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CONTINUOUS MODELS OF PERCOLATION THEORY. II

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Percolation models in which the centers of defects are distributed randomly in space in accordance with Poisson's law and the shape of each defect is also random are considered. Methods of obtaining rigorous estimates of the critical densities are described. It is shown that the number of infinite clusters can take only three values: 0, 1, or ∞. Models in which the defects have an elongated shape and a random orientation are investigated in detail. In the two-dimensional case, it is shown that the critical volume concentration of the defects is proportional to a/l, where l and a are, respectively, the major and minor axes of the defect; the mean number of (direct) bonds per defect when percolation occurs is bounded.

Introduction

The present paper continues our [1], which gave a rigorous formulation of continuous percolation problems. For convenience of reference, we continue the numbering of the sections begun in [1].

4. Methods of Estimating Critical Quantities

4.1. The Method of Generations. In $\mathbb{R}^d$, we consider the continuous problem of percolation in which the shape of the defect is fixed and only the orientation can be random. Let $\mathbf{n}$ be the direction vector of a defect (see Sec. 2 in [1]). For a fixed center, the orientation of a defect is determined by the probability measure μ on the sphere S^{d-1}. We restrict ourselves for the time being to measures such that for two independent random variables $\mathbf{n}_1$ and $\mathbf{n}_2$ subject to the distribution on S^{d-1} the difference $\mathbf{n}_2 - \mathbf{n}_1$ does not depend on $\mathbf{n}_1$. We shall say that such measures are symmetric. In particular, the uniform distribution is symmetric. It is readily seen that besides this the only symmetric distributions are those concentrated at k points that form the vertices of the regular (platonic) solids inscribed in S^{d-1}, so that the measure of each point is 1/k. For example, for d = 2 the class of symmetric distributions consists of the uniform distribution on the circle and the uniform distribution over the vertices of the regular k-gon (k = 1, 2, ...).

We shall say that defects are neighbors if they intersect. Let $K(\mathbf{r}, \mathbf{n}_1, \mathbf{n}_2)$ be the μ probability that a defect with center at 0 and direction $\mathbf{n}_1$ intersects a defect with center at $\mathbf{r}$ and direction $\mathbf{n}_2$. We denote $B(\mathbf{n}_1)=\int\limits_{S^{d-1}}\int\limits_{\mathbb{R}^d} K(\mathbf{r}, \mathbf{n}_1, \mathbf{n}_2)\, d\mathbf{r}\, \mu(d\mathbf{n}_2)$. We use the symmetry of the measure μ. Let μ' be the probability measure corresponding to the distribution of the difference $\mathbf{n}_2 - \mathbf{n}_1$. We distinguish in space a certain fixed direction e and denote by $A_{\mathbf{n}_1}$ the rotation about the origin that carries the direction $\mathbf{n}_1$ to e. Then

$$B(\mathbf{n}_1)=\int\limits_{S^{d-1}}\int\limits_{\mathbb{R}^d} K(\mathbf{r}, \mathbf{n}_1, \mathbf{n}_2)\, d\mathbf{r}\mu(d\mathbf{n}_2)=\int\limits_{S^{d-1}}\int\limits_{\mathbb{R}^d} K(A_{\mathbf{n}_1}\mathbf{r}, \mathbf{e}, \mathbf{n}_2-\mathbf{n}_1)\, d\mathbf{r}\mu(d\mathbf{n}_2)=$$

$$\int\limits_{S^{d-1}}\int\limits_{\mathbb{R}^d} K(\mathbf{r}', \mathbf{e}, \mathbf{n}')\, d\mathbf{r}\mu'(d\mathbf{n}')=B=\text{const}.$$

Thus, if the distribution of the orientation is symmetric, then for any defect (irrespective of its orientation) the mathematical expectation of the number of its neighbors is the same and equal to λB, where λ is the intensity of the Poisson field of the defect centers.

We distinguish some defect and call it the defect of generation 0. If we have already determined the defects of the generations 0, 1, ..., k − 1, then the defects of generation k are those that are neighbors of the defects of generation k − 1 and are not defects of the generation k − 2. Consider the random variable

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$\xi_k(\lambda)$, the number of defects of the k-th generation. Clearly, $\xi(\lambda)=\sum_{k=0}^{\infty}\xi_k(\lambda)$ is the power of the cluster containing the distinguished defect, and $E\xi_1(\lambda)=\lambda B$. Since the defects of the (k + 1)-th generation are neighbors of the defects of the k-th generation, for the conditional mathematical expectation the estimate

$$E(\xi_{k+1}(\lambda)\,|\,\xi_k(\lambda))\leqslant\lambda B\xi_k(\lambda) \tag{4.1}$$

is almost certainly valid. If follows from this that $E\xi_k(\lambda) \leq (\lambda B)^k$ for all k. Therefore, if $\lambda<1/B$,

$$E\xi(\lambda)=\sum_{k=0}^{\infty} E\xi_k(\lambda)\leqslant\sum_{k=0}^{\infty} (\lambda B)^k=1/(1-\lambda B)<\infty.$$

In accordance with the definition of λ_T, this means that

$$\lambda_H\geqslant\lambda_T\geqslant 1/B. \tag{4.2}$$

One can attempt to improve this result by obtaining an estimate of the form $E(\xi_{k+l}(\lambda)\,|\,\xi_k(\lambda))\leqslant\gamma(\lambda)\xi_k(\lambda)$ for all $k \geq N$. Then as soon as $\gamma < 1$

$$E\xi=\sum^{N-1}\times\xi_k+\sum^{N+l-1}\sum^{\infty} E\xi_{k+jl}\leqslant\sum^{N-1}E\xi_k+\sum_{k=N}^{N+l-1}\sum_{j=0}^{\infty}\gamma^j E\xi_k\leqslant$$

$$\sum_{k=0}^{N-1}(\lambda B)^k+\sum_{k=N}^{N+l-1}\frac{1}{1-\gamma}(\lambda B)^k=\frac{(\lambda B)^N-1}{\lambda B-1}+\frac{1}{1-\gamma}\frac{(\lambda B)^{N+l}-(\lambda B)^N}{\lambda B-1}<\infty,$$

and therefore there is no percolation.

This method is common to all percolation problems and makes it possible to obtain good lower bounds even in the problems for which the unique method hitherto is the Monte Carlo method. In problems on graphs, one need only regard as neighbors the defect vertices connected by a line of the graph. It is true that in continuous models the obtaining of estimates of the form $E(\xi_{k+l}\,|\,\xi_k)\leqslant\gamma\xi_k$ for $l\geqslant 2$ involves appreciable technical difficulties. For $l = 1$, the estimates (4.1) and (4.2) can be somewhat improved.

Example 4.1. We consider in $\mathbb{R}^3$ the problem of spheres of diameter 1. Here, the volume defect is $V=\pi/6$, and $B=4/3\pi$. Suppose the defect with center at the point $\mathbf{r}_k$ belongs to the k-th generation ($k \geq 1$). Then there exists a defect of the (k − 1)-th generation with center at some point $\mathbf{r}_{k-1}$, for which $|\mathbf{r}_k-\mathbf{r}_{k-1}|\leqslant 1$. The centers of the defects of the (k + 1)-th generation must be in the region of the points $\mathbf{r}$ determined by the inequalities $|\mathbf{r}-\mathbf{r}_k|\leqslant 1$, $|\mathbf{r}-\mathbf{r}_{k-1}|>1$. The volume of this region in $\mathbb{R}^3$ does not exceed $11\pi/12$. Therefore, for the given problem $E(\xi_{k+1}(\lambda)\,|\,\xi_k(\lambda))\leqslant {}^{11}/_{12}\pi\lambda\xi_k(\lambda)={}^{11}/_{16}\lambda B\xi_k(\lambda)$ and

$$\lambda_H\geqslant\lambda_T\geqslant\frac{16}{11}\cdot\frac{1}{B}. \tag{4.3}$$

A further strengthening of the estimates involves a refinement of the method.

4.2. Spectral Method. We recall that we associated with the defects a parameter a, which indicated the defect type (see Sec. 2 in [1]). The range of variation of the parameter was the space A with probability measure. Instead of the random variable ξ_k (we shall in what follows omit the index λ), we consider the random variable $\eta_k(a)=\xi_k\cdot\theta_k(a)$, where $\theta_k(a)$ is the density (fraction) of defects of the k-th generation with which the parameter a is associated. Let $\mathscr{L}_1(A)$ be the space of summable functions on A with norm $\|f\|=\int_A|f(a)|\nu(da)$. In the majority of problems, we shall take A to be an interval on the real numbers, and as ν we take the Lebesgue measure. In some cases, A will be a finite set and ν a counting measure. Note that $\xi_k=\|\eta_k\|$.

We seek the estimate $E\eta_1(a)\leqslant\lambda g(a)$, where $g(a)$ does not depend on λ and the integral operator $\Omega:\mathscr{L}_1(A)\to\mathscr{L}_1(A)$ is constructed with non-negative kernel ω satisfying for all k and a the condition

$$E(\eta_{k+1}(a)\,|\,\eta_k)\leqslant\lambda\Omega[\eta_k](a)=\lambda\int_A\omega(a,b)\eta_k(b)\nu(db). \tag{4.4}$$

Then for every k we have $E\eta_k(a)\leqslant\lambda^k\Omega^{k-1}[g](a)$, and, therefore, $E\xi_k=E\|\eta_k\|\leqslant\lambda^k\|\Omega^{k-1}[g]\|$. We denote by σ the spectral radius of the operator Ω:

$$\sigma=\lim_{k\to\infty}\Big[\sup_{\substack{g\in\mathscr{L}_1(A)\\ \|g\|\neq 0}}\frac{\|\Omega^k(g)\|}{\|g\|}\Big]^{1/k}.$$

Then if $\lambda\sigma<1$ we have $E\xi(\lambda)=\sum_{k=0}^{\infty}E\xi_k(\lambda)\leqslant\sum_{k=0}^{\infty}\lambda^k\|\Omega^{k-1}[g]\|<\infty$. Therefore

$$\lambda_H\geqslant\lambda_T\geqslant 1/\sigma. \tag{4.5}$$

Note that for an integral operator with non-negative kernel the spectral radius is equal to the largest eigenvalue (see, for example, [2]).

Example 4.2. We again consider the problem of spheres of diameter 1. We take the parameter a to be the distance from the center of a given defect to the nearest center of a defect of the preceding generation. Then A is the interval $(0, 1]$. Suppose the defect with center at the point $\mathbf{r}_k$ belongs to the k-th generation and that the defect of the $(k-1)$-th generation nearest it has center at the point $\mathbf{r}_{k-1}$, where $|\mathbf{r}_k-\mathbf{r}_{k-1}|=b$. Then the centers of the defects of the $(k+1)$-th generation separated from $\mathbf{r}_k$ by distance a must be on the surface defined by the conditions $|\mathbf{r}-\mathbf{r}_k|=a$, $|\mathbf{r}-\mathbf{r}_{k-1}|>1$. In $\mathbb{R}^3$, this surface has the area $\pi\frac{a}{b}((a+b)^2-1)$. Therefore

$$E(\eta_{k+1}(a)\mid\eta_k)\leqslant\lambda\Omega[\eta_k](a)=\lambda\int_{1-a}^{1}\pi\frac{a}{b}((a+b)^2-1)\eta_k(b)\,db.$$

In the given case, the spectral radius of the operator Ω is $4\pi/3\sqrt{3}=B/\sqrt{3}$. Thus, $\lambda_H\geq\lambda_T\geq\sqrt{3}/B$ (cf. (4.2)). Since here $V=B/8$, it follows that $\lambda_T V\geq\sqrt{3}/8$ and (see Sec.2 in [1]) $C_H\geqslant C_T=1-\exp\{-\lambda_T V\}\geqslant 1-\exp\{-\sqrt{3}/8\}\approx 0.19467$.

We now show how it is possible to construct the operator Ω in the more general case when the defects can have different shapes or the distribution of the orientation is not symmetric. Let there be a finite number of defect types, $a_1,\dots,a_n$ and p_i be the probability of realization of type i. We denote by $h(a_i, a_j)$ the volume of the region of $\mathbb{R}^d$ containing the vectors $\mathbf{r}$ for which a defect of type a_i with center at $\mathbf{r}$ and a defect of type a_j with center at $\mathbf{0}$ intersect. Let $\eta_k(a_i)$ be the number of defects of type a_i in the k-th generation. Then

$$E(\eta_{k+1}(a_i)\mid\eta_k(a_1),\dots,\eta_k(a_n))\leqslant\lambda\sum_{j=1}^{n}p_i h(a_i,a_j)\eta_k(a_j).$$

Thus, as the operator Ω we can choose the matrix operator determined by the matrix with elements $p_i h(a_i, a_j)$ $(i, j=1,\dots,n)$.

Now suppose the defect type is specified by a parameter a that ranges over the interval $(0, \infty)$, and $F(a)$ is the distribution function of the defect type. In this case $E(\eta_{k+1}(a)\mid\eta_k)\leqslant\lambda\int_0^\infty h(a,b)\eta_k(b)\,dF(b)$ and an integral operator plays the role of Ω.

We note that the calculation of the spectral radius of the integral operator Ω simplifies appreciably if the function h is represented in the form $h(a,b)=\sum_{j=1}^{k}V_j(a)U_j(b)$. At the same time, the kernel of the operator is degenerate (see, for example, [3]), and the problem reduces to determining the largest eigenvalue of the matrix of order k.

Example 4.3. Suppose each defect has the shape of a sphere whose radius is a random variable with density $f(a)$. Then $h(a, b)={}^4/_3\pi(a+b)^3$. The kernel is degenerate. An operator Ω with such kernel carries every function $g(b)$ into a linear combination of the functions $a^3f(a)$, $a^2f(a)$, $af(a)$, and $f(a)$. Therefore, to determine the spectral radius of Ω it is sufficient to consider as domain of definition of the operator the four-dimensional space generated by these function. Suppose

$$g(b)=\sum_{j=0}^{3} c_j b^j f(b), \quad m_i=\int_0^\infty a^i f(a)\,da=\int_0^\infty a^i\,dF(a).$$

Then

$$\Omega[g](a)=\int_0^\infty \frac{4}{3}\pi(a+b)^3 f(a)g(b)\,db=\int_0^\infty \frac{4}{3}\pi(a+b)^3 f(a)f(b)\sum_{j=0}^{3} c_j b^j\,db=$$

$$\frac{4}{3}\pi f(a)\sum_{j=0}^{3} c_j\int_0^\infty (a^3b^j+3a^2b^{j+1}+3ab^{j+2}+b^{j+3})f(b)\,db=\frac{4}{3}\pi f(a)\sum_{j=0}^{3} c_j(a^3m_j+3a^2m_{j+1}+3am_{j+2}+m_{j+3}).$$

It follows from this that the spectral radius of the operator Ω is equal to the spectral radius (largest eigenvalue) of the matrix

$$4\pi\begin{pmatrix} m_3 & 3m_2 & 3m_1 & m_0 \\ m_4 & 3m_3 & 3m_2 & m_1 \\ m_5 & 3m_4 & 3m_3 & m_2 \\ m_6 & 3m_5 & 3m_4 & m_3 \end{pmatrix},$$

where $m_i=\int_0^\infty a^i\,dF(a)$ is the i-th moment of the radius of the sphere.

By continuity, this result can also be extended to the case of a discontinuous distribution function F. Indeed, if the function F is discontinuous, there exist sequences of distribution functions $\{F'_n\}$ and $\{F''_n\}$ that converge weakly to F and are such that $F'_n \le F \le F''_n$ for all n. Further, see Remark 3.5 in [1].

5. Models with Random Orientation of Defects

Suppose all defects have the same convex shape D and fixed orientation. This is equivalent to the space A being trivial, and, therefore, the function L(r) takes only the values 0 or 1 (see Sec.2 in [1]). Then the function K(r) (see Sec.4) has the same property. In this case, neighboring defects can be defined in other terms: Two defects D(**x**) and D(**y**) with centers at the points **x** and **y** are neighbors if K(**x** − **y**) = 1. The equation K(**r**) = 1 defines a centrally symmetric region Q in $\mathbb{R}^d$. Note that its volume is B. Surrounding each center conceptually by a region of the same shape, we can say that D(**x**) and D(**y**) intersect if and only if the point **y** is in the region that surrounds the point **x**. It was in such a formulation that continuous percolation problems were considered in [4, 5]. Exact results were not obtained. Computer simulation showed (see [4]) that in the case of a convex region Q the quantity $\lambda_S B$ $(=\lambda_T B)$ depends weakly on the actual form of this region and is determined solely by the dimension of space. For two-dimensional space, it is 4.1 ± 0.4, and for three-dimensional 2.8 ± 0.2 (the deviations lie within the accuracy of the experiment). If the defect has convex shape, then Q is also convex. If, in addition, it is centrally symmetric, then the region Q is homothetic to the region of the defect with homothety coefficient 2. In such a case, $B = 2^d V$ (d is the dimension of space). Experiments were also made for nonconvex regions Q. A certain lowering (up to 20%) of $\lambda_S B$ was noted. However, such models are less interesting from the point of view of applications, and they cannot always be put into correspondence with the problem of intersecting defects. Thus, in a problem in which all the defects have the same convex shape and are equally oriented, the critical volume concentration is $C_T=1-\exp\{-\lambda_T V\}=1-\exp\{-\lambda_T B\cdot 2^{-d}\}\approx 0.3$ for d = 3. This agrees well with real experimental data. Thus, for the majority of polymer materials the mass loss associated with electric breakdown of an insulation covering is 30%. However, there are materials (for example, polyethylene) for which breakdown occurs much earlier. One of the reasons for this may be the high initial porosity of the material. Another not unimportant reason is that the supermolecular structure of polyethylene takes the form of randomly oriented thin filaments (see [6]), by virtue of which the defects also have an elongated shape. If the defects were oriented predominantly in one direction, the critical volume concentration would still be 30% (without allowance for the initial porosity). Therefore, an important factor that influences its reduction must be random orientation of the defects. A computer simulation for such situations has not been made. It is readily noted that on the transition from a fixed orientation to a random uniform one B increases, more strongly moreover, the more elongated the defects, the value of V remaining at the same time unchanged. Decrease in C_T is therefore associated with decrease in λ_T. In accordance with the inequality (4.2), $\lambda_H B \geq \lambda_T B \geq 1$. It can be assumed that the decrease of λ_T (and λ_H) compensates the increase of B, i.e., the products $\lambda_T B=\lambda_S B$ and $\lambda_H B$ remain bounded.

HYPOTHESIS 5.1. There exists a constant c_d that depends only on the dimension d of space and is such that for any problem with random symmetric orientation $1 \leqslant \lambda_T B = \lambda_S B \leqslant \lambda_H B \leqslant c_d$.

One can also put forward the stronger hypothesis that $\lambda_T B$ $(\lambda_H B)$ is approximately the same for all percolation problems with random symmetric orientation when the defects have the shape of a convex region. The theoretical or experimental confirmation of this hypothesis would be very important, since B can be calculated comparatively easily, and, knowing it, one could also determine the critical concentration.

In this section, we prove the validity of Hypothesis 5.1 in dimension d = 2. As a consequence, we find that for a random symmetric orientation the critical volume concentration is approximately proportional to a/l, where l is the major axis and a the minor axis of the region D. We cannot prove the validity of Hypothesis 5.1 in dimension d = 3, but we show that for regions D with linear dimensions $l \times l \times a$ $(l \gg a)$ the critical volume concentration is also proportional to a/l. Finally, making the assumption that Hypothesis 5.1 is valid, we prove that the critical volume concentration for a region D with dimensions $l \times a \times a$ has the same asymptotic behavior.

THEOREM 5.2. For d = 2, Hypothesis 5.1 is true.

Proof. We consider the continuous percolation problem in which all the defects have the shape of a convex figure D, and the distribution of the orientations is symmetric and has k outcomes (k = ∞ if there is a uniform random orientation). We first consider the case k = 1, i.e., when the orientation is fixed. In accordance with [7], there exist ellipses D_1 and D_2 such that $D_1 \subseteq D \subseteq D_2$ and D_1 is homothetic to D_2 with homothety coefficient 2. We consider the affine area-preserving transformation that carries D_1 and D_2 into disks of radius r_1 and $r_2 = 2r_1$ (this transformation does not change λ_H and B for the original problem). We denote by $\lambda_H(r)$ the value of λ_H for the problem having disks of radius r. The problem of disks of radius r_2 majorizes the problem with the region D′ (D′ is the result of the affine transformation of D), and that, in its turn, majorizes the problem of disks of radius r_1 (see Definition 3.4 in [1]). Therefore, $16\pi r_1^2 = 4\pi r_2^2 \geqslant B$, $\lambda_H \leqslant \lambda_H(r_1) = (r_1)^{-2}\lambda_H(1)$, whence $\lambda_H B \leqslant 16\pi\lambda_H(1)$.

We now consider the case k > 1. Let the diameter of D (i.e., the greatest segment of a straight line that it can cover) be l. Then the problem considered majorizes the problem of "needles" of length l with the same distribution of the orientation. We obtain an upper bound for λ_H in the needle problem; by Remark 3.5 in [1], it will also be an estimate for λ_H in the original problem. One can always choose a system of Cartesian coordinates for which the probability that the orientation vector forms with the abscissa an angle not greater than $\pi/6$ is bounded below by 2/7, the same estimate holding simultaneously for the ordinate. We now use a method analogous to the one used in [8]. We lay out on the plane a square grid of straight lines parallel to the coordinate axes with step $(\sqrt{3}-1)l/8$. We consider the graph whose vertices are the centers of the grid squares with each vertex connected to the eight vertices corresponding to the eight neighboring squares of the grid (the graph is a two-diagonal square lattice). A vertex is assumed to be defective if within the corresponding square there are the centers of two "needles," one inclined to the abscissa axis and the other to the ordinate axis at an angle not greater than $\pi/6$. Then from the existence of an infinite cluster in the discrete problem there follows the existence of an infinite cluster in the continuous problem. The probability that the vertex of a graph will be defective is bounded below by $\left[1-\exp\left\{-\frac{2}{7}\lambda\left(\frac{\sqrt{3}-1}{8}l\right)^2\right\}\right]^2$. Therefore $\left[1-\exp\left\{-\frac{2}{7}\lambda_H\left(\frac{\sqrt{3}-1}{8}l\right)^2\right\}\right]^2 \leqslant p_H$, where p_H is the critical probability for the two-diagonal square lattice ($p_H \approx 0.41$, see [4]). Hence

$$\lambda_H \leqslant \frac{112(2+\sqrt{3})}{l^2} \ln \frac{1}{1-\sqrt{p_H}}. \tag{5.1}$$

On the other hand, since the diameter of the region D is l, it can be covered by a disk of radius $\sqrt{3}l/3$ (see [9]). Therefore

$$B \leqslant 4\pi\left(\frac{\sqrt{3}}{3}l\right)^2 = \frac{4}{3}\pi l^2 \tag{5.2}$$

and $\lambda_H B \leqslant \frac{448(2+\sqrt{3})\pi}{3} \ln \frac{1}{1-\sqrt{p_H}}$, as was required.

COROLLARY 5.3. Let the plane convex figure D have diameter l and its projection onto the line perpendicular to the diameter be a. Then for the problem with random symmetric (not fixed) orientation $c'a/l \leqslant C_T = C_S \leqslant C_H \leqslant c''a/l$, where $C_T = 1-\exp\{-\lambda_T V\}$, $C_S = 1-\exp\{-\lambda_S V\}$, $C_H = 1-\exp\{-\lambda_H V\}$ are the critical volume concentrations and c′ and c″ are absolute constants.

Proof. In accordance with inequality (5.1), $\lambda_H\leqslant c/l^2$. It is obvious that the area of D is not less than $^1/_2 la$, and therefore $C_H\leqslant 1-\exp\left\{-\frac{c}{l^2}\frac{1}{2}la\right\}\leqslant\frac{1}{2}c\frac{a}{l}$. On the other hand, in accordance with (5.2), $B\leqslant{}^4/_3\pi l^2$. It follows from the inequality (4.2) that $\lambda_T\geqslant 1/B\geqslant 3/4\pi l^2$. The area of D does not exceed la. Since $a/l\leqslant 1$, it follows that $C_T\geqslant 1-\exp\left\{-\frac{3}{4\pi l^2}la\right\}=1-\exp\left\{-\frac{3}{4\pi}\frac{a}{l}\right\}\geqslant\left(1-\exp\left\{-\frac{3}{4\pi}\right\}\right)\frac{a}{l}$, as was to be proved.

We now turn to the case when the dimension of space is 3. If the linear dimensions of D are $l\times l\times a$ $(l\gg a)$, then, using the same method, we can prove an estimate analogous to (5.1): $\lambda_H\leqslant\frac{c_1}{l^3}\ln\frac{1}{1-\sqrt[3]{q_H}}$, where q_H is the critical probability for a cubic lattice. As in the case of (5.2), $B\leqslant c_2l^3$, and hence $1\leqslant\lambda_TB=\lambda_SB\leqslant\lambda_HB\leqslant c_3$. It then follows as in the case of Corollary 5.3 that $c'a/l\leqslant C_T=C_S\leqslant C_H\leqslant c''a/l$. But if the linear dimensions of D are $l\times a\times a$ $(l\gg a)$, then we cannot obtain an acceptable upper bound for λ_H. In this case, one can show that $B\approx c_4al^2$. If Hypothesis 5.1 is valid for d = 3, it then follows that $\lambda_T=\lambda_S\approx c_5\frac{1}{al^2}$, $V\approx c_6a^2l$, $C_H=1-\exp\{-\lambda_HV\}\approx c_7a/l$.

6. On the Number of Infinite Clusters

The possible number of infinite clusters in percolation models has frequently been discussed in the literature. It was only in 1981 that Newman and Schulman [10, 11] proved for lattice problems that the number of infinite clusters takes almost certainly one of three values: 0, 1, or ∞. This result was obtained assuming the validity of three hypotheses that the probability distribution of the defect vertices of the lattice must satisfy. Not all the hypotheses are valid for continuous models in the form in which they were formulated in [10, 11]. Nevertheless, an analogous theorem also holds for the continuous problems. We also formulate three hypotheses that the probability distribution P on the defect configurations must satisfy.

Hypothesis A: The probability distribution P is invariant with respect to the group of motions of $\mathbb{R}^d$, i.e., if T is a motion, then TP = P.

Hypothesis B: For any translation event, i.e., an event W such that TW = W for any T, we have P(W) = 0 or 1.

Let $\sigma(\Omega)$ denote the σ algebra of events generated by the defect configurations in the region $\Omega\subseteq\mathbb{R}^d$. Suppose $M\in\sigma(\Omega)$. The event M of nonzero probability generates the conditional distribution P_M on the defect configurations: $P_M(W) = P(W/M)$ for any event W.

Hypothesis C: For any bounded region Ω and all M, $\tilde{M}\in\sigma(\Omega)$ such that $P(M)\neq 0$ and $P(\tilde{M})\neq 0$ the measures P_M and $P_{\tilde{M}}$ are equivalent, i.e., if $P_M(W)\neq 0$ for some event W, then also $P_{\tilde{M}}(W)\neq 0$.

Hypotheses A and C are obviously satisfied for all continuous problems as we have defined them. The distribution P here depends on two parameters: λ, the density of the Poisson field, and μ, the defect type parameter. Hypothesis B is equivalent to ergodicity of the distribution P with respect to the group of spatial translations and was proved in Sec. 2 of [1] for the probability of existence of an infinite cluster. The situation is similar for other translation events.

THEOREM 6.1. The number of infinite clusters in continuous models almost certainly takes one of three values: 0, 1, or ∞.

Proof. Let H_k be the event that there exist exactly k (k = 0, 1, ..., ∞) infinite clusters. All these events are incompatible and invariant with respect to the group of motions of $\mathbb{R}^d$. Therefore, in accordance with Hypothesis B, one and only one of the events H_k has probability 1. We assume that $P(H_k)\neq 0$, where $k\neq 0, 1, \infty$. Then there exists a bounded region $\Omega\subset\mathbb{R}^d$ such that it intersects all k clusters. By the addition of a finite number of defects to Ω it is possible to connect all k clusters into one. Such an event obviously has nonzero probability; therefore, in accordance with Hypothesis C the event H_1 also has nonzero probability, i.e., we arrive at a contradiction. The theorem is proved.

The events H_0 and H_1 are obviously realized in continuous models. With regard to the situation H_∞, proofs of its impossibility (like an example of realizability) do not yet exist either for continuous problems or lattice problems. It is readily seen that in the percolation problem on a tree two situations are possible: H_0 and H_∞. This fact suggests H_∞ could be realized in spaces of sufficiently high dimension.

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QUASI-TAYLOR SERIES IN THE THEORY OF MAGNETISM

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Summation formulas are derived for quasi-Taylor series that arise in the diagram technique for spin operators and correspond to m-point correlations of the spins. In the approximation of self-consistent pair correlations, we obtain an equation of state of an Ising ferromagnet ($d = 3$) valid in a wide range of temperatures and magnetic fields except for a narrow neighborhood of the critical point. In the same approximation, we calculate the shape of the magnetic resonance line of the Ising ferromagnet; it is Gaussian. In the limit $T \to \infty$, complete summation of the quasi-Taylor series yields an exact expression for the line shape.

1. The most promising means of microscopic description of magnets is the diagram technique for spin operators [1, 2, 3, 4, 5]. In contrast to the traditional methods based on boson representations of the spin operators, this diagram technique holds in the complete range of temperatures, including the paramagnetic region and the neighborhood of the magnetic phase transition point T_c. Compared with the low-temperature region ($T \ll T_c$), where the diagram technique for the spin operators is almost identical to the standard technique for many-particle Bose systems (see, for example, [6]), at higher temperatures the structure of this diagram technique becomes more complicated. This is due to the appearance of correlated thermal fluctuations of the longitudinal spin components, taken into account by graphs with ovals [3, 5].

2. Besides geometric progressions, the diagram technique for spin operators has one further class of series that can be completely summed. These are quasi-Taylor series (in the terminology adopted in the book [3], Taylor series with gaps); they appear at higher temperatures and have the form

$$b_m(y) = \sum_{n=0}^{\infty} b^{[mn]}(y) \frac{x_m^{\,n}}{n!}, \qquad m = 1, 2, 3, \ldots \tag{1}$$

The function $b(y)$, for which expandability in a Taylor series is assumed, is usually the Brillouin function, and x_m corresponds to the diagrammatic m-point function (see [3]). For $m = 1$, the series (1) is an ordinary Taylor series, $b_1(y) = b(y + x_1)$. For $m \neq 1$, summation of the series (1) reduces to the action of some integral operator on the function $b(y)$. A corresponding formula for the case $m = 2$ was obtained in [7] (see also [3]). With regard to arbitrary values of m, the problem has not hitherto been adequately solved. The summation formulas obtained by a distinctive method in [8] and given in [3] are little suited for

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