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Replica scaling analysis of interfaces in random media

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Interfaces subject to random disorder are analyzed by a replica scaling theory. *Long*-range and *short*-range disorder corresponds to *repulsive* and *attractive* interaction among the replicas, respectively. If these interactions are treated properly, a scaling exponent is given for short-range disorder, such as random bonds; the Flory exponent for long-range disorder, such as random fields, is reproduced.

The problem of interfaces in random media arises in various experimental realizations in condensed-matter physics. Of particular interest are the interfaces subject to random fields (RF's) and random bonds (RB's). One is interested in the scaling behavior of interface deviations in the presence of disorder. A $(d-1)$-dimensional interface immersed in the d-dimensional space in equilibrium can be described by the partition function

$$Z=\int Dh \exp -H \equiv \exp(-F) . \tag{1}$$

F is the free energy and H is the disorder-dependent Hamiltonian

$$H=\int d^{d-1}\rho\{[\nabla h(\rho)]^2+V[h(\rho),\rho]\} , \tag{2}$$

where ρ is a $(d-1)$-dimensional vector on a base reference plane and $h(\rho)$ is the interface height or deviation from the flat position $h=0$. Note that overhangs are neglected in the above formulation. We consider a continuum approximation. Our results will be applicable to the strong-coupling region and for finite temperature which we set $T=1$. Our formalism does not permit us to probe possible crossover length scales and eventual roughening phase transitions. $V[h(\rho),\rho]$ is the random potential whose properties are specified by its correlation function,

$$\langle V(h,\rho)V(h',\rho')\rangle=\delta^{d-1}(\rho-\rho')R(h-h') , \tag{3}$$

where $\langle\cdots\rangle$ denotes the average over disorder. For the RF case $R(h)\sim h$; for the RB case $R(h)=\delta(h)$, which (viewed in Fourier space or under rescalings) is equivalent to $R(h)\sim 1/h$. In general, we assume $R(h)\sim h^{\alpha}$ with $-1\leq\alpha\leq 1$ interpolating between the RB and RF cases.

We are interested in the scaling exponent of the interface roughness $h\sim\rho^{\zeta}$ ($\rho=|\rho|$). The free-energy fluctuation $\Delta F\sim\rho^{\chi}$ is also of interest. There is a standard assumption that these two exponents are related by

$$\chi=2\zeta+d-3 , \tag{4}$$

which follows from a dimensionality argument about the kinetic term $(h^2\rho^{d-3}\sim\rho^{\chi})$ in the Hamiltonian (2).

The exponent ζ can be estimated in analogy with the Imry-Ma argument,[1] this was first proposed by Grinstein and Ma[2] and by Villain[3] for the RF case. Using a scaling argument they concluded that below the upper critical dimension $d_c=5$ the interface is rough and the exponent is

$$\zeta=(5-d)/3 . \tag{5}$$

Fisher[4] has developed a functional renormalization-group (FRG) scheme to treat the RF and RB problems systematically; his analysis confirms the above estimate for the RF case while giving the (perturbative) estimate of the RB case. It is straightforward to generalize the above scaling exponent (5) for continuous α in (3),

$$\zeta=(5-d)/(4-\alpha) , \tag{6}$$

which was first proposed by Kardar.[5] Equation (6) is obtained by requiring that the kinetic term in (2) $(h^2\rho^{d-3})$ and the random potential term $[\int V\sim\rho^{(d-1)/2}h^{\alpha/2}$, where typically $V\sim\rho^{-(d-1)/2}h^{\alpha/2}$ as implied by (3)] scale in the same way, in the spirit of the Flory argument[6] for polymer physics. For $d=2$, the interface is reduced to a line,[7] with the property of a directed polymer[8] (DP). The system is related to the noise-driven Burgers equation.[9] Extensive dynamical renormalization-group (DRG) analyses were carried out for the general correlation exponent[10] α in (3). A refined version of Fisher's functional RG for the general α was proposed by Halpin-Healy.[11]

There is now consensus on the exponent ζ (6) when disorder is sufficiently long-range correlated (i.e., large α); the RF case falls in this category. The functional RG (Refs. 4 and 11) and dynamical RG (Ref. 10) results agree with that of the Flory argument (6). This can be attributed to the fact that long-range-correlated disorder does not suffer anomalous renormalization under rescalings.[10] The large distance behavior of the disorder correlator determines the scaling of the interface, hence ζ. On the other hand, the case when disorder is short-range correlated is less understood. The RB case is an example. Straightforward extrapolation of the above Flory exponent (6) to $\alpha=-1$ (i.e., the RB case) is clearly wrong: in contrast to the exact exponent $\frac{2}{3}$ when $d=2$. Somewhere in the range $(1\geq\alpha\geq-1)$ the Flory result (6) must fail. This can be seen from analyzing perturbation series in the RG schemes[4,10,11] where one-loop calculation can give a reasonable estimate of ζ. For the RB case, Fisher[4] obtained $\zeta\simeq 0.2083\epsilon$ and $\epsilon=5-d$ while Halpin-Healy[11] advocated $\zeta=\frac{2}{9}\epsilon$. Nattermann,[12] using a straightforward generalization of the Imry-Ma argument,

has obtained $\zeta=(5-d)/4$. It is believed that RG schemes[4,10,11] give exact ζ for not too small α. The border value has been evaluated at one-loop perturbation level,[10] $\alpha_c=-\frac{1}{2}$. It is not clear if and how higher-order perturbative terms will change it.

In this Rapid Communication we propose a scaling analysis based on a replica scheme which permits us to treat long-range disorder (e.g., RF) and short-range disorder (e.g., RB) on the same footing. In this analysis we shall see that the dividing value is rather naturally $\alpha_c=0$. It will become clear why the Flory exponent, which is valid for long-range disorder, fails for short-range disorder. For $\alpha>0$ the Flory exponent is reproduced as it should be; for $\alpha<0$, in particular, for interfaces subject to RB disorder, we obtain a new result which is somewhat different from the previous[4,7,10] RG perturbative results.

Let us take the partition function (1) to the nth power and average it over disorder

$$\langle Z^n\rangle=\langle\exp(-nf)\rangle=\int Dh\exp\int d^{d-1}\rho'\left[-\sum_{i=1}^{n}[\nabla h_i(\rho)]^2+\sum_{i\neq j}^{n}R[h_i(\rho)-h_j(\rho)]\right], \tag{7}$$

where $F=-\ln Z$. It describes n identical interfaces, or replicas, with mutual interaction $R(h_i-h_j)$, with the same local ρ coordinates. The n interfaces are subject to the initial condition $h_i(\mathbf{0})=0$. For $d=2$ it reduces to a system of n directed polymers. When the interaction is $R=\delta(h_i-h_j)$ the above path integral can be carried out explicitly using Bethe ansatz techniques,[13] the right-hand side (RHS) of (7) is found to be $\exp(cn^3\rho)$ (assuming $\langle F\rangle=0$), where $-cn^3$ can be called the bound-state energy of the n replicas and c is a constant. For a particular case of very long-range disorder, Parisi was able to carry out the path integral directly.[14] His result also confirms the Flory result (6). Without explicit solutions for the more general cases it suffices to estimate the energy, as long as the exponent of n is correct. Indeed this can be done using a self-consistent Hartree-Fock approximation.[15] This approximation differs from the available exact results only in sub-leading-order terms[16] for large n.

Before actually tackling the above path integral, we show how to eventually use the results to obtain the exponent ζ. Suppose that (7) is of the form $\exp(n^\beta\rho^\gamma)$, i.e.,

$$\int dFP(F)\exp-nF\equiv\langle\exp(-nF)\rangle=\exp(n^\beta\rho^\gamma), \tag{8}$$

then the RHS of (8) can be regarded as the Laplace transform of the probability distribution density $P(F)$ of the free energy of a *single* interface in the *random* medium. The RHS of (8) plays the analogous role of a characteristic function. Knowing it permits us to invert the Laplace transform to find the free-energy distribution density,[17]

$$P(F)\sim\exp(-a|\Delta F|^\eta/\rho^\omega), \tag{9}$$

where a is a constant, $\eta=\beta/(\beta-1)$, $\omega=\gamma/(\beta-1)$, $\Delta F=F-F_0$, $F_0=\langle F\rangle$ is the mean free energy, and for our purpose it suffices to set F_0 zero. Even though $P(F)$ is defined to be symmetrical, only one sign ($F<0$) contributes in the above integral (8). Note that the above $P(F)$ is obtained by a steepest-descent approximation which retains only the leading-order contribution for both large n and large ρ. This is valid only if the physics in (7) produces the form (8) with both β,γ positive.

The knowledge of the $P(F)$ allows us to determine the free-energy fluctuation

$$\Delta F\sim\rho^{\omega/\eta}=\rho^{\gamma/\beta}, \tag{10}$$

where by definition we have $\chi=\gamma/\beta$. The last relation can be combined with (4) to yield ζ,

$$\zeta=\frac{3-d}{2}+\frac{\gamma}{2\beta}. \tag{11}$$

Now we show how to obtain (8) from the physics in (7). Observe that when $\alpha>0$ Eq. (7) describes n interfaces with the mutual *repulsive* interaction $R=(h_i-h_j)^\alpha$. That is, the potential energy tends to separate them. Only because of the kinetic-energy term and the initial condition $h_i(\mathbf{0})=0$, these interfaces do not immediately run away. However, they do not form a bound state in the usual sense since the typical distance between the interfaces increases with large ρ. The potential energy can be easily estimated giving $-\rho^{d-1}n^2h^\alpha$, the kinetic part $nh^2\rho^{d-3}$. Maximizing the exponential

$$\exp(-n\rho^{d-3}h^2+n^2\rho^{d-1}h^\alpha) \tag{12}$$

with respect to h we obtain the saddle-point solution $h\sim(n\rho^2)^{1/(2-\alpha)}$. As a matter of fact, the above solution is valid in the range of $0\leq\alpha<2$. Though for $\alpha>1$ we may not find physical realizations for general interfaces, the model (2) is nevertheless well defined and both the Flory (6) and the result here are still meaningful. The above expression is now of the form $\exp(n^\beta\rho^\gamma)$, with $\beta=(4-\alpha)/(2-\alpha)$ and $\gamma=d-3+4/(2-\alpha)$. Substituting the above γ and β into (11) we obtain $\zeta=(5-d)/(4-\alpha)$, which is the Flory result (6).

The above analysis reproduces the Flory exponent[5] for $\alpha>0$, thus reconfirming the conclusion reached for the interfaces in random fields.[2-4] What happens if $\alpha<0$? Now the potential energy in (7) is *attractive*. From physical considerations we expect that the n interfaces form a translationally invariant (in the directions of ρ), layered bound state. There is a well-defined bound-state energy $E(n)<0$ and the RHS of (7) or (8) should behave as $\exp[-\rho^{d-1}E(n)]$. The potential term would squeeze all the n interfaces to $h=0$, but this state has *least entropy*. It is clear that the estimate of the kinetic energy $h^2\rho^{d-3}$ is inadequate for attractive interactions. In polymer physics it is well known[6] that the kinetic term takes different forms: h^2/ρ for $h\gg\sqrt{\rho}$ and ρ/h^2 for $h\ll\sqrt{\rho}$.

Fisher and Fisher[18] were the first to consider the entropy loss of a $(d-1)$-dimensional interface between two rigid parallel walls separated by a distance h. They have

estimated it to be (in our present notation)

$$-\Delta S \sim \begin{cases} (\bar{h}/h)^{2(d-1)/(3-d)}, & \text{for } 1<d<3, \\ \exp[-(h/\bar{h})^2], & \text{for } d=3, \\ 0, & \text{for } d>3, \end{cases}$$

where $\bar{h}\sim\rho^{(3-d)/2}$, $\bar{h}\sim\sqrt{\log\rho}$, and $\bar{h}\sim$const, for $1<d<3$, $d=3$, and $d>3$, respectively. Note that for $d=2$ the above $-\Delta S$ reproduces for polymers ρ/h^2. It is the above entropy loss or kinetic-energy term that prevents the n interfaces from collapsing completely ($h=0$) in the presence of attractive interactions. This is possible only for $d<3$. For $d\geq 3$ unfortunately we have not found a mechanism that prevents the collapse. Combining the above kinetic term to the potential part we have

$$\exp\{\rho^{d-1}[-n(1/h)^{2(d-1)/(3-d)}+n^2h^{\alpha}]\}. \tag{13}$$

Maximizing (13) with respect to h, the saddle-point solution is

$$1/h = n^{(3-d)/[2(d-1)+(3-d)\alpha]}. \tag{14}$$

Note that the saddle-point solution only exists in the range

$$(2-3\alpha)/(2-\alpha) < d < 3; \tag{15}$$

for the RB case $\alpha=-1$, it is $\frac{5}{3}<d<3$. For d smaller than the lower bound the entropy loss term in (13) is not singular enough (when $h\ll 1$) to counter the attractive interaction. Substituting the solution (14) into the bound-state energy in (13) we obtain

$$\exp[-\rho^{d-1}E(n)],\quad E(n)=-n^{\beta}, \qquad \beta=\frac{4(d-1)+(3-d)\alpha}{2(d-1)+(3-d)\alpha}. \tag{16}$$

With the help of (11) we find ζ,

$$\zeta=\frac{(5-d)(d-1)+(3-d)\alpha}{4(d-1)+(3-d)\alpha}, \tag{17}$$

which is subject to the bound (15). Note that this result matches with that of the repulsive case $\alpha>0$ at $\alpha=0$. In fact, they both reduce to $\zeta=(5-d)/4$ at $\alpha=0$. One has to bear in mind that on the attractive side $\alpha=0^-$, the validity of the above matching relation is limited by (15) which is $1<d<3$ for $\alpha=0$. We do not know if the bound (15) is genuinely physical, or if it is due to the limitation of our formalism. Note also that though we do not have a saddle-point solution for the marginal case $d=3$, the result (17) for $d<3$ has a well-behaved limit $d\to 3$: $\zeta=\frac{1}{2}$ independent of α. Curiously, it coincides with the Imry-Ma-type result for RB *à la* Nattermann[12] for $d=3$. However, ours is not even a linear function of d, and (17) is different from Nattermann's $\zeta=(5-d)/4$ in the whole range of validity.

Let us compare our result for $d=2$ with DP analysis. Our result (17) reproduces the exact result[13] when $\alpha=-1$: $\zeta=\frac{2}{3}$. We plot our result together with that of RG perturbative result for general α in Fig. 1. The one-loop RG calculation[10] of ζ, or Flory argument, is believed to be exact for long-range disorder. However, the boundary or critical value of α_c has been determined perturbatively. From the present analysis it is clear that the conclusion about long-range disorder cannot be extended to $\alpha<0$, where the physics qualitatively changes—from repulsion to attraction.

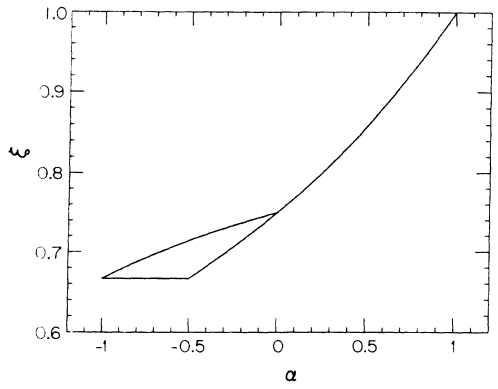

FIG. 1. The exponent ζ (vertical axis) vs α. The upper curve is our result $(3+\alpha)/(4+\alpha)$ for $\alpha<0$, obtained from (17) when $d=2$, compared to the perturbative result, the lower curve. For $\alpha>0$ ζ is the Flory exponent and the two curves overlap.

Consider another special case that is the RB interface in $d=3$. Besides Nattermann's scaling argument $\zeta=\frac{1}{2}$, Fisher's RG gives $\zeta\simeq 0.4166$ and Halpin-Healy argues for $\zeta=\frac{4}{9}\simeq 0.4444$. A recent transfer-matrix numerical simulation[19] of interfaces subject to random-bond disorder has estimated $\zeta=0.50\pm 0.08$ for $d=3$. It seems to support our scaling result. However, interface simulations for $d=3$ have proved to be very difficult to obtain reliable results at present stage.

One may wonder what is the justification of keeping only the large n leading order, whereas the traditional replica approaches consist of the $n\to 0$ limit; thus the smallest power contributes most. For our problems at hand we expect that the RHS of (8) is a single *positive* power on the exponent. However, within our scaling formalism we can never be sure if our procedure has generated artificially a subleading order. The key to this puzzle is that such subleading order always has the opposite sign and one can easily verify that a negative power does not contribute to the scaling of $P(F)$ defined in (8). Let us see how a subleading order arises. In our preceding discussions we assumed the kinetic terms being proportional to n. But one may argue that this factor should be $n-1$ since the motion of the center of mass should be ruled out. We have assumed also that potential terms have a factor n^2, which should really be $n(n-1)$ since an interface does not interact with itself. For more discussions on subleading orders[16] see an explicit calculation[20] for the DP $d=2$. With the above correction we would obtain $(n-1)n^{\beta-1}$ in place of n^{β} on the RHS of (8). On the other hand, positive "subleading" order can indeed appear in some circumstances when it plays a dominant role in the $P(F)$ scaling and thus cannot be neglected: in an external field as shown by Kardar,[13] and if the noise correlation function (3) is a Yukawa-like potential,[17] e.g., $1/h\exp(-h)$, which has different behaviors at the infrared and at the ultraviolet regions. In any event we do not expect these

complications in our present problems.

One may also wonder how the above replica scheme works; does the difference between attraction and repulsion manifest itself in the original random systems? Understanding this question will help us better appreciate the working of the present method and see its limitations. Let us call the optimal configuration of the interface in the random medium the ground state. Consider n identical interfaces being placed in a single-random environment without taking any average. For long-range disorder, such as RF, in the vicinity of the ground state it is not very likely that one would find energetically favorable (metastable) states[2,4]—they are more likely at large distances. The thermal fluctuations will take the interfaces wandering over large distances. This is the origin of the repulsive interaction among the n replicas. For short-range disorder, on the other hand, in the neighborhood of the ground state there are many favorable states.[21] The n replicas tend to stay together, and hence the attractive interaction. It is the non-Gaussian nature (9) of these fluctuations about the ground state that necessitates the detour into replica space. Our replica scheme serves to invert the moments to find the distribution function. In the above analysis we have tacitly assumed there is a unique ground state in the random systems; we have also assumed replica symmetry is unbroken in the replicated systems.

We can generalize the present analysis to the general case of d'-dimensional manifold imbedded in $d > d'$ dimensions,[11] which will not be discussed here. Of particular interest is the problem of directed polymers in $d > 2$ dimensions, which is related to other important physical problems, such as the surface growth.[22] The relevant attractive potential is $R \sim (h-h')^{\alpha}$, $\alpha = -d+1$. If we apply the above analysis to directed polymers for $\alpha \leq -d+1 = -2$, it can easily be verified that there is no (saddle point) physical solution. The failure to extend our analysis to DP of $d \leq 3$ shows some of our assumptions may break. Our conjecture is that the symmetry of replicas is broken. Derrida and co-workers have shown for finite or large d that replica symmetry is indeed broken.[23] Further investigations[24] into the possible replica breaking solutions for DP in high dimensions are worth pursuing.

In this paper we have introduced a replica scaling theory to analyze the interfaces in random media. We have shown that short-range-correlated disorder and long-range-correlated disorder have to be treated differently. The former corresponds to attractive interaction among the n replicas and the latter to repulsive interaction, respectively. From our analysis we conclude that the border value is $\alpha_c = 0$. For long-range disorder the Flory exponent is reproduced. For short-range disorder, in particular, for interfaces subject to random-bond disorder, a scaling exponent is found. Since they are based on the same level of assumptions we expect our result is as good as the Flory exponent for RF in the whole range of validity.

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