# An attraction-repulsion chemotaxis system with logistic source

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This paper deals with the attraction-repulsion chemotaxis system with logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + f(u), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n$ $(n\ge 1)$. Under a growth restriction on logistic source and suitable assumptions on the positive parameters $\chi$, $\xi$, $\alpha$, $\beta$, $\gamma$ and $\delta$, we show the existence of global bounded classical solutions. The global weak solution is also constructed if the logistic damping effect is rather mild. Furthermore, we obtain the asymptotic behavior of solutions for the logistic source $f(u)=\mu u(1-u)$.



## 1 Introduction

Chemotaxis is a mechanism by which cells and organisms efficiently respond to chemical stimuli in their environment, moving toward beneficial targets or environments and avoiding undesired ones [1]. In the present paper, we consider chemoattraction-repulsion process in which cells undergo random motion and chemotaxis towards attractant and away from repellent [12]. Moreover, we consider the model with proliferation and death of cells and assume that chemicals diffuse very quickly [6, 19, 20]. These lead to the model of partial differential equations as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + f(u), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n(n\ge 1)$ with smooth boundary, where $u(x,t)$ represents the population density, $v(x,t)$ and $w(x,t)$ denote chemical concentrations of attractant and repellent, respectively. The parameters $\chi$, $\xi$, $\alpha$, $\beta$, $\gamma$, and $\delta$ are all positive. The logistic function $f(s):\mathbb{R}\to\mathbb{R}$ is smooth and satisfies $f(0)\ge 0$ as well as

$$f(s)\le \mu s(1-s)\quad \text{for all } s\ge 0 \tag{1.2}$$

with some $\mu>0$.

The attraction-repulsion chemotaxis model (1.1) is a generalization of the famous Keller-Segel model [8]

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u),\\ \tau v_t = \Delta v - v + u,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0,\\ u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0. \end{cases} \tag{1.3}$$

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During the past four decades, much attention has been focused on this classical chemotaxis model. A large amount of efforts were devoted to making clear whether solutions to the so-called minimal model ($f(u) = 0$) are blowup or bounded [4,13,16,24,27].

Since the blowup of solutions is an extreme phenomenon in real application, a lot of papers consider the model variations in which the chemotactic collapse is precluded. The logistic function $f(u)$ in (1.3) under some growth restrictions is expected to prevent chemotactic collapse. In the case $\tau > 0$, if the logistic term satisfies

$$f(u) \le a - \mu u^2 \tag{1.4}$$

with some $a \ge 0$ and $\mu > 0$, problem (1.3) possesses a unique global and bounded solution provided that either the space dimensions $n \le 2$ [15,25] or $n \ge 3$ and $\mu > \mu_0(\tau, \chi, n)$ [25]. The variation of (1.3) has also been considered in [14], in which the production term $u$ in the second equation of (1.3) is replaced by $h(u)$. Under the assumptions $0 \le h(u) \le (u+1)^\beta$, $0 \le h'(u) \le \beta(u+1)^{\beta-1}$, and $f(u) \le 1 - \mu u^\alpha$ with $\frac{3}{2} \le \alpha \le 2$, and $0 < \beta < \alpha - 1$ or $(\alpha, \beta) = (\frac{3}{2}, \frac{1}{2})$, solutions are global and bounded.

The case $\tau = 0$ means essentially that chemical diffuses much faster than cells move. Under the assumption (1.4), Tello and Winkler [20] proved that if either $n \le 2$ or $n \ge 3$, $\mu > \frac{n-2}{n}\chi$, problem (1.3) with $\tau = 0$ possess a unique global bounded classical solution, and moreover for all $n \ge 1$, $\mu > 0$ there exist at least one global weak solution. When the degradation order of $f$ is weaker than quadratic, i.e., $-c_0(u + u^\alpha) \le f(u) \le c_1 - \mu u^\alpha$ for some $c_0 > 0$, $c_1 \ge 0$ and $\mu > 0$, Winkler [22] constructed a *very weak* solution and studied the boundedness properties of such solution if $\alpha > 2 - \frac{1}{n}$. In a recent paper, Wang et al. [21] obtained the boundedness of solutions to the nonlinear-diffusion model with such proliferation term. However, chemotactic collapse is not always ruled out by logistic term. Winkler [26] studied the system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \nu u^\kappa, \\ 0 = \Delta v - m(t) + u \end{cases} \tag{1.5}$$

in a ball $\Omega \subset \mathbb{R}^n$, where $m(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$. He proved when $n \ge 5$ and $\kappa < \frac{3}{2} + \frac{1}{2n-2}$, there exist initial data such that radially symmetric solutions of (1.5) blow up in finite time.

Recently, the attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \\ \tau v_t = \Delta v + \alpha u - \beta v, \\ \tau w_t = \Delta w + \gamma u - \delta w, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \\ u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0, \quad \tau w(x,0) = \tau w_0 \end{cases} \tag{1.6}$$

was studied by several authors [2,3,7,10,11,17,19]. Liu and Wang [10] studied the global existence, asymptotic behavior and steady states of solutions to system (1.6) with $\tau = 1$ in one dimension. In the two-dimensional case, if $\chi\alpha - \xi\gamma > 0$ (1.6) with $\tau = 0$ possesses a global classical solution provided that $\Omega$ is a disk and $\|u_0(x)\|_{L^1(\Omega)} < \frac{8\pi}{\chi\alpha - \xi\gamma}$; however, under the conditions $\|u_0(x)\|_{L^1(\Omega)} > \frac{8\pi}{\chi\alpha-\xi\gamma}$, $\beta = \delta$ and $\int_\Omega u_0(x)|x - x_0|^2 dx$ is sufficiently small, finite-time blow-up does occur. Moreover, (1.6) with $\tau > 0$ has a global in time classical solution for small initial mass [2]. In the case $\chi\alpha - \xi\gamma < 0$, Jin [7] proved the global existence of classical solution in two dimensions and weak solution in three dimensions with large initial data. Tao and Wang [19] considered the global solvability, boundedness, blow-up, existence of steady states, and asymptotic behavior of system (1.6). It has been shown that if $\chi\alpha - \xi\gamma < 0$, system (1.6) with $\tau = 0$ is globally well-posed in high dimensions $n \ge 2$, and system (1.6) with $\tau = 1$ is globally well-posed in two dimensions when $\chi\alpha - \xi\gamma < 0$ and $\beta = \delta$.

Our purpose in this paper is to investigate the global existence and asymptotic behavior of the solutions to system (1.1). The first result is concerning the existence of global bounded classical solution to system (1.1).

**Theorem 1.1.** *Assume $f$ satisfies (1.2) and one of the following assumptions holds:*

(a) $\chi\alpha - \xi\gamma \le \mu$,
(b) $n \le 2$,
(c) $\frac{n-2}{n}(\chi\alpha - \xi\gamma) < \mu$, $n \ge 3$.

*Then for any nonnegative $u_0(x) \in C^0(\bar{\Omega})$ the system (1.1) possesses a unique global classical solution $(u, v, w)$ which is uniformly bounded.*

 

The conditions from Theorem 1.1 are technical requirements. We have to leave open here whether they are optimal for the global existence of classical solutions. So in the case $n \geq 3$ and $\mu \in (0, \frac{n-2}{n}(\chi\alpha - \xi\gamma)]$, we give the existence of global weak solution in the following theorem.

**Theorem 1.2.** *Let $f$ satisfy (1.2). Moreover, we assume $f(u) \geq -C_0 u(1+u)$ with some $C_0 > 0$. Then for any nonnegative initial data $u_0(x) \in C^0(\bar{\Omega})$, the system (1.1) possesses at least one global weak solution.*

Finally, we study the asymptotic behavior of solutions to system (1.1) with the logistic source $f(u) = \mu u(1-u)$.

**Theorem 1.3.** *Let $u_0(x) \in C^0(\bar{\Omega})$ be nonnegative function and $f(u) = \mu u(1-u)$. Suppose that $\mu > 2\chi\alpha$. Then the global classical solution of (1.1) satisfies*

$$u(\cdot,t) \to 1, \quad v(\cdot,t) \to \frac{\alpha}{\beta}, \quad w(\cdot,t) \to \frac{\gamma}{\delta} \quad \text{as } t \to \infty$$

*uniformly with respect to $x \in \Omega$.*

The rest of the paper is organized as follows. In Sect. 2, we state some preliminaries and prove local-in-time existence of classical solution. In Sect. 3, we prove the boundedness of global solution. The global weak solution is constructed in Sect. 4. Finally, in Sect. 5 we consider the large time behavior of solutions. Throughout the rest of this paper, $C$ will denote a generic constant which may vary from line to line and we also denote it by $C_\delta$ or $C(\delta)$ to indicate that it depends on a real number $\delta$.

## 2 Preliminaries

In this paper we need the following well-known facts concerning the Laplacian in $\Omega$ supplemented with homogeneous Neumann boundary conditions (for instance, see [5, 23]). Firstly, the operator $A := -\Delta + 1$ is sectorial in $L^p(\Omega)$ and therefore possesses closed fractional powers $A^\theta$, $\theta \in (0,1)$, with dense domain $D(A^\theta)$. If $m \in \{0,1\}$, $p \in [1,\infty]$ and $q \in (1,\infty)$ with $m - \frac{n}{p} < 2\theta - \frac{n}{q}$, then we have

$$\|\omega\|_{W^{m,p}(\Omega)} \leq C\|A^\theta \omega\|_{L^q(\Omega)}, \quad \omega \in D(A^\theta). \tag{2.1}$$

Moreover, for $p < \infty$ the associated heat semigroup $(e^{t\Delta})_{t\geq 0}$ maps $L^p(\Omega)$ into $D(A^\theta)$ in any of the space $L^q(\Omega)$ for $q \geq p$, and there exist $C > 0$ and $\nu > 0$ such that

$$\|A^\theta \mathrm{e}^{-tA}\omega\|_{L^q(\Omega)} \leq Ct^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})}\mathrm{e}^{-\nu t}\|\omega\|_{L^p(\Omega)}, \quad \omega \in L^p(\Omega). \tag{2.2}$$

Finally, given $p \in (1,\infty)$, for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|A^\theta \mathrm{e}^{t\Delta}\nabla\cdot z\|_{L^p(\Omega)} \leq C_\delta t^{-\theta - \frac{1}{2} - \delta}\mathrm{e}^{-\nu t}\|z\|_{L^p(\Omega)}, \quad z \in (L^p(\Omega))^n. \tag{2.3}$$

The following lemma concerns the local solvability of (1.1).

**Lemma 2.1.** *Suppose that $f(u) \in W^{1,\infty}_{loc}(\mathbb{R})$, and that $u_0 \in C^0(\bar{\Omega})$ is nonnegative. Then there exist $T_{\max} \in (0,\infty]$ and a uniquely determined triple $(u, v, w)$ of nonnegative functions from $C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$ solves (1.1) in the classical sense. In addition, if $T_{\max} < \infty$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\max}.$$

P r o o f . The proof is based on the well-known and frequently used method (see [18, 25], for example). For completeness, we give the brief proof. With $R > 0$ and $T \in (0,1)$ to be specified below, in the Banach space

$$X := C^0([0,T]; C^0(\bar{\Omega})),$$

we consider the closed convex set

$$S := \left\{ u \in X \,\middle|\, \|u\|_{L^\infty((0,T);L^\infty(\Omega))} \leq R \right\}.$$

We define a mapping $\Phi$ by

$$\Phi(u)(t) := \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}\left\{-\chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + f(u)\right\}(s)ds$$

for $u \in S$ and $t \in [0,T]$, where $v$ is the solution of

$$\begin{cases} -\Delta v + \beta v = \alpha u, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

and $w$ denotes the solution of

$$\begin{cases} -\Delta w + \delta w = \gamma u, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

Applying the elliptic $L^p$ estimates to (2.4) and (2.5), respectively, we obtain

$$\|v(\cdot,t)\|_{W^{2,p}(\Omega)} \le C\|u(\cdot,t)\|_{L^p(\Omega)} \quad \text{and} \quad \|w(\cdot,t)\|_{W^{2,p}(\Omega)} \le C\|u(\cdot,t)\|_{L^p(\Omega)} \tag{2.6}$$

for all $t \in (0,T)$. We choose $\theta \in (\frac{n}{2p}, \frac{1}{2})$ for $p > n$, and then $\varepsilon \in (0, \frac{1}{2} - \theta)$. From the maximum principle, (2.1), (2.3) and (2.6) and $f \in W^{1,\infty}_{loc}(\mathbb{R})$, we obtain

$$\begin{aligned} \|\Phi(u)(t)\|_{L^\infty(\Omega)} &= \|\mathrm{e}^{t\Delta}u_0\|_{L^\infty(\Omega)} + \chi \int_0^t \|\mathrm{e}^{(t-s)\Delta}\nabla\cdot(u\nabla v)(s)\|_{L^\infty(\Omega)} ds \\ &\quad + \xi \int_0^t \|\mathrm{e}^{(t-s)\Delta}\nabla\cdot(u\nabla w)(s)\|_{L^\infty(\Omega)} ds + \int_0^t \|\mathrm{e}^{(t-s)\Delta} f(u)(s)\|_{L^\infty(\Omega)} ds \\ &\le \|u_0\|_{L^\infty(\Omega)} + C(|\Omega|, p)T^{\frac{1}{2}-\theta-\varepsilon}R^2 + T\|f\|_{L^\infty((-R,R)\times(-\frac{\gamma}{\delta}R, \frac{\gamma}{\delta}R))}. \end{aligned}$$

This proves that $\Phi$ maps $S$ into itself by choosing $R > 0$ sufficiently large and $T$ suitably small. According to a similar argument, using $f$ is locally Lipschitz continuous in $\mathbb{R}$, we deduce that $\Phi$ is a contraction on $S$. Therefore, by Banach fixed point theorem, we infer that there exists $u \in S$ such that $\Phi(u) = u$. From (2.4) and (2.5), we obtain the existence of $v$ and $w$. From the standard regularity argument, it follows that the solution $(u, v, w)$ satisfies the regularity properties listed in the lemma. The maximum principle to the first equation in (1.1) ensures $u \ge 0$. Then using elliptic maximum principle to the second equation and the third equation yields $v \ge 0$ and $w \ge 0$.

Now we prove the uniqueness of the solution. Suppose that $(u_1, v_1, w_1)$ and $(u_2, v_2, w_2)$ are two solutions of (1.1) in $\Omega \times (0,T)$ for some $T > 0$. We fix $T_0 \in (0,T)$, and let $U = u_1 - u_2$, $V = v_1 - v_2$, and $W = w_1 - w_2$. A direct calculation yields

$$U_t = \Delta U - \chi\nabla\cdot(U\nabla v_1) - \chi\nabla\cdot(u_2\nabla V) + \xi\nabla\cdot(U\nabla w_1) + \xi\nabla\cdot(u_2\nabla W) + f(u_1) - f(u_2). \tag{2.7}$$

Multiplying (2.7) by $U$ and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega U^2 + \int_\Omega |\nabla U|^2 &= \chi\int_\Omega U\nabla v_1\cdot\nabla U + \chi\int_\Omega u_2\nabla V\cdot\nabla U \\ &\quad - \xi\int_\Omega U\nabla w_1\cdot\nabla U - \xi\int_\Omega u_2\nabla W\cdot\nabla U \\ &\quad + \int_\Omega (f(u_1) - f(u_2))\,U. \end{aligned}$$

In view of elliptic $L^p$-estimates, the boundedness of $u_1, u_2$, the local Lipschitz continuity of $f$ and Young's inequality, we conclude

$$\frac{d}{dt}\int_\Omega U^2 \le C\int_\Omega U^2,$$

which yields $U \equiv 0$ in $\Omega \times (0, T_0)$. Since $T_0 \in (0,T)$ is arbitrary, we have the uniqueness of the solution to (1.1). □

## 3 Existence of global bounded solutions

In this section we shall show that the local solution constructed in Lemma 2.1 is global and uniformly bounded under the assumptions of Theorem 1.1. We start with some estimates for $w$. The idea of proof is based on [28], which obtained the similar results in the case $n = 1$.

**Lemma 3.1.** *Suppose that $f$ satisfies (1.2). Then the solution $(u, v, w)$ of (1.1) satisfies*

$$\int_\Omega u \le M \tag{3.1}$$

*and*

$$\int_\Omega w \le \frac{\gamma}{\delta}M \tag{3.2}$$

 

*as well as*

$$\|w\|_{L^p(\Omega)} \le \frac{\gamma}{\delta}\|u\|_{L^p(\Omega)}, \quad p > 1 \tag{3.3}$$

*for all $t \in (0, T_{\max})$, where $M := \max\left\{\int_\Omega u_0 dx, |\Omega|\right\}$.*

P r o o f. Integrating the first equation in (1.1) and using (1.2), we obtain

$$\frac{d}{dt}\int_\Omega u \le \int_\Omega f(u) \le \mu \int_\Omega (u - u^2).$$

Since $u^2 \ge 2u - 1$, we get

$$\frac{d}{dt}\int_\Omega u \le -\mu \int_\Omega u + \mu|\Omega|,$$

which yields (3.1). The estimate (3.2) immediately follows from (3.1) and an integration of the third equation in (1.1).

Now, we prove (3.3). Multiplying the third equation in (1.1) by $w(w^2+\varepsilon)^{\frac{p}{2}-1}$ with $\varepsilon > 0$ and integrating over $\Omega$, we get

$$\int_\Omega \left((p-1)w^2 + \varepsilon\right)(w^2+\varepsilon)^{\frac{p}{2}-2}|\nabla w|^2 + \delta\int_\Omega w^2(w^2+\varepsilon)^{\frac{p}{2}-1} = \gamma\int_\Omega uw(w^2+\varepsilon)^{\frac{p}{2}-1}.$$

Dropping the first nonnegative term on the left, we have

$$\delta\int_\Omega w^2(w^2+\varepsilon)^{\frac{p}{2}-1} \le \gamma\int_\Omega uw(w^2+\varepsilon)^{\frac{p}{2}-1}.$$

We conclude by the monotone convergence theorem in taking $\varepsilon \to 0$ that

$$\delta\int_\Omega w^p \le \gamma\int_\Omega uw^{p-1} \le \gamma\left(\int_\Omega u^p\right)^{\frac{1}{p}}\left(\int_\Omega w^p\right)^{\frac{p-1}{p}}.$$

This completes the proof. □

Based on Lemma 3.1, we have the following estimate.

**Lemma 3.2.** *Let (1.2) hold. Then for any $\varepsilon > 0$ one can find $C = C(\varepsilon, p, \|u_0\|_{L^1(\Omega)}) > 0$ such that*

$$\int_\Omega w^{p+1} \le \varepsilon\int_\Omega u^{p+1} + C, \quad p > 0 \tag{3.4}$$

*for all $t \in (0, T_{\max})$.*

P r o o f. Multiplying the third equation in (1.1) by $w^p$, integrating over $\Omega$ and applying Young's inequality, we see that

$$p\int_\Omega w^{p-1}|\nabla w|^2 + \delta\int_\Omega w^{p+1} = \gamma\int_\Omega uw^p \le \frac{\gamma^{p+1}}{\delta^p(p+1)}\int_\Omega u^{p+1} + \frac{p}{p+1}\delta\int_\Omega w^{p+1}.$$

Consequently, we obtain

$$\frac{4p}{p+1}\int_\Omega\left|\nabla w^{\frac{p+1}{2}}\right|^2 \le \frac{\gamma^{p+1}}{\delta^p}\int_\Omega u^{p+1}. \tag{3.5}$$

Since $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow L^{\frac{2}{p+1}}(\Omega)$ for any $p > 0$, by Ehrling's lemma, we have

$$\|\phi\|^2_{L^2(\Omega)} \le C\varepsilon\|\phi\|^2_{W^{1,2}(\Omega)} + C\|\phi\|^2_{L^{\frac{2}{p+1}}(\Omega)}, \quad \phi \in W^{1,2}(\Omega). \tag{3.6}$$

Using (3.2), (3.3) and (3.5), and (3.6) with $\phi = w^{\frac{p+1}{2}}$, we obtain

$$\begin{aligned}\int_\Omega w^{p+1} &\le C\varepsilon\left(\int_\Omega w^{p+1} + \int_\Omega\left|\nabla w^{\frac{p+1}{2}}\right|^2\right) + C\left(\int_\Omega w\right)^{p+1}\\ &\le \varepsilon\int_\Omega u^{p+1} + C.\end{aligned}$$

This completes the proof. □

 

We now give $L^p$-bounds for $u$.

**Lemma 3.3.** *Suppose $f$ satisfies (1.2). Let $1 < p < P$, where*

$$P := \begin{cases} \infty & \text{if } \chi\alpha - \xi\gamma \le \mu, \\ \frac{\chi\alpha-\xi\gamma}{\chi\alpha-\xi\gamma-\mu} & \text{if } \chi\alpha - \xi\gamma > \mu. \end{cases}$$

*Then there exists $C = C(\|u_0\|_{L^p(\Omega)}, p) > 0$ such that*

$$\|u(t)\|_{L^p(\Omega)} \le C, \quad t \in (0, T_{\max}). \tag{3.7}$$

P r o o f . We multiply the first equation in (1.1) by $u^{p-1}$ and integrate by parts over $\Omega$ to obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega u^p &\le \int_\Omega u^{p-1}\Delta u - \chi\int_\Omega u^{p-1}\nabla\cdot(u\nabla v) + \xi\int_\Omega u^{p-1}\nabla\cdot(u\nabla w) + \mu\int_\Omega u^p(1-u) \\ &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \chi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\ &\quad - \xi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla w + \mu\int_\Omega u^p(1-u) \\ &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{p}\int_\Omega u^p(-\chi\Delta v + \xi\Delta w) + \mu\int_\Omega u^p(1-u) \end{aligned}$$

for all $t \in (0, T_{\max})$. From the second and third equations in (1.1), We have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{p-1}{p}\int_\Omega u^p(\xi\delta w + (\chi\alpha-\xi\gamma)u - \chi\beta v) + \mu\int_\Omega u^p(1-u).$$

According to the fact that $v \ge 0$ by Lemma 2.1, we see that

$$\frac{d}{dt}\int_\Omega u^p \le \xi\delta(p-1)\int_\Omega u^p w - (\mu p - (\chi\alpha-\xi\gamma)(p-1))\int_\Omega u^{p+1} + \mu p\int_\Omega u^p.$$

Since $\lambda := \mu p - (\chi\alpha - \xi\gamma)(p-1) > 0$ for $1 < p < P$, by Young's inequality, we have

$$\xi\delta(p-1)\int_\Omega u^p w \le \frac{\lambda}{2}\int_\Omega u^{p+1} + C\int_\Omega w^{p+1}.$$

It follows that

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{\lambda}{2}\int_\Omega u^{p+1} + C\int_\Omega w^{p+1} + \mu p\int_\Omega u^p.$$

We apply (3.4) with $\varepsilon = \frac{\lambda}{4C}$ to have

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{\lambda}{4}\int_\Omega u^{p+1} + \mu p\int_\Omega u^p + C.$$

Using Young's inequality once more, we deduce that

$$(\mu p + 1)\int_\Omega u^p \le \frac{\lambda}{4}\int_\Omega u^{p+1} + \left(\frac{4p}{\lambda(p+1)}\right)^p \frac{|\Omega|}{p+1}(\mu p + 1)^{p+1},$$

which yields

$$\frac{d}{dt}\int_\Omega u^p \le -\int_\Omega u^p + C.$$

Hence (3.7) follows from an application of the Gronwall inequality. □

The following lemma implies that the $L^p$-bounds for all $p > 1$ are still true with some $\mu < \chi\alpha - \xi\gamma$.

**Lemma 3.4.** *Let (1.2) hold. Suppose that $\chi\alpha - \xi\gamma > \mu$ and*

$$\frac{n-2}{n}(\chi\alpha - \xi\gamma) < \mu. \tag{3.8}$$

 

*Then for all $p > 1$, there exists $C = C(\|u_0\|_{L^p(\Omega)}, p) > 0$ such that*

$$\|u(t)\|_{L^p(\Omega)} \le C, \quad t \in (0, T_{\max}). \tag{3.9}$$

P r o o f . We observe that (3.8) is equivalent to $\frac{\chi\alpha-\xi\gamma}{\chi\alpha-\xi\gamma-\mu} > \frac{n}{2}$. By means of Lemma 3.3 we can choose fixed $p_0 \in (\frac{n}{2}, \frac{\chi\alpha-\xi\gamma}{\chi\alpha-\xi\gamma-\mu})$ such that

$$\|u(t)\|_{L^{p_0}(\Omega)} \le C, \quad t \in (0, T_{\max}) \tag{3.10}$$

by Lemma 3.3. Thus, it is sufficient to prove (3.9) for $p > p_0$. Once more testing the first equation in (1.1) by $u^{p-1}$, from Young's inequality we obtain

$$\frac{d}{dt}\int_\Omega u^p \le -\frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + C\int_\Omega u^{p+1} + C. \tag{3.11}$$

It follows from (3.10) that

$$\|u^{\frac{p}{2}}(t)\|_{L^{\frac{2p_0}{p}}(\Omega)} \le C, \quad t \in (0, T_{\max}).$$

Applying the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}\int_\Omega u^{p+1} &= \|u^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\ &\le C_{GN}\big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^a \|u^{\frac{p}{2}}\|_{L^{\frac{2p_0}{p}}(\Omega)}^{1-a} + \|u^{\frac{p}{2}}\|_{L^{\frac{2p_0}{p}}(\Omega)}\big)^{\frac{2(p+1)}{p}} \\ &\le C\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1)}{p}a} + C\end{aligned}$$

where

$$a = \frac{\frac{p}{2p_0} - \frac{p}{2(p+1)}}{\frac{p}{2p_0} - \frac{1}{2} + \frac{1}{n}} \in (0, 1)$$

because $p > p_0 > \frac{n}{2}$. By a straightforward calculation using $p_0 > \frac{n}{2}$, we see that $\frac{2(p+1)}{p}a < 2$, and therefore from Young's inequality, we obtain

$$(C+1)\int_\Omega u^{p+1} \le \frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + C.$$

This, along with (3.11), yields

$$\frac{d}{dt}\int_\Omega u^p \le -\int_\Omega u^{p+1} + C.$$

Consequently, an application of Hölder's inequality gives

$$\frac{d}{dt}\int_\Omega u^p \le -|\Omega|^{-\frac{1}{p}}\left(\int_\Omega u^p\right)^{\frac{p+1}{p}} + C,$$

which implies (3.9). □

Combining Lemma 3.3 and Lemma 3.4 along with semigroup arguments (see [20, 25]), we can prove the following result concerning the existence of global bounded solutions to the problem (1.1).

P r o o f o f T h e o r e m 1 . 1 . It follows from Lemma 2.1 that if for some $\tau \in (0, T_{\max})$ we can find $C(\tau) > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C(\tau), \quad t \in (\tau, T_{\max}),$$

then the solution is global in time. To this end, we fix $\tau \in (0, T_{\max})$ such that $\tau < 1$ and $p > n$. Applying the elliptic $L^p$ estimates, there exists $C > 0$ such that

$$\|v(\cdot, t)\|_{W^{2,p}(\Omega)} \le C\|u(\cdot, t)\|_{L^p(\Omega)} \quad \text{and} \quad \|w(\cdot, t)\|_{W^{2,p}(\Omega)} \le C\|u(\cdot, t)\|_{L^p(\Omega)}, \quad t \in (0, T_{\max}).$$

 

Lemma 3.3, Lemma 3.4 and the embedding $W^{2,p} \hookrightarrow C^1(\bar{\Omega})$ for $p > n$ yield a uniform bounded for $\nabla v$ and $\nabla w$. The variation-of-constants formula associated with the first equation in (1.1) represents $u$ as

$$
\begin{aligned}
u(\cdot,t) = {} & \mathrm{e}^{-(t-\frac{\tau}{2})A} u(\cdot,\frac{\tau}{2}) - \chi \int_{\frac{\tau}{2}}^{t} \mathrm{e}^{-(t-s)A} \nabla \cdot \big(u(\cdot,s)\nabla v(\cdot,s)\big) ds \\
& + \xi \int_{\frac{\tau}{2}}^{t} \mathrm{e}^{-(t-s)A} \nabla \cdot \big(u(\cdot,s)\nabla w(\cdot,s)\big) ds \\
& + \int_{\frac{\tau}{2}}^{t} \mathrm{e}^{-(t-s)A} \left( f(u(\cdot,s)) + u(\cdot,s) \right) ds, \quad t \in (\frac{\tau}{2}, T_{\max}).
\end{aligned}
$$

Since the semigroup $\left(\mathrm{e}^{-tA}\right)_{t\geq 0}$ is order preserving and $u$, $w$ are both nonnegative, we have

$$
\begin{aligned}
u(\cdot,t) \leq {} & \mathrm{e}^{-(t-\frac{\tau}{2})A} u(\cdot,\frac{\tau}{2}) - \chi \int_{\frac{\tau}{2}}^{t} \mathrm{e}^{-(t-s)A} \nabla \cdot \big(u(\cdot,s)\nabla v(\cdot,s)\big) ds \\
& + \xi \int_{\frac{\tau}{2}}^{t} \mathrm{e}^{-(t-s)A} \nabla \cdot \big(u(\cdot,s)\nabla w(\cdot,s)\big) ds \\
& + (\mu+1) \int_{\frac{\tau}{2}}^{t} \mathrm{e}^{-(t-s)A} u(\cdot,s) ds, \\
=: {} & I_1(t) + I_2(t) + I_3(t) + I_4(t), \quad t \in (\frac{\tau}{2}, T_{\max}).
\end{aligned}
\tag{3.12}
$$

For $p > n$, we may choose $\theta \in (\frac{n}{2p}, \frac{1}{2})$ and then $\varepsilon \in (0, \frac{1}{2} - \theta)$. Using (2.1) and (2.2), we deduce that

$$
\begin{aligned}
\|I_1(t)\|_{L^\infty(\Omega)} & \leq C \left\| A^\theta \mathrm{e}^{-(t-\frac{\tau}{2})A} u(\cdot,\frac{\tau}{2}) \right\|_{L^p(\Omega)} \\
& \leq C \left(t - \frac{\tau}{2}\right)^{-\theta} \mathrm{e}^{-\nu(t-\frac{\tau}{2})} \left\| u(\cdot,\frac{\tau}{2}) \right\|_{L^p(\Omega)} \\
& \leq C\tau^{-\theta}, \quad t \in (\tau, T_{\max}).
\end{aligned}
\tag{3.13}
$$

Applying (2.1) and (2.3), it follows that

$$
\begin{aligned}
\|I_2(t)\|_{L^\infty(\Omega)} & \leq C \int_{\frac{\tau}{2}}^{t} \|A^\theta \mathrm{e}^{-(t-s)A} \nabla \cdot \big(u(\cdot,s)\nabla v(\cdot,s)\big)\|_{L^p(\Omega)} ds \\
& \leq C \int_{\frac{\tau}{2}}^{t} (t-s)^{-\theta-\frac{1}{2}-\varepsilon} \mathrm{e}^{-\nu(t-s)} \|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)} ds \\
& \leq C\Gamma\left(\frac{1}{2} - \theta - \varepsilon\right), \quad t \in (\tau, T_{\max}),
\end{aligned}
\tag{3.14}
$$

where $\Gamma(\frac{1}{2} - \theta - \varepsilon)$ is a finite positive constant due to $\varepsilon < \frac{1}{2} - \theta$. Similarly, we obtain

$$
\|I_3(t)\|_{L^\infty(\Omega)} \leq C\Gamma\left(\frac{1}{2} - \theta - \varepsilon\right), \quad t \in (\tau, T_{\max}).
\tag{3.15}
$$

We employ (2.1) and (2.2) once more to get

$$
\begin{aligned}
\|I_4(t)\|_{L^\infty(\Omega)} & \leq C \int_{\frac{\tau}{2}}^{t} \|A^\theta \mathrm{e}^{-(t-s)A} u(\cdot,s)\|_{L^p(\Omega)} ds \\
& \leq C \int_{\frac{\tau}{2}}^{t} (t-s)^{-\theta} \mathrm{e}^{-\nu(t-s)} \|u(\cdot,s)\|_{L^p(\Omega)} ds \\
& \leq C\Gamma(1-\theta), \quad t \in (\tau, T_{\max}),
\end{aligned}
\tag{3.16}
$$

where $\Gamma(1-\theta)$ is a positive constant because of $\theta \in (0, \frac{1}{2})$. Collecting (3.12)-(3.16) gives the desired result. □

 

## 4 Existence of global weak solutions for any $\mu > 0$

In the case $\frac{n-2}{n}(\chi\alpha - \xi\gamma) \geq \mu$, Theorem 1.1 may not be true. Hence the purpose of the present section is to construct the global weak solutions to (1.1) for arbitrary $\mu > 0$. In this section, we assume that $\chi\alpha - \xi\gamma > \mu$ and $n \geq 3$. The main idea of the proof is base on the so-called regularized methods [9, 20].

**Definition 4.1.** *By a global weak solution of (1.1) we mean a triple $(u, v, w)$ of nonnegative functions*

$$(u, v, w) \in \left(L^1_{loc}([0,\infty); W^{1,1}(\Omega))\right)^3$$

*such that*

$$u\nabla v, \quad u\nabla w, \quad \text{and} \quad f(u) \quad \text{belong to } L^1_{loc}([0,\infty); L^1(\Omega))$$

*and*

$$\begin{aligned} -\int_0^\infty \int_\Omega u\partial_t\phi_1 - \int_\Omega u_0\phi_1(0) + \int_0^\infty \int_\Omega \nabla u \cdot \nabla\phi_1 \\ = \chi \int_0^\infty \int_\Omega u\nabla v \cdot \nabla\phi_1 - \xi \int_0^\infty \int_\Omega u\nabla w \cdot \nabla\phi_1 + \int_0^\infty \int_\Omega f(u)\phi_1, \end{aligned} \tag{4.1}$$

$$\int_0^\infty \int_\Omega \nabla v \cdot \nabla\phi_2 = \alpha \int_0^\infty \int_\Omega u\phi_2 - \beta \int_0^\infty \int_\Omega v\phi_2, \tag{4.2}$$

$$\int_0^\infty \int_\Omega \nabla w \cdot \nabla\phi_3 = \gamma \int_0^\infty \int_\Omega u\phi_3 - \delta \int_0^\infty \int_\Omega w\phi_2 \tag{4.3}$$

*for all $\phi_1 \in C_0^\infty(\bar\Omega \times [0,\infty))$, $\phi_2 \in C_0^\infty(\bar\Omega \times [0,\infty))$, and $\phi_3 \in C_0^\infty(\bar\Omega \times [0,\infty))$.*

Motivated by the works [20], for $\varepsilon \in (0, 1)$, we introduce the following approximate problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi\nabla\cdot(u_\varepsilon\nabla v_\varepsilon) + \xi\nabla\cdot(u_\varepsilon\nabla w_\varepsilon) + f_\varepsilon(u_\varepsilon), & x \in \Omega,\ t > 0, \\ 0 = \Delta v_\varepsilon + \alpha u_\varepsilon - \beta v_\varepsilon, & x \in \Omega,\ t > 0, \\ 0 = \Delta w_\varepsilon + \gamma u_\varepsilon - \delta w_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = \frac{\partial w_\varepsilon}{\partial\nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4.4}$$

where $f_\varepsilon(u_\varepsilon) = f(u_\varepsilon) - \varepsilon u_\varepsilon^\kappa$ with fixed

$$\kappa \in \left(2, 1 + \min\left\{2, \frac{\chi\alpha - \xi\gamma}{\chi\alpha - \xi\gamma - \mu}\right\}\right). \tag{4.5}$$

Thanks to the term $-\varepsilon u_\varepsilon^\kappa$, proceeding similarly as in Lemma 3.3 and Theorem 1.1 we obtain for each $\varepsilon \in (0, 1)$ the problem (4.4) has a unique nonnegative global classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ which is uniformly bounded. However, the bound depends on $\varepsilon$. In order to show that the solutions of (4.4) converge to the global weak solutions of (1.1) as $\varepsilon \to 0$, we furthermore establish some $\varepsilon$-independent estimates in the following.

**Lemma 4.1.** *Let (1.2) hold and $p \in (1, \frac{\chi\alpha-\xi\gamma}{\chi\alpha-\xi\gamma-\mu})$. Then for any $\varepsilon \in (0, 1)$ there exists $C = C(\|u_0\|_{L^p(\Omega)}, p) > 0$ such that the solution of (4.4) satisfies*

$$\|u_\varepsilon(t)\|_{L^p(\Omega)} \leq C, \quad t \in (0,\infty) \tag{4.6}$$

*as well as*

$$\int_0^T \int_\Omega u_\varepsilon^{p+1} + \int_0^T \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}| \leq C(T+1) \tag{4.7}$$

*holds for each $T > 0$.*

P r o o f . Since $f_\varepsilon(u_\varepsilon) \leq \mu u_\varepsilon(1 - u_\varepsilon)$, arguing as in the proof of Lemma 3.3 we obtain (4.6) and

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{4(p-1)}{p}\int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \leq -\frac{\lambda}{4}\int_\Omega u_\varepsilon^{p+1} + \mu p \int_\Omega u_\varepsilon^p + C.$$

Using Young's inequality, it follows that

$$\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{4(p-1)}{p}\int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \le -\frac{\lambda}{8}\int_\Omega u_\varepsilon^{p+1} + C.$$

Hence, integrating with respect to $t \in (0, T)$ yields (4.7). □

Based on Lemma 4.1, we have the following estimates.

**Lemma 4.2.** *Assume that $p \in (1, \min\{2, \frac{\chi\alpha-\xi\gamma}{\chi\alpha-\xi\gamma-\mu}\})$ and that $f$ satisfies (1.2). Then for each $T > 0$ there exists $C > 0$ such that for any $\varepsilon \in (0, 1)$ the solution of (4.4) satisfies*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^{\iota(p+1)} \le C \tag{4.8}$$

*for $\iota \in (0, \frac{2}{3}]$, and moreover*

$$\int_0^T \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{p+1}{2}} \le C, \tag{4.9}$$

*and*

$$\int_0^T \int_\Omega |u_\varepsilon \nabla w_\varepsilon|^{\frac{p+1}{2}} \le C. \tag{4.10}$$

*In addition, if $f(u) \ge -C_0 u(1+u)$ with some $C_0 > 0$, then*

$$\int_0^T \int_\Omega |f_\varepsilon(u_\varepsilon)|^{\frac{p+1}{\kappa}} \le C. \tag{4.11}$$

P r o o f . Since $\iota \in (0, \frac{2}{3}]$ and $p < 2$ imply $\iota(p+1) \le \frac{2}{3}(p+1) < 2$, we can use Young's inequality to estimate

$$\begin{aligned} |\nabla u_\varepsilon|^{\iota(p+1)} &= \left|\frac{2}{p}\nabla u_\varepsilon^{\frac{p}{2}} u_\varepsilon^{\frac{2-p}{2}}\right|^{\iota(p+1)} \\ &\le C\left(|\nabla u_\varepsilon^{\frac{p}{2}}|^2 + u_\varepsilon^{\frac{\iota(p+1)(2-p)}{2-\iota(p+1)}}\right). \end{aligned}$$

Noting that

$$\rho := \frac{\iota(p+1)(2-p)}{2-\iota(p+1)} \le p+1$$

due to $\iota \in (0, \frac{2}{3}]$, it follows from Hölder's inequality that

$$\begin{aligned} \int_0^T \int_\Omega |\nabla u_\varepsilon|^{\iota(p+1)} &\le C\left(\int_0^T \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 + \int_0^T \int_\Omega u_\varepsilon^\rho\right) \\ &\le C\left(\int_0^T \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 + (\int_0^T \int_\Omega u_\varepsilon^{p+1})^{\frac{\rho}{p+1}}\right). \end{aligned}$$

This, along with (4.7), yields (4.8).

Next we prove (4.9). An application of the elliptic $L^p$ estimates to the second equation of (4.4) implies

$$\int_0^T \int_\Omega |\nabla v_\varepsilon|^{p+1} \le \int_0^T \|v_\varepsilon\|_{W^{2,p+1}}^{p+1} \le C\int_0^T \int_\Omega u_\varepsilon^{p+1}. \tag{4.12}$$

From Cauchy's inequality, we obtain

$$\int_0^T \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{\frac{p+1}{2}} \le \frac{1}{2}\int_0^T \int_\Omega u_\varepsilon^{p+1} + \frac{1}{2}\int_0^T \int_\Omega |\nabla v_\varepsilon|^{p+1},$$

which gives (4.9) due to (4.12) and (4.7). Inequality (4.10) can be checked in an analogous way.

In view of (4.5), we have

$$|f_\varepsilon(u_\varepsilon)| \le C(1 + u_\varepsilon^\kappa)$$

with $C$ independent of $\varepsilon$. From (4.7) it follows that (4.11). □

Using the estimates shown in this section, we are now prepared to prove the existence of global weak solutions to (1.1) for any $\mu > 0$.

*Proof of Theorem 1.2.* According to Lemmas 4.1 and 4.2, for each $T > 0$ and $\varepsilon \in (0,1)$, there exists $p_0 > 1$ such that

$$\int_0^T \int_\Omega u_\varepsilon^{p_0} \le C, \quad \int_0^T \int_\Omega v_\varepsilon^{p_0} \le C, \quad \int_0^T \int_\Omega w_\varepsilon^{p_0} \le C,$$
$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^{p_0} \le C, \quad \int_0^T \int_\Omega |\nabla v_\varepsilon|^{p_0} \le C, \quad \int_0^T \int_\Omega |\nabla w_\varepsilon|^{p_0} \le C,$$
$$\int_0^T \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{p_0} \le C, \quad \int_0^T \int_\Omega |u_\varepsilon \nabla w_\varepsilon|^{p_0} \le C, \quad \int_0^T \int_\Omega |f_\varepsilon(u_\varepsilon)|^{p_0} \le C.$$

Hence, we have

$$\|u_\varepsilon\|_{L^{p_0}((0,T);W^{1,p_0}(\Omega))} \le C.$$

Then multiplying the first equation in (4.4) by $\varphi \in C_0^\infty(\Omega)$ and integrating by parts over $\Omega$, we obtain

$$\begin{aligned}
\left|\int_0^T \int_\Omega u_{\varepsilon t}\varphi\right| &= \left| -\int_0^T \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi + \chi \int_0^T \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi \right. \\
&\quad \left. -\xi \int_0^T \int_\Omega u_\varepsilon \nabla w_\varepsilon \cdot \nabla\varphi + \int_0^T \int_\Omega f_\varepsilon(u_\varepsilon)\varphi \right| \\
&\le \left\{ \left(\int_0^T \int_\Omega |\nabla u_\varepsilon|^{p_0}\right)^{\frac{1}{p_0}} + \chi \left(\int_0^T \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^{p_0}\right)^{\frac{1}{p_0}} \right. \\
&\quad \left. +\xi \left(\int_0^T \int_\Omega |u_\varepsilon \nabla w_\varepsilon|^{p_0}\right)^{\frac{1}{p_0}} \right\} \left(\int_0^T \int_\Omega |\nabla\varphi|^{p_0'}\right)^{\frac{1}{p_0'}} \\
&\quad + \left(\int_0^T \int_\Omega |f_\varepsilon(u_\varepsilon)|^{p_0}\right)^{\frac{1}{p_0}} \left(\int_0^T \int_\Omega \varphi^{p_0'}\right)^{\frac{1}{p_0'}} \\
&\le C\|\varphi\|_{L^{p_0'}((0,T);W^{1,p_0'}(\Omega))},
\end{aligned}$$

which implies

$$\|u_{\varepsilon t}\|_{L^{p_0'}((0,T);(W^{1,p_0'}(\Omega))^*)} \le C.$$

By means of the Aubin-Lions Lemma, we may pick a subsequence $\varepsilon = \varepsilon_j \to 0$ such that

$$\begin{cases}
u_\varepsilon \to u & \text{in } L^{p_0}((0,T);L^{p_0}(\Omega)), \\
\nabla u_\varepsilon \rightharpoonup \nabla u & \text{in } L^{p_0}((0,T);L^{p_0}(\Omega)), \\
u_\varepsilon \nabla v_\varepsilon \rightharpoonup m & \text{in } L^{p_0}((0,T);L^{p_0}(\Omega)), \\
u_\varepsilon \nabla w_\varepsilon \rightharpoonup \bar{m} & \text{in } L^{p_0}((0,T);L^{p_0}(\Omega)), \\
f_\varepsilon(u_\varepsilon) \rightharpoonup z & \text{in } L^{p_0}((0,T);L^{p_0}(\Omega)), \\
u_\varepsilon \to u & \text{a.e. } \Omega \times (0,T).
\end{cases} \tag{4.13}$$

Using elliptic $L^p$ estimates to the second and third equations of (4.4) implies

$$v_\varepsilon \to v, \quad w_\varepsilon \to w \quad \text{in } L^{p_0}((0,T);W^{2,p_0}(\Omega)).$$

Then, passing to a subsequence we get

$$\nabla v_\varepsilon \to \nabla v, \quad \nabla w_\varepsilon \to \nabla w \quad \text{a.e. } \Omega \times (0,T).$$

This, along with (4.13), yields

$$u_\varepsilon \nabla v_\varepsilon \to u\nabla v, \quad u_\varepsilon \nabla w_\varepsilon \to u\nabla w \quad \text{a.e. } \Omega \times (0,\infty).$$

In view of [9, Lemma 1.3], it follows that

$$m = u\nabla v, \quad \bar{m} = u\nabla w. \tag{4.14}$$

According to the continuity of $f_\varepsilon$ and (4.13), we immediately obtain

$$z = f(u). \tag{4.15}$$

Consequently, according to (4.13)-(4.15), we pass to the limit in the identities

$$\begin{aligned}
&-\int_0^\infty \int_\Omega u_\varepsilon \partial_t \phi_1 - \int_\Omega u_0 \phi_1(0) + \int_0^\infty \int_\Omega \nabla u_\varepsilon \cdot \nabla \phi_1 \\
&\quad = \chi \int_0^\infty \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla \phi_1 - \xi \int_0^\infty \int_\Omega u_\varepsilon \nabla w_\varepsilon \cdot \nabla \phi_1 + \int_0^\infty \int_\Omega f_\varepsilon(u_\varepsilon)\phi_1, \\
&\int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla \phi_2 = \alpha \int_0^\infty \int_\Omega u_\varepsilon \phi_2 - \beta \int_0^\infty \int_\Omega v_\varepsilon \phi_2, \\
&\int_0^\infty \int_\Omega \nabla w_\varepsilon \cdot \nabla \phi_3 = \gamma \int_0^\infty \int_\Omega u_\varepsilon \phi_3 - \delta \int_0^\infty \int_\Omega w_\varepsilon \phi_2
\end{aligned}$$

with $\phi_1 \in C_0^\infty(\bar{\Omega} \times [0,\infty))$, $\phi_2 \in C_0^\infty(\bar{\Omega} \times [0,\infty))$, and $\phi_3 \in C_0^\infty(\bar{\Omega} \times [0,\infty))$ to arrive (4.1)-(4.3). This completes the proof of Theorem 1.2. □

## 5 Asymptotic behavior

According to the idea in [18], we study the large time behavior of solutions of (1.1) with $f(u) = \mu u(1-u)$. Throughout this section we assume $\mu > 2\chi\alpha$. Theorem 1.1 guarantees that the solution of (1.1) is global and uniformly bounded in this case. Therefore, we can assume that

$$\overline{u} := \limsup_{t\to\infty} \big(\max_{x\in\bar{\Omega}} u(x,t)\big)$$

and

$$\underline{u} := \liminf_{t\to\infty} \big(\min_{x\in\bar{\Omega}} u(x,t)\big),$$

then, it is easy to see that $0 \le \underline{u} \le \overline{u}$. Due to the following lemma, in order to prove Theorem 1.3, it is sufficient to verify $\underline{u} = \overline{u}$.

**Lemma 5.1.** *For each $t \in (0,\infty)$, we have*

$$\alpha \min_{y\in\bar{\Omega}} u(y,t) \le \beta v(x,t) \le \alpha \max_{y\in\bar{\Omega}} u(y,t) \tag{5.1}$$

*and*

$$\gamma \min_{y\in\bar{\Omega}} u(y,t) \le \delta w(x,t) \le \gamma \max_{y\in\bar{\Omega}} u(y,t) \tag{5.2}$$

*for all $x \in \bar{\Omega}$.*

Proof. Let $\varphi \in C^2(\bar{\Omega})$ satisfy $\frac{\partial\varphi}{\partial\nu} < 0$ on $\partial\Omega$. We denote $z := v(\cdot,t) + \varepsilon\varphi$. If $z$ achieves its maximum at some point $x_0 \in \bar{\Omega}$, we necessarily have $x_0 \in \Omega$. Hence, $\Delta z(x_0) \le 0$. Since

$$\beta z = \Delta z + \alpha u - \varepsilon(\Delta\varphi - \beta\varphi),$$

we conclude at the point $x_0$

$$\begin{aligned}
\beta z(x) \le \beta z(x_0) &\le \alpha u(x_0) - \varepsilon(\Delta\varphi - \beta\varphi)(x_0) \\
&\le \alpha \max_{y\in\bar{\Omega}} u(y,t) + \varepsilon \max_{y\in\bar{\Omega}} |\Delta\varphi - \beta\varphi|.
\end{aligned}$$

By letting $\varepsilon \to 0$, we get

$$\beta v(x,t) \le \alpha \max_{y\in\bar{\Omega}} u(y,t).$$

By similar arguments, we obtain the left inequality of (5.1) and the inequality (5.2). □

**Corollary 5.2.** *For each $\varepsilon > 0$ there exists $t_\varepsilon$ such that*

$$\underline{u} - \varepsilon \le u(x,t) \le \overline{u} + \varepsilon \tag{5.3}$$

 

*and*

$$\alpha(\underline{u}-\varepsilon) \le \beta v(x,t) \le \alpha(\overline{u}+\varepsilon) \tag{5.4}$$

*as well as*

$$\gamma(\underline{u}-\varepsilon) \le \delta w(x,t) \le \gamma(\overline{u}+\varepsilon) \tag{5.5}$$

*for all* $(x,t) \in \bar{\Omega} \times (t_\varepsilon, \infty)$.

Proof. This is an immediate consequence of Lemma 5.1. □

Based on Corollary 5.2, we can show the following lemma, which makes precise some relations between $\underline{u}$ and $\overline{u}$.

**Lemma 5.3.** *Let* $\mu > 2\chi\alpha$. *Then we have* $\underline{u} = \overline{u} = 1$.

Proof. We define a second-order elliptic differential operator by

$$\mathcal{L}u := \Delta u + (-\chi \nabla v(x,t) + \xi \nabla w(x,t)) \cdot \nabla u.$$

The first equation in (1.1) then reads as

$$u_t - \mathcal{L}u = -\chi u \Delta v + \xi u \Delta w + \mu u(1-u).$$

Using the second and third equations, we obtain

$$u_t - \mathcal{L}u = u\left\{\mu - (\mu - (\chi\alpha - \xi\gamma))u - \chi\beta v + \xi\delta w\right\}. \tag{5.6}$$

For fixed $\varepsilon > 0$, there exists $t_\varepsilon$ taken from Lemma 5.2 we obtain by (5.4) and (5.5) that

$$-\chi\beta v \le -\chi\alpha(\underline{u}-\varepsilon)$$

and

$$\xi\delta w \le \xi\gamma(\overline{u}+\varepsilon)$$

in $\Omega \times (t_\varepsilon, \infty)$. Therefore, (5.6) yields

$$u_t - \mathcal{L}u \le u\left\{\mu - (\mu - (\chi\alpha - \xi\gamma))u - \chi\alpha(\underline{u}-\varepsilon) + \xi\gamma(\overline{u}+\varepsilon)\right\}.$$

A parabolic comparison argument shows that

$$u(x,t) \le y_1(t) \tag{5.7}$$

for all $(x,t) \in \bar{\Omega} \times (t_\varepsilon, \infty)$, where $y_1$ is the solution of

$$\begin{cases} y_1' = y_1\left\{\mu - (\mu - (\chi\alpha - \xi\gamma))y_1 - \chi\alpha(\underline{u}-\varepsilon) + \xi\gamma(\overline{u}+\varepsilon)\right\}, \\ y_1(t_\varepsilon) = \max_{x\in\bar{\Omega}} u(x,t_\varepsilon). \end{cases}$$

Since $u(x,t_\varepsilon) > 0$ in $\bar{\Omega}$ by the strong maximum principle, we have

$$y_1 \to \max\left\{0, \frac{\mu - \chi\alpha(\underline{u}-\varepsilon) + \xi\gamma(\overline{u}+\varepsilon)}{\mu - (\chi\alpha - \xi\gamma)}\right\}$$

as $t \to \infty$. From (5.7), it follows that

$$\overline{u} \le \max\left\{0, \frac{\mu - \chi\alpha(\underline{u}-\varepsilon) + \xi\gamma(\overline{u}+\varepsilon)}{\mu - (\chi\alpha - \xi\gamma)}\right\},$$

then upon letting $\varepsilon \to 0$, we obtain

$$(\mu - (\chi\alpha - \xi\gamma))\,\overline{u} \le (\mu - \chi\alpha\underline{u} + \xi\gamma\overline{u})_+ . \tag{5.8}$$

We once more use (5.4) and (5.5) to obtain

$$-\chi\beta v \ge -\chi\alpha(\overline{u}+\varepsilon)$$

and

$$\xi\delta w \ge \xi\gamma(\underline{u}-\varepsilon)$$

in $\Omega \times (t_\varepsilon, \infty)$. This, along with (5.6), yields

$$u_t - \mathcal{L}u \ge u\left\{\mu - (\mu - (\chi\alpha - \xi\gamma))u - \chi\alpha(\overline{u}+\varepsilon) + \xi\gamma(\underline{u}-\varepsilon)\right\}.$$

Let $y_2$ be the solution of the initial-value problem

$$\begin{cases} y_2' = y_2\left\{\mu - (\mu - (\chi\alpha - \xi\gamma))y_2 - \chi\alpha(\overline{u} + \varepsilon) + \xi\gamma(\underline{u} - \varepsilon)\right\}, \\ y_2(t_\varepsilon) = \min_{x\in\bar{\Omega}} u(x, t_\varepsilon). \end{cases}$$

It follows from the comparison principle that $u(x,t) \geq y_2$ in $(x,t) \in \bar{\Omega} \times (t_\varepsilon, \infty)$. Similarly as for (5.8), we obtain

$$(\mu - (\chi\alpha - \xi\gamma))\underline{u} \geq \mu - \chi\alpha\overline{u} + \xi\gamma\underline{u}. \tag{5.9}$$

We claim that $\mu > \chi\alpha\underline{u} - \xi\gamma\overline{u}$. In fact, assume the contrary. In view of (5.8), we have $\overline{u} = \underline{u} = 0$, this is a contraction to (5.9). Hence, we get

$$(\mu - (\chi\alpha - \xi\gamma))\overline{u} \leq \mu - \chi\alpha\underline{u} + \xi\gamma\overline{u}. \tag{5.10}$$

We subtract (5.10) from (5.9) to obtain

$$(\mu - (\chi\alpha - \xi\gamma))(\overline{u} - \underline{u}) \leq \chi\alpha(\overline{u} - \underline{u}) + \xi\gamma(\overline{u} - \underline{u}).$$

Since $\mu > 2\chi\alpha$ according to our assumption, we obtain $\overline{u} = \underline{u}$. From (5.10), we have $\overline{u} \leq 1$, and by (5.9) we see $\underline{u} \geq 1$. Then it follows that $\overline{u} = \underline{u} = 1$. □

*Proof of Theorem 1.3.* In view of Lemma 5.1, Lemma 5.3 and the definitions of $\overline{u}$, $\underline{u}$, we complete the proof of Theorem 1.3. □

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