Rev Quant Finan Acc (2006) 27:285–296
DOI 10.1007/s11156-006-9432-9

An adjusted binomial model for pricing Asian options

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Abstract We propose a model for pricing both European and American Asian options based on the arithmetic average of the underlying asset prices. Our approach relies on a binomial tree describing the underlying asset evolution. At each node of the tree we associate a set of representative averages chosen among all the effective averages realized at that node. Then, we use backward recursion and linear interpolation to compute the option price.

Keywords Asian options · Binomial algorithms · Discrete-time models

1. Introduction

An Asian option is a path-dependent option whose payoff depends on a certain average of the underlying asset prices recorded during the option's lifetime. The types of averages usually considered are the geometric and the arithmetic average. The first type leads to a closed-form formula for the European option price within the classical Black-Scholes model (see Zhang (1998) for a detailed description). The reason is because the geometric average of lognormally distributed random variables has also a lognormal

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distribution and this simplifies deeply the mathematical tractability of the pricing problem. Unfortunately, for the second type, the arithmetic average of lognormal random variables is not lognormally distributed. This causes a huge complexity in the pricing problem and no closed form formula exists for pricing European Asian options based on the arithmetic average of the underlying asset prices.

On the other hand, arithmetic average-based derivatives are widely diffused in financial markets, hence several models have been developed for pricing Asian options on the arithmetic average of asset prices. Many of these models involve different kinds of approximations. In particular, the model proposed by Levy (1992) approximates the arithmetic average of the underlying asset prices with the corresponding geometric average while the model proposed by Turnbull and Wakeman (1991) uses Edgeworth expansions to approximate the probability distribution of the arithmetic average of the underlying stock prices. Although these approaches are straightforward, the approximations yield inaccurate option prices in some instances.

Geman and Yor (1993) proposed a pricing algorithm based on the Laplace transform of the price of a European Asian option. A numerical scheme to invert this Laplace transform was proposed by Fu et al. (1998) but the numerical results presented are not very satisfactory in the case of low volatility and/or short time to maturity.

Another class of models is based on numerical methods for the solution of the PDE governing the Asian option price. Both explicit and implicit finite difference schemes used for pricing Asian options may cause problems in the evaluation process. In particular, the explicit finite difference method is numerically unstable when applied to PDE arising from the context of path-dependent option pricing. On the other hand, the implicit finite difference method is stable but it produces accurate prices only for particular volatility structures. A comprehensive treatment of the PDE approach for pricing Asian options can be found in Wilmott et al. (1993).

More recently, Vecer (2001) introduced a new PDE approach for pricing European Asian options that overcomes the stability problems mentioned above. The performance of this method is quite good but it cannot be extended to price American Asian options.

Linetsky (2004) applied spectral expansions to value European Asian options based on a continuous monitoring of the contract.

The first lattice-based model for pricing Asian options was proposed by Hull and White in 1993. The main problem in developing a pricing algorithm based on a binomial tree that describes the underlying asset evolution is due to a huge number of arithmetic averages that needs to be tracked down. Indeed, the number of arithmetic averages grows exponentially when the number of time steps used to compute the option price increases. This makes the pricing problem computationally unmanageable. To overcome this obstacle Hull and White considered a set of representative averages at each node of the tree and use linear interpolation to compute the missing values of the option prices. Barraquand and Pudet (1996), Klassen (2001) and Chalasani et al. (1998, 1999) proposed similar approaches based on binomial trees but they used different tricks in choosing the set of representative averages.

We note that lattice-based models play a crucial role in pricing Asian options for their simplicity, flexibility and efficiency. Moreover, lattice-based models can easily be implemented in pricing both European and American-style Asian options, with continuous or discrete monitoring of the contract.

In this paper, we present a pricing algorithm that is also based on a binomial lattice describing the evolution of the underlying asset price. However, we propose a different technique of choosing the representative averages of the underlying asset prices at each node of the tree. While the models previously mentioned consider simulated averages, we consider effective averages realized at each node of the tree according to a scheme that we shall explain in details in Section 3. We test our model on continuously sampled Asian options of European and American type. The model is flexible and it can be easily extended to price Asian options with discrete monitoring of the contract.

The rest of the paper is organized as follows. Section 2 describes binomial tree-based models to value Asian options. In Section 3, we present the adjusted binomial model for pricing Asian options. A summary and some concluding remarks are given in Section 4.

2. Binomial tree based models for pricing Asian options

In this section, at first we briefly review the binomial tree-based model for pricing Asian options as proposed by Hull and White (1993). Consider a European call option written on the arithmetic average of the prices of an asset with value $S(t) = S$ at time t. At maturity $t + T$, the option payoff is $\max(A(t + T) - K, 0)$ where $A(t + T)$ is the average of the underlying asset prices attained during the option lifetime, T, and K is the strike price. This option is known as a fixed strike Asian call option. The Hull-White model can easily be adapted to evaluate a floating strike Asian option with payoff $\max(S(t + T) - A(t + T), 0)$. The arithmetic average of the underlying asset prices is calculated on $n + 1$ asset prices attained during the option's life, that is

$$A(t + T) = \frac{1}{n + 1} \sum_{i=0}^{n} S(t + i\Delta t),$$

where $\Delta t = T/n$ and n is the number of time steps used to compute the price. The Hull-White model postulates that the underlying asset price evolves according to the binomial model of Cox et al. (1979). According to this model, the asset price at each time step increases by the factor $u = \exp(\sigma\sqrt{\Delta t})$ if an up step occurs or decreases by the factor $d = 1/u$ if a down step takes place, where σ is the volatility of the underlying asset price. The probability of an up step is the risk-neutral probability $p = (\exp(r\Delta t) - d)/(u - d)$ while the probability of a down step is $q = 1 - p$ and r is the risk-free interest rate. Without loss of generality, we assume $t = 0$ and denote by $S(i, j)$ the underlying asset price at node (i, j) after j up steps and $i - j$ down steps, with $S(0, 0) = S$. In a binomial lattice model, the main problem for pricing an Asian option is the large number of possible payoffs at each node of the tree. In order to solve this problem, Hull and White considered a set of representative averages at each node, (i, j), of the tree. The minimum and the maximum representative average, $A_{\min}(i)$, $A_{\max}(i)$ at time $i\Delta t$, $i = 1, \ldots, n$ are of the form $Se^{\pm mh}$ where h is a fixed parameter and m is the smallest integer that simultaneously satisfies the following inequalities:

$$A_{\min}(i) = Se^{-mh} \leq \frac{1}{i+1}[iA_{\min}(i-1) + dS(i-1,0)], \tag{1}$$

$$A_{\max}(i) = Se^{mh} \geq \frac{1}{i+1}[iA_{\max}(i-1) + uS(i-1,i-1)]. \tag{2}$$

Once the integer m is found, the other representative averages at each node $(i, \cdot)$ of the tree at time $i\Delta t$ are of the form Se^{kh} where k assumes all integers in the interval $(-(m-1), m-1)$. It is worth noting that, following this procedure, the set of representative averages is the same for all the nodes at time $i\Delta t$, $i = 0, \ldots, n$.

Now, let $A(i, j; k)$ denote the k-th representative average at time i after j up steps and $i - j$ down steps. Then, the option price $C(i, j; k)$ associated to $A(i, j; k)$ may be computed via the usual backward induction scheme as

$$C(i, j; k) = e^{-r\Delta t}[pC(i+1, j+1; k_u) + qC(i+1, j; k_d)],$$

where $C(i+1, j+1; k_u)$ and $C(i+1, j; k_d)$ are, respectively, the option values associated to the averages $[(i+1)A(i, j; k) + uS(i, j)]/(i+2)$ and $[(i+1)A(i, j; k) + dS(i, j)]/(i+2)$. The problem is that at time $(i+1)\Delta t$ the representative averages, $A(i+1, j; k)$, are of the form Se^{kh} and, in general, do not coincide with the above values of the averages. To overcome this problem, Hull and White propose to compute $C(i+1, j+1; k_u)$ by using linear interpolation between the option price associated to the smallest representative average greater than $[(i+1)A(i, j; k) + uS(i, j)]/(i+2)$ and the option price associated to the greatest average smaller than $[(i+1)A(i, j; k) + uS(i, j)]/(i+2)$. The same interpolation technique is used to estimate the option price $C(i+1, j; k_d)$. The Asian option price at inception is then computed following this scheme working backward along the tree.

A crucial point in the implementation of the Hull-White model relies on the choice of the value for the parameter h. This choice influences strongly the evaluation process since it determines the number of representative averages to be calculated at each node. The numerical results illustrated by Hull and White show that the option price is very sensitive to the value of the parameter h. Table 1 illustrates the prices of a European Asian call option written on a stock with initial price $S = 50$, strike price $K = 50$, risk-free interest rate (continuously compounded) $r = 0.1$, volatility $\sigma = 0.3$ and time

Table 1 The Hull-White algorithm

	Number of time steps			
h	20	40	60	80
0.1	4.663	4.679	4.685	4.687
0.05	4.588	4.605	4.612	4.614
0.01	4.517	4.530	4.536	4.539
0.005	4.513	4.522	4.526	4.529
0.003	4.512	4.520	4.523	4.525

to maturity $T = 1$ year. As a benchmark, computed through the Monte Carlo method, Hull and White considered the option price equal to 4.515.

From Table 1, it is evident that the prices obtained from the Hull-White model are strongly dependent on the value of the parameter h. Moreover, two effects seem to emerge in the evaluation process. The first one is related to the declining pattern of the option prices when the values of the parameter h decrease. This is due to linear interpolation that overestimates the option values as they are convex functions of the averages. Clearly, since a smaller value of h implies a greater number of representative averages, the effect of linear interpolation decreases when h decreases. However, according to Hull and White the overestimation effect should disappear asymptotically.

The second effect is related to the convergence of the option values computed by the Hull-White algorithm to the true option prices obtained through the Monte Carlo method. The numerical results illustrated in Table 1 show that, for a given h, the option value is an increasing function of the number of time steps, n, used in the algorithm. By increasing the number of time steps, the option value also increases.

The increasing pattern is confirmed by the numerical results illustrated in Table 2, which contains the Asian call option prices computed using the Hull-White model for a larger number of time steps. We observe that as h decreases, the number of representative averages grows rapidly and the computational efficiency of the algorithm diminishes. This is the reason why, in Table 2, we compute option prices with at most 140 time steps. The lack of convergence of the Hull-White model is confirmed by Forsyth et al. (2002) who showed that, in order to achieve convergence when a linear interpolation procedure is applied, the parameter h has to be chosen proportional to Δt. On the contrary, Hull and White implemented their model by using fixed values for the parameter h. This fact could explain the increasing pattern of option values and the lack of convergence of the Hull-White model.

Forsyth et al. (2002) proposed to modify Hull-White model by choosing h equal to

$$\alpha\sqrt{\frac{0.25}{T}\sigma^2\Delta t},$$

where α is a parameter that influences the number of representative averages to be considered at each node. In other words, smaller values for α produce a finer grid and a large number of representative averages. Once the set of representative averages has been chosen, these averages are also applied to all the nodes at a given time-step $i\Delta t$. This point was criticized by Klassen (2001) who proposed an algorithm where the

Table 2 The Hull-White algorithm

h	Number of time steps 100	120	140
0.1	4.689	4.691	4.691
0.05	4.616	4.617	4.618
0.01	4.541	4.542	4.543
0.005	4.530	4.532	4.533
0.003	4.527	4.528	4.528

representative averages at node (i, j) are of the form

$$A(i, j; k) = A_{\min}(i, j)e^{kh}, \quad k = 0, \ldots, k_{\max},$$

where

$$A_{\min}(i, j) = \frac{1}{i+1}\left(\sum_{h=0}^{i-j} Sd^h + \sum_{h=0}^{j-1} Sd^{i-2j+h}\right) \tag{3}$$

is the smallest possible average at node (i, j), $k_{\max}$ is the smallest integer such that $A(i, j; k_{\max}) \geq A_{\max}(i, j)$, with $A_{\max}(i, j)$ the greatest possible average at node (i, j) given by

$$A_{\max}(i, j) = \frac{1}{i+1}\left(\sum_{h=0}^{j} Su^h + \sum_{h=0}^{i-j-1} Su^{h+2j-i}\right). \tag{4}$$

The rationale for the choice of these representative averages is to consider values closer to the true averages than the values used in the Hull-White algorithm at node (i, j). Moreover, Klassen proposed that the value h can be chosen as

$$h = h(N) = \frac{h_0}{1 + N/b},$$

where $b = 100$, $h_0 = \alpha\sigma\sqrt{T}$ and N is the number of time steps used for the option price computations. Again α is a parameter that influences the number of representative averages at each node. A greater number of representative averages correspond to smaller value of α. Finally, Klassen used Richardson extrapolation techniques to speed up convergence.

Both the models proposed by Forsyth et al. and Klassen produce accurate prices but, again, no fixed rule is specified on how to choose the extra parameter, α, needed to compute the representative averages.

A different approach has been proposed by Chalasani et al. (1998, 1999). They developed a pricing algorithm based on the "partition of each node (i, j) of the lattice into "nodelets", each of which represents binomial tree paths that reach (i, j) and have the same geometric stock price from time 0 to time i" and then they derived upper and lower bound for the option value on the refined lattice so defined. The evaluation procedure is similar to that one depicted in the Hull-White model (see Chalasani et al. (1998, 1999) for the details).

3. An adjusted binomial model to evaluate Asian options

In this section, we illustrate a new model for computing the price of Asian options written on the arithmetic average of the underlying asset prices. The framework is the same described in the previous section and is based on the binomial model of Cox, Ross and Rubinstein for the evolution of the underlying asset price. The key difference of the binomial approach proposed here with respect to the other approaches mentioned in Section 2 lies on the technique of choosing the representative averages used to compute the option price. We already pointed out that the arithmetic average of the underlying asset prices is not recombining. This means that, in general, in a binomial lattice, each trajectory of the asset price generates a different arithmetic average and, as a consequence, the total number of possible averages grows exponentially when the number of time steps used for price computations increases. This explains why the pricing algorithms mentioned in the previous section work on a set of values representing the set of averages associated to each node of the tree. The essential feature of our approach involves choosing a subset of true averages that we still call representative averages.

In order to give a clear description of how the model works, we consider the node (i, j) of the tree reached after j up steps and $i - j$ down steps. The set of the representative averages is obtained as follows. First, we compute the maximum average, $A_{\max}(i, j)$, associated with the node (i, j) that is produced by the trajectory $\tau_{\max}(i, j)$, i.e., the path with j up steps followed by $i - j$ down steps. The first average that we compute is $A_{\max}(i, j)$ and we shall denote it by $A(i, j; 1)$ being the first element in the set of the representative averages for the node (i, j). The last element of the set is the minimum average, $A_{\min}(i, j)$, that is produced by the trajectory $\tau_{\min}(i, j)$, i.e., the path with $i - j$ down steps followed by j up steps. The other representative averages for the node (i, j), labeled as $A(i, j; k)$, $k = 1, \ldots, j(i - j)$, are computed recursively in the following way. If $S_{\max}(i, j; k)$ is the greatest value of the underlying asset price, not belonging to $\tau_{\min}(i, j)$, in the trajectory that produces the average $A(i, j; k)$, then,

$$A(i, j; k + 1) = A(i, j; k) - \frac{1}{i + 1}[S_{\max}(i, j; k) - S_{\max}(i, j; k)d^2].$$

In other words, the $(k + 1)$-th representative path is obtained from the previous one by simply substituting $S_{\max}(i, j; k)$, the highest value reached in the k-th path, not belonging to $\tau_{\min}(i, j)$, with the value $S_{\max}(i, j; k)d^2$. This procedure continues until the last trajectory, $\tau_{\min}(i, j)$, is reached. The trajectory $\tau_{\min}(i, j)$ produces the minimum average, $A_{\min}(i, j)$, among those associated to the node (i, j). Clearly, starting from $\tau_{\max}(i, j)$, we reach $\tau_{\min}(i, j)$ after $j(i - j)$ substitutions, so that at the (i, j)-th node, we associate a set of representative averages made up of $1 + j(i - j)$ elements. The values of the maximum and minimum averages corresponding to the node (i, j), i.e., the first and the last element in the set of representative averages for the node (i, j), are, as in Section 2, given by,

$$A(i, j; 1) = A_{\max}(i, j) = \frac{1}{i + 1}\left(\sum_{h=0}^{j} Su^h + \sum_{h=0}^{i-j-1} Su^{h+2j-i}\right),$$

$$A(i, j; 1 + j(i - j)) = A_{\min}(i, j) = \frac{1}{i + 1}\left(\sum_{h=0}^{i-j} Sd^h + \sum_{h=0}^{j-1} Sd^{i-2j+h}\right).$$

The following example should further clarify how the algorithm works.

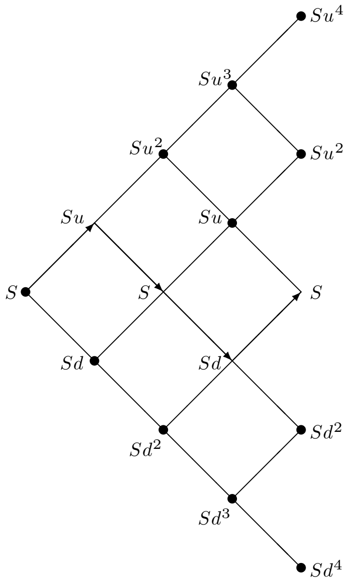

Fig. 1 The representative averages

Figure 1 illustrates a binomial tree that describes the evolution of the underlying asset price. The proposed algorithm considers the sets of representative averages defined as follows. Consider the trajectory that reaches the node (4, 4). Since there is only one trajectory involved, there is only one average considered and it is computed using the values $(S, Su, Su^2, Su^3, Su^4)$. For the node (4, 2), the first average of the set, $A(4, 2; 1) = A_{\max}(4, 2)$, is computed by using the values (S, Su, Su^2, Su, S). The highest value used to compute $A(4, 2; 1)$ is $S_{\max}(4, 2; 1) = Su^2$. The second average is computed by considering the path obtained through substitution of $S_{\max}(4, 2; 1)$ with $S_{\max}(4, 2; 1)d^2 = S$. Hence $A(4, 2; 2)$ is computed using the vector (S, Su, S, Su, S). The last three averages are computed using the vectors (S, Sd, S, Su, S), (S, Sd, S, Sd, S) and (S, Sd, Sd^2, Sd, S). Continuing in this manner, the set of representative averages associated with the node (4, 2) contains all the true averages but the one generated by the vector (S, Su, S, Sd, S) (the corresponding path is depicted in Fig. 1 through lines marked with terminal arrows). It is worth mentioning that there may exist some cases such that the average $A(i, j; k)$ is produced by a path in which $S_{\max}(i, j; k)$ is reached more than one time. In such cases, the next representative average $A(i, j; k + 1)$ is computed by substituting only the first value $S_{\max}(i, j; k)$ reached by the trajectory with the value $S_{\max}(i, j; k)d^2$. In the same way, the set of representative averages is constructed for the other nodes of the tree.[1]

[1] The sets of representative averages that we propose is generated starting from the greatest average, $A_{\max}(i, j)$, and ending to the smallest one, $A_{\min}(i, j)$, associated to the node (i, j). The same sets of representative averages could be generated in a symmetrical way simply by starting from the average associated to the minimum path, $\tau_{\min}(i, j)$, and ending to that associated to the highest one, $\tau_{\max}(i, j)$.

Table 3 The adjusted binomial model for European Asian options

n	$K = 40$	$K = 50$	$K = 60$
10	11.5276	4.5014	1.1176
15	11.5348	4.5082	1.1428
20	11.5384	4.5126	1.1548
30	11.5413	4.5165	1.167
40	11.5302	4.509	1.1665
50	11.5449	4.5209	1.1778
60	11.5458	4.522	1.1805
70	11.5463	4.5228	1.1824
80	11.5467	4.5233	1.1838
90	11.547	4.5237	1.1849
MC	11.544	4.515	1.185

Once the sets of representative averages are obtained, the option price is computed using the backward induction scheme described in the previous section, i.e.,

$$C(i, j; k) = e^{-r\Delta t}[pC(i+1, j+1; k_u) + qC(i+1, j; k_d)].$$

We emphasize that in many cases we do not need to compute the option values $C(i+1, j; k_d)$ and $C(i+1, j+1; k_u)$ via linear interpolation. In fact, they are associated, respectively, to the averages $[(i+1)A(i, j; k) + dS(i, j)]/(i+2)$ and $[(i+1)A(i, j; k) + uS(i, j)]/(i+2)$ that could be in the sets of representative averages $A(i+1, j; k)$ and $A(i+1, j+1; k)$. In other cases, $C(i+1, j; k_d)$ and $C(i+1, j+1; k_u)$ are computed by using linear interpolation as in the Hull-White model.

When the early exercise is allowed, i.e., in the case of an American Asian call option, the option price at the node (i, j, k) is given by

$$C(i, j; k) = \max\{e^{-r\Delta t}[pC(i+1, j+1; k_u) + qC(i+1, j; k_d)], A(i, j; k) - K\}.$$

We present now the numerical results of the adjusted binomial algorithm. In Table 3, we illustrate the price of European Asian call options for different strike price values. The underlying asset price at inception is $S = 50$, the time to maturity is $T = 1$ year, the risk-free interest rate is $r = 0.1$ (continuously compounded) and the volatility is $\sigma = 0.3$. In the last row of Table 3, we report the option values computed by Hull and White through the Monte Carlo method (MC).

In Table 4, we compare the numerical results of the model proposed here with those computed by Forsyth et al. (FVZ) (2002) reported in the last row. Both options are written on an asset with price $S = 100$, with strike price $K = 100$ and risk-free interest rate $r = 0.1$. Moreover, the first option has a volatility of $\sigma = 0.1$ and time to maturity $T = 0.25$ years while the second one has volatility $\sigma = 0.5$ and time to maturity $T = 5$ years.

Tables 5 and 6 illustrate the performance of the adjusted binomial algorithm for American Asian call options. In Table 5, we compare different option values computed through the adjusted binomial algorithm with those computed by using the re-

Table 4 The adjusted binomial model for European Asian options

	$\sigma = 0.1$	$\sigma = 0.5$
n	$T = 0.25$	$T = 5$
10	1.8381	28.3491
15	1.8418	28.3687
20	1.8442	28.2439
30	1.8466	28.3478
40	1.8475	28.3866
50	1.8485	28.3899
60	1.849	28.392
70	1.8492	26.8899
80	1.8497	28.3934
90	1.8499	28.3875
FVZ	1.8509	28.4003

fined binomial model of Chalasani et al. (1999). The option parameters are $S = 100$, $T = 1$, $\sigma = 0.3$ and $r = 0.1$. In the last row of Table 5, we report a lower bound (LB) and an upper bound (UB) for the option values computed by Chalasani et al. with $n = 40$ time steps. We observe that the prices computed by the adjusted binomial algorithm are very close to those computed through the Chalasani et al. algorithm but slightly smaller. This is beacuse the adjusted binomial model considers only "true averages" and, consequently, no interpolation is required in many cases for computing the option price and the overestimation effect due to linear interpolation is reduced.

In Table 6, we report the prices of American Asian options computed by the adjusted binomial model compared with those obtained by Zvan et al. (ZFV) (1997/98) and by Klassen (K) (2001). Zvan et al. used a numerical scheme based on the Crank-Nicolson model with flux limiter while Klassen implemented a binomial model with $n = 512$ time steps coupled with Richardson extrapolation. The numerical results contained in Table 6, show a slower convergence of the adjusted binomial model for pricing American Asian options with respect to its European counterpart. Indeed, as it is

Table 5 The adjusted binomial model for American Asian options

n	$K = 40$	$K = 45$	$K = 50$	$K = 55$	$K = 60$
10	12.6824	8.1766	4.7097	2.4391	1.1279
20	12.9562	8.3949	4.8134	2.4960	1.1772
30	13.0756	8.4312	4.8619	2.5194	1.1951
40	13.1416	8.5375	4.8793	2.5237	1.1975
50	13.1986	8.5844	4.9053	2.5411	1.2110
60	13.2347	8.6114	4.9175	2.5470	1.2152
70	13.2612	8.6322	4.9264	2.5513	1.2181
80	13.2820	8.6490	4.9334	2.5545	1.2203
	LB = 13.150	LB = 8.546	LB = 4.888	LB = 2.532	LB = 1.204
	UB 13.151	UB = 8.547	UB = 4.889	UB = 2.534	UB = 1.206

Table 6 The adjusted binomial model for American Asian options

	$S = 100$ $r = 0.1$ $\sigma = 0.2$ $T = 0.25$			$S = 100$ $r = 0.1$ $\sigma = 0.4$ $T = 1$		
n	$K = 95$	$K = 100$	$K = 105$	$K = 95$	$K = 100$	$K = 105$
10	6.9386	3.0378	0.9169	14.6269	11.6977	9.2388
20	7.1302	3.1035	0.9478	14.6204	11.5531	9.0188
30	7.2144	3.1343	0.9599	15.2235	12.1183	9.5461
40	7.2626	3.151	0.9664	15.2471	12.1449	9.5527
50	7.2951	3.162	0.9705	15.4162	12.2652	9.6597
60	7.318	3.1697	0.9732	15.4655	12.3000	9.6847
70	7.3347	3.1754	0.9753	15.4976	12.3206	9.6979
80	7.3484	3.1799	0.9768	15.5317	12.3461	9.7176
ZFV	7.521	3.224	1.009	15.749	12.497	9.825
K	7.4660	3.2159	0.9882	15.7747	12.5094	9.8305

shown in Tables 2 and 3 for $n = 80$, the margin of error of the prices computed by the adjusted binomial model is negligible while, in the American Asian option case, a large number of time steps is required to achieve convergence to the values used for comparison.

The computational cost of the algorithm presented here is easily determined once we know the total number of representative averages used in the binomial tree valuation for Asian options. For each node, we consider the highest path reaching that node. The number of all other paths is equal to the number of nodes touched by any path reaching that node, excluding the nodes located on the lowest path. Hence, for the node (i, j), we consider $1 + j(i - j)$ representative averages. Then, the number of representative averages considered at the i-th time step is

$$\sum_{j=0}^{i}(1 + j(i - j)) = 1 + \frac{i^3 + 5i}{6}.$$

The total number of representative averages in the binomial tree with n time steps is

$$\sum_{i=0}^{n}\left(1 + \frac{i^3 + 5i}{6}\right) = 1 + \frac{n^4 + 2n^3 + 11n^2 + 34n}{24}.$$

Since the backward induction procedure to compute the option price considers a number of option values equals to the number of representative averages, we can conclude that the number of mathematical operations needed to evaluate an Asian option is proportional to n^4.

4. Conclusions

We proposed a new algorithm for pricing Asian options. The algorithm is developed in the framework of the Cox-Ross-Rubinstein binomial tree model that describes the evolution of the underlying asset price. The key difference of our contribution from the other models based on binomial trees lies on the way we choose the representative averages used to compute the option price. Instead of considering sets of simulated averages, we propose to use sets of representative averages selected among the true averages associated at each node of the tree. We test the proposed method using fixed strike European and American Asian calls but, of course, it is straightforward extendable to fixed strike Asian put options and floating strike Asian calls and puts both of European and American type. The algorithm presented here is very easy to implement and, as confirmed by the numerical results, it produces accurate prices compared to the other existing pricing models.

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