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Invariant Measures for the Nonlinear Stochastic Heat Equation with No Drift Term

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Abstract
This paper deals with the long-term behavior of the solution to the nonlinear stochastic heat equation $\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = b(u)\dot{W}$, where b is assumed to be a globally Lipschitz continuous function and the noise $\dot{W}$ is a centered and spatially homogeneous Gaussian noise that is white in time. We identify a set of nearly optimal conditions on the initial data, the correlation measure of the noise, and the weight function ρ, which together guarantee the existence of an invariant measure in the weighted space $L^2_\rho(\mathbb{R}^d)$. In particular, our result covers the *parabolic Anderson model* (i.e., the case when $b(u) = \lambda u$) starting from the Dirac delta measure.

Keywords Stochastic heat equation · Parabolic Anderson model · Invariant measure · Dirac delta initial condition · Weighted L^2 space · Matérn class of correlation functions · Bessel kernel

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Contents

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1 Introduction

In this paper, we study the following *nonlinear stochastic heat equation* (SHE):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{1}{2}\Delta u(t,x) = b(x,u(t,x))\dot{W}(t,x) & x \in \mathbb{R}^d, t > 0, \\ u(0,\cdot) = \mu(\cdot). \end{cases} \tag{1.1}$$

The noise, $\dot{W}(t,x)$, is a centered Gaussian noise that is white in time and homogeneously colored in space defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the noise. Its covariance structure is given by:

$$J(\psi,\phi) := \mathbb{E}\left[W(\psi)\,W(\phi)\right] = \int_0^\infty \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\psi(s,\cdot) * \widetilde{\phi}(s,\cdot))(x), \tag{1.2}$$

where ψ and ϕ are continuous and rapidly decreasing functions, $\widetilde{\phi}(x) := \phi(-x)$, "$*$" refers to the convolution in the spatial variable, and Γ is a nonnegative and nonnegative-definite tempered measure on $\mathbb{R}^d$ that is commonly referred to as the *correlation measure*. The Fourier transform of Γ (in the generalized sense) is also a nonnegative and nonnegative-definite tempered measure, which is usually called the *spectral measure* and is denoted by $\widehat{f}(\mathrm{d}\xi)$ (see (1.21) for the convention of Fourier transform). Moreover, in the case where Γ has a density f, namely $\Gamma(\mathrm{d}x) = f(x)\mathrm{d}x$, we write $\widehat{f}(\mathrm{d}\xi)$ as $\widehat{f}(\xi)\mathrm{d}\xi$.

The initial condition, μ, is a deterministic, locally finite, regular, signed Borel measure that satisfies the following integrability condition at infinity:

$$\int_{\mathbb{R}^d} |\mu|\,(\mathrm{d}x) \exp\left(-a|x|^2\right) < \infty \quad \text{for all } a > 0, \tag{1.3}$$

where $|\mu| = \mu_+ + \mu_-$ and $\mu = \mu_+ - \mu_-$ refers to the *Hahn decomposition* of the measure μ. Initial conditions of this type, introduced in [1] and further explored in [2, 3], are called *rough initial conditions*.

The function $b(x, u)$ is uniformly bounded in the first variable and Lipschitz continuous in the second variable, i.e., for some constants $L_b > 0$ and $L_0 \geq 0$,

$$|b(x, u) - b(x, v)| \leq L_b |u - v| \text{ and } |b(x, 0)| \leq L_0 \text{ for all } u, v \in \mathbb{R} \text{ and } x \in \mathbb{R}^d. \quad (1.4)$$

In particular, our assumption allows the linear case $b(x, u) = \lambda u$, which is usually referred to as the *parabolic Anderson model* (PAM) [4].

The SPDE (1.1) is understood in its *mild form*:

$$u(t, x) = J_0(t, x; \mu) + \int_0^t \int_{\mathbb{R}^d} b(y, u(s, y)) G(t - s, x - y) W(\mathrm{d}s, \mathrm{d}y), \quad (1.5)$$

where $G(t, x) = (2\pi t)^{-d/2} \exp\left(- (2t)^{-1} |x|^2\right)$ is the heat kernel,

$$J_0(t, x) = J_0(t, x; \mu) := (G(t, \cdot) * \mu)(x) = \int_{\mathbb{R}^d} \mu(\mathrm{d}y)\, G(t, x - y) \quad (1.6)$$

is the solution to the homogeneous equation, and the stochastic integral is the *Walsh integral*. We refer the interested readers to [2, 5–7] for more details of this setup.

The aim of this paper is to investigate the conditions required to guarantee the existence of an invariant measure for the solution to (1.1), which is a crucial step for the study of the ergodicity of the system that requires the corresponding uniqueness. We direct the interested readers to [8–10] for more details about the invariant measure, its existence/uniqueness, and the ergodicity of the system. The general procedure for finding the invariant measure, especially in the setting of (1.1), has been laid out by Tessitore and Zabczyk [11], which involves two parts: first one needs to show that the laws of the solution to (1.1) form a family of *Markovian transition functions* on some Hilbert space, H, and the corresponding *Markovian semigroup* is Feller; and second one needs to establish that the moments of solution are bounded in time (see (1.8)). For the second point, it requires some substantial work (see Theorems 1.1 and 1.3). On the other hand, the first point has been shown to be true for our case of interest (see, e.g., [10, Chapter 9]) with the following weighted $L^2(\mathbb{R}^d)$ space as our underlying Hilbert space as in [11]:

Definition 1.1 ([11]) A function $\rho : \mathbb{R}^d \mapsto \mathbb{R}$ is called an admissible weight function if it is a strictly positive, bounded, continuous, and $L^1(\mathbb{R}^d)$-integrable function such that for all $T > 0$, there exists a constant $C_\rho(T)$ such that

$$\big(G(t, \cdot) * \rho(\cdot)\big)(x) \leq C_\rho(T)\rho(x) \quad \text{for all } t \in [0, T] \text{ and } x \in \mathbb{R}^d. \quad (1.7)$$

Moreover, we denote by $L^2_\rho(\mathbb{R}^d)$ the corresponding Hilbert space of ρ-weighted square integrable functions, and we use $\langle \cdot, \cdot \rangle_\rho$ and $\|\cdot\|_\rho$ to denote the inner product and norm on $L^2_\rho(\mathbb{R}^d)$:

$$\langle f, g \rangle_\rho := \int_{\mathbb{R}^d} \mathrm{d}x\, f(x) g(x) \rho(x) \quad \text{and} \quad \|f\|_\rho := \int_{\mathbb{R}^d} \mathrm{d}x\, |f(x)|^2 \rho(x).$$

Accordingly, we will prove the existence of the invariant measure following the same strategy as in [11]. Let $\mathscr{L}(u(t,\cdot;\mu))$ denote the law of $u(t,\cdot)$ starting from μ at $t=0$. We will first establish the tightness of $\{\mathscr{L}(u(t,\cdot;\mu))\}_{t>t_0}$ for some $t_0 \geq 0$. A critical step in obtaining this tightness result is to show that the following moment is uniformly bounded in time (see Theorem 1.3):

$$\sup_{t>0} \mathbb{E}\left(\|u(t,\cdot)\|_\rho^2\right) < \infty. \tag{1.8}$$

Then we will apply the *Krylov–Bogoliubov theorem* (see, e.g., [10, Theorem 11.7]) to construct an invariant measure via

$$\eta(A) = \lim_{n\to\infty} \frac{1}{T_n} \int_{t_0}^{T_n+t_0} \mathrm{d}t\, \mathscr{L}(u(t,\cdot;\mu))(A), \tag{1.9}$$

for some sequence $\{T_n\}_{n\geq 1}$ with $T_n \uparrow \infty$.

In the literature, the existence of an invariant measure for the stochastic heat equation is more commonly studied with a drift term; we will postpone a brief review of this case to Sect. 5.1. In contrast, the existence of an invariant measure under the setting of equation (1.1) has been much less studied. Tessitore and Zabczyk [11] established the existence of an invariant measure for the case where the spatial domain is the whole space $\mathbb{R}^d$, the diffusion term, $b(x,u)$, is globally Lipschitz in the second variable, uniformly bounded in the first variable (see (1.4)) and there is no additional negative drift term to help. Recently, in the linear case, that is, the case when $b(u) = \lambda u$, Gu and Li [12] established the weak convergence of $u(t,\cdot)$ to $Z(\cdot)$ in $C(\mathbb{R}^d)$ with $Z(\cdot)$ being a stationary random field. The convergence in [12] relies on the fact that the initial condition is to be stationary, i.e., $\mu(\mathrm{d}x) = \mathrm{d}x$; see Remark 3.6 *ibid.* for some perturbations that can be made on the constant initial condition. A similar stationary limit has also been obtained in Dunlap et al [13]. The current paper follows largely the work by Tessitore and Zabczyk [11]. The major challenge is to identify the right conditions so that the probability moments of the solution are bounded in time (see (1.8)). The solution to (1.1) is usually *intermittent*, namely its moments possess a certain exponential growth in t, see, e.g., [4, 14]. For that reason, one has to impose some additional assumptions on the initial conditions, the noise, the coefficients of (1.1), and the weight function ρ in order to control the growth of the moments in $L^2\left((0,\infty); L^2_\rho(\mathbb{R}^d)\right)$. The point-wise moment formulas obtained in [2, 3] play an important role in this context.

Here we emphasize that we study the invariant measure using the Walsh random field approach [7], whereas such studies are mostly carried out under the framework of the stochastic evolution in Hilbert spaces [10]. Even though both theories are equivalent (see [15]), the differences in many technical aspects are still substantial. As the random field approach often produces results that are more explicit, we try to use this approach to obtain more precise conditions for the existence of an invariant measure. *Firstly*, for the initial conditions, the results in [11] allow for bounded $L^2_\rho(\mathbb{R}^d)$ functions, although the authors proved their main result—Theorem 3.3 *ibid.*—only for the constant one initial condition. Here we give the precise conditions on the initial condition (see (1.17)), which allows a wider class of data, including unbounded functions and measures such as the Dirac delta measure (see Examples 5.6 and 5.7). Note that the Dirac delta initial measure plays a prominent role in the study of the stochastic heat equation; see, e.g., [16]. *Secondly*, regarding the noise, we identify the right conditions, (1.10a) and (1.13), or equivalently (1.19), on the noise to ensure the existence of an invariant measure. Our conditions are explicit, easy to verify, and in a compact form (see (1.19)). Although they are not necessary and sufficient conditions, we believe these conditions are toward to the optimal conditions; see Remark 1.2. Note that the correlation function in Gu and Li [12] is assumed to be a smooth function with compact support. The comparisons of our conditions with those obtained by Tessitore and Zabyczyk [11] are given in Sect. 5.2. *Thirdly*, our proof relies on a factorization representation for the solution $u(t, x)$ to (1.1) (see Lemma 3.3), which is obtained under the random field framework, whereas such factorization lemma has been widely used in the framework of the stochastic evolution equation in Hilbert spaces; see Sect. 3 for more details. *Finally*, we point out a miscellany of results in Sect. 5, which may have independent interest.

Now we are ready to motivate the conditions that we use and present the main results.

1.1 Main Results

As mentioned earlier, in order to have moments bounded in time as in (1.8), one should better first identify the sharp conditions under which the second moments as a function of t, namely $t \mapsto \mathbb{E}\left(u(t,x)^2\right)$, with x fixed, are bounded. This question has been answered in [2, Theorem 1.3 and Lemma 2.5], where necessary and sufficient conditions are given. More precisely, to have the second moment bounded in time with x fixed, one needs to have the spatial dimension $d \geq 3$, and in addition, the spectral measure $\hat{f}$ and Lipschitz constant L_b of $b(\cdot)$ need to satisfy the following two conditions:

$$\Upsilon(0) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^2} < \infty \tag{1.10a}$$

and

$$64 L_b^2 < \frac{1}{2\Upsilon(0)} . \tag{1.10b}$$

These two conditions will guarantee the existence of the following non-empty open interval:

$$\left(2^7 L_b^2 \Upsilon(0), 1\right) \neq \emptyset. \tag{1.11}$$

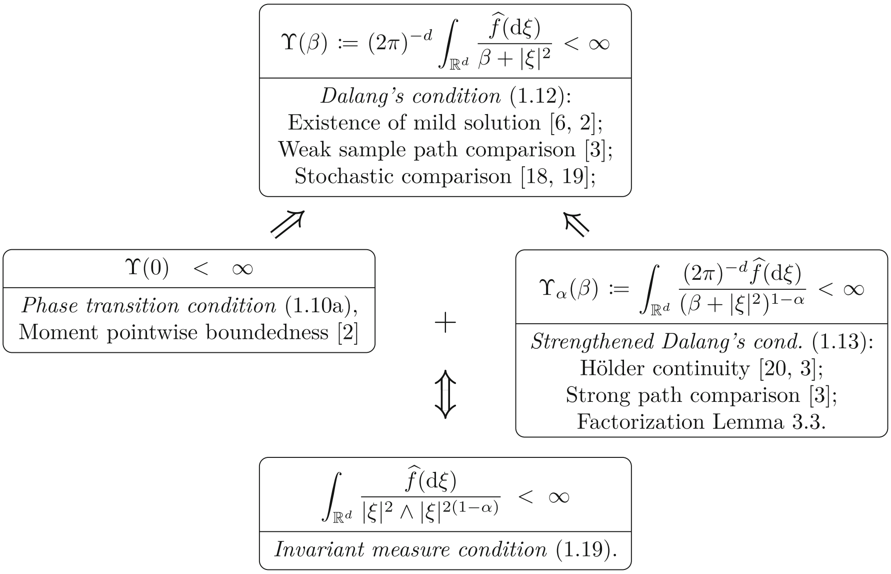


Fig. 1 Conditions on the spectral measure $\widehat{f}$ and their implications

Note that condition (1.10a) is a strengthened version of *Dalang's condition*:

$$\Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\widehat{f}(d\xi)}{\beta + |\xi|^2} < \infty, \quad \text{for some (and hence) all } \beta > 0. \tag{1.12}$$

It turns out that besides condition (1.10a) we need one more condition on the spectral measure $\widehat{f}$, which strengthens Dalang's condition (1.12) in a different way[1]: that for some $\alpha \in (0, 1]$,

$$\Upsilon_\alpha(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\widehat{f}(d\xi)}{\left(\beta + |\xi|^2\right)^{1-\alpha}} < \infty \quad \text{for some (hence all) } \beta > 0. \tag{1.13}$$

In some literature, condition (1.13) is called the *strengthened Dalang's condition*. These three conditions are illustrated in Fig. 1. The subtlety of these conditions is also illustrated in Example 5.10 and Fig. 5.

Remark 1.1 Dalang's condition (1.12) assures the existence and uniqueness of an $L^2(\Omega)$-continuous solution; see [6]. Moreover, Dalang's condition alone is sufficient for the *weak sample path comparison principle* [3] and the *stochastic comparison principle* [18, 19]. But Dalang's condition (1.12) alone is sometimes too weak to derive some useful properties. Instead, the strengthened Dalang's condition (1.13) is widely assumed in the literature. For example, condition (1.13) is used to derive the Hölder regularity of the solution; see [3, Theorem 1.8] or [20] for the random field approach

[1] In [17], the authors constructed a spectral measure $\widehat{f}$ such that $\Upsilon(0) < \infty$ and $\Upsilon_\alpha(\beta) = \infty$ for all $\alpha \in (0, 1)$.

and Theorem 2.3 of [21] and Theorem 4.4 of [11] for the infinite-dimensional SDE approach. One may also check the densely blowup solution obtained in [17] where Dalang's condition (1.12) is satisfied but condition (1.13) fails. As another example, condition (1.13) is also required in order to derive the sample path comparison principle in [3]. As shown in Lemma 3.3, condition (1.13) guarantees a factorization of the solution, which plays a key role in the tightness argument in the proof of the main result—Theorem 1.3.

We are now ready to state our two main results of the paper.

Theorem 1.1 *Let $u(t, x; \mu)$ be the solution to (1.1) starting from μ which satisfies (1.3). Assume that*

(i) *$\rho : \mathbb{R}^d \to \mathbb{R}_+$ is a nonnegative $L^1(\mathbb{R}^d)$ function;*
(ii) *for all $t > 0$, the initial condition μ satisfies $\mathcal{G}_\rho(t; |\mu|) < \infty$ where*

$$\mathcal{G}_\rho(t; \mu) := \int_{\mathbb{R}^d} \mathrm{d}x\, J_0^2(t, x; \mu)\, \rho(x); \tag{1.14}$$

(iii) *the spectral measure $\widehat{f}$ and the Lipschitz constant L_b satisfy the two conditions in (1.10).*

Then there exists a unique $L^2(\Omega)$-continuous solution $u(t, x)$ such that for some constant $C > 0$, which does not depend on t, the following holds:

$$\mathbb{E}\left(\|u(t, \cdot; \mu)\|_\rho^2\right) \le C\mathcal{G}_\rho(t; \mu^*) < \infty, \quad \text{for any } t > 0, \tag{1.15}$$

where $\mu^* := 1 + |\mu|$.

This theorem will be proved in Sect. 2. We now state and prove a corollary which shows that the solution to (1.1) starting from an $L^2_\rho(\mathbb{R}^d)$ initial condition will almost surely be in $L^2_\rho(\mathbb{R}^d)$ for all $t > 0$.

Corollary 1.2 *Under the same assumptions of Theorem 1.1, if in addition ρ is admissible (see Definition 1.1), then the solution $u(t, \cdot; \zeta)$ to (1.1) almost surely exists in $L^2_\rho(\mathbb{R}^d)$ for all $t > 0$, whenever the initial condition is also in $L^2_\rho(\mathbb{R}^d)$, i.e., $\zeta \in L^2_\rho(\mathbb{R}^d)$.*

Proof Choose and fix an arbitrary $\zeta \in L^2_\rho(\mathbb{R}^d)$ and set $\zeta^* = 1 + |\zeta|$. It is clear that $\zeta^* \in L^2_\rho(\mathbb{R}^d)$. By (ii) of Theorem 1.1, it suffices to show the finiteness of $\mathcal{G}_\rho(t, ; \zeta^*)$ for all $t > 0$. Indeed, by Hölder's inequality, $\mathcal{G}_\rho(t, ; \zeta^*)$ is equal to

$$\int_{\mathbb{R}^d} \mathrm{d}x \left(\int_{\mathbb{R}^d} G(t, x - y)\zeta^*(y)\mathrm{d}y\right)^2 \rho(x) \le \iint_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}y\, G(t, x - y)\zeta^*(y)^2\rho(x).$$

Now for any $t > 0$, choose $T > t$ and let $C_\rho(T)$ be as in (1.7). Then,

$$\iint_{\mathbb{R}^{2d}} \mathrm{d}x\mathrm{d}y\, G(t, x - y)\zeta^*(y)^2\rho(x) = \int_{\mathbb{R}^d} \mathrm{d}y\, (G(t, \cdot) * \rho)(y)|\zeta^*(y)|^2$$

$$\leq C_\rho(T) \left\| \zeta^* \right\|_\rho^2 < \infty.$$

Since t and T are arbitrary, this completes the proof of the corollary. □

Theorem 1.3 *Let $u(t,x)$ be the solution to (1.1) starting from μ and let ρ be an admissible weight function. Assume that*

(i) *there exists another admissible weight $\tilde{\rho}$ such that*

$$\int_{\mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} \mathrm{d}x < \infty; \tag{1.16}$$

(ii) *the weight function $\tilde{\rho}$ and the initial condition satisfy the following condition:*

$$\limsup_{t>0} \; \mathcal{G}_{\tilde{\rho}}(t; |\mu|) < \infty; \tag{1.17}$$

(iii) *the spectral measure $\widehat{f}$ and the Lipschitz constant L_b satisfy the two conditions in (1.10);*

(iv) *for some $\alpha \in \left(2^7 \Upsilon(0) L_b^2, 1\right)$ (see (1.11)), the spectral measure $\widehat{f}$ satisfies the strengthened Dalang's condition (1.13).*

Then, we have that

(a) *for any $\tau > 0$, the sequence of laws of $\{\mathcal{L}u(t,\cdot;\mu)\}_{t\geq\tau}$ is tight, i.e., for any $\epsilon \in (0,1)$, there exists a compact set $\mathscr{K} \subset L^2_\rho(\mathbb{R}^d)$ such that*

$$\mathscr{L}u(t,\cdot;\mu)(\mathcal{K}) \geq 1 - \epsilon, \qquad \textit{for all } t \geq \tau > 0; \tag{1.18}$$

(b) *there exists an invariant measure for the laws $\{\mathcal{L}u(t,\cdot;\mu)\}_{t>0}$ in $L^2_\rho(\mathbb{R}^d)$.*

This theorem will be proved in Sect. 4.

Remark 1.2 The conditions on the spectral measure $\widehat{f}$ for the existence of an invariant measure are given by conditions (iii) and (iv) of Theorem 1.3, which can be summarized in the following more compact form (thanks to Lemma 3.4):

$$\int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2 \wedge |\xi|^{2(1-\alpha)}} < \infty, \quad \text{for some } \alpha \in \left(2^7 \Upsilon(0) L_b^2, 1\right). \tag{1.19}$$

Even though condition (1.19) is not the necessary and sufficient condition for the existence of an invariant measure, it is almost the optimal condition. In particular, as proved in [2], if $\Upsilon(0) = \infty$, namely condition (iii) of Theorem 1.3 fails, then no matter how small the level of the noise is, the solution is always intermittent, i.e., the second moment Lyapunov exponent of the solution is always strictly positive. Note that intermittency is another way to characterize the chaotic behavior of the underlying system that is far from equilibrium.

1.2 Outline and Notation

The paper is organized as follows: we first prove Theorem 1.1 in Sect. 2. Then in Sect. 3, we study the factorization lemma and explore the relations of various conditions on the spectral measure $\widehat{f}$. Then, we proceed to prove Theorem 1.3 in Sect. 4. Finally, in Sect. 5 we make some further discussion on the main results and present various examples. In particular, in Sect. 5.1, we give a brief review of the problem of finding invariant measures for the SHE with a drift term; in Sect. 5.2, we compare our conditions on the spectral density with those obtained by Tessitore and Zabczyk [11]; in Sect. 5.3, we show that our results could include a wider class of initial conditions; in Sect. 5.4, we carry out some explicit computations for the Bessel and related kernels as the correlation functions; finally, in Sect. 5.5, we give a few examples of the admissible weight functions.

We conclude this Introduction by introducing some notation and formulas that we use throughout the paper. We will use $\|X\|_p$ to denote the $L^p(\Omega)$ norm, namely $\|X\|_p = (\mathbb{E}(|X|^p))^{1/p}$. We will also use the following factorization property of the heat kernel,

$$G(t,x)G(s,y) = G\left(\frac{ts}{t+s}, \frac{sx+ty}{t+s}\right) G\left(t+s, x-y\right), \tag{1.20}$$

which can be easily verified and has been used extensively and critically in [1–3]. Next, we will need the following spherical coordinate integration formula:

$$\int_{\mathbb{R}^d} \mathrm{d}x\ f(|x|) = \sigma(\mathbb{S}^{d-1}) \int_0^\infty \mathrm{d}r\ f(r) r^{d-1},$$

where $\sigma(\mathbb{S}^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ and $\Gamma(x)$ denotes the Gamma function. We use "$\sim$" to denote the standard asymptotic equivalent relation. Lastly, the convention of Fourier transform is given by (see Remark 5.1)

$$\widehat{\phi}(\xi) = \mathcal{F}\phi(\xi) := \int_{\mathbb{R}^d} \mathrm{d}x\ e^{-ix\cdot\xi}\phi(x) \quad \text{and} \quad \mathcal{F}^{-1}\psi(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{d}\xi\ e^{ix\cdot\xi}\psi(\xi). \tag{1.21}$$

2 Moment Estimates: Proof of Theorem 1.1

We first state some known results and prove a moment bound in Corollary 2.3.

Theorem 2.1 (Theorem 1.2 of [3]) *Suppose that*

(i) *the initial deterministic measure μ satisfies* (1.3)*;*
(ii) *the spectral measure $\widehat{f}$ satisfies Dalang's condition* (1.12)*,*

Then, (1.1) *has a unique random field solution starting from μ. Moreover, the solution is $L^2(\Omega)$ continuous and is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.*

Theorem 2.2 (Theorem 1.7 of [3]) *Under the assumptions of Theorem 2.1, for any $t > 0$, $x \in \mathbb{R}^d$ and $p \geq 2$, the solution to (1.1), $u(t,x)$, given by (1.5) is in $L^p(\Omega)$ and*

$$\|u(t,x)\|_p \leq \left[\bar{\varsigma} + \sqrt{2}(G(t,\cdot) * |\mu|)(x)\right] H(t;\gamma_p)^{1/2}, \tag{2.1}$$

where $\bar{\varsigma} = L_0/L_b$, $\gamma_p = 32pL_b^2$ (see (1.4) for L_0 and L_b) and the function $H(t;\gamma_p)$ is defined in (2.2) below.

Remark 2.1 Recall that the function $H(t;\gamma)$ is defined as:

$$H(t;\gamma) := \sum_{n=0}^{\infty} \gamma^n h_n(t), \qquad \text{for all } \gamma \geq 0, \tag{2.2}$$

where $h_0(t) := 1$, $h_n(t) := \int_0^t \mathrm{d}s\, h_{n-1}(s) k(t-s)$ for $n \geq 1$, and $k(t) := \int_{\mathbb{R}^d} \mathrm{d}z\, f(z) G(t,z)$; see [2, 3]. This function is a nondecreasing function with an exponential upper bound (see Lemma 2.5 of [2] and Lemma 3.8 of [22]):

$$\limsup_{t\to\infty} \frac{1}{t} \log H(t;\gamma) \leq \inf\left\{\beta > 0 : \Upsilon(2\beta) < \frac{1}{2\gamma}\right\}.$$

Corollary 2.3 *Under the same setting as Theorem 2.2, if the two conditions in (1.10) hold (see also (1.11)), then*

$$\|u(t,x)\|_p \leq C_p\left(1 + (G(t,\cdot) * |\mu|)(x)\right), \quad \textit{for all } p \textit{ s.t. } 1/p \in \left(64L_b^2\Upsilon(0), 1\right), \tag{2.3}$$

where $C_p = \left(\sqrt{2} \vee \bar{\varsigma}\right) \sup_{t\geq 0} H(t;\gamma_p)^{1/2} < \infty$.

Proof Lemma 2.5 of [2] gives one sufficient condition, namely $2\gamma_p\Upsilon(0) < 1$, under which the function $H((t;\gamma_p)$ is bounded in t. Therefore, by taking into account the expression of γ_p in Theorem 2.2, we see that as a direct consequence of (2.1), whenever

$$32pL_b^2 < \frac{1}{2\Upsilon(0)}, \tag{2.4}$$

we have the p-th moment bounded as given in (2.3). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Under condition (iii), we can apply Fubini's theorem and the moment bound (2.3) to see that for some constant $C > 0$ independent of t, which may vary from line to line, that

$$\mathbb{E}\left(\|u(t,\cdot;\mu)\|_\rho^2\right) \leq C\,\mathbb{E}\left[\int_{\mathbb{R}^d} \mathrm{d}x \left(1 + (G(t,\cdot) * |\mu|)(x)\right)^2 \rho(x)\right]$$

$$
\begin{aligned}
&= C \int_{\mathbb{R}^d} \mathrm{d}x \, \mathbb{E}\left[\left(\left(G(t,\cdot) * (1+|\mu|)\right)(x)\right)^2\right] \rho(x) \\
&= C\, \mathcal{G}_\rho(t;\mu^*) < \infty,
\end{aligned}
$$

where we recall that $\mu^* = 1 + |\mu|$. This proves Theorem 1.1. $\square$

Remark 2.2 (*Restarted SHE*) Recall that the Markov property of the solution to (1.1) implies that for any $t \geq t_0 > 0$,

$$
u(t+t_0, x; \mu) \overset{\mathcal{L}}{=} u\left(t, x; u\left(t_0, \cdot; \mu\right)\right) =: v(t,x), \tag{2.5}
$$

where $\mathcal{L}$ refers to the equality in law. Then, v satisfies the following restarted SPDE:

$$
\begin{cases}
\dfrac{\partial v}{\partial t}(t,x) - \dfrac{1}{2}\Delta v(t,x) = b(x, v(t,x))\dot{W}_{t_0}(t,x) & x \in \mathbb{R}^d, t > 0, \\
v(0,x) = u(t_0, x; \mu), & x \in \mathbb{R}^d,
\end{cases} \tag{2.6}
$$

where $\dot{W}_{t_0}(t,x) := \dot{W}(t+t_0, x)$ denotes the time-shifted noise, i.e.,

$$
\int_0^t \int_{\mathbb{R}^d} W_{t_0}(\mathrm{d}s, \mathrm{d}y) = \int_{t_0}^{t+t_0} \int_{\mathbb{R}^d} W(\mathrm{d}s, \mathrm{d}y). \tag{2.7}
$$

Under the conditions in (1.10), Theorem 2.2 and (2.5) imply immediately that

$$
\begin{aligned}
\|v(t,x)\|_q = \|u(t+t_0, x; \mu)\|_q &\leq C_q\left(1 + (G(t+t_0,\cdot) * |\mu|)(x)\right) \\
&= C_q J_0(t+t_0, x; \mu^*),
\end{aligned}
$$

for all $q \geq 2$ and $t > 0$, where the constant C_q does not depend on t. Moreover, under the assumptions of Theorem 1.1, we have $v(0,\cdot) \in L^2_\rho(\mathbb{R}^d)$ a.s. and

$$
\mathbb{E}\left(\|v(t,x)\|_\rho^2\right) = \mathbb{E}\left(\|u(t+t_0, x; \mu)\|_\rho^2\right) \leq C\mathcal{G}_\rho(t+t_0; \mu^*) < \infty.
$$

3 A Factorization Lemma and Various Conditions on Spectral Measure

3.1 A Factorization Lemma

In this part, we establish a factorization lemma with corresponding moment estimates; see Lemmas 3.2 and 3.3. This factorization lemma appeared in [23]; check also Section 5.3.1 of [10]. For $\alpha \in (0,1)$, $t > 0$ and $x \in \mathbb{R}^d$, define formally

$$
(F_\alpha f)(t,x) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, (t-s)^{\alpha-1} G(t-s, x-y) f(s,y) \tag{3.1}
$$

and

$$(Y_\alpha f)(t,x) := \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} G(t-s, x-y) f(s,y) W(\mathrm{d}s, \mathrm{d}y)\,. \tag{3.2}$$

For F_α, the first step of the proof of [11, Theorem 3.1] showed the following proposition:

Proposition 3.1 *Let ρ and $\tilde{\rho}$ be given as in condition (i) of Theorem 1.3 (see (1.16)). For any $q > 2$, $t_0 > 0$ and $\alpha \in (q^{-1}, 2^{-1})$, the operator F_α, as an operator from $L^q\big((0,t_0);\ L^2_{\tilde{\rho}}(\mathbb{R}^d)\big)$ to $L^2_\rho(\mathbb{R}^d)$, is compact.*

As for Y_α, we have the following two lemmas, which hold for both the non-restarted SHE ($t_0 = 0$) and the restarted SHE ($t_0 > 0$). We need first introduce some notation. Recall that Dalang's condition (1.12) can be strengthened condition (1.13), which can be further strengthened to

$$\Upsilon_\alpha(0) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}\xi)}{|\xi|^{2(1-\alpha)}} < \infty \quad \text{for some } \alpha \in (0,1]. \tag{3.3}$$

We will also need the following slightly different condition:

$$\mathcal{H}_{\alpha/2}(t) < \infty \quad \text{for some } \alpha \in [0,1) \text{ and for all } t > 0, \tag{3.4}$$

where

$$\mathcal{H}_\alpha(t) := \int_0^t \mathrm{d}r\, r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(\mathrm{d}\xi)\ \exp(-r|\xi|^2). \tag{3.5}$$

The quantity $\mathcal{H}_\alpha(t)$ will appear naturally in the proof of Lemma 3.2. We use the convention that when $\alpha = 0$, we simply drop it from $\Upsilon_\alpha(\beta)$ and $\mathcal{H}_{\alpha/2}(t)$, i.e., $\Upsilon(\beta) := \Upsilon_0(\beta)$ and $\mathcal{H}(t) := \mathcal{H}_0(t)$.

Lemma 3.2 *Suppose that μ—the initial condition for u—satisfies (1.3) and that $\widehat{f}$ satisfies Dalang's condition (1.12). Suppose that for some $\alpha \in (0, 1/2)$, $\mathcal{H}_\alpha(t)$ defined in (3.5) is finite for all $t > 0$. Fix an arbitrary $t_0 \geq 0$. Let $v(t,x)$ be the solution to the restarted SHE (2.6) and $\dot{W}_{t_0}$ be the time-shifted noise (see (2.7)) when $t_0 > 0$ and let $v = u$ when $t_0 = 0$. Then,*

$$\begin{aligned} Y_v(s,y) &:= [Y_\alpha b(\circ, v(\cdot,\circ))](s,y) \\ &= \int_0^s \int_{\mathbb{R}^d} (s-r)^{-\alpha} G(s-r, y-z) b(z, v(r,z)) W_{t_0}(\mathrm{d}r, \mathrm{d}z) \end{aligned} \tag{3.6}$$

has the following properties:

(1) *for all $q \geq 2$, $s > 0$, $y \in \mathbb{R}^d$,*

$$\|Y_v(s,y)\|_q^2 \leq H\left(s + t_0; 32qL_b^2\right)\ J_0^2(s+t_0, y; \mu^*)\, \mathcal{H}_\alpha(s) < \infty, \tag{3.7}$$

where we remind the reader that $\mu^*:=1+|\mu|$ *and the function* $H(t;\gamma)$ *is defined in* (3.5)*;*

(2) *under conditions in* (1.10)*, if* $\mathcal{H}_\alpha(t)$ *is finite for some* $\alpha \in \left(64L_b^2\Upsilon(0),\ 1/2\right)$*, then for any* q *with* $1/q \in \left(64L_b^2\Upsilon(0),\ \alpha\right)$*, the function* $H\left(t;32qL_b^2\right)$ *in* (3.7) *is uniformly bounded in* $t \geq 0$*, i.e.,* $\sup_{t\geq 0} H\left(t;32qL_b^2\right) < \infty$*;*

(3) *under conditions in* (1.10)*, if* $\mathcal{H}_\alpha(t)$ *is finite for some* $\alpha \in \left(64L_b^2\Upsilon(0),\ 1/2\right)$*, then for any* q *with* $1/q \in \left(64L_b^2\Upsilon(0),\ \alpha\right)$ *and for any nonnegative and* $L^1(\mathbb{R}^d)$*-function* ρ*, there exists a constant* $\Theta = \Theta(q, L_b, L_0, \alpha)$*, which does not depend on* t*, such that for* $t > 0$*,*

$$\mathbb{E}\left(\int_0^t \mathrm{d}s\ \|Y_v(s,\cdot)\|_\rho^q\right) \leq \Theta \int_0^t \mathrm{d}s\left[\mathcal{G}_\rho(s+t_0;\mu^*)\,\mathcal{H}_\alpha(s)\right]^{q/2}, \tag{3.8}$$

which is finite, provided that

$$\int_0^t \mathrm{d}s\left[\mathcal{G}_\rho(s+t_0;\mu)\,\mathcal{H}_\alpha(s)\right]^{q/2} < \infty. \tag{3.9}$$

Remark 3.1 Condition (3.9) is true for $t_0 > 0$ because $\mathcal{G}_\rho(t;\mu)$ is a continuous function for $t > 0$ and $\mathcal{H}_\alpha(s)$ is continuous and bounded for $s \in [0,t]$ thanks to (3.5). However, when $t_0 = 0$, the situation is much trickier. For example, when the initial condition is the delta initial condition, we have that

$$\mathcal{G}_{\tilde{\rho}}(t;\delta_0) = \int_{\mathbb{R}^d} \mathrm{d}x\ G(t,x)^2\rho(x) = G(2t,0)\int_{\mathbb{R}^d} \mathrm{d}x\ G(t/2,x)\rho(x) < \infty,$$

where one can obtain the second equality via (1.20). Hence, when $s \to 0$, $\mathcal{G}_{\tilde{\rho}}(s;\delta_0)$ blows up with a rate $s^{-d/2}$. Considering that $\mathcal{H}_\alpha(s)$ goes to zero with a different rate, one needs to combine these two rates to check if condition (3.9) holds. By introducing t_0 and restarting the heat equation, one can avoid this issue, that being the potential singularity of $\mathcal{G}_{\tilde{\rho}}$ at $s = 0$.

Proof In the proof, we use C to denote a generic constant that may change its value at each appearance. We first prove (3.7). By the Burkholder–Davis–Gundy inequality and Minkowski's integral inequality, we see that

$$\begin{aligned}\|Y_v(s,y)\|_q^2 \leq C\int_0^s \mathrm{d}r\ (s-r)^{-2\alpha}\iint_{\mathbb{R}^{2d}} \mathrm{d}z_1\mathrm{d}z_2\ G(s-r,y-z_1)\,\|b\,(z_1,v(r,z_1))\|_q \\ \times f(z_1-z_2)G(s-r,y-z_2)\,\|b\,(z_2,v(r,z_2))\|_q\,.\end{aligned}$$

Note that for the Lipschitz condition in (1.4), we have that

$$\begin{aligned}|b\,(x,v)| &\leq |b\,(x,v)-b\,(x,0)| + |b\,(x,0)| \\ &\leq L_b|v| + L_0 \leq C(1+|v|), \quad C:=L_b\vee L_0.\end{aligned}$$

We apply this and the moment bound (2.1) to $\|b(z_i,v(r,z_i))\|_q$ above to see that

$$\begin{aligned}\|b(z_i, v(r, z_i))\|_q &\le C\left(1 + \|v(r, z_i)\|_q\right)\\ &= C\left(1 + \|u(r + t_0, z_i)\|_q\right)\\ &\le CH\left(r + t_0; 32qL_b^2\right) J_0\left(r + t_0, z_i; \mu^*\right)\\ &\le CH\left(s + t_0; 32qL_b^2\right) J_0\left(r + t_0, z_i; \mu^*\right), \quad i = 1, 2,\ r \in (0, s),\end{aligned} \tag{3.10}$$

where the last step is due to the fact that $H(t; \gamma)$ is a nondecreasing function; see Lemma 2.6 of [2]. Therefore, by denoting $C_s := H\left(s + t_0; 32qL_b^2\right)$,

$$\begin{aligned}&\|Y_v(s, y)\|_q^2\\ &\le CC_s \int_0^s \mathrm{d}r\ (s - r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\\ &\quad \times f(z_1 - z_2) \prod_{i=1}^2 G(s - r, y - z_i) J_0\left(r + t_0, z_i; \mu^*\right)\\ &= CC_s \int_0^s \mathrm{d}r\ (s - r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}\sigma_1)\mu^*(\mathrm{d}\sigma_2) \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\\ &\quad \times f(z_1 - z_2) \prod_{i=1}^2 \Big(G(s - r, y - z_i)G(r + t_0, z_i - \sigma_i)\Big)\\ &= CC_s \int_0^s \mathrm{d}r\ (s - r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}\sigma_1)\mu^*(\mathrm{d}\sigma_2)\ G(s + t_0, y - \sigma_1)G(s + t_0, y - \sigma_2)\\ &\quad \times \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2 f(z_1 - z_2) \prod_{i=1}^2 G\left(\frac{(r + t_0)(s - r)}{s + t_0}, z_i - \sigma_i \frac{r + t_0}{s + t_0} - \frac{s - r}{s + t_0} y\right)\\ &\le CC_s \int_0^s \mathrm{d}r\ (s - r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}\sigma_1)\mu^*(\mathrm{d}\sigma_2)\ G(s + t_0, y - \sigma_1)G(s + t_0, y - \sigma_2)\\ &\quad \times (2\pi)^{-2d} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{(r + t_0)(s - r)}{s + t_0}|\xi|^2\right),\end{aligned}$$

where we have applied (1.20) and Plancherel's theorem. Hence, $\|Y_v(s, y)\|_q^2$ is bounded from above by

$$\begin{aligned}&CC_s (2\pi)^{-2d} J_0^2\left(s + t_0, y; \mu^*\right) \int_0^s \mathrm{d}r\ (s - r)^{-2\alpha}\\ &\quad \times \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{(r + t_0)(s - r)}{s + t_0}|\xi|^2\right).\end{aligned}$$

Because the function

$$t_0 \mapsto \frac{r + t_0}{s + t_0} = 1 - \frac{s - r}{s + t_0} \quad \text{for } t_0 > 0,$$

is nondecreasing in t_0 whenever $s > r > 0$, we can replace the two appearances of t_0 in the exponent of the above inequality by zero to see that

$$\begin{aligned}\|Y_v(s,y)\|_q^2 \le\; & CC_s(2\pi)^{-2d} J_0^2(s+t_0,y;\mu^*)\\ & \times \int_0^s \mathrm{d}r\,(s-r)^{-2\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)\exp\left(-\frac{r(s-r)}{s}|\xi|^2\right). \end{aligned}\tag{3.11}$$

Furthermore, by symmetry of $r(s-r)/s$ and the fact that $r(s-r)/s \ge r/2$ for all $r \in [0, s/2]$, we see that the above double integral is bounded by:

$$\begin{aligned}&\le 2\int_0^{s/2}\mathrm{d}r\; r^{-2\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)\exp\left(-\frac{r}{2}|\xi|^2\right)\\ &= 2^{2(1-\alpha)}\int_0^{s/4}\mathrm{d}r\; r^{-2\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)\exp\left(-r|\xi|^2\right)\\ &\le 2^{2(1-\alpha)}\int_0^{s}\mathrm{d}r\; r^{-2\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)\exp\left(-r|\xi|^2\right)\\ &= 2^{2(1-\alpha)}\mathcal{H}_\alpha(s).\end{aligned}$$

Plugging the above bound back to (3.11) proves (3.7).

Part (2) is a direct consequence of Theorem 2.2. It remains to prove (3.8). An application of Minkowski's inequality shows that

$$\mathbb{E}\left(\|Y_v(s,\cdot)\|_\rho^q\right) = \left\|\int_{\mathbb{R}^d}\mathrm{d}y\; Y_v(s,y)^2\rho(y)\right\|_{q/2}^{q/2} \le \left(\int_{\mathbb{R}^d}\mathrm{d}y\;\; \|Y_v(s,y)\|_q^2\,\rho(y)\right)^{q/2}. \tag{3.12}$$

By the definition of $\mathcal{G}_\rho(t;\mu)$ in (1.14) and by (3.7), we see that

$$\int_{\mathbb{R}^d}\mathrm{d}y\;\;\|Y_v(s,y)\|_q^2\;\rho(y) \le C\,\mathcal{G}_\rho(s+t_0;\mu^*)\mathcal{H}_\alpha(s).$$

Plugging the above expression to the far right-hand side of (3.12) proves (3.8). Finally, the finiteness of the upper bound in (3.8) is guaranteed by condition (3.9). This completes the proof of Lemma 3.2. □

Lemma 3.3 (Factorization lemma) *Suppose that μ—the initial condition for u—satisfies (1.3) and $\widehat{f}$ satisfies Dalang's condition (1.12). Assume that condition (3.5) is satisfied for some $\alpha \in (0, 1/2)$. Fix an arbitrary $t_0 \ge 0$. Let $v(t,x)$ be the solution to the restarted SHE (2.6) and $\dot{W}_{t_0}$ be the time-shifted noise (see (2.7)) when $t_0 > 0$ and let $v = u$ when $t_0 = 0$. Then, the following factorization holds*

$$\frac{\sin(\alpha\pi)}{\pi}\int_0^t \mathrm{d}s\ (t-s)^{\alpha-1}\left[G(t-s,\cdot)*Y_v(s,\cdot)\right](x)$$
$$=\int_0^t\int_{\mathbb{R}^d} G(t-r,x-z)b(z,v(r,z))W_{t_0}(\mathrm{d}r,\mathrm{d}z),$$

for all $t>0$ and $x\in\mathbb{R}^d$. As a consequence, for all $t>0$ and $x\in\mathbb{R}^d$,

$$v(t,x)=\left[G(t,\cdot)*u(t_0,\cdot;\mu)\right](x)+\frac{\sin(\alpha\pi)}{\pi}\left[F_\alpha Y_v\right](t,x). \tag{3.13}$$

Proof The lemma is straightforward, provided that one can switch the orders of stochastic and ordinary integrals:

$$\begin{aligned}
&\int_0^t \mathrm{d}s\ (t-s)^{\alpha-1}\left[G(t-s,\cdot)*Y_v(s,\cdot)\right](x)\\
&=\int_0^t \mathrm{d}s\ (t-s)^{\alpha-1}\int_{\mathbb{R}^d}\mathrm{d}y\ G(t-s,x-y)\\
&\quad\times\int_0^s\int_{\mathbb{R}^d}(s-r)^{-\alpha}G(s-r,y-z)b(z,v(r,z))W_{t_0}(\mathrm{d}r,\mathrm{d}z)\\
&=\int_0^t \mathrm{d}s\ (t-s)^{\alpha-1}\int_0^s\int_{\mathbb{R}^d}(s-r)^{-\alpha}G(t-r,x-z)b(z,v(r,z))W_{t_0}(\mathrm{d}r,\mathrm{d}z) \qquad (3.14)\\
&=\int_0^t\int_{\mathbb{R}^d}W(\mathrm{d}r,\mathrm{d}z)G(t-r,x-z)b(z,v(r,z))\int_r^t \mathrm{d}s\ (s-r)^{-\alpha}(t-s)^{\alpha-1}\\
&=\frac{\pi}{\sin(\alpha\pi)}\int_0^t\int_{\mathbb{R}^d}G(t-r,x-z)b(z,v(r,z))W_{t_0}(\mathrm{d}r,\mathrm{d}z), \qquad (3.15)
\end{aligned}$$

where the last step is the *Beta integral* which requires that $\alpha\in(0,1)$. It remains to justify the two applications of the stochastic Fubini's theorem (see Theorem 5.30 of Chapter one in [5], or also [7] or Theorem 4.33 of [10]) in (3.14) and (3.15) in the following two steps.

Step 1. In this step, we justify the change of orders in (3.14). Note that t, x and s are fixed. It suffices to prove the following condition:

$$\begin{aligned}
I_1:=&\int_{\mathbb{R}^d}\mathrm{d}y\ G(t-s,x-y)\int_0^s \mathrm{d}r\ (s-r)^{-2\alpha}\iint_{\mathbb{R}^{2d}}\mathrm{d}z_1\mathrm{d}z_2\\
&\times f(z_1-z_2)\left(\prod_{i=1}^2 G(s-r,y-z_i)\right)\mathbb{E}\left(\prod_{i=1}^2 b(z_i,v(r,z_i))\right)\\
=&\int_{\mathbb{R}^d}\mathrm{d}y\ G(t-s,x-y)\left\|Y_v(s,y)\right\|_2^2<+\infty,
\end{aligned}$$

which follows immediately from (3.7). Indeed,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}y \;\; G(t-s, x-y) \, \|Y_v(s,y)\|_2^2 \\
&\quad \le \; C \int_{\mathbb{R}^d} \mathrm{d}y \; G(t-s, x-y) J_0^2(s+t_0, y; \mu^*) \mathcal{H}_\alpha(s) \\
&\quad = \; C\mathcal{H}_\alpha(s) \int_{\mathbb{R}^d} \mathrm{d}y \; G(t-s, x-y) \iint_{\mathbb{R}^{2d}} \mu^*(\mathrm{d}z_1)\mu^*(\mathrm{d}z_2) \\
&\qquad \times G\left(s+t_0, y-z_1\right) G\left(s+t_0, y-z_2\right).
\end{aligned}
$$

Now we bound the three heat kernels using (1.20) as follows:

$$
\begin{aligned}
&G(t-s, x-y) \prod_{i=1}^{2} G(s+t_0, y-z_i) \\
&\quad = \frac{G\left(2(t-s), x-y\right)^2}{G\left(4(t-s, 0\right)} \prod_{i=1}^{2} G\left(s+t_0, y-z_i\right) \\
&\quad \le 2^d \frac{G\left(2(t-s), x-y\right)^2}{G\left(4(t-s), 0\right)} \prod_{i=1}^{2} G\left(2s+2t_0, y-z_i\right) \\
&\quad = 2^d [4(t-s)]^{d/2} \prod_{i=1}^{2} \Big[G\left(2s+2t_0, y-z_i\right) G\left(2(t-s), x-y\right) \Big] \\
&\quad = 2^{2d} (t-s)^{d/2} \prod_{i=1}^{2} \left[G(2(t+t_0), x-z_i) G\left(\frac{2(t-s)(s+t_0)}{t+t_0}, y - \frac{s+t_0}{t+t_0}(x-z_i) \right) \right] \\
&\quad \le 2^{2d} (t-s)^{d/2} \prod_{i=1}^{2} \left[G\left(2(t+t_0), x-z_i\right) G\left(\frac{2(t-s)(s+t_0)}{t+t_0}, 0 \right) \right] \\
&\quad \le C_{t,s,t_0} \prod_{i=1}^{2} G\left(2(t+t_0), x-z_i\right).
\end{aligned}
$$

Therefore, $I_1 \le C_{t,s,t_0} \, \mathcal{H}_\alpha(s) \, J_0^2\left(2(t+t_0), x; \mu^*\right) < \infty$.

Step 2. Similarly, as for (3.15), we need to show that

$$
\begin{aligned}
I_2 := &\int_0^t \mathrm{d}s \; (t-s)^{\alpha-1} \int_0^s \mathrm{d}r \; (s-r)^{-2\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2 \\
&\times f(z_1 - z_2) \left(\prod_{i=1}^{2} G(t-r, x-z_i) \right) \mathbb{E}\left(\prod_{i=1}^{2} b(z_i, v(r, z_i)) \right) < \infty.
\end{aligned}
$$

By the Cauchy–Schwarz inequality, (3.10) and because $\alpha \in (0, 1/2)$,

$$
\begin{aligned}
I_2 \leq C \int_0^t \mathrm{d}s\ (t-s)^{\alpha-1} \int_0^s \mathrm{d}r\ (s-r)^{-2\alpha} \\
\times \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\ f(z_1 - z_2) \prod_{i=1}^{2} \Big(G(t-r, x-z_i) J_0\left(r + t_0, z_i; \mu^*\right) \Big) \\
= C' \int_0^t \mathrm{d}r\ (t-r)^{-\alpha} \iint_{\mathbb{R}^{2d}} \mathrm{d}z_1 \mathrm{d}z_2\ f(z_1 - z_2) \prod_{i=1}^{2} G(t-r, x-z_i) J_0\left(r+t_0, z_i; \mu^*\right).
\end{aligned}
$$

Now by the same arguments as those leading to (3.7) (with 2α there replaced by α), we see that

$$
I_2 \leq C J_0^2\left(t, x; \mu^*\right) \int_0^t \mathrm{d}r\ r^{-\alpha} \int_{\mathbb{R}^d} \widehat{f}(\mathrm{d}\xi) \exp\left(-\frac{r(t-r)}{t} |\xi|^2\right),
$$

which is finite by (3.5) where we replace α with $\alpha/2$ and repeat the same steps right after (3.11). This completes the proof of Lemma 3.3. □

3.2 Relations of Various Conditions on Spectral Measure

The relations of various conditions on the spectral measure $\widehat{f}$ are illustrated in Fig. 2; see Lemma 3.4 for the proof:

If we denote the first row of four conditions in Fig. 2 by (a_1), (a_2), (a_3) and (a_4), and those in the second row by (b_1), (b_2), (b_3) and (b_4). The implications $(b_i) \Rightarrow (a_i)$ with $i = 1, \ldots, 4$ are clear. It remains to show the four equivalent relations. In particular, part (i) of Lemma 3.4 proves the bottom two equivalences in Fig. 2, i.e., $(b_1) \Leftrightarrow (b_2)$ and $(b_3) \Leftrightarrow (b_4)$. The top two, i.e., $(a_1) \Leftrightarrow (a_2)$ and $(a_3) \Leftrightarrow (a_4)$, are covered by part (ii) of Lemma 3.4.

Remark 3.2 By Lemma 3.4, if one assumes (1.10a), which is the case for Theorem 1.3, then all the right four conditions in Fig. 2 are equivalent.

Lemma 3.4 *For all* $\alpha \in [0, 1)$*, we have the following properties:*

(i) $\lim_{t\to\infty} (2\pi)^{-d}\, \mathcal{H}_{\alpha/2}(t) = \Gamma(1-\alpha)\, \Upsilon_\alpha(0)$*;*

$$
\begin{array}{ccccccc}
\mathcal{H}(t) < \infty & \Leftrightarrow & \begin{array}{c}\Upsilon(\beta) < \infty \\ \text{Dalang's} \\ \text{condition (1.12)}\end{array} & \Leftarrow & \begin{array}{c}\Upsilon_\alpha(\beta) < \infty \\ \text{Condition (1.13)}\end{array} & \Leftrightarrow & \begin{array}{c}\mathcal{H}_{\alpha/2}(t) < \infty \\ \text{Condition (3.4)}\end{array} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\lim_{t\to\infty} \mathcal{H}(t) < \infty & \Leftrightarrow & \begin{array}{c}\Upsilon(0) < \infty \\ \text{Condition (1.10a)}\end{array} & & \begin{array}{c}\Upsilon_\alpha(0) < \infty \\ \text{Condition (3.3)}\end{array} & \Leftrightarrow & \lim_{t\to\infty} \mathcal{H}_{\alpha/2}(t) < \infty
\end{array}
$$

Fig. 2 Relations among various conditions where $\alpha \in (0, 1)$ and $\beta > 0$

(ii) *$\gamma(1-\alpha, t\beta)(2\pi)^d\,\Upsilon_\alpha(\beta) \le \mathcal{H}_{\alpha/2}(t) \le \Gamma(1-\alpha)(2\pi)^d\,\Upsilon_\alpha(\beta)$ for all $t > 0$ and $\beta > 0$, where $\gamma(s,x) := \int_0^x t^{s-1}e^{-t}\mathrm{d}t$ refers to the incomplete gamma function (see* [24]*).*

(iii) *if $\Upsilon(0) < \infty$, then $\mathcal{H}_{\alpha/2}(t) < \infty$ for all $t > 0$ implies that $\Upsilon_\alpha(0) < \infty$.*

Proof Fix $\alpha \in [0, 1)$. For part (i), it is clear that the function $\mathcal{H}_\alpha(t)$ is nondecreasing. By Fubini's theorem,

$$\begin{aligned}\lim_{t\to\infty}\mathcal{H}_{\alpha/2}(t) &= \int_0^\infty \mathrm{d}r\, r^{-\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)e^{-r|\xi|^2}\\ &= \Gamma(1-\alpha)\int_{\mathbb{R}^d}\frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^{2(1-\alpha)}} = C\Upsilon_\alpha(0),\end{aligned} \tag{3.16}$$

with $C := \Gamma(1-\alpha)(2\pi)^d$. As for part (ii), notice that for all $t > 0$ and $\beta > 0$,

$$\mathcal{H}_{\alpha/2}(t) = \int_0^t \mathrm{d}r\, r^{-\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)e^{-r|\xi|^2} = \int_0^t \mathrm{d}r\, r^{-\alpha}e^{\beta r}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)e^{-r\left(|\xi|^2+\beta\right)}.$$

By replacing $e^{\beta r}$ by 1 and $e^{\beta t}$, we see that

$$I_\beta(t) \le \mathcal{H}_{\alpha/2}(t) \le e^{\beta t}I_\beta(t), \quad \text{with} \quad I_\beta(t) := \int_0^t \mathrm{d}r\, r^{-\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)e^{-r\left(|\xi|^2+\beta\right)}$$

Notice that

$$I_\beta(t) = \int_{\mathbb{R}^d}\frac{1}{\left(\beta+|\xi|^2\right)^{1-\alpha}}\widehat{f}(\mathrm{d}\xi)\int_0^{t\left(|\xi|^2+\beta\right)}\mathrm{d}r\, r^{-\alpha}e^{-r},$$

from which we see that

$$(2\pi)^d\,\Upsilon_\alpha(\beta)\int_0^{t\beta}\mathrm{d}r\, r^{-\alpha}e^{-r} \le I_\beta(t) \le (2\pi)^d\,\Upsilon_\alpha(\beta)\int_0^\infty \mathrm{d}r\, r^{-\alpha}e^{-r}.$$

Then by the definition of the (incomplete) gamma function, we obtain the inequalities in part (ii). As for part (iii), for any $t > 0$, by splitting the $\mathrm{d}r$ integral in (3.16) into two parts, we see that

$$\begin{aligned} C\Upsilon_\alpha(0) &= \int_0^\infty \mathrm{d}r\, r^{-\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)\exp\left(-r|\xi|^2\right) = \mathcal{H}_{\alpha/2}(t) + I_\alpha(t), \quad \text{with}\\ I_\alpha(t) &:= \int_t^\infty \mathrm{d}r\, r^{-\alpha}\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)\exp\left(-r|\xi|^2\right).\end{aligned}$$

Notice that

$$I_\alpha(t) \le t^{-\alpha}\int_t^\infty \mathrm{d}r\int_{\mathbb{R}^d}\widehat{f}(\mathrm{d}\xi)e^{-r|\xi|^2} = t^{-\alpha}\int_{\mathbb{R}^d}\frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2}e^{-t|\xi|^2}$$

$$\leq t^{-\alpha} \int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{|\xi|^2} = \frac{(2\pi)^d}{t^\alpha} \Upsilon(0).$$

Therefore,

$$\Upsilon_\alpha(0) \leq \frac{\mathcal{H}_{\alpha/2}(t)}{(2\pi)^d \Gamma(1-\alpha)} + \frac{\Upsilon(0)}{\Gamma(1-\alpha)\, t^\alpha} < \infty, \quad \text{for all } t > 0,$$

which proves part (iii). This completes the proof of Lemma 3.4. □

4 Tightness and Construction: Proof of Theorem 1.3

4.1 Proof of Part (a) of Theorem 1.3

Before we start the proof of part (a) of Theorem 1.3, we first recall the following result:

Proposition 4.1 (Proposition 2.1 of [11]) *For any admissible weight ρ, the operators on $L^2_\rho(\mathbb{R}^d)$ defined by $\varphi \mapsto \big(G(t,\cdot) * \varphi(\cdot)\big)(x)$ can be extended to a $C_0 - semigroup$ on $L^2_\rho(\mathbb{R}^d)$. Moreover, if $\tilde{\rho}$ is another admissible weight such that*

$$\int_{\mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} \mathrm{d}x < \infty,$$

then for any $t > 0$, the operators defined above are compact from $L^2_{\tilde{\rho}}(\mathbb{R}^d)$ to $L^2_\rho(\mathbb{R}^d)$.

Proof of Theorem 1.3 (a) In this proof, $u(t,x)$ refers to $u(t,x;\mu)$. Fix $\tau > 0$ and let $t_0 = \tau/2$. Throughout the proof, we have $t \geq \tau$. Let v be the solution to (2.6) that is restarted from $t - t_0$. Then (see Fig. 3 for an illustration)

$$v_t(s,x) \stackrel{\mathcal{L}}{=} u\left(s,x;u\left(t-t_0,\cdot;\mu\right)\right) \quad \text{for } s \geq 0 \text{ and } t \geq \tau. \tag{4.1}$$

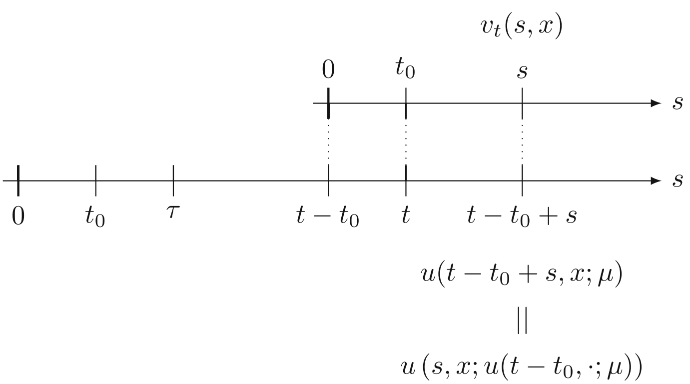


Fig. 3 An illustration for the restarted SHE in (4.1)

According to Assumption (i), we can choose and fix some admissible weight function $\tilde{\rho}$ such that (1.16) is satisfied. Hence, by Proposition 4.1, the following set

$$\mathscr{K}_1(\Lambda) := \left\{ \left(G(t_0,\cdot) * y(\cdot)\right)(x) : \ \|y\|_{\tilde{\rho}} \le \Lambda \right\} \quad \text{with } \Lambda > 0$$

is relatively compact in $L^2_\rho(\mathbb{R}^d)$.

Assumption (iii), i.e., (1.10), implies that the interval $\left(64L_b^2\Upsilon(0), 1/2\right)$ is not empty. Moreover, Assumption (iv), i.e., (3.3), guarantees that there exists a constant α in this interval, namely $64L_b^2\Upsilon(0) < \alpha < 1/2$, such that (3.3) holds with α replaced by 2α, i.e., $\Upsilon_{2\alpha}(0) < \infty$. Now we can apply part (3) of Lemma 3.4, thanks to (1.10a), to see that $\Upsilon_{2\alpha}(0) < \infty$ if and only if (3.5) holds. Therefore, both Lemmas 3.2 and 3.3 (more precisely part (3) of Lemma 3.2) are applicable. In particular, Lemma 3.3 ensures that the following factorization is well-defined:

$$v(t_0,x) = \Big(G(t_0,\cdot) * u(t-t_0,\cdot)\Big)(x) + \frac{\sin(\alpha\pi)}{\pi}\left[F_\alpha Y_v\right](t_0,x). \tag{4.2}$$

Part (3) of Lemma 3.2 shows that for any q in the following range,

$$64L_b^2\Upsilon(0) < \frac{1}{q} < \alpha < \frac{1}{2} \qquad \left(\text{or equivalently} \quad 2 < \frac{1}{\alpha} < q < \frac{1}{64L_b^2\Upsilon(0)}\right), \tag{4.3}$$

we can apply Proposition 3.1 to see that the set

$$\mathscr{K}_2(\Lambda) := \left\{ (F_\alpha h)(t_0,x) : \|h\|_{L^q\left((0,t_0);L^2_{\tilde{\rho}}(\mathbb{R}^d)\right)} \le \Lambda \right\}, \quad \text{with } \Lambda > 0,$$

is relatively compact in $L^2_\rho(\mathbb{R}^d)$. Now for any $\Lambda > 0$, define the set $\mathscr{K}(\Lambda)$ as

$$\begin{aligned}\mathscr{K}(\Lambda) &:= \mathscr{K}_1(\Lambda) + \mathscr{K}_2(\Lambda) \\ &= \left\{ \big(G(t_0,\cdot) * y(\cdot)\big)(x) + (F_\alpha h)(t_0,x) : \|y\|_{\tilde{\rho}} \le \Lambda \text{ and } \|h\|_{L^q\left((0,t_0);L^2_{\tilde{\rho}}(\mathbb{R}^d)\right)} \le \Lambda \right\}.\end{aligned}$$

Notice that from the factorization formula (4.2),

$$\begin{aligned}&\mathbb{P}\left[v(t_0,\cdot) \notin \mathscr{K}(\Lambda)\right] \\ &\le \mathbb{P}\left[\left(\int_0^{t_0} \mathrm{d}s \ \|Y_v(s,\cdot)\|_{\tilde{\rho}}^q\right)^{1/q} > \frac{\pi\Lambda}{\sin(\alpha\pi)}\right] + \mathbb{P}\left[\|u(t-t_0,\cdot)\|_{\tilde{\rho}} > \Lambda\right] \\ &=: I_1 + I_2.\end{aligned}$$

By Chebyshev's inequality and (1.15), we see that

$$I_2 \le \frac{1}{\Lambda^2}\mathbb{E}\left(\|u(t-t_0,\cdot)\|_{\tilde{\rho}}^2\right) \le \frac{1}{\Lambda^2}\mathcal{G}_{\tilde{\rho}}\left(t-t_0;\mu^*\right).$$

Because $\mathcal{G}_{\tilde{\rho}}(t;\mu^*)$ is a continuous function for $t>0$, and because it is also bounded at infinity, thanks to Assumption (ii) (see (1.17)), we have that

$$\mathcal{G}_{\tilde{\rho}}\left(t-t_0;\mu^*\right) \le \sup_{t\ge\tau} \mathcal{G}_{\tilde{\rho}}\left(t-t_0;\mu^*\right) = \sup_{t\ge t_0} \mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right) < \infty. \tag{4.4}$$

Therefore, we can bound I_2 from above with a constant that does not depend on $t\ge\tau$, namely,

$$I_2 \le \frac{1}{\Lambda^2} \sup_{t\ge t_0} \mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right) < \infty.$$

As for I_1, with the choice of α and q in (4.3), one can apply Chebyshev's inequality and part (3) of Lemma 3.2 to see that

$$\begin{aligned} I_1 &\le \frac{\sin^q(\alpha\pi)}{\pi^q\Lambda^q}\mathbb{E}\left[\int_0^{t_0} \mathrm{d}s\ \|Y_v(s,\cdot)\|_{\tilde{\rho}}^q\right] \\ &\le \frac{\sin^q(\alpha\pi)}{\pi^q\Lambda^q}\Theta\int_0^{t_0} \mathrm{d}s \left(\mathcal{G}_{\tilde{\rho}}(s+t-t_0;\mu^*)\mathcal{H}_\alpha(s)\right)^{q/2}, \end{aligned}$$

where the constant Θ does not depend on t. As explained in Remark 3.2, under condition $\Upsilon(0)<\infty$ and condition (1.13), we have $\Upsilon_{2\alpha}(0)<\infty$. Hence, we can apply Lemma 3.4 to bound $\mathcal{H}_\alpha(s)$ from above by the finite bound: $(2\pi)^d\,\Gamma(1-2\alpha)\Upsilon_{2\alpha}(0)$. Therefore, together with (4.4), we obtain the following upper bound for I_1 that is uniform in $t\ge\tau$:

$$I_1 \le \frac{\sin^q(\alpha\pi)\Theta(2\pi)^{dq/2}t_0}{\Gamma(1-2\alpha)^{q/2}\pi^q\Lambda^q}\left(\sup_{t\ge t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right)\right)^{q/2}\Upsilon_{2\alpha}^{q/2}(0).$$

Combining these two upper bounds, we see that

$$\begin{aligned} \mathbb{P}\left[v(t_0,\cdot)\notin\mathscr{K}(\Lambda)\right] \le{}& \frac{\sin^q(\alpha\pi)\Theta(2\pi)^{dq/2}t_0}{\Gamma(1-2\alpha)^{q/2}\pi^q\Lambda^q}\left(\sup_{t\ge t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right)\right)^{q/2}\Upsilon_{2\alpha}^{q/2}(0) \\ &+\frac{1}{\Lambda^2}\sup_{t\ge t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right) \\ <{}& +\infty, \end{aligned}$$

with the upper bound holding uniformly for all $t\ge\tau$. Hence, for any $\epsilon>0$, by choosing $\Lambda>0$ big enough such that

$$\frac{\sin^q(\alpha\pi)\Theta(2\pi)^{dq/2}t_0}{\Gamma(1-2\alpha)^{q/2}\pi^q\Lambda^q}\left(\sup_{t\ge t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right)\right)^{q/2}\Upsilon_{2\alpha}^{q/2}(0)+\frac{1}{\Lambda^2}\sup_{t\ge t_0}\mathcal{G}_{\tilde{\rho}}\left(t;\mu^*\right)<\epsilon,$$

we can ensure that

$$\mathbb{P}\left(u(t,\cdot)\in\mathscr{K}(\Lambda)\right) = \mathbb{P}\left(v(t_0,\cdot)\in\mathscr{K}(\Lambda)\right) \ge 1-\epsilon, \quad \text{for all } t\ge\tau,$$

which proves part (a) of Theorem 1.3. □

4.2 Proof of Part (b) of Theorem 1.3

Proof Fix an arbitrary $\tau > 0$ and denote

$$U(T):=\frac{1}{T}\int_{\tau}^{T+\tau} \mathrm{d}t\, \mathcal{L}\left(u(t,\cdot;\mu)\right), \quad T > 0.$$

We claim that the family of laws $U(T,\cdot)$ for $T > 0$ is tight in $L^2_\rho(\mathbb{R}^d)$. Indeed, for any $\epsilon \in (0,1)$, by part (a), there exists a compact set $\mathcal{K} \in L^2_\rho(\mathbb{R}^d)$ such that (1.18) holds. Hence, for all $T > 0$, it holds that

$$U(T)\left(\mathcal{K}\right) = \frac{1}{T}\int_{\tau}^{T+\tau} \mathrm{d}t\, \mathscr{L}(u(t,\cdot;\mu))\left(\mathcal{K}\right) \geq \frac{1}{T}\int_{\tau}^{T+\tau} \mathrm{d}t\, (1-\epsilon) = 1-\epsilon.$$

Let $\{T_n\}_{n\in\mathbb{N}}$ be any deterministic sequence such that $T_n \uparrow \infty$. Since $\{U(T_n)\}_{n\geq 1}$ is a tight sequence of measures, then there exists a subsequence $\{U\left(T_{n_m}\right)\}_{m\geq 1}$ that converges weakly to a measure, η, on $L^2_\rho(\mathbb{R}^d)$ (e.g. see [25, Theorem 5.1]). Then one can apply the Krylov–Bogoliubov existence theorem (see, e.g., [10, Theorem 11.7]) to conclude that the measure η is an invariant measure for $\mathscr{L}(u(t,\cdot;\mu))$, $t \geq \tau$. Finally, since τ can be arbitrarily close to zero, one can conclude part (b) of Theorem 1.3. □

5 Discussion and Examples

5.1 Invariant Measures for SHE with a Drift Term

In this part, we give a brief account of the case when the SHE has a drift term which plays a crucial role in controlling the moments. The equation usually takes the following form:

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) u(t,x) = g(x,u(t,x)) + b(x,u(t,x))\dot{W}(t,x) \quad x \in \mathcal{O}, t > 0. \tag{5.1}$$

The references in this part are far from being complete. The interested readers can find more references from the references below.

The first case is when the drift term $g(\cdot)$ in (5.1) satisfies certain dissipativity conditions, which push the solution toward zero, see, e.g., [8, 26–29]. Such a "negative" drift term helps to cancel the growth of the moments. Here is one example of such drift term: for some $m,\ k_i,\ c_i > 0$ as $|u| \to \infty$:

$$\begin{cases} g(u) \leq -k_1|u|^m + k_2 & u > 0, \\ g(u) \geq c_1|u|^m - c_2 & u < 0. \end{cases} \tag{5.2}$$

In particular, Cerrai [8, 28] and Brzeźniak and Gątarek [27] considered the case of a bounded spatial domain, while Assing and Manthey [26] and Eckmann and Hairer [29] considered the whole space $\mathbb{R}^d$. Note that Eckmann and Hairer [29] studied the additive noise case along with a bounded initial condition.

Several works that do not require an added drift term with dissipativity as in (5.2) include Misiats et al [30, 31]. In Theorem 1.2 of [31], they provide a result guaranteeing the existence of an invariant measure for the stochastic heat equation on the whole space $\mathbb{R}^d$. More precisely, they allow for a drift term, $g(x,u) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, such that for all $x \in \mathbb{R}^d$ and $u_1, u_2 \in \mathbb{R}$,

$$|g(x,0)| \le \varphi(x) \qquad \text{and} \qquad |g(x,u_1) - g(x,u_2)| \le L\varphi(x)|u_1 - u_2|,$$

for some $L > 0$ where $\varphi(x)$ must decay fast enough such that $\varphi/\sqrt{\rho} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and ρ is the admissible weight. Thus, $f \equiv 0$ is allowed. However, they require the following condition on the diffusion term b:

$$|b(x,u_1) - b(x,u_2)| \le L\varphi(x)|u_1 - u_2|,$$

which excludes the parabolic Anderson model. Lastly, Theorem 1.2 *ibid.* requires the initial condition to be in $L^2(\mathbb{R}^d)$, which excludes the two important cases, $u(0,x) = 1$ and $u(0,\cdot) = \delta_0(\cdot)$. Our Theorem 1.3 includes both of these initial conditions; see Sect. 5.3.

5.2 The Conditions for Spectral Measures by Tessitore and Zabczyk

Tessitore and Zabczyk [11] established the existence of an invariant measure for (1.1) in $L^2_\rho(\mathbb{R}^d)$ under the assumptions that (1) there exists a $\varphi \in L^2_\rho(\mathbb{R}^d) \cap L^2_{\widetilde{\rho}}(\mathbb{R}^d)$ where $\rho/\widetilde{\rho} \in L^1(\mathbb{R}^d)$ and the solution starting from φ is bounded in probability in $L^2_{\widetilde{\rho}}(\mathbb{R}^d)$ and (2) that the spectral density $\hat{f}$ satisfies

$$\widehat{f} \in L^p(\mathbb{R}^d) \qquad \text{for some } p \in [1, d/(d-2))\,; \tag{5.3}$$

see part (i) of Hypothesis 2.1 *ibid.* However, as was illustrated in Theorem 3.3 *ibid.*, in order to apply this theorem to a specific initial condition in L^2_ρ (or to have moments uniformly bounded in time), the following additional assumptions were imposed:

$$\begin{aligned} & d \ge 3 \quad \text{and} \\ & L_b^{-2} > \frac{\Gamma(d/2-1)2^{d/2-2}}{(2\pi)^{2d}} \int_{\mathbb{R}^d} \mathrm{d}\xi \left(|\mathcal{F}(\sqrt{\widehat{f}}\ )| * |\mathcal{F}(\sqrt{\widehat{f}}\ )| \right)(\xi)|\xi|^{2-d}, \end{aligned} \tag{5.4}$$

where the convention of the Fourier transform is given in Remark 5.1. With these assumptions, they were able to prove that (1.1) starting from the constant one initial condition satisfies (1.8) and thus is bounded in probability, verifying the existence of an invariant measure via the construction (1.9).

Remark 5.1 The Fourier transform may be defined differently depending on how one handles the 2π constant. In this paper (as in [2, 3]), we use the convention given in (1.21). Hence, Plancherel's theorem takes the form of $\int_{\mathbb{R}^d} \psi(x)\phi(x)\mathrm{d}x = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}\mathrm{d}\xi$. The authors in [11] did not explicitly mention their convention of the Fourier transform. However, the proof of Theorem 3.3 *ibid.* suggests that the following convention has been used:

$$\widehat{\phi}(\xi) = \mathcal{F}\phi(\xi):=(2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}x\, e^{-ix\cdot\xi}\phi(x) \quad \text{and}$$
$$\mathcal{F}^{-1}\psi(x):=(2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{d}\xi\, e^{ix\cdot\xi}\psi(\xi).$$

Hence, Plancherel's theorem takes the form, $\int_{\mathbb{R}^d} \mathrm{d}x\, \psi(x)\phi(x) = \int_{\mathbb{R}^d} \mathrm{d}\xi\, \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)}$, without the additional factor $(2\pi)^{-d}$. In particular, the spectral density γ *ibid.* corresponds to $(2\pi)^{-d/2}\widehat{f}$ in this paper. Our equation (5.4), which is condition (3.4) *ibid.*, takes into account this difference, therefore explaining the slightly different factor in front of the integral in (5.4) from that in (3.4) *ibid.*

Our condition (1.10a) should be compared with the second condition in (5.4). We will explain below why the former is more natural, easier to verify, and weaker than the latter.

The square root and absolute value in (5.4) make their condition more restrictive. Indeed, if $\mathcal{F}\big(\sqrt{\widehat{f}}\,\big)$ is nonnegative, then the absolute values in (5.4) can be removed without ambiguity, which will reduce to our condition (1.10) up to a constant factor. However, when $\widehat{f}$ is only nonnegative and not strictly positive, finding the right square root of $\widehat{f}$ so that $\mathcal{F}\big(\sqrt{\widehat{f}}\,\big)$ is nonnegative becomes tricky. This last point will be illustrated by the examples below. We first set up the notation in Example 5.1.

Example 5.1 Let $d = 1$ and $g(x) = \frac{1}{2}1_{[-1,1]}(x)$. Then we have that $\widehat{g}(\xi) = \xi^{-1}\sin(\xi)$. Now set $f(x) = (g * g)(x) = 2^{-2}\max(2 - |x|, 0)$. It is clear that f is nonnegative. It is also nonnegative-definite because $\widehat{f}(\xi) = \widehat{g}(\xi)^2 = \xi^{-2}\sin^2(\xi) \geq 0$.

Up to a constant, one may replace $\mathcal{F}$ by $\mathcal{F}^{-1}$. The following example shows that $\mathcal{F}^{-1}\big(\sqrt{\widehat{f}}\,\big)(x)$ is signed, and hence, the absolute value make spoil the oscillatory structure.

Example 5.2 Suppose that $d = 1$. Set $f(x) = (2\pi)^{-1}x^{-2}\sin^2(x)$. Hence, $\widehat{f}(\xi) = 2^{-2}\max\{2 - |\xi|, 0\}$. Then as explained in Example 5.1, both f and $\widehat{f}$ are nonnegative and nonnegative-definite. We claim that

$$\mathcal{F}^{-1}\big(\sqrt{\widehat{f}}\,\big)(x) \text{ takes both positive and negative values.} \tag{5.5}$$

Indeed, for all $x \in \mathbb{R}$,

$$\mathcal{F}^{-1}\big(\sqrt{\widehat{f}}\,\big)(x) = \frac{2}{2\pi}\int_0^2 \mathrm{d}\xi\, 2^{-1}\sqrt{2-\xi}\cos(x\xi) = \frac{1}{\pi}\int_0^{\sqrt{2}} \mathrm{d}\xi\, \xi^2 \cos\left(x\left(2-\xi^2\right)\right)$$

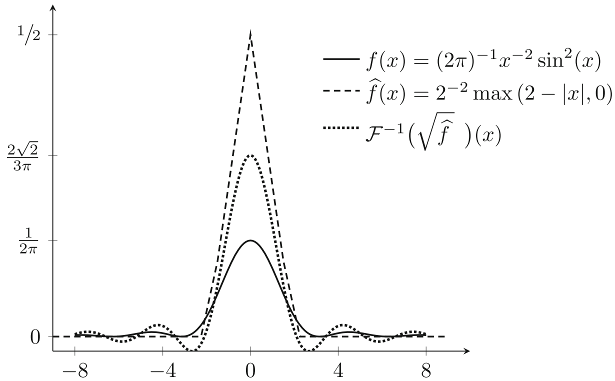


Fig. 4 One example where $\mathcal{F}^{-1}(\sqrt{\widehat{f}}\ )(x)$ assumes both positive and negative values

$$= \frac{1}{\pi}\cos(2x)\int_0^{\sqrt{2}} \mathrm{d}\xi\ \xi^2\cos(x\xi^2) + \frac{1}{\pi}\sin(2x)\int_0^{\sqrt{2}} \mathrm{d}\xi\ \xi^2\sin(x\xi^2)$$
$$= -\frac{1}{\pi}\cos(2x)\int_0^{\sqrt{2}} \mathrm{d}\xi\ \frac{\sin(x\xi^2)}{x} + \frac{1}{\pi}\sin(2x)\int_0^{\sqrt{2}} \mathrm{d}\xi\ \frac{\cos(x\xi^2)}{x},$$

where we have applied the change of variables $\xi' = \sqrt{2-\xi}$ and an integration by parts. By using the *Fresnel integrals* (see, e.g., [24, 7.2 (iii)])

$$\mathcal{S}(z) = \int_0^z \mathrm{d}t\ \sin\left(\frac{\pi t^2}{2}\right) \quad \text{and} \quad \mathcal{C}(z) = \int_0^z \mathrm{d}t\ \cos\left(\frac{\pi t^2}{2}\right),$$

the above expression can be further simplified via the change of variables $\xi' = \sqrt{2|x|/\pi}\ \xi$ to

$$\mathcal{F}^{-1}\left(\sqrt{\widehat{f}}\ \right)(x) = (8\pi)^{-1/2}|x|^{-3/2}\left(-\cos(2\,|x|)\mathcal{S}\left(\frac{2\sqrt{|x|}}{\sqrt{\pi}}\right) + \sin(2\,|x|)\mathcal{C}\left(\frac{2\sqrt{|x|}}{\sqrt{\pi}}\right)\right),$$

for all $x \in \mathbb{R}$. This proves the claim in (5.5); see also Fig. 4 for some plots.

On the other hand, the next example shows that the spectral density given in Example 5.2 can be easily handled by our condition—$\Upsilon_\alpha(0) < \infty$, i.e., (3.3).

Example 5.3 From the one-dimension case in Example 5.2, one can construct a d-dimensional counterpart: Let f_1 and $\widehat{f_1}$ be the f and $\widehat{f}$, respectively, in Example 5.2. Then define

$$f_d(x) := \prod_{i=1}^d f_1(x_i), \quad x \in \mathbb{R}^d, \quad \text{and hence} \quad \widehat{f_d}(\xi) := \prod_{i=1}^d \widehat{f_1}(\xi_i), \quad \xi \in \mathbb{R}^d.$$

It is straightforward to verify that

$$\begin{aligned}\Upsilon_\alpha(0) &= C\int_{\mathbb{R}^d} \mathrm{d}\xi\, \frac{\prod_{i=1}^d \max\{2-|\xi_i|,0\}}{|\xi|^{2(1-\alpha)}}\\ &\le C\int_{|\xi|\le 2\sqrt{d}} \mathrm{d}\xi\, \frac{2^d}{|\xi|^{2(1-\alpha)}} = C\int_0^{2\sqrt{d}} \mathrm{d}r\, \frac{r^{d-1}}{r^{2(1-\alpha)}}.\end{aligned}$$

Hence, if $\alpha > 1 - d/2$, then $\Upsilon_\alpha(0) < \infty$.

The next example illustrates the delicacy of choosing the right branches for the square root in (5.4).

Example 5.4 Let f and g be given as Example 5.1. In this case, $\widehat{f}(\cdot)$ is only nonnegative (not strictly positive) with infinitely many zeros. Hence, when taking the square root of $\widehat{f}(\xi)$ as in (5.4), one needs to wisely select the correct positive and negative branches: (1) Clearly, the signed version $\sqrt{\widehat{f}(\xi)} = \xi^{-1}\sin(\xi)$ is preferable since its inverse Fourier transform can be easily computed, which is equal to $g(x)$. Moreover, because this inverse Fourier transform $g(x)$ is nonnegative, the absolute value signs in (5.4) do not pose any additional restrictions. (2) However, if one chooses the positive branches, namely, $\sqrt{\widehat{f}(\xi)} = |\xi^{-1}\sin(\xi)|$, then it is not clear how to compute its Fourier transform. In general, some bad choices of the positive/negative branches may make the conditions in (5.4) fail. For example, such choice may turn $\sqrt{\widehat{f}(\xi)}$ into a distribution, and then taking the absolute value of a distribution (unless it is a measure) may be problematic. Another issue that may arise is when $\sqrt{\widehat{f}(\xi)}$ is a well-defined function, taking on both positive and negative values and after taking the absolute value, the integral in (5.4) may blow up.

5.3 Various Initial Conditions

In this part, we give some concrete examples of initial conditions.

Example 5.5 (*Bounded initial condition*) We emphasize that if the initial condition, μ, is deterministic and is such that $\mu(\mathrm{d}x) = \varphi(x)\mathrm{d}x$ with $\varphi \in L^\infty(\mathbb{R}^d)$, then all conditions related to $\mathcal{G}_\rho(\cdot)$ in both Theorems 1.1 and 1.3 are trivially satisfied. To be more precise, both Conditions (1.14) and (1.17) hold because

$$\mathcal{G}_\rho(t;|\varphi|) \le \|\varphi\|^2_{L^\infty(\mathbb{R}^d)}\, \|\rho\|_{L^1(\mathbb{R}^d)} < \infty \quad \text{uniformly for all } t \ge 0.$$

Example 5.6 (*Delta initial condition*) In this example, we study the case when the initial condition, μ, is the Dirac delta measure at zero, namely δ_0. Let ρ be a nonnegative $L^1(\mathbb{R}^d)$ function. Since

$$\mathcal{G}_\rho(t;|\delta_0|) = \int_{\mathbb{R}^d} \mathrm{d}x\, G(t,x)^2\rho(x) \le G(t,0)^2\, \|\rho\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0,$$

both conditions (1.14) and (1.17) are satisfied. In particular, $\limsup_{t>0} \mathcal{G}_\rho(t;\delta_0) = 0$.

Example 5.7 (*More initial conditions not in the weighted L^2 space*) In this example, we study the case when $\mu(\mathrm{d}x) = |x|^{-\alpha}\mathrm{d}x$ for some $\alpha \in (0, d)$. It is clear that when $\alpha \in (d/2, d)$, $\mu \notin L^2_\rho(\mathbb{R}^d)$. However, in this case, we have

$$J_0(t, x) = \left(G(t, \cdot) * |\cdot|^{-\alpha}\right)(x) \le \left(G(t, \cdot) * |\cdot|^{-\alpha}\right)(0).$$

On the other hand,

$$\left(G(t, \cdot) * |\cdot|^{-\alpha}\right)(0) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \times (2\pi t)^{-d/2} \int_0^\infty \mathrm{d}r\, e^{-\frac{r^2}{2t}} r^{-\alpha+d-1} = C_* t^{-\alpha/2},$$

with $C_* = 2^{-\alpha/2}\Gamma\left((d-\alpha)/2\right) / \Gamma(d/2)$, which implies that

$$\mathcal{G}_\rho\left(t; |\cdot|^{-\alpha}\right) \le \int_{\mathbb{R}^d} \mathrm{d}x\, J_0^2(t, 0)\rho(x) = C_*^2 t^{-\alpha} \|\rho\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0.$$

Therefore, we see that both conditions (1.14) and (1.17) are satisfied.

The following proposition shows that for initial conditions with unbounded tails, condition (1.14) may hold while condition (1.17) may fail.

Proposition 5.1 *Suppose that $\rho(x) = \exp(-|x|)$, which is an admissible weight function. Let the initial condition μ be given as $\mu(\mathrm{d}x) = |x|^\alpha \mathrm{d}x$ with $\alpha > 0$. Then for some constants $C, C' > 0$ that depend on d and α, it holds that*

$$C'(1 + t^\alpha) \le \mathcal{G}_\rho(t; |\mu|) \le C(1 + t^\alpha), \qquad \textit{for all } t > 0. \tag{5.6}$$

In particular, this implies that condition (1.14) is satisfied, but condition (1.17) fails.

Proof Notice that by scaling arguments,

$$\begin{aligned}
\mathcal{G}_\rho(t; |\mu|) &= \int_{\mathbb{R}^d} \mathrm{d}x \left(\int_{\mathbb{R}^d} G(t, x - y)|y|^\alpha \mathrm{d}y\right)^2 e^{-|x|} \\
&= t^{\alpha + d/2} \int_{\mathbb{R}^d} \mathrm{d}\xi \left(\int_{\mathbb{R}^d} \mathrm{d}z\, G\left(1, \xi - z\right)|z|^\alpha\right)^2 e^{-\sqrt{t}|\xi|}.
\end{aligned}$$

In the following, let C_d, C_α, C'_α, $C_{\alpha,d}$ and $C'_{\alpha,d}$ be generic constants that may depend on α and d and may change their value at each appearance.
Upper bound: Because

$$\int_{\mathbb{R}^d} \mathrm{d}z\, G\left(1, z\right)|\xi - z|^\alpha \le C_\alpha \int_{\mathbb{R}^d} \mathrm{d}z\, G\left(1, z\right)\left(|\xi|^\alpha + |z|^\alpha\right) \le C'_\alpha\left(1 + |\xi|^\alpha\right),$$

we see that $\mathcal{G}_\rho(t; \mu)$ is bounded from above by

$$C_\alpha t^{\alpha + d/2} \int_{\mathbb{R}^d} \mathrm{d}\xi \left(1 + |\xi|^{2\alpha}\right) e^{-\sqrt{t}|\xi|} = C_{\alpha,d}\left(t^\alpha \Gamma(d) + \Gamma(d + 2\alpha)\right) = C'_{\alpha,d}(1 + t^\alpha),$$

the far right-hand of which is finite. This proves the upper bound in (5.6).
Lower bound: Now we prove the lower bound in (5.6). Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}z\, G(1,z)\,|\xi - z|^\alpha &\geq \int_{\mathbb{R}^d} \mathrm{d}z\, G(1,z)\,||\xi| - |z||^\alpha \\ &\geq C_d \int_0^\infty \mathrm{d}x\, \left| |\xi| - x\right|^\alpha e^{-\frac{x^2}{2}} x^{d-1} \geq C_d \int_1^2 \mathrm{d}x\, \left||\xi| - x\right|^\alpha = \frac{C_d}{1+\alpha}\psi(|\xi|)\,,\end{aligned}$$

where, by considering three cases, we have

$$\psi(r) = \begin{cases} (2-r)^{\alpha+1} - (1-r)^{\alpha+1} & \text{if } 0 < r < 1, \\ (2-r)^{\alpha+1} + (r-1)^{\alpha+1} & \text{if } 1 \leq r \leq 2, \\ (r-1)^{\alpha+1} - (r-2)^{\alpha+1} & \text{if } r > 2, \end{cases}$$

which is equal to $\mathrm{sgn}(2-r)|r-2|^{\alpha+1} + \mathrm{sgn}(r-1)|r-1|^{\alpha+1}$. We claim that

$$\inf_{r\geq 0} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} > 0. \tag{5.7}$$

With (5.7), we have that

$$\int_{\mathbb{R}^d} \mathrm{d}z\, G(1,z)\,|\xi - z|^\alpha \geq C_{\alpha,d}\sqrt{1+|\xi|^{2\alpha}}.$$

Then, by the same arguments as above for the upper bound, we obtain the lower bound in (5.6). It remains to prove (5.7), which will be proved in three cases.

When $r > 2$, we see that

$$\begin{aligned}\frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} &\geq C_\alpha \frac{(r-1)^{\alpha+1} - (r-2)^{\alpha+1}}{(1+r)^\alpha} \geq C_\alpha \frac{(r-1)^\alpha (r-1) - (r-1)^\alpha (r-2)}{(1+r)^\alpha} \\ &= C_\alpha \left(\frac{r-1}{1+r}\right)^\alpha = C_\alpha \left(1 - \frac{2}{1+r}\right)^\alpha \geq C_\alpha \left(1 - \frac{2}{3}\right)^\alpha.\end{aligned}$$

Note that in the first inequality above, we have considered two cases: $2\alpha \geq 1$ and $2\alpha < 1$. When $2\alpha < 1$, we have used the concavity of $x^{2\alpha}$, namely, $(1 + r^{2\alpha})^{2\alpha}/2 \leq ((1+r)/2)^{2\alpha}$; when $2\alpha \geq 1$, we have used the super-additivity of $x^{2\alpha}$: namely, that for $a, b > 0$ that $(a+b)^{2\alpha} \geq a^{2\alpha} + b^{2\alpha}$. Therefore, $\inf_{r>2} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} > 0$.

When $r \in (1, 2]$, elementary calculations show that the minimum of $\psi(r)$ is achieved at $r = 3/2$. Hence, $\inf_{r\in(1,2]} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} \geq \frac{\psi(3/2)}{\sqrt{1+4^\alpha}} > 0$.

Similarly, when $r \in (0, 1]$, by differentiation, one finds that the function $\psi(r)$ is nonincreasing. Hence, the minimum is achieved at $r = 1$, $\inf_{r\in(0,1]} \frac{\psi(r)}{\sqrt{1+r^{2\alpha}}} \geq \frac{\psi(1)}{\sqrt{2}} > 0$.

Combining the above three cases proves (5.7) and hence, Proposition 5.1. □

5.4 Bessel and Other Related Kernels

In this part, we will make some explicit computations for Bessel and related kernels.

Example 5.8 (*Bessel kernel*) Let f_s denote the Bessel kernel with a strictly positive parameter $s > 0$. It is known that (see, e.g., Section 1.2.2 of [32])

1. $f_s(x) > 0$ for all $x \in \mathbb{R}^d$ and $\|f_s\|_{L^1(\mathbb{R}^d)} = 1$;
2. there exists a constant $C(s,d) > 0$ such that $f_s(x) \le C(s,d)\exp(-|x|/2)$ for $|x| \ge 2$.
3. there exists a constant $c(s,d) > 0$ such that

$$\frac{1}{c(s,d)} \le \frac{f_s(x)}{H_s(x)} \le c(s,d) \quad \text{for} \quad |x| \le 2,$$

   with

$$H_s(x) = \begin{cases} |x|^{s-d} + 1 + O(|x|^{s-d+2}) & \text{for } 0 < s < d, \\ \log\left(\frac{2}{|x|}\right) + 1 + O(|x|^2) & \text{for } s = d, \\ 1 + O(|x|^{s-d}) & \text{for } s > d; \end{cases}$$

4. the Fourier transform of f_s is strictly positive:

$$\mathcal{F} f_s(\xi) = \frac{1}{(1+|\xi|^2)^{s/2}}. \tag{5.8}$$

Note that one can use (5.8) as the definition of the Bessel kernel. Properties 1 and 4 ensure that f_s is a nonnegative and nonnegative-definite tempered measure for all $s > 0$.

Example 5.9 (*Matérn class of correlation functions*) The *Matérn class of correlation functions* has been widely used in spatial statistics; one may check the recent work [33] for references. Following Section 2.10 of [34], this class of correlation functions is given by:

$$K(x) = \phi \cdot (\alpha|x|)^{\nu} \mathcal{K}_{\nu}(\alpha|x|), \quad \text{for } x \in \mathbb{R}^d \text{ with } \phi > 0, \alpha > 0, \nu > 0, \tag{5.9}$$

where $\mathcal{K}_\nu(\cdot)$ is the modified Bessel function of second type, and α and ν refer to the *scaling and smoothness parameters*, respectively. From the inversion formula (see p. 46 *ibid.*), one sees that, for all $\xi \in \mathbb{R}^d$,

$$\mathcal{F}K(\xi) = (2\pi)^d \mathcal{F}^{-1}K(\xi) = (2\pi)^d f(|\xi|) \quad \text{with}$$
$$f(\xi) = \frac{2^{\nu-1}\phi\Gamma\,(\nu + d/2)\,\alpha^{2\nu}}{\pi^{d/2}\left(\alpha^2 + |\xi|^2\right)^{\nu+d/2}}.$$

Comparing the above expression with (5.8), we see that the class of Bessel kernels f_s, with $s > d-2$ and $d \geq 3$, includes the Matérn class (5.9) as a special case under the following choice of parameters:

$$\alpha = 1, \quad \nu = (s-d)/2, \quad \text{and} \quad \phi = 2^{(2-d-s)/2}\pi^{-d/2}\Gamma(s/2)^{-1}.$$

Note that the requirement of the smoothness parameter $\nu > 0$ for the Matérn class corresponds to the case of the Bessel kernel with $s > d$.

The following proposition shows what conditions (3.3), (1.10a), and (3.4) reduce to for the Bessel kernel as the correlation function in terms of its parameters.

Proposition 5.2 (Bessel kernel as correlation function) *If the correlation function f is given by the Bessel kernel $f_s(\cdot)$ with $s > 0$ defined in Example 5.8, then*

$$\Upsilon_\alpha(0) = \frac{\Gamma\left(\frac{d}{2} - 1 + \alpha\right)\Gamma\left(\frac{s-d}{2} + 1 - \alpha\right)}{2^d\pi^{d/2}\,\Gamma(d/2)\,\Gamma\,(s/2)} \qquad \textit{for all } s > d - 2(1-\alpha) > 0, \tag{5.10}$$

and in particular when $\alpha = 0$, (5.10) simplifies to the following:

$$\Upsilon(0) = \frac{\Gamma\left(\frac{2+s-d}{2}\right)}{2^{d-1}\pi^{d/2}(d-2)\Gamma(s/2)} \qquad \textit{for all } s > d-2 > 0. \tag{5.11}$$

In addition,

$$\mathcal{H}_\alpha(t) < \infty \quad \forall t > 0 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} - \frac{(d-s)_+}{4} \quad \textit{and} \quad s > d-2 > 0, \tag{5.12}$$

where $a_+ := \max(a, 0)$. Moreover, for $\alpha \in (0, 1/2)$, we have the following asymptotic behavior of $\mathcal{H}_\alpha(t)$ at $t \to 0$:

$$
\mathcal{H}_\alpha(t) = \begin{cases}
\dfrac{\pi^{d/2}\Gamma\left((d-s)/2\right) t^{(s-d)/2+1-2\alpha}}{((s-d)/2+1-2\alpha)\,\Gamma(d/2)} - \dfrac{\pi^{d/2}\Gamma\left((s-d)/2\right)}{(1-2\alpha)\Gamma(s/2)} t^{1-2\alpha} + O\left(t^{(s-d)/2+2(1-\alpha)}\right), & d-2<s<d \text{ and } \alpha < \dfrac{1}{2} - \dfrac{1}{4}(d-s) \qquad (5.13\text{-a}) \\[2ex]
\dfrac{\pi^{d/2}}{(1-2\alpha)\Gamma\left(d/2\right)} t^{1-2\alpha} \log\left(\dfrac{1}{t}\right) + \dfrac{\pi^{d/2}\left(1-(1-2\alpha)\left[\psi(d/2)+2\gamma\right]\right)}{(1-2\alpha)^2\Gamma\left(d/2\right)} t^{1-2\alpha} + O\left(t^2\log(t)\right), & s=d \qquad (5.13\text{-b}) \\[2ex]
\dfrac{\pi^{d/2}\Gamma\left((s-d)/2\right) t^{1-2\alpha}}{(1-2\alpha)\Gamma(s/2)} + O\left(t^{\frac{s-d}{2}+1-2\alpha}\right), & d<s<d+2 \qquad (5.13\text{-c}) \\[2ex]
\dfrac{\pi^{d/2} t^{1-2\alpha}}{(1-2\alpha)\Gamma\left(d/2+1\right)} + O\left(t^{2(1-\alpha)}\log(t)\right), & s=d+2 \qquad (5.13\text{-d}) \\[2ex]
\dfrac{\pi^{d/2} t^{1-2\alpha}}{(1-2\alpha)\Gamma\left(d/2+1\right)} + O\left(t^{2(1-\alpha)}\right), & s>d+2 \qquad (5.13\text{-e})
\end{cases} \tag{5.13}
$$

where $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ *refers to the* digamma function *and* $\gamma \approx 0.57721$ *to* Euler's constant*; see, e.g., 5.2.2 and 5.2.3 on p. 136 of* [24].

Proof By the spherical coordinate integration formula and (5.8), for all $\alpha \in [0, 1)$,

$$
\Upsilon_\alpha(0) = (2\pi)^{-d}\int_{\mathbb{R}^d} \frac{\mathrm{d}\xi}{|\xi|^{2(1-\alpha)}\left(1+|\xi|^2\right)^{s/2}} = (2\pi)^{-d}\, C_d \int_0^\infty \frac{\mathrm{d}r\; r^{d-1}}{r^{2(1-\alpha)}\left(1+r^2\right)^{s/2}},
$$

where $C_d := \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Now by the change of variables $z = r^2/(1+r^2)$, we can evaluate the above integral explicitly by transforming it to the Beta integral:

$$
\begin{aligned}
\int_0^\infty \frac{r^{d-1}}{\mathrm{d}r\; r^{2(1-\alpha)}\left(1+r^2\right)^{s/2}} &= \frac{1}{2}\int_0^1 \mathrm{d}z\; z^{d/2+\alpha-2}(1-z)^{(s-d)/2-\alpha} \\
&= \frac{\Gamma\left(d/2-1+\alpha\right)\Gamma\left((s-d)/2+1-\alpha\right)}{2\Gamma(s/2)},
\end{aligned}
$$

which is finite provided that $s > d - 2(1-\alpha) > 0$. This proves (5.10) and from this, we easily deduce (5.11) by letting $\alpha = 0$ in (5.10) and by applying the formula $\Gamma(z+1) = z\Gamma(z)$, which holds for $z \in \mathbb{C}$ such that $\Re(z) > 0$.

It remains to prove (5.13), which then implies (5.12). From (3.5) and by the spherical coordinate integration formula, for all $t > 0$,

$$\begin{aligned}\mathcal{H}_\alpha(t) &= \int_0^t \mathrm{d}r\ r^{-2\alpha} \int_{\mathbb{R}^d} \mathrm{d}\xi\ \frac{\exp(-r|\xi|^2)}{(1+|\xi|^2)^{s/2}} \\ &= \frac{C_d}{2} \int_0^t \mathrm{d}r\ r^{-2\alpha} \int_0^\infty \mathrm{d}u\ \exp(-ru)(1+u)^{-s/2} u^{d/2-1} \\ &=: \frac{C_d \Gamma(d/2)}{2} \int_0^t \mathrm{d}r\ r^{-2\alpha} I(r) = \pi^{d/2} \int_0^t \mathrm{d}r\ r^{-2\alpha} I(r).\end{aligned}$$

By [24, 13.4.4 on p.326], $I(r)$ is equal to the *confluent hypergeometric function*:

$$I(r) = U\left(\frac{d}{2}, \frac{2+d-s}{2}, r\right). \tag{5.14}$$

By 18.2.18–13.2.22 on p. 323 *ibid.*, we see that

$$I(r) = \begin{cases} \dfrac{\Gamma\left(\frac{d-s}{2}\right) r^{\frac{s-d}{2}}}{\Gamma(d/2)} + \dfrac{\Gamma\left(\frac{s-d}{2}\right)}{\Gamma(s/2)} + O\left(r^{\frac{s-d}{2}+1}\right) & d-2 < s < d \quad 18.2.18, \\ -\dfrac{1}{\Gamma(d/2)} \left(\log(r) + \psi(d/2) + 2\gamma\right) + O\left(r \log(r)\right) & s = d \quad 18.2.19, \\ \dfrac{\Gamma\left(\frac{s-d}{2}\right)}{\Gamma(s/2)} + O\left(r^{(s-d)/2}\right) & d < s < d+2 \quad 18.2.20, \\ \dfrac{1}{\Gamma\left(d/2+1\right)} + O\left(r \log(r)\right) & s = d+2 \quad 18.2.21, \\ \dfrac{\Gamma\left(\frac{s-d}{2}\right)}{\Gamma(s/2)} + O\left(r\right) & s > d+2 \quad 18.2.22. \end{cases}$$

Then integrating the right-hand side of the above expressions against $\pi^{d/2} r^{-2\alpha} \mathrm{d}r$ over $[0, t]$ gives the five cases in (5.13). This completes the proof of Proposition 5.2. □

Similarly, one can use the Bessel kernel as the spectral density. In this case, we have the following proposition:

Proposition 5.3 (Bessel kernel as spectral density) *Suppose that the spectral density $\widehat{f}$ is given by the Bessel kernel $f_s(\cdot)$ defined in Example 5.8 (see (5.8)), or in other words, suppose that the correlation function $f(x)$ is given by $f(x) = \left(1+|x|^2\right)^{-s/2}$ for $s > 0$. Then*

$$\Upsilon_\alpha(0) = \frac{\Gamma(1-\alpha)\,\Gamma(\alpha - 1 + s/2)}{2^{2d}\pi^{3d/2}\Gamma(d/2)\Gamma(s/2)} \qquad \textit{for all } s \wedge d > 2(1-\alpha) > 0, \tag{5.15}$$

and in particular when $\alpha = 0$, (5.15) simplifies to the following:

$$\Upsilon(0) = \frac{2^{1-2d}\pi^{-3d/2}}{(s-2)\Gamma(d/2)} \qquad \textit{for all } s \wedge d > 2. \tag{5.16}$$

In addition,

$$\mathcal{H}_\alpha(t) < \infty \quad \forall t > 0 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad s > 0. \tag{5.17}$$

Moreover, for $\alpha \in (0, 1/2)$*, we have the following asymptotic*

$$\mathcal{H}_\alpha(t) \sim \frac{t^{1-2\alpha}}{1-2\alpha}, \quad \text{as } t \downarrow 0. \tag{5.18}$$

Proof By similar arguments as Proposition 5.2, we have that

$$\begin{aligned}
\Upsilon_\alpha(0) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{d}\xi \, \frac{f_s(\xi)}{|\xi|^{2(1-\alpha)}} = (2\pi)^{-2d} \int_{\mathbb{R}^d} \frac{\mathrm{d}\xi \, \widehat{f_s}(\xi)}{|\xi|^{d-2(1-\alpha)}} \\
&= (2\pi)^{-2d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{\mathrm{d}r \, r^{d-1}}{\left(1+r^2\right)^{s/2} r^{d-2(1-\alpha)}} = \frac{\Gamma(1-\alpha)\Gamma(\alpha-1+s/2)}{2^{2d}\pi^{3d/2}\Gamma(d/2)\Gamma(s/2)},
\end{aligned}$$

which is finite provided $s > 2(1-\alpha)$. Note that in the second equality in the above chain of equalities, we need the condition $d > 2(1-\alpha)$. This proves both (5.15) and (5.16).

As for (5.17),

$$\begin{aligned}
\mathcal{H}_\alpha(t) &= \int_0^t \mathrm{d}r \, r^{-2\alpha} \int_{\mathbb{R}^d} \mathrm{d}\xi \; f_s(\xi) \exp\left(-r|\xi|^2\right) \\
&= (2\pi)^{-d}\pi^{d/2} \int_0^t \mathrm{d}r \, r^{-2\alpha} \int_{\mathbb{R}^d} \mathrm{d}\xi \; \widehat{f_s}(\xi) \exp\left(-\frac{|\xi|^2}{4r}\right) r^{-d/2} \\
&= (2\pi)^{-d}\pi^{d/2} \int_0^t \mathrm{d}r \, r^{-2\alpha} \int_{\mathbb{R}^d} \mathrm{d}\xi \; \left(1+|\xi|^2\right)^{-s/2} \exp\left(-\frac{|\xi|^2}{4r}\right) r^{-d/2} \\
&= (2\pi)^{-d}\pi^{d/2} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^t \mathrm{d}r \, r^{-2\alpha-d/2} \int_0^\infty \mathrm{d}z \, z^{d-1} \left(1+z^2\right)^{-s/2} \exp\left(-\frac{z^2}{4r}\right),
\end{aligned}$$

where we have used Plancherel's theorem and the following identities:

$$\mathcal{F}\left(\exp(-|\cdot|^2)\right)(\xi) = \pi^{d/2} \exp\left(-4^{-1}|\xi|^2\right) \quad \text{and} \quad \mathcal{F}(f(a\cdot))(\xi) = a^{-d}\mathcal{F}f(\xi/a).$$

Then, by the same arguments as Proposition 5.2,

$$\mathcal{H}_\alpha(t) = 2^{-d} \int_0^t \mathrm{d}r \, r^{-2\alpha-d/2} I\left(\frac{1}{4r}\right), \quad \text{with } I(r) = U\left(\frac{d}{2}, \frac{2+d-s}{2}, r\right),$$

where U is given in (5.14). Since $I(r) \sim r^{-d/2}$ as $r \to \infty$, we see that $I\left(\frac{1}{4r}\right) \sim (4r)^{d/2}$ as $r \to 0$ (see 13.2.6 on p. 322 of [24]). Hence, the above integral behave as follows:

$$\mathcal{H}_\alpha(t) \sim \int_0^t \mathrm{d}r\, r^{-2\alpha-\frac{d}{2}}(4r)^{d/2} = \int_0^t \mathrm{d}r\, r^{-2\alpha} = \frac{t^{1-2\alpha}}{1-2\alpha},$$

provided $\alpha \in (0, 1/2)$, which proves both (5.17) and (5.18). $\square$

The necessity of the finiteness of $\Upsilon(0)$ excludes the Riesz kernel as a choice for the spectral density. However, we can still construct a Riesz-type kernel which has polynomial growth at the origin and polynomial decay at infinity, but with different rates, using Propositions 5.2 and 5.3. This Riesz-type kernel gives another example of a kernel that is easily verifiable to be permissible under the conditions of our Theorem 1.3, while being demanding to verify using (5.4); see Example 5.10 for more details.

Example 5.10 (*Riesz-type kernel*) For $s_1, s_2 \in (0, d)$, let f_{s_1} and f_{s_2} be Bessel kernels as in Example 5.8. Define

$$r(x) := f_{s_1}(x) + \widehat{f_{s_2}}(x) \quad \text{or equivalently} \quad \widehat{r}(\xi) := \widehat{f_{s_1}}(x) + f_{s_2}(x).$$

It is easy to see that $r(\cdot)$ is both nonnegative and nonnegative-definite which follows immediately from the linearity of the Fourier transform and the fact that the Bessel kernel is both nonnegative and nonnegative-definite. Also, we easily deduce from properties (2)–(4) in Example 5.8 that

$$r(x) \sim \begin{cases} |x|^{s_1-d} & |x| \to 0, \\ |x|^{-s_2} & |x| \to \infty, \end{cases} \quad \text{and} \quad \widehat{r}(\xi) \sim \begin{cases} |\xi|^{s_2-d} & |\xi| \to 0, \\ |\xi|^{-s_1} & |\xi| \to \infty. \end{cases}$$

Propositions 5.2 and 5.3 imply that

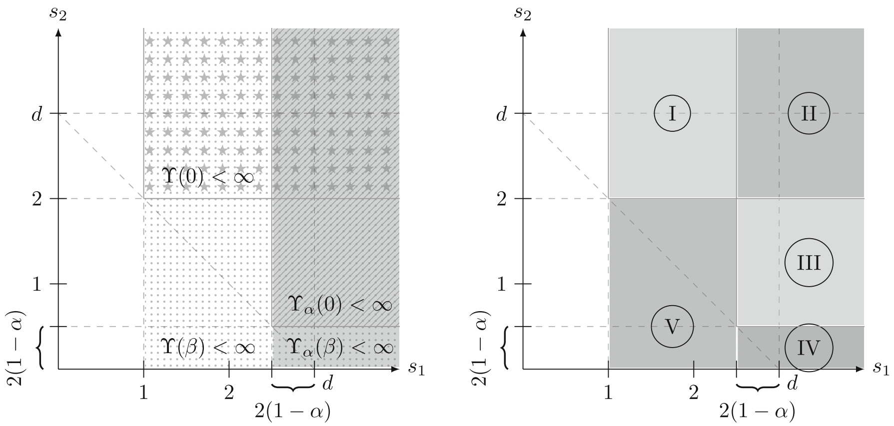
$$\Upsilon(0) = \frac{\Gamma(d/2-1)\,\Gamma((s_1-d)/2+1)}{2^d\pi^{d/2}\,\Gamma(d/2)\,\Gamma(s_1/2)} + \frac{\Gamma(s_2/2-1)}{2^{2d}\pi^{3d/2}\,\Gamma(d/2)\,\Gamma(s_2/2)} \quad \text{and} \tag{5.19}$$

$$\Upsilon_\alpha(0) = \frac{\Gamma(d/2-1+\alpha)\,\Gamma((s_1-d)/2+1-\alpha)}{2^d\pi^{d/2}\,\Gamma(d/2)\,\Gamma(s_1/2)} + \frac{\Gamma(1-\alpha)\,\Gamma(s_2/2+\alpha-1)}{2^{2d}\pi^{3d/2}\,\Gamma(d/2)\,\Gamma(s_2/2)}, \tag{5.20}$$

which are finite provided that

$$\begin{cases} 0 < d-2 < s_1 \\ s_2 > 2 \end{cases} \quad \text{and} \quad \begin{cases} 0 < d-2(1-\alpha) < s_1 \\ 0 < 2(1-\alpha) < s_2 \end{cases}, \quad \text{respectively.} \tag{5.21}$$

In contrast, it is not clear how to compute $\mathcal{F}\big(\sqrt{\widehat{r}}\big)$; see condition (5.4). This is the case when neither condition (1.10a) nor condition (3.3) implies each other. See Fig. 5 for an illustration of the parameter regions for Dalang's condition (1.12) (i.e., $\Upsilon(\beta) < \infty$)



Region	Dalang's condition (1.12) $\Upsilon(\beta) < \infty$	Strengthened Dalang's conditions (1.10a) $\Upsilon(0) < \infty$	(1.13) $\Upsilon_\alpha(\beta) < \infty$	(3.3) $\Upsilon_\alpha(0) < \infty$
I	✓	✓	✗	✗
II	✓	✓	✓	✓
III	✓	✗	✓	✓
IV	✓	✗	✓	✗
V	✓	✗	✗	✗

Fig. 5 The left figure: Parameter regions for the four conditions: $\Upsilon(\beta) < \infty$ (the dotted region), $\Upsilon(0) < \infty$ (the star-patterned one), $\Upsilon_\alpha(0) < \infty$ (the one with northeast lines), and $\Upsilon_\alpha(\beta) < \infty$ (the shaded one). The right figure and the table below: five parameter regions determined through Dalang's condition (1.12) (the first column) and three strengthened Dalang's conditions (1.10a) (the second column), (1.13) (the third column), and (3.3) (the forth column)

and three strengthened versions in (1.10a) (i.e., $\Upsilon(0) < \infty$), (1.13) (i.e., $\Upsilon_\alpha(\beta) < \infty$), and (3.3) (i.e., $\Upsilon_\alpha(0) < \infty$).

5.5 Examples of Admissible Weight Functions

In this part, we give some examples of the admissible weight functions. As given in Section 2 of [11], the following functions are admissible functions:

$$\begin{cases} \rho(x) = \exp(-a|x|) & a > 0, \\ \rho(x) = (1 + |x|^a)^{-1} & a > d. \end{cases} \tag{5.22}$$

The smaller the weight function $\rho(\cdot)$ (not necessarily admissible) is, the larger the space $L^2_\rho(\mathbb{R}^d)$ is. For example, one may choose ρ to be either a nonnegative function with compact support or the heat kernel itself $G(1, \cdot)$. In both cases, ρ is smaller than those in (5.22) (up to a constant). However, one can easily check that the admissible condition (1.7) excludes these two cases. However, the examples in Proposition 5.4 seem to be less obvious:

Proposition 5.4 *$\rho_b(\cdot)$ is admissible if and only if $b \in (0, 1]$ where*

$$\rho_b(x) := \exp\left(-|x|^b\right), \quad x \in \mathbb{R}^d, \quad \textit{with } b > 0.$$

Proof From Definition 1.1, we see that ρ_b is admissible if and only if for all $T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}y \;\, (2\pi t)^{-d/2}\, e^{-\frac{|x-y|^2}{2t}+(|x|^b-|y|^b)} < \infty.$$

Denote the above integral by $I(t, x)$. We will use C to denote a generic constant that does not depend on (t, x), which value may change at each occurrence.

We first assume that $b \in (0, 1]$. In this case,

$$|x|^b = |x - y + y|^b \le (|x - y| + |y|)^b \le |x - y|^b + |y|^b.$$

Hence,

$$I(t, x) \le C \int_{\mathbb{R}^d} \mathrm{d}y\, t^{-d/2} e^{-\frac{|x-y|^2}{2t}+|x-y|^b} = C \int_0^\infty \mathrm{d}r\, t^{-d/2} e^{-\frac{r^2}{2t}+r^b} r^{d-1}.$$

Then by applying the change of variables $r' = t^{-1/2} r$,

$$I(t, x) \le C \int_0^\infty \mathrm{d}r\; e^{-\frac{r^2}{2}+t^{b/2}r^b} r^{d-1} \le C \int_0^\infty \mathrm{d}r\; e^{-\frac{r^2}{2}+T^{b/2}r^b} r^{d-1} < \infty.$$

Next we assume that $b > 1$. We need to show that ρ_b is not admissible. Without loss of generality, we assume that $d \ge 2$. The case when $d = 1$ is easier and can be proved similarly as the proof below. It suffices to show that

$$\lim_{r\to\infty} I(1/2, x_r) = \infty, \quad \text{where } x_r := (r, 0, \cdots, 0) \in \mathbb{R}^d.$$

Without loss of generality, we may assume that $r \gg 2$. Denote $y = (y_1, \cdots, y_d) = (y_1, y_*)$ with $y_* \in \mathbb{R}^{d-1}$. Using the subadditivity (resp. convexity) of $(x + y)^{b/2}$ when $b \in (1, 2]$ (resp. $b > 2$), we see that

$$|y|^b = \left(y_1^2 + y_2^2 + \cdots + y_d^2\right)^{b/2} \le c|y_1|^b + \left(y_2^2 + \cdots + y_d^2\right)^{b/2} = c|y_1|^b + c|y_*|^b,$$

where $c := 1 \wedge 2^{b/2-1}$. Hence,

$$
\begin{aligned}
I(1/2, x_r) &= C \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-\left(\sum_{i=2}^d y_i^2\right) - |y_1 - r|^2 + (r^b - |y|^b)} \\
&\geq C \int_{\mathbb{R}^{d-1}} \mathrm{d}y_* \int_{\mathbb{R}} \mathrm{d}y_1\, e^{-|y_*|^2 - |y_*|^b - |y_1 - r|^2 + (r^b - c|y_1|^b)} \\
&= C \int_{\mathbb{R}^{d-1}} \mathrm{d}y_*\, e^{-|y_*|^2 - c|y_*|^b} \int_{\mathbb{R}} \mathrm{d}y_1\, e^{-|y_1 - r|^2 + r^b - c|y_1|^b} \\
&= C \int_{\mathbb{R}} \mathrm{d}y\, e^{-y^2 + r^b - c|y - r|^b} \geq C \int_0^r \mathrm{d}y\, e^{-y^2 + r^b - c|y - r|^b} =: C K(r).
\end{aligned}
$$

It suffices to show that $\lim_{r \to \infty} K(r) = \infty$, which is true when $b > 2$ because

$$
K(r) \geq \int_{r/2}^r \mathrm{d}y\, e^{-y^2 + r^b - c(r-y)^b} \geq \int_{r/2}^r \mathrm{d}y\, e^{-r^2 + r^b - c(r/2)^b} = \frac{r}{2} e^{\left(1 - 2^{-1-b/2}\right) r^b - r^2},
$$

which blows up as $r \to \infty$. Hence, we may assume that $b \in (1, 2]$. In this case, $c = 1$ and

$$
K'(r) = e^{r^b - r^2} + b \int_0^r \mathrm{d}y\, e^{-y^2 + r^b - (r-y)^b} \left(r^{b-1} - (r-y)^{b-1} \right).
$$

By the intermediate value theorem, we see that $r^{b-1} - (r-y)^{b-1} = (b-1) y \xi^{b-2}$ for some $\xi \in [r - y, r]$. Since $b - 1 \in (0, 1]$, this implies that $r^{b-1} - (r-y)^{b-1} \geq (b-1) y r^{b-2}$. Hence,

$$
\begin{aligned}
K'(r) &\geq e^{r^b - r^2} + b(b-1) \int_1^r \mathrm{d}y\, e^{-y^2 + r^b - (r-y)^b} y r^{b-2} \\
&\geq b(b-1) r^{b-2} \int_1^r \mathrm{d}y\, e^{-y^2 + r^b - (r-1)^b} \geq b(b-1) r^{b-2} e^{r^b - (r-1)^b} \int_1^2 \mathrm{d}y\, e^{-y^2}.
\end{aligned}
$$

Another application of the intermediate value theorem shows that $r^b - (r-1)^b = b \xi^{b-1}$ with $\xi \in [r-1, r]$. Hence, $r^b - (r-1)^b \geq b(r-1)^{b-1}$ and then

$$
K'(r) \geq C r^{b-2} \exp\left(b(r-1)^{b-1} \right).
$$

Hence, for $r \gg 2$, $K'(r)$ is positive and unbounded as $r \to \infty$. Therefore, this implies that $K(r)$ blows up as $r \to \infty$, which completes the proof of Proposition 5.4. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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