

# Pricing Asian Options on Commodities with GARCH Model

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# Commodities, Commodity Futures and Asian Options

- Commodities: basic good used in commerce that is interchangeable with other commodities of the same type.
- Commodity Futures: agreements to buy or sell a raw material at a specific date in the future at a particular price.

$$F_t = S_t - f_t \quad (1.1)$$

- Asian Options: average value options, an exotic option whose payoff is determined by the average underlying price over some pre-set period of time. This averaging feature provides
  - Risk reduction of market manipulation of the underlying instrument close to maturity [5].
  - Lower cost than European or American options with same strike and maturity.

$$\begin{aligned}
 C_T &= (A_T(t, T) - K)^+ \\
 P_T &= (K - A_T(t, T))^+ \\
 A_T(0, T) &= \frac{1}{T} \sum_{t=0}^T S_t = \frac{1}{T} \int_0^T S_t dt \\
 \tilde{A}_T(0, T) &= \left( \prod_{t=0}^T S_t \right)^{\frac{1}{T}} = \exp\left( \frac{1}{T} \int_0^T \log S_t dt \right)
 \end{aligned} \quad (1.2)$$

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# Prevailing Pricing Methods

- Semi-Analytical: fast Fourier transform (FFT) and convolutions [1] [2].
- Approximation: approximate the real distribution of the average underlying price at the maturity with tractable ones, such as Edgeworth series expansion [3] or lognormal distributions [4].
- Monte Carlo: combination of stochastic models and Monte Carlo simulations [5].

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# Proposed Methods

- GARCH model: Monte Carlo simulations with GJR-GARCH model.
- Non-GARCH models: binomial tree model and Monte Carlo simulations with constant volatility.
- Model comparison criteria: ARE criteria [6].

$$ARE = \frac{1}{N} \sum_{j=1}^N \frac{|V_j^{model} - V_j^{market}|}{V_j^{market}} \times 100 \quad (1.3)$$

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# Commodities

16 commodities from 3 broad categories are selected for analysis. For each of them, price data of the most liquid/major future contract is collected for analysis.

## Energy:

- Crude Oil: WTI Financial Futures
- Natural Gas: Henry Hub Natural Gas Futures
- Refined Products: RBOB Gasoline Futures
- Biofuels: Chicago Ethanol (Platts) Futures
- Coal: Coal (API2) CIF ARA (ARGUS-McCloskey) Futures

## Agriculture:

- Corns: Corns Futures
- Wheats: Chicago SRW Wheat Futures
- Soybean: Soybean Futures
- Soybean Meal: Soybean Meal Futures
- Soybean Oil: Soybean Oil Futures
- Livestock: Live Cattle Futures
- Livestock: Lean Hog Futures

## Metals:

- Gold: Gold Futures
- Silver: Silver Futures
- Platinum: Platinum Futures
- Copper: Copper Futures

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# Asian Options

Strike prices and maturities from 5 commodity option contracts are collected to construct numerical examples for proposed pricing models. There are 319 option contracts (call put pairs) in total, with maturity varying from 5 days to 5 years. Their contract specifications and settlement prices are collected as reference.

Here is the list:

- WTI Crude Oil Asian Option
- Chicago Ethanol (Platts) Asian Option
- Gold American Option
- Silver American Option
- Natural Gas European Option

# Data Source

- Commodity future prices: investing.com, with time span from 2000-01-03 to 2019-03-22, and there are 16 price time series in total.
- Commodity options: recorded manually from CME group website, and there are  $319 \times 2 = 638$  options in total (319 call put pairs).
- Risk free interest rates: US treasury yield curve on 2019-03-22 from US government website.

# Data Preprocess

Two major preprocessing steps:

- Time series concatenation: concat price series into one data table with respect to dates.
- Convert risk free spot curve to forward curve:

$$\begin{aligned} f(t_i, t_{i+1}) &= \frac{\log(e^{r_{t_{i+1}} t_{i+1}} / e^{r_{t_i} t_i})}{t_{i+1} - t_i} \\ &= \frac{r_{t_{i+1}} t_{i+1} - r_{t_i} t_i}{t_{i+1} - t_i} \end{aligned} \quad (2.1)$$

Missing value preprocessing:

- Time series analysis: for two time series selected for correlation analysis, all the date points with missing values will be removed.
- Pricing: date points with missing values will be dropped.

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# Correlation Analysis

Correlation analysis is a method of statistical evaluation used to study the strength of a relationship between two, numerically measured, continuous variables. In this step, the correlation coefficients among future prices and price log returns are calculated respectively by equation 2.2.

$$\begin{aligned}\rho_{xy} &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y} \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}}\end{aligned}\quad (2.2)$$

Findings from heat maps:

- Price correlations:
  - Many strong positive linear relationships.
  - Few commodities are less correlated to all other commodities.
  - Randomness of correlations in commodity prices.
- Log return correlations: strong (positive) correlations are only presented between commodities within the same category, especially in agriculture or metals.

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# Correlation Analysis

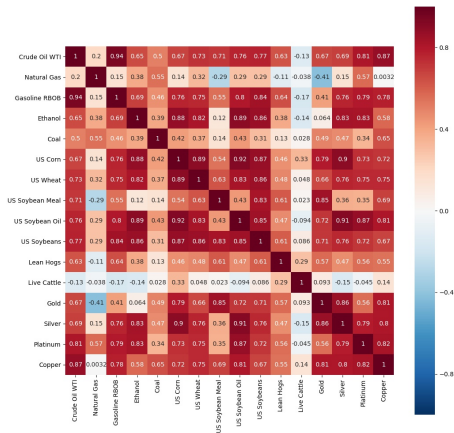


Figure: Correlation Analysis of Commodity Prices

# Correlation Analysis

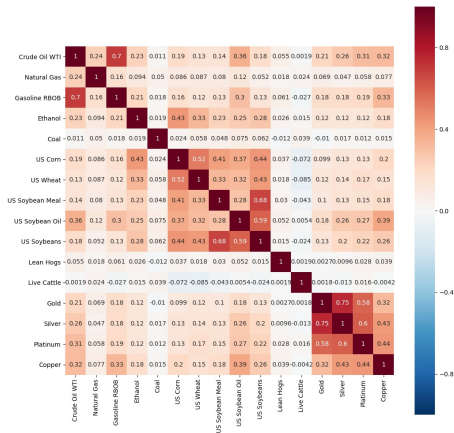


Figure: Correlation Analysis of Commodity Log Returns

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# GJR-GARCH Model

The Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model by Glosten, Jagannathan and Runkle (1993) models asymmetry in the ARCH process. For *GJR – GARCH*( $p, o, q$ ), the model is expressed as the following:

$$\begin{aligned}
 r_t &= \log S_t - \log S_{t-1} \\
 r_t &= \mu + \epsilon_t \\
 \epsilon_t &= \sigma_t Z_t \\
 \sigma_t^2 &= \omega + \sum_{i=1}^p \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^o \gamma_j \epsilon_{t-j}^2 I_{\epsilon_{t-j} < 0} + \sum_{k=1}^q \beta_k \sigma_{t-k}^2
 \end{aligned} \tag{3.1}$$

This model will be used to fit the historical log returns and predict the mean volatility along the simulated Monte Carlo path, since

$$\sigma_0(t) = E_0[\sigma(t)] \tag{3.2}$$

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# GJR-GARCH Model Volatility Predictions

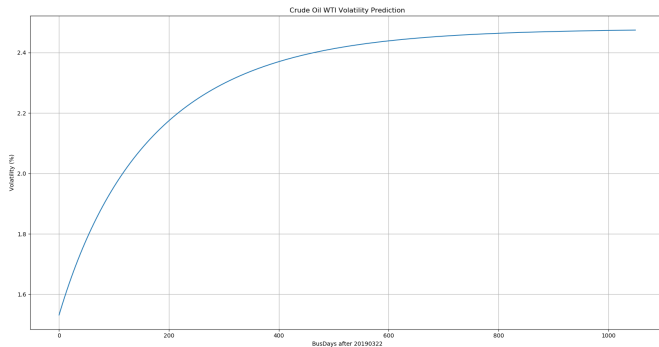


Figure: Volatility Prediction on Crude Oil

# GJR-GARCH Model Volatility Predictions

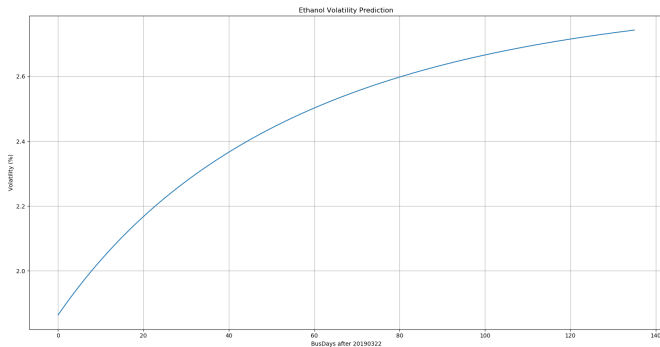


Figure: Volatility Prediction on Ethanol

# GJR-GARCH Model Volatility Predictions

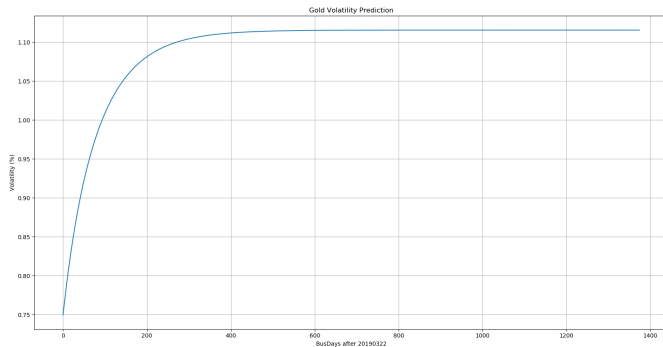


Figure: Volatility Prediction on Gold

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# GJR-GARCH Model Volatility Predictions

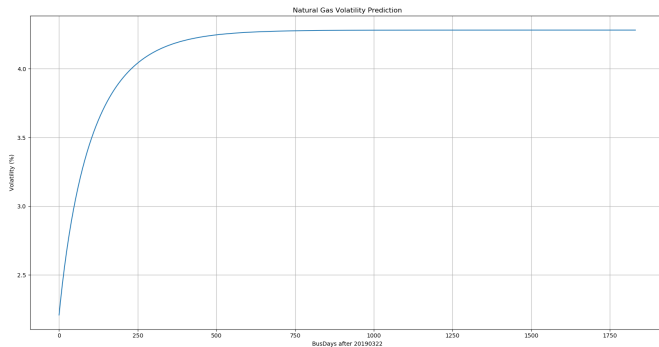


Figure: Volatility Prediction on Natural Gas

# GJR-GARCH Model Volatility Predictions

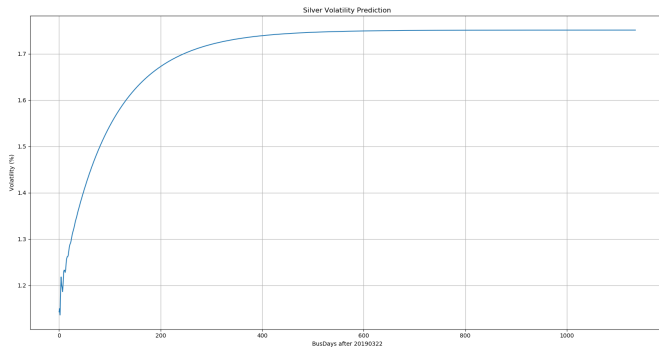


Figure: Volatility Prediction on Silver

# GJR-GARCH Model Selection Criteria

For each commodity, the best model configuration, say the number of  $\alpha$ ,  $\beta$ ,  $\gamma$  and whether using zero mean, is selected based on the following steps.

- Define a GJR-GARCH model: define  $p$ ,  $o$ ,  $q$  and fit historical with constant mean model.  $p$ ,  $o$ ,  $q$  are taken from 0 to 9, so there are 1000 possible models to fit.
- zero mean vs constant mean: if the p-value of  $\mu$  in constant mean model is larger than 0.05, then the mean is considered insignificant, and it will be removed to refit the corresponding zero mean model.
- Ljung-Box test: the fitted model will be plugged into historical data, and the series of  $\{Z_t\}$  is derived. If  $\{Z_t\}$  passes Ljung-Box test at a significance level of 0.05, then it proves that the model well explains the historical time series. Otherwise, the model will be rejected, and go to step 1.
- BIC and number of parameters: if one model has a less BIC and less number of parameters than the other one, it will be selected as the temporary best model. Alternatively, if BIC of one model is 5% less than the other one, it will also win the selection.

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# Monte Carlo Simulation Scheme

Given a commodity start price  $S_0$  and a fitted GJR-GARCH model  $GJR - GARCH(p, o, q)$ , the underlying price movements can be simulated by the dynamics of its log returns  $\{r_i\}$ :

$$\begin{aligned}
 \sigma_i &= \sqrt{\omega + \sum_{i=1}^p \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^o \gamma_j \epsilon_{t-j}^2 I_{\epsilon_{t-j} < 0} + \sum_{k=1}^q \beta_k \sigma_{t-k}^2} \\
 Z_i &\sim N(0, 1), \Delta t = 1/252 \\
 \epsilon_i &= \sigma_i Z_i \\
 r_i &= \mu \Delta t + \epsilon_i \\
 S_{t_i} &= S_{t_{i-1}} \exp(r_i \Delta t) = S_0 \exp\left(\sum_{k=0}^i r_k \Delta t\right)
 \end{aligned} \tag{3.3}$$

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# Monte Carlo Simulation Scheme

The final payoff is calculated as

$$\begin{aligned}
 A_{t_N}(0, t_N) &= \frac{1}{N} \sum_{i=0}^N S_{t_i} \\
 df(0, Tt_N) &= \exp\left(-\sum_{i=0}^{N-1} r_{t_i} \Delta t\right) \\
 V_{t_N} &= (A_{t_N}(0, t_N) - K)^+ \\
 \bar{V}_{t_N} &= \frac{1}{n} \sum_{i=1}^n V_{t_N}^i \\
 \bar{V}_0 &= df(0, Tt_N) \bar{V}_{t_N}
 \end{aligned} \tag{3.4}$$

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# Monte Carlo Constant Volatility Model

Constant volatility model assumes that the underlying price follows the following dynamics:

$$\begin{aligned}dS_t &= (r_t - \frac{1}{2}\sigma^2)S_t dt + \sigma S_t dW_t \\d \log S_t &= (r_t - \frac{1}{2}\sigma^2)dt + \sigma dW_t\end{aligned}\tag{3.5}$$

The constant volatility model shares the same Monte Carlo scheme with GJR-GARCH scheme, except for its path generating scheme for the underlying.

$$\begin{aligned}dW_t &\sim N(0, \Delta t), \Delta t = 1/252 \\r_i &= r_{i-1} + (r_t - \frac{1}{2}\sigma^2)\Delta t + \sigma dW_t \\S_{t_i} &= S_{t_{i-1}} \exp(r_i \Delta t) = S_0 \exp(\sum_{k=0}^i r_k \Delta t)\end{aligned}\tag{3.6}$$

The rest process follows that of GARCH scheme

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# Binomial Tree Model for Asian Options

- Forward shooting gird: approximate the average price.
- $S_j^n$  and  $A_k^n$ : the asset value jumping upward  $j$  times and average price with index  $k$  at  $n$ -th time level, respectively.

$$\begin{aligned} S_j^n &= S_0 e^{j\Delta W}, \quad A_k^n = S_0 e^{k\Delta Y}, \\ \Delta W &= \sigma\sqrt{\Delta t}, \quad \Delta Y = \rho\Delta W j, \quad k \in \mathbb{N} \end{aligned} \quad (3.7)$$

- $c_{j,l}^n$ : numerical approximation to the Asian call value at  $(n, j)$  node with the averaging state variable assuming the value  $A_l^n$ .
- Gird function:

$$k^\pm(j) = \frac{\ln \frac{(n+1)\exp(k\Delta Y) + \exp((j\pm 1)\Delta W)}{n+2}}{\Delta Y} \quad (3.8)$$

- Interpolation:

$$c_{j,l}^n = c_{j,l_{floor}}^n + \varepsilon_l (c_{j,l_{ceil}}^n - c_{j,l_{floor}}^n) \quad (3.9)$$

where  $\varepsilon_l$  is the fraction of one step  $\Delta Y$  between  $\ln A_{l_{ceil}}^n$  and  $A_{l_{floor}}^n$ :

$$\varepsilon_l = \frac{\ln \frac{A_l^n}{A_{l_{floor}}^n}}{\Delta Y}, \quad A_l^n = A_{l_{floor}}^n e^{\varepsilon_l \Delta Y} \quad (3.10)$$

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# Binomial Tree Model for Asian Options

Backward induction:

$$\begin{aligned}
 c_{j,k}^n &= \frac{1}{R} \left[ p c_{j+1,k^+(j)}^{n+1} + (1-p) c_{j-1,k^-(j)}^{n+1} \right] \\
 &\approx \frac{1}{R} \left\{ p \left[ \varepsilon_{k^+(j)} c_{j+1,k_{ceil}^+}^{n+1} + (1-\varepsilon_{k^+(j)}) c_{j+1,k_{floor}^+}^{n+1} \right] \right. \\
 &\quad \left. + (1-p) \left[ \varepsilon_{k^-(j)} c_{j-1,k_{ceil}^-}^{n+1} + (1-\varepsilon_{k^-(j)}) c_{j-1,k_{floor}^-}^{n+1} \right] \right\} \quad (3.11) \\
 n &= N-1, \dots, 0, j = -n, -n+2, \dots, n, \\
 k &\in \mathbb{N} \cap \left[ -\frac{n}{p}, \frac{n}{p} \right]
 \end{aligned}$$

where the risk neutral probability of jump upward is:

$$p = \frac{R-d}{u-d}, \quad R = e^{r_t(T-t)} \quad (3.12)$$

where  $r_t$  is the forward rate at time  $t$

# Binomial Tree Simulation

Parameter setting:

- $\sigma$ : historical volatility of log return.
- $r_t$ : forward rate at time  $t$ .
- Number of time steps  $N$ : total number of business days from the spot date to maturity date. If the time length reaches over 1 year, we restrict the time steps to 252.
- $\Delta t$ :  $\frac{1}{N}$ .
- Underlying: 1 unit.

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# Fitted GJR-GARCH Model

Underlying	$(p, o, q)$	$Q(20)$	p-value	BIC
Crude Oil WTI	(1,1,1)	13.69	0.84	21129
Ethanol	(1,1,1)	39.91	0.005	13715
Gold	(1,0,1)	27.53	0.12	14403
Natural gas	(1,0,1)	23.48	0.2	24608
Silver	(1,0,5)	29.42	0.07	12151

**Table:** Model result: GJR model for each underlying futures

# Model Pricing Results

## ■ ARE:

$$ARE = \frac{1}{N} \sum_{j=1}^N \frac{|V_j^{model} - V_j^{market}|}{V_j^{market}} \times 100 \quad (4.1)$$

Models	Crude Oil WTI		Ethanol		Gold	
	put	call	put	call	put	call
GARCH-MC	46.679	304.798	10.757	25.258	72.799	84.586
MC	83.121	122.238	56.035	73.732	52.874	91.129
BT	402.088	819.106	45.110	62.466	106.801	149.124

Models	Silver		Natural gas	
	put	call	put	call
GARCH-MC	17.455	26.597	215.232	324.858
MC	64.379	89.397	64.891	84.962
BT	83.969	107.557	264.242	72.697

**Table:** ARE for the Asian put and call option for each underlying futures.

- Fair price of European option: fair price of the Asian option < the European option value with the same specification of options: all average option values from 3 models < European option value with the same specification

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# Discussions

- Data Limitations: lack of underlying price and option information.
  - Using future prices rather than spot prices to fit models.
  - Only WTI Crude Oil Asian Option price and Chicago Ethanol (Platts) Asian Option are Asian options.
- Monte Carlo Limitations standard deviation of simulated payoffs is relatively large.
  - The number of simulated paths might not be enough: increase the number of paths.
  - Apply probability bounds analysis (PBA) to estimate upper and lower bounds of simulated payoffs.
- Advanced GARCH Models[7]: semi-analytical formula for geometric Asian options

$$\begin{aligned}r_t &= r - \frac{1}{2}\sigma_t^2 + \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha_1(Z_{t-1} - \lambda\sigma_{t-1})^2 + \beta_1\sigma_{t-1}^2\end{aligned}\tag{5.1}$$

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# Conclusions

- 16 commodity future prices are studied.
- 5 options are priced by three pricing methods, say binomial tree model, Monte Carlo simulations with constant volatility, and Monte Carlo simulations with GJR-GARCH volatility.
- Strong sector correlations are discovered in the 16 time series of commodity log returns.
- The ARE criterion shows that GJR-GARCH model is the most appropriate pricing model in three models.
- Limitations in data and models can be overcome in future studies.

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# References I



A. P. Carverhill and L. J. Clewlow.  
Valuing average rate (asian) options.  
*Risk*, 3:33–36, 1990.



Eric Benhamou.  
Fast fourier transform for discrete asian options.  
In *EFMA 2001 Lugano Meetings*, 2000.



Stuart M Turnbull and Lee Macdonald Wakeman.  
A quick algorithm for pricing european average options.  
*Journal of financial and quantitative analysis*, 26(3):377–389, 1991.



Edmond Levy.  
Pricing european average rate currency options.  
*Journal of International Money and Finance*, 11(5):474–491, 1992.



Angelen GZ Kemna and Antonius CF Vorst.  
A pricing method for options based on average asset values.  
*Journal of Banking & Finance*, 14(1):113–129, 1990.



Ke Zhu and Shiqing Ling.  
Model-based pricing for financial derivatives.  
*Journal of Econometrics*, 187(2):447–457, 2015.

<sup>1</sup>Department of Mathematics, the Hong Kong University of Science and Technology, Hong Kong SAR, China

# References II



Mercuri Lorenzo.

Pricing asian options in affine garch models.

*International Journal of Theoretical and Applied Finance*, 14(02):313–333, 2011.

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