n}, dW_r\right\rangle.$$

Apply Itô's formula to $\mathrm{e}^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2$ for a.e. $x \in \mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \mathrm{e}^{-Ks} \left|\bar{Y}_s^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx + \left(2\mu - K - 2C - \sum_{j=1}^{\infty} C_j\right) \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|\bar{Y}_r^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\,dr$$
$$+ \left(\frac{1}{2} - \sum_{j=1}^{\infty} \alpha_j\right) \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|\bar{Z}_r^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\,dr$$
$$\leqslant \int_{\mathbb{R}^d} \mathrm{e}^{-Kn} \left|Y_n^{t,x,m}\right|^2 \rho^{-1}(x)\,dx - \sum_{j=1}^{\infty} \int_s^n \int_{\mathbb{R}^d} 2\mathrm{e}^{-Kr} \bar{Y}_r^{t,x,m,n} \bar{g}_j^{m,n}(r,x) \rho^{-1}(x)\,dx\,d^\dagger \hat{\beta}_j(r)$$
$$- \int_s^n \left\langle \int_{\mathbb{R}^d} 2\mathrm{e}^{-Kr} \bar{Y}_r^{t,x,m,n} \bar{Z}_r^{t,x,m,n} \rho^{-1}(x)\,dx, dW_r\right\rangle. \quad (5.8)$$

Taking expectation on both sides of (5.8), as $n, m \to \infty$, using (5.7), we have

$$E\left[\int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|\bar{Y}_r^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\,dr\right] + E\left[\int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|\bar{Z}_r^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\,dr\right]$$
$$\leqslant C_p E\left[\sup_{n\leqslant s\leqslant m} \int_{\mathbb{R}^d} \mathrm{e}^{-Ks} \left|Y_s^{t,x,m}\right|^2 \rho^{-1}(x)\,dx\right] \to 0. \tag{5.9}$$

Also by the B–D–G inequality, (5.7)–(5.9), as $n, m \to \infty$, we have

$$E\left[\sup_{0\leqslant s\leqslant n} \int_{\mathbb{R}^d} \mathrm{e}^{-Ks} \left|\bar{Y}_s^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\right] \leqslant C_p E\left[\sup_{n\leqslant s\leqslant m} \int_{\mathbb{R}^d} \mathrm{e}^{-Ks} \left|Y_s^{t,x,m}\right|^2 \rho^{-1}(x)\,dx\right] \to 0.$$

Therefore taking a combination of cases (i) and (ii), as $n, m \to \infty$, we have

$$E\left[\sup_{s\geqslant 0} \int_{\mathbb{R}^d} \mathrm{e}^{-Ks} \left|\bar{Y}_s^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\right] + E\left[\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|\bar{Y}_r^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\,dr\right]$$
$$+ E\left[\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \left|\bar{Z}_r^{t,x,m,n}\right|^2 \rho^{-1}(x)\,dx\,dr\right] \to 0.$$

That is to say $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence. Take $(Y_s^{t,x}, Z_s^{t,x})$ as the limit of $(Y_s^{t,x,n}, Z_s^{t,x,n})$ in the space $S^{2,-K} \cap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and we will show that $(Y_s^{t,x}, Z_s^{t,x})$ is the solution of Eq. (5.1). We only need to verify that for arbitrary $\varphi \in C_c^0(\mathbb{R}^d;\mathbb{R}^1)$, $(Y_s^{t,x}, Z_s^{t,x})$ satisfies (2.14$_r$), where (2.14$_r$) means a more general form of (2.14) with f and g_j also depending on $r \in [0,\infty)$. Since $(Y_s^{t,x,n}, Z_s^{t,x,n})$ satisfies Eq. (5.4), so we verify that Eq. (5.4) converges to Eq. (2.14$_r$) in $L^2(\Omega)$ term by term as $n \to \infty$. We only show the infinite-dimensional stochastic integral term:

$$E\Bigg[\bigg|\sum_{j=1}^\infty \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)$$
$$- \sum_{j=1}^\infty \int_s^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]$$
$$\leqslant 2E\Bigg[\bigg|\sum_{j=1}^\infty \int_s^n \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} \Big(g_j\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big) - g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\Big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg]$$
$$+ 2E\Bigg[\bigg|\sum_{j=1}^\infty \int_n^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\,dx\,d^\dagger\hat{\beta}_j(r)\bigg|^2\Bigg].$$

We see that each term in the above formula tends to zero as $n \to \infty$ since

$$E\left[\left|\sum_{j=1}^{\infty}\int_s^n\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big(g_j\big(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}\big) - g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\big)\varphi(x)\,dx\,d^{\dagger}\hat{\beta}_j(r)\right|^2\right]$$

$$\leqslant C_p E\left[\int_0^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big(\big|Y_r^{t,x,n} - Y_r^{t,x}\big|^2 + \big|Z_r^{t,x,n} - Z_r^{t,x}\big|^2\big)\rho^{-1}(x)\,dx\,dr\right] \to 0, \quad \text{as } n \to \infty,$$

and

$$E\left[\left|\sum_{j=1}^{\infty}\int_n^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-Kr} g_j\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\varphi(x)\,dx\,d^{\dagger}\hat{\beta}_j(r)\right|^2\right]$$

$$\leqslant C_p E\left[\int_n^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big(\big|Y_r^{t,x}\big|^2 + \big|Z_r^{t,x}\big|^2\big)\rho^{-1}(x)\,dx\,dr\right]$$

$$+ C_p \int_n^{\infty}\int_{\mathbb{R}^d}\sum_{j=1}^{\infty} \mathrm{e}^{-Kr}\big|g_j(r, x, 0, 0)\big|^2\rho^{-1}(x)\,dx\,dr \to 0, \quad \text{as } n \to \infty.$$

That is to say $(Y_s^{t,x}, Z_s^{t,x})_{s\geqslant 0}$ satisfies Eq. (2.14_r). The proof of Theorem 5.1 is completed. □

By similar method as in the proof of existence part case (i) in Theorem 5.1, we have the following estimation.

Proposition 5.2. *Let* $(Y_s^{t,x,n}, Z_s^{t,x,n})$ *be the solution of Eq.* (3.1_n)*, then under the conditions of Theorem* 5.1*,*

$$\sup_n E\left[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-Ks}\big|Y_s^{t,x,n}(x)\big|^2\rho^{-1}(x)\,dx\right] + \sup_n E\left[\int_0^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big|Y_r^{t,x,n}(x)\big|^2\rho^{-1}(x)\,dx\,dr\right]$$

$$+ \sup_n E\left[\int_0^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-Kr}\big|Z_r^{t,x,n}(x)\big|^2\rho^{-1}(x)\,dx\,dr\right] < \infty.$$

6. The continuity of the solution of the infinite horizon BDSDEs as the solution of the corresponding SPDEs

Now we study BDSDE (2.13), a simpler form of Eq. (5.1).

Proof of Theorem 2.10. Since conditions here are stronger than those in Theorem 5.1, so there exists a unique solution $(Y_s^{t,x}, Z_s^{t,x})$. We only need to prove $E[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-pKs}|Y_s^{t,x}|^p \times$

$\rho^{-1}(x)\,dx] < \infty$. Let $\varphi_{N,p}(x) = x^{\frac{p}{2}} I_{\{0\leqslant x<N\}} + \frac{p}{2} N^{\frac{p-2}{2}}(x-N) I_{\{x\geqslant N\}}$. We apply generalized Itô's formula to $\mathrm{e}^{-pKr}\varphi_{N,p}(\psi_M(Y_r^{t,x}))$ for a.e. $x \in \mathbb{R}^d$ to have the following estimation:

$$\begin{aligned}
&\mathrm{e}^{-pKs}\varphi_{N,p}\big(\psi_M\big(Y_s^{t,x}\big)\big) - pK\int_s^T \mathrm{e}^{-pKr}\varphi_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big)\,dr \\
&\quad + \frac{1}{2}\int_s^T \mathrm{e}^{-pKr}\varphi''_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big)\big|\psi'_M\big(Y_r^{t,x}\big)\big|^2\big|Z_r^{t,x}\big|^2\,dr \\
&\quad + \int_s^T \mathrm{e}^{-pKr}\varphi'_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big) I_{\{-M\leqslant Y_r^{t,x}<M\}}\big|Z_r^{t,x}\big|^2\,dr \\
&\leqslant \mathrm{e}^{-pKT}\varphi_{N,p}\big(\psi_M\big(Y_T^{t,x}\big)\big) + \int_s^T \mathrm{e}^{-pKr}\varphi'_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big)\psi'_M\big(Y_r^{t,x}\big) f\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,dr \\
&\quad + \int_s^T \mathrm{e}^{-pKr}\varphi'_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big) I_{\{-M\leqslant Y_r^{t,x}<M\}} \sum_{j=1}^{\infty}\big|g_j\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\big|^2\,dr \\
&\quad + \frac{1}{2}\int_s^T \mathrm{e}^{-pKr}\varphi''_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big)\big|\psi'_M\big(Y_r^{t,x}\big)\big|^2 \sum_{j=1}^{\infty}\big|g_j\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\big|^2\,dr \\
&\quad - \sum_{j=1}^{\infty}\int_s^T \mathrm{e}^{-pKr}\varphi'_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big)\psi'_M\big(Y_r^{t,x}\big) g_j\big(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\big)\,d^{\dagger}\hat{\beta}_j(r) \\
&\quad - \int_s^T \big\langle \mathrm{e}^{-pKr}\varphi'_{N,p}\big(\psi_M\big(Y_r^{t,x}\big)\big)\psi'_M\big(Y_r^{t,x}\big) Z_r^{t,x}, dW_r\big\rangle.
\end{aligned} \tag{6.1}$$

Note that $\lim_{T\to\infty} \mathrm{e}^{-pKT}\varphi_{N,p}(\psi_M(Y_T^{t,x})) = 0$, so after taking limit as $T \to \infty$, we take the integration on $\Omega \times \mathbb{R}^d$. As $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2,-K} \cap M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)) \otimes M^{2,-K}([0,\infty); L_\rho^2(\mathbb{R}^d;\mathbb{R}^d))$ and $\varphi'_{N,p}(\psi_M(Y_r^{t,x}))\psi'_M(Y_r^{t,x})$ is bounded, we can use the stochastic Fubini theorem and all the stochastic integrals have zero expectation. Using conditions (A.1)′–(A.4)′, and taking the limit as $M \to \infty$ first, then the limit as $N \to \infty$, by the monotone convergence theorem, we have

$$\begin{aligned}
&\left(p\mu - pK - pC - \frac{p(p-1)}{2}\sum_{j=1}^{\infty} C_j - \left(3 + \frac{p(p-1)}{2}\sum_{j=1}^{\infty} C_j\right)\varepsilon\right) \\
&\quad \times E\left[\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}\big|Y_r^{t,x}\big|^p \rho^{-1}(x)\,dx\,dr\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{p}{4}\left(2p-3-(2p-2)\sum_{j=1}^{\infty}\alpha_j-(2p-2)\sum_{j=1}^{\infty}\alpha_j\varepsilon\right)\\
&\times E\left[\int_s^{\infty}\int_{\mathbb{R}^d}\mathrm{e}^{-pKr}\left|Y_r^{t,x}\right|^{p-2}\left|Z_r^{t,x}\right|^2\rho^{-1}(x)\,dx\,dr\right]\\
\leqslant{}& C_p\int_{\mathbb{R}^d}\left|f(x,0,0)\right|^p\rho^{-1}(x)\,dx+C_p\int_{\mathbb{R}^d}\sum_{j=1}^{\infty}\left|g_j(x,0,0)\right|^p\rho^{-1}(x)\,dx<\infty. \qquad (6.2)
\end{aligned}
$$

Note that the constant ε can be chosen to be sufficiently small such that all the terms on the left-hand side of (6.2) are positive. Also by the B–D–G inequality, Cauchy–Schwarz inequality and Young inequality, from (6.1) we have

$$
\begin{aligned}
&E\left[\sup_{s\geqslant 0}\int_{\mathbb{R}^d}\mathrm{e}^{-pKs}\left|Y_s^{t,x}\right|^p\rho^{-1}(x)\,dx\right]\\
&\leqslant C_p\int_{\mathbb{R}^d}\left|f(x,0,0)\right|^p\rho^{-1}(x)\,dx+C_p\int_{\mathbb{R}^d}\sum_{j=1}^{\infty}\left|g_j(x,0,0)\right|^p\rho^{-1}(x)\,dx\\
&\quad+C_pE\left[\int_0^{\infty}\int_{\mathbb{R}^d}\mathrm{e}^{-pKr}\left|Y_r^{t,x}\right|^{p-2}\left|Z_r^{t,x}\right|^2\rho^{-1}(x)\,dx\,dr\right]\\
&\quad+C_pE\left[\int_0^{\infty}\int_{\mathbb{R}^d}\mathrm{e}^{-pKr}\left|Y_r^{t,x}\right|^p\rho^{-1}(x)\,dx\,dr\right].
\end{aligned}
$$

So by (6.2), Theorem 2.10 is proved. □

We need to prove two lemmas before giving a proof of Theorem 2.11.

Lemma 6.1. *Under condition* (A.3)′*, for arbitrary* $T>0$*,* $t,t'\in[0,T]$*,*

$$
E\left[\int_0^{\infty}\int_{\mathbb{R}^d}\mathrm{e}^{-Kr}\left|X_r^{t',x}-X_r^{t,x}\right|^p\rho^{-1}(x)\,dx\,dr\right]\leqslant C_p|t'-t|^{\frac{p}{2}}\quad a.s.
$$

Proof. It is not difficult to deduce from Lemma 4.5.6 in [17], so we omit the proof. □

Lemma 6.2. *Under conditions* (A.1)′–(A.4)′*, for arbitrary* $T>0$*,* $t,t'\in[0,T]$*, let* $(Y_s^{t',x})_{s\geqslant 0}$*,* $(Y_s^{t,x})_{s\geqslant 0}$ *be the solutions of Eq.* (5.1)*, then*

$$
E\left[\sup_{s\geqslant 0}\int_{\mathbb{R}^d}\mathrm{e}^{-pKs}\left|Y_s^{t',x}-Y_s^{t,x}\right|^p\rho^{-1}(x)\,dx\right]\leqslant C_p|t'-t|^{\frac{p}{2}}.
$$

Proof. Let

$$\bar{Y}_s = Y_s^{t',x} - Y_s^{t,x}, \qquad \bar{Z}_s = Z_s^{t',x} - Z_s^{t,x},$$
$$\bar{f}(s) = f\big(X_s^{t',x}, Y_s^{t',x}, Z_s^{t',x}\big) - f\big(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\big),$$
$$\bar{g}_j(s) = g_j\big(X_s^{t',x}, Y_s^{t',x}, Z_s^{t',x}\big) - g_j\big(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\big), \quad s \geqslant 0.$$

Then

$$\begin{cases} d\bar{Y}_s = -\bar{f}(s)\,ds + \sum_{j=1}^{\infty} \bar{g}_j(s)\,d^{\dagger}\hat{\beta}_j(s) + \langle \bar{Z}_s, dW_s\rangle, \\ \lim_{T\to\infty} \mathrm{e}^{-KT}\bar{Y}_T = 0 \quad \text{for a.e. } x \in \mathbb{R}^d \text{ a.s.} \end{cases}$$

First note that from Theorem 2.10, we know $E[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-pKs}|\bar{Y}_s|^p\rho^{-1}(x)\,dx] < \infty$. Applying Itô's formula to $\mathrm{e}^{-pKr}|\bar{Y}_r|^p$ for a.e. $x \in \mathbb{R}^d$ (we leave out procedure of localization as in (6.1) for simplicity) and taking integration on $\mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{e}^{-pKs}|\bar{Y}_s|^p\rho^{-1}(x)\,dx \\
&\quad + \left(p\mu - pK - pC - \frac{p(p-1)}{2}\sum_{j=1}^{\infty} C_j - 3\varepsilon\right)\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^p\rho^{-1}(x)\,dx\,dr \\
&\quad + \frac{p}{4}\left(2p - 3 - (2p-2)\sum_{j=1}^{\infty}\alpha_j\right)\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2\rho^{-1}(x)\,dx\,dr \\
&\leqslant C_p\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{X}_r|^p\rho^{-1}(x)\,dx\,dr - p\sum_{j=1}^{\infty}\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^{p-2}\bar{Y}_r\bar{g}_j(r)\rho^{-1}(x)\,dx\,d^{\dagger}\hat{\beta}_j(r) \\
&\quad - p\int_s^{\infty}\left\langle \int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^{p-2}\bar{Y}_r\bar{Z}_r\rho^{-1}(x)\,dx, dW_r\right\rangle.
\end{aligned} \tag{6.3}$$

Note that the constant ε can be chosen to be sufficiently small such that all the terms on the left-hand side of (6.3) are positive. Taking integration on Ω on both sides of (6.3), by Lemma 6.1 we have

$$\begin{aligned}
&E\left[\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^p\rho^{-1}(x)\,dx\,dr\right] + E\left[\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2\rho^{-1}(x)\,dx\,dr\right] \\
&\quad \leqslant C_pE\left[\int_s^{\infty}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{X}_r|^p\rho^{-1}(x)\,dx\,dr\right] \\
&\quad \leqslant C_p|t'-t|^{\frac{p}{2}}.
\end{aligned} \tag{6.4}$$

Also by the B–D–G inequality, from (6.3) and (6.4), we have

$$
\begin{aligned}
&E\Big[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-pKs}|\bar{Y}_s|^p\rho^{-1}(x)\,dx\Big]\\
&\quad\leqslant C_pE\Big[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{X}_r|^p\rho^{-1}(x)\,dx\,dr\Big]+C_pE\Big[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^p\rho^{-1}(x)\,dx\,dr\Big]\\
&\qquad+C_pE\Big[\int_0^\infty\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}|\bar{Y}_r|^{p-2}|\bar{Z}_r|^2\rho^{-1}(x)\,dx\,dr\Big]\\
&\quad\leqslant C_p|t'-t|^{\frac{p}{2}}. \qquad \square
\end{aligned}
$$

Proof of Theorem 2.11. By Lemma 6.2, we have

$$
\begin{aligned}
&E\bigg(\Big[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-2Ks}\big|Y_s^{t',x}-Y_s^{t,x}\big|^2\rho^{-1}(x)\,dx\Big]\bigg)^{\frac{p}{2}}\\
&\quad\leqslant C_pE\Big[\sup_{s\geqslant 0}\int_{\mathbb{R}^d} \mathrm{e}^{-pKr}\big|Y_s^{t',x}-Y_s^{t,x}\big|^p\rho^{-1}(x)\,dx\Big]\bigg(\int_{\mathbb{R}^d}\rho^{-1}(x)\,dx\bigg)^{\frac{p-2}{2}}\\
&\quad\leqslant C_p|t'-t|^{\frac{p}{2}}.
\end{aligned}
$$

Noting $p>2$, by the Kolmogorov continuity theorem (see [17]), we have $t\to Y_s^{t,x}$ is a.s. continuous for $t\in[0,T]$ under the norm $(\sup_{s\geqslant 0}\int_{\mathbb{R}^d}\mathrm{e}^{-2Ks}|\cdot|^2\rho^{-1}(x)\,dx)^{\frac{1}{2}}$. Without losing any generality, assume that $t'\geqslant t$. Then we can see

$$
\begin{aligned}
&\lim_{t'\to t}\bigg(\int_{\mathbb{R}^d}\mathrm{e}^{-2Kt'}\big|Y_{t'}^{t',x}-Y_{t'}^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg)^{\frac{1}{2}}\\
&\quad\leqslant\lim_{t'\to t}\bigg(\sup_{s\geqslant 0}\int_{\mathbb{R}^d}\mathrm{e}^{-2Ks}\big|Y_s^{t',x}-Y_s^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg)^{\frac{1}{2}}=0\quad\text{a.s.}
\end{aligned}
$$

Notice $t'\in[0,T]$, so

$$
\lim_{t'\to t}\bigg(\int_{\mathbb{R}^d}\big|Y_{t'}^{t',x}-Y_{t'}^{t,x}\big|^2\rho^{-1}(x)\,dx\bigg)^{\frac{1}{2}}=0\quad\text{a.s.} \tag{6.5}
$$

Since $Y_\cdot^{t,\cdot}\in S^{2,-K}([0,\infty);L_\rho^2(\mathbb{R}^d;\mathbb{R}^1))$, $Y_{t'}^{t,\cdot}$ is continuous with respect to t' in $L_\rho^2(\mathbb{R}^d;\mathbb{R}^1)$. That is to say for each t,

$$\lim_{t'\to t}\left(\int_{\mathbb{R}^d}\left|Y_{t'}^{t,x}-Y_t^{t,x}\right|^2\rho^{-1}(x)\,dx\right)^{\frac{1}{2}}=0\quad\text{a.s.}\tag{6.6}$$

Now by (6.5) and (6.6)

$$\begin{aligned}&\lim_{t'\to t}\left(\int_{\mathbb{R}^d}\left|Y_{t'}^{t',x}-Y_t^{t,x}\right|^2\rho^{-1}(x)\,dx\right)^{\frac{1}{2}}\\&\leqslant\lim_{t'\to t}\left(\int_{\mathbb{R}^d}\left|Y_{t'}^{t',x}-Y_{t'}^{t,x}\right|^2\rho^{-1}(x)\,dx\right)^{\frac{1}{2}}+\lim_{t'\to t}\left(\int_{\mathbb{R}^d}\left|Y_{t'}^{t,x}-Y_t^{t,x}\right|^2\rho^{-1}(x)\,dx\right)^{\frac{1}{2}}\\&=0\quad\text{a.s.}\end{aligned}$$

For arbitrary $T>0$, $0\leqslant t\leqslant T$, define $u(t,\cdot)=Y_t^{t,\cdot}$, then $u(t,\cdot)$ is a.s. continuous with respect to t in $L^2_\rho(\mathbb{R}^d;\mathbb{R}^1)$. Since $Y_\cdot^{t,\cdot}\in S^{2,-K}([0,\infty);L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$, $Y_T^{T,x}$ is $\mathscr{F}^{\hat{B}}_{T,\infty}\otimes\mathscr{B}_{\mathbb{R}^d}$ measurable and $E[\int_{\mathbb{R}^d}|Y_T^{T,x}|^2\rho^{-1}(x)\,dx]<\infty$. It follows that condition (H.1) is satisfied. Moreover, conditions (A.1)′–(A.3)′ are stronger than conditions (H.2)–(H.4), so by Theorem 4.3, $u(t,x)$ is a weak solution of Eq. (2.16). Theorem 2.11 is proved. □

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