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Global existence and boundedness of solutions to a chemotaxis system with singular sensitivity and logistic-type source ☆

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Abstract

We consider the fully parabolic Keller–Segel system with singular sensitivity and logistic-type source: $u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + ru - \mu u^k$, $v_t = \Delta v - v + u$ under the non-flux boundary conditions in a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, $\chi, r, \mu > 0$, $k > 1$. A global very weak solution for the system with $n \geq 2$ is obtained under one of the following conditions: (i) $r > \frac{\chi^2}{4}$ for $0 < \chi \leq 2$, or $r > \max\{\frac{\chi^2}{4}(1 - p_0^2), \chi - 1\}$ for $\chi > 2$ with $p_0 = \frac{4(k-1)}{4+(2-k)k\chi^2}$ if $k \in (2 - \frac{1}{n}, 2]$; (ii) $\chi^2 < \min\left\{\frac{2(r+r^2)}{k}, \frac{4}{k(k-1)(k-2)}\right\}$ if $k > 2$. Furthermore, this global very weak solution should be globally bounded in fact provided $\frac{r}{\mu}$ and the initial data $\|u_0\|_{L^2(\Omega)}, \|\nabla v\|_{L^4(\Omega)}$ suitably small for $n = 2, 3$. In addition, if $k > \frac{3(n+2)}{n+4}$ replaces $k > 2$ in the condition (ii), the system admits globally bounded classical solutions. All these describe the influence of the exponent $k > 1$ in the logistic-type source $ru - \mu u^k$ to the behavior of solutions for the considered fully parabolic Keller–Segel system with singular sensitivity.

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1. Introduction

Chemotaxis, a spontaneous aggregative phenomenon, is a widespread cross-diffusion mechanism in a wide rage of biological processes. In 1970, Keller and Segel proposed a model to represent the chemotaxis phenomena, i.e., the oriented or partially oriented movement of cells in response to a chemical signal produced by the cells themselves [9]:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where the singular chemotactic sensitive function $\frac{\chi}{v}$ with $\chi > 0$ is derived by the Weber–Fechner laws on the response of the cells u to the chemical signal v. In the parabolic–elliptic case of (1.1), where the second parabolic equation in (1.1) is replaced by the elliptic equation $0 = \Delta v - v + u$. It is known that all radial classical solutions are global-in-time if either $n = 2$ with $\chi > 0$, or $n \geq 3$ with $\chi < \frac{2}{n-2}$ [14]. When $\chi < \frac{2}{n}$ with $n \geq 1$, there exists a unique globally bounded classical solution [7]. If $n \geq 2$ and $\chi < \frac{n}{n-2}$, there exists a generalized solution [3]. For the parabolic–parabolic case, all solutions are global in time when either $n = 1$ [16], or $n = 2$ and $\chi < \frac{5}{2}$ under the radial assumption, while $\chi < 1$ without the radial assumption [15]. For $n \geq 2$, there exist globally bounded classical solutions if $0 < \chi < \sqrt{\frac{2}{n}}$ [5], and a global weak solution if $0 < \chi < \sqrt{\frac{n+2}{3n+4}}$ [24]. Moreover, the system (1.6) possesses a global generalized solution for $n \geq 2$ [12], provided

$$\chi < \begin{cases} \infty, & n = 2, \\ \sqrt{8}, & n = 3, \\ \frac{n}{n-2}, & n \geq 4. \end{cases} \tag{1.2}$$

See [11,6] for more results with singular sensitivities.

Generally, in a real chemotaxis system the proliferation-death mechanism of the cells u should be included as well. This can be mathematically formulated by adding a logistic-type source $ru - \mu u^k$ to the u-equation [2,10]

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ru - \mu u^k, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $\chi, r, \mu > 0$, $k > 1$ and $\tau \in \{0, 1\}$. Such self-limiting growth source generally benefits the global existence-boundedness of solutions. For parabolic–elliptic case ($\tau = 0$ in (1.3)), it is known with $k = 2$ that the solution is globally bounded if $\mu > \frac{n-2}{n}\chi$, global weak solution is admitted for all $\mu > 0$, and the constant equilibrium $(1, 1)$ with $r = \mu$ is globally asymptotically stable in $L^\infty(\Omega)$ if $\mu > 2\chi$ [18]. When $k > 2 - \frac{1}{n}$ with $n \geq 1$, there exists a global very weak solution, which is globally bounded if μ sufficiently large and $\|u_0\|_{L^\infty(\Omega)}$ sufficiently small [21]. For the case with $0 = \Delta v - m(t) + u$, $m(t) := \frac{1}{|\Omega|} \int_\Omega u(x, t) dx$ instead of $0 = \Delta v - v + u$ in (1.3), there exists radial initial data such that the smooth solution blows up in finite time if $1 < k < \frac{3}{2} + \frac{1}{2n-2}$ with $n \geq 5$ [25]. For the parabolic–parabolic case ($\tau = 1$ in (1.3)), if $k = 2$, $n = 2$ [17], or $n \geq 3$ with $\mu > 0$ sufficiently large [22], the solutions are globally bounded.

When $k > 2 - \frac{1}{n}$ with $n \geq 1$, the problem possesses global very weak solutions [19], which are globally bounded if $\frac{r}{\mu}$ sufficiently small and the initial data sufficiently small in suitable norms for $n = 3$ [20]. Refer to [4,27,28] for results with nonlinear diffusion and sensitivities.

Consider the chemotaxis system with singular sensitivity and logistic source

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + ru - \mu u^2, & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

where $\chi, r, \mu > 0$ and $\tau \in \{0, 1\}$. For the parabolic–elliptic case ($\tau = 0$ in (1.4)) with $n = 2$, there exists a unique globally bounded classical solution [8], whenever

$$r > \begin{cases} \frac{\chi^2}{4}, & 0 < \chi \leq 2, \\ \chi - 1, & \chi > 2. \end{cases} \tag{1.5}$$

For the parabolic–parabolic case ($\tau = 1$ in (1.4)), there exists a global solution when $n = 2$ [1]. Currently, the authors have proved that the condition (1.5) ensures the global boundedness of classical solutions in fact as well [29].

In this paper, we continuously consider case with general exponent $k > 1$ in the logistic-type source

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + ru - \mu u^k, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \Omega, \end{cases} \tag{1.6}$$

where $\chi, r, \mu > 0$, smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $\frac{\partial}{\partial \nu}$ denotes the derivation with respect to the outer normal of $\partial\Omega$, and the initial data

$$\begin{cases} u_0(x) \in C^0(\overline{\Omega}),\ u_0(x) \geq 0 \text{ and } u_0(x) \not\equiv 0,\ x \in \overline{\Omega}, \\ v_0(x) \in W^{2,\infty}(\Omega),\ v_0(x) > 0,\ x \in \overline{\Omega}. \end{cases} \tag{1.7}$$

To treat the more general exponent $k > 1$, we have to deal with very weak solutions. Inspired by [21,19], we introduce the very weak solutions to (1.6) via the following definitions.

Definition 1.1. Let $T > 0$. A pair (u, v) of nonnegative functions

$$u \in L^1(\Omega \times (0, T)),\ v \in L^1((0, T); W^{1,1}(\Omega))$$

will be called a *very weak subsolution* to (1.6) in $\Omega \times (0, T)$ if

$$u^k \text{ and } \frac{u}{v} \nabla v \text{ belong to } L^1(\Omega \times (0, T)),$$

and moreover

$$-\int_0^T\int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot,0) \le \int_0^T\int_\Omega u\Delta\varphi + \chi\int_0^T\int_\Omega \frac{u}{v}\nabla v\cdot\nabla\varphi$$
$$+ r\int_0^T\int_\Omega u\varphi - \mu\int_0^T\int_\Omega u^k\varphi, \tag{1.8}$$

$$-\int_0^T\int_\Omega v\psi_t - \int_\Omega v_0\psi(\cdot,0) + \int_0^T\int_\Omega \nabla v\cdot\nabla\psi + \int_0^T\int_\Omega v\psi = \int_0^T\int_\Omega u\psi \tag{1.9}$$

hold for all

$$\varphi \in C_0^\infty(\bar\Omega\times(0,T)) \text{ with } \varphi\ge 0 \text{ and } \frac{\partial\varphi}{\partial\nu}=0 \text{ on } \partial\Omega\times(0,T), \tag{1.10}$$

$$\psi\in L^\infty(\Omega\times(0,T))\cap L^2((0,T);W^{1,2}(\Omega)), \text{ and } \psi_t\in L^2(\Omega\times(0,T)). \tag{1.11}$$

Definition 1.2. Let $T>0$ and $\gamma\in(0,1)$. A pair of nonnegative functions

$$u\in L^{\gamma+1}(\Omega\times(0,T)),\ v\in L^1((0,T);W^{1,1}(\Omega))\cap L^{\gamma+1}(\Omega\times(0,T))$$

form a *weak γ-entropy supersolution* to (1.6) in $\Omega\times(0,T)$ if

$$u^{\gamma-2}|\nabla u|^2 \text{ and } u^\gamma\frac{|\nabla v|^2}{v^2} \text{ belong to } L^1(\Omega\times(0,T)),$$

and moreover

$$-\int_0^T\int_\Omega u^\gamma\varphi_t - \int_\Omega u_0^\gamma\varphi(\cdot,0) \ge \gamma(1-\gamma)\int_0^T\int_\Omega u^{\gamma-2}|\nabla u|^2\varphi + \int_0^T\int_\Omega u^\gamma\Delta\varphi$$
$$+\chi\gamma(\gamma-1)\int_0^T\int_\Omega u^{\gamma-1}\nabla u\cdot\frac{\nabla v}{v}\varphi + \chi\gamma\int_0^T\int_\Omega u^\gamma\frac{\nabla v}{v}\cdot\nabla\varphi$$
$$+r\gamma\int_0^T\int_\Omega u^\gamma\varphi - \mu\gamma\int_0^T\int_\Omega u^{\gamma-1+k}\varphi \tag{1.12}$$

and the equality (1.9) hold for all φ and ψ satisfying (1.10) and (1.11).

Definition 1.3. Let $T>0$. We call a couple (u,v) a *very weak solution* to (1.6) in $\Omega\times(0,T)$ if it is both a very weak subsolution and a weak γ-entropy supersolution of (1.6) in $\Omega\times(0,T)$ for some $\gamma\in(0,1)$. A *global very weak solution* to (1.6) is a pair of functions defined in $\Omega\times(0,\infty)$ which is a very weak solution to (1.6) in $\Omega\times(0,T)$ for all $T>0$.

In order to study the dynamic behavior of solutions to the system (1.6), we will at first establish a positive uniform-in-time lower bound of the chemical signal v. Similarly to those proceeded in [29], this can be transformed to build the global boundedness for $\int_\Omega u^{-\alpha}dx$ with $\alpha > 0$ via a crucial weighted integral $\int_\Omega u^{-p}v^{-q}dx$ with $p, q > 0$ to be determined.

To deal with the global very weak solution of (1.3) in [19] (i.e., without singular sensitivity $\frac{1}{v}$ in (1.6)), the main step is to conclude that the solution $\{v_\epsilon\}_{\epsilon\in(0,1)}$ of the regularization problem is relatively compact in $L_{\mathrm{loc}}^{\frac{k}{k-1}}((0,\infty); W^{1,\frac{k}{k-1}}(\Omega))$ with respect to the strong topology for $k \in (2-\frac{1}{n}, 2)$. Differently, for the system (1.6), we will firstly show that $\{v_\epsilon\}_{\epsilon\in(0,1)}$ has a uniform-in-time lower bound (independent of $\epsilon \in (0,1)$), namely,

$$\left\{\frac{1}{v_\epsilon}\right\}_{\epsilon\in(0,1)} \text{ is relatively compact in } L^\infty_{\mathrm{loc}}(\Omega\times(0,\infty)) \\ \text{with respect to the weak-star topology.}$$

Secondly, it will be derived that for some $p > \frac{k}{k-1}$ with $k > 2 - \frac{1}{n}$ that

$$\left\{\frac{\nabla v_\epsilon}{v_\epsilon}\right\}_{\epsilon\in(0,1)} \text{ is relatively compact in } L^p_{\mathrm{loc}}(\Omega\times(0,\infty)) \\ \text{with respect to the weak topology.}$$

Thirdly, it will be essential that

$$\left\{\frac{\nabla v_\epsilon}{v_\epsilon}\right\}_{\epsilon\in(0,1)} \text{ is relatively compact in } L^2_{\mathrm{loc}}(\Omega\times(0,\infty)) \\ \text{with respect to the strong topology.}$$

Upon selecting a suitable subsequence, we will obtain a global very weak solution for $k > 2 - \frac{1}{n}$ with $n \ge 2$ by a standard compactness argument. Furthermore, it will be shown that for the carrying capacity $\frac{r}{\mu}$ and the initial data $\|u_0\|_{L^2(\Omega)} + \|\nabla v_0\|_{L^4(\Omega)}$ suitably small, this global very weak solution for $n = 2, 3$ is in fact globally bounded.

Finally, for the case $n \ge 2$ and $k > 2$, by estimating on $\int_\Omega u^p dx + \int_\Omega |\nabla v|^{2q}dx$ with the a priori estimate $\int_\Omega |\nabla v|^l dx < C$ for $l \in (k, \frac{nk}{(n+2-k)_+})$, we will conclude the global boundedness to classical solutions if $k > \frac{3(n+2)}{n+4}$ and $\chi > 0$ suitably small.

Now, we state the main results of this paper.

Theorem 1. *Let $n \ge 2$. There exists a global very weak solution to the system (1.6), if one of the following holds:*

(i) $k \in (2-\frac{1}{n}, 2]$ *and $r, \chi > 0$ satisfy*

$$r > \begin{cases} \frac{\chi^2}{4}, & 0 < \chi \le 2, \\ \max\{\frac{\chi^2}{4}(1-p_0^2), \chi - 1\} \text{ with } p_0 := \frac{4(k-1)}{4+(2-k)k\chi^2}, & \chi > 2. \end{cases} \tag{1.13}$$

(ii) *$k > 2$ and $r, \chi > 0$ satisfy*

$$\chi^2 < \min\left\{\frac{2(r+r^2)}{k}, \frac{4}{k(k-1)(k-2)}\right\}. \tag{1.14}$$

Theorem 2. *Let (u, v) be the global very weak solution established in Theorem 1 for $n = 2, 3$. There exist $\eta, \lambda > 0$ small such that this very weak solution is globally bounded provided $\frac{r}{\mu} < \eta$ and $\|u_0\|^2_{L^2(\Omega)} + \|\nabla v_0\|^4_{L^4(\Omega)} < \lambda$.*

Remark 1. It can be found that the global very weak solution to the system (1.6) with singular sensitivity obtained in Theorem 1 (i) corresponds to that for the classical chemotaxis system (1.3) in [21,19]. Theorem 2 says that the smallness of $\frac{r}{\mu}$ and the initial data (u_0, v_0) ensure the global boundedness of the very weak solution to the system (1.6), and the same were observed for the classical chemotaxis system (1.3) in [21,20]. In particular, Theorems 1 and 2 show that $k < 2$ is permitted for the global boundedness of solutions to (1.6). This extends the global boundedness results for (1.6) (with $k = 2$) obtained in [29].

Theorem 3. *Let $n \geq 2$, $k > \frac{3(n+2)}{n+4}$ and $r, \chi > 0$ satisfy (1.14). Then the problem (1.6) possesses a globally bounded classical solution.*

Remark 2. Recall from [4,28] that the classical chemotaxis system (1.4) with logistic-type source $ru - \mu u^k$ possesses globally bounded classical solutions if $k > 2$ or $k = 2$ with $\mu > 0$ sufficiently large. While Theorem 3 says that the global boundedness of classical solutions requires $k > \frac{3(n+2)}{n+4}$ for $n \geq 2$ with the chemotactic coefficient $\chi > 0$ suitably small to the present problem (1.6). Here the difficulty due to the singular sensitivity involved in (1.6) is substantial.

Remark 3. By [29, Theorem 1] it was known for $n = 2$ that $k = 2$ ensures the global boundedness of classical solutions to (1.6). On the other hand, it is easy to see that the value $\frac{3(n+2)}{n+4} = 2$ in Theorem 3 if $n = 2$. Thus, combining with [29, Theorem 1], we conclude for the case $n = 2$ that the classical solutions must be globally bounded if $k \geq 2$.

The rest part of the paper is arranged as follows. In Section 2, we introduce the local existence of classical solutions and establish a uniform-in-time lower bound estimate on v. Section 3 deals with the global existence of classical solutions to the regularization problem of (1.6). Then we prove the global existence and boundedness to the very weak solutions in Sections 4–6. Finally, we study the global boundedness of classical solutions in Section 7.

2. Lower estimate of v for $k > 1$

We introduce a lemma on the local existence of classical solutions to the system (1.6) without proof, which can be obtained by the standard contraction argument as that in [29, Lemma 2.1].

Lemma 2.1. *With $k > 1$ and $q > n$, there exist $T_{\max} \in (0, +\infty]$ and a unique pair (u, v) of functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ v \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \cap L^{\infty}_{\mathrm{loc}}\big([0, T_{\max});\ W^{1,q}(\Omega)\big), \end{cases}$$

satisfying (1.6) *in the classical sense with* $u, v > 0$ *in* $\overline{\Omega} \times (0, T_{\max})$. *Moreover, either* $T_{\max} = \infty$, *or* $T_{\max} < \infty$ *with* $\lim_{t \to T_{\max}} (\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)}) = \infty$. □

Let (u, v) be the local classical solution established in this section. We give some simple a priori estimates for (u, v).

Lemma 2.2. *For* $k > 1$, *it holds that*

$$\int_\Omega u dx \le m^* := \max\left\{\int_\Omega u_0 dx, |\Omega|\Big(\frac{r}{\mu}\Big)^{\frac{1}{k-1}}\right\},\ t \in (0, T_{\max}), \tag{2.1}$$

$$\int_\Omega v dx \le \max\left\{\int_\Omega v_0 dx, m^*\right\},\ t \in (0, T_{\max}). \tag{2.2}$$

Proof. Integrate $(1.6)_1$ to get

$$\frac{d}{dt}\int_\Omega u dx = r\int_\Omega u dx - \mu\int_\Omega u^k dx \le r\int_\Omega u dx - \frac{\mu}{|\Omega|^{k-1}}\Big(\int_\Omega u dx\Big)^k,\ t \in (0, T_{\max}) \tag{2.3}$$

by the Hölder inequality. This entails (2.1) by the Bernoulli inequality. Moreover, it is known by integrating $(1.6)_2$ that

$$\frac{d}{dt}\int_\Omega v dx = -\int_\Omega v dx + \int_\Omega u dx,\quad t \in (0, T_{\max}). \tag{2.4}$$

This concludes (2.2), again by the Bernoulli inequality with (2.1). □

To study the dynamic behavior of solutions to (1.6) for $k > 1$, the key step is to establish a positive uniform-in-time lower bound for v. This will be realized by a uniform-in-time upper estimate for the weighted integral $\int_\Omega u^{-p} v^{-q} dx$ with some $p, q > 0$. Repeating the procedure in the proof of [29, Lemma 3.2], we can get the following differential inequality on $\int_\Omega u^{-p} v^{-q} dx$.

Lemma 2.3. *With* $k > 1$, *we have for* $p > 0$ *and* $q > q_+ = q_+(p) := \frac{p+1}{2}(\sqrt{1 + p\chi^2} - 1)$ *that*

$$\frac{d}{dt}\int_\Omega u^{-p} v^{-q} dx \le (q - rp)\int_\Omega u^{-p} v^{-q} dx + \mu p\int_\Omega u^{-p-1+k} v^{-q} dx - q\int_\Omega u^{-p+1} v^{-q-1} dx \tag{2.5}$$

for $t \in (0, T_{\max})$. □

In [29, Lemma 3.3] the authors proved that the chemical signal v for (1.4) (i.e. $k = 2$ in (1.6)) has the uniform-in-time lower bound if the growth rate $r > 0$ is suitably large with respect to the chemotaxis coefficient $\chi > 0$, or, in other words, $\chi > 0$ is suitably small with respect to $r > 0$. This mechanism is also valid for the general $k > 1$, even with more complicated representations.

At first consider the case $k \in (1, 2]$.

Lemma 2.4. *Under the condition of Theorem 1 (i), there exists $\delta_1 > 0$ such that*

$$v(x,t) \geq \delta_1 \quad for\ all\ t \in (0, T_{\max})\ and\ x \in \Omega. \tag{2.6}$$

Proof. The case of $k = 2$ was treated in [29, Lemma 3.3] already.

Assume $k \in (1, 2)$. Let $\beta_0 := \frac{1}{2} \inf_{x\in\Omega} v_0(x)$. By Lemma 2.1, there exists $t_0 \in (0, T_{\max})$, such that $v(x,t) \geq \beta_0$ for all $x \in \Omega$ and $t \in (0, t_0]$, and $u(x, t_0) \geq \gamma_0$ for all $x \in \Omega$ with some $\gamma_0 > 0$. So we only need to prove (2.6) for $t \in (t_0, T_{\max})$.

By Young's inequality, we have

$$\mu p \int_\Omega u^{-p-1+k} v^{-q} dx \leq q \int_\Omega u^{-p+1} v^{-q-1} dx + (\frac{\mu p}{q+1})^{q+1} \int_\Omega u^{1-p+(k-2)(q+1)} dx \tag{2.7}$$

for $t \in (t_0, T_{\max})$. Notice that $p, q > 0$ with $k \in (1, 2)$ implies $1 - p + (k-2)(q+1) < 1$. Let

$$\begin{aligned} f(p) &:= 1 - p + (k-2)(q_+ + 1) \\ &= -p + k - 1 + \frac{1}{2}(k-2)(p+1)(\sqrt{1 + p\chi^2} - 1), \ \ p \in (0, k-1). \end{aligned}$$

With the fact of $\sqrt{1+s} < 1 + \frac{s}{2}$ for $s > 0$, we know for $p \in (0, \frac{4(k-1)}{4+(2-k)k\chi^2}) \subset (0, k-1)$ that

$$f(p) > -p + k - 1 + \frac{1}{4}(k-2)(p+1)p\chi^2 > -p + k - 1 + \frac{1}{4}p(k-2)k\chi^2 > 0,$$

and hence there exists some $q > q_+$ such that $1 - p + (k-2)(q+1) > 0$. By the Hölder inequality with (2.1), (2.5) and (2.7), we have

$$\frac{d}{dt} \int_\Omega u^{-p} v^{-q} dx \leq (q - rp) \int_\Omega u^{-p} v^{-q} dx + C_1 \mu^{q+1}, \quad t \in (t_0, T_{\max}) \tag{2.8}$$

for some $C_1 > 0$.

Next, denote

$$A(p) := q_+ - rp = \frac{p+1}{2}(\sqrt{1 + p\chi^2} - 1) - rp, \ \ p > 0.$$

It is easy to see that $A(p) < 0$ is equivalent to

$$B(p) := \chi^2 p^2 + 2(\chi^2 - 2r - 2r^2)p + \chi^2 - 4r < 0, \ \ p > 0.$$

A direct calculation shows

$$\Delta_p = 16r^2[(1+r)^2 - \chi^2] > 0,$$

whenever $r > \max\{0, \chi - 1\}$. Denote

$$p_{1,2} := \frac{2r^2 + 2r - \chi^2}{\chi^2} \mp \frac{2r\sqrt{(1+r)^2 - \chi^2}}{\chi^2}. \tag{2.9}$$

Then $A(p) < 0$ for $p \in (p_1, p_2)$. By the Viète formula, we know $p_1 \le 0 < p_2$ if $r \ge \frac{\chi^2}{4}$ with $\chi > 0$, and moreover $p_1 \cdot p_2 = \frac{\chi^2 - 4r}{\chi^2} > 0$ if $\chi - 1 < r < \frac{\chi^2}{4}$ with $\chi > 2$, and thus

$$0 < p_1 < \sqrt{p_1 \cdot p_2} \le \sqrt{\frac{\chi^2 - 4r}{\chi^2}} < p_0,$$

if $\max\{\chi - 1, \frac{\chi^2}{4}(1 - p_0^2)\} < r < \frac{\chi^2}{4}$. Take $p_* \in (0, p_0)$ and $q_* > q_+$ such that $1 - p_* + (k-2)(q_* + 1) \in (0, 1)$ and $q_* - rp_* < 0$ for $r, \chi > 0$ and $k \in (1, 2)$ satisfying (1.13). Let $\alpha := \frac{p_*}{1+q_*} \in (0, p_*)$. Then $\frac{q_*\alpha}{p_* - \alpha} = 1$, and hence

$$\int_\Omega u^{-\alpha} dx \le \Big(\int_\Omega u^{-p_*} v^{-q_*} dx\Big)^{\frac{\alpha}{p_*}} \Big(\int_\Omega v dx\Big)^{\frac{p_* - \alpha}{p_*}}, \quad t \in (t_0, T_{\max}) \tag{2.10}$$

by the Höder inequality. Integrating (2.8) from t_0 to t yields

$$\begin{aligned}\int_\Omega u^{-p_*} v^{-q_*} dx &\le \mathrm{e}^{(q_* - rp_*)(t - t_0)} \int_\Omega u(x, t_0)^{-p_*} v(x, t_0)^{-q_*} dx + \frac{C_1}{rp_* - q_*} \mu^{q_*+1} \\ &\le C_2(1+\mu)^{q_*+1}, \quad t \in (t_0, T_{\max})\end{aligned} \tag{2.11}$$

with some $C_2 > 0$. Combining (2.10), (2.11) with (2.2), we have

$$\int_\Omega u^{-\alpha} dx \le C_3(1+\mu)^{\frac{(q_*+1)\alpha}{p_*}}, \quad t \in (t_0, T_{\max})$$

with some $C_3 > 0$, and hence

$$\int_\Omega u dx \ge |\Omega|^{\frac{\alpha+1}{\alpha}} \Big(\int_\Omega u^{-\alpha} dx\Big)^{-\frac{1}{\alpha}} \ge C_3^{-\frac{1}{\alpha}} |\Omega|^{\frac{\alpha+1}{\alpha}} (1+\mu)^{-\frac{q_*+1}{p_*}} =: \eta_0 > 0 \tag{2.12}$$

by the Hölder inequality for $t \in (t_0, T_{\max})$. By the pointwise lower bound estimate for the Neumann heat semigroup $\{\mathrm{e}^{t\Delta}\}_{t \ge 0}$, we obtain from (2.12) that

$$
\begin{aligned}
v(x,t) &= \mathrm{e}^{t(\Delta-1)}v_0 + \int_0^t \mathrm{e}^{(t-s)(\Delta-1)}u(x,s)ds \\
&\geq \int_0^t \frac{1}{4\pi(t-s)}\mathrm{e}^{-((t-s)+\frac{(\mathrm{diam}\Omega)^2}{4(t-s)})}\Big(\int_\Omega u(x,s)dx\Big)ds \\
&\geq \eta_0 \int_0^{t_0} \frac{1}{4\pi r}\mathrm{e}^{-(r+\frac{(\mathrm{diam}\Omega)^2}{4r})}dr =: \beta_1 > 0
\end{aligned}
$$

for all $t \in (t_0, T_{\max})$, $x \in \Omega$. This completes the proof with $\delta_1 := \max\{\beta_0, \beta_1\}$. □

Next consider the case of $k > 2$.

Lemma 2.5. *Under the condition of Theorem 1 (ii), there exists $\delta_2 > 0$ such that*

$$
v(x,t) \geq \delta_2 \quad \textit{for all } t \in (0,T) \textit{ and } x \in \Omega. \tag{2.13}
$$

Proof. Notice that $k > 2$ with $p \in (0, k-1)$ and $q > 0$ implies $1 - p + (k-2)(q+1) > 0$. Let

$$
\begin{aligned}
g(p) &:= -p + (k-2)(q_+ + 1) \\
&= -p + k - 2 + \frac{1}{2}(k-2)(p+1)(\sqrt{1+p\chi^2} - 1), \ \ p \in (0, k-1).
\end{aligned}
$$

Since $\chi^2 < \frac{4}{k(k-1)(k-2)}$, we have

$$
g(p) < -p + k - 2 + \frac{1}{4}p\chi^2(p+1)(k-2) < -p + k - 2 + \frac{1}{4}p\chi^2 k(k-2) < 0
$$

for $p \in (\frac{4(k-2)}{4-(k-2)k\chi^2}, k-1)$, and hence there exists $q > q_+$ such that $1 - p + (k-2)(q+1) \in (0,1)$. Thus, we obtain from (2.7) and (2.1) that

$$
\mu p \int_\Omega u^{-p-1+k}v^{-q}dx \leq q \int_\Omega u^{-p+1}v^{-q-1}dx + C_4\mu^{q+1} \tag{2.14}
$$

by Young's inequality with $C_4 > 0$. It is easy to know that $p_1 < 1$ by (2.9) for $0 < \chi < r + 1$. Therefore, $p_2 > \frac{4(k-2)}{4-(k-2)k\chi^2}$ is sufficient to ensure $(\frac{4(k-2)}{4-(k-2)k\chi^2}, k-1) \cap (p_1, p_2) \neq \emptyset$. A simple calculation shows

$$
p_2 \geq \frac{2r^2 + 2r - \chi^2}{\chi^2} > k - 1 > \frac{4(k-2)}{4-(k-2)k\chi^2}
$$

due to $\chi^2 < \frac{2(r+r^2)}{k}$ with $k > 2$ and $r > 0$. Take $p_* \in (\frac{4(k-2)}{4-(k-2)k\chi^2}, p_2)$ with some $q_* > q_+$. Similarly to (2.10)–(2.12) with (2.14), we conclude (2.13) with some $\delta_2 > 0$. □

In summary, we have

Corollary 2.1. *Under the conditions of Theorem 1, there exists* $\bar{C} > 0$ *such that*

$$v(x,t) \geq \bar{C}(1+\mu)^{-\frac{q_*+1}{p_*}} \tag{2.15}$$

with $p_*, q_* > 0$ *selected in Lemmas 2.4 and 2.5.* □

3. Regularization problem

We introduce an appropriate regularization to deal with the very weak solution of (1.6). That is

$$\begin{cases} u_{\epsilon t} = \Delta u_\epsilon - \chi \nabla \cdot (\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon} \nabla v_\epsilon) + r u_\epsilon - \mu u_\epsilon^k, & x \in \Omega, \quad t > 0, \\ v_{\epsilon t} = \Delta v_\epsilon - v_\epsilon + u_\epsilon, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u_\epsilon}{\partial \nu} = \dfrac{\partial v_\epsilon}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ (u_\epsilon(x,0), v_\epsilon(x,0)) = (u_0(x), v_0(x)), & x \in \Omega \end{cases} \tag{3.1}$$

for $\epsilon \in (0,1)$. In [29, Lemma 2.1] the authors established the local existence of classical solutions to the regularization problem with $k = 2$. The local classical solutions of the regularization problem (3.1) with general $k > 1$ can be obtained in the same arguments.

Lemma 3.1. *Let* $\Omega \subset \mathbb{R}^n (n \geq 2)$ *be a bounded domain,* $k > 1$ *and* $q > n$*. If* (u_0, v_0) *satisfies (1.7), then for each* $\epsilon \in (0,1)$ *there exist* $T_{\max,\epsilon} \leq \infty$ *and a unique pair function*

$$\begin{cases} u_\epsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\epsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\epsilon})), \\ v_\epsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\epsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\epsilon})) \cap L^\infty_{\mathrm{loc}}([0, T_{\max,\epsilon}); W^{1,q}(\Omega)), \end{cases} \tag{3.2}$$

solving (3.1) in the classical sense with $u_\epsilon, v_\epsilon > 0$ *in* $\overline{\Omega} \times (0, T_{\max,\epsilon})$*. Moreover, either* $T_{\max,\epsilon} = \infty$ *or* $T_{\max,\epsilon} < \infty$ *with* $\lim_{t \to T_{\max,\epsilon}} (\|u_\epsilon\|_{L^\infty(\Omega)} + \|v_\epsilon\|_{W^{1,q}(\Omega)}) = \infty$. □

By the comparison principle with the positivity of u_ϵ, we know from $(3.1)_2$ that

$$v_\epsilon \geq \inf_{y \in \Omega} v_0(y) e^{-t} =: \delta(t) \ \text{ for all } (x,t) \in \Omega \times (0, T_{\max,\epsilon}) \text{ and } \epsilon \in (0,1). \tag{3.3}$$

Lemma 3.2. *For* $k > 1$*, it holds that*

$$\int_\Omega u_\epsilon dx \leq m^*, \quad t \in (0, T_{\max,\epsilon}), \tag{3.4}$$

$$\mu \int_0^t \int_\Omega u_\epsilon^k dx ds \leq M_1(1+t), \quad t \in (0, T_{\max,\epsilon}) \tag{3.5}$$

with $M_1 = m^* \max\{1, r\}$ *for all* $\epsilon \in (0,1)$.

Proof. Integrate (3.1)$_1$ over Ω with the Hölder inequality to know that

$$\frac{d}{dt}\int_\Omega u_\epsilon dx = r\int_\Omega u_\epsilon dx - \mu\int_\Omega u_\epsilon^k dx \tag{3.6}$$

$$\leq r\int_\Omega u_\epsilon dx - \frac{\mu}{|\Omega|^{k-1}}\Big(\int_\Omega u_\epsilon dx\Big)^k, \quad t\in(0,T_{\max,\epsilon}). \tag{3.7}$$

By the Bernoulli inequality with (3.7), we get (3.4). The estimate (3.5) comes from by integrating (3.6) from 0 to t and (3.4). □

Lemma 3.3. *Let $n\geq 2$ and $k>1$. If $p\in(1,\frac{n}{n-2})$, there exists $M_2>0$ such that*

$$\int_\Omega v_\epsilon^p dx \leq M_2, \quad t\in(0,T_{\max,\epsilon}) \tag{3.8}$$

for all $\epsilon\in(0,1)$. Moreover, if $q\in(1,\frac{n}{n-1})$, there exists $M_3>0$ such that

$$\int_\Omega |\nabla v_\epsilon|^q dx \leq M_3, \quad t\in(0,T_{\max,\epsilon}) \tag{3.9}$$

for all $\epsilon\in(0,1)$.

Proof. Denote $\bar{u}_\epsilon=\bar{u}_\epsilon(t):=\frac{1}{|\Omega|}\int_\Omega u_\epsilon(x,t)dx$. By the semigroup estimates in [23, Lemma 1.3], we have from (3.1)$_2$ with $p\in(1,\frac{n}{n-2})$ that

$$\begin{aligned}\|v_\epsilon\|_{L^p(\Omega)} &\leq \|v_0\|_{L^\infty(\Omega)} + \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)}(u_\epsilon-\bar{u}_\epsilon)\|_{L^p(\Omega)}ds + \int_0^t \|\mathrm{e}^{(t-s)(\Delta-1)}\bar{u}_\epsilon\|_{L^p(\Omega)}ds\\
&\leq \|v_0\|_{L^\infty(\Omega)} + K_1\int_0^t (1+(t-s)^{-(1-\frac{1}{p})})\mathrm{e}^{-\lambda_1(t-s)}\|u_\epsilon-\bar{u}_\epsilon\|_{L^1(\Omega)}ds\\
&\quad + m^*\int_0^t \mathrm{e}^{-(t-s)}ds\\
&\leq M_2, \quad t\in(0,T_{\max,\epsilon})\end{aligned}$$

for some $\lambda_1, K_1, M_2>0$ and all $\epsilon\in(0,1)$. Similarly, we know for $q\in(1,\frac{n}{n-1})$ that

$$\|\nabla v_\epsilon\|_{L^q(\Omega)} \leq \|\nabla\mathrm{e}^{t(\Delta-1)}v_0\|_{L^q(\Omega)} + \int_0^t \|\nabla\mathrm{e}^{(t-s)(\Delta-1)}u_\epsilon\|_{L^q(\Omega)}ds$$

$$\leq K_3\|\nabla v_0\|_{L^q(\Omega)} + K_2\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})})\|u_\epsilon\|_{L^1(\Omega)}ds$$

$$\leq M_3, \quad t\in(0,T_{\max,\epsilon})$$

by [23, Lemma 1.3] and (3.4) with $K_2, K_3, M_3 > 0$ for all $\epsilon \in (0,1)$. □

In addition, based on the Maximal L^p-L^q estimates for parabolic equations in [13, Theorem 3.1], we have directly the following lemma.

Lemma 3.4. *Let (u_ϵ, v_ϵ) be the classical solution of (3.1). Then we have for $\bar{p}, \bar{q} > 1$ that*

$$\int_0^{\bar{T}} \|v_\epsilon\|^{\bar{p}}_{W^{2,\bar{q}}(\Omega)}ds + \int_0^{\bar{T}} \|v_{\epsilon t}\|^{\bar{p}}_{L^{\bar{q}}(\Omega)}ds \leq \bar{K}\Big(1+\int_0^{\bar{T}} \|u_\epsilon\|^{\bar{p}}_{L^{\bar{q}}(\Omega)}ds\Big), \quad \bar{T}>0 \tag{3.10}$$

with $\bar{K} > 0$ independent of ϵ. □

Now, we deal with the L^p-norm of u_ϵ for each $\epsilon \in (0,1)$.

Lemma 3.5. *Let $n \geq 2$ and $k > 1$. Then for $p > 2$, there exists $H_1 = H_1(\epsilon, t) > 0$ such that*

$$\int_\Omega u_\epsilon^p dx \leq H_1, \quad t \in (0, T_{\max,\epsilon}). \tag{3.11}$$

Proof. It follows from $(3.1)_1$ and (3.3) for $p > 2$ that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u_\epsilon^p dx &= -(p-1)\int_\Omega u_\epsilon^{p-2}|\nabla u_\epsilon|^2dx + \chi(p-1)\int_\Omega \frac{u_\epsilon^{p-1}}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla u_\epsilon\cdot\nabla v_\epsilon dx \\
&\quad + r\int_\Omega u_\epsilon^p dx - \mu\int_\Omega u_\epsilon^{p+k-1}dx \\
&\leq \frac{\chi^2(p-1)}{4\epsilon^2\delta(t)^2}\int_\Omega u_\epsilon^{p-2}|\nabla v_\epsilon|^2dx + r\int_\Omega u_\epsilon^p dx - \mu\int_\Omega u_\epsilon^{p+k-1}dx \\
&\leq 2r\int_\Omega u_\epsilon^p dx + C_5\Big(\frac{1}{\epsilon\delta(t)}\Big)^p\int_\Omega |\nabla v_\epsilon|^p dx - \mu\int_\Omega u_\epsilon^{p+k-1}dx \\
&\leq 2r\int_\Omega u_\epsilon^p dx + \frac{\mu}{\bar{K}}\int_\Omega |\nabla v_\epsilon|^{p+k-1}dx \\
&\quad - \mu\int_\Omega u_\epsilon^{p+k-1}dx + C_6\Big(\frac{1}{\epsilon\delta(t)}\Big)^{\frac{p(p+k-1)}{k-1}}, \quad t\in(0,T_{\max,\epsilon})
\end{aligned} \tag{3.12}$$

by Young's inequality twice with $C_5, C_6 > 0$. By Lemma 3.4 with $\bar{p} = \bar{q} = p + k - 1$, we have

$$\frac{\mu}{\bar{K}} \int_0^t \int_\Omega |\nabla v_\epsilon|^{p+k-1} dxds \le \mu\Big(1 + \int_0^t \int_\Omega u_\epsilon^{p+k-1} dxds\Big), \quad t \in (0, T_{\max,\epsilon}). \tag{3.13}$$

By Gronwall's inequality with (3.12) and (3.13), we arrive at

$$\int_\Omega u_\epsilon^p dx \le p\mathrm{e}^{2rpt}\Big[\frac{1}{p}\int_\Omega u_0^p dx + \mu + C_6 \int_0^t \big(\frac{1}{\epsilon\delta(s)}\big)^{\frac{p(p+k-1)}{k-1}} ds\Big] =: H_1, \; t \in (0, T_{\max,\epsilon}). \tag{3.14}$$

The proof is complete. □

Lemma 3.6. *Let $n \ge 2$, $k > 1$. Then there exists $H_2 = H_2(\epsilon, t) > 0$ such that*

$$\|\nabla v_\epsilon\|_{L^\infty(\Omega)} \le H_2, \; t \in (0, T_{\max,\epsilon}). \tag{3.15}$$

Proof. By the semigroup estimates in [23, Lemma 1.3] and (3.11), we have for $(3.1)_2$ that

$$\|\nabla v_\epsilon\|_{L^\infty(\Omega)} \le \|\nabla \mathrm{e}^{t(\Delta-1)} v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla \mathrm{e}^{(t-s)(\Delta-1)} u_\epsilon\|_{L^\infty(\Omega)} ds \tag{3.16}$$

$$\begin{aligned}
&\le K_3\|\nabla v_0\|_{L^\infty(\Omega)} + K_2 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2(n+1)}})\|u_\epsilon\|_{L^{n+1}(\Omega)} ds \\
&\le K_3\|\nabla v_0\|_{L^\infty(\Omega)} + K_2 H_1 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2(n+1)}}) ds \\
&=: H_2, \quad t \in (0, T_{\max,\epsilon})
\end{aligned} \tag{3.17}$$

with $K_2, K_3 > 0$. □

Lemma 3.7. *Under the conditions in Lemma 3.1, for each $\epsilon \in (0, 1)$ the system (3.1) has a globally classical solution.*

Proof. With $k > 1$, a simple calculation shows

$$ra - \mu a^k \le r\big(\frac{r}{\mu k}\big)^{\frac{1}{k-1}} \frac{k-1}{k} =: c_k, \; a \ge 0. \tag{3.18}$$

It is known from $(3.1)_1$ by using the estimates in [23, Lemma 1.3] with (3.3), (3.15) and (3.18) that

$$
\begin{aligned}
\|u_\epsilon\|_{L^\infty(\Omega)} \le & \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + \chi \int_0^t \left\|e^{(t-s)\Delta}\nabla\cdot(\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon)\right\|_{L^\infty(\Omega)} \\
& + \int_0^t \|e^{(t-s)\Delta}(ru-\mu u^k)\|_{L^\infty(\Omega)}ds \\
& \le \|u_0\|_{L^\infty(\Omega)} + c_k t + \chi K_4 \int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2(n+1)}})\left\|\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon\right\|_{L^{n+1}(\Omega)}ds \\
& \le \|u_0\|_{L^\infty(\Omega)} + c_k t + \frac{\chi K_4}{\epsilon}\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2(n+1)}})\|\nabla v_\epsilon\|_{L^{n+1}(\Omega)}\left\|\frac{1}{v_\epsilon}\right\|_{L^\infty(\Omega)}ds \\
& \le \|u_0\|_{L^\infty(\Omega)} + c_k t + H_2\frac{\chi K_4}{\epsilon\delta(t)}\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2(n+1)}})ds \\
& =: H_3 = H_3(\epsilon,t), \quad t\in(0,T_{\max,\epsilon})
\end{aligned}
$$

with $K_4 > 0$. This concludes for each $\epsilon\in(0,1)$ that $T_{\max,\epsilon}=\infty$ by Lemma 3.1. □

4. Estimates for (u_ϵ, v_ϵ)

To arrive at the global existence of very weak solutions to (1.6), we should also establish a positive uniform-in-time lower bound of v_ϵ for all $\epsilon\in(0,1)$.

At first consider the case of $k\in(1,2]$.

Lemma 4.1. *Under the condition of Theorem 1 (i), we have*

$$
v_\epsilon(x,t) \ge \delta_1 \quad \text{for all } \epsilon\in(0,1),\ x\in\Omega \text{ and } t>0 \tag{4.1}
$$

with δ_1 defined in Lemma 2.4.

Proof. With $p, q > 0$ to be determined, a direct computation with (3.1) shows

$$
\begin{aligned}
& \frac{d}{dt}\int_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx \\
& = -p\int_\Omega u_\epsilon^{-p-1}v_\epsilon^{-q}[\Delta u_\epsilon - \chi\nabla\cdot(\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon)]dx \\
& \quad - rp\int_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx + \mu p\int_\Omega u_\epsilon^{-p-1+k}v_\epsilon^{-q}dx \\
& \quad - q\int_\Omega u_\epsilon^{-p}v_\epsilon^{-q-1}\Delta v_\epsilon dx + q\int_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx - q\int_\Omega u_\epsilon^{-p+1}v_\epsilon^{-q-1}dx
\end{aligned}
$$

$$
\begin{aligned}
&= -p(p+1)\int\limits_\Omega u_\epsilon^{-p-2}v_\epsilon^{-q}|\nabla u_\epsilon|^2dx + \int\limits_\Omega \Big[\frac{p(p+1)\chi}{1+\epsilon u_\epsilon} - 2pq\Big]u_\epsilon^{-p-1}v_\epsilon^{-q-1}\nabla u_\epsilon\cdot\nabla v_\epsilon dx \\
&\quad + \int\limits_\Omega \Big[\frac{pq\chi}{1+\epsilon u_\epsilon} - q(q+1)\Big]u_\epsilon^{-p}v_\epsilon^{-q-2}|\nabla v_\epsilon|^2dx + (q-rp)\int\limits_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx \\
&\quad + \mu p\int\limits_\Omega u_\epsilon^{-p-1+k}v_\epsilon^{-q}dx - q\int\limits_\Omega u_\epsilon^{-p+1}v_\epsilon^{-q-1}dx \\
&\le \int\limits_\Omega \Big\{\frac{1}{4p(p+1)}\Big[\frac{p(p+1)\chi}{1+\epsilon u_\epsilon} - 2pq\Big]^2 + \frac{pq\chi}{1+\epsilon u_\epsilon} - q(q+1)\Big\}u_\epsilon^{-p}v_\epsilon^{-q-2}|\nabla v_\epsilon|^2dx \\
&\quad + (q-rp)\int\limits_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx + \mu p\int\limits_\Omega u_\epsilon^{-p-1+k}v_\epsilon^{-q}dx \\
&\quad - q\int\limits_\Omega u_\epsilon^{-p+1}v_\epsilon^{-q-1}dx,\quad t>0
\end{aligned}
\tag{4.2}
$$

by Young's inequality. Denote

$$
f(q;p,\chi,u_\epsilon) := \frac{p}{4(p+1)}\Big[\frac{(p+1)\chi}{1+\epsilon u_\epsilon} - 2q\Big]^2 + \frac{pq\chi}{1+\epsilon u_\epsilon} - q(q+1),
$$

and rewrite it as the quadric expression in q to get

$$
\begin{aligned}
4(p+1)f(q;p,\chi,u_\epsilon) &= p\frac{(p+1)^2\chi^2}{(1+\epsilon u_\epsilon)^2} + 4pq^2 - 4(p+1)q(q+1) \\
&\le -4q^2 - 4(p+1)q + p(p+1)^2\chi^2.
\end{aligned}
$$

Since $\Delta_q = 16(p+1)^2(1+p\chi^2) > 0$, we know $f(q;p,\chi,u_\epsilon) < 0$ if $q > q_+$. This together with (4.2) yields

$$
\frac{d}{dt}\int\limits_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx \le (q-rp)\int\limits_\Omega u_\epsilon^{-p}v_\epsilon^{-q}dx + \mu p\int\limits_\Omega u_\epsilon^{-p-1+k}v_\epsilon^{-q}dx - q\int\limits_\Omega u_\epsilon^{-p+1}v_\epsilon^{-q-1}dx
\tag{4.3}
$$

for all $\epsilon \in (0,1)$ and $t > 0$. Repeating the procedure in the proof of Lemma 2.4, we conclude the desired estimate (4.1). □

Next consider the case of $k > 2$. Correspondingly, there is the following lemma.

Lemma 4.2. *Under the condition of Theorem 1 (ii), we have*

$$
v_\epsilon(x,t) \ge \delta_2 \text{ for all } \epsilon \in (0,1),\ x \in \Omega \text{ and } t > 0 \tag{4.4}
$$

with δ_2 defined in Lemma 2.5.

Proof. By means of the same procedure as that in the proof of Lemma 4.1, we can get (4.3). Together with the arguments used for the proof of Lemma 2.5, we arrive at the lower bound (4.4). □

We deal with a spatio-temporal integral estimate on ∇v_ϵ for all $\epsilon \in (0, 1)$.

Lemma 4.3. *Let $n \geq 2$ and $k > 2 - \frac{2}{n+1}$. Then there exists $M_4 > 0$ such that*

$$\int_0^T \int_\Omega |\nabla v_\epsilon|^2 dxds \leq M_4(1+T), \quad T > 0 \tag{4.5}$$

for all $\epsilon \in (0, 1)$.

Proof. Multiply $(3.1)_2$ by v_ϵ and integrate by part to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v_\epsilon^2 dx &= -\int_\Omega |\nabla v_\epsilon|^2 dx - \int_\Omega v_\epsilon^2 dx + \int_\Omega v_\epsilon u_\epsilon dx \\ &\leq -\int_\Omega |\nabla v_\epsilon|^2 dx - \int_\Omega v_\epsilon^2 dx + \frac{1}{k}\int_\Omega u_\epsilon^k dx + \frac{k-1}{k}\int_\Omega v_\epsilon^{\frac{k}{k-1}} dx, \ t > 0 \end{aligned} \tag{4.6}$$

by Young's inequality. Applying the Gaglirado–Nirenberg inequality and the Poincaré inequality, we have

$$\|v_\epsilon\|_{L^{\frac{k}{k-1}}(\Omega)} \leq C_7 \|v_\epsilon\|^a_{W^{1,2}(\Omega)} \|v_\epsilon\|^{1-a}_{L^l(\Omega)} \leq C_8(\|\nabla v_\epsilon\|^a_{L^2(\Omega)}\|v_\epsilon\|^{1-a}_{L^l(\Omega)} + \|v_\epsilon\|_{L^l(\Omega)}) \tag{4.7}$$

with $C_7, C_8 > 0$, where

$$a := \frac{\frac{n}{l} - \frac{(k-1)n}{k}}{1 - \frac{n}{2} + \frac{n}{l}}.$$

For $k > 2 - \frac{2}{n+1}$, there exists $l \in (\frac{(2-k)+n}{2(k-1)}, \frac{n}{n-1})$ such that $\frac{k}{k-1}a < 2$ with $a \in (0, 1)$. Combining (4.6), (4.7) and (3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\epsilon^2 dx \leq -\frac{1}{2}\int_\Omega |\nabla v_\epsilon|^2 dx - \int_\Omega v_\epsilon^2 dx + \frac{1}{k}\int_\Omega u_\epsilon^k dx + C_9, \ t > 0 \tag{4.8}$$

by Young's equality with $C_9 > 0$. Integrating (4.8) from 0 to T, we conclude (4.5) by (3.5) with some $M_4 > 0$ for all $\epsilon \in (0, 1)$. □

We proceed to derive another spatio-temporal integral estimate on u_ϵ for all $\epsilon \in (0, 1)$.

Lemma 4.4. *Under the condition of Theorem 1 (i), there exists $\gamma \in (0, k-1)$ that*

$$\int_0^T \int_\Omega u_\epsilon^{\gamma-2} |\nabla u_\epsilon|^2 dx dt \le M_5(1+T), \quad T > 0 \tag{4.9}$$

with some $M_5 > 0$ for all $\epsilon \in (0, 1)$.

Proof. A direct calculation with $(3.1)_1$ and $\gamma \in (0, k-1)$ shows

$$\begin{aligned} \frac{1}{\gamma}\frac{d}{dt}\int_\Omega u_\epsilon^\gamma dx &= \int_\Omega u_\epsilon^{\gamma-1}[\Delta u_\epsilon - \chi \nabla \cdot (\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon) + r u_\epsilon - \mu u_\epsilon^k] dx \\ &= (1-\gamma)\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2 dx + \chi(\gamma-1)\int_\Omega \frac{u_\epsilon^{\gamma-1}}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla u_\epsilon \cdot \nabla v_\epsilon dx \\ &\quad + r\int_\Omega u_\epsilon^\gamma dx - \mu \int_\Omega u_\epsilon^{\gamma-1+k} dx, \quad t > 0. \end{aligned} \tag{4.10}$$

By Young's inequality, we have

$$\chi(1-\gamma)\int_\Omega \frac{u_\epsilon^{\gamma-1}}{(1+u_\epsilon)v_\epsilon}\nabla u_\epsilon \cdot \nabla v_\epsilon dx \le \frac{1-\gamma}{2}\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2 dx + \frac{(1-\gamma)\chi^2}{2}\int_\Omega \frac{u_\epsilon^\gamma}{v_\epsilon^2}|\nabla v_\epsilon|^2 dx. \tag{4.11}$$

It is known by (4.10) and (4.11) that

$$\begin{aligned} \frac{(1-\gamma)}{2}\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2 dx &\le \frac{1}{\gamma}\frac{d}{dt}\int_\Omega u_\epsilon^\gamma dx + \frac{(1-\gamma)\chi^2}{2}\int_\Omega \frac{u_\epsilon^\gamma}{v_\epsilon^2}|\nabla v_\epsilon|^2 dx \\ &\quad - r\int_\Omega u_\epsilon^\gamma dx + \mu\int_\Omega u_\epsilon^{\gamma-1+k} dx, \quad t > 0. \end{aligned} \tag{4.12}$$

Using Young's inequality with Lemma 4.1, then

$$\int_\Omega \frac{u_\epsilon^\gamma}{v_\epsilon^2}|\nabla v_\epsilon|^2 dx \le \frac{1}{\delta_1^2}\int_\Omega u_\epsilon^\gamma |\nabla v_\epsilon|^2 dx \le C_{10}\int_\Omega u_\epsilon^k dx + C_{10}\int_\Omega |\nabla v_\epsilon|^{\frac{2k}{k-\gamma}} dx \tag{4.13}$$

with $C_{10} > 0$. By the Gaglirado–Nirenberg inequality and the Poincaré inequality,

$$\begin{aligned} \|\nabla v_\epsilon\|_{L^{\frac{2k}{k-\gamma}}(\Omega)} &\le C_{11}\|\nabla v_\epsilon\|_{W^{1,k}(\Omega)}^a \|\nabla v_\epsilon\|_{L^l(\Omega)}^{1-a} \\ &\le C_{12}(\|\Delta v_\epsilon\|_{L^k(\Omega)}^a \|\nabla v_\epsilon\|_{L^l(\Omega)}^{1-a} + \|\nabla v_\epsilon\|_{L^l(\Omega)}), \end{aligned} \tag{4.14}$$

with $C_{11}, C_{12} > 0$, where

$$a := \frac{\frac{n}{l} - \frac{(k-\gamma)n}{2k}}{1 - \frac{n}{k} + \frac{n}{l}}.$$

For $n \geq 2$ and $k \in (2 - \frac{2}{n}, 2]$, taking $l \in (\frac{(2-k)n}{k}, \frac{n}{n-1})$ and $\gamma \in (0, k - \frac{2n}{l+n}) \subset (0, k-1)$, we get

$$\frac{2k}{k-\gamma} a < k.$$

Together with (4.13), (4.14) and (3.9), we have

$$\int_\Omega \frac{u_\epsilon^\gamma}{v_\epsilon^2} |\nabla v_\epsilon|^2 dx \leq C_{13}\Big(1 + \int_\Omega u_\epsilon^k dx + \int_\Omega |\Delta v_\epsilon|^k dx\Big) \tag{4.15}$$

by Young's inequality with $C_{13} > 0$. Combining (4.12), (4.15) with (3.4) and (3.5), we obtain

$$\begin{aligned} \frac{1-\gamma}{2} \int_0^T \int_\Omega u_\epsilon^{\gamma-2} |\nabla u_\epsilon|^2 dxds &\leq \frac{1}{\gamma} \int_\Omega u_\epsilon^\gamma(\cdot, t) dx - \frac{1}{\gamma} \int_\Omega u_0^\gamma dx + \frac{(1-\gamma)\chi^2}{2} \int_0^T \int_\Omega \frac{u_\epsilon^\gamma}{v_\epsilon^2} |\nabla v_\epsilon|^2 dxds \\ &\quad + r \int_0^T \int_\Omega u_\epsilon^\gamma dxds - \mu \int_0^T \int_\Omega u_\epsilon^{\gamma-1+k} dxds \\ &\leq C_{14}(1+T) + C_{14} \int_0^T \int_\Omega |\Delta v_\epsilon|^k dxds, \quad T > 0 \end{aligned} \tag{4.16}$$

with $C_{14} > 0$. In addition, we know by Lemma 3.4 with $\bar{p} = \bar{q} = k$ and (3.5) that

$$\int_0^T \int_\Omega |\Delta v_\epsilon|^k dxds \leq \bar{K}\Big(1 + \int_0^T \int_\Omega u_\epsilon^k dxds\Big) \leq C_{15}(1+T), \quad T > 0$$

with $C_{15} > 0$. Together with (4.16), this yields the conclusion (4.9). □

Next, we deal with the estimate on the time derivative of u_ϵ for all $\epsilon \in (0, 1)$.

Lemma 4.5. *Under the condition of Theorem 1 (i), there exists $M_6 > 0$ such that*

$$\int_0^T \Big\| \frac{d}{dt}(1+u_\epsilon)^{\frac{\gamma}{2}} \Big\|_{(W_0^{n+1,2}(\Omega))^*} ds \leq M_6(1+T), \quad T > 0 \tag{4.17}$$

for all $\epsilon \in (0, 1)$.

Proof. Let $\gamma \in (0, k-1)$ and $\phi \in W_0^{n+1,2}(\Omega)$. Then we have from $(3.1)_1$ that

$$\begin{aligned}
\frac{2}{\gamma}\int_\Omega \frac{d}{dt}(1+u_\epsilon)^{\frac{\gamma}{2}}\phi dx &= \int_\Omega (1+u_\epsilon)^{\frac{\gamma-2}{2}}\phi[\Delta u_\epsilon - \chi\nabla\cdot(\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon) + ru_\epsilon - \mu u_\epsilon^k]dx \\
&= -\frac{\gamma-2}{2}\int_\Omega (1+u_\epsilon)^{\frac{\gamma-4}{2}}|\nabla u_\epsilon|^2\phi dx - \int_\Omega (1+u_\epsilon)^{\frac{\gamma-2}{2}}\nabla u_\epsilon\cdot\nabla\phi dx \\
&\quad + \chi\int_\Omega \frac{u_\epsilon(1+u_\epsilon)^{\frac{\gamma-2}{2}}}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon\cdot\nabla\phi dx + \frac{(\gamma-2)\chi}{2}\int_\Omega \frac{u_\epsilon(1+u_\epsilon)^{\frac{\gamma-4}{2}}}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla u_\epsilon\cdot\nabla v_\epsilon\phi dx \\
&\quad + r\int_\Omega (1+u_\epsilon)^{\frac{\gamma-2}{2}}u_\epsilon\phi dx - \mu\int_\Omega (1+u_\epsilon)^{\frac{\gamma-2}{2}}u_\epsilon^k\phi dx \\
&\le \frac{2-\gamma}{2}(\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2dx)\|\phi\|_{L^\infty(\Omega)} + (\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2dx)^{\frac{1}{2}}\|\nabla\phi\|_{L^2(\Omega)} \\
&\quad + \frac{\chi}{\delta_1}(\int_\Omega (1+u_\epsilon)^k dx)^{\frac{\gamma}{2k}}(\int_\Omega |\nabla v_\epsilon|^k dx)^{\frac{1}{k}}\|\nabla\phi\|_{L^{\frac{2k}{2k-2-\gamma}}(\Omega)} \\
&\quad + \frac{(2-\gamma)\chi}{2\delta_1}(\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2dx)^{\frac{1}{2}}(\int_\Omega |\nabla v_\epsilon|^2dx)^{\frac{1}{2}}\|\phi\|_{L^\infty(\Omega)} \\
&\quad + r(\int_\Omega (1+u_\epsilon)^k dx)^{\frac{\gamma}{2k}}\|\phi\|_{L^{\frac{2k}{2k-\gamma}}(\Omega)} + \mu(\int_\Omega (1+u_\epsilon)^k dx)^{\frac{\gamma-2+2k}{2k}}\|\phi\|_{L^{\frac{2k}{2-\gamma}}(\Omega)}
\end{aligned} \tag{4.18}$$

by the Hölder inequality for $t > 0$. Since $W_0^{n+1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, it is known by Young's inequality with (4.18) and the fact $(1+a)^b \le 2^b(1+a^b)$ for $a, b > 0$ that

$$\begin{aligned}
\Big|\int_\Omega \frac{d}{dt}(1+u_\epsilon)^{\frac{\gamma}{2}}\phi dx\Big| \le C_{16}\Big(1 + \int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2dx + \int_\Omega u_\epsilon^k dx \\
+ \int_\Omega |\nabla v_\epsilon|^k dx + \int_\Omega |\nabla v_\epsilon|^2 dx\Big)\|\phi\|_{W_0^{n+1,2}(\Omega)}, \quad t > 0
\end{aligned} \tag{4.19}$$

with $C_{16} > 0$. Integrating (4.19) from 0 to T, we obtain from (3.5), (3.8)–(3.10), (4.5) and (4.9) that

$$\begin{aligned}
\int_0^T \Big\|\frac{d}{dt}(1+u_\epsilon)^{\frac{\gamma}{2}}\Big\|_{(W_0^{n+1,2}(\Omega))^*}ds &\le \sup_{\phi\in W_0^{n+1,2}(\Omega), \|\phi\|_{W_0^{n+1,2}(\Omega)}\le 1}\int_0^T\Big|\int_\Omega \frac{d}{dt}(1+u_\epsilon)^{\frac{\gamma}{2}}\phi dx\Big|ds \\
&\le C_{16}\Big(T + \int_0^T\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2dxds + \int_0^T\int_\Omega u_\epsilon^k dxds
\end{aligned}$$

$$+\int_0^T\int_\Omega|\nabla v_\epsilon|^k dxds+\int_0^T\int_\Omega|\nabla v_\epsilon|^2dxds\Big)$$
$$\le M_6(1+T),\quad T>0$$

with some $M_6>0$. □

Based on Lemma 3.4, we further prove the following estimates on v_ϵ for all $\epsilon\in(0,1)$.

Lemma 4.6. *Let $k>1$ and $p\in(1,k)$. Then there exists $M_7>0$ such that*

$$\int_0^T\|v_{\epsilon t}\|_{L^p(\Omega)}^{\frac{p(k-1)}{p-1}}ds+\int_0^T\|v_\epsilon\|_{W^{2,p}(\Omega)}^{\frac{p(k-1)}{p-1}}ds\le M_7(1+T),\quad T>0 \tag{4.20}$$

for all $\epsilon\in(0,1)$. Moreover, there exists $M_8>0$ such that

$$\int_0^T\left\|\frac{\nabla v_\epsilon}{v_\epsilon}\right\|_{L^p(\Omega)}^{\frac{p(k-1)}{p-1}}ds\le M_8(1+T),\ T>0 \tag{4.21}$$

for all $\epsilon\in(0,1)$.

Proof. By the Hölder inequality with $p\in(1,k)$, we have

$$\int_\Omega u_\epsilon^p dx\le\Big(\int_\Omega u_\epsilon dx\Big)^{\frac{k-p}{k-1}}\Big(\int_\Omega u_\epsilon^k dx\Big)^{\frac{p-1}{k-1}},\quad t>0.$$

This together with (3.4) indicates

$$\Big(\int_\Omega u_\epsilon^p dx\Big)^{\frac{k-1}{p-1}}\le\Big(\int_\Omega u_\epsilon dx\Big)^{\frac{k-p}{p-1}}\int_\Omega u_\epsilon^k dx\le C_{17}\int_\Omega u_\epsilon^k dx,\quad t>0,$$

and hence shows by (3.5) that

$$\int_0^T\|u_\epsilon\|_{L^p(\Omega)}^{\frac{p(k-1)}{p-1}}ds\le C_{17}\int_0^T\int_\Omega u_\epsilon^k dxds\le C_{18}(1+T),\quad T>0$$

with $C_{17},C_{18}>0$. Consequently, it follows from (3.10) that

$$\int_0^T\|v_{\epsilon t}\|_{L^p(\Omega)}^{\frac{p(k-1)}{p-1}}ds+\int_0^T\|v_\epsilon\|_{W^{2,p}(\Omega)}^{\frac{p(k-1)}{p-1}}ds\le C_{18}\int_0^T\|u_\epsilon\|_{L^p(\Omega)}^{\frac{p(k-1)}{p-1}}ds\le M_7(1+T),\quad T>0 \tag{4.22}$$

with $M_7>0$. By (4.22) and (4.1) (or (4.4)), we conclude (4.21) with some $M_8>0$. □

We now perform a subsequence extraction procedure to obtain a limit object (u, v), i.e., a very weak solution to the problem (1.6).

Lemma 4.7. *Under the condition of Theorem 1 (i), for $0 < \gamma \le \frac{k-1}{2}$ and $p \in (1, k)$, there exist $u \in L^1_{\mathrm{loc}}(\Omega \times (0, \infty))$ and $v \in L^2_{\mathrm{loc}}((0, \infty), W^{1,2}(\Omega))$ with some $m > \frac{k}{k-1}$ such that*

$$u_\epsilon^\gamma \rightharpoonup u^\gamma, \qquad in\ L^2_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)), \tag{4.23}$$

$$u_\epsilon \rightharpoonup u, \qquad in\ L^k_{\mathrm{loc}}(\Omega \times [0,\infty)), \tag{4.24}$$

$$u_\epsilon \to u, \qquad a.e.\ in\ \Omega \times (0,\infty)\ and\ L^p_{\mathrm{loc}}(\Omega \times [0,\infty)), \tag{4.25}$$

$$v_\epsilon \to v, \qquad a.e.\ in\ \Omega \times (0,\infty)\ and\ L^m_{\mathrm{loc}}([0,\infty); W^{1,m}(\Omega)), \tag{4.26}$$

$$\frac{\nabla v_\epsilon}{v_\epsilon} \rightharpoonup \frac{\nabla v}{v}, \qquad in\ L^m_{\mathrm{loc}}(\Omega \times (0,\infty)), \tag{4.27}$$

$$\frac{1}{v_\epsilon} \overset{\star}{\rightharpoonup} \frac{1}{v}, \qquad in\ L^\infty_{\mathrm{loc}}(\Omega \times (0,\infty)) \tag{4.28}$$

for $\epsilon = \epsilon_j \searrow 0$.

Proof. The conclusions (4.23), (4.24) and (4.28) are the direct results from (4.9), (3.5) and (4.1). Since $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we have by the Aubin–Lions lemma with (4.9) and (4.17) that $(1+u_\epsilon)^{\frac{\gamma}{2}} \to (1+u)^{\frac{\gamma}{2}}$ in $L^2(\Omega \times (0,T))$, and moreover $u_\epsilon \to u$ a.e. in $\Omega \times (0,T)$, as $\epsilon = \epsilon_j \searrow 0$. This together with (4.24) indicates (4.25) by a similar argument as that for [21, Lemma 1.4]. Since $W^{2,p}(\Omega) \hookrightarrow\hookrightarrow W^{1,q}(\Omega)$ for $q < \frac{np}{n-p}$, it is known by applying again Aubin–Lions lemma with (4.20) that $v_\epsilon \to v$ in $L^{\frac{(k-1)p}{p-1}}((0,T), W^{1,q}(\Omega))$ for $p \in (1,k)$ and $q < \frac{np}{n-p}$, and moreover $v_\epsilon \to v$ a.e. in $\Omega \times (0,\infty)$, as $\epsilon = \epsilon_j \searrow 0$. A simple calculation shows that there exists $p \in (1,k)$ such that $\frac{(k-1)p}{p-1} > \frac{k}{k-1}$ and $\frac{np}{n-p} > \frac{k}{k-1}$ due to $k \in (2 - \frac{1}{n}, 2]$. This concludes (4.26) and (4.27) with $m \in (\frac{k}{k-1}, \min\{\frac{(k-1)p}{p-1}, \frac{np}{n-p}\})$. $\square$

5. Strong convergence of $\{\nabla \ln v_\epsilon\}_{\epsilon \in (0,1)}$

Obviously, we should establish the strong precompactness of $\left\{\frac{\nabla v_\epsilon}{v_\epsilon}\right\}_{\epsilon \in (0,1)}$. Inspired by [26], it is sufficient to show that $\int_0^T \int_\Omega \frac{|\nabla v|^2}{v^2} dxdt$ satisfies a required estimate from below for $T > 0$. We begin with the entropy identity

$$\frac{d}{dt}\int_\Omega \ln v dx = \int_\Omega \frac{|\nabla v|^2}{v^2} dx - |\Omega| + \int_\Omega \frac{u}{v} dx.$$

By a suitable choice of test functions in (1.11), we will derive a qualified inequality.

Lemma 5.1. *There exists a null set $N \subset (0, \infty)$ such that the functions u and v obtained in Lemma 4.7 satisfy the inequality*

$$\int_0^T \int_\Omega \frac{|\nabla v|^2}{v^2} dxdt \geq |\Omega|T - \int_0^T \int_\Omega \frac{u}{v} dxdt + \int_\Omega \ln v(x,T)dx - \int_\Omega \ln v_0 dx \tag{5.1}$$

for all $T \in (0,\infty) \setminus N$.

Proof. Due to $v \in L^1_{loc}(\Omega \times (0,\infty))$, we know that the function $z(t) := \int_\Omega \ln v(x,t)dx$ belongs to $L^1_{loc}(0,\infty)$. Therefore, there exists a null set $N \subset (0,\infty)$ such that each $T \in (0,\infty) \setminus N$ is a Lebesgue point of $z(t)$, i.e.,

$$\frac{1}{\delta} \int_T^{T+\delta} \int_\Omega \ln v(x,t)dxdt \to \int_\Omega \ln(x,T)dx \text{ for all } T \in (0,\infty) \setminus N \text{ as } \delta \searrow 0. \tag{5.2}$$

For $T \in (0,\infty) \setminus N$ and $\delta \in (0,1)$, let

$$\zeta_\delta(t) := \begin{cases} 1, & t \in [0,T], \\ 1 - \frac{t-T}{\delta}, & t \in (T, T+\delta), \\ 0, & t \geq T+\delta, \end{cases}$$

and denote

$$\tilde{v}(x,t) := \begin{cases} \frac{1}{v(x,t)}, & (x,t) \in \Omega \times (0,\infty), \\ \frac{1}{v_0(x)}, & (x,t) \in \Omega \times (-1,0]. \end{cases}$$

Now, for $\delta \in (0,1)$, $h \in (0,\delta)$, introduce

$$\psi(x,t) := \zeta_\delta(t) \cdot (A_h\tilde{v})(x,t), \quad (x,t) \in \Omega \times (0,\infty),$$

with

$$(A_h\tilde{v})(x,t) := \frac{1}{h} \int_{t-h}^t \tilde{v}(x,s)ds, \ (x,t) \in \Omega \times (0,\infty).$$

By Lemma 4.7 with (4.1), we know that $\frac{1}{v}$ belongs $L^\infty_{loc}(\Omega \times (0,\infty)) \cap L^2_{loc}((0,\infty); W^{1,2}(\Omega))$, and so does ψ. In addition, we also know that ψ is supported in $\bar{\Omega} \times [0, T+\delta]$ and

$$\psi_t(x,t) = \zeta'_\delta(t) \cdot (A_h\tilde{v})(x,t) + \zeta_\delta(t) \cdot \frac{1}{h}(\tilde{v}(x,t) - \tilde{v}(x,t-h)), \ (x,t) \in \Omega \times (0,\infty).$$

This indicates $\psi_t \in L^2_{loc}(\Omega \times (0,\infty))$. Therefore, we can insert ψ into (1.9) to obtain

$$
\begin{aligned}
I(\delta,h) :&= -\int_0^\infty\int_\Omega \zeta_\delta(t)\nabla v(x,t)\cdot\nabla(A_h\tilde{v})(x,t)dxdt \\
&= \int_0^\infty\int_\Omega \zeta_\delta(t)v(x,t)(A_h\tilde{v})(x,t)dxdt - \int_0^\infty\int_\Omega \zeta_\delta(t)u(x,t)(A_h\tilde{v})(x,t)dxdt \\
&\quad - \int_0^\infty\int_\Omega \zeta_\delta'(t)v(x,t)(A_h\tilde{v})(x,t)dxdt - |\Omega| \\
&\quad - \int_0^\infty\int_\Omega \zeta_\delta(t)v(x,t)\cdot\frac{1}{h}(\tilde{v}(x,t)-\tilde{v}(x,t-h))dxdt \\
&=: I_1(\delta,h)+I_2(\delta,h)+I_3(\delta,h)+I_4(\delta,h)+I_5(\delta,h),
\end{aligned}
\tag{5.3}
$$

where we have used that

$$
\psi(x,0)=\zeta_\delta(0)\cdot\frac{1}{h}\int_{-h}^0 \tilde{v}(x,s)ds=\frac{1}{v_0(x)},\ x\in\Omega
$$

by the definitions of $\zeta_\delta(t)$ and $\tilde{v}$. Since $v_0\in W^{2,\infty}(\Omega)$, it is known by [26, Lemma A.2(a)] with $\nabla\tilde{v}\in L^2(\Omega\times(-1,T+\delta))$ that

$$
\nabla(A_h\tilde{v})=A_h(\nabla\tilde{v})\rightharpoonup\nabla\tilde{v}=-\frac{\nabla v}{v^2}\text{ in } L^2(\Omega\times(0,T+\delta))\ \text{ as } h\searrow 0,
$$

and hence

$$
I(\delta,h)\to\int_0^{T+\delta}\int_\Omega \zeta_\delta(t)\frac{|\nabla v|^2}{v^2}dxdt\ \text{ as } h\searrow 0.
\tag{5.4}
$$

Similarly, by [26, Lemma A.2(b)] with $\tilde{v}\in L^\infty(\Omega\times(-1,T+\delta))$,

$$
A_h\tilde{v}\overset{\star}{\rightharpoonup}\tilde{v}=\frac{1}{v}\text{ in }\ L^\infty(\Omega\times(0,T+\delta))\ \text{ as } h\searrow 0,
\tag{5.5}
$$

and then

$$
I_1(\delta,h)\to\int_0^{T+\delta}\int_\Omega \zeta_\delta(t)dxdt\ \text{ as } h\searrow 0,
\tag{5.6}
$$

$$
I_2(\delta,h)\to-\int_0^{T+\delta}\int_\Omega \zeta_\delta(t)\frac{u}{v}dxdt\ \text{ as } h\searrow 0,
\tag{5.7}
$$

$$I_3(\delta, h) \to -\int_T^{T+\delta}\int_\Omega \zeta_\delta'(t)dxdt \ \text{ as } h \searrow 0. \tag{5.8}$$

We know

$$\begin{aligned}
I_5(\delta, h) &= -\frac{1}{h}\int_0^{T+\delta}\int_\Omega \zeta_\delta(t)dxdt + \frac{1}{h}\int_0^{h}\int_\Omega \zeta_\delta \frac{v(x,t)}{v_0(x)}dxdt \\
&\quad + \frac{1}{h}\int_h^{T+\delta}\int_\Omega \zeta_\delta \frac{v(x,t)}{v(x,t-h)}dxdt \\
&=: I_{5_1}(\delta, h) + I_{5_2}(\delta, h) + I_{5_3}(\delta, h).
\end{aligned} \tag{5.9}$$

Now, we deal with the last term in (5.9). With the fact $x \geq \ln x + 1$ for $x > 0$, we obtain

$$\begin{aligned}
I_{5_3}(\delta, h) &\geq \frac{1}{h}\int_h^{T+\delta} \zeta_\delta(t)\Big(\ln \frac{v(x,t)}{v(x,t-h)} + 1\Big)dxdt \\
&= \frac{1}{h}\int_h^{T+\delta}\int_\Omega \zeta_\delta(t)dxdt + \frac{1}{h}\int_h^{T+\delta}\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt \\
&\quad - \frac{1}{h}\int_0^{T+\delta-h}\int_\Omega \zeta_\delta(t+h)\ln v(x,t)dxdt \\
&= \frac{1}{h}\int_h^{T+\delta}\int_\Omega \zeta_\delta(t)dxdt + \int_0^{T+\delta-h}\int_\Omega \frac{\zeta_\delta(t) - \zeta_\delta(t+h)}{h}\ln v(x,t)dxdt \\
&\quad - \frac{1}{h}\int_0^{h}\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt + \frac{1}{h}\int_{T+\delta-h}^{T+\delta}\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt.
\end{aligned} \tag{5.10}$$

Combine (5.9) with (5.10) to get

$$\begin{aligned}
I_5(\delta, h) &\geq -\frac{1}{h}\int_0^{h}\int_\Omega \zeta_\delta(t)dxdt + \frac{1}{h}\int_0^{h}\int_\Omega \zeta_\delta \frac{v(x,t)}{v_0(x)}dxdt \\
&\quad - \frac{1}{h}\int_0^{h}\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt + \frac{1}{h}\int_{T+\delta-h}^{T+\delta}\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt \\
&\quad + \int_0^{T+\delta-h}\int_\Omega \frac{\zeta_\delta(t) - \zeta_\delta(t+h)}{h}\ln v(x,t)dxdt.
\end{aligned} \tag{5.11}$$

With $\zeta_\delta(0) = 1$, we know

$$-\frac{1}{h}\int_0^h\int_\Omega \zeta_\delta(t)dxdt \to -|\Omega| \ \text{ as } h \searrow 0, \tag{5.12}$$

$$\frac{1}{h}\int_0^h\int_\Omega \zeta_\delta(t)\frac{v(x,t)}{v_0(x)}dxdt \to |\Omega| \ \text{ as } h \searrow 0, \tag{5.13}$$

$$-\frac{1}{h}\int_0^h\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt \to -\int_\Omega \ln v_0(x)dx \ \text{ as } h \searrow 0. \tag{5.14}$$

By the dominated convergence theorem with a simple calculation, we conclude

$$\int_0^{T+\delta-h}\int_\Omega \frac{\zeta_\delta(t)-\zeta_\delta(t+h)}{h}\ln v(x,t)dxdt \to \int_T^{T+\delta}\int_\Omega \zeta_\delta'(t)\ln v(x,t)dxdt \ \text{ as } h \searrow 0, \tag{5.15}$$

$$\frac{1}{h}\int_{T+\delta-h}^{T+\delta}\int_\Omega \zeta_\delta(t)\ln v(x,t)dxdt \to 0 \qquad \text{as } h \searrow 0. \tag{5.16}$$

Insert (5.12)–(5.16) into (5.11) to get

$$\liminf_{h\to 0} I_5(\delta,h) \geq -\int_\Omega \ln v_0(x)dx - \int_T^{T+\delta}\int_\Omega \zeta_\delta'(t)\ln v(x,t)dxdt. \tag{5.17}$$

Combining (5.3) with (5.4), (5.6)–(5.8) and (5.17) by $h \searrow 0$, we obtain

$$\begin{aligned}\int_0^{T+\delta}\int_\Omega \zeta_\delta(t)\frac{|\nabla v|^2}{v^2}dxdt \geq& \int_0^{T+\delta}\int_\Omega \zeta_\delta(t)dxdt - \int_0^{T+\delta}\int_\Omega \zeta_\delta(t)\frac{u}{v}dxdt \\ &+ \int_0^{T+\delta}\int_\Omega \zeta_\delta'(t)dxdt + |\Omega| - \int_\Omega \ln v_0(x)dx \\ &- \int_T^{T+\delta}\int_\Omega \zeta_\delta'(t)\ln v(x,t)dxdt.\end{aligned} \tag{5.18}$$

By the definition of $\zeta_\delta(t)$ with the Lebesgue point property of T, we know

$$\int_0^{T+\delta}\int_\Omega \zeta_\delta'(t)dxdt = -\frac{1}{\delta}\int_T^{T+\delta}\int_\Omega dxdt \to -|\Omega| \ \text{ as } \delta \searrow 0, \tag{5.19}$$

$$-\int_T^{T+\delta}\int_\Omega \zeta_\delta'(t)\ln v(x,t)dxdt = \frac{1}{\delta}\int_T^{T+\delta}\int_\Omega \ln v(x,t)dxdt$$
$$\to \int_\Omega \ln v(x,T)dx \ \text{ as } \delta \searrow 0. \tag{5.20}$$

Applying the monotone convergence theorem to (5.18) with (5.19) and (5.20) by $\delta \searrow 0$, we conclude (5.1). □

Now, we deal with the desired strong convergence result.

Lemma 5.2. *Let $\{\epsilon_j\}_{j\in\mathbb{N}}$ be taken in Lemma 4.7. Then there exists a subsequence, again denoted by $\{\epsilon_j\}_{j\in\mathbb{N}}$, such that for each $T>0$*

$$\frac{\nabla v_\epsilon}{v_\epsilon} \to \frac{\nabla v}{v} \ \text{ in } L^2(\Omega\times(0,T)) \ \text{ as } \epsilon=\epsilon_j \searrow 0. \tag{5.21}$$

Proof. We know from (4.26) that $\ln v_\epsilon \to \ln v$ in $L^1_{\text{loc}}(\Omega\times(0,\infty))$. Upon selecting a subsequence, still denoted by $\{\epsilon_j\}_{j\in\mathbb{N}}$, we have a null set $N_1 \subset (0,\infty)$ such that

$$\int_\Omega \ln v_\epsilon(x,T)dx \to \int_\Omega \ln v(x,T)dx \quad \text{for all } T\in(0,\infty)\setminus N_1 \ \text{ as } \epsilon=\epsilon_j\searrow 0. \tag{5.22}$$

Taking $N\subset(0,\infty)$ as in Lemma 5.1, we only need to verify (5.21) for $T\in(0,\infty)\setminus(N\cup N_1)$. Now, given such T, we know from (4.25) and (4.28) that

$$\int_0^T\int_\Omega \frac{u_\epsilon}{v_\epsilon}dxdt \to \int_0^T\int_\Omega \frac{u}{v}dxdt \ \text{ in } L^1(\Omega\times(0,T)) \text{ as } \epsilon_j\searrow 0. \tag{5.23}$$

By Lemma 5.1 with (5.22) and (5.23), we have

$$\begin{aligned}\int_0^T\int_\Omega \frac{|\nabla v|^2}{v^2}dxdt &\geq |\Omega|T - \int_0^T\int_\Omega \frac{u}{v}dxdt + \int_\Omega \ln v(x,T)dx - \int_\Omega \ln v_0 dx \\ &= \lim_{\epsilon=\epsilon_j\searrow 0}\left\{|\Omega|T - \int_0^T\int_\Omega \frac{u_\epsilon}{v_\epsilon}dxdt + \int_\Omega \ln v_\epsilon(x,T)dx - \int_\Omega \ln v_0 dx\right\}.\end{aligned} \tag{5.24}$$

In addition, test $(3.1)_2$ by $\frac{1}{v_\epsilon}$ to obtain

$$\int_0^T\int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2}dxdt = |\Omega|T - \int_0^T\int_\Omega \frac{u_\epsilon}{v_\epsilon}dxdt + \int_\Omega \ln v_\epsilon(x,T)dx - \int_\Omega \ln v_0 dx,$$

which together with (5.24) entails

$$\int_0^T\int_\Omega \frac{|\nabla v|^2}{v^2}dxdt \geq \liminf_{\epsilon=\epsilon_j\searrow 0}\int_0^T\int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2}dxdt. \tag{5.25}$$

On the other hand, we get by the lower semicontinuity of the norm in $L^2(\Omega\times(0,T))$ with respect to weak convergence that

$$\int_0^T\int_\Omega \frac{|\nabla v|^2}{v^2}dxdt \leq \liminf_{\epsilon=\epsilon_j\searrow 0}\int_0^T\int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2}dxdt. \tag{5.26}$$

Consequently, the conclusion (5.21) results from (5.25) and (5.26). □

6. Global existence and boundedness to very weak solutions

In this section we begin with proving that the function (u, v) determined in the last two sections just is the global very weak solution of (1.6).

Proof of Theorem 1. (i) For the case of $k \in (2-\frac{1}{n}, 2]$ with (1.13), we will demonstrate that the function (u, v) obtained in Lemma 4.7 is a very weak subsolution of (1.6) in $\Omega\times(0,T)$ for all $T>0$. Let φ satisfy (1.10). Multiplying $(3.1)_1$ by φ and integrating by parts, we have for all $\epsilon \in (0,1)$ that

$$\begin{aligned}-\int_0^T\int_\Omega u_\epsilon\varphi_t - \int_\Omega u_0\varphi(\cdot,0) = \int_0^T\int_\Omega u_\epsilon\Delta\varphi + \chi\int_0^T\int_\Omega \frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon\cdot\nabla\varphi \\ + r\int_0^T\int_\Omega u_\epsilon\varphi - \mu\int_0^T\int_\Omega u_\epsilon^k\varphi.\end{aligned} \tag{6.1}$$

By (4.24), we know

$$-\int_0^T\int_\Omega u_\epsilon\varphi_t \to -\int_0^T\int_\Omega u\varphi_t, \tag{6.2}$$

$$\int_0^T\int_\Omega u_\epsilon\Delta\varphi \to \int_0^T\int_\Omega u\Delta\varphi, \tag{6.3}$$

$$r\int_0^T\int_\Omega u_\epsilon\varphi \to r\int_0^T\int_\Omega u\varphi \tag{6.4}$$

as $\epsilon = \epsilon_j \searrow 0$. In addition, by (4.25) with (4.27),

$$\chi \int_0^T \int_\Omega \frac{u_\epsilon \nabla v_\epsilon}{(1+\epsilon u_\epsilon) v_\epsilon} \cdot \nabla \varphi \to \chi \int_0^T \int_\Omega u \frac{\nabla v}{v} \cdot \nabla \varphi \quad \text{as } \epsilon = \epsilon_j \searrow 0. \tag{6.5}$$

Consequently, we obtain by (6.2)–(6.5) with the Fatou lemma that

$$\begin{aligned}\mu \int_0^T \int_\Omega u^k \varphi &\le \mu \liminf_{\epsilon=\epsilon_j \searrow 0} \int_0^T \int_\Omega u_\epsilon^k \varphi \\ &= \int_0^T \int_\Omega u\varphi_t + \int_\Omega u_0 \varphi(\cdot, 0) + \int_0^T \int_\Omega u \Delta \varphi \\ &\quad + \chi \int_0^T \int_\Omega u \frac{\nabla v}{v} \cdot \nabla \varphi + r \int_0^T \int_\Omega u \varphi. \end{aligned} \tag{6.6}$$

Take ψ satisfying (1.11). Multiply $(3.1)_2$ by ψ and integrate by parts,

$$-\int_0^T \int_\Omega v_\epsilon \psi_t - \int_\Omega v_0 \psi(\cdot, 0) + \int_0^T \int_\Omega \nabla v_\epsilon \cdot \nabla \psi + \int_0^T \int_\Omega v_\epsilon \psi = \int_0^T \int_\Omega u_\epsilon \psi. \tag{6.7}$$

According to (4.26) and (4.24), we get (1.9) by taking $\epsilon = \epsilon_j \searrow 0$. This together with (6.6) indicates that (u, v) is a very weak subsolution of (1.6).

Taking φ in (1.10) and multiplying $(1.6)_1$ by $\gamma u_\epsilon^{\gamma-1} \varphi$, we have

$$\begin{aligned}-\int_0^T \int_\Omega u_\epsilon^\gamma \varphi_t - \int_\Omega u_0^\gamma \varphi(\cdot, 0) &= \gamma(1-\gamma) \int_0^T \int_\Omega u_\epsilon^{\gamma-2} |\nabla u_\epsilon|^2 \varphi + \int_0^T \int_\Omega u_\epsilon^\gamma \Delta \varphi \\ &\quad + \chi\gamma(\gamma-1) \int_0^T \int_\Omega \frac{u_\epsilon^{\gamma-1} \nabla u_\epsilon}{(1+\epsilon u_\epsilon)} \cdot \frac{\nabla v_\epsilon}{v_\epsilon} \varphi \\ &\quad + \chi\gamma \int_0^T \int_\Omega \frac{u_\epsilon^\gamma}{(1+\epsilon u_\epsilon)} \frac{\nabla v_\epsilon}{v_\epsilon} \cdot \nabla \varphi \\ &\quad + r\gamma \int_0^T \int_\Omega u_\epsilon^\gamma \varphi - \mu\gamma \int_0^T \int_\Omega u_\epsilon^{\gamma-1+k} \varphi. \end{aligned} \tag{6.8}$$

By (4.24), we know

$$-\int_0^T\int_\Omega u_\epsilon^\gamma \varphi_t \to -\int_0^T\int_\Omega u^\gamma \varphi_t, \tag{6.9}$$

$$\int_0^T\int_\Omega u_\epsilon^\gamma \Delta\varphi \to \int_0^T\int_\Omega u^\gamma \Delta\varphi, \tag{6.10}$$

$$r\gamma\int_0^T\int_\Omega u_\epsilon^\gamma \varphi \to r\gamma\int_0^T\int_\Omega u^\gamma \varphi, \tag{6.11}$$

$$-\mu\gamma\int_0^T\int_\Omega u_\epsilon^{\gamma-1+k}\varphi \to -\mu\gamma\int_0^T\int_\Omega u^{\gamma-1+k}\varphi \tag{6.12}$$

as $\epsilon = \epsilon_j \searrow 0$. Combine (4.23) and (5.21) in Lemma 5.2 to get

$$\chi\gamma(\gamma-1)\int_0^T\int_\Omega \frac{u_\epsilon^{\gamma-1}\nabla u_\epsilon}{(1+\epsilon u_\epsilon)}\cdot\frac{\nabla v_\epsilon}{v_\epsilon}\varphi \to \chi\gamma(\gamma-1)\int_0^T\int_\Omega u^{\gamma-1}\nabla u\cdot\frac{\nabla v}{v}\varphi \tag{6.13}$$

as $\epsilon = \epsilon_j \searrow 0$. In addition, we know from (4.25) and (4.27) that

$$\chi\gamma\int_0^T\int_\Omega \frac{u_\epsilon^\gamma}{1+\epsilon u_\epsilon}\frac{\nabla v_\epsilon}{v_\epsilon}\cdot\nabla\varphi \to \chi\gamma\int_0^T\int_\Omega u^\gamma\frac{\nabla v}{v}\cdot\nabla\varphi, \tag{6.14}$$

as $\epsilon = \epsilon_j \searrow 0$. Consequently, we obtain from (6.8) with (6.9)–(6.14) and the Fatou lemma that

$$\begin{aligned}
\gamma(1-\gamma)\int_0^T\int_\Omega u^{\gamma-2}|\nabla u|^2\varphi &\le \liminf_{\epsilon=\epsilon_j\searrow 0}\gamma(1-\gamma)\int_0^T\int_\Omega u_\epsilon^{\gamma-2}|\nabla u_\epsilon|^2\varphi \\
&= -\int_0^T\int_\Omega u^\gamma\varphi_t - \int_\Omega u_0^\gamma\varphi(\cdot,0) - \int_0^T\int_\Omega u^\gamma\Delta\varphi \\
&\quad - \chi\gamma(\gamma-1)\int_0^T\int_\Omega \frac{u^{\gamma-1}}{v}\nabla u\cdot\nabla v\varphi - \chi\gamma\int_0^T\int_\Omega \frac{u_\epsilon^\gamma}{v_\epsilon}\nabla v_\epsilon\cdot\nabla\varphi \\
&\quad - r\gamma\int_0^T\int_\Omega u^\gamma\varphi + \mu\gamma\int_0^T\int_\Omega u^{\gamma-1+k}\varphi
\end{aligned} \tag{6.15}$$

as $\epsilon = \epsilon_j \searrow 0$. This together with (6.7) yields that (u, v) is in fact a γ-entropy supersolution of (1.6) as well.

(ii) For the case of $k > 2$ with $r, \chi > 0$ satisfying (1.14), we can find that the conclusions (4.9) and (4.17) for some $\gamma \in (0, \frac{1}{2})$ are also valid by repeating the arguments for proving Lemmas 4.4 and 4.5, where Lemmas 4.2 and 4.3 are useful. This implies the corresponding results (4.23)–(4.28). Therefore, we can conclude that (u, v) is also a very weak solution of (1.6) by using the same procedures as those for the case (i).

The proof is complete. □

Next, we will prove that the global very weak solution to (1.6) is globally bounded for $n = 2, 3$. At first, we establish the following uniform-in-time estimate on $\int_\Omega u_\epsilon^2 dx$ for all $\epsilon \in (0, 1)$, with the initial data and $\frac{r}{\mu}$ suitably small.

Lemma 6.1. *Let (u_ϵ, v_ϵ) be the global very weak solution of the problem (1.6) established in Theorem 1 with $n = 2, 3$. There exist $\eta, \lambda > 0$ such that*

$$\int_\Omega u_\epsilon^2 dx \le M_9, \quad t > 0, \tag{6.16}$$

provided $\frac{r}{\mu} < \eta$ and $\int_\Omega u_0^2 dx + \int_\Omega |\nabla v_0|^4 dx < \lambda$, for all $\epsilon \in (0, 1)$ with $M_9 > 0$.

Proof. Assume $n = 3$. It follows from $(3.1)_1$ and (4.1) (or (4.4)) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_\epsilon^2 dx &= \int_\Omega u_\epsilon[\Delta u_\epsilon - \chi\nabla\cdot(\frac{u_\epsilon}{(1+\epsilon u_\epsilon)v_\epsilon}\nabla v_\epsilon) + ru_\epsilon - \mu u_\epsilon^k]dx \\
&\le -\frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2 dx + \frac{\chi^2}{2}\int_\Omega \frac{u_\epsilon^2}{v_\epsilon^2}|\nabla v_\epsilon|^2 dx + r\int_\Omega u_\epsilon^2 dx - \mu\int_\Omega u_\epsilon^{k+1} dx \\
&\le -\frac{1}{2}\int_\Omega |\nabla u_\epsilon|^2 dx - 2\int_\Omega u_\epsilon^2 dx + \frac{\chi^2}{2\min\{\delta_1^2, \delta_2^2\}}\int_\Omega u_\epsilon^2|\nabla v_\epsilon|^2 dx + C_{19}\int_\Omega u_\epsilon
\end{aligned} \tag{6.17}$$

by Young's inequality with positive constant $C_{19} := (r+2)^{\frac{2k}{k-1}}\mu^{-\frac{k+1}{k-1}}$ for $t > 0$. In addition, due to $\Delta|\nabla v_\epsilon|^2 = 2\nabla v_\epsilon \cdot \nabla\Delta v_\epsilon + 2|D^2 v_\epsilon|^2$ and $|\Delta v_\epsilon|^2 \le 3|D^2 v_\epsilon|^2$, we have from $(3.1)_2$ with the convexity of Ω and Young's inequality that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\epsilon|^4 dx &= 2\int_\Omega |\nabla v_\epsilon|^2 \nabla v_\epsilon \cdot \nabla(\Delta v_\epsilon - v_\epsilon + u_\epsilon)dx \\
&= \int_\Omega \Delta|\nabla v_\epsilon|^2 |\nabla v_\epsilon|^2 dx - 2\int_\Omega |D^2 v_\epsilon|^2 |\nabla v_\epsilon|^2 dx - 2\int_\Omega |\nabla v_\epsilon|^4 dx \\
&\quad + 2\int_\Omega \nabla u_\epsilon \cdot \nabla v_\epsilon |\nabla v_\epsilon|^2 dx
\end{aligned}$$

$$= -\int_\Omega |\nabla|\nabla v_\epsilon|^2|^2 dx - 2\int_\Omega |D^2 v_\epsilon|^2|\nabla v_\epsilon|^2 dx - 2\int_\Omega |\nabla v_\epsilon|^4 dx$$
$$- 2\int_\Omega u_\epsilon \nabla v_\epsilon \cdot \nabla|\nabla v_\epsilon|^2 dx - 2\int_\Omega u_\epsilon \Delta v_\epsilon |\nabla v_\epsilon|^2 dx$$
$$\le -\frac{1}{2}\int_\Omega |\nabla|\nabla v_\epsilon|^2|^2 dx - 2\int_\Omega |\nabla v_\epsilon|^4 dx + \frac{7}{2}\int_\Omega u_\epsilon^2 |\nabla v_\epsilon|^2 dx, \ t > 0.$$

This together with (6.17) indicates

$$\frac{d}{dt}\Big(\int_\Omega u_\epsilon^2 dx + \int_\Omega |\nabla v_\epsilon|^4 dx\Big) \le -4\Big(\int_\Omega u_\epsilon^2 dx + \int_\Omega |\nabla v_\epsilon|^4 dx\Big) - \int_\Omega |\nabla u_\epsilon|^2 dx - \int_\Omega |\nabla|\nabla v_\epsilon|^2|^2 dx$$
$$+ C_{20}\int_\Omega u_\epsilon^3 dx + C_{20}\int_\Omega |\nabla v_\epsilon|^6 dx + 2C_{19}m^*, \ t > 0 \tag{6.18}$$

with $C_{20} := 7 + \frac{\chi^2}{\min\{\delta_1^2, \delta_2^2\}}$. By the Gaglirado–Nirenberg inequality with the corresponding constant $C_{21} > 0$ and Young's inequality, we have

$$\int_\Omega u_\epsilon^3 dx = \|u_\epsilon\|_{L^3(\Omega)}^3 \le C_{21}\|u_\epsilon\|_{W^{1,2}(\Omega)}^{\frac{3}{2}} \|u_\epsilon\|_{L^2(\Omega)}^{\frac{3}{2}}$$
$$\le 3C_{21}(\|\nabla u_\epsilon\|_{L^2(\Omega)}^{\frac{3}{2}} \|u_\epsilon\|_{L^2(\Omega)}^{\frac{3}{2}} + \|u_\epsilon\|_{L^2(\Omega)}^3)$$
$$\le \frac{1}{C_{20}}\int_\Omega |\nabla u_\epsilon|^2 dx + 3C_{21}\Big(\int_\Omega u_\epsilon^2 dx\Big)^{\frac{3}{2}} + C_{22}\Big(\int_\Omega u_\epsilon^2 dx\Big)^3 \tag{6.19}$$

with $C_{22} := 81C_{20}^3 C_{21}^4$. Replacing u_ϵ by $|\nabla v_\epsilon|^2$ in (6.19), we know by a similar argument that

$$\int_\Omega |\nabla v_\epsilon|^6 dx = \||\nabla v_\epsilon|^2\|_{L^3(\Omega)}^3$$
$$\le \frac{1}{C_{20}}\int_\Omega |\nabla|\nabla v_\epsilon|^2|^2 dx + 3C_{21}\Big(\int_\Omega |\nabla v_\epsilon|^4 dx\Big)^{\frac{3}{2}} + C_{22}\Big(\int_\Omega |\nabla v_\epsilon|^4 dx\Big)^3. \tag{6.20}$$

Let $F_\epsilon(t) := \int_\Omega u_\epsilon^2 dx + \int_\Omega |\nabla v_\epsilon|^4 dx$. We have from (6.18)–(6.20) that

$$\begin{cases} F_\epsilon'(t) \le -4F_\epsilon(t) + 3C_{20}C_{21}F_\epsilon(t)^{\frac{3}{2}} + C_{20}C_{22}F_\epsilon(t)^3 + 2C_{19}m^*, & t > 0, \\ F_\epsilon(0) = \int_\Omega u_0^2 dx + \int_\Omega |\nabla v_0|^4 dx. \end{cases} \tag{6.21}$$

Denote

$$h(s, m^*) := -4s + 3C_{20}C_{21}s^{\frac{3}{2}} + C_{20}C_{22}s^3 + 2C_{19}m^*, \quad s > 0. \tag{6.22}$$

Then there exists $m_0^* > 0$ such that $h(s, m_0^*)$ has the unique positive root s_0. Obviously, $F_\epsilon(t) \equiv s_0$ satisfies the ODE problem

$$\begin{cases} F_\epsilon'(t) = h(F_\epsilon(t), m_0^*), \quad t > 0, \\ F_\epsilon(0) = s_0. \end{cases} \tag{6.23}$$

Now let

$$\eta := \left(\frac{m_0^*}{|\Omega|}\right)^{k-1} \text{ and } \lambda := \min\left\{s_0, \frac{m_0^{*2}}{|\Omega|}\right\}$$

with $\frac{r}{\mu} < \eta$ and $\int_\Omega u_0^2 dx + \int_\Omega |\nabla v_0|^4 dx < \lambda$. Then

$$\int_\Omega u_0 dx < |\Omega|^{\frac{1}{2}} \Big(\int_\Omega u_0^2 dx\Big)^{\frac{1}{2}} < m_0^*,$$

and thus

$$\int_\Omega u_\epsilon dx \le m^* = \max\left\{\int_\Omega u_0 dx, (\frac{r}{\mu})^{\frac{1}{k-1}}|\Omega|\right\} < m_0^*$$

for all $\epsilon \in (0, 1)$. Observe that $m^* < m_0^*$ implies $h(s, m^*) < h(s, m_0^*)$ for $s > 0$, and the function $h = h(s, m^*)$ has exactly two positive roots $0 < s_1 < s_0 < s_2$. We know from (6.22) that $h(s, m^*) < 0$ whenever $s \in (s_1, s_0)$. By (6.21) with $F_\epsilon(0) \le s_0$, this concludes that

$$F_\epsilon(t) = \int_\Omega u_\epsilon^2 dx + \int_\Omega |\nabla v_\epsilon|^4 dx \le s_0, \quad t > 0$$

for all $\epsilon \in (0, 1)$ by comparison.

The case $n = 2$ can be treated similarly, and even more simply. We omit the details.

The proof is complete. □

Proof of Theorem 2. Based on Lemma 6.1, we obtain the global boundedness of solutions to the regularization problem by a similar argument as that in [29] that

$$\|u_\epsilon\|_{L^\infty(\Omega)} + \|v_\epsilon\|_{L^\infty(\Omega)} \le C, \quad t > 0 \tag{6.24}$$

with some $C > 0$ for all $\epsilon \in (0, 1)$. Consequently, we conclude that the very weak solution (u, v) is globally bounded in time as well by taking $\epsilon = \epsilon_j \searrow 0$. □

7. Global boundedness to classical solutions

In this section we deal with classical solutions to (1.6) with $k > 2$ ensured by Lemma 2.1, and begin with the known estimate below.

Lemma 7.1. *([27, Lemma 2.3]) Let $n \geq 1$, $k > 2$, and (u, v) be the local classical solution of (1.6). Then there exists $M_{10} > 0$ such that for $k < m < \frac{nk}{(n+2-k)_+}$*

$$\|\nabla v\|_{L^m(\Omega)} \leq M_{10}, \; t \in (0, T_{\max}). \tag{7.1}$$

Due to the singularity of the sensitivity function in (1.6), we use the energy functional of the form $E(t) := \frac{1}{p}\int_\Omega u^p dx + \frac{1}{q}\int_\Omega |\nabla v|^{2q} dx$ with $p, q > 1$, instead of that for the classical Keller–Segel system with logistic source [4]. Firstly, we estimate $\frac{d}{dt}E(t)$.

Lemma 7.2. *Let (u, v) be the local classical solution of (1.6). Then for $p, q > 1$, there exists $M_{11} > 0$ such that*

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx \leq & -\frac{1}{p}\int_\Omega u^p dx - \frac{1}{q}\int_\Omega |\nabla v|^{2q} dx \\ & - \frac{2q-2}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx + M_{11}\int_\Omega \Big(\frac{|\nabla v|}{v}\Big)^{\frac{2(p+k-1)}{k-1}} dx \\ & + M_{11}\int_\Omega |\nabla v|^{2(q-1)\frac{p+k-1}{p+k-3}} dx + M_{11}\end{aligned} \tag{7.2}$$

for $t \in (0, T_{\max})$.

Proof. Multiply $(1.6)_1$ by u^{p-1} and integrate by parts to know

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx =& \int_\Omega u^{p-1}[\Delta u - \chi\nabla\cdot(\frac{u}{v}\nabla v) + ru - \mu u^k]dx \\ =& -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx + \chi(p-1)\int_\Omega \frac{u^{p-1}}{v}\nabla u\cdot\nabla v dx \\ & + r\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1} dx \\ \leq & \chi^2(p-1)\int_\Omega u^p\frac{|\nabla v|^2}{v^2} dx + r\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1} dx \\ \leq & -\frac{1}{p}\int_\Omega u^p dx + C_{23}\int_\Omega \Big(\frac{|\nabla v|}{v}\Big)^{\frac{2(p+k-1)}{k-1}} dx - \frac{\mu}{2}\int_\Omega u^{p+k-1} dx + C_{23}\end{aligned} \tag{7.3}$$

by Young's inequality and (2.13) with some $C_{23} > 0$ for $t \in (0, T_{\max})$. In addition, noticing $\Delta|\nabla v|^2 = 2\nabla v \cdot \nabla\Delta v + 2|D^2 v|^2$, we have from $(1.6)_2$ that

$$\begin{aligned} \frac{d|\nabla v|^2}{dt} &= 2\nabla v \cdot \nabla\Delta v - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v \\ &= \Delta|\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v. \end{aligned}$$

Testing this by $|\nabla v|^{2q-2}$ and integrating by part with Ω convex, we get

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx &= \int_\Omega |\nabla v|^{2q-2}[\Delta|\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v] dx \\ &\le -(q-1)\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 dx - 2\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 dx \\ &\quad - 2\int_\Omega |\nabla v|^{2q} dx + 2\int_\Omega |\nabla v|^{2q-2}\nabla u \cdot \nabla v dx, \quad t \in (0, T_{\max}). \end{aligned} \tag{7.4}$$

Apply Young's inequality to the last term of (7.4) to obtain

$$\begin{aligned} 2\int_\Omega |\nabla v|^{2q-2}\nabla u \cdot \nabla v dx &= -2(q-1)\int_\Omega u|\nabla v|^{2q-4}\nabla v \cdot \nabla|\nabla v|^2 dx - 2\int_\Omega u|\nabla v|^{2q-2}\Delta v dx \\ &\le \frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 dx + 2(q-1)\int_\Omega u^2|\nabla v|^{2q-2} dx \\ &\quad + \frac{2}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 dx + \frac{n}{2}\int_\Omega u^2|\nabla v|^{2q-2} dx \\ &\le \frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 dx + 2\int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 dx \\ &\quad + (2(q-1) + \frac{n}{2})\int_\Omega u^2|\nabla v|^{2q-2} dx. \end{aligned}$$

Together with (7.4), this yields

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx &\le -\frac{2q-2}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx \\ &\quad - 2\int_\Omega |\nabla v|^{2q} dx + (2(q-1) + \frac{n}{2})\int_\Omega u^2|\nabla v|^{2q-2} dx \end{aligned}$$

$$\leq -\frac{2q-2}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx - 2\int_\Omega |\nabla v|^{2q} dx$$
$$+\frac{\mu}{2}\int_\Omega u^{p+k-1} dx + C_{24}\int_\Omega |\nabla v|^{2(q-1)\frac{p+k-1}{p+k-3}} dx, \quad t \in (0, T_{\max}) \tag{7.5}$$

by Young's inequality with some $C_{24} > 0$. Combining (7.3) with (7.5), we conclude (7.2) with $M_{11} = \max\{C_{23}, C_{24}\}$. □

Based on Lemmas 7.2 and 2.5, we can obtain the L^p-boundedness of solutions for $p > 1$. That is the following lemma.

Lemma 7.3. *Under the condition of Theorem 3, there exists $p > n$ with $M_{12} > 0$ such that*

$$\int_\Omega u^p dx \leq M_{12}, \ t \in (0, T_{\max}). \tag{7.6}$$

Proof. For $k > 2$ with $r, \chi > 0$ satisfying (1.14), we have from Lemma 2.5 with (7.2) that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx$$
$$\leq -\frac{1}{p}\int_\Omega u^p dx - \frac{1}{q}\int_\Omega |\nabla v|^{2q} dx - \frac{2q-2}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx$$
$$+ M_{11}\delta_2^{-\frac{2(p+k-1)}{k-1}}\int_\Omega |\nabla v|^{\frac{2(p+k-1)}{k-1}} dx + M_{11}\int_\Omega |\nabla v|^{2(q-1)\frac{p+k-1}{p+k-3}} dt + M_{11} \tag{7.7}$$

for $t \in (0, T_{\max})$. By the Gaglirado–Nirenberg inequality and the Poincaré inequality with (7.1), we get

$$\|\nabla v\|_{L^{\frac{2(p+k-1)}{k-1}}(\Omega)}^{\frac{2(p+k-1)}{k-1}} \leq C_{25}\||\nabla v|^q\|_{W^{1,2}(\Omega)}^{\frac{2(p+k-1)}{(k-1)q}a}\||\nabla v|^q\|_{L^{\frac{m}{q}}(\Omega)}^{\frac{2(p+k-1)}{(k-1)q}(1-a)}$$
$$\leq C_{26}(\|\nabla|\nabla v|^q\|_{L^2(\Omega)}^{\frac{2(p+k-1)}{(k-1)q}a}\||\nabla v|^q\|_{L^{\frac{m}{q}}(\Omega)}^{\frac{2(p+k-1)}{(k-1)q}(1-a)} + \||\nabla v|^q\|_{L^{\frac{m}{q}}(\Omega)}^{\frac{2(p+k-1)}{(k-1)q}})$$
$$\leq C_{26}\Big(M_{10}^{\frac{2(p+k-1)(1-a)}{k-1}}\|\nabla|\nabla v|^q\|_{L^2(\Omega)}^{\frac{2(p+k-1)}{(k-1)q}a} + M_{10}^{\frac{2(p+k-1)}{k-1}}\Big) \tag{7.8}$$

with $C_{25}, C_{26} > 0$, and

$$a := \frac{\frac{qn}{m} - \frac{(k-1)qn}{2(p+k-1)}}{1 - \frac{n}{2} + \frac{qn}{m}}.$$

Let $q > \frac{(n-2)m}{2n}$ and $\frac{m-2}{2}(k-1) < p < \frac{(qn+m)(k-1)}{n} - (k-1)$. Then

$$a \in (0,1) \text{ and } \frac{2(p+k-1)}{(k-1)q}a < 2.$$

Therefore, we know from (7.8) by Young's inequality that

$$\int_\Omega |\nabla v|^{\frac{2(p+k-1)}{k-1}} dx \le \epsilon_1 \int_\Omega |\nabla|\nabla v|^q|^2 dx + C_{\epsilon_1} \tag{7.9}$$

with $C_{\epsilon_1} := C_{27}(1+\epsilon_1^{-\frac{(p+k-1)a}{(k-1)q-(p+k-1)a}})$ and $C_{27} > 0$. Again by the Gaglirado–Nirenberg inequality and the Poincaré inequality with (7.1),

$$\begin{aligned}
\|\nabla v\|^{\frac{2(q-1)(p+k-1)}{p+k-3}}_{L^{\frac{2(q-1)(p+k-1)}{p+k-3}}(\Omega)} &\le C_{28}\||\nabla v|^q\|^{\frac{2(q-1)(p+k-1)}{(p+k-3)q}b}_{W^{1,2}(\Omega)} \||\nabla v|^q\|^{\frac{2(q-1)(p+k-1)}{(p+k-3)q}(1-b)}_{L^{\frac{m}{q}}(\Omega)} \\
&\le C_{29}(\|\nabla|\nabla v|^q\|^{\frac{2(q-1)(p+k-1)}{(p+k-3)q}b}_{L^2(\Omega)} \||\nabla v|^q\|^{\frac{2(q-1)(p+k-1)}{(p+k-3)q}(1-b)}_{L^{\frac{m}{q}}(\Omega)} \\
&\quad + \||\nabla v|^q\|^{\frac{2(q-1)(p+k-1)}{(p+k-3)q}}_{L^{\frac{m}{q}}(\Omega)}) \\
&\le C_{29}\big(M_{10}^{\frac{2(q-1)(p+k-1)}{(p+k-3)m}(1-b)} \|\nabla|\nabla v|^q\|^{\frac{2(q-1)(p+k-1)}{(p+k-3)q}b}_{L^2(\Omega)} + M_{10}^{\frac{2(q-1)(p+k-1)}{(p+k-3)m}}\big)
\end{aligned} \tag{7.10}$$

with $C_{28}, C_{29} > 0$, and

$$b := \frac{\frac{qn}{m} - \frac{(p+k-3)qn}{2(q-1)(p+k-1)}}{1 - \frac{n}{2} + \frac{qn}{m}}.$$

Let $q > \frac{m+2}{2}$ and $p > \frac{2nq-n(k-1)-m(k-3)}{m+n}$. Then

$$b \in (0,1) \text{ and } \frac{2(q-1)(p+k-1)}{(p+k-3)q}b < 2.$$

Therefore, we know from (7.10) by Young's inequality that

$$\int_\Omega |\nabla v|^{\frac{2(q-1)(p+k-1)}{(p+k-3)}} dx \le \epsilon_2 \int_\Omega |\nabla|\nabla v|^q|^2 dx + C_{\epsilon_2} \tag{7.11}$$

with $C_{\epsilon_2} := C_{30}(1+\epsilon_2^{-\frac{(q-1)(q+k-1)b}{(p+k-3)q-(q-1)(q+k-1)b}})$ and $C_{30} > 0$. For $k > \frac{3(n+2)}{n+4}$, there exists $\hat{m} \in (k, \frac{nk}{n+2-k})$ such that $(3-k)n - \hat{m}(k-1) < 0$. If $q > \frac{\hat{m}+2}{2}$, then the interval for

$$p \in \big(\max\{\frac{\hat{m}-2}{2}(k-1), \frac{2nq-n(k-1)-\hat{m}(k-3)}{\hat{m}+n}\}, \frac{qn+\hat{m}-n}{n}(k-1)\big)$$

is well-defined. Let $\epsilon_1, \epsilon_2 > 0$ satisfy $\frac{2q-2}{q^2} - M_{11}\delta_2^{-\frac{2(p+k-1)}{k-1}}\epsilon_1 - M_{11}\epsilon_2 \geq 0$. We obtain by (7.7), (7.9) and (7.11) that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx \\
&\leq -\frac{1}{p}\int_\Omega u^p dx - \frac{1}{q}\int_\Omega |\nabla v|^{2q} dx \\
&\quad - \Big(\frac{2q-2}{q^2} - M_{11}\delta_2^{-\frac{2(p+k-1)}{k-1}}\epsilon_1 - M_{11}\epsilon_2\Big)\int_\Omega |\nabla|\nabla v|^q|^2 dx + C_{31} \\
&\leq -\frac{1}{p}\int_\Omega u^p dx - \frac{1}{q}\int_\Omega |\nabla v|^{2q} dx + C_{31} \qquad (7.12)
\end{aligned}$$

with some $C_{31} > 0$ for $t \in (0, T_{\max})$. If $q > n+1$, there exists some $p > n$ due to $\frac{qn+\hat{m}-n}{n}(k-1) > n$. We complete the proof from (7.12) by the Bernoulli inequality with some $M_{12} > 0$. □

Proof of Theorem 3. Applying [23, lemma 1.3] to $(1.6)_2$ with (7.6), we have

$$\begin{aligned}
\|\nabla v\|_{L^\infty(\Omega)} &\leq \|\nabla e^{t(\Delta-1)}v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)}u\|_{L^\infty(\Omega)} ds \\
&\leq 2K_3\|\nabla v_0\|_{L^\infty(\Omega)} + K_2\int_0^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\Big)e^{-\lambda_1(t-s)}\|u\|_{L^p(\Omega)} ds \\
&\leq C_{32}, \quad t \in (0, T_{\max}) \qquad (7.13)
\end{aligned}$$

with $C_{32} > 0$. Again by [23, Lemma 1.3] with $(1.6)_1$, we know from the Hölder inequality with (7.6) and (7.13) that

$$\begin{aligned}
\|u\|_{L^\infty(\Omega)} &\leq \|e^{t(\Delta-1)}u_0\|_{L^\infty(\Omega)} + \chi\int_0^t \|e^{(t-s)(\Delta-1)}\nabla\cdot(\frac{u}{v}\nabla v)\|_{L^\infty(\Omega)} ds \\
&\quad + (r+1)\int_0^t \|e^{(t-s)(\Delta-1)}u\|_{L^\infty(\Omega)} ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + \frac{\chi K_4}{\delta_2}\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{p}})e^{-\lambda_1(t-s)}\|u\nabla v\|_{L^p(\Omega)} ds \\
&\quad + K_1(r+1)\int_0^t (1 + (t-s)^{-\frac{n}{2p}})e^{-\lambda_1(t-s)}\|u - \overline{u}\|_{L^p(\Omega)} ds + m^*(r+1)\int_0^t e^{-(t-s)} ds
\end{aligned}$$

$$\leq \frac{\chi K_4}{\delta_2} \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}) e^{-\lambda_1 (t-s)} \|u\|_{L^p(\Omega)} \|\nabla v\|_{L^\infty(\Omega)} ds + C_{33}$$

$$\leq C_{34}, \ t \in (0, T_{\max})$$

with $C_{33}, C_{34} > 0$. This concludes $T_{\max} = \infty$ by Lemma 2.1, and thus the classical solution (u, v) is globally bounded. $\square$

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