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A Reflected Stochastic Heat Equation as Symmetric Dynamics with Respect to the 3-d Bessel Bridge

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We prove that a stochastic heat equation with reflection at 0, on the spatial interval $[0, 1]$ with Dirichlet boundary conditions and additive white-noise, admits an explicit symmetrizing invariant measure on $C([0, 1])$: the 3-d Bessel Bridge, i.e., the law of the modulus of a 3-dimensional Brownian motion conditioned to be 0 at time 1, a classical measure in probability theory, also connected with the theory of excursions of Brownian motion. This is a non-trivial example of a Gibbs-type measure being singular with respect to the reference Gaussian measure and concentrated on the convex set of positive, continuous functions on $[0, 1]$.

1. INTRODUCTION

In [6], Nualart and Pardoux studied existence and uniqueness of a solution to a reflected stochastic heat equation, namely of a pair (u, η), where u is a continuous function of $(t, \xi) \in \mathcal{O} := [0, +\infty) \times [0, 1]$ and η is a positive measure on $\mathcal{O}$, satisfying

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial \xi^2} - f(\xi, u(t, \xi)) + \dfrac{\partial^2 W}{\partial t\, \partial \xi} + \dfrac{\eta(dt, d\xi)}{dt\, d\xi} \\ u(0, \xi) = x(\xi), \qquad u(t, 0) = u(t, 1) = 0 \\ u \geqslant 0, \qquad \displaystyle\int_{\mathcal{O}} u\, d\eta = 0. \end{cases} \tag{1}$$

where $x : [0, 1] \mapsto [0, \infty)$ is continuous with $x(0) = x(1) = 0$, $\{W(t, \xi) : (t, \xi) \in \mathcal{O}\}$ is a Brownian sheet, and $f : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$.

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The aim of this paper is to prove that Eq. (1) admits, on the space $C_0(0,1)$ of continuous functions of $\xi\in[0,1]$, satisfying Dirichlet boundary conditions, an explicit invariant and symmetrizing measure, given by

$$\exp\left\{-\int_0^1 d\xi\int_0^{x(\xi)} 2f(\xi,s)\,ds\right\}\nu(dx),\qquad x\in C_0(0,1),\tag{2}$$

where ν is a well-known probability measure on $C_0(0,1)$: the 3-d Bessel bridge, namely the law of the modulus of a 3-dimensional Brownian motion $(B_\tau)_{\tau\in[0,1]}$, conditioned to be 0 at $\tau=1$. This measure plays an important role in the study of Brownian motion (see [7]).

On the other hand, Da Prato proved in [2] the existence of a symmetric semigroup $(P_t)_{t\geqslant 0}$ on $L^2(H,(\int_H e^{-2U}\,d\mu)^{-1}\,e^{-2U}\,d\mu)$, associated with the stochastic differential inclusion

$$dX\in(AX-\partial U(X))\,dt+dW,\qquad X(0)=x\in H,\tag{3}$$

where H is a separable Hilbert space, $A:=D(A)\subset H\mapsto H$ is a strictly negative self-adjoint operator such that $Q:=(-2A)^{-1}$ is trace-class, μ is the Gaussian measure $\mathcal{N}(0,Q)$ on H with 0 mean and covariance operator Q, and W is a cylindrical white-noise on H. Moreover, $U\colon H\mapsto\mathbb{R}\cup\{+\infty\}$ is a convex, lower semicontinuous function, satisfying suitable integrability conditions with respect to μ and

$$\mu(x: U(x)<+\infty \text{ and } \partial U(x)\neq\varnothing)=1,$$

where $\partial U(x)$, the subdifferential of U at x, is defined as the subset of H:

$$\partial U(x):=\{y\in H: U(x+h)\geqslant U(x)+\langle h, y\rangle,\ \forall h\in H\}.$$

Equation (1) can be interpreted as an example of (3), setting $H:=L^2(0,1)$, $A:=(1/2)\,d^2/d\xi^2$ on H with Dirichlet boundary conditions,

$$U(x):=\begin{cases}\displaystyle\int_0^1 d\xi\int_0^{x(\xi)} f(\xi,s)\,ds & \text{if}\quad x\geqslant 0\\ +\infty & \text{otherwise}\end{cases}\tag{4}$$

and defining, for all non-negative $x\in H$, $\partial U(x)$ as the subset of the dual space M of $C_0(0,1)$, $M:=\{\text{signed measures on }(0,1)\}$,

$$\begin{aligned}\partial U(x)&:=\{m\in M: U(x+z)\geqslant U(x)+\langle z,m\rangle,\ \forall z\in C_0(0,1)\}\\ &= f(\xi,x(\xi))\,d\xi-\left\{m\in M: m\geqslant 0, \int_{(0,1)} x(\xi)\,m(d\xi)=0\right\}.\end{aligned}$$

Then (1) can be written formally as a differential inclusion of measures:

$$\left(\frac{\partial u}{\partial t} - Au - \frac{\partial^2 W}{\partial t\, \partial \xi}\right) d\xi \in \partial U(u(t, \cdot)).$$

However, in this case $\mu(U = +\infty) = 1$ (see Lemma 5 below) and the measure "$(\int_H e^{-2U}\, d\mu)^{-1}\, e^{-2U}\, d\mu$" is not well defined. Our result shows that a natural Gibbs-type measure for (1) is provided by (2). Notice that ν is not Gaussian and is even singular with respect to the reference Gaussian measure μ. Moreover, the support of ν is a closed convex set having empty interior both in the topologies of $L^2(0, 1)$ and $C([0, 1])$.

The proof we give relies on the construction of solutions to (1) given by Nualart and Pardoux in [6] and on a result of Biane on a connection between the law of the Brownian Bridge and the law of the 3-d Bessel Bridge (see [1] and Theorem 4 below).

2. DEFINITIONS

Our aim is to find a symmetrizing invariant measure for the process $x \mapsto u(t, \cdot)$, $t \geqslant 0$, where (u, η) satisfies:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \xi^2} - f(\xi, u(t, \xi)) + \dfrac{\partial^2 W}{\partial t\, \partial \xi} + \dfrac{\eta(dt, d\xi)}{dt\, d\xi} \\ u(0, \xi) = x(\xi), \qquad u(t, 0) = u(t, 1) = 0 \\ u + \alpha \geqslant 0, \qquad \displaystyle\int_{\mathcal{O}} (u + \alpha)\, d\eta = 0, \end{cases} \tag{5}$$

where $\alpha \geqslant 0$, $x\colon [0, 1] \mapsto [-\alpha, +\infty)$ is continuous and $x(0) = x(1) = 0$.

We introduce the following notations: $(t, \xi) \in \mathcal{O} := [0, +\infty) \times [0, 1]$, $H := L^2(0, 1)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$,

$$\langle h, k \rangle := \int_0^1 h(\xi)\, k(\xi)\, d\xi, \qquad \|h\|^2 := \langle h, h \rangle,$$

$$C_0(0, 1) := \{c\colon [0, 1] \mapsto \mathbb{R} \text{ continuous}, \ c(0) = c(1) = 0\},$$

$$A\colon D(A) \subset H \mapsto H, \qquad D(A) := H^2 \cap H_0^1(0, 1), \qquad A := \frac{1}{2}\frac{d^2}{d\xi^2}.$$

We set $K_\alpha := \{h \in H : h \geqslant -\alpha\}$ with $\alpha \geqslant 0$, and we denote by $\Pi_{K_\alpha}: H \mapsto K_\alpha$ the projection from H onto the closed convex set $K_\alpha \subset H$. Recall that Π_{K_α} is 1-Lipschitz continuous. If $D \subseteq H$, we denote by $C_b(D)$ the space of all $\varphi: D \mapsto \mathbb{R}$ being bounded and uniformly continuous with respect to $\|\cdot\|$. If $D \subseteq H$ and $\varphi \in C_b(D)$, we denote by ω_φ the modulus of continuity of φ:

$$\omega_\varphi: [0, \infty) \mapsto [0, 1], \qquad \omega_\varphi(r) := \sup\{|\varphi(x) - \varphi(x')| \wedge 1 : \|x - x'\| \leqslant r\}.$$

We identify $C_b(K_\alpha)$ with a subspace of $C_b(H)$ by means of the injection: $C_b(K_\alpha) \ni \varphi \mapsto \varphi \circ \Pi_{K_\alpha} \in C_b(H)$. If $0 \leqslant \alpha \leqslant \beta$, then $C_b(K_\alpha) \subseteq C_b(K_\beta)$.

If $\{m_n\}_n \cup \{m\}$ is a sequence of probability measures on $(H, \mathscr{B}(H))$, where $\mathscr{B}(H)$ is the Borel σ-field of H, we say that m_n converges weakly to m, if:

$$\lim_{n \to \infty} \int_H \varphi \, dm_n = \int_H \varphi \, dm, \qquad \forall \varphi \in C_b(H).$$

Given a Markov process $\{Y(t, x) : t \geqslant 0, x \in D\}$ on $D \subseteq H$, we say that a probability measure m on D is symmetrizing for Y, if, setting for all $\varphi \in C_b(D) : R_t^Y \varphi(x) := \mathbb{E}[\varphi(Y(t, x))]$, $x \in D$, we have:

$$\int_D \varphi \, R_t^Y \psi \, dm = \int_D \psi \, R_t^Y \varphi \, dm, \qquad \forall \varphi, \psi \in C_b(D).$$

A symmetrizing measure is in particular invariant; i.e.,

$$\int_D R_t^Y \varphi \, dm = \int_D \varphi \, dm, \qquad \forall \varphi \in C_b(D).$$

We denote by $1_D(\cdot)$ the characteristic function of a set D. We sometimes write $m(\varphi)$ for $\int_H \varphi \, dm$, $\varphi \in C_b(H)$.

By $W = \{W(t, \xi) : (t, \xi) \in \mathcal{O}\}$ we denote a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e., W is a Gaussian process with zero mean and covariance function

$$\mathbb{E}[W(t, \xi) \, W(t', \xi')] = (t \wedge t')(\xi \wedge \xi'), \qquad (t, \xi), (t', \xi') \in \mathcal{O}.$$

We denote by $\mathscr{F}_t$ the σ-field generated by the random variables $\{W(s, \xi) : (s, \xi) \in [0, t] \times [0, 1]\}$ and by $C_c^\infty(0, 1)$ the subset of $C_0(0, 1)$ of all C^∞ functions with support being compact in $(0, 1)$.

We always assume in the following that:

(H1) $f = f_1 + f_2$, $f_1, f_2: [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ are jointly measurable.

(H2) There exists $c > 0$ such that $|f_1(\xi, y) - f_1(\xi, y')| \leqslant c \, |y - y'|$ for all $\xi \in [0, 1]$, $y, y' \in \mathbb{R}$.

(H3) $f_2(\xi, \cdot)$ is non-decreasing and continuous, for all $\xi \in [0, 1]$.

(H4) There exist a constant $\lambda < \pi^2$, $a \in H$ and a polynomial p of degree $N \in \mathbb{N}$, such that for all $\xi \in [0, 1]$, $y \in \mathbb{R}$:

$$-a(\xi) - \lambda\, |y| \leqslant 2f(\xi, y) \leqslant a(\xi) + p(|y|).$$

Following [6], we set the following:

Definition 1. A pair (u, η) is said to be a solution of Eq. (5) with reflection in $-\alpha \leqslant 0$ and initial value $x \in K_\alpha \cap C_0(0, 1)$, if:

(i) $\{u(t, \xi) : (t, \xi) \in \mathcal{O}\}$ is a continuous and adapted process, i.e., $u(t, \xi)$ is $\mathcal{F}_t$-measurable for all $(t, \xi) \in \mathcal{O}$, and a.s. $u(\cdot, \cdot)$ is continuous on $\mathcal{O}$, $u(t, \cdot) \in K_\alpha \cap C_0(0, 1)$ for all $t \geqslant 0$, and $u(0, \cdot) = x$.

(ii) $\eta(dt, d\xi)$ is a random positive measure on $\mathcal{O}$ such that $\eta([0, T] \times [\delta, 1-\delta]) < +\infty$ for all $T, \delta > 0$, and η is adapted; i.e., $\eta(B)$ is $\mathcal{F}_t$-measurable for every Borel set $B \subset [0, t] \times [0, 1]$.

(iii) For all $t \geqslant 0$ and $\varphi \in C_c^\infty(0, 1)$, setting $u_s := u(s, \cdot)$,

$$\langle u_t, \varphi \rangle - \int_0^t \langle u_s, A\varphi \rangle\, ds + \int_0^t \langle f(\cdot, u_s), \varphi \rangle\, ds$$

$$= \langle x, \varphi \rangle + \int_0^t \int_0^1 \varphi(\xi)\, dW_{s,\xi} + \int_0^t \int_0^1 \varphi(\xi)\, \eta(ds, d\xi).$$

(iv) $\int_{\mathcal{O}} (u + \alpha)\, d\eta = 0$.

Finally, we will use the following:

Lemma 1. *Let T be a Polish metric space, and let $\{m_n\}_n \cup \{m\}$, respectively $\{\varphi_n\}_n$, be a sequence of probability measures, resp. of real-valued continuous functions, on T, satisfying*:

- *m_n converges weakly to m.*
- *The family $\{\varphi_n\}_n$ is uniformly bounded and equicontinuous on T.*
- *$\varphi_n(x)$ has a limit $\varphi(x)$ as $n \to \infty$, for all $x \in S$, with $S \subseteq T$ Borel and $m(S) = 1$.*

Then:

$$\lim_{n \to \infty} \int_T \varphi_n\, dm_n = \int_S \varphi\, dm.$$

Proof. We can suppose that $0 \leqslant \varphi_n \leqslant 1$ for all n. By Prokhorov's theorem, there exists for every $\delta > 0$ a compact set $Q_\delta \subset T$ such that eventually $m_n(Q_\delta) \geqslant 1 - \delta$. Let $\{\varphi_{n_k}\}_k$ be any subsequence of $\{\varphi_n\}_n$. On Q_δ we can apply the Ascoli–Arzelà theorem and obtain uniform convergence of a sub-subsequence $\{\varphi_{n_{k(l)}}\}_l$ to a continuous function f: $Q_\delta \mapsto \mathbb{R}$. Then:

$$\int_T \varphi_{n_{k(l)}} \, dm_{n_{k(l)}} - \int_T \varphi_{n_{k(l)}} \, dm \leqslant m_{n_{k(l)}}(T - Q_\delta) + \int_{Q_\delta} \varphi_{n_{k(l)}} [dm_{n_{k(l)}} - dm]$$

$$\leqslant \delta + 2\delta + \int_{Q_\delta} f \, [dm_{n_{k(l)}} - dm],$$

where for $l \geqslant l_0$, $\sup_{Q_\delta} |\varphi_{n_{k(l)}} - f| \leqslant \delta$. Since $m(T - S) = 0$ and Q_δ is closed,

$$\lim_{n \to \infty} \int_T \varphi_n \, dm = \int_S \varphi \, dm,$$

$$\limsup_{n \to \infty} \int_{Q_\delta} f \, dm_n \leqslant \int_{Q_\delta} f \, dm, \qquad \text{and therefore:}$$

$$\limsup_{l \to \infty} \int_T \varphi_{n_{k(l)}} \, dm_{n_{k(l)}} \leqslant \int_S \varphi \, dm.$$

Changing φ_n with $1 - \varphi_n$, we obtain the thesis. ∎

3. THE PROCESS X_α, $\alpha \geqslant 0$

In [6], the following theorem is proved:

THEOREM 1. *Assume that f satisfies* (H1), (H2), (H3), (H4) *and let $x \in K_\alpha \cap C_0(0, 1)$. Then there exists a unique solution (u, η) to Eq.* (5) *with reflection in $-\alpha$ and initial value x.*

We recall the strategy of the proof, given in [6], of the existence of solutions. First, the following approximating problem is introduced:

$$\begin{cases} \dfrac{\partial u_\alpha^\varepsilon}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u_\alpha^\varepsilon}{\partial \xi^2} - f(\cdot, u_\alpha^\varepsilon(t, \cdot)) + \dfrac{\partial^2 W}{\partial t \, \partial \xi} + \dfrac{(\alpha + u_\alpha^\varepsilon)^-}{\varepsilon} \\ u_\alpha^\varepsilon(0, \cdot) = x \in H, \qquad u_\alpha^\varepsilon(t, 0) = u_\alpha^\varepsilon(t, 1) = 0, \quad \forall t \geqslant 0. \end{cases} \tag{6}$$

with $\varepsilon > 0$, $(r)^- := \sup\{-r, 0\}$, and $\alpha \geqslant 0$. This is now a SPDE in $L^2(0,1)$ with additive noise and monotone or Lipschitz-continuous drift terms, for which existence and uniqueness of a solution are well known (see [3]). Then, if $x \in K_\alpha \cap C_0(0,1)$, the following is proved:

(a) $u_\alpha^\varepsilon(t,\cdot) \in C_0(0,1)$ for all $t \geqslant 0$, and u_α^ε is continuous on $\mathcal{O}$.

(b) The map $0 < \varepsilon \mapsto u_\alpha^\varepsilon(t,\xi)$ is non-decreasing for all $(t,\xi) \in \mathcal{O}$. The limit $\lim_{\varepsilon \downarrow 0} u_\alpha^\varepsilon(t,\xi) = \sup_{\varepsilon > 0} u_\alpha^\varepsilon(t,\xi) =: u_\alpha(t,\xi)$ is finite for all $(t,\xi) \in \mathcal{O}$, $u_\alpha(t,\cdot) \in K_\alpha \cap C_0(0,1)$ for all $t \geqslant 0$, and u_α is continuous on $\mathcal{O}$.

(c) The measure on $\mathcal{O}$, $\eta_\alpha^\varepsilon(dt, d\xi) := (1/\varepsilon)(\alpha + u_\alpha^\varepsilon)^- \, dt \, d\xi$, converges distributionally to a measure $\eta_\alpha(dt, d\xi)$ on $\mathcal{O}$.

(d) The pair (u_α, η_α) is the solution to (5) with reflection in $-\alpha$ and initial value $x \in K_\alpha \cap C_0(0,1)$.

We set for all $t \geqslant 0$, $\alpha \geqslant 0$, $\varepsilon > 0$:

- $X_\alpha(t,x) \in C_0(0,1)$, $X_\alpha(t,x)(\xi) := u_\alpha(t,\xi)$, $x \in K_\alpha \cap C_0(0,1)$,
- $X_\alpha^\varepsilon(t,x) \in H$, $X_\alpha^\varepsilon(t,x)(\xi) := u_\alpha^\varepsilon(t,\xi)$, $x \in H$.

LEMMA 2. *For all $\alpha \geqslant 0$, $\varepsilon > 0$, $t \geqslant 0$, we have*: $\forall x, x' \in C_0(0,1)$,

$$\|X_\alpha^\varepsilon(t,x) - X_\alpha^\varepsilon(t,x')\| \leqslant e^{ct} \, \|x - x'\|, \tag{7}$$

where $\|\cdot\|$ denotes the norm in H and $c > 0$ is the constant of (H2).

Proof. By the monotonicity properties of $(\cdot)^-$ and $f_2(\xi,\cdot)$ we have

$$\frac{1}{2}\frac{d}{dt} \|X_\alpha^\varepsilon(t,x) - X_\alpha^\varepsilon(t,x')\|^2 \leqslant c \, \|X_\alpha^\varepsilon(t,x) - X_\alpha^\varepsilon(t,x')\|^2$$

and the thesis follows from Gronwall's lemma. ∎

Therefore, the same estimate holds for X_α, $\alpha \geqslant 0 : \forall x, x' \in K_\alpha \cap C_0(0,1)$,

$$\|X_\alpha(t,x) - X_\alpha(t,x')\| \leqslant e^{ct} \, \|x - x'\| \tag{8}$$

and we can uniquely extend $X_\alpha^\varepsilon(t,\cdot)$, respectively $X_\alpha(t,\cdot)$, to maps from H to H, resp. from K_α to K_α, that we denote by the same symbols, satisfying (7) for all $x, x' \in H$, resp. (8) for all $x, x' \in K_\alpha$. We set for all $\alpha \geqslant 0$, $\varepsilon > 0$, $\varphi \in C_b(H)$, $t \geqslant 0$:

$$P_\alpha^\varepsilon(t)\,\varphi : H \mapsto \mathbb{R}, \qquad P_\alpha^\varepsilon(t)\,\varphi(x) := \mathbb{E}[\varphi(X_\alpha^\varepsilon(t,x))], \qquad x \in H, \tag{9}$$

$$P_\alpha(t)\,\varphi : K_\alpha \mapsto \mathbb{R}, \qquad P_\alpha(t)\,\varphi(x) := \mathbb{E}[\varphi(X_\alpha(t,x))], \qquad x \in K_\alpha. \tag{10}$$

LEMMA 3. *For all* $\alpha \geqslant 0$, $\varepsilon > 0$, $\varphi \in C_b(H)$, $t \geqslant 0$, *we have*:

$$P^\varepsilon_\alpha(t)\,\varphi \in C_b(H), \qquad \omega_{P^\varepsilon_\alpha(t)\,\varphi}(r) \leqslant \omega_\varphi(e^{ct} r) \qquad \forall r \geqslant 0, \tag{11}$$

$$P_\alpha(t)\,\varphi \in C_b(K_\alpha), \qquad \omega_{P_\alpha(t)\,\varphi}(r) \leqslant \omega_\varphi(e^{ct} r) \qquad \forall r \geqslant 0, \tag{12}$$

$$\lim_{\varepsilon \downarrow 0} P^\varepsilon_\alpha(t)\,\varphi(x) = P_\alpha(t)\,\varphi(x) \qquad \forall x \in K_\alpha, \tag{13}$$

$$P_\alpha(s)\,P_\alpha(t)\,\varphi(x) = P_\alpha(t+s)\,\varphi(x), \qquad \forall x \in K_\alpha. \tag{14}$$

In particular, $(P_\alpha(t))_{t \geqslant 0}$ *is a Markov semigroup acting on* $C_b(K_\alpha)$.

Proof. For (12), notice that, by (8), for all $x, x' \in K_\alpha$:

$$\begin{aligned} |P_\alpha(t)\,\varphi(x) - P_\alpha(t)\,\varphi(x')| &\leqslant \mathbb{E}[\,|\varphi(X_\alpha(t,x)) - \varphi(X_\alpha(t,x'))|\,] \\ &\leqslant \mathbb{E}[\,\omega_\varphi(\|X_\alpha(t,x) - X_\alpha(t,x')\|)] \\ &\leqslant \omega_\varphi(e^{ct}\,\|x - x'\|), \end{aligned}$$

and (11) follows analogously. Equation (13) is a consequence of (b) in the proof of Theorem 1 and (11). It is well known that $(P^\varepsilon_\alpha(t))_{t \geqslant 0}$ is a semigroup acting on $C_b(H)$: since the family of probability measures $\{m^\varepsilon\}_{\varepsilon > 0}$, where m^ε is the law of $X^\varepsilon_\alpha(s,x)$, and the family of functions $\{P^\varepsilon_\alpha(t)\,\varphi\}_{\varepsilon > 0}$ satisfy the Hypothesis of Lemma 1; Eq. (14) follows. ∎

LEMMA 4. *For all* $\varphi \in C_b(H)$, $\lim_{\alpha \downarrow 0} P_\alpha(t)\,\varphi(x) = P_0(t)\,\varphi(x)$, $t \geqslant 0$, $x \in K_0$.

Proof. If $x \in K_0 \cap C_0(0,1)$, then the map $0 < \alpha \mapsto X^\varepsilon_\alpha(t,x)(\xi)$ is non-decreasing for all $(t,\xi) \in \mathcal{O}$, $\varepsilon > 0$. Therefore,

$$\begin{aligned} \lim_{\alpha \downarrow 0} X_\alpha(t,x)(\xi) &= \sup_{\alpha > 0} X_\alpha(t,x)(\xi) = \sup_{\alpha > 0}\, \sup_{\varepsilon > 0} X^\varepsilon_\alpha(t,x)(\xi) \\ &= \sup_{\varepsilon > 0}\, \sup_{\alpha > 0} X^\varepsilon_\alpha(t,x)(\xi) = \sup_{\varepsilon > 0} X^\varepsilon_0(t,x)(\xi) \\ &= X_0(t,x)(\xi), \end{aligned}$$

since $\sup_{\alpha > 0} X^\varepsilon_\alpha(t,x)(\cdot) = X^\varepsilon_0(t,x)(\cdot)$ by the uniqueness of solutions to (6). The general case follows by (12) and a density argument. ∎

4. THE BROWNIAN BRIDGE RESTRICTED TO K_α AS SYMMETRIZING MEASURE FOR X_α, $\alpha > 0$

Recall that the Ornstein–Uhlenbeck process

$$Z(t,x) := e^{tA}x + \int_0^t e^{(t-s)A}\,dW_s \qquad t \geqslant 0, \qquad x \in H,$$

is, under our assumptions, a continuous Markov process with values in H, admitting as symmetrizing measure $\mu := \mathcal{N}(0, (-2A)^{-1})$, i.e., the Gaussian measure on H with 0 mean and covariance operator $(-2A)^{-1}$.

Our next lemma identifies μ with a well-known probability measure on $C_0(0, 1)$: the law of the Brownian bridge. Recall that the Brownian bridge is defined as a linear Brownian motion $(w_\tau)_{\tau \in [0,1]}$, conditioned to be 0 at $\tau = 1$, and can be realized as $[0, 1] \ni \tau \mapsto w_\tau - \tau w_1$. The law of the Brownian bridge is concentrated on $C_0(0, 1)$ and is the unique Gaussian measure on $\mathbb{R}^{[0,1]}$ with 0 mean and covariance function: $\Gamma(\tau, \sigma) = \tau \wedge \sigma - \tau\sigma$, $\sigma, \tau \in [0, 1]$, (see [7, Chap. I]).

Lemma 5. *The measure μ coincides with the law of the Brownian bridge.*

Proof. Recall that the measure μ is concentrated on $C([0, 1]) \subset H$. By definition of Gaussian measures, the following holds for all $h, k \in H$:

$$\int_H \langle x, h \rangle \langle x, k \rangle \; \mathcal{N}(0, (-2A)^{-1})(dx) = \langle (-2A)^{-1} h, k \rangle. \tag{15}$$

Since the operator $(-2A)^{-1}$ can be expressed as an integral operator with kernel: $\xi \wedge \sigma - \xi\sigma$, $\xi, \sigma \in [0, 1]$, then setting in (19) $h = \chi_{[0,t]}$, $k = \chi_{[0,s]}$, $s, t \in [0, 1]$, and differentiating with respect to t and s, we obtain:

$$\int_{C([0,1])} x(t)\, x(s)\, d\mu(x) = t \wedge s - ts. \quad \blacksquare$$

Lemma 5 allows us to calculate explicitly $\mu(K_\alpha)$:

$$\mu(K_\alpha) = 1 - \exp\{-2\alpha^2\}, \qquad \alpha \geqslant 0, \tag{16}$$

(see [7, Chap. III, Exercise (3.14)]). We introduce the functions

$$F: L^{N+1}(0, 1) \subset H \mapsto \mathbb{R}, \qquad F(x) := \int_0^1 d\xi \int_0^{x(\xi)} f(\xi, s)\, ds,$$

$$V_\alpha: H \mapsto [0, +\infty),$$

$$V_\alpha(x) := \tfrac{1}{2} \int_0^1 [(\alpha + x(\xi))^-]^2\, d\xi = \tfrac{1}{2} [d(x, K_\alpha)]^2,$$

where N is the degree of p in (H4) and $d(x, K_\alpha)$ denotes the distance in H of x from the closed convex set K_α, $\alpha \geqslant 0$. Notice that $\mu(L^{N+1}(0, 1)) = 1$, and by (H2), (H3), and (H4), $\exp(-F)$ is well defined and satisfies for all $x \in H$,

$$\exp(-2F(x)) \leqslant \exp\left(\langle a, x\rangle + \frac{\lambda}{2}\|x\|^2\right) \leqslant \exp(C\|a\|^2)\exp\left(\frac{\rho}{2}\|x\|^2\right) \tag{17}$$

for some $\lambda < \rho < \pi^2$, $C > 0$. Since $-2A \geqslant \pi^2 > \rho$, we have:

$$\int_H \exp(-2F(x))\,\mu(dx) \leqslant e^{C\|a\|^2}\int_H \exp\left(\frac{\rho}{2}\|x\|^2\right)\mathcal{N}(0, (-2A)^{-1})$$
$$= e^{C\|a\|^2}\frac{1}{\sqrt{\det(I + \rho(2A)^{-1})}} < \infty. \tag{18}$$

V_α is Fréchet differentiable on H with Lipschitz-continuous gradient $\nabla V_\alpha(x) = -(\alpha + x(\cdot))^-$, $x \in H$. F belongs to the Sobolev space $W^{1,p}(H, \mu)$ for all $p \in [1, \infty[$, with gradient $\nabla F(x) = f(\cdot, x(\cdot))$, $x \in L^{2N}(0, 1)$, and $\mu(L^{2N}(0, 1)) = 1$. Equation (6) can now be written in the following form:

$$\begin{cases} dX_\varepsilon^\alpha = \left(AX_\varepsilon^\alpha - \nabla F(X_\varepsilon^\alpha) - \dfrac{1}{\varepsilon}\nabla V_\alpha(X_\varepsilon^\alpha)\right)dt + dW \\ X_\varepsilon^\alpha(0, x) = x \in H \end{cases} \tag{19}$$

If $\varepsilon > 0$, (19) is a gradient system, namely (see [4, Sect. 8.6], and [5, Chap. 2]):

PROPOSITION 1. *If $\varepsilon > 0$, then $\forall \alpha \geqslant 0$, setting*

$$Z_{\alpha,\varepsilon} := \int_H e^{-2F-(2V_\alpha/\varepsilon)}\,d\mu > 0,$$

the probability measure on H,

$$\nu_{\alpha,\varepsilon}^F(dx) := \frac{1}{Z_{\alpha,\varepsilon}}\exp\left\{-2F(x) - \frac{2}{\varepsilon}V_\alpha(x)\right\}\mu(dx),$$

is symmetrizing for the process $\{X_\alpha^\varepsilon(t, x) : t \geqslant 0, x \in H\}$.

By (16), if $\alpha > 0$ then $\mu(K_\alpha) > 0$, and we can introduce the probability measures ν_α and ν_α^F on K_α, with $Z_\alpha := \int_{K_\alpha} e^{-2F}\,d\mu > 0$:

$$d\nu_\alpha := \frac{1}{\mu(K_\alpha)}1_{K_\alpha}\,d\mu, \qquad d\nu_\alpha^F := \frac{1}{Z_\alpha}1_{K_\alpha}e^{-2F}\,d\mu. \tag{20}$$

THEOREM 2. *If $\alpha > 0$, ν_α^F is a symmetrizing measure for the processes $\{X_\alpha(t, x) : t \geqslant 0, x \in K_\alpha\}$ and $\{X_\alpha(t, x) : t \geqslant 0, x \in K_\alpha \cap C_0(0, 1)\}$.*

Proof. First, we have:

$$\frac{1}{Z_\alpha}\, 1_{K_\alpha}(x)\, e^{-2F(x)} = \lim_{\varepsilon \downarrow 0} \frac{1}{Z_{\alpha,\varepsilon}} \exp\left\{-2F(x) - \frac{2}{\varepsilon} V_\alpha(x)\right\} \quad \forall\, x \in H,$$

$$\frac{1}{Z_{\alpha,\varepsilon}} \exp\left\{-2F - \frac{2}{\varepsilon} V_\alpha\right\} \leqslant \frac{1}{Z_{\alpha,1}} \exp\left\{\langle a, x\rangle + \frac{\lambda}{2}\|x\|^2\right\} \qquad \forall\, \varepsilon \in \,]0, 1].$$

Then, by (13), (17), and (18) and by the dominated convergence theorem, we obtain:

$$\nu_\alpha^F(\psi\, P_\alpha(t)\, \varphi) = \lim_{\varepsilon \downarrow 0} \nu_{\alpha,\varepsilon}^F(\psi\, P_\alpha^\varepsilon(t)\, \varphi) = \lim_{\varepsilon \downarrow 0} \nu_{\alpha,\varepsilon}^F(\varphi\, P_\alpha^\varepsilon(t)\, \psi) = \nu_\alpha^F(\varphi\, P_\alpha(t)\, \psi)$$

for all $\varphi, \psi \in C_b(H)$. Therefore, ν_α^F is symmetrizing measure for $\{X_\alpha(t, x) : t \geqslant 0,\ x \in K_\alpha\}$. Finally, $\nu_\alpha^F(K_\alpha \cap C_0(0,1)) = 1$ and, by Theorem 1, the set $K_\alpha \cap C_0(0,1)$ is invariant for $\{X_\alpha(t,\cdot) : t \geqslant 0\}$, i.e., $x \in K_\alpha \cap C_0(0,1)$ implies $X_\alpha(t,x) \in K_\alpha \cap C_0(0,1)$ for all $t \geqslant 0$, a.s. ∎

5. CONVERGENCE OF ν_α TO ν

Let $(B_\tau)_{\tau \in [0,1]}$ be a 3-dimensional Brownian motion. We denote by ν the law of the 3-d Bessel Bridge, namely of the modulus of B, conditioned to be equal to 0 at $\tau = 1$. The probability measure ν is concentrated on $K_0 \cap C_0(0,1)$. In this section we prove the following:

THEOREM 3. *The measures* $d\nu_\alpha = (1/\mu(K_\alpha))\, 1_{K_\alpha}\, d\mu$, $\alpha > 0$, *converge weakly as* $\alpha \downarrow 0$ *to the law* ν *of the* 3-*d Bessel bridge.*

We recall the following result from [1]:

THEOREM 4. *Let* $(e_\tau)_{\tau \in [0,1]}$ *be a* 3-*d Bessel bridge, and let* ζ *be a random variable with uniform distribution on* $[0,1]$ *and independent of* e. *Then the process*:

$$(\beta_\tau)_{\tau \in [0,1]}, \qquad \beta_\tau := e_{\tau \oplus \zeta} - e_\zeta,$$

where $\oplus$ *denotes the sum* mod 1, *is a Brownian bridge.*

Theorem 3 was proved in [8]. We give here a proof, based on Theorem 4, which seems to be promising for further developments.

We set $e_\tau \colon C_0(0,1) \mapsto \mathbb{R}$, $e_\tau(x) := x(\tau)$, $\tau \in [0,1]$. Then $(e_\tau)_{\tau \in [0,1]}$ is a 3-d Bessel bridge under ν. By Theorem 4 and (16), we have for $\alpha > 0$, $\varphi \in C_b(H)$,

$$\nu_\alpha(\varphi) = \frac{1}{1 - \exp\{-2\alpha^2\}} \int_0^1 \nu(\varphi(e_{(\cdot \oplus r)} - e_r)\, 1_{(e_r \leqslant \alpha)})\, dr, \tag{21}$$

since $\{e_{(\cdot \oplus r)} - e_r \geqslant -\alpha\} = \{e_r \leqslant \alpha\}$.

LEMMA 6. *For all $r \in [0, 1]$, there exists a regular conditional distribution $\{\nu(\cdot \mid e_r = y) : y \geqslant 0\}$ of ν given e_r, such that, setting*

$$\delta_\varphi(r, y) := \nu(\varphi(e_{(\cdot \oplus r)} - e_r) \mid e_r = y), \qquad \varphi \in C_b(H), \quad r \in]0, 1[, \quad y \geqslant 0,$$

we have for all $y \geqslant 0$:

$$\lim_{r \downarrow 0} \delta_\varphi(r, \sqrt{r(1-r)}\, y) = \lim_{r \uparrow 1} \delta_\varphi(r, \sqrt{r(1-r)}\, y) = \nu(\varphi).$$

Proof. Let $(B_\tau)_{\tau \in [0, \infty)}$ and $(\hat{B}_\tau)_{\tau \in [0, \infty)}$ be two independent 3-d Brownian motions and $r \in]0, 1[$. Denoting by $|\cdot|$ the euclidean norm in $\mathbb{R}^3$, we set $b := |B|$, $\hat{b} := |\hat{B}|$,

$$\beta(z)(\tau) := B_\tau - \tau B_1 + \tau z, \qquad \hat{\beta}(z)(\tau) := \hat{B}_\tau - \tau \hat{B}_1 + \tau z, \quad \tau \in [0, 1], \quad z \in \mathbb{R}^3,$$

$$\pi_r, \hat{\pi}_r : L^2(0, \infty) \times L^2(0, \infty) \mapsto L^2(0, 1),$$

$$\pi_r(c, d)(\tau) := 1_{[0, r]}(\tau)\, c(\tau) + 1_{]r, 1]}(\tau)\, d(1 - \tau),$$

$$\hat{\pi}_r(c, d)(\tau) := 1_{[0, 1-r]}(\tau)\, d(1 - r - \tau) + 1_{]1-r, 1]}(\tau)\, c(\tau + r - 1). \tag{22}$$

For all $\varphi \in C_b(H)$, we set

$$\nu(\varphi(e) \mid e_r = y) := \mathbb{E}[\varphi(\pi_r(b, \hat{b})) \mid b_r = y = \hat{b}_{1-r}], \qquad y \geqslant 0, \tag{23}$$

$$\mathbb{E}[\varphi(b) \mid b(1) = y] := \int_{S^2} \sigma(dn)\, \mathbb{E}[\varphi(|\beta(yn)|)], \qquad y \geqslant 0, \tag{24}$$

where S^2 is the unitary sphere in $\mathbb{R}^3$ and $\sigma(dn)$ is the normalized uniform distribution on S^2. Then (23), respectively (24), is a regular conditional distribution of ν given e_r, resp. of $\mathbb{P}(b \in \cdot)$ given $b(1)$. In particular, the law of $|\beta(0)|$ is equal to ν. By (22) and (23) we have:

$$\nu(\varphi(e_{(\cdot \oplus r)} - e_r) \mid e_r = y) = \mathbb{E}[\varphi(\hat{\pi}_r(b, \hat{b}) - y) \mid b_r = y = \hat{b}_{1-r}]. \tag{25}$$

Identifying $h \in L^2(0, 1)$ with $h 1_{[0, 1]} \in L^2(0, \infty)$, we set $\varphi_r : H \times H \mapsto \mathbb{R}$,

$$\varphi_r(h, k) := \varphi(\hat{\pi}_r(\sqrt{r}\, h(\,\cdot\,/r), \sqrt{1-r}\, k(\cdot/(1-r))) - \sqrt{r(1-r)}\, y).$$

Since for $\gamma > 0$, $\sqrt{\gamma}\, B_{(\cdot/\gamma)}$ is still a 3-d Brownian motion, we obtain by (25):

$$\begin{aligned}
&\delta_\varphi(r, \sqrt{r(1-r)}\, y) \\
&\quad = \nu(\varphi(e_{(\cdot \oplus r)} - e_r) \mid e_r = \sqrt{r(1-r)}\, y) \\
&\quad = \mathbb{E}[\varphi(\hat{\pi}_r(b, \hat{b}) - \sqrt{r(1-r)}\, y) \mid b_r = \sqrt{r(1-r)}\, y = \hat{b}_{1-r}] \\
&\quad = \mathbb{E}[\varphi_r(b, \hat{b}) \mid b_1 = \sqrt{1-r}\, y, \hat{b}_1 = \sqrt{r}\, y].
\end{aligned} \tag{26}$$

Since for all $n \in S^2$ and $y \geqslant 0$,

$$\lim_{r \downarrow 0} \varphi_r(|\beta(\sqrt{1-r}\; yn)|, |\hat{\beta}(\sqrt{r}\; yn)|) = \varphi(|\hat{\beta}(0)(1-\cdot)|),$$

$$\lim_{r \uparrow 1} \varphi_r(|\beta(\sqrt{1-r}\; yn)|, |\hat{\beta}(\sqrt{r}\; yn)|) = \varphi(|\beta(0)|),$$

and since ν is invariant by the time-change $\tau \mapsto 1-\tau$, the thesis follows by (24)–(26) and by the dominated convergence theorem. ▮

Proof of Theorem 3. We split the integral on $[0, 1]$ in (21) into two integrals on $[0, 1/2]$ and $[1/2, 1]$, respectively. Conditioning with respect to e_r and setting $c_\alpha := (1-\exp\{-2\alpha^2\})$, we obtain

$$\begin{aligned}
&\frac{1}{c_\alpha}\int_0^{1/2} dr\; \nu(\varphi(e_{(\cdot \oplus r)} - e_r)\, 1_{(e_r \leqslant \alpha)}) \\
&\quad = \frac{1}{c_\alpha}\int_0^{1/2} dr \int_0^{\alpha} dy \sqrt{\frac{2}{\pi[r(1-r)]^3}}\; y^2 \exp\left\{-\frac{y^2}{2r(1-r)}\right\} \delta_\varphi(r, y) \\
&\quad = \frac{1}{c_\alpha}\int_0^{1/2} dr \int_0^{\alpha/\sqrt{r(1-r)}} dy \sqrt{\frac{2}{\pi}}\; y^2 \exp\left\{-\frac{y^2}{2}\right\} \delta_\varphi(r, \sqrt{r(1-r)}\; y) \\
&\quad = \frac{1}{c_\alpha}\sqrt{\frac{2}{\pi}} \int_0^{2\alpha} dy \exp\left\{-\frac{y^2}{2}\right\} y^2 \int_0^{1/2} dr\; \delta_\varphi(r, \sqrt{r(1-r)}\; y) \\
&\qquad + \frac{\alpha^2}{c_\alpha}\sqrt{\frac{2}{\pi}} \int_{2\alpha}^{+\infty} dy \exp\left\{-\frac{y^2}{2}\right\}\left(\frac{y}{\alpha}\right)^2 \int_0^{\rho(\alpha, y)} dr\; \delta_\varphi(r, \sqrt{r(1-r)}\; y) \\
&\quad =: I_1(\alpha) + I_2(\alpha), \qquad \rho(\alpha, y) := \frac{1}{2}\left(1-\sqrt{1-\left(\frac{2\alpha}{y}\right)^2}\right) \sim \left(\frac{\alpha}{y}\right)^2
\end{aligned}$$

as $\alpha \downarrow 0$, $y > 0$. It is easy to see that $\lim_{\alpha \downarrow 0} I_1(\alpha) = 0$, while $I_2(\alpha)$ tends to $(1/2)\, \nu(\varphi)$ by Lemma 6 and the dominated convergence theorem. Since analogous computations hold for the integral on $[1/2, 1]$, we obtain that $\nu_\alpha(\varphi)$ converges to $\nu(\varphi)$ and Theorem 3 is proved. ▮

6. 3-*D* BESSEL BRIDGE AS SYMMETRIZING MEASURE FOR X_0

In this section we prove that the probability measure on K_0:

$$d\nu^F := \frac{1}{\int_{K_0} \exp\{-2F\}\, d\nu} \exp\{-2F\}\, d\nu, \tag{27}$$

is well defined, that ν_α^F, defined in (20), converges weakly to ν^F as $\alpha \downarrow 0$, and that X_0 is symmetric with respect to ν^F. The difficulty is that $\exp(-2F)$ is not bounded, so that we cannot apply directly Theorem 3.

LEMMA 7. *$\nu(e^{-2F}) \in]0, \infty[$ and ν_α^F converges weakly to ν^F as $\alpha \downarrow 0$.*

Proof. We retain the notations of the proof of Lemma 6. Moreover, we set for all $\varphi \in C_b(H)$, $\varphi^F := \varphi e^{-2F}$. By Lemma 5 the law of $\beta(0)$ on $H \times H \times H$ is $\mu \otimes \mu \otimes \mu = \mathcal{N}(0, Q)$, $Q := (-2A)^{-1} \oplus (-2A)^{-1} \oplus (-2A)^{-1}$. Since $-2A \geqslant \pi^2 > \lambda$, we have by (17), (H4) and by $\mu(L^{N+1}(0,1)) = 1$:

$$\int \exp\{-2F\}\, d\nu = \mathbb{E}[\exp\{-2F(|\beta(0)|)\}]$$

$$= \int_{H^3} \exp\{-2F(|z|)\}\, \mathcal{N}(0, Q)(dz) \in]0, \infty[.$$

Notice that

$$\exp\left\{\frac{\lambda}{2}\|\beta(yn)\|^2\right\} \leqslant e^{Cy^2} \exp\left\{\frac{\rho}{2}\|\beta(0)\|^2\right\} \tag{28}$$

for some $\lambda < \rho < \pi^2$, $C > 0$, and:

$$\mathbb{E}\left[\exp\left\{\frac{\rho}{2}\|\beta(0)\|^2\right\}\right] = \frac{1}{\sqrt{\det(I - \rho Q)}} < \infty. \tag{29}$$

By (23), (24), (28), and (29), $\nu(e^{-2F}(e) \mid e(r) = y) < \infty$, and therefore $\delta_{(\varphi^F)}(r, y)$ is well defined for all $r \in]0, 1[$ and $y \geqslant 0$. Arguing as in the proof of Lemma 6, by the dominated convergence theorem we have for all $\varphi \in C_b(H)$ and $y \geqslant 0$,

$$\lim_{r \downarrow 0} \delta_{(\varphi^F)}(r, \sqrt{r(1-r)}\, y) = \lim_{r \uparrow 1} \delta_{(\varphi^F)}(r, \sqrt{r(1-r)}\, y) = \nu(\varphi^F), \tag{30}$$

and the thesis follows proceeding as in the proof of Theorem 3. ∎

THEOREM 5. *ν^F is a symmetrizing measure for the processes $\{X_0(t, x) : t \geqslant 0, x \in K_0\}$ and $\{X_0(t, x) : t \geqslant 0, x \in K_0 \cap C_0(0, 1)\}$.*

Proof. Arguing as in the proof of Theorem 2, the thesis follows from Lemmas 1, 3, 4, and 7, Theorems 1 and 2, and from $\nu^F(K_0 \cap C_0(0, 1)) = \nu(K_0 \cap C_0(0, 1)) = 1$. ∎

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