Pricing Asian Options on Commodities with GARCH Model

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Commodities, Commodity Futures and Asian Options

- Commodities: basic good used in commerce that is interchangeable with other commodities of the same type.
- Commodity Futures: agreements to buy or sell a raw material at a specific date in the future at a particular price.

$$F_t = S_t - f_t \tag{1.1}$$

- Asian Options: average value options, an exotic option whose payoff is determined by the average underlying price over some pre-set period of time. This averaging feature provides
  - Risk reduction of market manipulation of the underlying instrument close to maturity [5].
  - Lower cost than European or American options with same strike and maturity.

$$\begin{aligned}
C_T &= (A_T(t,T) - K)^+ \\
P_T &= (K - A_T(t,T))^+ \\
A_T(0,T) &= \frac{1}{T}\sum_{t=0}^{T} S_t = \frac{1}{T}\int_0^T S_t dt \qquad (1.2) \\
\tilde{A}_T(0,T) &= (\prod_{t=0}^{T} S_t)^{\frac{1}{T}} = \exp(\frac{1}{T}\int_0^T \log S_t dt)
\end{aligned}$$

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Prevailing Pricing Methods

- Semi-Analytical: fast Fourier transform (FFT) and convolutions [1] [2].
- Approximation: approximate the real distribution of the average underlying price at the maturity with tractable ones, such as Edgeworth series expansion [3] or lognormal distributions [4].
- Monte Carlo: combination of stochastic models and Monte Carlo simulations [5].

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Proposed Methods

- GARCH model: Monte Carlo simulations with GJR-GARCH model.
- Non-GARCH models: binomial tree model and Monte Carlo simulations with constant volatility.
- Model comparison criteria: ARE criteria [6].

$$ARE = \frac{1}{N} \sum_{j=1}^{N} \frac{|V_j^{model} - V_j^{market}|}{V_j^{market}} \times 100 \tag{1.3}$$

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Commodities

16 commodities from 3 broad categories are selected for analysis. For each of them, price data of the most liquid/major future contract is collected for analysis.

Energy:

- Crude Oil: WTI Financial Futures
- Natural Gas: Henry Hub Natural Gas Futures
- Refined Products: RBOB Gasonline Futures
- Biofuels: Chicago Ethanol (Platts) Futures
- Coal: Coal (API2) CIF ARA (ARGUS-McCloskey) Futures

Agriculture:

- Corns: Corns Futures
- Wheats: Chicago SRW Wheat Futures
- Soybean: Soybean Futures
- Soybean Meal: Soybean Meal Futures
- Soybean Oil: Soybean Oil Futures
- Livestock: Live Cattle Futures
- Livestock: Lean Hog Futures

Metals:

- Gold: Gold Futures
- Silver: Silver Futures
- Platinum: Platinum Futures
- Copper: Copper Futures

Asian Options

Strike prices and maturities from 5 commodity option contracts are collected to construct numerical examples for proposed pricing models. There are 319 option contracts (call put pairs) in total, with maturity varying from 5 days to 5 years. Their contract specifications and settlement prices are collected as reference.
Here is the list:

- WTI Crude Oil Asian Option
- Chicago Ethanol (Platts) Asian Option
- Gold American Option
- Silver American Option
- Natural Gas European Option

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Data Source

- Commodity future prices: investing.com, with time span from 2000-01-03 to 2019-03-22, and there are 16 price time series in total.
- Commodity options: recorded manually from CME group website, and there are 319*2=638 options in total (319 call put pairs).
- Risk free interest rates: US treasury yield curve on 2019-03-22 from US government website.

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Data Preprocess

Two major preprocessing steps:

- Time series concatenation: concat price series into one data table with respect to dates.
- Convert risk free spot curve to forward curve:

$$
\begin{aligned}
f(t_i, t_{i+1}) &= \frac{\log(e^{r_{t_{i+1}} t_{i+1}} / e^{r_{t_i} t_i})}{t_{i+1} - t_i} \\
&= \frac{r_{t_{i+1}} t_{i+1} - r_{t_i} t_i}{t_{i+1} - t_i}
\end{aligned} \tag{2.1}
$$

Missing value preprocessing:

- Time series analysis: for two time series selected for correlation analysis, all the date points with missing values will be removed.
- Pricing: date points with missing values will be dropped.

Correlation Analysis

Correlation analysis is a method of statistical evaluation used to study the strength of a relationship between two, numerically measured, continuous variables. In this step, the correlation coefficients among future prices and price log returns are calculated respectively by equation 2.2.

$$\begin{aligned}\rho_{xy} &= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{(n-1)s_x s_y} \\ &= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}}\end{aligned} \tag{2.2}$$

Findings from heat maps:

- Price correlations:
  - Many strong positive linear relationships.
  - Few commodities are less correlated to all other commodities.
  - Randomness of correlations in commodity prices.
- Log return correlations: strong (positive) correlations are only presented between commodities within the same category, especially in agriculture or metals.

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Correlation Analysis

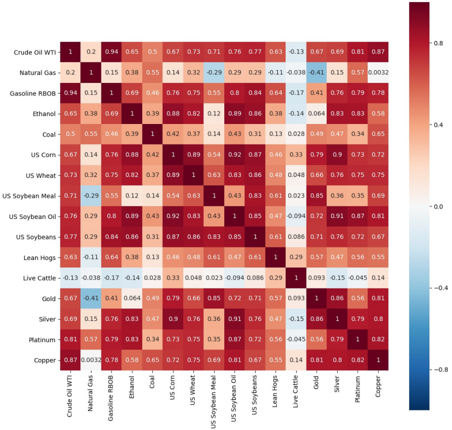

Figure: Correlation Analysis of Commodity Prices

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Correlation Analysis

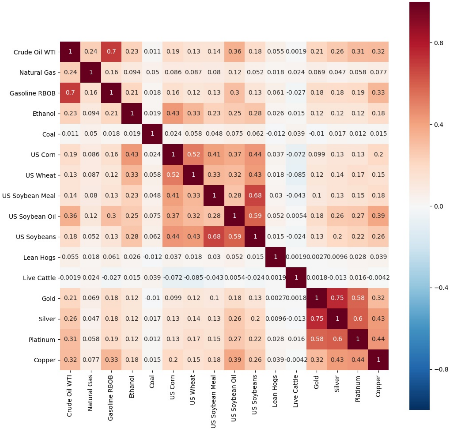

Figure: Correlation Analysis of Commodity Log Returns

GJR-GARCH Model

The Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) model by Glosten, Jagannathan and Runkle (1993) models asymmetry in the ARCH process. For $GJR - GARCH(p, o, q)$, the model is expressed as the following:

$$
\begin{aligned}
r_t &= \log S_t - \log S_{t-1} \\
r_t &= \mu + \epsilon_t \\
\epsilon_t &= \sigma_t Z_t \\
\sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{o} \gamma_j \epsilon_{t-j}^2 I_{\epsilon_{t-j}<0} + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2
\end{aligned} \tag{3.1}
$$

This model will be used to fit the historical log returns and predict the mean volatility along the simulated Monte Carlo path, since

$$\sigma_0(t) = E_0[\sigma(t)] \tag{3.2}$$

GJR-GARCH Model Volatility Predictions

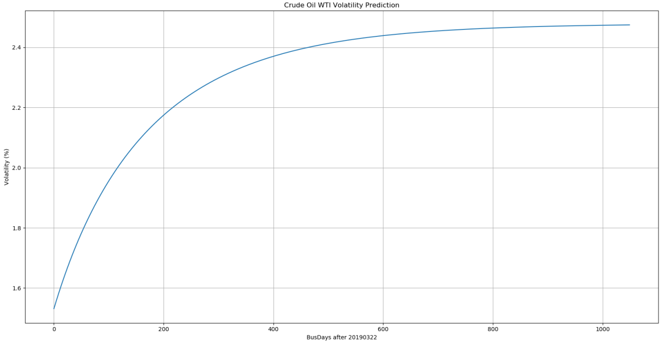

Figure: Volatility Prediction on Crude Oil

GJR-GARCH Model Volatility Predictions

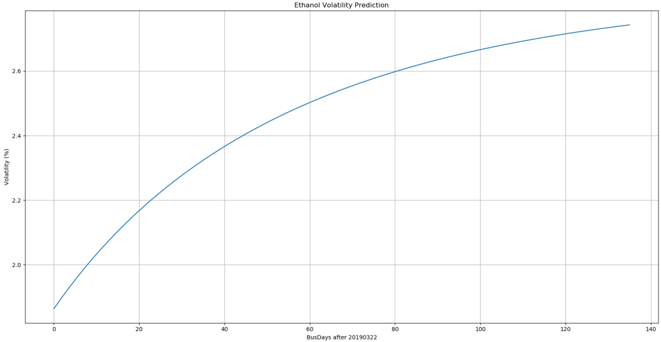

Figure: Volatility Prediction on Ethanol

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GJR-GARCH Model Volatility Predictions

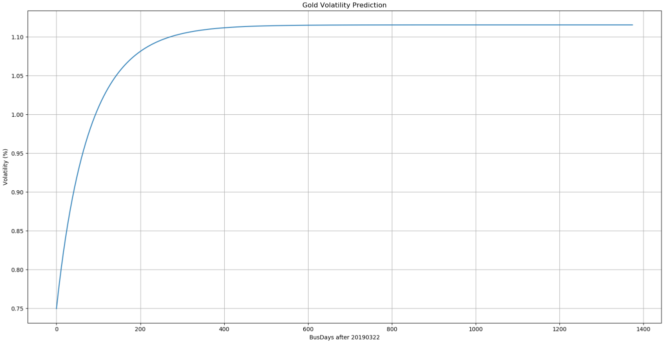

Figure: Volatility Prediction on Gold

GJR-GARCH Model Volatility Predictions

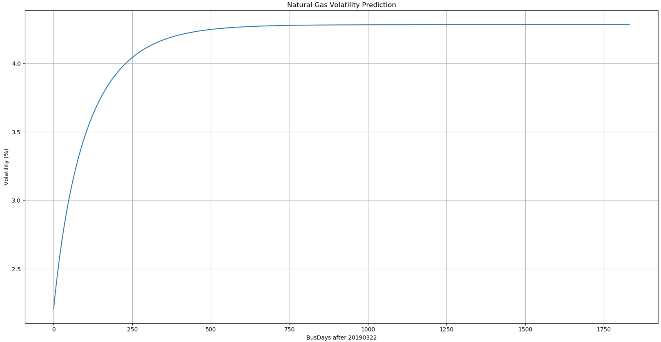

Figure: Volatility Prediction on Natural Gas

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GJR-GARCH Model Volatility Predictions

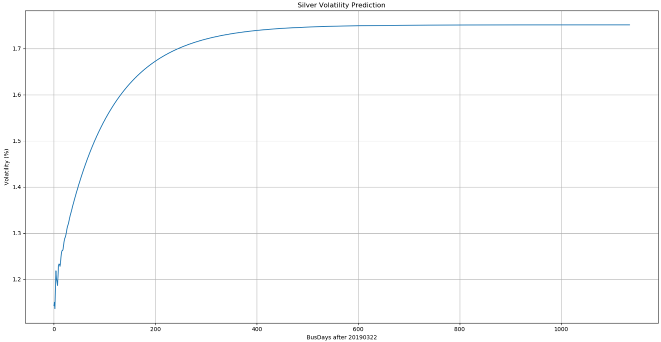

Figure: Volatility Prediction on Silver

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GJR-GARCH Model Selection Criteria

For each commodity, the best model configuration, say the number of α, β, γ and whether using zero mean, is selected based on the following steps.

- Define a GJR-GARCH model: define p, o, q and fit historical with constant mean model. p, o, q are taken from 0 to 9, so there are 1000 possible models to fit.
- zero mean vs constant mean: if the p-value of μ in constant mean model is larger than 0.05, then the mean is considered insignificant, and it will be removed to refit the corresponding zero mean model.
- Ljung-Box test: the fitted model will be plugged into historical data, and the series of $\{Z_t\}$ is derived. If $\{Z_t\}$ passes Ljung-Box test at a significance level of 0.05, then it proves that the model well explains the historical time series. Otherwise, the model will be rejected, and go to step 1.
- BIC and number of parameters: if one model has a less BIC and less number of parameters than the other one, it will be selected as the temporary best model. Alternatively, if BIC of one model is 5% less than the other one, it will also win the selection.

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Monte Carlo Simulation Scheme

Given a commodity start price S_0 and a fitted GJR-GARCH model $GJR - GARCH(p, o, q)$, the underlying price movements can be simulated by the dynamics of its log returns $\{r_i\}$:

$$
\begin{aligned}
\sigma_i &= \sqrt{\omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{o} \gamma_j \epsilon_{t-j}^2 I_{\epsilon_{t-j}<0} + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2} \\
Z_i &\sim N(0,1), \Delta t = 1/252 \\
\epsilon_i &= \sigma_i Z_i \\
r_i &= \mu \Delta t + \epsilon_i \\
S_{t_i} &= S_{t_{i-1}} \exp(r_i \Delta t) = S_0 \exp(\sum_{k=0}^{i} r_k \Delta t)
\end{aligned}
\tag{3.3}
$$

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Monte Carlo Simulation Scheme

The final payoff is calculated as

$$
\begin{aligned}
A_{t_N}(0, t_N) &= \frac{1}{N}\sum_{i=0}^{N} S_{t_i} \\
df(0, Tt_N) &= \exp(-\sum_{i=0}^{N-1} r_{t_i}\Delta t) \\
V_{t_N} &= (A_{t_N}(0, t_N) - K)^+ \\
\bar{V}_{t_N} &= \frac{1}{n}\sum_{i=1}^{n} V_{t_N}^i \\
\bar{V}_0 &= df(0, Tt_N)\bar{V}_{t_N}
\end{aligned}
\tag{3.4}
$$

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Monte Carlo Constant Volatility Model

Constant volatility model assumes that the underlying price follows the following dynamics:

$$\begin{aligned} dS_t &= (r_t - \frac{1}{2}\sigma^2)S_t dt + \sigma S_t dW_t \\ d\log S_t &= (r_t - \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned} \tag{3.5}$$

The constant volatility model shares the same Monte Carlo scheme with GJR-GARCH scheme, except for its path generating scheme for the underlying.

$$\begin{aligned} dW_t &\sim N(0, \Delta t), \Delta t = 1/252 \\ r_i &= r_{i-1} + (r_t - \frac{1}{2}\sigma^2)\Delta t + \sigma dW_t \\ S_{t_i} &= S_{t_{i-1}} \exp(r_i \Delta t) = S_0 \exp(\sum_{k=0}^{i} r_k \Delta t) \end{aligned} \tag{3.6}$$

The rest process follows that of GARCH scheme

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Binomial Tree Model for Asian Options

- Forward shooting gird: approximate the average price.
- S_j^n and A_k^n: the asset value jumping upward j times and average price with index k at n-th time level, respectively.

$$\begin{aligned} S_j^n &= S_0 e^{j\Delta W}, \quad A_k^n = S_0 e^{k\Delta Y}, \\ \Delta W &= \sigma\sqrt{\Delta t}, \ \Delta Y = \rho \Delta W j, k \in \mathbb{N} \end{aligned} \tag{3.7}$$

- $c_{j,l}^n$: numerical approximation to the Asian call value at (n, j) node with the averaging state variable assuming the value A_l^n.
- Gird function:

$$k^{\pm}(j) = \frac{\ln \frac{(n+1)\exp(k\Delta Y) + \exp((j\pm 1)\Delta W)}{n+2}}{\Delta Y} \tag{3.8}$$

- Interpolation:

$$c_{j,l}^n = c_{j,l_{floor}}^n + \varepsilon_l (c_{j,l_{ceil}}^n - c_{j,l_{floor}}^n) \tag{3.9}$$

where ε_l is the fraction of one step ΔY between $\ln A_{l_{ceil}}^n$ and $A_{l_{floor}}^n$:

$$\varepsilon_l = \frac{\ln \frac{A_l^n}{A_{l_{floor}}^n}}{\Delta Y}, A_l^n = A_{l_{floor}}^n e^{\varepsilon_l \Delta Y} \tag{3.10}$$

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Binomial Tree Model for Asian Options

Backward induction:

$$
\begin{aligned}
c_{j,k}^{n} &= \frac{1}{R}\left[p c_{j+1,k^{+}(j)}^{n+1} + (1-p) c_{j-1,k^{-}(j)}^{n+1}\right] \\
&\approx \frac{1}{R}\{p\left[\varepsilon_{k^{+}(j)} c_{j+1,k_{ceil}^{+}}^{n+1} + (1-\varepsilon_{k^{+}(j)}) c_{j+1,k_{floor}^{+}}^{n+1}\right] \\
&+ (1-p)\left[\varepsilon_{k^{-}(j)} c_{j-1,k_{ceil}^{-}}^{n+1} + (1-\varepsilon_{k^{-}(j)}) c_{j-1,k_{floor}^{-}}^{n+1}\right]\} \\
& n = N-1, \cdots, 0, j = -n, -n+2, \cdots, n, \\
& k \in \mathbb{N} \cap [-\frac{n}{p}, \frac{n}{p}]
\end{aligned}
\tag{3.11}
$$

where the risk neutral probability of jump upward is:

$$
p = \frac{R-d}{u-d}, \quad R = e^{r_t(T-t)} \tag{3.12}
$$

where r_t is the forward rate at time t

Binomial Tree Simulation

Parameter setting:

- σ: historical volatility of log return.
- r_t: forward rate at time t.
- Number of time steps N: total number of business days from the spot date to maturity date. If the time length reaches over 1 year, we restrict the time steps to 252.
- Δt: $\frac{1}{N}$.
- Underlying: 1 unit.

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Fitted GJR-GARCH Model

Underlying	(α, γ, β)	$Q(20)$	p-value	BIC
Crude Oil WTI	(1,1,1)	13.69	0.84	21129
Ethanol	(1,1,1)	39.91	0.005	13715
Gold	(1,0,1)	27.53	0.12	14403
Natural gas	(1,0,1)	23.48	0.2	24608
Silver	(1,0,5)	29.42	0.07	12151

Table: Model result: GJR model for each underlying futures

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Model Pricing Results

- ARE:

$$ARE = \frac{1}{N}\sum_{j=1}^{N}\frac{|V_j^{model} - V_j^{market}|}{V_j^{market}} \times 100 \quad (4.1)$$

Models	Crude Oil WTI		Ethanol		Gold	
	put	call	put	call	put	call
GARCH-MC	46.679	304.798	10.757	25.258	72.799	84.586
MC	83.121	122.238	56.035	73.732	52.874	91.129
BT	402.088	819.106	45.110	62.466	106.801	149.124
Models	**Silver**		**Natural gas**			
	put	call	put	call		
GARCH-MC	17.455	26.597	215.232	324.858		
MC	64.379	89.397	64.891	84.962		
BT	83.969	107.557	264.242	72.697		

Table: ARE for the Asian put ans call option for each underlying futures.

- Fair price of European option: fair price of the Asian option < the European option value with the same specification of options: all average option values from 3 models < European option value with the same specification

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Discussions

- Data Limitations: lack of underlying price and option information.
  - Using future prices rather than spot prices to fit models.
  - Only WTI Crude Oil Asian Option price and Chicago Ethanol (Platts) Asian Option are Asian options.
- Monte Carlo Limitations standard deviation of simulated payoffs is relatively large.
  - The number of simulated paths might not be enough: increase the number of paths.
  - Apply probability bounds analysis (PBA) to estimate upper and lower bounds of simulated payoffs.
- Advanced GARCH Models[7]: semi-analytical formula for geometric Asian options

$$\begin{aligned} r_t &= r - \frac{1}{2}\sigma_t^2 + \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha_1 (Z_{t-1} - \lambda \sigma_{t-1})^2 + \beta_1 \sigma_{t-1}^2 \end{aligned} \tag{5.1}$$

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Conclusions

- 16 commodity future prices are studied.
- 5 options are priced by three pricing methods, say binomial tree model, Monte Carlo simulations with constant volatility, and Monte Carlo simulations with GJR-GARCH volatility.
- Strong sector correlations are discovered in the 16 time series of commodity log returns.
- The ARE criterion shows that GJR-GARCH model is the most appropriate pricing model in three models.
- Limitations in data and models can be overcomed in future studies.

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