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RANDOM DETERMINANTS, MIXED VOLUMES OF ELLIPSOIDS, AND ZEROS OF GAUSSIAN RANDOM FIELDS

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Consider a $d \times d$ matrix M whose rows are independent, centered, nondegenerate Gaussian vectors $\xi_1, \dots, \xi_d$ with covariance matrices $\Sigma_1, \dots, \Sigma_d$. Denote by $\mathcal{E}_i$ the dispersion ellipsoid of $\xi_i : \mathcal{E}_i = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \Sigma_i^{-1} \mathbf{x} \le 1\}$. We show that

$$\mathbb{E}\,|\det M| = \frac{d!}{(2\pi)^{d/2}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_d),$$

where $V_d(\cdot, \dots, \cdot)$ denotes the mixed volume. We also generalize this result to the case of rectangular matrices. As a direct corollary, we get an analytic expression for the mixed volume of d arbitrary ellipsoids in $\mathbb{R}^d$.

As another application, we consider a smooth, centered, nondegenerate Gaussian random field $X = (X_1, \dots, X_k)^\top : \mathbb{R}^d \to \mathbb{R}^k$. Using the Kac–Rice formula, we obtain a geometric interpretation of the intensity of zeros of X in terms of the mixed volume of dispersion ellipsoids of the gradients of $X_i/\sqrt{\mathbf{Var} X_i}$. This relates zero sets of equations to mixed volumes in a way which is reminiscent of the well-known Bernstein theorem about the number of solutions of a typical system of algebraic equations. Bibliography: 10 titles.

1. Main results

1.1. Random determinant and mixed volume of ellipsoids. Consider independent, centered, nondegenerate Gaussian random vectors $\xi_1, \dots, \xi_k \in \mathbb{R}^d$, $k \le d$, with covariance matrices $\Sigma_1, \dots, \Sigma_k$. Denote by $\mathcal{E}_i$ the dispersion ellipsoid of ξ_i:

$$\mathcal{E}_i = \left\{\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d : \mathbf{x}^\top \Sigma_i^{-1} \mathbf{x} \le 1\right\}, \quad i = 1, \dots, k. \tag{1.1}$$

Denote by M the $k \times d$ matrix whose rows are $\xi_1, \dots, \xi_k$.

Theorem 1.1. *The following relation holds:*

$$\mathbf{E}\sqrt{\det(MM^\top)} = \frac{(d)_k}{(2\pi)^{k/2}\kappa_{d-k}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_k, B, \dots, B), \tag{1.2}$$

where $V_d(\cdot, \dots, \cdot)$ denotes the mixed volume of d convex bodies in $\mathbb{R}^d$ (see Sec. 2 for details), B is the unit ball in $\mathbb{R}^d$, $(d)_k = d(d-1)\cdots(d-k+1)$ is the Pochhammer symbol, and $\kappa_n = \pi^{n/2}/\Gamma(1+n/2)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

The left-hand side of (1.2) can be interpreted as the average k-dimensional volume of a Gaussian random parallelotope.

Corollary 1.2. *In the case $k = d$, the following relation holds:*

$$\mathbf{E}\,|\det M| = \frac{d!}{(2\pi)^{d/2}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_d).$$

As another direct corollary, we can calculate the mixed volume of d arbitrary ellipsoids in $\mathbb{R}^d$.

Corollary 1.3. *If $\mathcal{E}_1, \dots, \mathcal{E}_d$ are arbitrary ellipsoids defined by symmetric positive definite matrices $\Sigma_1, \dots, \Sigma_d$ as in* (1.1)*, then*

$$V_d(\mathcal{E}_1, \dots, \mathcal{E}_d) = \frac{1}{d!}\prod_{i=1}^{d} (\det \Sigma_i)^{-1/2} \int\limits_{\mathbb{R}^{d^2}} |\det(x_{ij})| \prod_{i=1}^{d} \exp\left(-\frac{1}{2}\mathbf{x}_i^\top \Sigma_i^{-1} \mathbf{x}_i\right) dx_{11} \dots dx_{dd},$$

where

$$\mathbf{x}_i = (x_{i1}, \dots, x_{id})^\top.$$

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The only estimate of the mixed volume of ellipsoids which we know is due to Barvinok [2]. He showed that

$$\frac{\kappa_d}{3^{(d-1)/2}}\sqrt{D_d(\Sigma_1,\dots,\Sigma_d)} \le V_d(\mathcal{E}_1,\dots,\mathcal{E}_d) \le \kappa_d\sqrt{D_d(\Sigma_1,\dots,\Sigma_d)},$$

where $D_d(\cdot,\dots,\cdot)$ denotes the mixed discriminant of d symmetric $d\times d$ matrices:

$$D_d(A_1,\dots,A_d) = \frac{1}{d!}\frac{\partial^d}{\partial\lambda_1\dots\partial\lambda_d}\det(\lambda_1A_1+\dots+\lambda_dA_d)\Big|_{\lambda_1=\dots=\lambda_d=0}.$$

If $\xi_1,\dots,\xi_k$ are independent, standard Gaussian vectors, then $MM^\top$ is a Wishart matrix, and (1.2) turns into (see [5, 10])

$$\mathbf{E}\sqrt{\det(MM^\top)} = \frac{(d)_k\kappa_d}{(2\pi)^{k/2}\kappa_{d-k}}.$$

1.2. Intrinsic volumes. If $\xi_1,\xi_2,\dots,\xi_k\in\mathbb{R}^d, k\le d$, are identically distributed with the common covariance matrix Σ and dispersion ellipsoid $\mathcal{E}$, then (1.2) turns into

$$\mathbf{E}\sqrt{\det(MM^\top)} = \frac{k!}{(2\pi)^{k/2}}V_k(\mathcal{E}), \tag{1.3}$$

where $V_k(\cdot)$ denotes the kth *intrinsic volume* of a convex body in $\mathbb{R}^d$:

$$V_k(K) = \frac{\binom{d}{k}}{\kappa_{d-k}}V_d(\underbrace{K,\dots,K}_{k\text{ times}},B,\dots,B).$$

The normalization is chosen so that $V_k(K)$ depends only on K and not on the dimension of the surrounding space, i.e., if $\dim K<d$, then the computation of $V_k(K)$ in $\mathbb{R}^d$ leads to the same result as the computation in the affine span of K. In particular, if $\dim K=k$, then $V_k(K)=\mathrm{Vol}_k(K)$, the k-dimensional volume of K.

It is known that $V_1(K)$ is proportional to the mean width of K:

$$V_1(K) = \frac{d\kappa_d}{2\kappa_{d-1}}w(K).$$

Taking $k=1$ in (1.3), we see that for any centered Gaussian vector ξ with dispersion ellipsoid $\mathcal{E}$,

$$\mathbf{E}\|\xi\| = \frac{1}{\sqrt{2\pi}}V_1(\mathcal{E}). \tag{1.4}$$

It was pointed out by M. Lifshits that (1.4) is a special case of the following remarkable result of Sudakov.

1.3. Connection with Sudakov's result. For our purposes, the following finite-dimensional version of Sudakov's theorem suffices. The result in full generality can be found in [9, Proposition 14].

Proposition 1.4. *For an arbitrary subset $A\subset\mathbb{R}^d$,*

$$\mathbf{E}\sup_{\mathbf{x}\in A}\langle\mathbf{x},\eta\rangle = \frac{1}{\sqrt{2\pi}}V_1(\mathrm{conv}(A)), \tag{1.5}$$

where η is a standard Gaussian vector in $\mathbb{R}^d$ and $\mathrm{conv}(A)$ *is the convex hull of A.*

Let us deduce (1.4) from (1.5). Consider a matrix U such that $\Sigma=U^{-1}(U^{-1})^\top$ and $U\xi$ is a standard Gaussian vector. Using (1.5) with $A=\mathcal{E}$ and $\eta=U\xi$, we get

$$\mathbf{E}\|\xi\| = \mathbf{E}\sup_{\|\mathbf{x}\|\le1}\langle\mathbf{x},\xi\rangle = \mathbf{E}\sup_{\|\mathbf{x}\|\le1}\langle(U^{-1})^\top\mathbf{x},U\xi\rangle = \mathbf{E}\sup_{\|U^\top\mathbf{x}\|\le1}\langle\mathbf{x},U\xi\rangle = \mathbf{E}\sup_{\mathbf{x}\in\mathcal{E}}\langle\mathbf{x},U\xi\rangle = \frac{1}{\sqrt{2\pi}}V_1(\mathcal{E}).$$

1.4. Zeros of Gaussian random fields. Let $X(\mathbf{t}) = (X_1(\mathbf{t}), \dots, X_k(\mathbf{t}))^\top : \mathbb{R}^d \to \mathbb{R}^k$, $k \le d$, be a random field. Following Azaïs and Wschebor [1], we always assume that the following conditions hold:

(a) X is Gaussian;
(b) almost surely, the function $X(\cdot)$ is of class $\mathcal{C}^1$;
(c) for all $\mathbf{t} \in \mathbb{R}^d, X(\mathbf{t})$ has a nondegenerate distribution;
(d) almost surely, if $X(\mathbf{t}) = 0$, then $X'(\mathbf{t})$, the Jacobian matrix of $X(\mathbf{t})$, has the full rank.

Then, almost surely, the level set $X^{-1}(0)$ is a $\mathcal{C}^1$-manifold of dimension $d-k$, and for any Borel set F, the Lebesgue measure $\mathrm{Vol}_{d-k}(X^{-1}(0) \cap F)$ is well-defined ($\mathrm{Vol}_0(\cdot)$ denotes the counting measure).

It was shown in [1, p. 177] that

$$\mathbf{E}\,\mathrm{Vol}_{d-k}(X^{-1}(0) \cap F) = \int\limits_F \mathbf{E}\left(\sqrt{\det\left(X'(\mathbf{t})X'(\mathbf{t})^\top\right)} \,\middle|\, X(\mathbf{t}) = 0\right) p_{X(\mathbf{t})}(0)\, d\mathbf{t}, \tag{1.6}$$

where $p_{X(\mathbf{t})}(\cdot)$ is the density of $X(\mathbf{t})$. Thus, the integrand in (1.6) can be interpreted as the intensity of zeros of X.

In this paper, we consider the special case where X is centered and its coordinates $X_1, \dots, X_k$ are independent. Denote by $\mathcal{E}_i(\mathbf{t})$ the dispersion ellipsoid of $\nabla[X_i(\mathbf{t})/\sqrt{\mathrm{Var}\, X_i(\mathbf{t})}]$.

Theorem 1.5. *Let X be a centered random field with independent coordinates defined as above and satisfying conditions* (a)–(d). *Then*

$$\mathbf{E}\,\mathrm{Vol}_{d-k}(X^{-1}(0) \cap F) = \frac{(d)_k}{(2\pi)^k \kappa_{d-k}} \int\limits_F V_d(\mathcal{E}_1(\mathbf{t}), \dots, \mathcal{E}_k(\mathbf{t}), B, \dots, B)\, d\mathbf{t}. \tag{1.7}$$

Formula (1.7) relates zero sets of random equations to mixed volumes. In the case $k = d$, it is therefore reminiscent of the well-known fact from the algebraic geometry which we formulate in the next subsection.

1.5. Bernstein's theorem. Consider a complex polynomial in d variables,

$$f(z_1, \dots, z_d) = \sum c_{j_1,\dots,j_d} z_1^{j_1} \dots z_d^{j_d}.$$

The Newton polytope of f is the subset of $\mathbb{R}^d$ defined as follows:

$$\mathrm{Nw}(f) = \mathrm{conv}\left\{(j_1, \dots, j_d) \in \mathbb{Z}^d \,:\, c_{j_1,\dots,j_d} \ne 0\right\}.$$

Let $K_1, \dots, K_d$ be compact convex polytopes in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. Consider a system of algebraic equations

$$\begin{cases} f_1(z_1, \dots, z_d) = 0, \\ \qquad \dots \\ f_d(z_1, \dots, z_d) = 0, \end{cases}$$

such that $\mathrm{Nw}(f_i) = K_i$. Bernstein showed [3] that for almost all such systems (with respect to Lebesgue measure in the space of coefficients of the polynomials), the number of nonzero solutions is equal to

$$\mathrm{Vol}_0(f_1^{-1}(0) \cap \dots \cap f_d^{-1}(0) \setminus \{\mathbf{0}\}) = d!\,V_d(K_1, \dots, K_d).$$

2. Some essential tools from geometry

For the basic facts from integral and convex geometry we refer the reader to [4] and [8].

2.1. Mixed volumes. Consider arbitrary convex bodies $K_1, \dots, K_d \subset \mathbb{R}^d$. Minkowski showed [7] that $\mathrm{Vol}_d(\lambda_1 K_1 + \dots + \lambda_d K_d)$, where $\lambda_1, \dots, \lambda_d \ge 0$, is a homogeneous polynomial of degree d with nonnegative coefficients:

$$\mathrm{Vol}_d(\lambda_1 K_1 + \dots + \lambda_d K_d) = \sum_{i_1=1}^{d} \cdots \sum_{i_d=1}^{d} \lambda_{i_1} \dots \lambda_{i_d} V_d(K_{i_1}, \dots, K_{i_d}). \tag{2.1}$$

The coefficients $V_d(K_{i_1}, \dots, K_{i_d})$ are uniquely determined by the assumption that they are symmetric with respect to permutations of $K_{i_1}, \dots, K_{i_d}$. The coefficient $V_d(K_1, \dots, K_d)$ is called the mixed volume of $K_1, \dots, K_d$. Differentiating (2.1), we get an alternative definition of the mixed volume:

$$V_d(K_1, \dots, K_d) = \frac{1}{d!} \frac{\partial^d}{\partial \lambda_1 \dots \partial \lambda_d} \mathrm{Vol}_d(\lambda_1 K_1 + \dots + \lambda_d K_d)\big|_{\lambda_1 = \dots = \lambda_d = 0}.$$

For any affine transformation L,

$$V_d(LK_1,\dots,LK_d) = |\det L| \cdot V_d(K_1,\dots,K_d). \tag{2.2}$$

The following relation can also be stated:

$$\int\limits_{\mathbb{S}^{d-1}} V_{d-1}(P_{\mathbf{u}}K_1,\dots,P_{\mathbf{u}}K_{d-1})\,d\mathbf{u} = \frac{\kappa_{d-1}}{\kappa_d} V_d(K_1,\dots,K_{d-1},B), \tag{2.3}$$

where $d\mathbf{u}$ is the surface measure on $\mathbb{S}^{d-1}$ normalized to have total mass 1 and $P_{\mathbf{u}}$ denotes the orthogonal projection to the linear hyperplane $\mathbf{u}^{\perp}$.

2.2. Volumes of parallelotopes. For any $A \subset \mathbb{R}^d$ and $\mathbf{x}_1,\dots,\mathbf{x}_k \in \mathbb{R}^d$ denote by $P_{\mathbf{x}_1,\dots,\mathbf{x}_k}A$ the orthogonal projection of A to $\operatorname{span}^{\perp}\{\mathbf{x}_1,\dots,\mathbf{x}_k\}$ (the orthogonal complement of the linear span of $\mathbf{x}_1,\dots,\mathbf{x}_k$). Denote by $H_{\mathbf{x}_1,\dots,\mathbf{x}_k}$ the parallelotope generated by the vectors $\mathbf{x}_1,\dots,\mathbf{x}_k$. It is known that

$$\operatorname{Vol}_k(H_{\mathbf{x}_1,\dots,\mathbf{x}_k}) = \sqrt{\det(AA^{\top})}, \tag{2.4}$$

where A is the matrix whose rows are $\mathbf{x}_1,\dots,\mathbf{x}_k$.

For any $\mathbf{x}_1,\dots,\mathbf{x}_d \in \mathbb{R}^d$ and $k = 1,\dots,d-1$,

$$\operatorname{Vol}_d(H_{\mathbf{x}_1,\dots,\mathbf{x}_d}) = \operatorname{Vol}_k(H_{\mathbf{x}_1,\dots,\mathbf{x}_k}) \operatorname{Vol}_{d-k}(P_{\mathbf{x}_1,\dots,\mathbf{x}_k} H_{\mathbf{x}_{k+1},\dots,\mathbf{x}_d}). \tag{2.5}$$

2.3. Ellipsoids. There is a bijection $A \mapsto \mathcal{E}$ between $d \times d$ symmetric positive definite matrices and d-dimensional nondegenerate ellipsoids centered at the origin (see [6] for details):

$$\mathcal{E} = \left\{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{x}^{\top} A^{-1} \mathbf{x} \le 1\right\}.$$

Any nondegenerate linear coordinate transformation of the form $\mathbf{x} \mapsto L\mathbf{x}$ is reflected by a change of the corresponding representing matrix A to the matrix A_L given by

$$A_L = LAL^{\top}. \tag{2.6}$$

Let $\mathcal{E}'$ be the orthogonal projection of $\mathcal{E}$ onto an k-dimensional subspace with some orthonormal basis $\mathbf{x}_1,\dots,\mathbf{x}_k \in \mathbb{R}^d$. Denote by A' the $k \times k$ matrix representing the ellipsoid $\mathcal{E}'$ in this basis. If C is the $k \times d$ matrix whose rows are $\mathbf{x}_1,\dots,\mathbf{x}_k$, then

$$A' = CAC^{\top}. \tag{2.7}$$

3. Proofs

3.1. *Proof of Theorem 1.1. Case $k = d$.* We proceed by induction on d. First let us assume that ξ_d is a standard Gaussian vector. Denote by χ_d a random variable having the chi distribution with d degrees of freedom and independent from $\xi_1,\dots,\xi_{d-1}$. Using (2.4) and (2.5) with $k = 1$, we get the relations

$$\begin{aligned}
\mathbf{E}\,|\det M| = \mathbf{E}\,\operatorname{Vol}_d(H_{\xi_1,\dots,\xi_d}) &= \int\limits_{\mathbb{S}^{d-1}} \mathbf{E}\,\operatorname{Vol}_d(H_{\xi_1,\dots,\xi_{d-1},\chi_d\mathbf{u}})\,d\mathbf{u} \\
&= \mathbf{E}\chi_d \int\limits_{\mathbb{S}^{d-1}} \mathbf{E}\operatorname{Vol}_{d-1}(P_{\mathbf{u}}H_{\xi_1,\dots,\xi_{d-1}})\,d\mathbf{u} \\
&= \frac{d\kappa_d}{\sqrt{2\pi}\kappa_{d-1}} \int\limits_{\mathbb{S}^{d-1}} \mathbf{E}\operatorname{Vol}_{d-1}(H_{P_{\mathbf{u}}\xi_1,\dots,P_{\mathbf{u}}\xi_{d-1}})\,d\mathbf{u}.
\end{aligned}$$

It follows from (2.7) that $P_{\mathbf{u}}\xi_i$ has dispersion ellipsoid $P_{\mathbf{u}}\mathcal{E}_i$. By the induction assumption,

$$\mathbf{E}\operatorname{Vol}_{d-1}\left(H_{P_{\mathbf{u}}\xi_1,\dots,P_{\mathbf{u}}\xi_{d-1}}\right) = \frac{(d-1)!}{(2\pi)^{(d-1)/2}} V_{d-1}(P_{\mathbf{u}}\mathcal{E}_1,\dots,P_{\mathbf{u}}\mathcal{E}_{d-1}).$$

Combining the latter two relations with (2.3), we obtain the equality

$$\mathbf{E}\,|\det M| = \frac{d!}{(2\pi)^{d/2}} V_d(\mathcal{E}_1,\dots,\mathcal{E}_{d-1},B). \tag{3.1}$$

If ξ_d is an arbitrary nondegenerate Gaussian vector, then there exists a linear transformation L such that $L\xi_d$ is a standard Gaussian vector. It follows from (2.6) that $L\mathcal{E}_i$ is the dispersion ellipsoid of $L\xi_i$, and, in particular, $L\mathcal{E}_d = B$. Applying (3.1) to the matrix $LM^\top$ and using (2.2), we get the equalities

$$\begin{aligned}\mathbf{E}\,|\det M| = |\det L|^{-1}\,\mathbf{E}\,|\det LM^\top| &= \frac{d!}{(2\pi)^{d/2}}|\det L|^{-1}V_d(L\mathcal{E}_1,\dots,L\mathcal{E}_{d-1},B)\\ &= \frac{d!}{(2\pi)^{d/2}}V_d(\mathcal{E}_1,\dots,\mathcal{E}_{d-1},\mathcal{E}_d).\end{aligned}$$

□

3.2. *Proof of Theorem 1.1. Case $k < d$.* Consider a $d\times d$ matrix M' whose first k rows form the matrix M and the last $d-k$ rows are independent standard Gaussian vectors $\xi_{k+1},\dots,\xi_d$ (independent from M). By the previous case,

$$\mathbf{E}\,|\det M'| = \frac{d!}{(2\pi)^{d/2}}V_d(\mathcal{E}_1,\dots,\mathcal{E}_k,B,\dots,B).$$

On the other hand, by (2.5),

$$\begin{aligned}\mathbf{E}\,|\det M'| = \mathbf{E}\ \mathrm{Vol}_d(H_{\xi_1,\dots,\xi_d}) &= \mathbf{E}\ \mathrm{Vol}_k(H_{\xi_1,\dots,\xi_k})\,\mathrm{Vol}_{d-k}(P_{\xi_1,\dots,\xi_k}H_{\xi_{k+1},\dots,\xi_d})\\ &= \mathbf{E}\,\sqrt{\det(MM^\top)}\,\mathbf{E}\ \mathrm{Vol}_{d-k}(H_{\eta_1,\dots,\eta_{d-k}}),\end{aligned}$$

where $\eta_1,\dots,\eta_{d-k}$ are independent, standard Gaussian vectors in $\mathbb{R}^{d-k}$. By the previous case,

$$\mathbf{E}\ \mathrm{Vol}_{d-k}(H_{\eta_1,\dots,\eta_{d-k}}) = \frac{(d-k)!}{(2\pi)^{(d-k)/2}}\kappa_{d-k}.$$

Combining the latter three relations completes the proof. □

3.3. *Proof of Theorem 1.5.* First we assume that X_j has a unit variance: $\operatorname{Var} X_j(\mathbf{t}) \equiv 1$ for all $j = 1,\dots,k$. Differentiating the relation $\mathbf{E}X_j(\mathbf{t})X_j(\mathbf{t}) = 1$ with respect to t_i, we obtain the equality

$$\mathbf{E}\frac{\partial X_j}{\partial t_i}(\mathbf{t})X_j(\mathbf{t}) = 0,$$

which, together with the independence of the coordinates of X, implies that $X'(\mathbf{t})$ and $X(\mathbf{t})$ are independent. This means that the conditioning on $X(\mathbf{t}) = 0$ in (1.6) may be dropped. To complete the proof of the theorem in the case $\operatorname{Var} X_j(\mathbf{t}) \equiv 1$, it remains to combine (1.6) with (1.2).

To cover the general case, it suffices to note that $X_j/\sqrt{\operatorname{Var} X_j}$ has the same zero set as X_j. □

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