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Stochastic Volterra equations in Banach spaces and stochastic partial differential equation

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Abstract

In this paper, we study the existence-uniqueness and large deviation estimate for stochastic Volterra integral equations with singular kernels in 2-smooth Banach spaces. Then we apply them to a large class of semilinear stochastic partial differential equations (SPDE), and obtain the existence of unique maximal strong solutions (in the sense of SDE and PDE) under local Lipschitz conditions. Moreover, stochastic Navier–Stokes equations are also investigated.

Keywords: Stochastic Volterra equation; Large deviation; Stochastic Navier–Stokes equation

Contents

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1. Introduction

The aims of this paper are three folds: First of all, we prove the existence and uniqueness of solutions with continuous paths for stochastic Volterra integral equations with singular kernels in a 2-smooth Banach space. Secondly, the large deviation principles (abbrev. LDP) of Freidlin–Wentzell's type for stochastic Volterra equations are established under small perturbations of multiplicative noises. Thirdly, we apply them to several classes of semilinear stochastic partial differential equations (abbrev. SPDE). Compared with the well-known results on SPDEs, the main contributions of the present paper are that we can prove the existence and uniqueness of strong solutions (in the sense of SDE and PDE) for SPDEs, and give a unified treatment for the LDPs to a large class of SPDEs.

In finite-dimensional space, stochastic Volterra integral equations with regular kernels and driven by Brownian motions were first studied by Berger and Mizel [3]. Later, Protter [52] studied stochastic Volterra equations driven by general semimartingales. Using the Skorohod integral, Pardoux and Protter [47] also investigated stochastic Volterra equations with anticipating coefficients. The study of stochastic Volterra equations with singular kernels can be found in [14,16, 65,36,44], etc. Recently, the present author [68] studied the approximation of Euler's type and the LDP of Freidlin–Wentzell's type for stochastic Volterra equations with singular kernels. In particular, the kernels in [68] can be used to deal with fractional Brownian motion kernels as well as fractional order integral kernels. The study of LDP for stochastic Volterra equations is also referred to [44,36].

Since the work of Freidlin and Wentzell [21], the theory of small perturbation large deviations for stochastic differential equations (abbrev. SDE) has been studied extensively (cf. [2,62], etc.). In the classical method, to establish such an LDP for SDE, one usually needs to discretize the time variable and then prove various necessary exponential continuity and tightness for approximation equations in different spaces by using comparison principle. However, such verifications would become rather complicated and even impossible in some cases, e.g., stochastic evolution equations with multiplicative noises.

Recently, Dupuis and Ellis [19] systematically developed a weak convergence approach to the theory of large deviation. The central idea is to prove some variational representation formula for the Laplace transform of bounded continuous functionals, which will lead to proving a Laplace principle which is equivalent to the LDP. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué and Dupuis [5] and Budhiraja and Dupuis [10]. A simplified proof was given by the present author [67]. This variational representation has already been proved to be very effective for various finite and infinite-dimensional stochastic dynamical systems even with irregular coefficients (cf. [54,55,11,68,56], etc.). One of the main advantages of this argument is that one only needs to make some simple moment estimates (see Section 4 below).

On the other hand, it is well known that in the deterministic case, many PDE problems of parabolic and hyperbolic types can be written as Volterra type integral equations in Banach spaces by using the corresponding semigroup and the variation-of-constants formula (cf. [22, 28,48]). An obvious merit of this procedure is that the unbounded operators in PDEs no longer appear and the analysis is entirely analogous to the ODE case. Thus, one naturally expects to take the same advantages for SPDEs in Banach spaces. However, it is not all Banach spaces in which stochastic integrals are well defined. One can only work in a class of 2-smooth Banach spaces. The definition of stochastic integrals in 2-smooth Banach spaces and related properties such as Burkholder–Davis–Gundy's (abbrev. BDG) inequality, Girsanov's theorem, stochastic Fubini's theorem and the distribution of stochastic integrals can be found in [43,6,7,45], etc. Thus, similar to the deterministic case, we can develop a parallel theory in 2-smooth Banach spaces for SPDEs. It should be emphasized that besides the usual SPDEs driven by multiplicative Brownian noises, a class of stochastic evolutionary integral equations appearing in viscoelasticity and heat conduction with memory (cf. [53]) can also be written as abstract stochastic Volterra equations in Banach spaces.

In the past three decades, the theory of general SPDEs has been developed extensively by numerous authors mainly based on two different approaches: semigroup method based on the variation-of-constants formula (cf. [64,15,6–8,66], etc.) and variation method based on Galerkin's finite-dimensional approximation (cf. [46,35,58,34,41,51,69,26], etc.). A new regularization method is given in [71]. An overview for the classification and applications of SPDEs are referred to the recent book of Kotelenez [33]. In the author's knowledge, most of the well-known results are primarily concentrated on the mild or weak solutions, even measure-valued solutions. Such notions of solutions naturally appear in the study of SPDEs driven by the space–time white noises, and in this case one cannot obtain any differentiability of solutions with respect to the spatial variable.

Nevertheless, when one considers an SPDE driven by the spatial regular and time white noises, it is reasonable to require the existence of spatial regular solutions or classical solutions in the sense of PDE. For linear SPDEs, such regular solutions are relatively easy and well known (cf. [35,58,20], etc.). However, for non-linear SPDEs, there seems to be few results (cf. [34,39,67, 71]). A major difficulty to prove the spatial regularity of solutions is that one cannot use the usual bootstrap method in the theory of PDE since there is no differentiability of solutions with respect to the time variable. The present author [67] solved this problem by using a non-linear interpolation result due to Tartar [63]. Obviously, for the regularity theory of SPDEs, by using Sobolev's embedding theorem (cf. [1]), it is natural to consider the L^p-solutions of SPDEs. This is also why we need to work in 2-smooth Banach spaces. It should be noticed that the L^p-theory for SPDEs has been established in [6–8,34,17,18,66], etc. But, there are few results to deal with the L^p-strong solution in the sense of PDE. In the present paper, we shall prove a general result about the existence of strong solutions in the sense of both SDE and PDE (see Theorem 6.6 below).

We now describe the structure of this paper: In Section 2, we prepare some preliminaries for later use, and divide it into three subsections. In Section 2.1, we prove a Gronwall's lemma of

Volterra type under rather weak assumptions on kernel functions. Moreover, two simple examples are provided to show this lemma. In Section 2.2, we recall the Itô integral in 2-smooth Banach spaces and Burkholder–Davies–Gundy's inequality as well as Kolmogorov's continuity criterion of random fields in random intervals. In Section 2.3, we recall a criterion of Laplace principle established by Budhiraja and Dupuis [5,10] (see also [70]).

In Section 3, using the Gronwall inequality of Volterra type in Section 2.1, we first prove the existence and uniqueness of solutions for stochastic Volterra equations in 2-smooth Banach spaces under global Lipschitz conditions and singular kernels. Next, in Section 3.2, we study the regularity of solutions under slightly stronger assumptions on kernels. Moreover, a BDG type of inequality for stochastic Volterra type integral is also proved. In Section 3.3, employing the usual localizing method, we prove the existence of a unique maximal solution for stochastic Volterra equation under local Lipschitz conditions. Lastly, in Section 3.4, we discuss the continuous dependence of solutions with respect to the coefficients.

In Section 4, using the weak convergence method, we prove the Freidlin–Wentzell large deviation principle for the small perturbations of stochastic Volterra equations under a compactness assumption and some uniform non-explosion conditions for the controlled equations. We also refer to [38,56] for the application of weak convergence approach in the LDPs of stochastic evolution equations (the case of evolution triple). In the proof of Section 4, we need to use the Yamada–Watanabe Theorem in infinite-dimensional space, which has been established by Ondreját [45] (see also [57] for the case of evolution triple). We want to say that although Ondreját only considered the case of convolution semigroup, their proofs are also adapted to more general stochastic Volterra equations. Moreover, since we are considering the path continuous solution, the proof in [45] can be simplified.

In Section 5, a simple application in a class of semilinear stochastic evolutionary integral equations is presented, which has been studied in [13,4,31], etc., for additive noises. Such type of stochastic evolution equations appears in viscoelasticity, heat conduction in materials with memory, and electrodynamics with memory [53].

In Section 6, we apply our general results to a large class of semilinear stochastic evolution equations driven by multiplicative Brownian noises. A basic result in semigroup theory states that if f is a Hölder continuous function in Banach space $\mathbb{X}$, then

$$t \mapsto \int_0^t \mathfrak{T}_{t-s} f(s)\,\mathrm{d}s \text{ is continuous in } \mathscr{D}(\mathfrak{L}),$$

where $\mathfrak{T}_t$ is an analytic semigroup and $\mathfrak{L}$ is the generator of $\mathfrak{T}_t$. We will use this result to prove the existence of strong solutions (in the sense of PDE) for semilinear SPDEs. The corresponding LDPs are also obtained (see also [61,49,11,56,38], etc., for the study of LDPs of stochastic evolution equations). More applications can be found in an uncompressed version [72].

In Section 7, we prove the existence and uniqueness of local L^p-strong solutions for stochastic Navier–Stokes equations (SNSE) in any dimensional case. In the two-dimensional case, we also obtain the non-explosion of solutions. Moreover, the LDPs for two-dimensional SNSEs are established in the case of both Dirichlet boundary and periodic boundary. We remark that the L^p-solutions for SNSEs have been studied by Brzeźniak and Peszat [9] (bounded domain) and Mikulevicius and Rozovskii [40] (the whole space). The large deviation result for two-dimensional SNSEs with additive noise was proved by Chang [12] using Girsanov's transformation. In [60], the authors also used the weak convergence method to prove the large deviation estimate for two-dimensional SNSEs with multiplicative noises. But, they worked in the L^2-space and only for weak solutions. Here we can do it for strong solutions in L^p-space.

We conclude this introduction by making the following CONVENTION: Throughout this paper, the letter C with or without subscripts will denote a positive constant, whose value may change from one place to another. Moreover, we also use the notation $E_1 \preccurlyeq E_2$ to denote $E_1 \leqslant C \cdot E_2$, where $C > 0$ is an unimportant constant.

2. Preliminaries

2.1. Gronwall's inequality of Volterra type

Let $\Delta := \{(t,s) \in \mathbb{R}_+^2 \colon s \leqslant t\}$. We first recall the following result due to Gripenberg [27, Theorem 1 and p. 88].

Lemma 2.1. *Let $\kappa : \Delta \to \mathbb{R}_+$ be a measurable function. Assume that for any $T > 0$,*

$$t \mapsto \int_0^t \kappa(t,s)\,\mathrm{d}s \in L^\infty(0,T)$$

and

$$\limsup_{\epsilon \downarrow 0} \left\| \int_{\cdot}^{\cdot+\epsilon} \kappa(\cdot+\epsilon, s)\,\mathrm{d}s \right\|_{L^\infty(0,T)} < 1.$$

Define

$$r_1(t,s) := \kappa(t,s), \qquad r_{n+1}(t,s) := \int_s^t \kappa(t,u) r_n(u,s)\,\mathrm{d}u, \quad n \in \mathbb{N}. \tag{2.1}$$

Then for any $T > 0$, there exist constants $C_T > 0$ and $\gamma \in (0,1)$ such that

$$\left\| \int_0^{\cdot} r_n(\cdot, s)\,\mathrm{d}s \right\|_{L^\infty(0,T)} \leqslant C_T n \gamma^n, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

In particular, the series

$$r(t,s) := \sum_{n=1}^{\infty} r_n(t,s) \tag{2.3}$$

converges for almost all $(t,s) \in \Delta$, and

$$r(t,s)-k(t,s)=\int_s^t k(t,u)r(u,s)\,\mathrm{d}u=\int_s^t r(t,u)k(u,s)\,\mathrm{d}u \tag{2.4}$$

and for any $T>0$*,*

$$t\mapsto\int_0^t r(t,s)\,\mathrm{d}s\in L^\infty(0,T). \tag{2.5}$$

The function r defined by (2.3) is called the resolvent of κ. All the functions κ in Lemma 2.1 will be denoted by $\mathscr{K}$. In what follows, we shall denote by $\mathscr{K}_0$ the subclass of $\mathscr{K}$ with the property that

$$\limsup_{\epsilon\downarrow 0}\left\|\int_{\cdot}^{\cdot+\epsilon}\kappa(\cdot+\epsilon,s)\,\mathrm{d}s\right\|_{L^\infty(0,T)}=0.$$

We also denote by $\mathscr{K}_{>1}$ the set of all nonnegative measurable functions κ on Δ with the property that for any $T>0$ and some $\beta=\beta(T)>1$,

$$t\mapsto\int_0^t \kappa^\beta(t,s)\,\mathrm{d}s\in L^\infty(0,T). \tag{2.6}$$

It is clear that $\mathscr{K}_{>1}\subset\mathscr{K}_0\subset\mathscr{K}$ and for any $\kappa_1,\kappa_2\in\mathscr{K}_0$ (resp. $\mathscr{K}_{>1}$) and $C_1,C_2\geqslant 0$,

$$C_1\kappa_1+C_2\kappa_2\in\mathscr{K}_0\quad(\text{resp. }\mathscr{K}_{>1}).$$

Let $0\leqslant h\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. If $\kappa(t,s)=h(s)$, then $\kappa\in\mathscr{K}_0$ and

$$r(t,s)=h(s)\exp\left\{\int_s^t h(u)\,\mathrm{d}u\right\};$$

if $\kappa(t,s)=h(t-s)$, then $\kappa\in\mathscr{K}_0$ and

$$r(t,s)=a(t-s):=\sum_{n=1}^\infty a_n(t-s), \tag{2.7}$$

where

$$a_1(t)=h(t),\qquad a_{n+1}(t):=\int_0^t h(t-s)a_n(s)\,\mathrm{d}s.$$

When $0 \leqslant h \in L^1(\mathbb{R}_+)$, a classical result due to Paley and Wiener (cf. [42, p. 207, Theorem 5.2]) says that

$$a \in L^1(\mathbb{R}_+) \quad \text{if and only if} \quad \int_0^\infty h(t)\,\mathrm{d}t < 1. \tag{2.8}$$

In this case, $\hat{a}(s) = \hat{h}(s)/(1 - \hat{h}(s))$, where the hat denotes the Laplace transform, i.e.:

$$\hat{h}(s) := \int_0^\infty e^{-st} h(t)\,\mathrm{d}t, \quad s \geqslant 0.$$

We want to say that (2.8) is useful in the study of large time asymptotic behavior of solutions for Volterra equations. An important extension to nonintegrable convolution kernel can be found in [59,29] (see also [27]). A simple example is provided in Example 3.2 below.

We now prove the following Gronwall's lemma of Volterra type (see also [28, Lemma 7.1.1] for a case of special convolution kernel).

Lemma 2.2. *Let $\kappa \in \mathscr{K}$ and r_n and r be defined respectively by* (2.1) *and* (2.3). *Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ be two measurable functions satisfying that for any $T > 0$ and some $n \in \mathbb{N}$,*

$$t \mapsto \int_0^t r_n(t,s) f(s)\,\mathrm{d}s \in L^\infty(0,T) \tag{2.9}$$

and for almost all $t \in (0,\infty)$,

$$\int_0^t r(t,s) g(s)\,\mathrm{d}s < +\infty. \tag{2.10}$$

If for almost all $t \in (0,\infty)$,

$$f(t) \leqslant g(t) + \int_0^t \kappa(t,s) f(s)\,\mathrm{d}s, \tag{2.11}$$

then for almost all $t \in (0,\infty)$,

$$f(t) \leqslant g(t) + \int_0^t r(t,s) g(s)\,\mathrm{d}s. \tag{2.12}$$

Proof. First of all, if we define

$$h(t) := g(t) + \int_0^t r(t,s)g(s)\,\mathrm{d}s,$$

then by (2.4) and (2.10)

$$h(t) = g(t) + \int_0^t \kappa(t,s)h(s)\,\mathrm{d}s \quad \text{for a.a. } t \in (0,\infty).$$

Thus, by (2.11) we have

$$f(t) - h(t) \leqslant \int_0^t \kappa(t,s)\big(f(s) - h(s)\big)\,\mathrm{d}s \quad \text{for a.a. } t \in (0,\infty). \tag{2.13}$$

Set $\tilde{f}(t) := f(t) - h(t)$ and define

$$\tilde{f}^*(t) := \operatorname*{ess\,sup}_{s\in[0,t]} \tilde{f}(s), \quad t > 0,$$

and

$$\tau_0 := \inf\big\{t > 0\colon\ \tilde{f}^*(t) > 0\big\}.$$

Clearly, $t \mapsto \tilde{f}^*(t)$ is non-decreasing and

$$\tilde{f}(t) \leqslant 0 \quad \text{for a.a. } t \in [0,\tau_0). \tag{2.14}$$

We want to prove that

$$\tau_0 = +\infty.$$

Iterating inequality (2.13), we have

$$\tilde{f}(t) \leqslant \int_0^t r_n(t,s)\tilde{f}(s)\,\mathrm{d}s \leqslant \int_0^t r_n(t,s) f(s)\,\mathrm{d}s, \quad \forall n \in \mathbb{N}.$$

By (2.9), one knows that $\tilde{f}^*(T) < +\infty$ for any $T > 0$. Moreover, for almost all $t > 0$,

$$\tilde{f}(t) \overset{(2.14)}{\leqslant} \int_{\tau_0}^t r_n(t,s)\tilde{f}(s)\,\mathrm{d}s \leqslant \tilde{f}^*(t)\int_{\tau_0}^t r_n(t,s)\,\mathrm{d}s, \quad \forall n \in \mathbb{N}.$$

Suppose $\tau_0 < +\infty$. Then, for any $T > \tau_0$, we have

$$0 < \tilde{f}^*(T) \leqslant \tilde{f}^*(T) \cdot \left\| \int_{\tau_0}^{\cdot} r_n(\cdot, s)\,\mathrm{d}s \right\|_{L^\infty(\tau_0,T)} \xrightarrow{(2.2)} 0$$

as $n \to \infty$, which is impossible. So, $\tau_0 = +\infty$. $\square$

The following two examples show that (2.12) is sensitive to $\kappa \in \mathscr{K}$.

Example 2.3. For $C_0 > 0$, set

$$\kappa_{C_0}(t,s) := \frac{C_0}{\sqrt{t^2 - s^2}}, \quad s < t.$$

It is clear that

$$\int_s^t \kappa_{C_0}(t,u)\,\mathrm{d}u = C_0\big((\pi/2) - \arcsin(s/t)\big).$$

From this, one sees that

$$\begin{cases} \kappa_{C_0} \notin \mathscr{K}, & \text{if } C_0 \geqslant 2/\pi; \\ \kappa_{C_0} \in \mathscr{K} \cap \mathscr{K}_0^c, & \text{if } 0 < C_0 < 2/\pi. \end{cases}$$

Consider the following Volterra equation

$$x(t) = \int_0^t \kappa_{C_0}(t,s)x(s)\,\mathrm{d}s, \quad t \geqslant 0.$$

If $C_0 = 1$, there are at least two solutions $x(t) \equiv 0$ and $x(t) = t$; if $C_0 = \frac{2}{\pi}$, there are infinitely many solutions $x(t) \equiv$ constant; if $0 < C_0 < 2/\pi$, by Lemma 2.2 there is only one solution $x(t) \equiv 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$.

Example 2.4. For $C_0 > 0$ and $\alpha, \beta \in [0,1)$, set

$$\kappa^{\alpha,\beta}_{C_0}(t,s) := \frac{C_0}{(t-s)^\alpha s^\beta}, \quad s < t.$$

It is clear that

$$\int_u^t \kappa^{\alpha,\beta}_{C_0}(t,s)\,\mathrm{d}s = C_0 t^{1-\alpha-\beta} \int_{u/t}^1 \frac{1}{(1-s)^\alpha s^\beta}\,\mathrm{d}s. \tag{2.15}$$

From this, one sees that

$$\begin{cases} \kappa_{C_0}^{\alpha,\beta} \notin \mathscr{K}, & \text{if } \alpha+\beta>1 \text{ and } C_0>0; \\ \kappa_{C_0}^{\alpha,\beta} \notin \mathscr{K}, & \text{if } \alpha+\beta=1 \text{ and } C_0 \geqslant \int_0^1 \frac{1}{(1-s)^\alpha s^\beta}\,\mathrm{d}s; \\ \kappa_{C_0}^{\alpha,\beta} \in \mathscr{K} \cap \mathscr{K}_0^c, & \text{if } \alpha+\beta=1 \text{ and } C_0 < \int_0^1 \frac{1}{(1-s)^\alpha s^\beta}\,\mathrm{d}s; \\ \kappa_{C_0}^{\alpha,\beta} \in \mathscr{K}_{>1}, & \text{if } \alpha+\beta<1 \text{ and } C_0>0. \end{cases}$$

Consider the following Volterra equation

$$x(t) = \int_0^t k_{C_0}^{\alpha,\beta}(t,s)x(s)\,\mathrm{d}s, \quad t \geqslant 0.$$

If $\alpha+\beta<1$, by Lemma 2.2 there is only one solution $x(t) \equiv 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$; if $\alpha=\beta=C_0=1/2$, there are at least two solutions $x(t)\equiv 0$ and $x(t)=\sqrt{t}$.

2.2. Itô's integral in 2-smooth Banach spaces

Throughout this paper, we shall fix a stochastic basis $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$, i.e., a complete probability space with a family of right-continuous filterations. In what follows, without special declarations, all expectations $\mathbb{E}$ are taken with respect to the probability measure P.

Let $\{W^k(t)\colon t \geqslant 0,\ k \in \mathbb{N}\}$ be a sequence of independent one-dimensional standard Brownian motions on $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$. Let l^2 be the usual Hilbert space of all square summable real number sequences, $\{e_k, k \in \mathbb{N}\}$ the usual orthonormal basis of l^2. Let $\mathbb{X}$ be a separable Banach space, and $L(l^2;\mathbb{X})$ the set of all bounded linear operators from l^2 to $\mathbb{X}$. For an operator $B \in L(l^2;\mathbb{X})$, we also write

$$B = (B_1, B_2, \dots) \in \mathbb{X}^{\mathbb{N}}, \qquad B_k = Be_k.$$

Definition 2.5. An operator $B \in L(l^2;\mathbb{X})$ is called radonifying if

$$\text{the series } \sum_k Be_k \cdot W^k(1) \text{ converges in } L^2(\Omega;\mathbb{X}).$$

We shall denote by $L_2(l^2;\mathbb{X})$ the space of all radonifying operators, and write for $B \in L_2(l^2;\mathbb{X})$,

$$\|B\|_{L_2(l^2;\mathbb{X})} := \big(\mathbb{E}\big\|Be_k \cdot W^k(1)\big\|^2_{\mathbb{X}}\big)^{1/2}. \tag{2.16}$$

Here and below, we use the convention that the repeated indices will be summed.

The following proposition is well known, and a detailed proof was given in [45, Proposition 2.5].

Proposition 2.6. *The space $L_2(l^2;\mathbb{X})$ with norm* (2.16) *is a separable Banach space.*

In order to introduce the stochastic integral of an $\mathbb{X}$-valued measurable $(\mathcal{F}_t)$-adapted process with respect to W, in the sequel, we assume that $\mathbb{X}$ is 2-smooth (cf. [50]), i.e., there exists a constant $C_{\mathbb{X}} \geqslant 2$ such that for all $x, y \in \mathbb{X}$,

$$\|x+y\|_{\mathbb{X}}^2 + \|x-y\|_{\mathbb{X}}^2 \leqslant 2\|x\|_{\mathbb{X}}^2 + C_{\mathbb{X}}\|y\|_{\mathbb{X}}^2.$$

Let now $s \mapsto B(s)$ be an $L_2(l^2;\mathbb{X})$-valued measurable and $(\mathcal{F}_t)$-adapted process with

$$\int_0^T \|B(s)\|_{L_2(l^2;\mathbb{X})}^2 \,\mathrm{d}s < +\infty \quad \text{a.s., } \forall T > 0.$$

One can define the Itô stochastic integral (cf. [45, Section 3])

$$t \mapsto \mathcal{I}_t(B) := \int_0^t B(s)\,\mathrm{d}W(s) = \int_0^t B_k(s)\,\mathrm{d}W^k(s) \in \mathbb{X}$$

such that $t \mapsto \mathcal{I}_t(B)$ is an $\mathbb{X}$-valued continuous local $(\mathcal{F}_t)$-martingale. Moreover, let τ be any $(\mathcal{F}_t)$-stopping time, then

$$\int_0^{t\wedge\tau} B(s)\,\mathrm{d}W(s) = \int_0^t 1_{\{s<\tau\}} \cdot B(s)\,\mathrm{d}W(s).$$

The following BDG inequality for $\mathcal{I}_t(B)$ holds (cf. [45, Section 5]).

Theorem 2.7. *For any $p > 0$, there exists a constant $C_p > 0$ depending only on p such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t B(s)\,\mathrm{d}W(s)\right\|_{\mathbb{X}}^p\right) \leqslant C_p\mathbb{E}\left(\int_0^T \|B(s)\|_{L_2(l^2;\mathbb{X})}^2\,\mathrm{d}s\right)^{p/2}. \tag{2.17}$$

The following two typical examples of 2-smooth Banach spaces are usually met in applications.

Example 2.8. Let $\mathbb{X}$ be a separable Hilbert space. Clearly, $\mathbb{X}$ is 2-smooth. In this case, $L_2(l^2;\mathbb{X})$ consists of all Hilbert–Schmidt operators of mapping l^2 into $\mathbb{X}$, and

$$\|B\|_{L_2(l^2;\mathbb{X})} = \left(\sum_{k=1}^{\infty} \|Be_k\|_{\mathbb{X}}^2\right)^{1/2}.$$

Example 2.9. Let $(E,\mathcal{E},\mu)$ be a measure space, $\mathbb{H}$ a separable Hilbert space. For $p\geqslant 2$, let $L^p(E,\mu;\mathbb{H})$ be the usual $\mathbb{H}$-valued L^p-space over $(E,\mathcal{E},\mu)$. Then $\mathbb{X}=L^p(E,\mu;\mathbb{H})$ is 2-smooth (cf. [50,6]). In this case, by BDG's inequality for Hilbert valued martingale we have

$$
\begin{aligned}
\|B\|^2_{L_2(l^2;\mathbb{X})} &= \mathbb{E}\left(\int_E \big\|B_k(x)\cdot W^k(1)\big\|^p_{\mathbb{H}}\mu(\mathrm{d}x)\right)^{2/p} \\
&\leqslant \left(\int_E \mathbb{E}\big\|B_k(x)\cdot W^k(1)\big\|^p_{\mathbb{H}}\mu(\mathrm{d}x)\right)^{2/p} \\
&\leqslant C_p\left(\int_E \left(\sum_{k=1}^\infty \big\|B_k(x)\big\|^2_{\mathbb{H}}\right)^{p/2}\mu(\mathrm{d}x)\right)^{2/p} \\
&= C_p\|B\|^2_{L^p(E,\mu;l^2\otimes\mathbb{H})}.
\end{aligned}
\tag{2.18}
$$

Hence,

$$
L^p\big(E,\mu;l^2\otimes\mathbb{H}\big)\hookrightarrow L_2\big(l^2;\mathbb{X}\big)=L_2\big(l^2;L^p(E,\mu;\mathbb{H})\big).
$$

We also recall the following Kolmogorov's continuity criterion, which can be derived directly by Garsia's inequality (cf. [64]).

Theorem 2.10. *Let $\{X(t),t\geqslant 0\}$ be an $\mathbb{X}$-valued stochastic process, and τ a bounded random time. Suppose that for some C_0, $p>0$ and $\delta>1$,*

$$
\mathbb{E}\big\|\big(X(t)-X(s)\big)\cdot 1_{\{s,t\in[0,\tau]\}}\big\|^p_{\mathbb{X}}\leqslant C_0|t-s|^\delta.
$$

Then there exist constants $C_1>0$ and $a\in(0,(\delta-1)/p)$ independent of C_0 and a continuous version $\tilde{X}$ of X such that

$$
\mathbb{E}\left(\sup_{s\neq t\in[0,\tau]}\frac{\|\tilde{X}(t)-\tilde{X}(s)\|^p_{\mathbb{X}}}{|t-s|^{ap}}\right)\leqslant C_1\cdot C_0.
$$

2.3. A criterion for Laplace principles

It is well known that there exists a Hilbert space so that $l^2\subset\mathbb{U}$ is Hilbert–Schmidt with embedding operator J and $\{W^k(t),\ k\in\mathbb{N}\}$ is a Brownian motion with values in $\mathbb{U}$, whose covariance operator is given by $Q=J\circ J^*$. For example, one can take $\mathbb{U}$ as the completion of l^2 with respect to the norm generated by scalar product

$$
\langle h,h'\rangle_{\mathbb{U}}:=\left(\sum_{k=1}^\infty\frac{h_kh'_k}{k^2}\right)^{\frac{1}{2}},\quad h,h'\in l^2.
$$

For $T > 0$ and a Banach space $\mathbb{B}$, we denote by $\mathcal{B}(\mathbb{B})$ the Borel σ-field, and by $\mathbb{C}_T(\mathbb{B})$ the space of all continuous functions from $[0, T]$ to $\mathbb{B}$, which is endowed with the uniform norm. Define

$$\ell_T^2 := \left\{ h = \int_0^{\cdot} \dot{h}(s)\,\mathrm{d}s \colon \dot{h} \in L^2\big(0, T; l^2\big) \right\} \tag{2.19}$$

with the norm

$$\|h\|_{\ell_T^2} := \left(\int_0^T \big\|\dot{h}(s)\big\|_{l^2}^2 \,\mathrm{d}s \right)^{1/2},$$

where the dot denotes the generalized derivative. Let μ be the law of the Brownian motion W in $\mathbb{C}_T(\mathbb{U})$. Then

$$\big(\mathbb{C}_T(\mathbb{U}), \ell_T^2, \mu\big)$$

forms an abstract Wiener space.

For $T, N > 0$, set

$$\mathbb{D}_N := \big\{ h \in \ell_T^2 \colon \|h\|_{\ell_T^2} \leqslant N \big\}$$

and

$$\mathcal{A}_N^T := \left\{ \begin{array}{c} h : [0, T] \to l^2 \text{ is a continuous and } (\mathcal{F}_t)\text{-adapted} \\ \text{process, and for almost all } \omega,\ h(\cdot, \omega) \in \mathbb{D}_N \end{array} \right\}. \tag{2.20}$$

It is well known that with respect to the weak convergence topology in ℓ_T^2 (cf. [32]),

$$\mathbb{D}_N \text{ is metrizable as a compact Polish space.} \tag{2.21}$$

Let $\mathbb{S}$ be a Polish space. A function $I : \mathbb{S} \to [0, \infty]$ is given.

Definition 2.11. The function I is called a rate function if for every $a < \infty$, the set $\{f \in \mathbb{S}\colon I(f) \leqslant a\}$ is compact in $\mathbb{S}$.

Let $\{Z_\epsilon : \mathbb{C}_T(\mathbb{U}) \to \mathbb{S},\ \epsilon \in (0, 1)\}$ be a family of measurable mappings. Assume that there is a measurable map $Z_0 : \ell_T^2 \mapsto \mathbb{S}$ such that:

(LD)$_1$ For any $N > 0$, if a family $\{h^\epsilon, \epsilon \in (0, 1)\} \subset \mathcal{A}_N^T$ (as random variables in $\mathbb{D}_N$) converges in distribution to $h \in \mathcal{A}_N^T$, then for some subsequence ϵ_k, $Z_{\epsilon_k}(\cdot + \frac{h^{\epsilon_k}(\cdot)}{\sqrt{\epsilon_k}})$ converges in distribution to $Z_0(h)$ in $\mathbb{S}$.

(LD)$_2$ For any $N > 0$, if $\{h_n, n \in \mathbb{N}\} \subset \mathbb{D}_N$ weakly converges to $h \in \ell_T^2$, then for some subsequence h_{n_k}, $Z_0(h_{n_k})$ converges to $Z_0(h)$ in $\mathbb{S}$.

For each $f \in \mathbb{S}$, define

$$I(f) := \frac{1}{2} \inf_{\{h \in \ell_T^2:\ f = Z_0(h)\}} \|h\|_{\ell_T^2}^2, \tag{2.22}$$

where $\inf \emptyset = \infty$ by convention. Then under $(\mathrm{LD})_2$, $I(f)$ is a rate function. In fact, assume that $I(f_n) \leqslant a$. By the definition of $I(f_n)$, there exists a sequence $h_n \in \ell_2$ such that $Z_0(h_n) = f_n$ and

$$\frac{1}{2}\|h_n\|_{\ell_T^2}^2 \leqslant a + \frac{1}{n}.$$

By the weak compactness of $\mathbb{D}_{2a+2}$, there exist a subsequence n_k (still denoted by n) and $h \in \ell_T^2$ such that h_n weakly converges to h and

$$\|h\|_{\ell_T^2}^2 \leqslant \varliminf_{n\to\infty} \|h_n\|_{\ell_T^2}^2 \leqslant 2a.$$

Hence, by $(\mathrm{LD})_2$ we have

$$\lim_{k\to\infty} \big\| Z_0(h_{n_k}) - Z_0(h) \big\|_{\mathbb{S}} = 0$$

and

$$I\big(Z_0(h)\big) \leqslant a.$$

We recall the following result due to [5,10] (see also [67, Theorem 4.4]).

Theorem 2.12. *Under* $(\mathrm{LD})_1$ *and* $(\mathrm{LD})_2$, $\{Z_\epsilon,\ \epsilon \in (0,1)\}$ *satisfies the Laplace principle with the rate function* $I(f)$ *given by* (2.22). *More precisely, for each real bounded continuous function* g *on* $\mathbb{S}$:

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}^\mu\left(\exp\left[-\frac{g(Z_\epsilon)}{\epsilon}\right]\right) = -\inf_{f\in\mathbb{S}}\big\{g(f) + I(f)\big\}. \tag{2.23}$$

In particular, the family of $\{Z_\epsilon, \epsilon \in (0,1)\}$ *satisfies the large deviation principle in* $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ *with the rate function* $I(f)$. *More precisely, let* ν_ϵ *be the law of* Z_ϵ *in* $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, *then for any* $A \in \mathcal{B}(\mathbb{S})$:

$$-\inf_{f\in A^o} I(f) \leqslant \liminf_{\epsilon\to 0} \epsilon \log \nu_\epsilon(A) \leqslant \limsup_{\epsilon\to 0} \epsilon \log \nu_\epsilon(A) \leqslant -\inf_{f\in \bar{A}} I(f),$$

where the closure and the interior are taken in $\mathbb{S}$, *and* $I(f)$ *is defined by* (2.22).

3. Abstract stochastic Volterra integral equations

In this section, we consider the following stochastic Volterra integral equation in 2-smooth Banach space $\mathbb{X}$:

$$X(t) = g(t) + \int_0^t A\big(t, s, X(s)\big)\,\mathrm{d}s + \int_0^t B\big(t, s, X(s)\big)\,\mathrm{d}W(s), \tag{3.1}$$

where $g(t)$ is an $\mathbb{X}$-valued measurable and $(\mathcal{F}_t)$-adapted process, and

$$A : \Delta \times \Omega \times \mathbb{X} \to \mathbb{X} \in \mathcal{M}_\Delta \times \mathcal{B}(\mathbb{X})/\mathcal{B}(\mathbb{X})$$

and

$$B : \Delta \times \Omega \times \mathbb{X} \to L_2\big(l^2; \mathbb{X}\big) \in \mathcal{M}_\Delta \times \mathcal{B}(\mathbb{X})/\mathcal{B}\big(L_2\big(l^2; \mathbb{X}\big)\big).$$

Here and below, $\Delta := \{(t, s) \in \mathbb{R}_+^2\colon\ s \leqslant t\}$, and $\mathcal{M}_\Delta$ denotes the progressively measurable σ-field on $\Delta \times \Omega$ generated by the sets $E \in \mathcal{B}(\Delta) \times \mathcal{F}$ with properties: $1_E(t, s, \cdot) \in \mathcal{F}_s$ for all $(t, s) \in \Delta$, and $s \mapsto 1_E(t, s, \omega)$ is right continuous for any $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

We start with the global existence and uniqueness of solutions for Eq. (3.1) under global Lipschitz conditions and singular kernels.

3.1. Global existence and uniqueness

In this subsection, we make the following global Lipschitz and linear growth conditions on the coefficients:

(H1) For some $p \geqslant 2$ and any $T > 0$,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_0^t \big[\kappa_1(t, s) + \kappa_2(t, s)\big] \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s < +\infty,$$

where κ_1 and κ_2 are from (H2) and (H3) below.

(H2) There exists $\kappa_1 \in \mathscr{K}_0$ such that for all $(t, s) \in \Delta$, $\omega \in \Omega$ and $x \in \mathbb{X}$,

$$\big\|A(t, s, \omega, x)\big\|_{\mathbb{X}} \leqslant \kappa_1(t, s) \cdot \big(\|x\|_{\mathbb{X}} + 1\big)$$

and

$$\big\|B(t, s, \omega, x)\big\|_{L_2(l^2;\mathbb{X})}^2 \leqslant \kappa_1(t, s) \cdot \big(\|x\|_{\mathbb{X}}^2 + 1\big).$$

(H3) There exists $\kappa_2 \in \mathscr{K}_0$ such that for all $(t, s) \in \Delta$, $\omega \in \Omega$ and $x, y \in \mathbb{X}$,

$$\big\|A(t, s, \omega, x) - A(t, s, \omega, y)\big\|_{\mathbb{X}} \leqslant \kappa_2(t, s) \cdot \|x - y\|_{\mathbb{X}}$$

and

$$\big\|B(t,s,\omega,x)-B(t,s,\omega,y)\big\|^2_{L_2(l^2;\mathbb{X})}\leqslant \kappa_2(t,s)\cdot\|x-y\|^2_{\mathbb{X}}.$$

We now prove the following basic existence and uniqueness result.

Theorem 3.1. *Assume that* (H1)–(H3) *hold. Then there exists a unique measurable* $(\mathcal{F}_t)$*-adapted process* $X(t)$ *such that for almost all* $t\geqslant 0$*,*

$$X(t)=g(t)+\int_0^t A\big(t,s,X(s)\big)\,\mathrm{d}s+\int_0^t B\big(t,s,X(s)\big)\,\mathrm{d}W(s),\quad P\text{-a.s.},\tag{3.2}$$

and for any $T>0$ *and some* $C_{T,p,\kappa_1}>0$*,*

$$\mathbb{E}\big\|X(t)\big\|^p_{\mathbb{X}}\leqslant C_{T,p,\kappa_1}\left[\mathbb{E}\big\|g(t)\big\|^p_{\mathbb{X}}+\operatorname*{ess\,sup}_{t\in[0,T]}\int_0^t \kappa_1(t,s)\cdot\mathbb{E}\big\|g(s)\big\|^p_{\mathbb{X}}\,\mathrm{d}s\right]\tag{3.3}$$

for almost all $t\in[0,T]$*, where* p *is from* (H1)*. Moreover, if*

$$t\mapsto\int_0^t \kappa_1(t,s)\,\mathrm{d}s\in L^\infty(\mathbb{R}_+),\tag{3.4}$$

then for almost all $t\geqslant 0$*,*

$$\begin{aligned}\mathbb{E}\big\|X(t)\big\|^p_{\mathbb{X}}\leqslant C_{p,\kappa_1}\Bigg(&\mathbb{E}\big\|g(t)\big\|^p_{\mathbb{X}}+\int_0^t \tilde\kappa_1(t,s)\cdot\mathbb{E}\big\|g(s)\big\|^p_{\mathbb{X}}\,\mathrm{d}s\\&+\int_0^t r_{\tilde\kappa_1}(t,u)\cdot\left[\int_0^u \tilde\kappa_1(u,s)\cdot\mathbb{E}\big\|g(s)\big\|^p_{\mathbb{X}}\,\mathrm{d}s\right]\mathrm{d}u\Bigg),\end{aligned}\tag{3.5}$$

where $\tilde\kappa_1=\tilde C_{p,\kappa_1}\cdot\kappa_1$*,* $r_{\tilde\kappa_1}$ *is defined by* (2.3) *in terms of* $\tilde\kappa_1$*, and* $C_{p,\kappa_1},\tilde C_{p,\kappa_1}$ *are constants only depending on* p,κ_1*.*

Proof. We use Picard's iteration to prove the existence. Let $X_1(t):=g(t)$ and define recursively for $n\in\mathbb{N}$,

$$X_{n+1}(t)=g(t)+\int_0^t A\big(t,s,X_n(s)\big)\,\mathrm{d}s+\int_0^t B\big(t,s,X_n(s)\big)\,\mathrm{d}W(s).\tag{3.6}$$

Fix $T > 0$ below. By (H2), BDG's inequality (2.17) and Hölder's inequality we have

$$\begin{aligned}
\mathbb{E}\|X_{n+1}(t)\|_{\mathbb{X}}^{p} &\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^{p} + \mathbb{E}\left(\int_0^t \|A(t,s,X_n(s))\|_{\mathbb{X}}\,\mathrm{d}s\right)^{p} \\
&\quad + \mathbb{E}\left\|\int_0^t B(t,s,X_n(s))\,\mathrm{d}W(s)\right\|_{\mathbb{X}}^{p} \\
&\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^{p} + \mathbb{E}\left(\int_0^t \kappa_1(t,s)\cdot(\|X_n(s)\|_{\mathbb{X}}+1)\,\mathrm{d}s\right)^{p} \\
&\quad + \mathbb{E}\left(\int_0^t \|B(t,s,X_n(s))\|_{L_2(l^2;\mathbb{X})}^{2}\,\mathrm{d}s\right)^{\frac{p}{2}} \\
&\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^{p} + \int_0^t \kappa_1(t,s)\cdot\mathbb{E}(\|X_n(s)\|_{\mathbb{X}}^{p}+1)\,\mathrm{d}s\cdot\left(\int_0^t \kappa_1(t,s)\,\mathrm{d}s\right)^{p-1} \\
&\quad + \int_0^t \kappa_1(t,s)\cdot\mathbb{E}(\|X_n(s)\|_{\mathbb{X}}^{p}+1)\,\mathrm{d}s\cdot\left(\int_0^t \kappa_1(t,s)\,\mathrm{d}s\right)^{\frac{p}{2}-1} \\
&\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^{p} + C_{T,p}\cdot C_T + C_{T,p}\int_0^t \kappa_1(t,s)\cdot\mathbb{E}\|X_n(s)\|_{\mathbb{X}}^{p}\,\mathrm{d}s, \qquad (3.7)
\end{aligned}$$

where $C_T := \operatorname{ess\,sup}_{t\in[0,T]} |\int_0^t \kappa_1(t,s)\,\mathrm{d}s|$ and $C_{T,p} := C_T^{p-1} + C_T^{(p-2)/2}$.

Set

$$f_m(t) := \sup_{n=1,\dots,m} \mathbb{E}\|X_n(t)\|_{\mathbb{X}}^{p}.$$

Then

$$f_m(t) \leqslant C_{T,p,\kappa_1}\left(\mathbb{E}\|g(t)\|_{\mathbb{X}}^{p}+1\right) + \int_0^t \tilde{\kappa}_1(t,s)\cdot f_m(s)\,\mathrm{d}s,$$

where $\tilde{\kappa}_1 = C_{T,p,\kappa_1}\cdot\kappa_1$ and the constant C_{T,p,κ_1} is independent of m.

Let $r_{\tilde{\kappa}_1}$ be defined by (2.3) in terms of $\tilde{\kappa}_1$. Note that by (2.4)

$$\int_0^t r_{\tilde{\kappa}_1}(t,s)\cdot\mathbb{E}\|g(s)\|_{\mathbb{X}}^{p}\,\mathrm{d}s - \int_0^t \tilde{\kappa}_1(t,s)\cdot\mathbb{E}\|g(s)\|_{\mathbb{X}}^{p}\,\mathrm{d}s$$

$$= \int_0^t \left(\int_s^t r_{\tilde{\kappa}_1}(t,u)\tilde{\kappa}_1(u,s)\,\mathrm{d}u \right) \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s$$

$$= \int_0^t r_{\tilde{\kappa}_1}(t,u) \left(\int_0^u \tilde{\kappa}_1(u,s) \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s \right) \mathrm{d}u.$$

Hence, by Lemma 2.2 and (H1), we obtain that for almost all $t \in [0,T]$,

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \mathbb{E}\big\|X_n(t)\big\|_{\mathbb{X}}^p &= \lim_{m\to\infty} f_m(t) \leqslant C_{T,p,\kappa_1} \left(\mathbb{E}\big\|g(t)\big\|_{\mathbb{X}}^p + \int_0^t r_{\tilde{\kappa}_1}(t,s) \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s \right) \\
&\leqslant C_{T,p,\kappa_1} \left(\mathbb{E}\big\|g(t)\big\|_{\mathbb{X}}^p + \int_0^t \tilde{\kappa}_1(t,s) \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s \right. \\
&\qquad \left. + \int_0^t r_{\tilde{\kappa}_1}(t,u) \left(\int_0^u \tilde{\kappa}_1(u,s) \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s \right) \mathrm{d}u \right) \qquad (3.8) \\
&\overset{(2.5)}{\leqslant} C_{T,p,\kappa_1} \left[\mathbb{E}\big\|g(t)\big\|_{\mathbb{X}}^p + \operatorname*{ess\,sup}_{t\in[0,T]} \int_0^t \kappa_1(t,s) \cdot \mathbb{E}\big\|g(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s \right]. \qquad (3.9)
\end{aligned}$$

On the other hand, set

$$Z_{n,m}(t) := X_n(t) - X_m(t)$$

and

$$f(t) := \limsup_{n,m\to\infty} \mathbb{E}\big\|Z_{n,m}(t)\big\|_{\mathbb{X}}^2.$$

As the above calculations, by (H3) we have

$$\begin{aligned}
\mathbb{E}\big\|Z_{n+1,m+1}(t)\big\|_{\mathbb{X}}^2 &\preccurlyeq \mathbb{E}\left\| \int_0^t \big(A\big(t,s,X_n(s)\big) - A\big(t,s,X_m(s)\big)\big)\,\mathrm{d}s \right\|_{\mathbb{X}}^2 \\
&\quad + \mathbb{E}\left\| \int_0^t \big(B\big(t,s,X_n(s)\big) - B\big(t,s,X_m(s)\big)\big)\,\mathrm{d}W(s) \right\|_{\mathbb{X}}^2 \\
&\preccurlyeq \int_0^t \kappa_2(t,s) \cdot \mathbb{E}\big\|Z_{n,m}(s)\big\|_{\mathbb{X}}^2\,\mathrm{d}s.
\end{aligned}$$

By (3.9), (H1) and using Fatou's lemma, we get

$$f(t) \preccurlyeq \int_0^t \kappa_2(t,s) \cdot f(s)\,\mathrm{d}s.$$

By Lemma 2.2 again, we have for almost all $t \in [0, T]$,

$$f(t) = \limsup_{n,m\to\infty} \mathbb{E}\big\|Z_{n,m}(t)\big\|_{\mathbb{X}}^2 = 0.$$

Hence, there exists an $\mathbb{X}$-valued $(\mathcal{F}_t)$-adapted process $X(t)$ such that for almost all $t \in [0, T]$,

$$\lim_{n\to\infty} \mathbb{E}\big\|X_n(t) - X(t)\big\|_{\mathbb{X}}^2 = 0.$$

Taking limits for (3.6), one finds that (3.2) holds.

Moreover, estimate (3.3) follows from (3.9). Note that when (3.4) is satisfied, the constant $C_{T,p}$ in (3.7) is independent of T. Hence, estimate (3.5) is direct from (3.8). The uniqueness follows by similar calculations as above. □

Example 3.2. Let for $\delta > 0$,

$$h(s) := \frac{e^{-\delta s}}{s \log^2 s}, \quad t > s \geqslant 0.$$

It is easy to see that $h \in L^1(\mathbb{R}_+)$. Consider the following stochastic Volterra equation:

$$X(t) = x_0\sqrt{\big|\log(t \wedge 1)\big|} + \int_0^t h(t-s)A\big(X(s)\big)\,\mathrm{d}s + \int_0^t \sqrt{h(t-s)}B\big(X(s)\big)\,\mathrm{d}W(s),$$

where $A : \mathbb{X} \to \mathbb{X}$ and $B : \mathbb{X} \to L_2(l^2; \mathbb{X})$ are global Lipschitz continuous functions. By elementary calculations, one finds that

$$\sup_{t\geqslant 0} \int_0^t \frac{e^{-\delta(t-s)}|\log(s \wedge 1)|}{(t-s)\log^2(t-s)}\,\mathrm{d}s < +\infty.$$

So, (H1)–(H3) are satisfied with $p = 2$. Moreover, by (2.8) and (3.5), one finds that if δ is large enough, then for any $T > 0$,

$$\sup_{t\geqslant T} \mathbb{E}\big\|X(t)\big\|_{\mathbb{X}}^2 < +\infty.$$

We remark that in this example, $X(0) = \infty$.

3.2. Path continuity of solutions

In this subsection, in addition to (H2) and (H3), we also assume that:

(H1)′ The process $t \mapsto g(t)$ is continuous and $(\mathcal{F}_t)$-adapted, and for any $p \geqslant 2$ and $T > 0$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big\|g(t)\big\|_{\mathbb{X}}^p\Big) < +\infty.$$

(H4) For all $s < t < t'$, $\omega \in \Omega$ and $x \in \mathbb{X}$,

$$\big\|A\big(t', s, \omega, x\big) - A(t, s, \omega, x)\big\|_{\mathbb{X}} \leqslant \lambda\big(t', t, s\big) \cdot \big(\|x\|_{\mathbb{X}} + 1\big)$$

and

$$\big\|B\big(t', s, \omega, x\big) - B(t, s, \omega, x)\big\|_{L_2(l^2;\mathbb{X})}^2 \leqslant \lambda\big(t', t, s\big) \cdot \big(\|x\|_{\mathbb{X}}^2 + 1\big),$$

where λ is a positive measurable function satisfying that for any $T > 0$ and some $\gamma = \gamma(T)$, $C = C(T) > 0$,

$$\int_0^t \lambda\big(t', t, s\big)\,\mathrm{d}s \leqslant C\big|t' - t\big|^{\gamma}, \quad 0 \leqslant t < t' \leqslant T. \tag{3.10}$$

Theorem 3.3. *Assume that* (H1)′ *and* (H2)–(H4) *hold, and the kernel function κ_1 in* (H2) *belongs to $\mathscr{K}_{>1}$. Then there exists a unique $\mathbb{X}$-valued continuous $(\mathcal{F}_t)$-adapted process $X(t)$ such that P-a.s., for all $t \geqslant 0$,*

$$X(t) = g(t) + \int_0^t A\big(t, s, X(s)\big)\,\mathrm{d}s + \int_0^t B\big(t, s, X(s)\big)\,\mathrm{d}W(s) \tag{3.11}$$

and for any $p \geqslant 2$ and $T > 0$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big\|X(t)\big\|_{\mathbb{X}}^p\Big) < +\infty. \tag{3.12}$$

Moreover, if for some $\delta > 0$ and any $p \geqslant 2$, $T > 0$,

$$\mathbb{E}\big\|g\big(t'\big) - g(t)\big\|_{\mathbb{X}}^p \leqslant C_{T,p}\big|t' - t\big|^{\delta p},$$

then, $t \mapsto X(t)$ admits a Hölder continuous modification and for any $p \geqslant 2$, $T > 0$ and some $a > 0$,

$$\mathbb{E}\bigg(\sup_{t\neq t'\in[0,T]} \frac{\|X(t') - X(t)\|_{\mathbb{X}}^p}{|t' - t|^{ap}}\bigg) \leqslant C_{T,p,a}.$$

Proof. First of all, for any $p \geqslant 2$ and $T > 0$, by (H1)$'$ and (3.3) we have

$$\operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\|X(t)\|_{\mathbb{X}}^p < +\infty. \tag{3.13}$$

Set

$$J(t) := \int_0^t B\big(t, s, X(s)\big)\,\mathrm{d}W(s)$$

and write for $0 \leqslant t < t' \leqslant T$,

$$\begin{aligned} J\big(t'\big) - J(t) &= \int_0^t \big[B\big(t', s, X(s)\big) - B\big(t, s, X(s)\big)\big]\,\mathrm{d}W(s) \\ &\quad + \int_t^{t'} B\big(t', s, X(s)\big)\,\mathrm{d}W(s) =: J_1\big(t', t\big) + J_2\big(t', t\big). \end{aligned}$$

In view of $\kappa_1 \in \mathscr{K}_{>1}$, (2.6) holds for some $\beta > 1$. Fix $p \geqslant 2\beta^*$ ($\beta^* := \beta/(\beta-1)$). By BDG inequality (2.17), (H2) and Hölder's inequality, we have

$$\begin{aligned} \mathbb{E}\big\|J_2\big(t', t\big)\big\|_{\mathbb{X}}^p &\preccurlyeq \mathbb{E}\bigg(\int_t^{t'} \kappa_1\big(t', s\big)\cdot\big(\|X(s)\|_{\mathbb{X}}^2 + 1\big)\,\mathrm{d}s\bigg)^{\frac{p}{2}} \\ &\preccurlyeq \bigg(\int_t^{t'} k_1^\beta\big(t', s\big)\,\mathrm{d}s\bigg)^{\frac{p}{2\beta}} \mathbb{E}\bigg(\int_t^{t'} \big(\|X(s)\|_{\mathbb{X}}^{2\beta^*} + 1\big)\,\mathrm{d}s\bigg)^{\frac{p}{2\beta^*}} \\ &\overset{(2.6)}{\preccurlyeq} \big|t' - t\big|^{\frac{p}{2\beta^*}-1} \int_t^{t'} \big(\mathbb{E}\|X(s)\|_{\mathbb{X}}^p + 1\big)\,\mathrm{d}s \overset{(3.13)}{\preccurlyeq} \big|t' - t\big|^{\frac{p}{2\beta^*}}, \end{aligned}$$

and by (H4) and Minkowski's inequality,

$$\begin{aligned} \mathbb{E}\big\|J_1\big(t', t\big)\big\|_{\mathbb{X}}^p &\preccurlyeq \mathbb{E}\bigg(\int_0^t \lambda\big(t', t, s\big)\cdot\big(\|X(s)\|_{\mathbb{X}}^2 + 1\big)\,\mathrm{d}s\bigg)^{\frac{p}{2}} \\ &\preccurlyeq \bigg(\int_0^t \lambda\big(t', t, s\big)\cdot\big(\big(\mathbb{E}\|X(s)\|_{\mathbb{X}}^p\big)^{\frac{2}{p}} + 1\big)\,\mathrm{d}s\bigg)^{\frac{p}{2}} \\ &\overset{(3.13)}{\preccurlyeq} \bigg(\int_0^t \lambda\big(t', t, s\big)\,\mathrm{d}s\bigg)^{\frac{p}{2}} \overset{(3.10)}{\preccurlyeq} \big|t' - t\big|^{\frac{\gamma p}{2}}. \end{aligned}$$

Hence, for all $0 \leqslant t < t' \leqslant T$,

$$\mathbb{E}\big\|J(t') - J(t)\big\|_{\mathbb{X}}^p \preccurlyeq \big|t-t'\big|^{\frac{\gamma p}{2}} + \big|t-t'\big|^{\frac{p}{2\beta^*}}.$$

Similarly, we may prove that for all $0 \leqslant t < t' \leqslant T$ and $p \geqslant \beta^*$,

$$\mathbb{E}\left\|\int_0^{t'} A\big(t', s, X(s)\big)\,\mathrm{d}s - \int_0^t A\big(t, s, X(s)\big)\,\mathrm{d}s\right\|_{\mathbb{X}}^p \preccurlyeq \big|t-t'\big|^{\gamma p} + \big|t-t'\big|^{\frac{p}{\beta^*}}.$$

The desired conclusions follow from Theorem 2.10. □

We conclude this subsection by proving a lemma, which will be used frequently later. We put it here since the proof is similar to Theorem 3.3.

Lemma 3.4. *Let τ be an $(\mathcal{F}_t)$-stopping time and*

$$G : \Delta \times \Omega \to L_2\big(l^2; \mathbb{X}\big) \in \mathcal{M}_\Delta / \mathcal{B}\big(L_2\big(l^2; \mathbb{X}\big)\big).$$

Assume that for all $0 \leqslant s < t < t'$ and $\omega \in \Omega$,

$$\big\|G(t,s,\omega)\big\|_{L_2(l^2;\mathbb{X})}^2 \leqslant \kappa(t,s)\cdot f^2(s,\omega), \tag{3.14}$$

$$\big\|G\big(t',s,\omega\big) - G(t,s,\omega)\big\|_{L_2(l^2;\mathbb{X})}^2 \leqslant \lambda\big(t',t,s\big)\cdot f^2(s,\omega), \tag{3.15}$$

where $\kappa \in \mathscr{K}_{>1}$ and for any $T > 0$ and some $\alpha > 1$ and $\gamma > 0$,

$$\int_0^t \lambda^\alpha\big(t',t,s\big)\,\mathrm{d}s \leqslant C_T\big|t'-t\big|^\gamma, \quad \forall 0 \leqslant t < t' \leqslant T,$$

and $(s,\omega) \mapsto f(s,\omega)$ is a positive measurable process with

$$\mathbb{E}\left(\int_0^{T\wedge\tau} f^p(s)\,\mathrm{d}s\right) < +\infty, \quad \forall p \geqslant 2.$$

Then $t \mapsto J(t) := \int_0^t G(t,s)\,\mathrm{d}W(s) \in \mathbb{X}$ admits a continuous modification on $[0,\tau)$, and for any $T > 0$ and p large enough

$$\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau]}\left\|\int_0^t G(t,s)\,\mathrm{d}W(s)\right\|_{\mathbb{X}}^p\right) \leqslant C_T\mathbb{E}\left(\int_0^{T\wedge\tau} f^p(s)\,\mathrm{d}s\right),$$

where the constant C_T is independent of f and τ.

Proof. Fix $T > 0$ and write for $0 \leqslant t < t' \leqslant T$,

$$\begin{aligned} J(t') - J(t) &= \int_t^{t'} G(t', s)\, \mathrm{d}W(s) + \int_0^t \big[G(t', s) - G(t, s)\big]\, \mathrm{d}W(s) \\ &=: J_1(t', t) + J_2(t', t). \end{aligned}$$

In view of $\kappa \in \mathscr{K}_{>1}$ and (2.6), by BDG's inequality (2.17) and Hölder's inequality we have, for some $\beta > 1$ and $p \geqslant 2\beta^*$ ($\beta^* = \beta/(\beta - 1)$),

$$\begin{aligned} \mathbb{E}\big\| J_1(t', t) \cdot 1_{\{t', t \in [0, \tau)\}} \big\|_{\mathbb{X}}^p &\leqslant \mathbb{E}\left\| \int_{t\wedge\tau}^{t'\wedge\tau} G(t', s)\, \mathrm{d}W(s) \right\|_{\mathbb{X}}^p \\ &\preccurlyeq \mathbb{E}\left(\int_{t\wedge\tau}^{t'\wedge\tau} \big\| G(t', s) \big\|_{L_2(l^2; \mathbb{X})}^2\, \mathrm{d}s \right)^{p/2} \\ &\overset{(3.14)}{\preccurlyeq} \mathbb{E}\left(\int_{t\wedge\tau}^{t'\wedge\tau} \kappa(t', s) \cdot f^2(s)\, \mathrm{d}s \right)^{p/2} \\ &\preccurlyeq \left(\int_t^{t'} \kappa^\beta(t', s)\, \mathrm{d}s \right)^{\frac{p}{2\beta}} \cdot \mathbb{E}\left(\int_{t\wedge\tau}^{t'\wedge\tau} f^{2\beta^*}(s)\, \mathrm{d}s \right)^{\frac{p}{2\beta^*}} \\ &\preccurlyeq \left| t' - t \right|^{\frac{p}{2\beta^*} - 1} \cdot \mathbb{E}\left(\int_0^{T\wedge\tau} f^p(s)\, \mathrm{d}s \right) \end{aligned}$$

and for $p \geqslant 2\alpha^*$ ($\alpha^* = \alpha/(\alpha - 1)$),

$$\begin{aligned} \mathbb{E}\big\| J_1(t', t) \cdot 1_{\{t', t \in [0, \tau)\}} \big\|_{\mathbb{X}}^p &\preccurlyeq \mathbb{E}\left(\int_0^{t\wedge\tau} \big\| G(t', s) - G(t, s) \big\|_{L_2(l^2; \mathbb{X})}^2\, \mathrm{d}s \right)^{p/2} \\ &\overset{(3.15)}{\preccurlyeq} \mathbb{E}\left(\int_0^{t\wedge\tau} \lambda(t', t, s) \cdot f^2(s)\, \mathrm{d}s \right)^{p/2} \\ &\preccurlyeq \left(\int_0^t \lambda^\alpha(t', t, s)\, \mathrm{d}s \right)^{\frac{p}{2\alpha}} \cdot \mathbb{E}\left(\int_0^{t\wedge\tau} f^{2\alpha^*}(s)\, \mathrm{d}s \right)^{\frac{p}{2\alpha^*}} \\ &\preccurlyeq \left| t' - t \right|^{\frac{\gamma p}{2\alpha}} \cdot \mathbb{E}\left(\int_0^{T\wedge\tau} f^p(s)\, \mathrm{d}s \right). \end{aligned}$$

Hence, for any $p \geqslant 2(\alpha^* \vee \beta^*)$ and $0 \leqslant t < t' \leqslant T$,

$$\mathbb{E}\big\|\big(J(t') - J(t)\big) \cdot 1_{\{t',t\in[0,\tau)\}}\big\|_{\mathbb{X}}^p \preccurlyeq \big|t' - t\big|^{(\frac{p}{2\beta^*}-1)\wedge\frac{\gamma p}{2\alpha}} \cdot \mathbb{E}\left(\int_0^{T\wedge\tau} f^p(s)\,\mathrm{d}s\right).$$

The desired result now follows by Theorem 2.10. $\square$

3.3. Local existence and uniqueness

In this subsection, we assume that:

(H2)$'$ For any $R > 0$, there exists $\kappa_{1,R} \in \mathscr{K}_{>1}$ such that for all $(t,s) \in \Delta$, $\omega \in \Omega$ and $x \in \mathbb{X}$ with $\|x\|_{\mathbb{X}} \leqslant R$,

$$\big\|A(t,s,\omega,x)\big\|_{\mathbb{X}} + \big\|B(t,s,\omega,x)\big\|^2_{L_2(l^2;\mathbb{X})} \leqslant \kappa_{1,R}(t,s).$$

(H3)$'$ For any $R > 0$, there exists $\kappa_{2,R} \in \mathscr{K}_0$ such that for all $(t,s) \in \Delta$, $\omega \in \Omega$ and $x, y \in \mathbb{X}$ with $\|x\|_{\mathbb{X}}, \|y\|_{\mathbb{X}} \leqslant R$,

$$\big\|A(t,s,\omega,x) - A(t,s,\omega,y)\big\|_{\mathbb{X}} \leqslant \kappa_{2,R}(t,s) \cdot \|x-y\|_{\mathbb{X}}$$

and

$$\big\|B(t,s,\omega,x) - B(t,s,\omega,y)\big\|^2_{L_2(l^2;\mathbb{X})} \leqslant \kappa_{2,R}(t,s) \cdot \|x-y\|^2_{\mathbb{X}}.$$

(H4)$'$ For any $R > 0$, there exists a measurable function λ_R satisfying that for any $T > 0$ and some $\gamma, C > 0$,

$$\int_0^t \lambda_R\big(t',t,s\big)\,\mathrm{d}s \leqslant C\big|t'-t\big|^\gamma, \quad 0 \leqslant t < t' \leqslant T,$$

such that for all $s < t < t'$, $\omega \in \Omega$ and $x \in \mathbb{X}$ with $\|x\|_{\mathbb{X}} \leqslant R$,

$$\big\|A\big(t',s,\omega,x\big) - A(t,s,\omega,x)\big\|_{\mathbb{X}} + \big\|B\big(t',s,\omega,x\big) - B(t,s,\omega,x)\big\|^2_{L_2(l^2;\mathbb{X})} \leqslant \lambda_R\big(t',t,s\big).$$

We first introduce the following notion of local solutions.

Definition 3.5. Let τ be an $(\mathcal{F}_t)$-stopping time, and $\{X(t);\ t \in [0,\tau)\}$ an $\mathbb{X}$-valued continuous $(\mathcal{F}_t)$-adapted process. The pair of (X,τ) is called a local solution of Eq. (3.1) if P-a.s., for all $t \in [0,\tau)$,

$$X(t) = g(t) + \int_0^t A\big(t,s,X(s)\big)\,\mathrm{d}s + \int_0^t B\big(t,s,X(s)\big)\,\mathrm{d}W(s);$$

(X, τ) is called a maximal solution of Eq. (3.1) if

$$\lim_{t\uparrow\tau(\omega)} \big\|X(t,\omega)\big\|_{\mathbb{X}} = +\infty \quad \text{on } \big\{\omega\colon\ \tau(\omega) < +\infty\big\},\ P\text{-a.s.}$$

We call (X, τ) a non-explosion solution of Eq. (3.1) if

$$P\big\{\omega\colon\ \tau(\omega) < +\infty\big\} = 0.$$

Remark 3.6. The stochastic integral in the above definition is defined on $[0, \tau)$ by

$$\int_0^t B\big(t,s,X(s)\big)\,\mathrm{d}W(s) = \lim_{n\to\infty} \int_0^{t\wedge\tau_n} B\big(t,s,X(s)\big)\,\mathrm{d}W(s), \quad t<\tau,$$

where $\tau_n := \inf\{t > 0\colon \|X(t)\|_{\mathbb{X}} > n\} \nearrow \tau$ as $n \to \infty$.

We now prove the following main result in this section.

Theorem 3.7. *Under* (H1)′–(H4)′*, there exists a unique maximal solution* (X, τ) *for Eq.* (3.1) *in the sense of Definition* 3.5.

Proof. For $n \in \mathbb{N}$, let χ_n be a positive smooth function on $\mathbb{R}_+$ with $\chi_n(s) = 1$, $s \leqslant n$, and $\chi_n(s) = 0$, $s \geqslant n + 1$. Define

$$A_n(t,s,\omega,x) := A(t,s,\omega,x)\cdot\chi_n\big(\|x\|_{\mathbb{X}}\big),$$
$$B_n(t,s,\omega,x) := B(t,s,\omega,x)\cdot\chi_n\big(\|x\|_{\mathbb{X}}\big).$$

It is easy to see that for A_n and B_n, (H2) holds with $\kappa_{1,n+1}$, (H4) holds with λ_{n+1}, and (H3) holds with some $\kappa_{3,n} \in \mathscr{K}_0$. Thus, by Theorem 3.3 there exists a unique continuous $(\mathcal{F}_t)$-adapted process $X_n(t)$ such that for any $p \geqslant 2$ and $T > 0$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big\|X_n(t)\big\|_{\mathbb{X}}^p\Big) \leqslant C_{T,p,n}$$

and

$$X_n(t) = g(t) + \int_0^t A_n\big(t,s,X_n(s)\big)\,\mathrm{d}s + \int_0^t B_n\big(t,s,X_n(s)\big)\,\mathrm{d}W(s). \tag{3.16}$$

We have the following claim:

Let τ be any stopping time. The uniqueness holds for (3.16) *on* $[0, \tau)$.

We remark that when $\tau = T$ is non-random, it follows from Theorem 3.1. Let $X_i(t)$, $i = 1, 2$, be two $\mathbb{X}$-valued continuous $(\mathcal{F}_t)$-adapted processes and satisfy on $[0, \tau)$,

$$X_i(t) = g(t) + \int_0^t A_n\big(t, s, X_i(s)\big)\,\mathrm{d}s + \int_0^t B_n\big(t, s, X_i(s)\big)\,\mathrm{d}W(s), \quad i = 1, 2.$$

Set

$$Z(t) := X_1(t) - X_2(t).$$

Since $\kappa_{3,n} \in \mathscr{K}_0$, as the calculations in (3.7), by BDG's inequality (2.17) and (H3) for A_n and B_n, we have

$$\begin{aligned}
&\mathbb{E}\big\|Z(t) \cdot 1_{\{t<\tau\}}\big\|_{\mathbb{X}}^p \\
&\quad \preccurlyeq \mathbb{E}\left(\int_0^{t\wedge\tau} \kappa_{3,n}(t, s) \cdot \big\|Z(s)\big\|_{\mathbb{X}}\,\mathrm{d}s\right)^p + \mathbb{E}\left(\int_0^{t\wedge\tau} \kappa_{3,n}(t, s) \cdot \big\|Z(s)\big\|_{\mathbb{X}}^2\,\mathrm{d}s\right)^{\frac{p}{2}} \\
&\quad = \mathbb{E}\left(\int_0^{t} \kappa_{3,n}(t, s) \cdot 1_{\{s<\tau\}} \cdot \big\|Z(s)\big\|_{\mathbb{X}}\,\mathrm{d}s\right)^p + \mathbb{E}\left(\int_0^{t} \kappa_{3,n}(t, s) \cdot 1_{\{s<\tau\}} \cdot \big\|Z(s)\big\|_{\mathbb{X}}^2\,\mathrm{d}s\right)^{\frac{p}{2}} \\
&\quad \preccurlyeq \int_0^t \kappa_{3,n}(t, s) \cdot \mathbb{E}\big\|Z(s) \cdot 1_{\{s<\tau\}}\big\|_{\mathbb{X}}^p\,\mathrm{d}s.
\end{aligned} \tag{3.17}$$

By Lemma 2.2, we get

$$\mathbb{E}\big\|Z(t) \cdot 1_{\{t<\tau\}}\big\|_{\mathbb{X}}^p = 0 \quad \text{for almost all } t \in [0, T],$$

which implies by the arbitrariness of T and the continuities of $X_i(t)$, $i = 1, 2$,

$$X_1(\cdot)|_{[0,\tau)} = X_2(\cdot)|_{[0,\tau)}.$$

The claim is proved.

Now, for $n \in \mathbb{N}$, define the stopping times

$$\tau_n := \inf\big\{t > 0\colon \big\|X_n(t)\big\|_{\mathbb{X}} > n\big\}$$

and

$$\sigma_n := \inf\big\{t > 0\colon \big\|X_{n+1}(t)\big\|_{\mathbb{X}} > n\big\}.$$

By the above claim, we have

$$X_n(\cdot)|_{[0,\tau_n\wedge\sigma_n)} = X_{n+1}(\cdot)|_{[0,\tau_n\wedge\sigma_n)},$$

which implies

$$\tau_n \leqslant \sigma_n \leqslant \tau_{n+1}, \quad \text{a.s.}$$

Hence, we may define

$$\tau(\omega) := \lim_{n\to\infty} \tau_n(\omega)$$

and for all $t < \tau(\omega)$,

$$X(t,\omega) := X_n(t,\omega), \quad \text{if } t < \tau_n(\omega).$$

Clearly, (X,τ) is a maximal solution of Eq. (3.1) in the sense of Definition 3.5.

We next prove the uniqueness. Let $(\tilde{X},\tilde{\tau})$ be another maximal solution of Eq. (3.1) in the sense of Definition 3.5. Define the stopping times

$$\tilde{\tau}_n := \inf\bigl\{t > 0:\ \bigl\|\tilde{X}(t)\bigr\|_{\mathbb{X}} > n\bigr\}$$

and

$$\hat{\tau}_n := \tau_n \wedge \tilde{\tau}_n, \qquad \hat{\tau} := \tau \wedge \tilde{\tau}.$$

It is clear that

$$\hat{\tau}_n \nearrow \hat{\tau} \quad \text{a.s. as } n \to \infty,$$

and

$$\begin{aligned}
1_{[0,\hat{\tau}_n)}(t)\cdot \tilde{X}(t) &= 1_{[0,\hat{\tau}_n)}(t)\cdot g(t) + 1_{[0,\hat{\tau}_n)}(t)\cdot \int_0^t A\bigl(t,s,\tilde{X}(s)\bigr)\,\mathrm{d}s \\
&\quad + 1_{[0,\hat{\tau}_n)}(t)\cdot \int_0^t B\bigl(t,s,\tilde{X}(s)\bigr)\,\mathrm{d}W(s) \\
&= 1_{[0,\hat{\tau}_n)}(t)\cdot g(t) + 1_{[0,\hat{\tau}_n)}(t)\cdot \int_0^t A_n\bigl(t,s,\tilde{X}(s)\bigr)\,\mathrm{d}s \\
&\quad + 1_{[0,\hat{\tau}_n)}(t)\cdot \int_0^t B_n\bigl(t,s,\tilde{X}(s)\bigr)\,\mathrm{d}W(s).
\end{aligned}$$

By the above claim again, we have

$$X(\cdot)|_{[0,\hat{\tau}_n)} = \tilde{X}(\cdot)|_{[0,\hat{\tau}_n)}.$$

So

$$X(\cdot)|_{[0,\hat{\tau})} = \tilde{X}(\cdot)|_{[0,\hat{\tau})}.$$

By the definition of maximal solution we must have $\hat{\tau} = \tau = \tilde{\tau}$. $\square$

We have the following simple criterion of non-explosion.

Theorem 3.8. *Assume that* (H1)′*,* (H2) *and* (H4) *hold, and* κ_1 *in* (H2) *belongs to* $\mathscr{K}_{>1}$. *Then there is no explosion for Eq.* (3.1)*.*

Proof. Let (X, τ) be a maximal solution of Eq. (3.1). Define

$$\tau_n := \inf\{t > 0\colon \|X(t)\|_{\mathbb{X}} \geqslant n\}.$$

By BDG's inequality (2.17) and Hölder's inequality, and using the same method as estimating (3.17), we have, for any $T > 0$ and some $\beta > 1$ and $p \geqslant 2\beta^*$ ($\beta^* = \beta/(\beta - 1)$),

$$\begin{aligned}
\mathbb{E}\|X(t)\cdot 1_{\{t\leqslant\tau_n\}}\|_{\mathbb{X}}^p &\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^p + \mathbb{E}\left(\int_0^{t\wedge\tau_n} \|A(t,s,X(s))\|_{\mathbb{X}}\,\mathrm{d}s\right)^p \\
&\quad + \mathbb{E}\left\|\int_0^{t\wedge\tau_n} B(t,s,X(s))\,\mathrm{d}W(s)\right\|_{\mathbb{X}}^p \\
&\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^p + \mathbb{E}\left(\int_0^{t\wedge\tau_n} \kappa_1(t,s)\cdot\left(\|X(s)\|_{\mathbb{X}} + 1\right)\mathrm{d}s\right)^p \\
&\quad + \mathbb{E}\left(\int_0^{t\wedge\tau_n} \|B(t,s,X(s))\|_{L_2(l^2;\mathbb{X})}^2\,\mathrm{d}s\right)^{\frac{p}{2}} \\
&\preccurlyeq \mathbb{E}\|g(t)\|_{\mathbb{X}}^p + \mathbb{E}\left(\int_0^{t\wedge\tau_n} \left(\|X(s)\|_{\mathbb{X}}^{\beta^*} + 1\right)\mathrm{d}s\right)^{\frac{p}{\beta^*}} \\
&\quad + \mathbb{E}\left(\int_0^{t\wedge\tau_n} \left(\|X(s)\|_{\mathbb{X}}^{2\beta^*} + 1\right)\mathrm{d}s\right)^{\frac{p}{2\beta^*}} \\
&\leqslant C_{T,p}\left[\mathbb{E}\|g(s)\|_{\mathbb{X}}^p + 1 + \int_0^t \mathbb{E}\|X(s)\cdot 1_{\{s\leqslant\tau_n\}}\|_{\mathbb{X}}^p\,\mathrm{d}s\right],
\end{aligned}$$

where the constant $C_{T,p}$ is independent of n.

By Gronwall's inequality, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\left\| X(t)\cdot 1_{\{t\leqslant\tau_n\}}\right\|_{\mathbb{X}}^p \leqslant C_{T,p}.$$

Using this estimate, as in the proofs of Theorem 3.3 and Lemma 3.4, we can prove that for any $T>0$ and $p\geqslant 2$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_n]} \left\|X(t)\right\|_{\mathbb{X}}^p\Big) \leqslant C_{T,p}.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} P\{\tau_n\leqslant T\} &= \lim_{n\to\infty} P\Big\{\sup_{t\in[0,T\wedge\tau_n]}\left\|X(t)\right\|_{\mathbb{X}}\geqslant n\Big\}\\ &\leqslant \lim_{n\to\infty}\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_n]}\left\|X(t)\right\|_{\mathbb{X}}^p\Big)/n^p\\ &\leqslant \lim_{n\to\infty} C_{T,p}/n^p = 0,\end{aligned}$$

which produces the non-explosion, i.e., $P\{\tau<\infty\}=0$. □

Remark 3.9. One cannot directly prove

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left\|X(t\wedge\tau_n)\right\|_{\mathbb{X}}^p<+\infty, \quad \forall t\geqslant 0,$$

to obtain the non-explosion, because it does not in general make sense to write

$$\int_0^{t\wedge\tau_n} B\big(t\wedge\tau_n, s, X(s)\big)\,\mathrm{d}W(s).$$

3.4. Continuous dependence of solutions with respect to data

In this subsection, we study the continuous dependence of solutions for Eq. (3.1) with respect to the coefficients.

Let $\{(g_m, A_m, B_m), m\in\mathbb{N}\}$ be a sequence of coefficients associated to Eq. (3.1). Assume that for each $m\in\mathbb{N}$, (g_m, A_m, B_m) satisfies (H1)′–(H4)′ with the same $\kappa_{1,R}, \kappa_{2,R}$ and λ_R as (g, A, B), and for each $p\geqslant 2$,

$$\lim_{m\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left\|g_m(t)-g(t)\right\|_{\mathbb{X}}^p = 0 \tag{3.18}$$

and for each $T, R>0$,

$$\lim_{m\to\infty} \sup_{t\in[0,T],\|x\|_{\mathbb{X}}\leqslant R} \int_0^t \big\| A_m(t,s,x) - A(t,s,x) \big\|_{\mathbb{X}} \, \mathrm{d}s = 0, \tag{3.19}$$

$$\lim_{m\to\infty} \sup_{t\in[0,T],\|x\|_{\mathbb{X}}\leqslant R} \int_0^t \big\| B_m(t,s,x) - B(t,s,x) \big\|^2_{L_2(l^2;\mathbb{X})} \, \mathrm{d}s = 0. \tag{3.20}$$

Let (X_m, τ_m) (resp. (X, τ)) be the unique maximal solution associated with (g_m, A_m, B_m) (resp. (g, A, B)). For each $R > 0$ and $m \in \mathbb{N}$, define

$$\tau_m^R := \inf\big\{t > 0\colon\ \big\|X(t)\big\|_{\mathbb{X}}, \big\|X_m(t)\big\|_{\mathbb{X}} > R\big\}.$$

Suppose that for each $t > 0$,

$$\lim_{R\to\infty} \sup_m P\big\{\tau_m^R < t\big\} = 0. \tag{3.21}$$

Then we have:

Theorem 3.10. *For each $t > 0$ and $\epsilon > 0$,*

$$\lim_{m\to\infty} P\big\{\big\|X_m(t) - X(t)\big\|_{\mathbb{X}} \geqslant \epsilon\big\} = 0.$$

Proof. For $R > 0$ and $m \in \mathbb{N}$, set

$$Z_m^R(t) := \big(X_m(t) - X(t)\big) \cdot 1_{\{t\leqslant\tau_m^R\}}.$$

Then

$$Z_m^R(t) = J_{1,m}^R(t) + J_{2,m}^R(t) + J_{3,m}^R(t) + J_{4,m}^R(t) + J_{5,m}^R(t),$$

where

$$J_{1,m}^R(t) := 1_{\{t\leqslant\tau_m^R\}} \cdot \big[g_m(t) - g(t)\big],$$

$$J_{2,m}^R(t) := 1_{\{t\leqslant\tau_m^R\}} \cdot \int_0^{t\wedge\tau_m^R} \big[A_m\big(t,s,X_n(s)\big) - A_m\big(t,s,X(s)\big)\big] \, \mathrm{d}s,$$

$$J_{3,m}^R(t) := 1_{\{t\leqslant\tau_m^R\}} \cdot \int_0^{t\wedge\tau_m^R} \big[A_m\big(t,s,X(s)\big) - A\big(t,X(s)\big)\big] \, \mathrm{d}s,$$

$$J_{4,m}^R(t) := 1_{\{t\leqslant\tau_m^R\}} \cdot \int_0^{t\wedge\tau_m^R} \big[B_m\big(t,s,X_m(s)\big) - B_m\big(t,s,X(s)\big)\big] \, \mathrm{d}W(s),$$

$$J_{5,m}^R(t) := 1_{\{t\leqslant\tau_m^R\}}\cdot\int_0^{t\wedge\tau_m^R}\big[B_m\big(t,s,X(s)\big)-B\big(t,s,X(s)\big)\big]\,\mathrm{d}W(s).$$

Fix $T>0$. Clearly, for any $p\geqslant 2$ and $t\in[0,T]$,

$$\mathbb{E}\big\|J_{1,m}^R(t)\big\|_{\mathbb{X}}^p\leqslant\sup_{t\in[0,T]}\mathbb{E}\big\|g_m(t)-g(t)\big\|_{\mathbb{X}}^p=:\mathcal{J}_{1,m}.$$

For $J_{2,m}^R(t)$, by (H3)$'$ and Hölder's inequality we have, for p large enough ($\kappa_{2,R}\in\mathscr{K}_{>1}$)

$$\begin{aligned}\mathbb{E}\big\|J_{2,m}^R(t)\big\|_{\mathbb{X}}^p&\leqslant\mathbb{E}\left(\int_0^{t\wedge\tau_m^R}\kappa_{2,R}(t,s)\cdot\big\|X_m(s)-X(s)\big\|_{\mathbb{X}}\,\mathrm{d}s\right)^p\\&\leqslant\left[\int_0^t\kappa_{2,R}^\beta(t,s)\,\mathrm{d}s\right]^{\frac{p}{\beta}}\cdot\mathbb{E}\left[\int_0^t\big\|Z_m^R(s)\big\|_{\mathbb{X}}^{\beta^*}\,\mathrm{d}s\right]^{\frac{p}{\beta^*}}\\&\leqslant C\int_0^t\mathbb{E}\big\|Z_m^R(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s.\end{aligned}$$

For $J_{3,m}^R(t)$, we have

$$\begin{aligned}\mathbb{E}\big\|J_{3,m}^R(t)\big\|_{\mathbb{X}}^p&\leqslant\mathbb{E}\left(\sup_{\|x\|_{\mathbb{X}}\leqslant R}\int_0^{t\wedge\tau_m^R}\big\|A_m(t,s,x)-A(t,s,x)\big\|_{\mathbb{X}}\,\mathrm{d}s\right)^p\\&\leqslant\left(\sup_{t\in[0,T]}\sup_{\|x\|_{\mathbb{X}}\leqslant R}\int_0^t\big\|A_m(t,s,x)-A(t,s,x)\big\|_{\mathbb{X}}\,\mathrm{d}s\right)^p\\&=:\mathcal{J}_{3,m}^R.\end{aligned}$$

Similarly, by BDG's inequality (2.17) we have, for p large enough

$$\mathbb{E}\big\|J_{4,m}^R(t)\big\|_{\mathbb{X}}^p\leqslant C\int_0^t\mathbb{E}\big\|Z_m^R(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s$$

and

$$\mathbb{E}\big\|J_{5,m}^R(t)\big\|_{\mathbb{X}}^p\leqslant C_p\left(\sup_{t\in[0,T]}\sup_{\|x\|_{\mathbb{X}}\leqslant R}\int_0^t\big\|B_m(t,s,x)-B(t,s,x)\big\|_{L_2(l^2;\mathbb{X})}^2\,\mathrm{d}s\right)^{\frac{p}{2}}=:\mathcal{J}_{5,m}^R.$$

Combining the above calculations, we get

$$\mathbb{E}\big\|Z_m^R(t)\big\|_{\mathbb{X}}^p \leqslant \mathcal{J}_{1,m} + \mathcal{J}_{3,m}^R + \mathcal{J}_{5,m}^R + C\int_0^t \mathbb{E}\big\|Z_m^R(s)\big\|_{\mathbb{X}}^p\,\mathrm{d}s.$$

By Gronwall's inequality and (3.18)–(3.20) we get, for any $R > 0$ and p large enough

$$\lim_{m\to\infty} \mathbb{E}\big\|Z_m^R(t)\big\|_{\mathbb{X}}^p = 0.$$

Hence

$$\begin{aligned} P\big\{\big\|X_m(t) - X(t)\big\|_{\mathbb{X}} \geqslant \epsilon\big\} &\leqslant P\big\{\big\|X_m(t) - X(t)\big\|_{\mathbb{X}} \cdot 1_{\{t\leqslant\tau_m^R\}} \geqslant \epsilon\big\} + P\big\{\tau_m^R < t\big\} \\ &\leqslant \mathbb{E}\big\|Z_m^R(t)\big\|_{\mathbb{X}}^p/\epsilon^p + P\big\{\tau_m^R < t\big\}. \end{aligned}$$

First letting $m \to \infty$ and then $R \to \infty$, we then get the desired limit by (3.21). □

4. Large deviation for stochastic Volterra equations

In this section, we study the large deviation of small perturbations for stochastic Volterra equations and work in the finite time interval $[0, T]$.

In what follows, we fix a densely defined closed linear operator $\mathfrak{L}$ on $\mathbb{X}$ for which

$$S_\phi := \big\{\lambda \in \mathbb{C}\colon\ 0 < \phi \leqslant |\arg\lambda| \leqslant \pi\big\} \subset \rho(\mathfrak{L}), \tag{4.1}$$

and for some $C \geqslant 1$,

$$\big\|(\lambda - \mathfrak{L})^{-1}\big\|_{L(\mathbb{X})} \leqslant \frac{C}{1+|\lambda|}, \quad \lambda \in S_\phi,$$

where $\rho(\mathfrak{L})$ denotes the resolvent set of $\mathfrak{L}$. The above operator $\mathfrak{L}$ is also called sectorial (cf. [28, p. 18]). It is well known that $\mathfrak{L}$ generates an analytic semigroup

$$\mathfrak{T}_t = e^{-\mathfrak{L}t}, \quad t \geqslant 0.$$

Moreover, we also assume that $\mathfrak{L}^{-1}$ is a bounded linear operator on $\mathbb{X}$, i.e.,

$$0 \in \rho(\mathfrak{L}).$$

Thus, for any $\alpha \in \mathbb{R}$, the fractional power $\mathfrak{L}^\alpha$ is well defined (cf. [28,48]). For $\alpha > 0$, we define the fractional Sobolev space $\mathbb{X}_\alpha$ by

$$\mathbb{X}_\alpha := \mathscr{D}\big(\mathfrak{L}^\alpha\big)$$

with the norm

$$\|x\|_{\mathbb{X}_\alpha} := \big\|\mathfrak{L}^\alpha x\big\|_{\mathbb{X}}.$$

For $\alpha < 0$, $\mathbb{X}_\alpha$ is defined as the completion of $\mathbb{X}$ with respect to the above norm. It is clear that $\mathbb{X}_\alpha$ is still 2-smooth, and $B \in L_2(l^2; \mathbb{X}_\alpha)$ if and only if $\mathfrak{L}^\alpha B \in L_2(l^2; \mathbb{X})$, i.e.,

$$\|B\|_{L_2(l^2;\mathbb{X}_\alpha)} = \big\|\mathfrak{L}^\alpha B\big\|_{L_2(l^2;\mathbb{X})}. \tag{4.2}$$

We recall the following well-known properties about $\mathfrak{T}_t$ for later use (cf. [28, pp. 24–27] or [48, p. 74]).

Proposition 4.1.

(i) $\mathfrak{T}_t : \mathbb{X} \to \mathbb{X}_\alpha$ *for each* $t > 0$ *and* $\alpha > 0$.
(ii) *For each* $t > 0$, $\alpha \in \mathbb{R}$ *and every* $x \in \mathbb{X}_\alpha$, $\mathfrak{T}_t \mathfrak{L}^\alpha x = \mathfrak{L}^\alpha \mathfrak{T}_t x$.
(iii) *For some* $\delta > 0$ *and each* $t, \alpha > 0$, *the operator* $\mathfrak{L}^\alpha \mathfrak{T}_t$ *is bounded in* $\mathbb{X}$ *and*

$$\big\|\mathfrak{L}^\alpha \mathfrak{T}_t x\big\|_{\mathbb{X}} \leqslant C_\alpha t^{-\alpha} e^{-\delta t} \|x\|_{\mathbb{X}}, \quad \forall x \in \mathbb{X}.$$

(iv) *Let* $\alpha \in (0, 1]$ *and* $x \in \mathbb{X}_\alpha$, *then*

$$\|\mathfrak{T}_t x - x\|_{\mathbb{X}} \leqslant C_\alpha t^\alpha \|x\|_{\mathbb{X}_\alpha}.$$

(v) *For any* $0 \leqslant \beta < \alpha$,

$$\|x\|_{\mathbb{X}_\beta} \leqslant C_{\alpha,\beta} \|x\|^{1-\frac{\beta}{\alpha}} \|x\|^{\frac{\beta}{\alpha}}_{\mathbb{X}_\alpha}, \quad \forall x \in \mathbb{X}_\alpha.$$

In addition to (H2)′, (H3)′ and (H4)′, in this section we assume that g and A, B are non-random, and:

(H1)″ For some $\delta > 0$,

$$\big\|g(t) - g\big(t'\big)\big\|_{\mathbb{X}} \leqslant C_T \big|t - t'\big|^\delta, \quad t, t' \in [0, T],$$

and for some $\alpha > 0$,

$$\sup_{t\in[0,T]} \big\|g(t)\big\|_{\mathbb{X}_\alpha} < +\infty.$$

(H2)″ For the same α as in (H1)″ and any $R > 0$, there exists a kernel function $\kappa_{\alpha,R} \in \mathscr{K}_0$ such that for all $(t, s) \in \Delta$ and $x \in \mathbb{X}$ with $\|x\|_{\mathbb{X}} \leqslant R$,

$$\big\|A(t, s, x)\big\|_{\mathbb{X}_\alpha} + \big\|B(t, s, x)\big\|^2_{L_2(l^2;\mathbb{X}_{\frac{\alpha}{2}})} \leqslant \kappa_{\alpha,R}(t, s).$$

Remark 4.2. If the $\kappa_{\alpha,R}$ in (H2)″ belongs to $\mathscr{K}_{>1}$, then (H2)″ implies (H2)′ in view of $\mathbb{X}_\alpha \hookrightarrow \mathbb{X}$.

Consider the following small perturbation of stochastic Volterra equation (3.1)

$$X_\epsilon(t) = g(t) + \int_0^t A\big(t, s, X_\epsilon(s)\big)\,\mathrm{d}s + \sqrt{\epsilon}\int_0^t B\big(t, s, X_\epsilon(s)\big)\,\mathrm{d}W(s), \tag{4.3}$$

where $\epsilon \in (0, 1)$. By Theorem 3.7, there exists a unique maximal solution $(X_\epsilon, \tau_\epsilon)$ for Eq. (4.3). Below, we fix $T > 0$ and work in the finite time interval $[0, T]$, and assume that for each $\epsilon \in (0, 1)$,

$$\tau_\epsilon > T, \quad \text{a.s.}$$

By Yamada–Watanabe's theorem (cf. [45,57]), there exists a measurable mapping

$$\Phi_\epsilon : \mathbb{C}_T(\mathbb{U}) \to \mathbb{C}_T(\mathbb{X})$$

such that

$$X_\epsilon(t, \omega) = \Phi_\epsilon\big(W(\cdot, \omega)\big)(t).$$

It should be noticed that although the equation considered in [45] is a little different from Eq. (3.1), the proof is still adapted to our more general equation.

We now fix a family of processes $\{h^\epsilon,\ \epsilon \in (0, 1)\}$ in $\mathcal{A}_N^T$ (see (2.20) for the definition of $\mathcal{A}_N^T$), and put

$$X^\epsilon(t, \omega) := \Phi_\epsilon\left(W(\cdot, \omega) + \frac{h^\epsilon(\cdot, \omega)}{\sqrt{\epsilon}}\right)(t).$$

Here, we have used a little confused notations X_ϵ and X^ϵ, but they are clearly different. By Girsanov's theorem (cf. [45, Section 7]), $X^\epsilon(t)$ solves the following stochastic Volterra equation (also called control equation):

$$\begin{aligned} X^\epsilon(t) = g(t) &+ \int_0^t A\big(t, s, X^\epsilon(s)\big)\,\mathrm{d}s + \int_0^t B\big(t, s, X^\epsilon(s)\big)\dot{h}^\epsilon(s)\,\mathrm{d}s \\ &+ \sqrt{\epsilon}\int_0^t B\big(t, s, X^\epsilon(s)\big)\,\mathrm{d}W(s). \end{aligned} \tag{4.4}$$

Although h is defined only on $[0, T]$, we can extend it to $\mathbb{R}_+$ by setting $\dot{h}(t) = 0$ for $t > T$ so that Eq. (4.4) can be considered on $\mathbb{R}_+$. We shall always use this extension below. Let τ^ϵ be the explosion time of Eq. (4.4). For $n \in \mathbb{N}$, define

$$\tau_n^\epsilon := \inf\big\{t \geqslant 0:\ \big\|X^\epsilon(t)\big\|_{\mathbb{X}} > n\big\}. \tag{4.5}$$

Then $\tau_n^\epsilon \nearrow \tau^\epsilon$ as $n \to \infty$. We have:

Lemma 4.3. *For any $\alpha_0 \in (0, \alpha)$, there is an $a > 0$ such that for p sufficiently large*

$$\sup_{\epsilon\in(0,1)} \mathbb{E}\left(\sup_{t\neq t'\in[0,T\wedge\tau_n^\epsilon]} \frac{\|X^\epsilon(t') - X^\epsilon(t)\|^p_{\mathbb{X}_{\alpha_0}}}{|t'-t|^{ap}} \right) \leqslant C_{N,n,T,p,\kappa_{\alpha,n},\alpha_0}.$$

Proof. Note that

$$\begin{aligned}
&\big\|X^\epsilon(t)\cdot 1_{\{t\leqslant\tau_n^\epsilon\}}\big\|_{\mathbb{X}_\alpha} \\
&\quad\leqslant \big\|g(t)\big\|_{\mathbb{X}_\alpha} + \int_0^{t\wedge\tau_n^\epsilon} \big\|A\big(t,s,X^\epsilon(s)\big)\big\|_{\mathbb{X}_\alpha}\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_n^\epsilon} \big\|B\big(t,s,X^\epsilon(s)\big)\dot{h}^\epsilon(s)\big\|_{\mathbb{X}_\alpha}\,\mathrm{d}s + \sqrt{\epsilon}\left\| \int_0^{t\wedge\tau_n^\epsilon} B\big(t,s,X^\epsilon(s)\big)\,\mathrm{d}W(s)\right\|_{\mathbb{X}_\alpha} \\
&\quad=: J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}$$

By (H2)$''$ and (4.5) we have

$$\mathbb{E}\big|J_2(t)\big|^p \leqslant C_n\mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon} \kappa_{\alpha,n}(t,s)\,\mathrm{d}s\right)^p \leqslant C_{n,T,p,\kappa_{\alpha,n}}$$

and by Hölder's inequality

$$\begin{aligned}
\mathbb{E}\big|J_3(t)\big|^p &\leqslant \mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon} \big\|B\big(t,s,X^\epsilon(s)\big)\dot{h}^\epsilon(s)\big\|_{\mathbb{X}_\alpha}\,\mathrm{d}s\right)^p \\
&\leqslant \mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon} \big\|B\big(t,s,X^\epsilon(s)\big)\big\|_{L_2(l^2;\mathbb{X}_\alpha)}\cdot\big\|\dot{h}^\epsilon(s)\big\|_{l^2}\,\mathrm{d}s\right)^p \\
&\leqslant N^{\frac{p}{2}}\mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon} \big\|B\big(t,s,X^\epsilon(s)\big)\big\|^2_{L_2(l^2;\mathbb{X}_\alpha)}\,\mathrm{d}s\right)^{\frac{p}{2}} \\
&\leqslant C_{N,n,T,p,\kappa_{\alpha,n}},
\end{aligned}$$

where we have used that $h^\epsilon \in \mathcal{A}_N^T$.

Similarly, by BDG's inequality (2.17) and (H2)$''$ we have

$$\mathbb{E}\big|J_4(t)\big|^p \leqslant C_p\mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon} \big\|B\big(t,s,X^\epsilon(s)\big)\big\|^2_{L_2(l^2;\mathbb{X}_\alpha)}\,\mathrm{d}s\right)^{\frac{p}{2}} \leqslant C_{n,T,p,\kappa_{\alpha,n}}.$$

Combining the above calculations, we get

$$\sup_{\epsilon\in(0,1)}\sup_{t\in[0,T]}\mathbb{E}\big\|X^\epsilon(t)\cdot 1_{\{t\leqslant\tau_n^\epsilon\}}\big\|_{\mathbb{X}_\alpha}^p\leqslant C_{N,n,T,p,\kappa_{\alpha,n}},\quad p\geqslant 2. \tag{4.6}$$

Moreover, as in the proofs of Theorem 3.3 and Lemma 3.4, by (H1)″, (H2)′ and (H4)′, for some $\beta_3>1$ and $p\geqslant 2\beta_3^*$ ($\beta_3^*:=\beta_3/(\beta_3-1)$), we have that for any $0\leqslant t<t'\leqslant T$,

$$\sup_{\epsilon\in(0,1)}\mathbb{E}\big\|\big(X^\epsilon(t')-X^\epsilon(t)\big)\cdot 1_{\{t',t\leqslant\tau_n^\epsilon\}}\big\|_{\mathbb{X}}^p\leqslant C_{T,p,n}\big(\big|t-t'\big|^{\delta p}+\big|t-t'\big|^{\frac{\gamma p}{2}}+\big|t-t'\big|^{\frac{p}{2\beta_3^*}}\big).$$

Thus, by (v) of Proposition 4.1 and (4.6), for any $\alpha_0\in(0,\alpha)$ and p large enough we have

$$\begin{aligned}&\sup_{\epsilon\in(0,1)}\mathbb{E}\big\|\big(X^\epsilon(t')-X^\epsilon(t)\big)\cdot 1_{\{t',t\leqslant T\wedge\tau_n^\epsilon\}}\big\|_{\mathbb{X}_{\alpha_0}}^p\\&\quad\leqslant C_{N,n,T,p,\kappa_{\alpha,n},\alpha_0}\big(\big|t-t'\big|^{\delta p}+\big|t-t'\big|^{\frac{\gamma p}{2}}+\big|t-t'\big|^{\frac{p}{2\beta^*}}\big)^{1-\frac{\alpha_0}{\alpha}}.\end{aligned}$$

The desired estimate now follows by Theorem 2.10. □

In order to obtain the tightness of the laws of $\{X^\epsilon,\epsilon\in(0,1)\}$ in $\mathbb{C}_T(\mathbb{X})$, we assume that:

(C1) $\mathfrak{L}^{-1}$ is a compact operator on $\mathbb{X}$.
(C2) $\lim_{n\to\infty}\sup_{\epsilon\in(0,1)}P\{\omega\colon \tau_n^\epsilon(\omega)<T\}=0$.

Note that (C2) implies

$$P\big\{\omega\colon \tau^\epsilon(\omega)>T\big\}=1.$$

We now prove the following key lemma for the large deviation principle of Eq. (4.3).

Lemma 4.4. *Under* (C1) *and* (C2)*, there exist subsequence $\epsilon_k\downarrow 0$, a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde P)$ and a sequence $\{(\tilde h^k,\tilde X^k,\tilde W^k)\}_{k\in\mathbb{N}}$ as well as $(h,X^h,\tilde W)$ defined on this probability space and taking values in $\mathbb{D}_N\times\mathbb{C}_T(\mathbb{X})\times\mathbb{C}_T(\mathbb{U})$ such that*:

(i) *$(\tilde h^k,\tilde X^k,\tilde W^k)$ has the same law as $(h^{\epsilon_k},X^{\epsilon_k},W)$ for each $k\in\mathbb{N}$;*
(ii) *$(\tilde h^k,\tilde X^k,\tilde W^k)\to(h,X^h,\tilde W)$ in $\mathbb{D}_N\times\mathbb{C}_T(\mathbb{X})\times\mathbb{C}_T(\mathbb{U})$, $\tilde P$-a.s. as $k\to\infty$;*
(iii) *(h,X^h) uniquely solves the following Volterra equation*:

$$X^h(t)=g(t)+\int_0^t A\big(t,s,X^h(s)\big)\,\mathrm{d}s+\int_0^t B\big(t,s,X^h(s)\big)\dot h(s)\,\mathrm{d}s. \tag{4.7}$$

In particular, (LD)$_1$ *in Section* 2.3 *holds.*

Proof. Let $\alpha_0 \in (0,\alpha)$ and $a>0$ be as in Lemma 4.3. For $R>0$, set

$$K_R := \left\{ x \in \mathbb{C}_T(\mathbb{X})\colon \sup_{t\in[0,T]} \left\| x(t) \right\|_{\mathbb{X}} + \sup_{s\neq t\in[0,T]} \frac{\|x(t)-x(s)\|_{\mathbb{X}_{\alpha_0}}}{|t-s|^a} \leqslant R \right\}.$$

By (C1), $\mathbb{X}_{\alpha_0} \hookrightarrow \mathbb{X}$ is compact (cf. [28, p. 29, Theorem 1.4.8]). Thus, by Ascoli–Arzelà's theorem (cf. [30]), the set K_R is compact in $\mathbb{C}_T(\mathbb{X})$. For any $\delta>0$, by (C2) we can choose n sufficiently large such that

$$\sup_{\epsilon\in(0,1)} P\left\{\omega\colon \tau_n^\epsilon(\omega) < T\right\} \leqslant \delta.$$

By Lemma 4.3 and Chebyschev's inequality, for any $R>n$ we have

$$\begin{aligned} P\left\{X^\epsilon(\cdot)\notin K_R\right\} &= P\left\{X^\epsilon(\cdot)\notin K_R, \tau_n^\epsilon \geqslant T\right\} + P\left\{X^\epsilon(\cdot)\notin K_R, \tau_n^\epsilon < T\right\} \\ &\leqslant P\left\{ \sup_{s\neq t\in[0,T\wedge\tau_n^\epsilon]} \frac{\|X^\epsilon(t)-X^\epsilon(s)\|_{\mathbb{X}_{\alpha_0}}}{|t-s|^a} \geqslant R-n \right\} + P\left\{\tau_n^\epsilon < T\right\} \\ &\leqslant \mathbb{E}\left[\sup_{s\neq t\in[0,T\wedge\tau_n^\epsilon]} \frac{\|X^\epsilon(t)-X^\epsilon(s)\|^p_{\mathbb{X}_{\alpha_0}}}{|t-s|^{ap}} \right] / (R-n)^p + \delta \\ &\leqslant C_{N,n,T,p,\kappa_{\alpha,n},\alpha_0}/(R-n)^p + \epsilon'. \end{aligned}$$

Therefore, for R large enough we have

$$\sup_{\epsilon\in(0,1)} P\left\{X^\epsilon(\cdot)\notin K_R\right\} \leqslant 2\delta.$$

Thus, by the compactness of $\mathbb{D}_N$ (see (2.21)), the laws of $(h^\epsilon, X^\epsilon, W)$ in $\mathbb{D}_N \times \mathbb{C}_T(\mathbb{X}) \times \mathbb{C}_T(\mathbb{U})$ is tight. By Skorohod's embedding theorem (cf. [30]), the conclusions (i) and (ii) hold.

We now prove (iii). Note that by (i) (cf. [45, Section 8])

$$\begin{aligned} \tilde{X}^k(t) &= g(t) + \int_0^t A\left(t,s,\tilde{X}^k(s)\right) \mathrm{d}s + \int_0^t B\left(t,s,\tilde{X}^k(s)\right) \dot{\tilde{h}}^k(s)\, \mathrm{d}s \\ &\quad + \sqrt{\epsilon_k} \int_0^t B\left(t,s,\tilde{X}^k(s)\right) \mathrm{d}\tilde{W}^k(s) \\ &=: g(t) + J_1^k(t) + J_2^k(t) + J_3^k(t), \quad \tilde{P}\text{-a.s.} \end{aligned}$$

Set

$$\tilde{\tau}_n^k := \inf\left\{t \geqslant 0\colon \left\|\tilde{X}^k(t)\right\|_{\mathbb{X}} > n\right\}.$$

Then for any $\delta > 0$, by (i) and (C2) there exists an n large enough such that

$$\begin{aligned}\sup_{k\in\mathbb{N}} \tilde{P}\{\tilde{\tau}_n^k < T\} &= \sup_{k\in\mathbb{N}} \tilde{P}\Big\{\sup_{s\in[0,T)} \|\tilde{X}^k(s)\|_{\mathbb{X}} > n\Big\} \\ &= \sup_{k\in\mathbb{N}} P\Big\{\sup_{s\in[0,T)} \|X^{\epsilon_k}(s)\|_{\mathbb{X}} > n\Big\} \\ &= \sup_{k\in\mathbb{N}} P\{\tau_n^{\epsilon_k} < T\} \leqslant \delta.\end{aligned}$$

Hence, for any $\delta' > 0$, by BDG's inequality (2.17) and (H2)$'$ we have

$$\begin{aligned}\tilde{P}\{\|J_3^k(t)\|_{\mathbb{X}} \geqslant \delta'\} &\leqslant \tilde{P}\{J_3^k(t) \geqslant \delta'; \tilde{\tau}_n^k \geqslant T\} + \tilde{P}\{\tilde{\tau}_n^k < T\} \\ &\leqslant \frac{\mathbb{E}^{\tilde{P}} \|J_3^k(t) \cdot 1_{\{t\leqslant\tilde{\tau}_n^k\}}\|_{\mathbb{X}}^2}{\delta'^2} + \delta \\ &\leqslant \frac{\epsilon_k \cdot C_n \mathbb{E}^{\tilde{P}}(\int_0^{t\wedge\tilde{\tau}_n^k} \kappa_{1,n}(t,s)\,\mathrm{d}s)}{\delta'^2} + \delta \\ &\leqslant \frac{\epsilon_k \cdot C_{n,t}}{\delta'^2} + \delta.\end{aligned}$$

Thus, we get

$$\lim_{k\to\infty} \tilde{P}\{\|J_3^k(t)\|_{\mathbb{X}} \geqslant \delta'\} = 0.$$

Let $J_i(t)$, $i = 1, 2$, be the corresponding terms in Eq. (4.7). In order to prove that X^h solves Eq. (4.7), it is now enough to show that for any $t \in [0, T]$ and $y \in \mathbb{X}^*$,

$$\lim_{k\to\infty} {}_{\mathbb{X}}\langle J_i^k(t) - J_i(t), y\rangle_{\mathbb{X}^*} = 0, \quad i = 1, 2, \ \tilde{P}\text{-a.s.}$$

Observe that

$$\begin{aligned}\big|{}_{\mathbb{X}}\langle J_2^k(t) - J_2(t), y\rangle_{\mathbb{X}^*}\big| &\leqslant \|y\|_{\mathbb{X}^*} \cdot \int_0^t \big\|\big[B\big(t,s,\tilde{X}^k(s)\big) - B\big(t,s,X^h(s)\big)\big]\dot{\tilde{h}}^k(s)\big\|_{\mathbb{X}}\,\mathrm{d}s \\ &\quad + \left|\int_0^t {}_{\mathbb{X}}\langle B\big(t,s,X^h(s)\big)\big[\dot{\tilde{h}}^k(s) - \dot{h}(s)\big], y\rangle_{\mathbb{X}^*}\,\mathrm{d}s\right| \\ &=: \|y\|_{\mathbb{X}^*} \cdot J_{21}^k(t) + J_{22}^k(t).\end{aligned}$$

By the weak convergence of $\tilde{h}^k$ to h in $\mathbb{D}_N$, we have

$$\lim_{k\to\infty} J_{22}^k(t) = 0.$$

Noting that by (ii), for almost all $\tilde{\omega} \in \tilde{\Omega}$ and some $K(\tilde{\omega}) \in \mathbb{N}$,

$$n(\tilde{\omega}) := \sup_{s\in[0,T]} \left\| X^h(s, \tilde{\omega}) \right\|_{\mathbb{X}} \vee \sup_{k \geqslant K(\tilde{\omega})} \sup_{s\in[0,T]} \left\| \tilde{X}^k(s, \tilde{\omega}) \right\|_{\mathbb{X}} < +\infty,$$

we have, by Hölder's inequality and (H3)$'$

$$\begin{aligned} J^k_{21}(t, \tilde{\omega}) &\leqslant \left\| \tilde{h}^k(\tilde{\omega}) \right\|_{\ell^2_T} \cdot \left(\int_0^t \left\| B\big(t, s, \tilde{X}^k(s, \tilde{\omega})\big) - B\big(t, s, X^h(s, \tilde{\omega})\big) \right\|^2_{L_2(l^2;\mathbb{X})} \mathrm{d}s \right)^{1/2} \\ &\leqslant N \cdot \left(\int_0^t \kappa_{2,n(\tilde{\omega})}(t, s) \cdot \left\| \tilde{X}^k(s, \tilde{\omega}) - X^h(s, \tilde{\omega}) \right\|^2_{\mathbb{X}} \mathrm{d}s \right)^{1/2} \\ &\overset{\text{(ii)}}{\to} 0 \quad \text{as } k \to \infty, \end{aligned}$$

where we have used $\tilde{h}^k(\tilde{\omega}) \in \mathbb{D}_N$.

Similarly, we have

$$\lim_{k\to\infty} \left\| J^k_1(t) - J_1(t) \right\|_{\mathbb{X}} = 0, \quad \tilde{P}\text{-a.s.}$$

Combining the above estimates, we find that X^h solves Eq. (4.7). □

Let $I(f)$ be defined by

$$I(f) := \frac{1}{2} \inf_{\{h \in \ell^2_T:\ f = X^h\}} \|h\|^2_{\ell^2_T}, \quad f \in \mathbb{C}_T(\mathbb{X}), \tag{4.8}$$

where X^h is defined by Eq. (4.7). In order to identify $I(f)$, we assume that:

(C3) For any $N \in \mathbb{N}$,

$$\sup_{h\in\mathbb{D}_N} \sup_{t\in[0,T]} \left\| X^h(t) \right\|_{\mathbb{X}} < +\infty.$$

Similar to the proof of Lemma 4.4, we can prove that:

Lemma 4.5. *Under* (C3)*,* $(\mathrm{LD})_2$ *in Section* 2.3 *holds.*

Thus, by Theorem 2.12 we have proven:

Theorem 4.6. *Assume that* (H1)$''$–(H2)$''$, (H2)$'$–(H4)$'$ *and* (C1)–(C3) *hold. Then,* $\{X_\epsilon, \epsilon \in (0, 1)\}$ *satisfies the large deviation principle in* $\mathbb{C}_T(\mathbb{X})$ *with the rate function* $I(f)$ *given by* (4.8).

Remark 4.7. Conditions (C2) and (C3) are satisfied if (H1)$''$, (H2) and (H4) hold, and κ_1 in (H2) belongs to $\mathscr{K}_{>1}$. In fact, we can prove as in the proof of Theorem 3.8

$$\sup_{n\in\mathbb{N}}\sup_{\epsilon\in(0,1)}\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_n^\epsilon]}\big\|X^\epsilon(t)\big\|_{\mathbb{X}}^p\Big)\leqslant C_{T,p,\kappa_1},$$

which then implies (C2). Condition (C3) is more direct in this case.

5. Semilinear stochastic evolutionary integral equations

In this section, we consider the following semilinear stochastic evolutionary integral equation:

$$X(t)=x_0-\int_0^t a(t-s)\mathfrak{L}X(s)\,\mathrm{d}s+\int_0^t \Phi\big(s,X(s)\big)\,\mathrm{d}s+\int_0^t \Psi\big(s,X(s)\big)\,\mathrm{d}W(s),\tag{5.1}$$

where $a:\mathbb{R}_+\to\mathbb{R}_+$ is a measurable function, and

$$\Phi:\mathbb{R}_+\times\Omega\times\mathbb{X}\to\mathbb{X}\in\mathcal{M}\times\mathcal{B}(\mathbb{X})/\mathcal{B}(\mathbb{X})$$

and

$$\Psi:\mathbb{R}_+\times\Omega\times\mathbb{X}\to L_2\big(l^2;\mathbb{X}\big)\in\mathcal{M}\times\mathcal{B}(\mathbb{X})/\mathcal{B}\big(L_2\big(l^2;\mathbb{X}\big)\big).$$

Here and below, $\mathcal{M}$ stands for the progressively measurable σ-algebra over $\mathbb{R}_+\times\Omega$.

Consider first the following deterministic integral equation:

$$x(t)=x_0-\int_0^t a(t-s)\mathfrak{L}x(s)\,\mathrm{d}s.\tag{5.2}$$

The solution of this equation is called the resolvent of $(a,\mathfrak{L})$, and denoted by $\mathfrak{S}_t x_0=x(t)$. Note that in general

$$\mathfrak{S}_{t+s}\neq\mathfrak{S}_t\circ\mathfrak{S}_s.$$

We make the following assumptions:

(S1) The resolvent $\{\mathfrak{S}_t\colon t\geqslant 0\}$ is of analyticity type (ω_0,θ_0) in the sense of [53, Definition 2.1], where $\omega_0\in\mathbb{R}$ and $\theta_0\in(0,\pi/2]$.

(S2) For any $R>0$, there exist $C_R>0$ and $\beta\in[0,1)$ such that for all $s>0$, $\omega\in\Omega$ and $x,y\in\mathbb{X}$ with $\|x\|_{\mathbb{X}},\|y\|_{\mathbb{X}}\leqslant R$,

$$\big\|\Phi(s,\omega,x)\big\|_{\mathbb{X}}+\big\|\Psi(s,\omega,x)\big\|_{L_2(l^2;\mathbb{X})}^2\leqslant\frac{C_R}{(s\wedge 1)^\beta},$$

and

$$\big\|\Phi(s,\omega,x)-\Phi(s,\omega,y)\big\|_{\mathbb{X}} \leqslant \frac{C_R}{(s\wedge 1)^\beta}\|x-y\|_{\mathbb{X}},$$

$$\big\|\Psi(s,\omega,x)-\Psi(s,\omega,y)\big\|^2_{L_2(l^2;\mathbb{X})} \leqslant \frac{C_R}{(s\wedge 1)^\beta}\|x-y\|^2_{\mathbb{X}}.$$

(S3) For all $s>0$, $\omega\in\Omega$ and $x\in\mathbb{X}$, it holds that

$$\big\|\Phi(s,\omega,x)\big\|_{\mathbb{X}} \leqslant \frac{C}{(s\wedge 1)^\beta}\big(1+\|x\|_{\mathbb{X}}\big),$$

$$\big\|\Psi(s,\omega,x)\big\|^2_{L_2(l^2;\mathbb{X})} \leqslant \frac{C}{(s\wedge 1)^\beta}\big(1+\|x\|^2_{\mathbb{X}}\big).$$

The following property of analytic resolvent $\{\mathfrak{S}_t\colon t>0\}$ is crucial for the proof of Theorem 5.2 below (cf. [53, Corollary 2.1]).

Proposition 5.1. *Let $\mathfrak{S}_t$ be an analytic resolvent of type (ω_0,θ_0). Then for any $T>0$,*

$$\sup_{t\in[0,T]}\|\mathfrak{S}_t\|_{L(\mathbb{X};\mathbb{X})} \leqslant C_T \tag{5.3}$$

and for any $t\in(0,T]$,

$$\|\dot{\mathfrak{S}}_t\|_{L(\mathbb{X};\mathbb{X})} \leqslant C_T t^{-1}, \tag{5.4}$$

where the dot denotes the operator derivative and $\|\cdot\|_{L(\mathbb{X};\mathbb{X})}$ denotes the norm of bounded linear operators.

By a solution of Eq. (5.1) we mean that $X(t)$ satisfies the following stochastic Volterra equation:

$$X(t)=\mathfrak{S}_t x_0+\int_0^t \mathfrak{S}_{t-s}\Phi\big(s,X(s)\big)\,\mathrm{d}s+\int_0^t \mathfrak{S}_{t-s}\Psi\big(s,X(s)\big)\,\mathrm{d}W(s). \tag{5.5}$$

Let us define

$$A(t,s,\omega,x):=\mathfrak{S}_{t-s}\Phi(s,\omega,x),\qquad B(t,s,\omega,x):=\mathfrak{S}_{t-s}\Psi(s,\omega,x).$$

We have:

Theorem 5.2. *Under* (S1) *and* (S2)*, there exists a unique maximal solution (X,τ) for Eq.* (5.5) *in the sense of Definition* 3.5*. Moreover, if* (S3) *holds, then $\tau=+\infty$, a.s.*

Proof. First of all, it is easy to see by (5.3) that (H2)′ and (H3)′ hold with

$$\kappa_{1,R}(t,s)=\kappa_{2,R}(t,s)=\frac{C_R}{(s\wedge 1)^\beta}\in\mathscr{K}_{>1}.$$

For $0 \leqslant s < t < t'$, $\omega \in \Omega$ and $x \in \mathbb{X}$ with $\|x\|_{\mathbb{X}} \leqslant R$, we have

$$\begin{aligned}
&\big\|A\big(t', s, \omega, x\big) - A(t, s, \omega, x)\big\|_{\mathbb{X}} \\
&\quad = \big\|(\mathfrak{S}_{t'-s} - \mathfrak{S}_{t-s})\Phi(s, \omega, x)\big\|_{\mathbb{X}} \leqslant \frac{C_R}{(s \wedge 1)^\beta} \|\mathfrak{S}_{t'-s} - \mathfrak{S}_{t-s}\|_{L(\mathbb{X};\mathbb{X})} \\
&\quad \leqslant \frac{C_R}{(s \wedge 1)^\beta} \int_{t-s}^{t'-s} \|\dot{\mathfrak{S}}_r\|_{L(\mathbb{X};\mathbb{X})} \, \mathrm{d}r \overset{(5.4)}{\leqslant} \frac{C_R}{(s \wedge 1)^\beta} \int_{t-s}^{t'-s} \frac{1}{r} \, \mathrm{d}r \\
&\quad = \frac{C_R}{(s \wedge 1)^\beta} \log\left(\frac{t'-s}{t-s}\right)
\end{aligned}$$

and

$$\big\|B\big(t', s, \omega, x\big) - B(t, s, \omega, x)\big\|^2_{L_2(l^2;\mathbb{X})} \leqslant \frac{C_R}{(s \wedge 1)^\beta} \log^2\left(\frac{t'-s}{t-s}\right).$$

Note that the following elementary inequality holds for any $\gamma \in (0, 1)$,

$$\log(1+s) \leqslant C s^\gamma, \quad \forall s > 0.$$

Therefore, for $0 \leqslant s < t < t'$, $\omega \in \Omega$ and $x \in \mathbb{X}$ with $\|x\|_{\mathbb{X}} \leqslant R$,

$$\begin{aligned}
&\big\|A\big(t', s, \omega, x\big) - A(t, s, \omega, x)\big\|_{\mathbb{X}} + \big\|B\big(t', s, \omega, x\big) - B(t, s, \omega, x)\big\|^2_{L_2(l^2;\mathbb{X})} \\
&\quad \leqslant \frac{C_R (t'-t)^\gamma}{(s \wedge 1)^\beta (t-s)^\gamma} \left[1 + \frac{(t'-t)^\gamma}{(t-s)^\gamma}\right] =: \lambda_R\big(t', t, s\big).
\end{aligned}$$

Thus, we find that (H4)$'$ holds if $\gamma \in (0, (1-\beta)/2)$.

Lastly, if (S3) is satisfied, it is clear that (H2) holds with $\kappa_1(t, s) = \frac{C}{(s \wedge 1)^\beta} \in \mathscr{K}_{>1}$, and (H4) also holds from the above calculations. The non-explosion then follows from Theorem 3.8. □

We now turn to the small perturbation of Eq. (5.5) and assume that Φ and Ψ are non-random. Consider

$$X_\epsilon(t) = \mathfrak{S}_t x_0 + \int_0^t \mathfrak{S}_{t-s} \Phi\big(s, X_\epsilon(s)\big) \, \mathrm{d}s + \sqrt{\epsilon} \int_0^t \mathfrak{S}_{t-s} \Psi\big(s, X_\epsilon(s)\big) \, \mathrm{d}W(s).$$

In order to use Theorem 4.6 to get the LDP for $\{X_\epsilon, \epsilon \in (0, 1)\}$, we also assume:

(S4) Let $\{\mathfrak{S}_t \colon t \geqslant 0\}$ be an analytic resolvent of type (ω_0, θ_0). Assume that for some $\omega_1 > \omega_0$, $0 < \theta_1 < \theta_0$, $C > 0$ and $\alpha_1 > 0$,

$$\big|\hat{a}(\lambda)\big| \geqslant C\big(|\lambda - \omega_1|^{\alpha_1} + 1\big)^{-1}, \quad \forall \lambda \in \mathbb{C} \text{ with } \big|\arg(\lambda - \omega)\big| < \theta_1, \tag{5.6}$$

where $\hat{a}$ denotes the Laplace transform of a. Moreover, we also assume that

$$\int_0^r a(s)\,\mathrm{d}s + \int_0^t \big|a(r+s) - a(s)\big|\,\mathrm{d}s \leqslant C_T|r|^\delta, \tag{5.7}$$

where $r, t \in [0, T]$ and $T, \delta > 0$.

We have:

Theorem 5.3. *Under* (S1)–(S4) *and* (C1)*, for any $x_0 \in \mathscr{D}(\mathfrak{L})$, $\{X_\epsilon, \epsilon \in (0, 1)\}$ satisfies the large deviation principle in $\mathbb{C}_T(\mathbb{X})$ with the rate function $I(f)$ given by* (4.8).

Proof. From the proof of Theorem 5.2, it is enough to check (H1)$''$ and (H2)$''$. By (5.6) and [53, p. 57, Theorem 2.2 (ii)], we have

$$\|\mathfrak{L}\mathfrak{S}_t\|_{L(\mathbb{X};\mathbb{X})} \leqslant Ce^{\omega_1 t}\big(1 + t^{-\alpha_1}\big), \quad \forall t > 0,$$

which together with (v) of Proposition 4.1 yields that for any $\alpha \in (0, 1)$ and $T > 0$,

$$\big\|\mathfrak{L}^\alpha \mathfrak{S}_t\big\|_{L(\mathbb{X};\mathbb{X})} \leqslant C_T\big(1 + t^{-\alpha_1\cdot\alpha}\big), \quad \forall t \in (0, T].$$

Thus, (H2)$''$ holds by choosing $\alpha < \frac{1-\beta}{\alpha_1}$, where β is from (S3).

For (H1)$''$, since $x_0 \in \mathscr{D}(\mathfrak{L}) = \mathbb{X}_1$, by (5.3) we have

$$\|\mathfrak{L}\mathfrak{S}_t x_0\|_{\mathbb{X}} = \|\mathfrak{S}_t\mathfrak{L}x_0\|_{\mathbb{X}} \leqslant C\|\mathfrak{L}x_0\|_{\mathbb{X}}.$$

On the other hand, by the resolvent equation (5.2) and (5.7) we have, for any $0 \leqslant t < t' \leqslant T$,

$$\begin{aligned}\|\mathfrak{S}_{t'}x_0 - \mathfrak{S}_t x_0\|_{\mathbb{X}} &\leqslant \int_0^t \big|a\big(t'-s\big) - a(t-s)\big| \cdot \|\mathfrak{L}\mathfrak{S}_s x_0\|_{\mathbb{X}}\,\mathrm{d}s \\ &\quad + \int_t^{t'} \big|a\big(t'-s\big)\big| \cdot \|\mathfrak{L}\mathfrak{S}_s x_0\|_{\mathbb{X}}\,\mathrm{d}s \\ &\leqslant C_T\|\mathfrak{L}x_0\|_{\mathbb{X}} \cdot \big|t'-t\big|^\delta.\end{aligned}$$

The proof is thus complete by Theorem 4.6 and Remark 4.7. □

Example 5.4. Let a be a completely monotonic kernel function, i.e.,

$$a(t) = \int_0^\infty e^{-st}\,\mathrm{d}\rho(s), \quad t > 0, \tag{5.8}$$

where $s \mapsto \rho(s)$ is non-decreasing and such that $\int_1^\infty \mathrm{d}\rho(s)/s < \infty$. Then the resolvent $\{\mathfrak{S}_t: t \geqslant 0\}$ associated with a is of analyticity type $(0,\theta)$ for some $\theta \in (0,\pi/2)$ (cf. [53, p. 55, Example 2.2]), i.e., (S1) holds. For (S4), besides (5.8) and (5.7), we also assume that for some $C, \alpha_1 > 0$,

$$C(1+\lambda)^{-\alpha_1} \leqslant \int_0^\infty e^{-\lambda t} \cdot a(t)\,\mathrm{d}t < +\infty, \quad \forall \lambda > 0, \tag{5.9}$$

which implies by [53, p. 221, Lemma 8.1(v)] that (5.6) holds. In particular,

$$a_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \in (0,1],$$

is completely monotonic and satisfies (5.7) and (5.9), where Γ denotes the usual Gamma function.

Moreover, for the kernel function a_α, if

$$1 < \alpha < 2 - \frac{2\phi}{\pi} < 2,$$

where ϕ comes from (4.1), then $\mathfrak{S}_t$ is analytic (cf. [53, p. 55, Example 2.1]). Notice that in [53], $-\mathfrak{L}$ is considered. In this case, (5.6) and (5.7) clearly hold since $\hat{a}_\alpha(\lambda) = \lambda^{-\alpha}$, $\operatorname{Re}\lambda > 0$.

6. Semilinear stochastic partial differential equations

When $a = 1$ in Eq. (5.1), one sees that Eq. (5.1) contains a class of semilinear SPDEs. However, it cannot deal with the equation like stochastic Navier–Stokes equation. In this section, we shall discuss strong solutions of a large class of semilinear SPDEs by using the properties of analytic semigroups.

Consider the following semilinear stochastic partial differential equation:

$$\mathrm{d}X(t) = \big[-\mathfrak{L}X(t) + \Phi\big(t, X(t)\big)\big]\,\mathrm{d}t + \Psi\big(t, X(t)\big)\,\mathrm{d}W(t), \quad X(0) = x_0. \tag{6.1}$$

We introduce the following assumptions on the coefficients:

(M1) For some $\alpha \in (0,1)$,

$$\Phi: \mathbb{R}_+ \times \Omega \times \mathbb{X}_\alpha \to \mathbb{X} \in \mathcal{M} \times \mathcal{B}(\mathbb{X}_\alpha)/\mathcal{B}(\mathbb{X})$$

and

$$\Psi: \mathbb{R}_+ \times \Omega \times \mathbb{X}_\alpha \to L_2\big(l^2; \mathbb{X}_{\frac{\alpha}{2}}\big) \in \mathcal{M} \times \mathcal{B}(\mathbb{X}_\alpha)/\mathcal{B}\big(L_2\big(l^2; \mathbb{X}_{\frac{\alpha}{2}}\big)\big).$$

(M2) For any $R > 0$, there exist $C_R > 0$ and $\beta \in [0,1)$ with

$$\alpha + \beta < 1$$

such that for all $s > 0$, $\omega \in \Omega$ and $x, y \in \mathbb{X}_\alpha$ with $\|x\|_{\mathbb{X}_\alpha}, \|y\|_{\mathbb{X}_\alpha} \leqslant R$,

$$\big\|\Phi(s,\omega,x)\big\|_{\mathbb{X}} + \big\|\Psi(s,\omega,x)\big\|^2_{L_2(l^2;\mathbb{X}_{\frac{\alpha}{2}})} \leqslant \frac{C_R}{(s\wedge 1)^\beta}$$

and

$$\big\|\Phi(s,\omega,x) - \Phi(s,\omega,y)\big\|_{\mathbb{X}} \leqslant \frac{C_R}{(s\wedge 1)^\beta}\|x-y\|_{\mathbb{X}_\alpha},$$
$$\big\|\Psi(s,\omega,x) - \Psi(s,\omega,y)\big\|^2_{L_2(l^2;\mathbb{X}_{\frac{\alpha}{2}})} \leqslant \frac{C_R}{(s\wedge 1)^\beta}\|x-y\|^2_{\mathbb{X}_\alpha}.$$

(M3) For all $s > 0$, $\omega \in \Omega$ and $x \in \mathbb{X}_\alpha$, it holds that

$$\big\|\Phi(s,\omega,x)\big\|_{\mathbb{X}} \leqslant \frac{C}{(s\wedge 1)^\beta}\big(1 + \|x\|_{\mathbb{X}_\alpha}\big),$$
$$\big\|\Psi(s,\omega,x)\big\|^2_{L_2(l^2;\mathbb{X}_{\frac{\alpha}{2}})} \leqslant \frac{C}{(s\wedge 1)^\beta}\big(1 + \|x\|^2_{\mathbb{X}_\alpha}\big).$$

By a *mild solution* of equation (6.1), we mean that $X(t)$ solves the following stochastic Volterra integral equation:

$$X(t) = \mathfrak{T}_t x_0 + \int_0^t \mathfrak{T}_{t-s}\Phi\big(s, X(s)\big)\,\mathrm{d}s + \int_0^t \mathfrak{T}_{t-s}\Psi\big(s, X(s)\big)\,\mathrm{d}W(s). \tag{6.2}$$

Theorem 6.1. *Under* (M1) *and* (M2)*, for any* $x_0 \in \mathbb{X}_\alpha$ (α *is from* (M1))*, there exists a unique maximal solution* (X,τ) *for Eq.* (6.2) *so that*:

(i) $t \mapsto X(t) \in \mathbb{X}_\alpha$ *is continuous on* $[0,\tau)$ *almost surely*;
(ii) $\lim_{t\uparrow\tau}\|X(t)\|_{\mathbb{X}_\alpha} = +\infty$ *on* $\{\omega\colon \tau(\omega) < +\infty\}$;
(iii) *it holds that, P-a.s., on* $[0,\tau)$*,*

$$X(t) = \mathfrak{T}_t x_0 + \int_0^t \mathfrak{T}_{t-s}\Phi\big(s, X(s)\big)\,\mathrm{d}s + \int_0^t \mathfrak{T}_{t-s}\Psi\big(s, X(s)\big)\,\mathrm{d}W(s).$$

Moreover, if (M3) *holds, then* $\tau = +\infty$*, a.s.*

Proof. We first consider the following stochastic Volterra integral equation

$$Y(t) = \mathfrak{L}^\alpha\mathfrak{T}_t x_0 + \int_0^t \mathfrak{L}^\alpha\mathfrak{T}_{t-s}\Phi\big(s, \mathfrak{L}^{-\alpha}Y(s)\big)\,\mathrm{d}s + \int_0^t \mathfrak{L}^\alpha\mathfrak{T}_{t-s}\Psi\big(s, \mathfrak{L}^{-\alpha}Y(s)\big)\,\mathrm{d}W(s). \tag{6.3}$$

Define

$$
\begin{aligned}
g(t) &:= \mathfrak{L}^{\alpha}\mathfrak{T}_t x_0,\\
A(t,s,\omega,y) &:= \mathfrak{L}^{\alpha}\mathfrak{T}_{t-s}\Phi\big(s,\omega,\mathfrak{L}^{-\alpha}y\big),\\
B(t,s,\omega,y) &:= \mathfrak{L}^{\alpha}\mathfrak{T}_{t-s}\Psi\big(s,\omega,\mathfrak{L}^{-\alpha}y\big).
\end{aligned}
$$

Let us verify (H1)′–(H4)′. Clearly, (H1)′ holds since $x_0 \in \mathbb{X}_\alpha$.

By (iii) of Proposition 4.1 and (M2), for all $t > s > 0$, $\omega \in \Omega$ and $x, y \in \mathbb{X}$ with $\|x\|_{\mathbb{X}}, \|y\|_{\mathbb{X}} \leqslant R$ we have

$$
\begin{aligned}
&\big\|A(t,s,\omega,x)\big\|_{\mathbb{X}} + \big\|B(t,s,\omega,x)\big\|^2_{L_2(l^2;\mathbb{X})}\\
&\quad\preccurlyeq \frac{1}{(t-s)^{\alpha}}\Big(\big\|\Phi\big(s,\omega,\mathfrak{L}^{-\alpha}x\big)\big\|_{\mathbb{X}} + \big\|\Psi\big(s,\omega,\mathfrak{L}^{-\alpha}x\big)\big\|^2_{L_2(l^2;\mathbb{X}_{\frac{\alpha}{2}})}\Big)\\
&\quad\leqslant \frac{C_R}{(t-s)^{\alpha}(s\wedge 1)^{\beta}},
\end{aligned}
\tag{6.4}
$$

and

$$
\begin{aligned}
&\big\|A(t,s,\omega,x) - A(t,s,\omega,y)\big\|_{\mathbb{X}}\\
&\quad\preccurlyeq \frac{1}{(t-s)^{\alpha}}\big\|\Phi\big(s,\omega,\mathfrak{L}^{-\alpha}x\big) - \Phi\big(s,\omega,\mathfrak{L}^{-\alpha}y\big)\big\|_{\mathbb{X}}\\
&\quad\leqslant \frac{C_R}{(t-s)^{\alpha}(s\wedge 1)^{\beta}}\big\|\mathfrak{L}^{-\alpha}x - \mathfrak{L}^{-\alpha}y\big\|_{\mathbb{X}_\alpha} = \frac{C_R}{(t-s)^{\alpha}(s\wedge 1)^{\beta}}\|x-y\|_{\mathbb{X}},
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\big\|B(t,s,\omega,x) - B(t,s,\omega,y)\big\|^2_{L_2(l^2;\mathbb{X})}\\
&\quad\preccurlyeq \frac{1}{(t-s)^{\alpha}}\big\|\Psi\big(s,\omega,\mathfrak{L}^{-\alpha}x\big) - \Psi\big(s,\omega,\mathfrak{L}^{-\alpha}y\big)\big\|^2_{L_2(l^2;\mathbb{X}_{\frac{\alpha}{2}})}\\
&\quad\leqslant \frac{C_R}{(t-s)^{\alpha}(s\wedge 1)^{\beta}}\|x-y\|^2_{\mathbb{X}}.
\end{aligned}
$$

Hence, if we take

$$
\kappa_{1,R}(t,s) = \kappa_{2,R}(t,s) := \frac{C_R}{(t-s)^{\alpha}(s\wedge 1)^{\beta}} \in \mathscr{K}_{>1},
$$

then (H2)′ and (H3)′ hold.

Let $0 < \gamma < 1 - (\alpha+\beta)$. By (iv) of Proposition 4.1 and (M2) we have

$$
\begin{aligned}
&\big\|A\big(t',s,\omega,x\big) - A(t,s,\omega,x)\big\|_{\mathbb{X}}\\
&\quad= \big\|(\mathfrak{T}_{t'-t} - 1)\mathfrak{L}^{\alpha}\mathfrak{T}_{t-s}\Phi\big(s,\omega,\mathfrak{L}^{-\alpha}x\big)\big\|_{\mathbb{X}}\\
&\quad\preccurlyeq \big(t'-t\big)^{\gamma}\big\|\mathfrak{L}^{\alpha+\gamma}\mathfrak{T}_{t-s}\Phi\big(s,\omega,\mathfrak{L}^{-\alpha}x\big)\big\|_{\mathbb{X}} \leqslant \frac{C_R(t'-t)^{\gamma}}{(t-s)^{\alpha+\gamma}(s\wedge 1)^{\beta}}
\end{aligned}
$$

and

$$\begin{aligned}
&\left\|B\left(t', s, \omega, x\right) - B(t, s, \omega, x)\right\|^2_{L_2(l^2;\mathbb{X})} \\
&\quad \preccurlyeq \left\|(\mathfrak{T}_{t'-t} - 1)\mathfrak{L}^{\alpha}\mathfrak{T}_{t-s}\Psi\left(s, \omega, \mathfrak{L}^{-\alpha}x\right)\right\|^2_{L_2(l^2;\mathbb{X})} \\
&\quad \preccurlyeq \left(t'-t\right)^{\gamma}\left\|\mathfrak{L}^{\alpha+\gamma/2}\mathfrak{T}_{t-s}\Psi\left(s, \mathfrak{L}^{-\alpha}x\right)\right\|^2_{L_2(l^2;\mathbb{X})} \\
&\quad \preccurlyeq \frac{(t'-t)^{\gamma}}{(t-s)^{\alpha+\gamma}}\left\|\mathfrak{L}^{\frac{\alpha}{2}}\Psi\left(s, \mathfrak{L}^{-\alpha}x\right)\right\|^2_{L_2(l^2;\mathbb{X})} \leqslant \frac{C_R(t'-t)^{\gamma}}{(t-s)^{\alpha+\gamma}(s\wedge 1)^{\beta}}.
\end{aligned}$$

So, if we take

$$\lambda_R\left(t', t, s\right) := \frac{C_R(t'-t)^{\gamma}}{(t-s)^{\alpha+\gamma}(s\wedge 1)^{\beta}},$$

then (H4)$'$ holds.

Hence, by Theorem 3.7 there is a unique maximal solution (Y, τ) for Eq. (6.3) in the sense of Definition 3.5. Set

$$X(t) = \mathfrak{L}^{-\alpha}Y(t).$$

It is easy to see that (X, τ) is a unique maximal solution for Eq. (6.2), which satisfies (i), (ii) and (iii) in the theorem.

Lastly, if (M3) is satisfied, then as estimating (6.4), for the above A and B, (H2) holds with some $\kappa_1 \in \mathscr{K}_{>1}$, and also (H4) holds. So, by Theorem 3.8 we have $\tau = \infty$ a.s. $\square$

Remark 6.2. The solution (X, τ) in Theorem 6.1 is clearly a local solution of Eq. (6.2) in $\mathbb{X}$. However, it may be not a maximal solution in $\mathbb{X}$ because it may happen that

$$\lim_{t\uparrow\tau(\omega)}\left\|X(t,\omega)\right\|_{\mathbb{X}} < +\infty \quad \text{on } \left\{\omega\colon \tau(\omega) < +\infty\right\}.$$

Next, we study the large deviation estimate for Eq. (6.1), and assume that Φ and Ψ are non-random. Consider the following small perturbation of Eq. (6.1):

$$\mathrm{d}X_\epsilon(t) = \left[-\mathfrak{L}X_\epsilon(t) + \Phi\left(t, X_\epsilon(t)\right)\right]\mathrm{d}t + \sqrt{\epsilon}\Psi\left(t, X_\epsilon(t)\right)\mathrm{d}W(t), \qquad X_\epsilon(0) = x_0. \quad (6.5)$$

In order to apply Theorem 4.6 to this situation, we need the non-explosion assumptions as (C2) and (C3). For a family of processes $\{h^\epsilon, \epsilon \in (0,1)\}$ in $\mathcal{A}_N^T$ (see (2.20) for the definition of $\mathcal{A}_N^T$), consider

$$\begin{aligned}
X^\epsilon(t) = {}& \mathfrak{T}_t x_0 + \int_0^t \mathfrak{T}_{t-s}\Phi\left(s, X^\epsilon(s)\right)\mathrm{d}s + \int_0^t \mathfrak{T}_{t-s}\Psi\left(s, X^\epsilon(s)\right)\dot{h}^\epsilon(s)\,\mathrm{d}s \\
&+ \sqrt{\epsilon}\int_0^t \mathfrak{T}_{t-s}\Psi\left(s, X^\epsilon(s)\right)\mathrm{d}W(s),
\end{aligned}$$

and for $h \in \ell_T^2$ (see (2.19))

$$X^h(t) = \mathfrak{T}_t x_0 + \int_0^t \mathfrak{T}_{t-s}\Phi\big(s, X^h(s)\big)\,\mathrm{d}s + \int_0^t \mathfrak{T}_{t-s}\Psi\big(s, X^h(s)\big)\dot{h}(s)\,\mathrm{d}s.$$

Below, for $n \in \mathbb{N}$ we define

$$\tau_n^\epsilon := \inf\big\{t > 0\colon\ \big\|X^\epsilon(t)\big\|_{\mathbb{X}_\alpha} > n\big\}.$$

Our large deviation principle can be stated as follows:

Theorem 6.3. *Assume* (M1) *and* (M2). *Let $x_0 \in \mathbb{X}_\delta$ for some $1 \geqslant \delta > \alpha$, where α is from* (M1). *We also assume that $\mathscr{D}(\mathfrak{L}) = \mathbb{X}_1 \subset \mathbb{X}$ is compact, and*

$$\lim_{n\to\infty} \sup_{\epsilon\in(0,1)} P\big\{\omega\colon\ \tau_n^\epsilon(\omega) < T\big\} = 0 \tag{6.6}$$

and for any $N > 0$,

$$\sup_{h\in\mathbb{D}_N} \sup_{t\in[0,T]} \big\|X^h(t)\big\|_{\mathbb{X}_\alpha} < +\infty. \tag{6.7}$$

Then $\{X_\epsilon, \epsilon \in (0,1)\}$ satisfies the large deviation principle in $\mathbb{C}_T(\mathbb{X}_\alpha)$ with the rate function $I(f)$ given by

$$I(f) := \frac{1}{2} \inf_{\{h\in\ell_T^2\colon\ f = X^h\}} \|h\|_{\ell_T^2}^2, \qquad f \in \mathbb{C}_T(\mathbb{X}_\alpha). \tag{6.8}$$

Proof. By Theorem 4.6, it only needs to check (H1)$''$ and (H2)$''$ for Eq. (6.3). Since $x_0 \in \mathbb{X}^\delta$ with $\delta > \alpha$, by (iv) of Proposition 4.1, (H1)$''$ holds with $\delta' = \delta - \alpha$ and $\alpha' \in (0, \delta - \alpha)$. As the calculations given in (6.4), one finds that (H2)$''$ holds with $\alpha' \in (0, 1 - \alpha - \beta)$. $\square$

Remark 6.4. If (M3) is satisfied, one can see that (6.6) and (6.7) hold by Remark 4.7.

Below we study the existence of strong solutions for Eq. (6.1). For this aim, in addition to (M1) and (M2) with $\beta = 0$, we also assume:

(M4) For any $R, T > 0$, there exist $\delta > 0$ and $\alpha' > 1$ such that for all $s, s' \in [0,T]$, $\omega \in \Omega$ and $x \in \mathbb{X}_\alpha$ with $\|x\|_{\mathbb{X}_\alpha} \leqslant R$,

$$\big\|\Phi\big(s', \omega, x\big) - \Phi(s, \omega, x)\big\|_{\mathbb{X}} \leqslant C_{T,R}|s' - s|^\delta, \tag{6.9}$$

$$\big\|\Psi(s, \omega, x)\big\|_{L_2(l^2;\mathbb{X}_{\frac{\alpha'}{2}})}^2 \leqslant C_{T,R}. \tag{6.10}$$

Let us recall the following result (cf. [28, Theorem 3.2.2] or [48, p. 114, Theorem 3.5]).

Lemma 6.5. *Let* $[0,T] \ni s \mapsto f(s) \in \mathbb{X}$ *be a Hölder continuous function. Then*

$$t \mapsto \int_0^t \mathfrak{T}_{t-s} f(s)\,\mathrm{d}s \in C\big([0,T];\mathbb{X}_1\big).$$

Using this lemma, we can prove the following result.

Theorem 6.6. *Assume that* (M1)*,* (M2) *and* (M4) *hold. For any* $x_0 \in \mathbb{X}_1$*, let* (X,τ) *be the unique maximal solution of Eq.* (6.2) *in Theorem* 6.1*. Then*:

(i) $t \mapsto X(t) \in \mathbb{X}_1$ *is continuous on* $[0,\tau)$ *a.s.*;
(ii) *it holds that in* $\mathbb{X}$*,*

$$X(t) = x_0 - \int_0^t \mathfrak{L}X(s)\,\mathrm{d}s + \int_0^t \Phi\big(s,X(s)\big)\,\mathrm{d}s + \int_0^t \Psi\big(s,X(s)\big)\,\mathrm{d}W(s)$$

for all $t \in [0,\tau)$*,* P*-a.s.*

We shall call (X,τ) *the unique maximal strong solution of Eq.* (6.1).

Proof. For $n \in \mathbb{N}$, set

$$\tau_n := \inf\big\{t > 0\colon\ \|X(t)\|_{\mathbb{X}_\alpha} > n\big\}$$

and

$$G(t,s) := \mathfrak{T}_{t-s}\Psi\big(s,X(s)\big).$$

Then by (iii) and (iv) of Proposition 4.1 we have

$$\big\|G(t,s)\big\|^2_{L_2(l^2;\mathbb{X}_1)} \preccurlyeq \frac{1}{(t-s)^{2-\alpha'}}\big\|\Psi\big(s,X(s)\big)\big\|^2_{L_2(l^2;\mathbb{X}_{\alpha'/2})},$$

and in view of $\alpha' > 1$,

$$\big\|G\big(t',s\big) - G(t,s)\big\|^2_{L_2(l^2;\mathbb{X}_1)} \preccurlyeq \frac{(t'-t)^{(\alpha'-1)/2}}{(t-s)^{(3-\alpha')/2}}\big\|\Psi\big(s,X(s)\big)\big\|^2_{L_2(l^2;\mathbb{X}_{\alpha'/2})}.$$

Hence, by Lemma 3.4 and (6.10),

$$t \mapsto \int_0^t \mathfrak{T}_{t-s}\Psi\big(s,X(s)\big)\,\mathrm{d}W(s) \in \mathbb{X}_1$$

admits a continuous modification on $[0,\tau_n)$.

Moreover, starting from (6.3), as in the proof of Theorem 3.3, there exists an $a > 0$ such that for p sufficiently large

$$\mathbb{E}\left(\sup_{t \neq t' \in [0, T \wedge \tau_n]} \frac{\|X(t') - X(t)\|^p_{\mathbb{X}^\alpha}}{|t' - t|^{ap}} \right) \leqslant C_{n,T,p}.$$

Thus, by (M2) and (M4) we know that

$$s \mapsto \Phi\big(s, X(s)\big) \in \mathbb{X} \text{ is Hölder continuous on } [0, T \wedge \tau_n] \ P\text{-a.s.}$$

Therefore, by Lemma 6.5 we have

$$t \mapsto \int_0^t \mathfrak{T}_{t-s} \Phi\big(s, X(s)\big)\,\mathrm{d}s \in C\big([0, T \wedge \tau_n], \mathbb{X}_1\big), \quad P\text{-a.s.}$$

Noting that $x_0 \in \mathbb{X}_1$ and

$$1_{\{t \leqslant \tau_n\}} \cdot X(t) = 1_{\{t \leqslant \tau_n\}} \cdot \mathfrak{T}_t x_0 + 1_{\{t \leqslant \tau_n\}} \cdot \int_0^t \mathfrak{T}_{t-s} \Phi\big(s, X(s)\big)\,\mathrm{d}s$$
$$+ 1_{\{t \leqslant \tau_n\}} \cdot \int_0^t \mathfrak{T}_{t-s} \Psi\big(s, X(s)\big)\,\mathrm{d}W(s), \quad \forall t \geqslant 0,\ P\text{-a.s.},$$

by $\tau_n \nearrow \tau$, we therefore have that $t \mapsto X(t) \in \mathbb{X}_1$ is continuous on $[0, \tau)$ P-a.s.

Lastly, by stochastic Fubini's theorem (cf. [45, Section 6]) we have

$$\begin{aligned}
\int_0^t \mathfrak{L} X(s)\,\mathrm{d}s &= \int_0^t \mathfrak{L}\mathfrak{T}_s x_0\,\mathrm{d}s + \int_0^t \int_0^s \mathfrak{L}\mathfrak{T}_{s-r} \Phi\big(r, X(r)\big)\,\mathrm{d}r\,\mathrm{d}s \\
&\quad + \int_0^t \int_0^s \mathfrak{L}\mathfrak{T}_{s-r} \Psi\big(r, X(r)\big)\,\mathrm{d}W(r)\,\mathrm{d}s \\
&= x_0 - \mathfrak{T}_t x_0 + \int_0^t \int_r^t \mathfrak{L}\mathfrak{T}_{s-r} \Phi\big(r, X(r)\big)\,\mathrm{d}s\,\mathrm{d}r \\
&\quad + \int_0^t \int_r^t \mathfrak{L}\mathfrak{T}_{s-r} \Psi\big(r, X(r)\big)\,\mathrm{d}s\,\mathrm{d}W(r) \\
&= x_0 - \mathfrak{T}_t x_0 + \int_0^t \big[\Phi\big(r, X(r)\big) - \mathfrak{T}_{t-r} \Phi\big(r, X(r)\big)\big]\,\mathrm{d}r
\end{aligned}$$

$$+\int_0^t \big[\Psi\big(r, X(r)\big) - \mathfrak{T}_{t-r}\Psi\big(r, X(r)\big)\big]\,\mathrm{d}W(r)$$

$$= x_0 - X(t) + \int_0^t \Phi\big(s, X(s)\big)\,\mathrm{d}s + \int_0^t \Psi\big(s, X(s)\big)\,\mathrm{d}W(s)$$

on $\{t \leqslant \tau_n\}$. The proof is thus complete by letting $n \to \infty$. □

7. Application to stochastic Navier–Stokes equations

7.1. Unique maximal strong solution for SNSEs

Let $\mathcal{O}$ be a bounded smooth domain in $\mathbb{R}^d (d \geqslant 2)$, or the whole space $\mathbb{R}^d$, or d-dimensional torus $\mathbb{T}^d$. Let

$$\mathbf{W}^{m,p}(\mathcal{O}) := \big(W^{m,p}(\mathcal{O})\big)^d, \qquad \mathbf{W}_0^{m,p}(\mathcal{O}) := \big(W_0^{m,p}(\mathcal{O})\big)^d$$

and

$$\mathbf{C}_{0,\sigma}^\infty(\mathcal{O}) := \big\{\mathbf{u} \in \big(C_0^\infty(\mathcal{O})\big)^d\colon\ \mathrm{div}(\mathbf{u}) = 0\big\}.$$

Notice that $\mathbf{W}^{m,p}(\mathbb{R}^d) = \mathbf{W}_0^{m,p}(\mathbb{R}^d)$ and $\mathbf{W}^{m,p}(\mathbb{T}^d) = \mathbf{W}_0^{m,p}(\mathbb{T}^d)$.

Let $\mathbf{L}_\sigma^p(\mathcal{O})$ be the closure of $\mathbf{C}_{0,\sigma}^\infty(\mathcal{O})$ with respect to the norm in $\mathbf{L}^p(\mathcal{O}) := (L^p(\mathcal{O}))^d$. Let $\mathscr{P}_2$ be the orthonormal projection from $\mathbf{L}^2(\mathcal{O})$ to $\mathbf{L}_\sigma^2(\mathcal{O})$. It is well known that $\mathscr{P}_2$ can be extended to a bounded linear operator from $\mathbf{L}^p(\mathcal{O})$ to $\mathbf{L}_\sigma^p(\mathcal{O})$ (cf. [23]) so that for every $\mathbf{u} \in \mathbf{L}^p(\mathcal{O})$,

$$\mathbf{u} = \mathscr{P}_p\mathbf{u} + \nabla\pi, \quad \pi \in \big(L_{\mathrm{loc}}^p(\mathcal{O})\big)^d.$$

The stokes operator is defined by

$$A_p\mathbf{u} := -\mathscr{P}_p\Delta\mathbf{u}, \qquad \mathscr{D}(A_p) := \mathbb{H}_2^p \cap \mathbf{L}_\sigma^p(\mathcal{O}), \tag{7.1}$$

where

$$\mathbb{H}_2^p := \mathbf{W}^{2,p}(\mathcal{O}) \cap \mathbf{W}_0^{1,p}(\mathcal{O}) = \mathscr{D}(I - \Delta_p)$$

and Δ_p is the Laplace operator on $\mathbf{L}^p(\mathcal{O})$.

It is well known that $(A_p, \mathscr{D}(A_p))$ is a sectorial operator on $\mathbf{L}_\sigma^p(\mathcal{O})$ (cf. [24]). It should be noticed that when $\mathcal{O} = \mathbb{R}^d$ or $\mathbb{T}^d$, since the projection $\mathscr{P}_p$ can commute with ∇ (cf. [37, p. 84]), we have

$$A_p\mathbf{u} = -\Delta\mathscr{P}_p\mathbf{u} = -\Delta\mathbf{u}, \quad \mathbf{u} \in \mathscr{D}(A_p).$$

That is, the stokes operator is just the restriction of $-\Delta_p$ on $\mathbf{W}^{2,p}(\mathcal{O}) \cap \mathbf{L}_\sigma^p(\mathcal{O})$, where $\mathcal{O} = \mathbb{R}^d$ or $\mathbb{T}^d$.

Below, we write

$$\mathfrak{L}_p := I + A_p$$

and

$$\mathbf{H}^p_\alpha := \mathscr{D}\big(\mathfrak{L}_p^{\alpha/2}\big).$$

Giga [25] proved that for $\alpha \in [0, 1]$,

$$\mathbf{H}^p_\alpha = \big[\mathbf{L}^p_\sigma(\mathcal{O}), \mathscr{D}(A_p)\big]_\alpha = \mathbb{H}^p_\alpha \cap \mathbf{L}^p_\sigma(\mathcal{O}), \tag{7.2}$$

where $\mathbb{H}^p_\alpha = [\mathbf{L}^p(\mathcal{O}), \mathbb{H}^p_2]_\alpha$ and $[\cdot,\cdot]_\alpha$ stands for the complex interpolation space between two Banach spaces. In particular, the following embedding results hold (cf. [48]): for $p > 1$ and $0 \leqslant \alpha' < \frac{1}{2} < \alpha \leqslant 1$,

$$\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha'}} \preccurlyeq \|\mathbf{u}\|_{1,p} \preccurlyeq \|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}, \quad \mathbf{u} \in \mathbf{H}^p_\alpha, \tag{7.3}$$

and for $q \geqslant p$, $k - \frac{d}{q} < 2\alpha - \frac{d}{p}$,

$$\mathbf{H}^p_{2\alpha} \hookrightarrow \mathbf{W}^{k,q}(\mathcal{O}), \tag{7.4}$$

and for $\alpha > \frac{d}{p}$,

$$\mathbf{H}^p_\alpha \hookrightarrow C_b(\mathcal{O}). \tag{7.5}$$

In what follows, we fix

$$p > d, \qquad \frac{1}{2} < \alpha < 1, \tag{7.6}$$

and consider the following stochastic Navier–Stokes equation with Dirichlet boundary (only for bounded smooth domain):

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = \big[\Delta\mathbf{u}(t) + \big(\mathbf{u}(t)\cdot\nabla\big)\mathbf{u}(t) + \nabla\pi(t)\big]\,\mathrm{d}t + F\big(t,\mathbf{u}(t)\big)\,\mathrm{d}t + \Psi\big(t,\mathbf{u}(t)\big)\,\mathrm{d}W(t), \\ \mathbf{u}(t,\cdot)|_{\partial\mathcal{O}} = 0, \qquad \operatorname{div}\mathbf{u}(t) = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \end{cases} \tag{7.7}$$

where $\mathbf{u}$ and π are unknown functions, and

$$F : \mathbb{R}_+ \times \mathbf{H}^p_{2\alpha} \to \mathbf{H}^p_0 \quad \text{and} \quad \Psi : \mathbb{R}_+ \times \mathbf{H}^p_{2\alpha} \to \mathbf{H}^p_\alpha$$

are two measurable functions.

Assume that:

(N1) For each $T, R > 0$, there exist $\delta > 0$ and $C_{T,R,\delta} > 0$ such that for all $t, s \in [0, T]$ and $\mathbf{u}, \mathbf{v} \in \mathbf{H}^p_{2\alpha}$ with $\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}, \|\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}} \leqslant R$,

$$\big\|F(t,\mathbf{u}) - F(s,\mathbf{v})\big\|_{\mathbf{H}^p_0} \leqslant C_{T,R,\delta}\big(|t-s|^\delta + \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}}\big).$$

(N2) For each $T, R > 0$, there exist $\alpha' > 1$ and $C_{T,R} > 0$ such that for all $t \in [0, T]$ and $\mathbf{u}, \mathbf{v} \in \mathbf{H}^p_{2\alpha}$ with $\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}, \|\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}} \leqslant R$,

$$\big\|\Psi(t,\mathbf{u}) - \Psi(t,\mathbf{v})\big\|_{L_2(l^2;\mathbf{H}^p_\alpha)} \leqslant C_{T,R}\|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}}$$

and

$$\big\|\Psi(t,\mathbf{u})\big\|_{L_2(l^2;\mathbf{H}^p_{\alpha'})} \leqslant C_{T,R}. \tag{7.8}$$

Set

$$\Phi(t,\mathbf{u}) := \mathbf{u} + \mathscr{P}_p\big[(\mathbf{u}\cdot\nabla)\mathbf{u}\big] + F(t,\mathbf{u}). \tag{7.9}$$

Then Eq. (7.7) can be written as the following abstract form:

$$\mathrm{d}\mathbf{u}(t) = \big[-\mathfrak{L}_p\mathbf{u}(t) + \Phi(t,\mathbf{u})\big]\,\mathrm{d}t + \Psi(t,\mathbf{u})\,\mathrm{d}W(s), \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{7.10}$$

Theorem 7.1. *Let $p > d$ and $\frac{1}{2} < \alpha < 1$. Under* (N1) *and* (N2), *for any $\mathbf{u}_0 \in \mathbf{H}^p_2$, there exists a unique maximal strong solution $(\mathbf{u}, \tau)$ for Eq.* (7.10) *so that*:

(i) *$t \mapsto \mathbf{u}(t) \in \mathbf{H}^p_2$ is continuous on $[0, \tau)$ a.s.*;
(ii) *$\lim_{t\uparrow\tau} \|\mathbf{u}(t)\|_{\mathbf{H}^p_{2\alpha}} = \infty$ on $\{\tau < +\infty\}$*;
(iii) *it holds that in $\mathbf{L}^p_\sigma(\mathcal{O}) = \mathbf{H}^p_0$,*

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}_0 + \int_0^t \big[-\mathfrak{L}_p\mathbf{u}(s) + \Phi\big(s,\mathbf{u}(s)\big)\big]\,\mathrm{d}s + \int_0^t \Psi\big(s,\mathbf{u}(s)\big)\,\mathrm{d}W(s)\\ &= \mathbf{u}_0 + \int_0^t \big[A_p\mathbf{u}(s) + \mathscr{P}_p\big(\big(\mathbf{u}(s)\cdot\nabla\big)\mathbf{u}(s)\big)\big]\,\mathrm{d}s\\ &\quad + \int_0^t F\big(s,\mathbf{u}(s)\big)\,\mathrm{d}s + \int_0^t \Psi\big(s,\mathbf{u}(s)\big)\,\mathrm{d}W(s),\end{aligned}$$

for all $t \in [0, \tau)$, P-a.s.

Proof. In view of (7.6), (7.3) and (7.5), for any $\mathbf{u},\mathbf{v}\in\mathbf{H}^p_{2\alpha}$ we have

$$\begin{aligned}\big\|\mathscr{P}_p\big[(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{v}\cdot\nabla)\mathbf{v}\big]\big\|_{\mathbf{L}^p_\sigma} &\preccurlyeq \big\|(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{v}\cdot\nabla)\mathbf{v}\big\|_{\mathbf{L}^p}\\ &\preccurlyeq \|\mathbf{u}-\mathbf{v}\|_{\mathbf{L}^\infty}\cdot\|\nabla\mathbf{u}\|_{\mathbf{L}^p}+\|\mathbf{v}\|_{\mathbf{L}^\infty}\cdot\big\|\nabla(\mathbf{u}-\mathbf{v})\big\|_{\mathbf{L}^p}\\ &\preccurlyeq \|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}}\cdot\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}+\|\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}}\cdot\|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}^p_{2\alpha}}.\end{aligned}$$

Thus, by (N1) and (N2), it is easy to see that (M2) and (M4) hold for the above Φ and Ψ. The result now follows by Theorem 6.6. □

7.2. Non-explosion and large deviation for 2D SNSEs

In this subsection, we study the non-explosion and large deviation for SNSE in the case of two dimensions. For this aim, in addition to (N1) and (N2), we also suppose that:

(N3) For any $T>0$, there exists $C_T>0$ such that for all $t\in[0,T]$ and $\mathbf{u}\in\mathbf{H}^p_2$,

$$\begin{aligned}\big\|F(t,\mathbf{u})\big\|_{\mathbf{H}^2_0}&\leqslant C_T\big(\|\mathbf{u}\|_{\mathbf{H}^2_1}+1\big),\\ \big\|F(t,\mathbf{u})\big\|_{\mathbf{H}^p_0}&\leqslant C_T\big(\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}+1\big)\end{aligned}$$

and for $i=0,1$,

$$\begin{aligned}\big\|\Psi(s,\mathbf{u})\big\|_{L_2(l^2;\mathbf{H}^2_i)}&\leqslant C_T\big(1+\|\mathbf{u}\|_{\mathbf{H}^2_i}\big),\\ \big\|\Psi(s,\mathbf{u})\big\|_{L_2(l^2;\mathbf{H}^p_\alpha)}&\leqslant C_T\big(1+\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}\big),\end{aligned}$$

where p and α satisfy (7.6).

We have the following result, the proof will be given in Lemma 7.7 below.

Theorem 7.2. *Let $p>d$ and $\frac{1}{2}<\alpha<1$. Assume that* (N1)–(N3) *hold. Let* $(\mathbf{u},\tau)$ *be the unique maximal solution of Eq.* (7.11) *in Theorem* 7.1. *Then $\tau=+\infty$ a.s.*

We now consider the small perturbation for 2D stochastic Navier–Stokes equation:

$$\mathrm{d}\mathbf{u}_\epsilon(t)=\big[-\mathfrak{L}_p\mathbf{u}_\epsilon(t)+\Phi\big(t,\mathbf{u}_\epsilon(t)\big)\big]\,\mathrm{d}t+\sqrt{\epsilon}\Psi\big(t,\mathbf{u}_\epsilon(t)\big)\,\mathrm{d}W(t),\qquad \mathbf{u}_\epsilon(0)=\mathbf{u}_0$$

as well as the control equation:

$$\begin{aligned}&\mathrm{d}\mathbf{u}^\epsilon(t)=\big[-\mathfrak{L}_p\mathbf{u}^\epsilon(t)+\Phi\big(t,\mathbf{u}^\epsilon(t)\big)+\Psi\big(t,\mathbf{u}^\epsilon(t)\big)\dot{h}^\epsilon(t)\big]\,\mathrm{d}t+\sqrt{\epsilon}\Psi\big(t,\mathbf{u}^\epsilon(t)\big)\,\mathrm{d}W(t),\\ &\mathbf{u}^\epsilon(0)=\mathbf{u}_0,\end{aligned}\tag{7.11}$$

where $h^\epsilon\in\mathcal{A}^T_N$ (see (2.20) for the definition of $\mathcal{A}^T_N$), and $T>0$ is fixed below.

Let $(\mathbf{u}^\epsilon, \tau^\epsilon)$ be the unique maximal strong solution of Eq. (7.11) with the properties:

$$\lim_{t\uparrow\tau^\epsilon}\big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}^p_{2\alpha}} = +\infty \quad \text{on } \{\tau^\epsilon < \infty\},$$

and $t \mapsto \mathbf{u}^\epsilon(t) \in \mathbf{H}^p_2$ is continuous on $[0, \tau^\epsilon)$.

Before proving the non-explosion result (Lemma 7.7), we first prepare a series of lemmas.

Lemma 7.3. *There exists a constant $C_T > 0$ such that for any $t \in [0, T]$ and $\mathbf{u} \in \mathbf{H}^2_2$,*

$$\big\langle \mathbf{u}, -\mathfrak{L}_2\mathbf{u} + \Phi(s, \mathbf{u})\big\rangle_{\mathbf{H}^2_0} \leqslant -\frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{H}^2_1} + C_T\big(\|\mathbf{u}\|^2_{\mathbf{H}^2_0} + 1\big), \tag{7.12}$$

$$\big\langle \mathfrak{L}_2\mathbf{u}, -\mathfrak{L}_2\mathbf{u} + \Phi(s, \mathbf{u})\big\rangle_{\mathbf{H}^2_0} \leqslant C\|\mathbf{u}\|^2_{\mathbf{H}^2_0}\|\mathbf{u}\|^4_{\mathbf{H}^2_1} + C_T\big(1 + \|\mathbf{u}\|^2_{\mathbf{H}^2_1}\big) \tag{7.13}$$

and

$$\big\|\Phi(t, \mathbf{u})\big\|_{\mathbf{H}^p_0} \leqslant C_T\big(1 + \|\mathbf{u}\|_{\mathbf{H}^2_1}\big)\cdot\big(1 + \|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}\big). \tag{7.14}$$

Proof. Let $\mathbf{u} \in \mathbf{H}^2_2$. Noting that

$$\big\langle \mathbf{u}, \mathscr{P}_2\big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)\big\rangle_{\mathbf{H}^2_0} = \big\langle \mathbf{u}, (\mathbf{u}\cdot\nabla)\mathbf{u}\big\rangle_{\mathbf{L}^2} = \frac{1}{2}\int_{\mathcal{O}} \mathbf{u}(x)\cdot\nabla\big|\mathbf{u}(x)\big|^2\,\mathrm{d}x = 0,$$

by (N3) and Young's inequality we have

$$\big\langle \mathbf{u}, -\mathfrak{L}_2\mathbf{u} + \Phi(s, \mathbf{u})\big\rangle_{\mathbf{H}^2_0} = -\|\mathbf{u}\|^2_{\mathbf{H}^2_1} + \big\langle \mathbf{u}, \mathbf{u} + F(t, \mathbf{u})\big\rangle_{\mathbf{H}^2_0} \leqslant -\frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{H}^2_1} + C_T\big(\|\mathbf{u}\|^2_{\mathbf{H}^2_0} + 1\big).$$

Thus, (7.12) is proved.

For (7.13), noting that by Gagliado–Nirenberge's inequality (cf. [22, p. 24 Theorem 9.3]) and (7.2)

$$\|\mathbf{u}\|^2_{\mathbf{L}^\infty} \preccurlyeq \|\mathbf{u}\|_{\mathbb{H}^2_2}\cdot\|\mathbf{u}\|_{\mathbb{H}^2_0} \preccurlyeq \|\mathbf{u}\|_{\mathbf{H}^2_2}\cdot\|\mathbf{u}\|_{\mathbf{H}^2_0},$$

by Young's inequality we have

$$\begin{aligned}
\big\langle \mathfrak{L}_2\mathbf{u}, \mathscr{P}_2\big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)\big\rangle_{\mathbf{H}^2_0} &\leqslant \frac{1}{4}\|\mathbf{u}\|^2_{\mathbf{H}^2_2} + \big\|\mathscr{P}_2\big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)\big\|^2_{\mathbf{H}^2_0} \\
&\leqslant \frac{1}{4}\|\mathbf{u}\|^2_{\mathbf{H}^2_2} + C\big\|(\mathbf{u}\cdot\nabla)\mathbf{u}\big\|^2_{\mathbf{L}^2} \\
&\leqslant \frac{1}{4}\|\mathbf{u}\|^2_{\mathbf{H}^2_2} + C\|\mathbf{u}\|^2_{\mathbf{L}^\infty}\cdot\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2} \\
&\leqslant \frac{1}{4}\|\mathbf{u}\|^2_{\mathbf{H}^2_2} + C\|\mathbf{u}\|_{\mathbf{H}^2_0}\cdot\|\mathbf{u}\|_{\mathbf{H}^2_2}\cdot\|\mathbf{u}\|^2_{\mathbf{H}^2_1} \\
&\leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{H}^2_2} + C\|\mathbf{u}\|^2_{\mathbf{H}^2_0}\cdot\|\mathbf{u}\|^4_{\mathbf{H}^2_1},
\end{aligned}$$

and by (N3)

$$\big\langle \mathfrak{L}_2\mathbf{u}, F(s,\mathbf{u})\big\rangle_{\mathbf{H}^2_0} \leqslant \frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{H}^2_2} + C_T\big(1+\|\mathbf{u}\|^2_{\mathbf{H}^2_1}\big).$$

Thus, (7.13) holds.

Let

$$p < q < \frac{d}{1+\frac{d}{p}-2\alpha}, \qquad q^* = \frac{qp}{q-p}.$$

By Hölder's inequality we have

$$\big\|\mathscr{P}_p(\mathbf{u}\cdot\nabla)\mathbf{u}\big\|_{\mathbf{H}^p_0} \preccurlyeq \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{\mathbf{L}^p} \preccurlyeq \|\mathbf{u}\|_{\mathbf{L}^{q^*}}\cdot\|\nabla\mathbf{u}\|_{\mathbf{L}^q} \overset{(7.4)}{\preccurlyeq} \|\mathbf{u}\|_{\mathbf{H}^2_1}\cdot\|\mathbf{u}\|_{\mathbf{H}^p_{2\alpha}}.$$

Estimate (7.14) now follows by (N3). $\square$

Below, set for $n\in\mathbb{N}$,

$$\tau^\epsilon_n := \inf\big\{t\geqslant 0:\ \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}^p_{2\alpha}} > n\big\}.$$

Lemma 7.4. *There exists a constant $C_T>0$ such that for all $\epsilon\in(0,1)$ and $n\in\mathbb{N}$,*

$$\mathbb{E}\Big(\sup_{s\in[0,T\wedge\tau^\epsilon_n]}\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_0}\Big) + \mathbb{E}\left(\int_0^{T\wedge\tau^\epsilon_n}\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_1}\,\mathrm{d}s\right) \leqslant C_T.$$

Proof. By Itô's formula we have

$$\begin{aligned}
\big\|\mathbf{u}^\epsilon(t)\big\|^2_{\mathbf{H}^2_0} =\ & \|\mathbf{u}_0\|^2_{\mathbf{H}^2_0} + 2\int_0^t \big\langle \mathbf{u}^\epsilon(s), -\mathfrak{L}_2\mathbf{u}^\epsilon(s) + \Phi\big(s,\mathbf{u}^\epsilon(s)\big)\big\rangle_{\mathbf{H}^2_0}\,\mathrm{d}s \\
& + 2\int_0^t \big\langle \mathbf{u}^\epsilon(s), \Psi\big(s,\mathbf{u}^\epsilon(s)\big)\dot h^\epsilon(s)\big\rangle_{\mathbf{H}^2_0}\,\mathrm{d}s \\
& + 2\sqrt{\epsilon}\sum_k\int_0^t \big\langle \mathbf{u}^\epsilon(s), \Psi_k\big(s,\mathbf{u}^\epsilon(s)\big)\big\rangle_{\mathbf{H}^2_0}\,\mathrm{d}W^k(s) \\
& + \epsilon\sum_k\int_0^t \big\|\Psi_k\big(s,\mathbf{u}^\epsilon(s)\big)\big\|^2_{\mathbf{H}^2_0}\,\mathrm{d}s \\
=:\ & \|\mathbf{u}_0\|^2_{\mathbf{H}^2_1} + J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned}$$

Set

$$f(t) := \mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_n^\epsilon]}\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_0^2}^2\Big).$$

First of all, noting that by (7.12)

$$J_1(t) \leqslant -\int_0^t \big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_1^2}^2 + C_T\int_0^t \big(\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_0^2}^2 + 1\big)\,\mathrm{d}s,$$

we have

$$\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_n^\epsilon]} J_1(s)\Big) + \mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon}\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_1^2}^2\,\mathrm{d}s\right) \leqslant C_T\int_0^t \big(f(s)+1\big)\,\mathrm{d}s.$$

By (N3) and Young's inequality we have

$$\begin{aligned}
\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_n^\epsilon]} J_2(s)\Big) &\leqslant 2\mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon}\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_0^2}\cdot\big\|\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\big\|_{L_2(l^2;\mathbf{H}_0^2)}\cdot\big\|\dot{h}^\epsilon(s)\big\|_{l^2}\,\mathrm{d}s\right)\\
&\leqslant 2N\mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon}\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_0^2}^2\cdot\big\|\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\big\|_{L_2(l^2;\mathbf{H}_0^2)}^2\,\mathrm{d}s\right)^{1/2}\\
&\leqslant \frac{1}{4}f(t) + C_N\mathbb{E}\left(\int_0^{t\wedge\tau_n^\epsilon}\big(1+\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_0^2}^2\big)\,\mathrm{d}s\right)\\
&\leqslant \frac{1}{4}f(t) + C_N\int_0^t\big(1+f(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Similarly, we also have

$$\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_n^\epsilon]} J_3(s)\Big) \leqslant \frac{1}{4}f(t) + C\int_0^t\big(1+f(s)\big)\,\mathrm{d}s$$

and

$$\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_n^\epsilon]} J_4(s)\Big) \leqslant C\int_0^t\big(1+f(s)\big)\,\mathrm{d}s.$$

Combining the above calculations we get

$$f(t) + 2\mathbb{E}\int_0^{t\wedge\tau_n^\epsilon} \big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_1}\,\mathrm{d}s \leqslant 2\|\mathbf{u}_0\|^2_{\mathbf{H}^2_0} + C_N + C_N\int_0^t \big(1 + f(s)\big)\,\mathrm{d}s.$$

The desired estimate follows by Gronwall's inequality. □

Set for $n \in \mathbb{N}$,

$$\eta_n^\epsilon(t) := \int_0^{t\wedge\tau_n^\epsilon} \big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_1}\cdot\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_0}\,\mathrm{d}s + t$$
$$= \int_0^t \big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_1}\cdot\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_0}\cdot 1_{[0,\tau_n^\epsilon]}(s)\,\mathrm{d}s + t$$

and

$$\theta_n^\epsilon(t) := \inf\big\{s \geqslant 0:\ \eta_n^\epsilon(s) \geqslant t\big\}.$$

Clearly, $t \mapsto \eta_n^\epsilon(t)$ is a continuous and strictly increasing function, and the inverse function of $t \mapsto \theta_n^\epsilon(t)$ is just given by η_n^ϵ. Moreover, since $\eta_n^\epsilon(t) > t$, we have

$$\theta_n^\epsilon(t) < t.$$

Lemma 7.5. *For any $K > 0$, there exists a constant $C_{K,N} > 0$ such that for all $\epsilon \in (0,1)$ and $n \in \mathbb{N}$,*

$$\mathbb{E}\Big(\sup_{s\in[0,\theta_n^\epsilon(K)\wedge\tau_n^\epsilon]}\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}^2_1}\Big) \leqslant C_{K,N}.$$

Proof. Consider the following evolution triple

$$\mathbf{H}^2_2 \subset \mathbf{H}^2_1 \subset \mathbf{H}^2_0.$$

By Itô's formula (cf. [58]), we have

$$\big\|\mathbf{u}^\epsilon(t)\big\|^2_{\mathbf{H}^2_1} = \|\mathbf{u}_0\|^2_{\mathbf{H}^2_1} + 2\int_0^t \big\langle\mathfrak{L}_2\mathbf{u}^\epsilon(s), -\mathfrak{L}_2\mathbf{u}^\epsilon(s) + \Phi\big(s,\mathbf{u}^\epsilon(s)\big)\big\rangle_{\mathbf{H}^2_0}\,\mathrm{d}s$$
$$+ 2\int_0^t \big\langle\mathfrak{L}_2\mathbf{u}^\epsilon(s), \Psi\big(s,\mathbf{u}^\epsilon(s)\big)\dot{h}^\epsilon(s)\big\rangle_{\mathbf{H}^2_0}\,\mathrm{d}s$$

$$+2\sqrt{\epsilon}\sum_k\int_0^t\big\langle\mathbf{u}^\epsilon(s),\Psi_k\big(s,\mathbf{u}^\epsilon(s)\big)\big\rangle_{\mathbf{H}_1^2}\,\mathrm{d}W^k(s)$$

$$+\epsilon\sum_k\int_0^t\big\|\Psi_k\big(s,\mathbf{u}^\epsilon(s)\big)\big\|^2_{\mathbf{H}_1^2}\,\mathrm{d}s$$

$$=:\|\mathbf{u}_0\|^2_{\mathbf{H}_1^2}+J_1(t)+J_2(t)+J_3(t)+J_4(t).$$

Set

$$f(t):=\mathbb{E}\Big(\sup_{s\in[0,t]}\big\|\mathbf{u}^\epsilon\big(\theta_n^\epsilon(s)\wedge\tau_n^\epsilon\big)\big\|^2_{\mathbf{H}_1^2}\Big)=\mathbb{E}\Big(\sup_{s\in[0,\theta_n^\epsilon(t)\wedge\tau_n^\epsilon]}\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}_1^2}\Big).$$

For $J_1(t)$, by (7.13) we have, for $t\in[0,K]$,

$$\begin{aligned}J_1\big(\theta_n^\epsilon(t)\wedge\tau_n^\epsilon\big)&\leqslant\int_0^{\theta_n^\epsilon(t)\wedge\tau_n^\epsilon}\big[C\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}_0^2}\cdot\big\|\mathbf{u}^\epsilon(s)\big\|^4_{\mathbf{H}_1^2}+C_K\big(1+\big\|\mathbf{u}^\epsilon(s)\big\|^2_{\mathbf{H}_1^2}\big)\big]\,\mathrm{d}s\\&\leqslant C\int_0^{\theta_n^\epsilon(t)}\big\|\mathbf{u}^\epsilon\big(s\wedge\tau_n^\epsilon\big)\big\|^2_{\mathbf{H}_1^2}\,\mathrm{d}\eta_n^\epsilon(s)+C_K\\&=C\int_0^t\big\|\mathbf{u}^\epsilon\big(\theta_n^\epsilon(s)\wedge\tau_n^\epsilon\big)\big\|^2_{\mathbf{H}_1^2}\,\mathrm{d}s+C_K,\end{aligned}$$

where the last step is due to the substitution of variable formula. So,

$$\mathbb{E}\Big(\sup_{s\in[0,t]}J_1\big(\theta_n^\epsilon(s)\wedge\tau_n^\epsilon\big)\Big)\leqslant C\int_0^t f(s)\,\mathrm{d}s+C_K.$$

Using the same trick as used in Lemma 7.4 and by (N3), we also have

$$\mathbb{E}\Big(\sup_{s\in[0,t]}J_i\big(\theta_n^\epsilon(s)\wedge\tau_n^\epsilon\big)\Big)\leqslant\frac{1}{2}f(t)+C_{N,K}\int_0^t\big(f(s)+1\big)\,\mathrm{d}s,\quad i=2,3,4.$$

Thus, we get

$$f(t)\leqslant 2\|\mathbf{u}_0\|^2_{\mathbf{H}_1^2}+C_{N,K}\int_0^t\big(f(s)+1\big)\,\mathrm{d}s,$$

which yields the desired estimate by Gronwall's inequality. □

Set for $M > 0$,

$$\zeta_n^\epsilon(M) := \inf\big\{t \geqslant 0\colon \ \big\|\mathbf{u}^\epsilon\big(t \wedge \tau_n^\epsilon\big)\big\|_{\mathbf{H}_1^2} \geqslant M\big\}.$$

Lemma 7.6. *For any $M > 0$ and $q \geqslant 2$, there exists a constant $C_{T,M,N} > 0$ such that for all $\epsilon \in (0,1)$ and $n \in \mathbb{N}$,*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_n^\epsilon\wedge\zeta_n^\epsilon(M)]}\big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_{2\alpha}^p}^q\Big] \leqslant C_{T,M,N}.$$

Proof. Set for $t \in [0,T]$,

$$\xi_n^\epsilon(t) := t \wedge \tau_n^\epsilon \wedge \zeta_n^\epsilon(M)$$

and for $q \geqslant 2$,

$$f(t) := \mathbb{E}\Big[\sup_{t'\in[0,\xi_n^\epsilon(t)]}\big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_{2\alpha}^p}^q\Big].$$

Note that

$$\mathbf{u}^\epsilon(t) = \mathfrak{T}_t\mathbf{u}_0 + \int_0^t \mathfrak{T}_{t-s}\Phi\big(s,\mathbf{u}^\epsilon(s)\big)\,\mathrm{d}s + \int_0^t \mathfrak{T}_{t-s}\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\dot{h}^\epsilon(s)\,\mathrm{d}s$$
$$+ \sqrt{\epsilon}\int_0^t \mathfrak{T}_{t-s}\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\,\mathrm{d}W(s).$$

By (iii) of Proposition 4.1, Hölder's inequality and Lemma 7.14, we have, for $q > \frac{1}{1-\alpha}$,

$$\mathbb{E}\Bigg[\sup_{t'\in[0,\xi_n^\epsilon(t)]}\Bigg\|\int_0^{t'}\mathfrak{T}_{t'-s}\Phi\big(s,\mathbf{u}^\epsilon(s)\big)\,\mathrm{d}s\Bigg\|_{\mathbf{H}_{2\alpha}^p}^q\Bigg]$$
$$\preccurlyeq \mathbb{E}\Bigg[\sup_{t'\in[0,\xi_n^\epsilon(t)]}\Bigg(\int_0^{t'}\frac{1}{(t'-s)^\alpha}\big\|\Phi\big(s,\mathbf{u}^\epsilon(s)\big)\big\|_{\mathbf{H}_0^p}\,\mathrm{d}s\Bigg)^q\Bigg] \preccurlyeq \mathbb{E}\Bigg[\int_0^{\xi_n^\epsilon(t)}\big\|\Phi\big(s,\mathbf{u}^\epsilon(s)\big)\big\|_{\mathbf{H}_0^p}^q\,\mathrm{d}s\Bigg]$$
$$\overset{(7.14)}{\preccurlyeq} \mathbb{E}\Bigg[\int_0^{\xi_n^\epsilon(t)}\big[\big(1+\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_1^2}^q\big)\cdot\big(1+\big\|\mathbf{u}^\epsilon(s)\big\|_{\mathbf{H}_{2\alpha}^p}^q\big)\big]\,\mathrm{d}s\Bigg] \leqslant C_M\int_0^t\big(f(s)+1\big)\,\mathrm{d}s.$$

On the other hand, set

$$G(t,s) := \mathfrak{T}_{t-s}\Psi\big(s,\mathbf{u}^\epsilon(s)\big).$$

Then by (iii) and (iv) of Proposition 4.1, we have

$$\|G(t,s)\|^2_{\mathbf{H}^p_{2\alpha}} \leqslant \frac{C}{(t-s)^\alpha}\|\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\|^2_{L_2(l^2;\mathbf{H}^p_\alpha)}$$

and for $\gamma \in (0,(1-\alpha)/2)$,

$$\|G\big(t',s\big)-G(t,s)\|^2_{\mathbf{H}^p_{2\alpha}} \leqslant \frac{|t'-t|^\gamma}{(t-s)^{\alpha+2\gamma}}\|\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\|^2_{L_2(l^2;\mathbf{H}^p_\alpha)}.$$

Therefore, using Lemma 3.4 for q large enough, we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t'\in[0,T\wedge\xi_n^\epsilon(t)]}\left\|\int_0^{t'} G\big(t',s\big)\,\mathrm{d}W(s)\right\|^q_{\mathbf{H}^p_{2\alpha}}\right)\\
&\quad\leqslant C_T\mathbb{E}\left(\int_0^{T\wedge\xi_n^\epsilon(t)}\|\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\|^q_{L_2(l^2;\mathbf{H}^p_\alpha)}\,\mathrm{d}s\right)\\
&\quad\overset{\text{(N3)}}{\leqslant} C_T\int_0^t\big(f(s)+1\big)\,\mathrm{d}s.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t'\in[0,T\wedge\xi_n^\epsilon(t)]}\left\|\int_0^t \mathfrak{T}_{t-s}\Psi\big(s,\mathbf{u}^\epsilon(s)\big)\dot{h}^\epsilon(s)\,\mathrm{d}s\right\|^q_{\mathbf{H}^p_{2\alpha}}\right)\\
&\quad\leqslant C_{T,N}\int_0^t\big(f(s)+1\big)\,\mathrm{d}s.
\end{aligned}$$

Combining the above calculations, we obtain

$$f(t)\leqslant C_{T,M,N}\int_0^t f(s)\,\mathrm{d}s + C_{T,M,N},$$

which yields the desired estimate by Gronwall's inequality. □

Lemma 7.7. *It holds that*

$$\lim_{n\to\infty}\sup_{\epsilon\in(0,1)} P\big\{\omega\colon\ \tau_n^\epsilon(\omega)\leqslant T\big\}=0. \tag{7.15}$$

Proof. First of all, for any $M, K > 0$ we have

$$\begin{aligned}
P\{\zeta_n^\epsilon(M) < T\} &\leqslant P\{\zeta_n^\epsilon(M) < T; \theta_n^\epsilon(K) \geqslant T\} + P\{\theta_n^\epsilon(K) < T\} \\
&= P\Big\{\sup_{t\in[0,T)} \big\|\mathbf{u}^\epsilon\big(t\wedge\tau_n^\epsilon\big)\big\|_{\mathbf{H}_1^2} > M; \theta_n^\epsilon(K) \geqslant T\Big\} \\
&\quad + P\Big\{\sup_{s\in[0,T)} \eta_n^\epsilon(s) > K\Big\} \\
&\leqslant P\Big\{\sup_{t\in[0,\theta_n^\epsilon(K)\wedge\tau_n^\epsilon]} \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_1^2} > M\Big\} + P\big\{\eta_n^\epsilon(T) > K\big\} \\
&\leqslant \mathbb{E}\Big(\sup_{t\in[0,\theta_n^\epsilon(K)\wedge\tau_n^\epsilon)} \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_1^2}^2\Big)/M^2 + \mathbb{E}\big(\eta_n^\epsilon(T)\big)/K.
\end{aligned}$$

Hence, by Lemmas 7.4 and 7.5 we have

$$\lim_{M\to\infty} \sup_{n,\epsilon} P\{\zeta_n^\epsilon(M) < T\} = 0.$$

Secondly, we also have

$$P\{\tau_n^\epsilon < T\} \leqslant P\{\tau_n^\epsilon < T; \zeta_n^\epsilon(M) \geqslant T\} + P\{\zeta_n^\epsilon(M) < T\}. \tag{7.16}$$

For the first term, by Lemma 7.6 we have

$$\begin{aligned}
P\{\tau_n^\epsilon < T; \zeta_n^\epsilon(M) \geqslant T\} &= P\Big\{\sup_{t\in[0,T)} \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_{2\alpha}^p} > n; \zeta_n^\epsilon(M) \geqslant T\Big\} \\
&\leqslant P\Big\{\sup_{t\in[0,T\wedge\tau_n^\epsilon]} \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_{2\alpha}^p} \geqslant n; \zeta_n^\epsilon(M) \geqslant T\Big\} \\
&\leqslant P\Big\{\sup_{s\in[0,T\wedge\zeta_n^\epsilon(M)\wedge\tau_n^\epsilon]} \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_{2\alpha}^p} \geqslant n\Big\} \\
&\leqslant \mathbb{E}\Big(\sup_{s\in[0,T\wedge\zeta_n^\epsilon(M)\wedge\tau_n^\epsilon]} \big\|\mathbf{u}^\epsilon(t)\big\|_{\mathbf{H}_{2\alpha}^p}^q\Big)/n^q \leqslant \frac{C_{T,M,N}}{n^q},
\end{aligned}$$

where $C_{T,M,N}$ is independent of ϵ and n. The desired limit now follows by taking limits for (7.16), first $n \to \infty$, then $M \to \infty$. $\square$

Thus, using Theorem 6.3 we get:

Theorem 7.8. *Let $\mathcal{O} = \mathbb{T}^2$ or a bounded smooth domain in $\mathbb{R}^2$. Under* (N1)–(N3)*, for $\mathbf{u}_0 \in \mathbf{H}_2^p$, $\{\mathbf{u}_\epsilon, \epsilon \in (0,1)\}$ satisfies the large deviation principle in $\mathbb{C}_T(\mathbf{H}_{2\alpha}^p)$ with the rate function $I(f)$ given by*

$$I(f) := \frac{1}{2} \inf_{\{h\in\ell_T^2:\ f=\mathbf{u}^h\}} \|h\|_{\ell_T^2}^2, \quad f \in \mathbb{C}_T\big(\mathbf{H}_{2\alpha}^p\big),$$

where $\mathbf{u}^h$ *solves the following equation*:

$$\mathbf{u}^h(t) = \mathbf{u}_0 + \int_0^t \Delta\mathbf{u}^h(s)\,\mathrm{d}s + \int_0^t \mathscr{P}_p\big(\big(\mathbf{u}^h(s)\cdot\nabla\big)\mathbf{u}^h(s)\big)\,\mathrm{d}s$$
$$+ \int_0^t F\big(s, \mathbf{u}^h(s)\big)\,\mathrm{d}s + \int_0^t \Psi\big(s, \mathbf{u}^h(s)\big)\dot{h}(s)\,\mathrm{d}s.$$

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References

[1] R.A. Adams, Sobolev Space, Academic Press, 1975.
[2] R. Azencott, Grandes déviations et applications, in: Lecture Notes in Math., vol. 779, Springer-Verlag, New York, 1980, pp. 1–176.
[3] M. Berger, V. Mizel, Volterra equations with Itô integrals, I and II, J. Integral Equations 2 (1980) 187–245, 319–337.
[4] S. Bonaccorsi, M. Fantozzi, Large deviation principle for semilinear stochastic Volterra equations, Dynam. Systems Appl. 13 (2) (2004) 203–219.
[5] M. Boué, P. Dupuis, A variational representation for certain functionals of Brownian motion, Ann. Probab. 26 (4) (1998) 1641–1659.
[6] Z. Brzeźniak, Stochastic partial differential equations in M-type 2 Banach spaces, Potential Anal. 4 (1) (1995) 1–45.
[7] Z. Brzeźniak, On stochastic convolution in Banach spaces and applications, Stochastics Stochastics Rep. 61 (3–4) (1997) 245–295.
[8] Z. Brzeźniak, S. Peszat, Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process, Studia Math. 137 (3) (1999) 261–299.
[9] Z. Brzeźniak, S. Peszat, Strong local and global solutions to stochastic Navier–Stokes equations, in: Infinite Dimensional Stochastic Analysis, Proceedings of the Colloquium of the Royal Netherlands Academy of Sciences, Amsterdam, 1999, North-Holland, Amsterdam, 2000, pp. 85–98.
[10] A. Budhiraja, P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, Probab. Math. Statist. 20 (1) (2000) 39–61, Acta Univ. Wratislav. No. 2246.
[11] A. Budhiraja, P. Dupuis, V. Maroulas, Large deviations for infinite dimensional stochastic dynamical systems, Ann. Probab. 36 (4) (2008) 1390–1420.
[12] M.-H. Chang, Large deviation for Navier–Stokes equations with small stochastic perturbation, Appl. Math. Comput. 76 (1996) 65–93.
[13] P. Clément, G. Da Prato, Some results on stochastic convolutions arising in Volterra equations perturbed by noise, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 7 (3) (1996) 147–153.
[14] L. Coutin, L. Decreusefond, Stochastic Volterra equations with singular kernels, in: Stochastic Analysis and Mathematical Physics, in: Progr. Probab., vol. 50, Birkhäuser Boston, Boston, MA, 2001, pp. 39–50.
[15] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.
[16] L. Decreusefond, Regularity properties of some stochastic Volterra integrals with singular kernel, Potential Anal. 16 (2002) 139–149.
[17] L. Denis, L. Stoica, A general analytical result for non-linear SPDE's and applications, Electron. J. Probab. 9 (2004) 674–709.
[18] L. Denis, A. Matoussi, L. Stoica, L^p estimates for the uniform norm of solutions of quasilinear SPDE's, Probab. Theory Related Fields 133 (4) (2005) 437–463.
[19] P. Dupuis, R.S. Ellis, A Weak Convergence Approach to the Theory of Large Deviations, Wiley, New York, 1997.

[20] F. Flandoli, Regularity Theory and Stochastic Flows for Parabolic SPDEs, Stochastic Monogr., vol. 9, Gordon and Breach Publishers, 1995.
[21] M.I. Freidlin, A.D. Wentzell, On small random perturbations of dynamical system, Russian Math. Surveys 25 (1970) 1–55.
[22] A. Friedman, Partial Differential Equations, Holt, Reihart and Winston, New York, 1969.
[23] D. Fujiwara, H. Morimoto, An L_r-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977) 685–700.
[24] Y. Giga, Analyticity of the semigroup generated by the stokes operator in L_r spaces, Math. Z. 178 (1981) 297–329.
[25] Y. Giga, Domains of fractional powers of the Stokes operator in L_r spaces, Arch. Ration. Mech. Anal. 89 (3) (1985) 251–265.
[26] B. Goldys, M. Röckner, X. Zhang, Martingale solutions and Markov selections for stochastic partial differential equations, Stochastic Process. Appl. 119 (5) (2009) 1725–1764.
[27] G. Gripenberg, On the resolvents of nonconvolution Volterra kernels, Funkcial. Ekvac. 23 (1980) 83–95.
[28] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, 1980.
[29] G.S. Jordan, R.L. Wheeler, A generalization of the Wiener–Lévy theorem applicable to some Volterra equations, Proc. Amer. Math. Soc. 57 (1) (1976) 109–114.
[30] O. Kallenberg, Foundations of Modern Probability, second edition, Springer-Verlag, New York, Berlin, 2001.
[31] A. Karczewska, C. Lizama, Stochastic Volterra equations driven by cylindrical Wiener process, J. Evol. Equ. 7 (2) (2007) 373–386.
[32] A. Kolmogorov, S. Fomin, Elements of the Theory of Functions and Functional Analysis, Graylock, Rochester, 1957.
[33] P. Kotelenez, Stochastic Ordinary and Stochastic Partial Differential Equations: Transition from Microscopic to Macroscopic Equations, Stoch. Model. Appl. Probab., vol. 58, Springer-Verlag, New York, 2008, x+458 pp.
[34] N.V. Krylov, An analytic approach to SPDEs, in: Stochastic Partial Differential Equations, Six Perspectives, in: Math. Surveys Monogr., vol. 64, American Mathematical Society, Providence, 1999, pp. 185–242.
[35] N.V. Krylov, B.L. Rozovskii, Stochastic evolution equations, J. Soviet Math. (1979) 71–147 (in Russian); transl. in: 16 (1981) 1233–1277.
[36] E.H. Lakhel, Large deviation for stochastic Volterra equation in the Besov–Orlicz space and application, Random Oper. Stoch. Equ. 11 (4) (2003) 333–350.
[37] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. I: Incompressible Models, Oxford Lecture Ser. Math. Appl., Clarendon Press, Oxford, 1996.
[38] W. Liu, Large deviations for stochastic evolution equations with small multiplicative noise, Appl. Math. Optim., doi:10.1007/s00245-009-9072-2, in press.
[39] R. Mikulevicius, B. Rozovskii, A note of Krylov's L^p-theory for systems of SPDEs, Electron. J. Probab. 6 (2001) 1–35.
[40] R. Mikulevicius, B.L. Rozovskii, Stochastic Navier–Stokes equations for turbulent flows, SIAM J. Math. Anal. 35 (5) (2004) 1250–1310.
[41] R. Mikulevicius, B.L. Rozovskii, Martingale problems for stochastic PDE's, in: Stochastic Partial Differential Equations, Six Perspectives, in: Math. Surveys Monogr., vol. 64, AMS, Providence, 1999, pp. 185–242.
[42] R. Miller, Nonlinear Volterra Integral Equations, W.A. Benjamin, Inc. Publishers, California, 1971.
[43] A.L. Neidhardt, Stochastic Integrals in 2-Uniformly Smooth Banach Spaces, University of Wisconsin, 1978.
[44] D. Nualart, C. Rovira, Large deviation for stochastic Volterra equation, Bernoulli 6 (2000) 339–355.
[45] M. Ondreját, Uniqueness for stochastic evolution equations in Banach spaces, Dissertationes Math. (Rozprawy Mat.) 426 (2004), 63 pp.
[46] E. Pardoux, Equations aux dérivées partielles stochastiques non lineaires monotones: Etude de solutions fortes de type Itô, Thése Doct. Sci. Math. Univ. Paris Sud., 1975.
[47] E. Pardoux, P. Protter, Stochastic Volterra equations with anticipating coefficients, Ann. Probab. 18 (1990) 1635–1655.
[48] A. Pazy, Semi-Groups of Linear Operators and Applications, Springer-Verlag, Berlin, 1985.
[49] S. Peszat, Large deviation estimates for stochastic evolution equations, Probab. Theory Related Fields 98 (1994) 113–136.
[50] G. Pisier, Probabilistic methods in the geometry of Banach spaces, in: Probability and Analysis, Varenna, in: Lecture Notes in Math., vol. 1206, Springer-Verlag, Berlin, 1985, pp. 167–241.
[51] C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations, Lecture Notes in Math., vol. 1905, Springer-Verlag, Berlin, 2007, vi+144 pp.

[52] P. Protter, Volterra equations driven by semimartingales, Ann. Probab. 13 (1985) 519–530.
[53] J. Prüss, Evolutionary Integral Equations and Applications, Monogr. Math., Birkhäuser, 1993.
[54] J. Ren, X. Zhang, Schilder theorem for the Brownian motion on the diffeomorphism group of the circle, J. Funct. Anal. 224 (1) (2005) 107–133.
[55] J. Ren, X. Zhang, Freidlin–Wentzell's large deviations for homeomorphism flows of non-Lipschitz SDEs, Bull. Malays. Math. Sci. Soc. (2) 129 (8) (2005) 643–655.
[56] J. Ren, X. Zhang, Freidlin–Wentzell's large deviations for stochastic evolution equations, J. Funct. Anal. 254 (2008) 3148–3172.
[57] M. Röckner, B. Schmuland, X. Zhang, Yamada–Watanabe theorem for stochastic evolution equations in infinite dimensions, Eur. Phys. J. B 11 (2(54)) (2008) 247–259.
[58] B.L. Rozovskii, Stochastic Evolution Systems: Linear Theory and Applications to Nonlinear Filtering, Math. Appl. (Soviet Ser.), vol. 35, Kluwer Academic Publishers, 1990.
[59] D.F. Shea, S. Wainger, Variants of the Wiener–Lévy theorem, with application to stability problems for some Volterra integral equations, Amer. J. Math. 97 (1975) 312–343.
[60] S.S. Sritharan, P. Sundar, Large deviations for the two-dimensional Navier–Stokes equations with multiplicative noise, Stochastic Process. Appl. 116 (11) (2006) 1636–1659.
[61] R.B. Sower, Large deviations for a reaction-diffusion equation with non-Gaussian perturbations, Ann. Probab. 20 (1) (1992) 504–537.
[62] D.W. Stroock, An Introduction to the Theory of Large Deviations, Springer-Verlag, New York, 1984.
[63] L. Tartar, Interpolaiton non linéarie et régularité, J. Funct. Anal. 9 (1972) 469–489.
[64] J.B. Walsh, An introduction to stochastic partial differential equations, in: Lecture Notes in Math., vol. 1180, Springer-Verlag, 1986, pp. 266–437.
[65] Z. Wang, Existence-uniqueness of solutions to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients, Statist. Probab. Lett. 78 (9) (2008) 1062–1071.
[66] X. Zhang, L^p-theory of semi-linear SPDEs on general measure spaces and applications, J. Funct. Anal. 239 (1) (2006) 44–75.
[67] X. Zhang, Regularities for semilinear stochastic partial differential equations, J. Funct. Anal. 249 (2) (2007) 454–476.
[68] X. Zhang, Euler schemes and large deviations for stochastic Volterra equations with singular kernels, J. Differential Equations 224 (2008) 2226–2250.
[69] X. Zhang, On stochastic evolution equations with non-Lipschitz coefficients, Stoch. Dyn. (2009), in press.
[70] X. Zhang, A variational representation for random functionals on abstract Wiener spaces, J. Math. Kyoto Univ. 49 (3) (2009) 475–490.
[71] X. Zhang, Smooth solutions of non-linear stochastic partial differential equations, http://aps.arxiv.org/abs/0801.3883.
[72] X. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equations, http://arxiv.org/abs/0812.0834.