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CONTINUOUS MODELS OF PERCOLATION THEORY. I

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Percolation models in which defect centers are distributed randomly in space in accordance with Poisson's law and the shape of each defect is also random are considered. Coincidence of two critical points is proved. One of these corresponds to the time when the mean number of defects connected to a given defect becomes infinite. The other corresponds to the existence of percolation in an arbitrarily large region of space.

Introduction

Percolation theory is used to describe various physical phenomena. These include the Mott transition in extrinsic semiconductors (see [1, 2]), a polymer's loss of insulating properties as a result of chemical destruction (see [3]), etc. The following model corresponds to these phenomena. Points are chosen randomly in space and surrounded by certain regions, which are called defects. Depending on the mutual disposition and shape, some pairs of defects may form bonds. When the number of defects is sufficiently high, an infinite cluster arises, i.e., a connected set of defects that permeates the entire space. In such a case, one says that there is percolation. One of the main problems of the theory is to establish the conditions under which percolation occurs.

Almost all the studies on percolation theory have dealt with discrete (lattice) models. In the discrete models, the centers of the defects are chosen at the points of some lattice. Each point forms the center of a defect with probability p irrespective of the others. Defects are assumed to be connected if they are situated at neighboring points of the lattice.

With this paper we commence a cycle of papers devoted exclusively to continuous models. In these models, the centers of the defects form a Poisson ensemble. The shape and orientation of the defects can be fixed or random.

The contents of the papers of the cycle are as follows. In Sec. 1 (Part I) we give the basic definitions and results relating to discrete models (they are not available in the literature in Russian). In Sec. 2, continuous models are described. Three critical values of the density parameter of the Poisson field are distinguished: λ_H, λ_T, and λ_S. The first, λ_H, is the least density of defect centers for which an infinite cluster arises when this density is exceeded; λ_T is the least density such that when it is exceeded the mean number of defects connected (directly or in a chain) to a given defect becomes infinite; λ_S is the least density such that when it is exceeded it is almost always possible to find in any large cube (square) a path through connected defects from one of its sides to the opposite. It is easy to show that $\lambda_T \leq \lambda_H$. The quantity λ_T can be estimated by rigorous mathematical methods. The quantity λ_S can be found approximately by numerical simulation on a computer. In Sec. 3, we show that $\lambda_T = \lambda_S$ (the analogous result for discrete models was proved in [4]). In Sec. 4 (Part II), we shall describe methods for estimating λ_H, λ_T, λ_S and find estimates for specific problems. In Sec. 5, we consider the case when the defects are elongated and their orientation random. We show that in this case the critical volume concentration of defects is approximately proportional to a/l, where l and a are, respectively, the major and minor axes of the defect. For two-dimensional problems, we show that the mean number of (direct) bonds per defect is bounded at the instant of percolation. This fact was previously established in [1] for problems with fixed orientation. In Sec. 6, we investigate the question of the number of infinite clusters. We show that this number is almost certainly equal either to zero, or to unity, or to infinity (for discrete problems, this fact was proved in [5, 6]).

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1. Discrete Models

Our investigation will make essential use of results obtained for discrete problems. Therefore, we formulate the basic definitions and theorems of percolation theory for graphs.

Let $\mathcal{G}$ be some unoriented graph with set of vertices V. Two vertices v and w are said to be neighbors if there exists a line of the graph joining them. We denote this fact by $v\mathcal{G}w$. A path on $\mathcal{G}$ is a sequence $r=(v_0, v_1, \ldots, v_n)$, where $v_i \in V$ and $v_i\mathcal{G}v_{i+1} \forall i=0, 1, \ldots, n-1$. A graph is said to be connected if any two of its vertices (points) can be connected by a path.

DEFINITION 1.1. Let M be a set of vertices of the graph $\mathcal{G}$. Its boundary on $\mathcal{G}$ is the set $\partial M=\{v\in V\setminus M \mid \exists w\in M : w\mathcal{G}v\}$.

Generalized problem of points on a graph: Suppose the graph $\mathcal{G}$ satisfies the following four conditions:

1) $\mathcal{G}$ is embedded in $\mathbb{R}^d$ in such a way that each coordinate vector $\mathbb{R}^d$ is a period of the graph;

2) there exists a finite z such that the degree of each vertex (i.e., the number of lines of the graph that emanate from it) does not exceed z;

3) all lines of the graph $\mathcal{G}$ have finite length, and every compactum in Euclidean space $\mathbb{R}^d$ contains only a finite number of lines of the graph;

4) $\mathcal{G}$ is connected.

We assume that the set of vertices V of the graph is divided into m pairwise nonintersecting subsets $V_1, \ldots, V_m$ with $v \in V_i$ if and only if $v+\sum_{j=1}^{d} k_j\xi_j \in V_i, 1\leqslant i\leqslant m, k_j\in \mathbb{Z}$ (ξ_j are coordinate vectors of $\mathbb{R}^d$).

Each vertex $v_i \in V_i$ can, independently of the others, be in two states: It can be "defective" with probability p(i) and "nondefective" with probability 1 − p(i). Thus, the vector $\mathbf{p}=(p(1),\ldots, p(m))\in[0, 1]^m$ specifies a probability distribution P_p of defective vertices on the set V.

We do not consider the case when some component of the vector p is 0 or 1. It is obvious that in this case one can simply consider a different graph by eliminating certain lines or vertices from the original graph.

A path r is said to be defective if all its vertices are defective. Two defective vertices are said to be connected if there exists a defective path connecting them. A cluster on $\mathcal{G}$ is by definition a set of defective vertices connected on $\mathcal{G}$ and maximal in the sense of inclusion. A cluster is said to be infinite if it contains an infinite number of vertices of the graph.

We denote by $\theta(\mathbf{p}, v)$ the probability that vertex $v \in V$ belongs to an infinite cluster. It is shown in [4] that for any fixed vertex v the inequality $\theta(\mathbf{p}, v) > 0$ holds if and only if $\theta(\mathbf{p}, v) > 0$ for all $v \in V$.

We define the set

$$H=\{\mathbf{p}\in(0, 1)^m \mid \theta(\mathbf{p}, v)=0, \forall v\in V\}. \tag{1.1}$$

It has the property that for $\mathbf{p} \in H$ there is almost certainly no infinite cluster, while for $\mathbf{p}\in(0, 1)^m\setminus H$ an infinite cluster almost certainly exists (and one says that there is percolation).

We also introduce the set

$$T=\{\mathbf{p}\in(0, 1)^m \mid E_\mathbf{p}\{|W(v)|\}<\infty \ \forall v\in V\}, \tag{1.2}$$

where $|W(v)|$ denotes the number of vertices in the cluster W containing the vertex v and E_p is the operator of the mathematical expectation with respect to the measure P_p.

Let $\sigma_\mathbf{p}(B(n, i))$ be the probability that in the parallelepiped $B(n, i)=[0, 3n]\times[0, 3n]\times\ldots\times[0, n]\times\ldots\times[0, 3n]$ (here $[0, n]$ stands in the i-th component of the Cartesian product) there exists a cluster that intersects B(n, i) in the direction of the i-th coordinate, i.e., from the face $[0, 3n]\times\ldots\times\{0\}\times\ldots\times[0, 3n]$ to the face $[0, 3n]\times\ldots\times\{n\}\times\ldots\times[0, 3n]$. Let

$$S=\{\mathbf{p}\in(0,1)^m \mid \lim_{n\to\infty}\sigma_\mathbf{p}(B(n, i))=0, 1\leqslant i\leqslant d\}. \tag{1.3}$$

It is readily noted that $H\supseteq T$. Kesten's [4] fundamental result states that T = S. In the case m = 1, this result can be formulated in terms of the three critical points: $p_T=p_S\leqslant p_H$, where

$$p_T=\sup\{p\in[0,1]\,|\,E_p\{|W(v)|\}<\infty\}, \tag{1.4}$$

$$p_S=\sup\{p\in[0,1]\,|\,\lim_{n\to\infty}\sigma_p(B(n,i))=0,\ 1\leqslant i\leqslant d\}, \tag{1.5}$$

$$p_H=\sup\{p\in[0,1]\,|\,\theta(p,v)=0\}. \tag{1.6}$$

In this paper, we prove an analogous theorem for a large class of continuous problems.

2. Formulation of Continuous Percolation Problems

We specify a family of sets $\mathfrak{D}=\{\mathscr{D}(\alpha)\}_{\alpha\in A}$, where A is some space of parameters with probability measure μ on the standard Borel σ algebra $\sigma(A)$ [7]. With respect to all elements of $\mathscr{D}(\alpha)$, we assume the following conditions are satisfied: 1) $\mathscr{D}(\alpha)$ is compact in Euclidean space $\mathbb{R}^d$; 2) there exists a distinguished point $x_0\in\mathscr{D}(\alpha)$, called the center, with respect to which $\mathscr{D}(\alpha)$ is starlike, i.e., $\forall x\in\mathscr{D}(\alpha)$, the segment $[x_0, x]$ also belongs to $\mathscr{D}(\alpha)$. Accordingly, we denote $\mathscr{D}(\alpha)=\mathscr{D}(x_0,\alpha)$. We shall call the elements of $\mathfrak{D}$ defects. It follows from their definition that all sets $\mathscr{D}(\alpha)$ are connected. Frequently, we shall also ascribe to a defect not only a center but also a unit vector, called the direction.

We consider in $\mathbb{R}^d$ a set of points that form a Poisson field of intensity λ. With each such point x we associate, independently of the others, a definite defect $\mathscr{D}(x,\alpha)$ with center at this point. The parameter α is determined by the probability distribution on A, so that the μ-probability of having at the fixed point x_0 of the Poisson field the defect $\mathscr{D}(x_0,\alpha)$, where $\alpha\in B\in\sigma(A)$, is $\mu(B)$.

The simplest example of a continuous problem is the problem of spheres. In this case, the defects are spheres of fixed radius whose centers form in $\mathbb{R}^d$ a Poisson field. In this case, the space A is trivial. Another example is the problem of spheres of random radius: the radius is distributed with density $\rho(r)$ such that $\int_0^\infty \rho(r)\,dr=1$. Here, $A=[0,\infty)$, and the probability measure is given by the generalized function $\rho(r)$. A further example is the problem with random orientation of the defects. The defects are mutually congruent figures. Besides the center, we associate with a defect a unit vector $\mathbf{n}$, the direction. We consider the problem in which the centers form a Poisson ensemble in $\mathbb{R}^d$, and the directions are distributed with density $\varphi(\mathbf{n})$, where φ is a generalized function on the sphere S^{d-1} and $\int_{S^{d-1}}\varphi(\mathbf{n})\,d\mathbf{n}=1$. Note that if the defects are centrally symmetric with respect to the center, $\mathbf{n}$ can be regarded as distributed over a hemisphere (or rather, over the projective space $\mathbb{R}P^{d-1}$) with density $\varphi(\mathbf{n})+\varphi(-\mathbf{n})$.

We define a path S as a set of defects $\{\mathscr{D}_0,\mathscr{D}_1,\ldots\}$ satisfying the condition $\mathscr{D}_i\cap\mathscr{D}_{i+1}\neq\varnothing$, $i=0,1,\ldots$. Two defects $\mathscr{D}'$ and $\mathscr{D}''$ are said to be connected if there exists a finite path $\{\mathscr{D}_0,\mathscr{D}_1,\ldots,\mathscr{D}_k\}$ such that $\mathscr{D}_0=\mathscr{D}'$ and $\mathscr{D}_k=\mathscr{D}''$. A set of connected defects that is maximal in the sense of inclusion is called a cluster. Maximality is understood in the sense that all defects connected to at least one defect of the set also belong to the set. The power $|W|$ of the cluster W is defined as the number of defects in it. A cluster is said to be infinite if $|W|=\infty$.

We divide $\mathbb{R}^d$ into equal cubes K_ξ with edge of length ξ; $\theta_\xi(\lambda)$ is the probability that there exists a point $x\in K_\xi$ belonging to an infinite cluster. It follows from the homogeneity that $\theta_\xi(\lambda)$ does not depend on the position of K_ξ in $\mathbb{R}_p$. Let $\lambda_H=\sup\{\lambda\,|\,\theta_\xi(\lambda)=0\}$. Then, since the number of cubes is countable, for $\lambda<\lambda_H$ an infinite cluster almost certainly does not exist, while for $\lambda>\lambda_H$ an infinite cluster almost certainly does. One then says that there is percolation. Note that $\theta_\xi(\lambda)=0$ implies $\theta_\eta(\lambda)=0$ for all $\eta\leq\xi$. Since any cube K_ζ for $\zeta>\xi$ is contained in the union of a finite number of cubes K_ξ, $\theta_\xi(\lambda)=0$ also implies $\theta_\zeta(\lambda)=0$ for all $\zeta>\xi$. Therefore, the definition of λ_H does not depend on ξ. In particular,

$$\lambda_H=\sup\{\lambda\,|\,\theta(\lambda,x)=0\}, \tag{2.1}$$

where $\theta(\lambda,x)$ is the probability that the fixed point $x\in\mathbb{R}^d$ belongs to an infinite cluster. We take the last equation as the definition of λ_H.

We define a crossing of the parallelepiped $[a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_d,b_d]$ in the i-th direction as an event for which there exists a path $\{\mathscr{D}_1(x_1),\ldots,\mathscr{D}_n(x_n)\}$ such that $[x_1]_i\leqslant a_i$, $[x_n]_i\geqslant b_i$ and $\bigcup_{j=1}^n\mathscr{D}_j(x_j)$ is contained in the strip $[a_1, b_1]\times\ldots\times[a_{i-1}, b_{i-1}]\times\mathbb{R}\times[a_{i+1}, b_{i+1}]\times\ldots\times[a_d, b_d]$ ($[x]_i$ denotes the i-th component of the vector $x\in\mathbb{R}^d$).

We also introduce

$$\lambda_T=\sup\{\lambda|E_\lambda\{|W(x)|\}<\infty\}, \tag{2.2}$$

where $|W(x)|$ is the power of the cluster W, which contains the point x, and

$$\lambda_S=\sup\{\lambda|\lim_{n\to\infty}\sigma(n,\lambda,i)=0, 1\leqslant i\leqslant d\}, \tag{2.3}$$

where $\sigma(n, \lambda, i)$ is the probability of a crossing in the i-th direction of the parallelepiped $[0, 3n]\times[0, 3n]\times\ldots\underline{\ }[0, n]\times\ldots\times[0, 3n]$ (the factor $[0, n]$ corresponds to the i-th coordinate vector).

Let $L(\mathbf{r})$ be the μ-probability that the point displaced from the center of the defect by the vector $\mathbf{r}$ is covered by the defect; $V=\int L(\mathbf{r})d\mathbf{r}$ is the mean volume of the defect. The Poisson probability for the absence of a defect center in the infinitesimal volume $d\mathbf{r}$ is $\exp\{-\lambda d\mathbf{r}\}$. Therefore, the probability that the point $\mathbf{s}$ is not covered by a defect with center in $d\mathbf{r}$ near the point $\mathbf{r}$ is $\exp\{-L(\mathbf{s}-\mathbf{r})\lambda d\mathbf{r}\}$. Therefore, the probability that the point $\mathbf{s}$ is not covered by any defect is

$$\exp\left\{\int -L(\mathbf{s}-\mathbf{r})\lambda d\mathbf{r}\right\}=\exp\left\{-\lambda\int L(\mathbf{r})\,d\mathbf{r}\right\}=\exp\{-\lambda V\}.$$

Thus, the critical quantities can also be described in terms of the defect volume concentration $C=1-\exp\{-\lambda V\}$.

The recognition of the three different critical points has a deep meaning. The physics literature contains many results on the finding of the percolation point by computer simulation. It must be borne in mind that in this way one estimates, not λ_H, but λ_S, and the equality of these too in the general case has in no way been proved.

3. Critical Point Theorem

In the previous section, we introduced three critical points: λ_H, λ_S, and λ_T. The main result of this section will be to prove a theorem which establishes the equivalence of the points λ_T and λ_S. It will be convenient to give the proof first for the simplest problem – spheres of constant radius – and then generalize it to more complicated cases.

THEOREM 3.1. For the problem of congruent spheres in $\mathbb{R}^d$ the relation $\lambda_T=\lambda_S\leqslant\lambda_H$ holds.

Proof. We denote by $\lambda_H(l)$, $\lambda_T(l)$, $\lambda_S(l)$ the critical densities of the Poisson field for the problem of spheres of diameter l. We also denote $\lambda_H=\lambda_H(1), \lambda_T=\lambda_T(1), \lambda_S=\lambda_S(1)$. It is readily seen that $\lambda_H(l)=\lambda_H\cdot l^{-d}$, $\lambda_T(l)=\lambda_T\cdot l^{-d}$, $\lambda_S(l)=\lambda_S\cdot l^{-d}$. Therefore, it is sufficient to consider the case $l=1$.

The idea of the proof is to represent the continuous percolation problem as a limit of discrete problems for which the analogous equality holds. For this, we associate with the ensemble of defect centers distributed in $\mathbb{R}^d$ in accordance with the Poisson law with parameter λ a distribution of defective vertices in a discrete percolation problem. We consider in $\mathbb{R}^d$ the graph $\mathcal{G}_n$ with set of vertices $V_n=\{(k_1/n, k_2/n,\ldots, k_d/n)|k_i\in\mathbb{Z}, i=1,\ldots,d\}$. Let $x_0=(k_1/n,\ldots,k_d/n)\in V_n$. We denote by $\Omega_n(x_0)$ the set $[(k_1-{}^1/_2)/n, (k_1+{}^1/_2)/n)\times\ldots\times[(k_d-{}^1/_2)/n, (k_d+{}^1/_2)/n)$. Two vertices x_0 and y_0 of $\mathcal{G}_n$ are joined by a line if $\inf\limits_{\substack{x\in\Omega_n(x_0)\\ y\in\Omega_n(y_0)}}\|x-y\|\leqslant 1$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

We define on $\mathcal{G}_n$ a percolation problem as follows: The vertex $x_0\in\mathcal{G}_n$ is defective if at least one defect center of the continuous problem is contained in $\Omega_n(x_0)$. Thus, every vertex is defective with probability $p=1-\exp\{-\lambda/n^d\}\overset{\text{def}}{=}f_n(\lambda)$ and nondefective with probability $1-p$, and for different vertices x_0 and y_0 these events are independent, since $\Omega_n(x_0)\cap\Omega_n(y_0)=\varnothing$. Thus, for each defect $\mathcal{D}(x)$ with center at the point x there corresponds a defective vertex $x_0\in V_n$ such that $x\in\Omega_n(x_0)$. Such a vertex always exists and is unique. Thus, we have defined a mapping π of the defect centers of the continuous problem onto the set of defective vertices of the graph $\mathcal{G}_n$.

Consider two intersecting defects with centers at the points x and y. Then $\|x-y\|\leq 1$, and therefore the defective vertices of the graph corresponding to them either coincide or are joined by a line in accordance with the construction of $\mathcal{G}_n$. Thus, to each cluster $W^{(c)}$ in the continuous problem there corresponds a cluster $W^{(d)}$ in the discrete one (for brevity, we shall call them c-clusters and d-clusters). Since in any finite volume there is almost certainly a finite number of defect centers, for each realization of the Poisson field an infinite cluster in the continuous problem is associated with an infinite cluster on $\mathcal{G}_n$. Therefore, percolation in the continuous problem also implies percolation on $\mathcal{G}_n$. Therefore, for all n the inequality $p_H(\mathcal{G}_n)\leqslant f_n(\lambda_H)$ holds. Since $f_n(\lambda)$ is an increasing function, there exists an inverse function $f_n^{-1}(p)$, and $\lambda_H\geqslant f_n^{-1}(p_H(\mathcal{G}_n))$.

We show that such an inequality also holds for λ_T. We have defined the mapping π of the defect centers onto the set of defective vertices of the graph $\mathcal{G}_n$. We consider the complete pre-image $\pi^{-1}(W^{(d)})$ for each cluster $W^{(d)}$ on $\mathcal{G}_n$. All the defects of the continuous problem can be decomposed into nonintersecting classes, and it follows from the construction of $\mathcal{G}_n$ that no c-cluster belongs to more than one class. Thus, the set of c-clusters also decays into nonintersecting classes. We denote by $\Gamma(W^{(c)})$ such a class containing the cluster $W^{(c)}$. By definition, $|\Gamma|=\sum_{W^{(c)}\in\Gamma}|W^{(c)}|$. Thus, we always have $|W^{(c)}|\leqslant|\Gamma(W^{(c)})|$. Hence, $E\{|W^{(c)}|\}\leqslant E\{|\Gamma(W^{(c)})|\}$. Let $D(k, l)=D(k, l, x_0, \mathcal{G}_n)$ be the set of subsets $W\subseteq V_n$ that are connected on $\mathcal{G}_n$, contain the vertex x_0, and are such that $|W|=k$, $|\partial W|=l$ (see Definition 1.1). Let the elements of the set $D(k, l)$ be $W_1(k, l)$, $W_2(k, l),\dots, W_{d(k,l)}$, where $d(k, l)=|D(k, l)|$. We consider the d-cluster $W^{(d)}$, which contains the vertex x_0, and denote by $\Gamma=\pi^{-1}(W^{(d)})$ its complete pre-image. Let $A_i(k, l)$ be the event $W^{(d)}=W_i(k, l)$. All such events are mutually incompatible, and

$$\bigcup_{k,l}\bigcup_{i=1}^{d(k,l)}A_i(k,l)=\mathrm{I},$$

where I is an event of probability 1. Then in accordance with the formula for the total mathematical expectation

$$E\{|\Gamma|\}=\sum_{k,l=1}^{\infty}\sum_{i=1}^{d(k,l)}E\{|\Gamma|/A_i(k,l)\}P\{A_i(k,l)\}=\sum_{k,l}\sum_{i=1}^{d(k,l)}E\left\{\sum_{j=1}^{k}\xi_j/\xi_j>0\ \forall j\right\}P\{W_i(k,l)\},$$

where ξ_j is the complete pre-image of the j-th vertex of the d-cluster $W_i(k, l)$. Note that ξ_j are independent, equally distributed Poisson random variables, whence

$$E\{\xi_1+\ldots+\xi_k/\xi_1>0,\ldots,\xi_k>0\}=kE\{\xi_1/\xi_1>0\}=k\frac{E\xi_1}{P\{\xi_1>0\}}=k\nu,$$

where $\nu=\frac{\lambda}{n^d}\left(1-\frac{\lambda}{n^d}\right)^{-1}$. Then

$$E\{|\Gamma|\}=\sum_{k,l}\sum_{i=1}^{d(k,l)}k\nu P\{W_i(k,l)\}=\sum_{k,l}k\nu d(k,l)\,(f_n(\lambda))^k(1-f_n(\lambda))^l=\nu E\{|W^{(d)}|\}.$$

Thus, $E\{|W^{(c)}|\}\leqslant\nu E\{|W^{(d)}|\}$. Then $\lambda_T\geqslant f_n^{-1}(p_T(\mathcal{G}_n))$ for all n.

Similarly, it is easy to show that if there is percolation in the first direction of the parallelepiped B_ξ (see Sec. 2) in the continuous problem then there is also percolation in the first direction through $\mathcal{G}_n$ of the parallelepiped

$$[1/2n,\ \xi-1/2n]\times[1/2n,\ 3\xi+1/2n-1]\times\ldots\times[1/2n,\ 3\xi+1/2n-1],$$

and hence, of the parallelepiped $[1/2n,\ \xi-1/2n]\times[1/2n,\ 3(\xi-1/2n)]\times\ldots\times[1/2n,\ 3(\xi-1/2n)]$. Then, since all directions are on an equal footing,

$$p_S(\mathcal{G}_n)=\sup\{p\,|\lim_{\xi\to\infty}P_p\ \{\text{there is percolation of } B_\xi \text{ through } \mathcal{G}_n\}=0\}=$$

$$\sup\{p\,|\lim_{\xi\to\infty}P_p\ \{\text{there is percolation of } B_{\xi-1/n} \text{ through } \mathcal{G}_n\}=0\}\leqslant f_n(\lambda_S).$$

Therefore, for all n we have $\lambda_S\geqslant f_n^{-1}(p_S(\mathcal{G}_n))$.

We set

$$1\leqslant l_n=\sup\{\|x-y\|\,|\,x\in\Omega_n(x_0),\ y\in\Omega_n(y_0),\ x_0\mathcal{G}_ny_0\}\leqslant1+2\sqrt{d/n},$$

where d is the dimension of space, and describe around the defect centers of the original problem a sphere of diameter l_n. We have thus obtained a realization of the continuous problem with defects $\tilde{\mathscr{D}}(x)$ that are spheres of diameter l_n. Note that if there is a cluster $W^{(d)}$ on $\mathscr{G}_n$, then the set of defects $\bigcup_{x\in\pi^{-1}(W^{(d)})} \tilde{\mathscr{D}}(x)$ corresponding to it also forms a cluster $\tilde{W}^{(c)}$ in accordance with the choice of l_n, and $|\tilde{W}^{(c)}|\geqslant|W^{(d)}|$. Hence, by similar arguments it is easily shown that $\lambda_H(l_n)\leqslant f_n^{-1}(p_H(\mathscr{G}_n))$, $\lambda_T(l_n)\leqslant f_n^{-1}(p_T(\mathscr{G}_n))$, $\lambda_S(l_n)\leqslant f_n^{-1}(p_S(\mathscr{G}_n))$ for all n.

Thus, for every critical point we have the chain of inequalities $\lambda_H\geqslant f_n^{-1}(p_H(\mathscr{G}_n))\geqslant\lambda_n(l_n)=\lambda_H\cdot l_n^{-d}$; $\lambda_T\geqslant f_n^{-1}(p_T(\mathscr{G}_n))\geqslant\lambda_T(l_n)=\lambda_T\cdot l_n^{-d}$; $\lambda_S\geqslant f_n^{-1}(p_S(\mathscr{G}_n))\geqslant\lambda_S(l_n)=\lambda_S\cdot l_n^{-d}$.

Since $\lim_{n\to\infty} l_n=1$, there exist the limits

$$\lim_{n\to\infty} f_n^{-1}(p_H(\mathscr{G}_n))=\lambda_H;\ \lim_{n\to\infty} f_n^{-1}(p_T(\mathscr{G}_n))=\lambda_T;\ \lim_{n\to\infty} f_n^{-1}(p_S(\mathscr{G}_n))=\lambda_S.$$

Since for all graphs $\mathscr{G}_n$ it follows from Kesten's theorem (see Sec. 1) that $p_T(\mathscr{G}_n)=p_S(\mathscr{G}_n)\leqslant p_H(\mathscr{G}_n)$, and therefore for all n we have $f_n^{-1}(p_T(\mathscr{G}_n))=f_n^{-1}(p_S(\mathscr{G}_n))\leqslant f_n^{-1}(p_H(\mathscr{G}_n))$, whence $\lambda_T=\lambda_S\leqslant\lambda_H$, which is what we wanted to prove.

Note that if we could prove the equality $p_H = p_T = p_S$ for the constructed graphs $\mathscr{G}_n$, then we would obtain $\lambda_H=\lambda_T=\lambda_S$. Unfortunately, such a proof does not yet exist even for the case d = 2.

Remark 3.2. It is obvious that the relation $\lambda_T=\lambda_S\leqslant\lambda_H$ can be proved by the same method for all problems in which the defects are congruent figures of nonzero d-volume and the same orientation.

We now consider a more general case, namely, the case when the space A (see Sec. 2) is finite.

THEOREM 3.3. For continuous problems with defect set $\{\mathscr{D}_i\}_{i=1,\ldots,m}$ the relation $\lambda_T=\lambda_S\leqslant\lambda_H$ holds.

Proof. The only difference from the previous theorem will be in the construction of the graphs $\mathscr{G}_n$ and the second continuous problem. We recall that the defect centers form in $\mathbb{R}^d$ a Poisson ensemble of density λ. The probability that at a fixed point of the ensemble there is a defect $\mathscr{D}_i$ of type i is p_i, $\sum_{i=1}^{m} p_i=1$.

We construct the sequence of graphs $\mathscr{G}_n$. In each cell

$$\Omega_n(\mathbf{k})=\left[\frac{k_1-{}^1/_2}{n},\ \frac{k_1+{}^1/_2}{n}\right)\times\ldots\times\left[\frac{k_d-{}^1/_2}{n},\ \frac{k_d+{}^1/_2}{n}\right),\quad k_i\in\mathbb{Z},\quad i=1,\ldots,d,$$

there are precisely m vertices $v_1(\mathbf{k}), \ldots, v_m(\mathbf{k})$ of the graph $\mathscr{G}_n$. Their positions can be arbitrary provided only that the period of the set of vertices V_n of the graph $\mathscr{G}_n$ along each coordinate axis is 1/n. None of the vertices belonging to a given Ω_n are connected by lines of the graph, but the vertices $v_i(\mathbf{k})$ and $v_j(\mathbf{l})$ $(\mathbf{k} \neq \mathbf{l})$ are connected if there exist $x \in \Omega_n(\mathbf{k})$ and $y \in \Omega_n(\mathbf{l})$ such that the defects $\mathscr{D}_i(x)$ and $\mathscr{D}_j(y)$ with centers at x and y, respectively, have a nonempty intersection. Thus, the graphs $\mathscr{G}_n$ have been constructed. We define on them a percolation problem: The vertex $v_i(\mathbf{k})$ is defective if for a given realization of the continuous problem there exists at least one defect of the type $\mathscr{D}_i$ with center at the point $x \in \Omega_n(\mathbf{k})$. Thus, we have obtained a generalized percolation problem on the graphs $\mathscr{G}_n$ (see Sec. 1).

Now, as in Theorem 3.1, we consider a different continuous problem with slightly "inflated" defects, more precisely, the "new" defects are obtained from the "old" by a homothety Γ_k with center at the defect center and coefficient $k=k(n)=\max_{v_i(\mathbf{k})\mathscr{G}_n v_j(\mathbf{l})} k(v_i(\mathbf{k}), v_j(\mathbf{l}))$, where

$$k(v_i(\mathbf{k}),\ v_j(\mathbf{l}))=\inf\{k|\Gamma_k\mathscr{D}_i(x)\cap\Gamma_k\mathscr{D}_j(y)\neq\varnothing\quad\forall x\in\Omega_n(\mathbf{k}),\ \forall y\in\Omega_n(\mathbf{l})\}.$$

If all defects are starlike with respect to the center and have nonzero d-volume, then there exists $\lim_{n\to\infty} k(n)=1$. Otherwise, the proof remains almost the same.

For subsequent generalization of this result, we require the following definition.

DEFINITION 3.4. Suppose that in one and the same space $\mathbb{R}^d$ we consider two continuous problems: (P_1) with set of defects $\{\mathscr{D}_1(a_1)\}_{a_1\in A_1}$, and (P_2) with set of defects $\{\mathscr{D}_2(a_2)\}_{a_2\in A_2}$. In each problem, the type of defect at a fixed point of the Poisson field is determined by the probability measure μ_i on the standard σ algebra $\sigma(A_i)$ of the subsets A_i, i = 1, 2 [7]. Suppose that on $\sigma(A_1)$ we are given the homomorphism φ:

$\sigma(A_1)\to\sigma(A_2)$, i.e., a mapping with the properties $\varphi(\Omega_1\cup\Omega_2)=\varphi(\Omega_1)\cup\varphi(\Omega_2)$ and $\varphi(\Omega_1\cap\Omega_2)=\varphi(\Omega_1)\cap\varphi(\Omega_2)\,\forall\Omega_1$, $\Omega_2\in\sigma(A_1)$. We shall say that problem (P_2) majorizes problem (P_1) (and write $(P_1)\le(P_2)$), if for all $\Omega\in\sigma(A_1)$ we have $\mu_1(\Omega)\leqslant\mu_2(\varphi(\Omega))$ and $\bigcup_{a_1\in\Omega}\mathscr{D}_1(a_1)\subseteq\bigcup_{a_2\in\varphi(\Omega)}\mathscr{D}_2(a_2)$.

Remark 3.5. Suppose problem (P_2) majorizes problem (P_1) in the sense of Definition 3.4. Then $\lambda_{cr}^{(1)}\geqslant\lambda_{cr}^{(2)}$, where $\lambda_{cr}^{(i)}$ is one of the critical densities $\lambda_H, \lambda_T, \lambda_S$ for the problem (P_i), $i=1, 2$.

DEFINITION 3.6. We consider in $\mathbb{R}^d$ the continuous problem (P) with set of defects $\{\mathscr{D}(a)\}_{a\in A}$, where A is arbitrary. We say that this problem is approximable if there exist sequences of problems (P_n'): $\{\mathscr{D}_n'(a)\}_{a\in A_n}$ and (P_n''): $\{\mathscr{D}_n''(a)\}_{a\in A_n}$ such that for all n the following conditions are satisfied:

1) A_n are finite;

2) for any $a\in A_n$ the defect $\mathscr{D}_n''(a)$ is obtained from $\mathscr{D}_n'(a)$ by a homothety $\Gamma_{k(n)}$ with constant coefficient $k(n)\ge 1$;

3) $(P_n')\le(P)\le(P_n'')$ in the sense of Definition 3.4;

4) there exists $\lim_{n\to\infty}k(n)=1$.

Example 3.7. We consider a problem with continuous random orientation of the defects. All defects are starlike mutually congruent figures having nonzero d-volume. The defects have a unit vector, the direction. Thus, each defect is characterized by a pair $(x, \mathbf{l})$, where x is the coordinate of its center and $\mathbf{l}$ the direction vector, determined for fixed center by a probability distribution on the sphere S^{d-1} (see Sec. 2). We denote by $S_n(m)$ the subset of S^{d-1} that in the spherical coordinates $(\varphi_1,\dots,\varphi_{d-1})$ has the form

$$\left[\frac{2\pi(m_1-{}^1/_2)}{n},\frac{2\pi(m_1+{}^1/_2)}{n}\right)\times\left[\frac{\pi(m_2-{}^1/_2)}{n},\frac{\pi(m_2+{}^1/_2)}{n}\right)\times\ldots\times\left[\frac{\pi(m_{d-1}-{}^1/_2)}{n},\frac{\pi(m_{d-1}+{}^1/_2)}{n}\right);\quad m_i=0,\dots,n-1;$$

$$i=1,\dots,d-1.$$

Besides the original problem, we consider the continuous problems (P_n') and (P_n'') with defects $\mathscr{D}_n'$ and $\mathscr{D}_n''$, respectively. With each defect $\mathscr{D}(x, \mathbf{l})$ with $\mathbf{l}\in S_n(m)$ we associate defects $\mathscr{D}_n'(x, \mathbf{m})$ and $\mathscr{D}_n''(x, \mathbf{m})$, where $\mathscr{D}_n'(x, \mathbf{m})$ is obtained from $\mathscr{D}(x, \mathbf{m})$ by the homothety $\Gamma_{k'(n)}$ with center at x and coefficient

$$k'(n)=\sup\{k\,|\,\Gamma_k\mathscr{D}(x,\mathbf{m})\subseteq\bigcap_{\mathbf{l}\in S_n(\mathbf{m})}\mathscr{D}(x,\mathbf{l})\}.$$

Accordingly, $\mathscr{D}_n''(x, \mathbf{m})=\Gamma_{k''(n)}\mathscr{D}(x, \mathbf{m})$, where

$$k''(n)=\inf\{k\,|\,\Gamma_k\mathscr{D}(x,\mathbf{m})\supseteq\bigcup_{\mathbf{l}\in S_n(\mathbf{m})}\mathscr{D}(x,\mathbf{l})\}.$$

It is readily seen that $k(n)=k'(n)/k''(n)\to 1$ as $n\to\infty$, so that the original problem is approximable in the sense of Definition 3.6.

It can be shown that very many continuous problems are approximable. This is the case, for example, for problems with continuous random orientation, the problem of spheres of bounded random radius, etc. Therefore, the theorem that summarizes the section is fairly general.

THEOREM 3.8. For continuous problems approximable in the sense of Definition 3.6 the relation $\lambda_T=\lambda_S\leqslant\lambda_H$ holds.

Proof. Suppose the problem (P) is approximable. Then there exist sequences of problems (P_n') and (P_n'') such that $(P_n')\le(P)\le(P_n'')$ with $\Gamma_{k(n)}\mathscr{D}_n'(a)=\mathscr{D}_n''(a)\,\forall a\in A_n$ (see Definition 3.6). Then by Remark 3.5, for all n

$$\lambda_H(P_n')\geqslant\lambda_H(P)\geqslant\lambda_H(P_n'')=\lambda_H(P_n')\,(k(n))^{-d};\qquad \lambda_S(P_n')\geqslant\lambda_S(P)\geqslant\lambda_S(P_n'')=\lambda_S(P_n')\,(k(n))^{-d};$$

$$\lambda_T(P_n')\geqslant\lambda_T(P)\geqslant\lambda_T(P_n'')=\lambda_T(P_n')\,(k(n))^{-d}.$$

Since A_n are finite, we can apply to the problems (P_n') Theorem 3.3, by virtue of which $\lambda_T(P_n')=\lambda_S(P_n')\le\lambda_H(P_n')$ and since there exists $\lim_{n\to\infty}k(n)=1$, we also have $\lambda_T(P)=\lambda_S(P)\le\lambda_H(P)$. The theorem is proved.

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CONDITIONS FOR CERTAIN HIGGS POTENTIALS TO BE BOUNDED BELOW

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Necessary and sufficient conditions are obtained for Higgs potentials to be bounded below when they are constructed from: 1) two doublets, and also two doublets and a singlet of SU(2), 2) the adjoint and vector representations of SO(n). For a potential constructed from the adjoint and fundamental multiplets of SU(n), the problem of necessary and sufficient conditions has been only partly solved.

1. Introduction

In the construction of unified models of elementary-particle interaction the original symmetry group must be spontaneously broken. This is generally done by a mechanism based on the introduction of one or several multiplets of scalar Higgs fields. In accordance with the general principles of the theory, the Higgs potential must satisfy conditions that ensure it is renormalizable and bounded below, i.e., it must have the form of a polynomial of not higher than fourth degree and be positive at large values of the fields.

In the simplest cases with a small number of coupling constants (an example is the Salam–Weinberg model with one Higgs doublet), it is not difficult to find the only values for which the Higgs potential is bounded below. In models of the weak and electromagnetic interactions containing several scalar doublets, and also in grand unification theories based on groups of higher rank and having several multiplets of scalar particles, the Higgs potentials have much more complicated structures. As a result, the problem of finding the necessary and sufficient conditions on the parameters of the potential for which it is bounded below becomes nontrivial.

It should be noted that in a number of studies [1-5] and for some potentials only sufficient conditions have been found; these far from exhaust the complete region of parameters in which the requirement of positivity at large values of the scalar fields is satisfied.

In the present paper, we consider, in the tree approximation, a number of Higgs potentials that are interesting from the point of view of physical applications and obtain necessary and sufficient conditions for them to be bounded below. We also require this property to hold for all values of the dimensional constants of the coupling of the scalar particles with one another, this being equivalent to positive definiteness of the part of the potential that contains only fourth powers of the fields.

From the methodological point of view, the problem of necessary and sufficient conditions for boundedness below is a generalization of the problem of finding residual invariance groups in the presence of spontaneous symmetry breaking* (see, for example, [5-8]). The point is that all information about possible symmetry types of the theory can be obtained by studying the invariant properties of the point of

* This applies only to potentials that do not contain terms of the type φ^3.

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