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PRICING ASIAN OPTIONS IN AFFINE GARCH MODELS

MERCURI LORENZO

Dipartimento di Metodi Quantitativi
per le Scienze Economiche ed Aziendali
University of Milano-Bicocca
P.zza Ateneo Nuovo 1, 20126 Milano
lorenzo.mercuri@unimib.it

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We derive recursive relationships for the m.g.f. of the geometric average of the underlying within some affine Garch models [Heston and Nandi (2000), Christoffersen *et al.* (2006), Bellini and Mercuri (2007), Mercuri (2008)] and use them for the semi-analytical valuation of geometric Asian options. Similar relationships are obtained for low order moments of the arithmetic average, that are used for an approximated valuation of arithmetic Asian options based on truncated Edgeworth expansions, following the approach of Turnbull and Wakeman (1991). In both cases the accuracy of the semi-analytical procedure is assessed by means of a comparison with Monte Carlo prices. The results are quite good in the geometric case, while in the arithmetic case the proposed methodology seems to work well only in the Heston and Nandi case.

Keywords: Asian options; affine Garch models; Fourier transform; semi-analytical valuation; Edgeworth series.

1. Introduction

The aim of this paper is to price a discretely monitored Asian option when the underlying asset follows an affine Garch process. Asian options are options in which the underlying variable is an average (geometric or arithmetic); they are quite popular among derivative traders and risk managers for several reasons. First of all, Asian options smoothen possible market instabilities occurring near the expiry date. Moreover, these options often provide a suitable hedge for firms — this can be the case, for instance, of commodity end-users which are financially exposed to average prices.

Several approaches have been proposed for pricing Asian options (see Fusai and Roncoroni [12] for a recent review and numerical comparisons). These approaches can be broadly classified into three categories: semi-analytical, approximation and Monte Carlo.

The first approach, proposed by Carverhill and Clewlow [9] under a geometric Brownian motion assumption, is based on the Fourier transform. These authors obtain a recursive procedure for the computation of the density of the arithmetic average of the underlying. Benhamou [5] applies the same approach to some non-lognormal densities, e.g. Student t. Recently, for arithmetic Asian options under Lévy processes, Fusai and Meucci [11] solve the valuation problem by recursive integration and derive a recursive theoretical formula for the moments. Moreover, for the geometric Asian option, these authors provide a closed form formula in terms of a Fourier transform.

The second approach is based on approximating the true distribution of the average with a more tractable one that matches some low order moments. Following this idea, Lévy [18] proposed to approximate an arithmetic average of lognormals with a lognormal while Turnbull and Wakeman [24] proposed to use an Edgeworth series approximation. In this way, the authors could capture the skewness and kurtosis present in the log-returns, retaining the lognormal approximation as a special case. Recently Albrecher [1] explored these approaches under more general Lévy processes (see Albrecher and Predota [2]). The main drawback of the approximation method is that, in general, it is quite difficult to evaluate the approximation error. Furthermore the Edgeworth series expansion may have negative values (see Barton and Dennis [3] for conditions under which the Edgeworth is a positive and unimodal function and Ju [16] for a discussion of this problem in Asian option pricing).

The last approach combines the Monte Carlo simulation with the analytical formula for geometric Asian options. Indeed, in order to price an arithmetic Asian option, it is possible to increase the accuracy of the Monte Carlo simulation using the geometric Asian option price as control variate, as shown by Kemna and Vorst [17]. In the continuous time case Milewsky and Posner [20] proved that the stationary density of the arithmetic average of a geometric brownian motion is a reciprocal gamma. Other approaches are based on the numerical solutions of the pricing equation see for example Rogers and Shi [21] that also provided an explicit determination of upper and lower bounds.

From an empirical point of view, it is well known that the Black-Scholes model is not able to capture some "stylized facts" such as skewness and heavy tails in the distribution of log-returns. For this reason, Lévy processes have been introduced in finance (see among others Geman [13], Schoutens [22], Carr *et al.* [8], for an introduction to Lévy processes in finance). Although these processes are able to account for the skewness and the excess kurtosis, they are still based on an I.I.D. assumption that does not capture the dependence structure observed in real financial data.

The most common discrete time models of non I.I.D. data are Garch-like models, that are indeed very popular in Finance. For option pricing purposes, a very suitable class is constituted by the so-called affine Garch models, that yield the possibility of semi-analytical computation of option prices, since it is possible to compute the characteristic function of log-prices by a recursive procedure (see Heston and Nandi [14] for the normal case, Christoffersen *et al.* [10] for Inverse Gaussian innovations, Bellini and Mercuri [4] for Gamma innovations and Mercuri [19] for Tempered Stable innovations).

The contribution of this paper is two-fold. First, we provide a semi-analytical formula for geometric Asian options when the underlying follows an affine Garch process of the above mentioned types. Second we discuss the case of the arithmetic Asian option and provide a recursive procedure for the computation of the moments of the arithmetic average that will be the basis of an approximated procedure in the same spirit of Turnbull and Wakeman [24].

The paper is organized as follows: in Sec. 2 we quickly review some common affine Garch models. In Sec. 3 we obtain the valuation formula for a geometric Asian option and we check the accuracy of the proposed procedure by a comparison with Monte Carlo prices. In the last section we present the approximation method for arithmetic Asian options.

2. Affine Garch Models

The aim of this section is to review some of the existing affine Garch models. The main feature of these models is that they yield a semi-analytical formula for option prices in terms of Fourier transform. The models are written in an exponential form

$$S_t = S_{t-1} \exp(X_t) \tag{2.1}$$

where the log-returns X_t follow an affine Garch process.

We write the dynamics directly under the risk neutral measure. These models may also be introduced under the objective measure, and the change of measure may be performed by means of the conditional Esscher transform, as discussed in Siu *et al.* [23] and Bellini and Mercuri [4].

The first model was proposed by Heston and Nandi (HN henceforth) in which the dynamics of X_t under the martingale measure is given by:

$$\begin{cases} X_t = r - \dfrac{1}{2}h_t + \sqrt{h_t}Z_t \\ h_t = \alpha_0 + \alpha_1(Z_{t-1} - \lambda\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1} \end{cases} \tag{2.2}$$

where Z_t are i.i.d. standard normal.

Christoffersen *et al.* [10] pointed out that this model seems to be not sufficiently flexible, particularly if we consider options with short maturities.

These authors suggested that normal innovations are not able to capture the conditional skewness and the conditional kurtosis of the log-returns. For this reason, they proposed an affine Garch model with Inverse Gaussian innovations (CHJ model henceforth), where the risk-neutral dynamics of log-returns is given by:

$$\begin{cases} X_t = r + \lambda h_t - \varepsilon_t \\ \varepsilon_t | F_{t-1} = \eta Y_t \\ Y_t \sim IG(\delta_t) \quad \text{with } \delta_t = \dfrac{h_t}{\eta^2} \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1} + \dfrac{\gamma h_{t-1}^2}{\varepsilon_{t-1}}. \end{cases} \tag{2.3}$$

With similar motivations, Bellini and Mercuri [4] suggested a model with Gamma innovations (BM model):

$$\begin{cases} X_t = r + \lambda h_t - \varepsilon_t \\ \varepsilon_t | F_{t-1} = \dfrac{b}{\sqrt{a}} Y_t \\ Y_t \sim Ga(a h_t, b) \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1}. \end{cases} \tag{2.4}$$

Finally, Mercuri [19] proposed an affine Garch model with Tempered Stable innovations (M model) that encompasses both the CHJ and the BM models as special cases:

$$\begin{cases} X_t = r + \lambda h_t - \varepsilon_t \\ \varepsilon_t | F_{t-1} = \dfrac{Y_t}{2\sqrt{\alpha a (1-\alpha)(b)^{(\alpha-2)/\alpha}}} \\ Y_t \sim TS(\alpha, a h_t, b) \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1}. \end{cases} \tag{2.5}$$

We recall that Y_t has a $\mathrm{TS}(\alpha, a, b)$ distribution with $\alpha \; \epsilon \; (0,1)$, $a > 0$ and $b \geq 0$, if its density function is of the form

$$p(x; \alpha, a, b) = \exp(ab) s\left(x; \alpha, 1, \frac{a}{2^\alpha \sec(\alpha \frac{\pi}{2})}, 0\right) \exp\left(-\frac{1}{2} b^{1/\alpha} x\right) \tag{2.6}$$

where $s(x; \alpha, \beta, c, \delta)$ is a positively skewed α-stable density with $\alpha \; \epsilon \; (0,1)$.

3. Geometric Asian Options

In this section we provide a semi-analytic procedure for the pricing of a geometric Asian option where the underlying is observed at equally-spaced times (for notational simplicity we will consider unitary steps).

The payoff of the geometric Asian call option with fixed strike K is given by:

$$C(K, T) = \max\{G_T - K, 0\} \tag{3.1}$$

where

$$G_T = \left(\prod_{t=0}^{T} S_t\right)^{\frac{1}{T+1}}. \tag{3.2}$$

In order to price this option, we have to evaluate the expected payoff under the martingale measure:

$$C(K,t) = e^{-r(T-t)} E_t^Q[\max\{G_T - K, 0\}]. \tag{3.3}$$

By defining

$$Y_T := \ln(G_T) \tag{3.4}$$

we can rewrite the expected value in (3.3) as:

$$C(e^k, t) = e^{-r(T-t)} \int_k^{+\infty} (e^{Y_T} - e^k) dF(Y_T) \tag{3.5}$$

where $k = \ln(K)$.

In a general Garch setup this integral cannot be computed analytically since the distribution of Y_T is unknown. In the next section we show how to compute in a recursive fashion the m.g.f. of the variable Y_T.

3.1. *Recursive computation of the m.g.f. of average log-returns* $\mathbf{Y_T}$

We write the m.g.f. of Y_T in exponential form

$$\begin{aligned}\varphi_t(u) &= E_t[\exp(uY_T)] \\ &= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + A(t:T,u) + B(t:T,u)h_{t+1} + C(t:T,u)\ln(S_t)\right]\end{aligned} \tag{3.6}$$

with time-dependent coefficients $A(t:T,u)$, $B(t:T,u)$ and $C(t:T,u)$.

From the iteration property of conditional expectations we have that:

$$\begin{aligned}\varphi_t(u) = {} & \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right] \\ & * \exp[C(t+1:T,u)r + A(t+1:T,u) + \alpha_0 B(t+1:T,u)] \\ & * \exp\left\{-\frac{1}{2}\ln(1-2\alpha_1 B(t+1:T,u)) + \left[C(t+1:T,u)\left(\lambda - \frac{1}{2}\right)\right.\right. \\ & \left.\left. - \frac{\lambda^2}{2} + \beta_1 B(t+1:T,u) + \frac{\frac{1}{2}(C(t+1:T,u)-\lambda)^2}{1-2\alpha_1 B(t+1:T,u)}\right] h_{t+1}\right\}\end{aligned} \tag{3.7}$$

substituting and equating terms of the same order in log-price and volatility we get the following recursive system for the coefficients:

$$\begin{cases} A(t:T,u) = C(t+1:T,u)r + A(t+1:T,u) + \alpha_0 B(t+1:T,u) \\ \qquad\qquad - \dfrac{1}{2}\ln(1-2\alpha_1 B(t+1:T,u)) \\ B(t:T,u) = C(t+1:T,u)\left(\lambda - \dfrac{1}{2} - \dfrac{\lambda^2}{2} + \beta_1 B(t+1:T,u)\right) \\ \qquad\qquad + \dfrac{\frac{1}{2}(C(t+1:T,u)-\lambda)^2}{1-2\alpha_1 B(t+1:T,u)} \\ C(t:T,u) = \dfrac{u}{T+1} + C(t+1:T,u). \end{cases} \tag{3.8}$$

The same approach can be pursued in other affine Garch models. In the CHJ model, we obtain the following recursive system:

$$\begin{cases} A(t:T,u) = -\dfrac{1}{2}\ln[1-2\gamma\eta^4 B(t+1,T,u)] + C(t+1:T,u)r \\ \qquad\qquad + A(t+1,T,u) + \alpha_0 B(t+1,T,u) \\ B(t:T,u) = \beta_1 B(t+1,T,u) + C(t+1:T,u)\lambda + \dfrac{1}{\eta^2} \\ \qquad\qquad - \dfrac{1}{\eta^2}\sqrt{\begin{array}{l}[1-2\gamma\eta^4 B(t+1,T,u)][1-2C(t+1:T,u)\eta \\ -2\alpha_1 B(t+1,T,u)]\end{array}} \\ C(t:T,u) = \dfrac{u}{T+1} + C(t+1:T,u). \end{cases} \tag{3.9}$$

In BM model, we have:

$$\begin{cases} A(t:T,u) = C(t+1:T,u)r + A(t+1:T,u) + B(t+1:T,u)\omega \\ B(t:T,u) = \beta B(t+1:T,u) + C(t+1:T,u)\lambda \\ \qquad\qquad - a\log\left(1 + \dfrac{C(t+1:T,u)}{\sqrt{a}} - \dfrac{\alpha_1}{\sqrt{a}}B(t+1:T,u)\right) \\ C(t:T,u) = \dfrac{u}{T+1} + C(t+1:T,u) \end{cases} \tag{3.10}$$

and in M model we get:

$$\begin{cases} A(t,T,u) = C(t+1:T,u)r + A(t+1,T,u) + \alpha_0 B(t+1,T,u) \\ B(t,T,\phi) = C(t+1:T,u)\lambda + \beta_1 B(t+1,T,u) \\ \qquad\qquad + ab\left[1 - \left(1 - \dfrac{(\alpha_1 B(t+1,T,u) - C(t+1:T,u))}{\sqrt{\alpha ba(1-\alpha)}}\right)^{\alpha}\right] \\ C(t:T,u) = \dfrac{u}{T+1} + C(t+1:T,u) \end{cases} \tag{3.11}$$

see the Appendix for the computations.

Remark 3.1. The BM model can be obtained as a special case of the M model by setting $b = 1, a = \frac{a_1}{\alpha}$ and computing the limit for $\alpha \to 0^+$. The case of CHJ model can be recovered by imposing $\alpha = \frac{1}{2}, a = \frac{1}{\eta^2}, b = 1$ and considering the following variance dynamics

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1} + \beta_1 h_{t-1}. \tag{3.12}$$

where ε_t is conditionally distributed as $\mathrm{IG}(\frac{h_t}{\eta^2})$.

In all cases the recursive relations have to be numerically implemented with the terminal conditions:

$$\begin{cases} A(T:T,u) = 0 \\ B(T:T,u) = 0 \\ C(T:T,u) = \dfrac{u}{T+1}. \end{cases} \tag{3.13}$$

3.2. *Option pricing*

In order to price a geometric Asian call option, we follow the approach proposed in Carr and Madan [8].

Denoting with $C(e^k, t)$ the call price a time t, we define

$$c_\delta(e^k, t) := e^{\delta k} C(e^k, t) \tag{3.14}$$

where $\delta > 0$ is a dumping parameter usually chosen between 1.5 and 2.

The Fourier transform of $c_\delta(e^k, t)$ is given by

$$\begin{aligned} \mathcal{F}[c_\delta(e^k, t)](\gamma) &= \int_{-\infty}^{+\infty} e^{i\gamma k} c_\delta(e^k, t) dk \\ &= \frac{e^{-r(T-t)} \varphi_t(\gamma - \delta i - i)}{\delta^2 + \delta - \gamma^2 + i(2\delta + 1)\gamma} \end{aligned} \tag{3.15}$$

where $\varphi_t(\cdot)$ is the characteristic function of the underlying.

The option price may then be obtained by Fourier inversion:

$$C(e^k, t) = \frac{e^{-\delta k}}{\pi} \int_0^{+\infty} e^{i\gamma k} \mathcal{F}[c_\delta(e^k, t)](\gamma) d\gamma. \tag{3.16}$$

In order to assess the feasibility of our approach, we compare the semi-analytical option prices with Monte Carlo prices with $\mathrm{N} = 100000$ simulations. The confidence intervals for Monte Carlo prices are obtained as in Boyle [6], by means of a normal approximation. The parameters chosen for the comparison are from Mercuri [19]; we report the results in Tables 1–4.

We see in general a good accordance between semi-analytical and Monte Carlo prices; the former are outside 95% confidence intervals only in 3 cases.

Table 1. Comparison between Monte Carlo and semi-analytical formula for geometric Asian options in the HN model with parameters: $\alpha_0 = 4.23 * 10^{-5}, \alpha_1 = 2.8 * 10^{-5}, \beta_1 = 4.86 * 10^{-1}, \lambda = 4.67$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Fourier Transform
1 m	0.8	0.20035	0.20017	0.20053	0.20037
	0.85	0.15064	0.15046	0.15082	0.15047
	0.9	0.10036	0.10038	0.10074	0.10058
	0.95	0.05118	0.05101	0.05136	0.05111
	1	0.01196	0.01185	0.01207	0.01194
	1.05	$6.12 * 10^{-4}$	$5.89 * 10^{-4}$	$6.35 * 10^{-4}$	$6.21 * 10^{-4}$
	1.1	$6.36 * 10^{-6}$	$4.20 * 10^{-6}$	$8.52 * 10^{-6}$	$6.17 * 10^{-6}$
	1.15	$8.60 * 10^{-8}$	0	$2.55 * 10^{-7}$	$1.92 * 10^{-9}$
	1.2	0	0	0	0
3 m	0.8	0.20110	0.20079	0.20142	0.20117
	0.85	0.15145	0.15113	0.15176	0.15149
	0.9	0.10204	0.10173	0.10235	0.10211
	0.95	0.05591	0.05563	0.05619	0.05587
	1	0.02152	0.02133	0.02172	0.02153
	1.05	0.00511	0.00501	0.00520	0.00518
	1.1	$7.49 * 10^{-4}$	$7.15 * 10^{-4}$	$7.83 * 10^{-4}$	$7.41 * 10^{-4}$
	1.15	$6.48 * 10^{-5}$	$5.53 * 10^{-5}$	$7.42 * 10^{-5}$	$6.42 * 10^{-5}$
	1.2	$4.34 * 10^{-6}$	$2.37 * 10^{-6}$	$6.31 * 10^{-6}$	$3.56 * 10^{-6}$
1 y	0.8	0.20502	0.20440	0.20564	0.20504
	0.85	0.15758	0.15697	0.15819	0.15773
	0.9	0.11311	0.11254	0.11368	0.11349
	0.95	0.07477	0.07427	0.07527	0.07515
	1	0.04494	0.04453	0.04535	0.04524
	1.05	0.02466	0.02435	0.02497	0.02459
	1.1	0.01205	0.01183	0.01227	0.01205
	1.15	0.00546	0.00531	0.00560	0.00534
	1.2	0.00211	0.00202	0.00220	0.00215

Table 2. Comparison between Monte Carlo and semi-analytical formula for geometric Asian options in the CHJ model with parameters: $\alpha_0 = 17.131 * 10^{-6}, \alpha_1 = 0.515 * 10^{-4}, \beta_1 = 0.017, \lambda = 110.648, \eta = 0.033, \gamma = 0.033$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Fourier Transform
1 m	0.8	0.20055	0.20042	0.20068	0.20047
	0.85	0.15058	0.15045	0.15071	0.15058
	0.9	0.10083	0.10070	0.10095	0.10075
	0.95	0.05145	0.05133	0.05156	0.05145
	1	0.00826	0.00821	0.00831	0.00824
	1.05	$1.24 * 10^{-7}$	0	$2.53 * 10^{-7}$	$-4.38 * 10^{-7}$
	1.1	0	0	0	
3 m	0.8	0.20152	0.20129	0.20175	0.20149
	0.85	0.15196	0.15174	0.15219	0.15185
	0.9	0.10257	0.10235	0.10278	0.10252
	0.95	0.05507	0.05488	0.05526	0.05495
	1	0.01586	0.01575	0.01597	0.01584
	1.05	$3.48 * 10^{-4}$	$3.36 * 10^{-4}$	$3.59 * 10^{-4}$	$3.39 * 10^{-4}$
	1.1	0	0	0	0

Table 2. (*Continued*)

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Fourier Transform
1 y	0.8	0.20596	0.20551	0.20641	0.20612
	0.85	0.15822	0.15778	0.15866	0.15804
	0.9	0.11136	0.11094	0.11177	0.11160
	0.95	0.06952	0.06916	0.06988	0.06931
	1	0.03540	0.03513	0.03567	0.03523
	1.05	0.01304	0.01288	0.01320	0.01313
	1.1	0.00302	0.00295	0.00309	0.00303
	1.15	$3.39 * 10^{-4}$	$3.19 * 10^{-4}$	$3.59 * 10^{-4}$	$3.32 * 10^{-4}$
	1.2	$9.41 * 10^{-6}$	$6.78 * 10^{-6}$	$1.20 * 10^{-5}$	$1.07 * 10^{-5}$

Table 3. Comparison between Monte Carlo and semi-analytical formula for geometric Asian options in the BM model with parameters: $\lambda = 68.220, \alpha_0 = 25.77 * 10^{-6}, \alpha_1 = 67.051 * 10^{-4}, \beta = 0.7341 * 10^{-3}, a = 4721.8$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Fourier Transform
1 m	0.8	0.20053	0.20045	0.20061	0.20055
	0.85	0.15059	0.15051	0.15067	0.15066
	0.9	0.10082	0.10075	0.10090	0.10082
	0.95	0.05118	0.05111	0.05124	0.05120
	1	0.00426	0.00424	0.00428	0.00427
	1.05	0	0	0	0
3 m	0.8	0.20192	0.20178	0.20206	0.20171
	0.85	0.15194	0.15181	0.15208	0.15206
	0.9	0.10245	0.10232	0.10258	0.10254
	0.95	0.05361	0.05349	0.05372	0.05368
	1	0.00952	0.00947	0.00957	0.00954
	1.05	0	0	0	0
1 y	0.8	0.20683	0.20656	0.20711	0.20685
	0.85	0.15812	0.15785	0.15839	0.15828
	0.9	0.11011	0.10985	0.11036	0.11026
	0.95	0.06415	0.06393	0.06437	0.06414
	1	0.02431	0.02431	0.02460	0.02442
	1.05	0.00215	0.00215	0.00222	0.00220
	1.1	0	0	0	0

4. Arithmetic Asian Options

We follow the approximation approach that is based on replacing the true distribution of the arithmetic average with a more tractable function matching some low order moments (for example Fusai and Roncoroni [12]). Albrecher [1] extended this method to exponential Lévy processes. The idea is to compute the moments of the true distribution by means of a recursive procedure, much in the spirit of the preceding section.

Table 4. Comparison between Monte Carlo and semi-analytical formula for geometric Asian options in the M model with parameters: $\lambda = 147.29, \alpha_0 = 3.021 * 10^{-6}, \beta = 0.828, \alpha_1 = 7.946 * 10^{-4}$, $k = 0.33, a = 12411, b = 3.599$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Fourier Transform
1 m	0.8	0.20021	0.20010	0.20031	0.20025
	0.85	0.15036	0.15025	0.15046	0.15036
	0.9	0.10041	0.10031	0.10051	0.10046
	0.95	0.05067	0.05057	0.05077	0.05070
	1	0.00676	0.00676	0.00681	0.00671
	1.05	0	0	0	0
3 m	0.8	0.20048	0.20027	0.20069	0.20044
	0.85	0.15078	0.15057	0.15100	0.15076
	0.9	0.10114	0.10093	0.10135	0.10122
	0.95	0.05307	0.05288	0.05288	0.05313
	1	0.01440	0.01429	0.01429	0.01431
	1.05	$4.20 * 10^{-4}$	$4.05 * 10^{-4}$	$4.05 * 10^{-4}$	$3.89 * 10^{-4}$
	1.1	0	0	0	0
1 y	0.8	0.20087	0.20040	0.20133	0.20101
	0.85	0.15280	0.15234	0.15325	0.15303
	0.9	0.10675	0.10633	0.10718	0.10700
	0.95	0.06582	0.06546	0.06619	0.06574
	1	0.03347	0.03320	0.03374	0.03332
	1.05	0.01291	0.01274	0.01308	0.01280
	1.1	0.00336	0.00328	0.00344	0.00333
	1.2	0	0	0	0

We have to evaluate the following expected values:

$$\mu_n = E_0^Q\left[\left(\frac{1}{T+1}A_T\right)^n\right] \tag{4.1}$$

where $n = 1, \ldots, N$, (usually $N = 4$) and the variable A_T is defined as:

$$A_T := \sum_{t=0}^{T+1} S_t. \tag{4.2}$$

We can rewrite A_T using the log-returns

$$\begin{aligned} A_T &= S_0[1 + \exp(X_1) + \cdots + \exp(X_1 + \cdots + X_T)] \\ &= S_0[1 + \exp(X_1)[\cdots[1 + \exp(X_T)]]] \end{aligned} \tag{4.3}$$

and by a straightforward calculation we have

$$A_T^n = S_0^n \sum_{j_1=0}^{n} \binom{n}{j_1} \exp(j_1 X_1)[\cdots[1 + \exp(X_T)]]^{j_1} \tag{4.4}$$

and with repeated applications of the binomial theorem we get:

$$E_0^Q[A_T^n] = S_0^n \sum_{j_1=0}^{n} \sum_{j_2=0}^{j_1} \cdots \sum_{j_T=0}^{j_{T-1}} \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{T-1}}{j_T} E_0^Q \\ \times [\exp(j_1 X_1 + j_2 X_2 + \cdots + j_T X_T)]. \tag{4.5}$$

In the Heston-Nandi model it is possible to compute recursively the quantity:

$$\varphi_0(j_1, \ldots, j_T) = E_0^Q[\exp(j_1 X_1 + j_2 X_2 + \cdots + j_T X_T)] \tag{4.6}$$

indeed by defining:

$$\varphi_t(j_1, \ldots, j_T) = \exp[j_1 X_1 + j_2 X_2 + \cdots + A(t : T, j_{t+1}, \ldots, j_T) \\ + B(t : T, j_{t+1}, \ldots, j_T) h_{t+1}] \tag{4.7}$$

we have from the iteration property of conditional expectations:

$$\begin{aligned}\varphi_t(j_1, \ldots, j_T) &= E_t^Q[\varphi_{t+1}(j_1, \ldots, j_T)] \\ &= \exp\Bigg[j_1 X_1 + \cdots + j_t X_t + j_{t+1} r + A(t+1 : T, j_{t+2}, \ldots, j_T) \\ &\quad + \alpha_0 B(t+1 : T, j_{t+2}, \ldots, j_T) \\ &\quad - \frac{1}{2}\ln(1 - 2\alpha_1 B(t+1 : T, j_{t+2}, \ldots, j_T))\Bigg] \\ &\quad * \exp\Bigg[\Bigg(\lambda j_{t+1} - \frac{j_{t+1}}{2} - \frac{\lambda^2}{2} + \beta_1 B(t+1 : T, j_{t+2}, \ldots, j_T) \\ &\quad + \frac{\frac{1}{2}(\lambda - j_{t+1})^2}{1 - 2\alpha_1 B(t+1 : T, j_{t+2}, \ldots, j_T)}\Bigg) h_{t+1}\Bigg].\end{aligned} \tag{4.8}$$

Hence we obtain the following recursive system for the coefficients:

$$\begin{cases} A(t : T, , j_{t+1}, \ldots, j_T) \\ \quad = j_{t+1} r + A(t+1 : T, j_{t+2}, \ldots, j_T) + \alpha_0 B(t+1 : T, j_{t+2}, \ldots, j_T) \\ \quad\quad - \dfrac{1}{2}\ln(1 - 2\alpha_1 B(t+1 : T, j_{t+2}, \ldots, j_T)) \\ B(t : T, j_{t+1}, \ldots, j_T) \\ \quad = \lambda j_{t+1} - \dfrac{j_{t+1}}{2} - \dfrac{\lambda^2}{2} + \beta_1 B(t+1 : T, j_{t+2}, \ldots, j_T) \\ \quad\quad + \dfrac{\frac{1}{2}(\lambda - j_{t+1})^2}{1 - 2\alpha_1 B(t+1 : T, j_{t+2}, \ldots, j_T)}. \end{cases} \tag{4.9}$$

As for the recursive determination of the m.g.f. of the geometric average, this approach works also in the other considered affine Garch models.

In CHJ, BM, M model we obtain respectively (4.10), (4.11) and (4.12):

$$\begin{cases} A(t:T, j_{t+2}, \ldots, j_T) \\ \quad = -\dfrac{1}{2}\ln[1 - 2\gamma\eta^4 B(t+1, T, j_{t+2}, \ldots, j_T)] + j_{t+1}r \\ \qquad + A(t+1, T, j_{t+2}, \ldots, j_T) + \alpha_0 B(t+1, T, j_{t+2}, \ldots, j_T) \\ B(t:T, j_{t+2}, \ldots, j_T) \\ \quad = \beta_1 B(t+1, T, j_{t+2}, \ldots, j_T) + j_{t+1}\lambda + \dfrac{1}{\eta^2} - \dfrac{1}{\eta^2} \\ \qquad * \left\{[1 - 2\gamma\eta^4 B(t+1, T, j_{t+2}, \ldots, j_T)]\right. \\ \qquad \left. * [1 - 2j_{t+1}\eta - 2\alpha_1 B(t+1, T, j_{t+2}, \ldots, j_T)]\right\}^{1/2}. \end{cases} \tag{4.10}$$

$$\begin{cases} A(t:T, j_{t+2}, \ldots, j_T) \\ \quad = j_{t+1}r + A(t+1:T, j_{t+2}, \ldots, j_T) + \alpha_0 B(t+1:T, j_{t+2}, \ldots, j_T) \\ B(t:T, j_{t+2}, \ldots, j_T) \\ \quad = \beta B(t+1:T, j_{t+2}, \ldots, j_T) + j_{t+1}\lambda \\ \qquad - a\log\left(1 + \dfrac{j_{t+1}}{\sqrt{a}} - \dfrac{\alpha_1}{\sqrt{a}} B(t+1:T, j_{t+2}, \ldots, j_T)\right). \end{cases} \tag{4.11}$$

$$\begin{cases} A(t, T, j_{t+2}, \ldots, j_T) \\ \quad = j_{t+1}r + A(t+1, T, j_{t+2}, \ldots, j_T) + \alpha_0 B(t+1, T, j_{t+2}, \ldots, j_T) \\ B(t, T, j_{t+2}, \ldots, j_T) \\ \quad = j_{t+1}\lambda + \beta_1 B(t+1, T, j_{t+2}, \ldots, j_T) \\ \qquad + ab\left[1 - \left(1 - \dfrac{(\alpha_1 B(t+1, T, j_{t+2}, \ldots, j_T) - j_{t+1})}{\sqrt{\alpha ba(1-\alpha)}}\right)^{\alpha}\right]. \end{cases} \tag{4.12}$$

Remark 4.1. As in the geometric case, the recursive systems (4.10) and (4.11) can be obtained as special cases of M model by imposing the same condition introduced in Sec. 3.1.

Being able to compute recursively the first moments of the arithmetic average, we price the arithmetic Asian option following the approach of Turnbull and Wakeman [24] that is based on a fourth-order Edgeworth expansion. As remarked also by a referee, these Edgeworth expansions do not converge in general; however their use as a tractable numerical approximation is well established in the option pricing literature from Jarrow and Rudd [15] and Turnbull and Wakeman [24].

Let $f_{\log}(y; m, v^2)$ the lognormal density where the parameters m and v^2 match the mean and the variance of the variable A_T:

$$m = 2\log(E_0^Q[A_T]) - \frac{1}{2}\log(E_0^Q[A_T^2])$$
$$v^2 = \log(E_0^Q[A_T^2]) - 2\log(E_0^Q[A_T]).$$

The fourth-order Edgeworth approximation $f_{edg}(y; m, v^2)$ is given by:

$$f_{edg}(y; m, v^2) = f_{\log}(y; m, v^2) + \sum_{i=1}^{4}(-1)^i \frac{k_i}{i!}\frac{\partial^i f_{\log}(y; m, v^2)}{(\partial y)^i} + e(y) \tag{4.13}$$

where k_i is the difference in the ith cumulant[1] between the exact distribution and lognormal distribution, namely $k_i = \chi_i(f) - \chi_i(f_{\log}(y; m, v^2))$, with

$$\chi_1(f) = E_0^Q[A_T]$$
$$\chi_2(f) = E_0^Q[(A_T - E_0^Q[A_T])^2]$$
$$\chi_3(f) = E_0^Q[(A_T - E_0^Q[A_T])^3]$$
$$\chi_4(f) = E_0^Q[(A_T - E_0^Q[A_T])^4] - 3\chi_2(F).$$

Therefore the approximate Asian option price is given by:

$$\begin{aligned} c_{edg}(K, 0) = e^{-rT}\frac{S_0}{T+1}&\left[e^{m+\frac{v^2}{2}} N\left(\frac{m + v^2 - \log(\frac{nK}{S_0})}{v}\right)\right. \\ &\left. - \frac{(T+1)K}{S_0} N\left(\frac{m - \log(\frac{nK}{S_0})}{v}\right)\right] + e^{-rT}\frac{S_0}{T+1} \\ &* \left[-\frac{k_3}{3!}\frac{\partial f_{\log}(y; m, v^2)}{\partial y} + \frac{k_4}{4!}\frac{\partial^2 f_{\log}(y; m, v^2)}{(\partial y)^2}\right]_{y=\frac{nK}{S_0}}. \end{aligned} \tag{4.14}$$

If we consider the first two cumulants we obtain the approximation proposed by Lévy [18]. As in the case of the geometric Asian option, we compare the approximate arithmetic Asian option prices with Monte Carlo prices with N = 100000 simulations the results for the different models are shown in Tables 5–8.

In comparison with the geometric case, the proposed approximation is quite worse. The best case is the Heston and Nandi model, where we have a good accordance between approximated and Monte Carlo prices; the worst case is the tempered stable model, where there seems to be a sistematic positive bias in the approximated

[1] The cumulants of random variable X with distribution function F are defined by:

$$\chi_i(F) = \left[\frac{\partial^i \ln(E(e^{tX}))}{(\partial t)^i}\right]_{t=0}, \quad i = 1, 2, \ldots$$

Table 5. Comparison between Monte Carlo and approximate formula for arithmetic Asian options in the HN model with parameters: $\alpha_0 = 4.23 * 10^{-5}, \alpha_1 = 2.8 * 10^{-5}, \beta_1 = 4.86 * 10^{-1}, \lambda = 4.67$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Edg. Approx
1 m	0.8	0.20069	0.20051	0.20087	0.20056
	0.85	0.15066	0.15048	0.15084	0.15066
	0.9	0.10087	0.10069	0.10105	0.10077
	0.95	0.05134	0.05117	0.05152	0.05121
	1	0.01174	0.01163	0.01185	0.01166
	1.05	$6.35 * 10^{-4}$	$6.11 * 10^{-4}$	$6.59 * 10^{-4}$	$5.55 * 10^{-4}$
	1.1	$6.74 * 10^{-6}$	$4.70 * 10^{-6}$	$8.78 * 10^{-6}$	$4.39 * 10^{-6}$
	1.15	$1.07 * 10^{-7}$	0	$3.16 * 10^{-7}$	$5.66 * 10^{-9}$
	1.2	0	0	0	0
2 m	0.8	0.20136	0.20110	0.20161	0.20160
	0.85	0.15154	0.15128	0.15180	0.15170
	0.9	0.10192	0.10166	0.10218	0.10186
	0.95	0.05371	0.05347	0.05395	0.05369
	1	0.01745	0.01729	0.01761	0.01732
	1.05	0.00281	0.00275	0.00288	0.00270
	1.1	$2.10 * 10^{-4}$	$1.94 * 10^{-4}$	$2.26 * 10^{-4}$	$1.86 * 10^{-4}$
	1.15	$7.48 * 10^{-6}$	$4.68 * 10^{-6}$	$1.03 * 10^{-5}$	$5.82 * 10^{-6}$
	1.2	$1.46 * 10^{-7}$	0	$3.49 * 10^{-7}$	$8.47 * 10^{-8}$

Table 6. Comparison between Monte Carlo and approximate formula for arithmetic Asian options in CHJ model with parameters: $\alpha_0 = 17.131 * 10^{-6}, \alpha_1 = 0.515 * 10^{-4}, \beta_1 = 0.017, \lambda = 110.648,\ \eta = 0.033,\ \gamma = 0.033$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Edg. Approx
1 m	0.8	0.20059	0.20052	0.20067	0.20056
	0.85	0.15072	0.15064	0.15079	0.15066
	0.9	0.10084	0.10077	0.10092	0.10076
	0.95	0.05129	0.05122	0.05135	0.05089
	1	0.00428	0.00426	0.00429	0.00540
	1.05	0	0	0	0
2 m	0.8	0.20128	0.20117	0.20139	0.20118
	0.85	0.15141	0.15130	0.15152	0.15139
	0.9	0.10190	0.10179	0.10200	0.10159
	0.95	0.05257	0.05248	0.05266	0.05264
	1	0.00719	0.00716	0.00723	0.00199
	1.05	0	0	0	0

prices. In order to investigate the source of this bias, we perform two checks:

(1) We compare the moments of the arithmetic average with corresponding Monte Carlo moments, assessing the accuracy of the iterative procedure.
(2) As suggested by a referee, we compare approximated Edgeworth series with a "true" density obtained by means of Monte Carlo simulation. The results are reported respectively in Table 9 and Fig. 1.

Table 7. Comparison between Monte Carlo and approximate formula for arithmetic Asian options in the BM model with parameters: $\lambda = 68.220, \alpha_0 = 25.77 * 10^{-6}, \alpha_1 = 67.051 * 10^{-4}, \beta = 0.7341 * 10^{-3}, a = 4721.8$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Edg. Approx
1 m	0.8	0.20058	0.20045	0.20071	0.20056
	0.85	0.15077	0.15064	0.15089	0.15066
	0.9	0.10099	0.10087	0.10112	0.10076
	0.95	0.05151	0.05140	0.05163	0.05162
	1	0.00823	0.00818	0.00828	0.00658
	1.05	$5.1 * 10^{-8}$	0	$1.5 * 10^{-7}$	$2.08 * 10^{-4}$
	1.1	0	0	0	0
2 m	0.8	0.20119	0.20101	0.20138	0.20126
	0.85	0.15145	0.15127	0.15164	0.15146
	0.9	0.10182	0.10164	0.10200	0.10173
	0.95	0.05336	0.05321	0.05352	0.05414
	1	0.01253	0.01245	0.01262	0.01150
	1.05	$5.22 * 10^{-6}$	$4.11 * 10^{-6}$	$6.33 * 10^{-6}$	$1.64 * 10^{4}$
	1.1	0	0	0	0

Table 8. Comparison between Monte Carlo and approximate formula for arithmetic Asian options in the M model with parameters: $\lambda = 147.29, \alpha_0 = 3.021 * 10^{-6}, \beta = 0.828, \alpha_1 = 7.946 * 10^{-4}$, $k = 0.33$, $a = 12411, b = 3.599$.

Time	Strike	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Edg. Approx
1 m	0.8	0.20045	0.20035	0.20055	0.20055
	0.85	0.15048	0.15037	0.15058	0.15066
	0.9	0.10051	0.10041	0.10062	0.10076
	0.95	0.05076	0.05066	0.05086	0.05096
	1	0.00692	0.00687	0.00697	0.00660
	1.05	0	0	0	0
2 m	0.8	0.20069	0.20053	0.20085	0.20143
	0.85	0.15070	0.15054	0.15087	0.15164
	0.9	0.10101	0.10084	0.10117	0.10185
	0.95	0.05188	0.05172	0.05203	0.05289
	1	0.01113	0.01104	0.01121	0.01123
	1.05	$1.63 * 10^{-5}$	$1.44 * 10^{-5}$	$1.82 * 10^{-5}$	$5.41 * 10^{-5}$
	1.1	0	0	0	0

We see that the accuracy in computing moments is quite good, while the truncated Edgeworth approximations are close to the actual (simulated) density only in the Heston and Nandi case. In the other cases the strong asymmetry of the innovations slows down the convergence in the central limit theorem, thus worsening the Edgeworth approximation. A possible remedy could be the consideration of more general Edgeworth approximations altough in this case the numerical tractability could be worsened.

Table 9. Comparison between Monte Carlo and semi-analytical moments in the four considered models.

Model	Time	moments	Monte Carlo	Lower Bound (95% CI)	Upper Bound (95% CI)	Recursive Proc.
HN model	2 m	I	1.0022	1.0019	1.0024	1.0020
		II	1.0061	1.0056	1.0066	1.0057
		III	1.0118	1.0110	1.0126	1.0111
		IV	1.0193	1.0182	1.0204	1.0183
BM model	2 m	I	1.0021	1.0020	1.0023	1.0021
		II	1.0052	1.0048	1.0055	1.0051
		III	1.0091	1.0085	1.0096	1.0089
		IV	1.0138	1.0131	1.0145	1.0136
CHJ model	2 m	I	1.0022	1.0021	1.0023	1.0020
		II	1.0047	1.0045	1.0049	1.0044
		III	1.0075	1.0072	1.0078	1.0070
		IV	1.0106	1.0102	1.0110	1.0099
M model	2 m	I	1.0024	1.0022	1.0025	1.0023
		II	1.0055	1.0051	1.0058	1.0052
		III	1.0092	1.0087	1.0097	1.0089
		IV	1.0137	1.0130	1.0143	1.0132

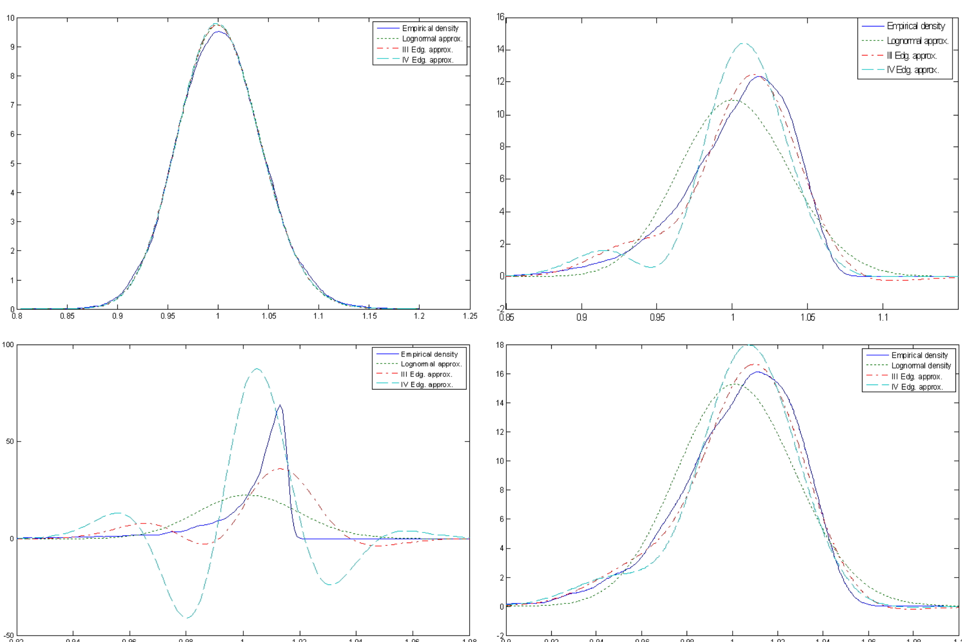

Fig. 1. Comparison between the simulated densities and the truncated Edgeworth series expansions in the four considered models.

Appendix A. Derivation of Conditional m.g.f. in HN Model for Y_T

Given the formula

$$\begin{aligned}\varphi_t(u) &= E_t[\exp(uY_T)]\\ &= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + A(t:T,u) + B(t:T,u)h_{t+1} + C(t:T,u)\ln(S_t)\right],\end{aligned} \tag{A.1}$$

we suppose that the relation (A.1) holds a time $t+1$ and by iteration property of the conditional expected value we compute the conditional m.g.f a time t:

$$\begin{aligned}\varphi_t(u) &= E_t\left[\varphi_{t+1}(u)\right]\\ &= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right]\\ &\quad * \exp[\alpha_0 B(t+1:T,u) + rC(t+1:T,u) + A(t+1:T,u)]\\ &\quad * \exp\left[\left(\beta_1 B(t+1:T,u) - \frac{1}{2}C(t+1:T,u)\right)h_{t+1}\right]\\ &\quad * E_t[\exp[\alpha_1 B(t+1:T,u)(Z_t - \lambda\sqrt{h_t})^2 + C(t+1:T,u)\sqrt{h_t}Z_t]].\end{aligned} \tag{A.2}$$

$$\begin{aligned}\varphi_t(u) &= E_t\left[\varphi_{t+1}(u)\right]\\ &= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right]\\ &\quad * \exp[\alpha_0 B(t+1:T,u) + rC(t+1:T,u) + A(t+1:T,u)]\\ &\quad * \exp\left[(\beta_1 B(t+1:T,u) + \left(\lambda - \frac{1}{2}\right)C(t+1:T,u)\right.\\ &\quad \left. - \frac{C^2(t+1:T,u)}{4\alpha_1 B(t+1:T,u)})h_{t+1}\right]\\ &\quad * E_t\left[\exp\left[\alpha_1 B(t+1:T,u)\left(Z_t - \left(\lambda - \frac{C(t+1:T,u)}{2\alpha_1 B(t+1:T,u)}\right)\sqrt{h_{t+1}}\right)^2\right]\right]\end{aligned} \tag{A.3}$$

using the moment generating function of the non central Chi-square

$$E[\exp(a(z+b)^2)] = \exp\left(-\frac{1}{2}\ln(1-2a) + \frac{ab^2}{1-2a}\right) \tag{A.4}$$

we obtain

$$\begin{aligned}\varphi_t(u) &= E_t[\varphi_{t+1}(u)]\\ &= \exp\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right.\\ &\quad \left. + \alpha_0 B(t+1:T,u)\right]\end{aligned}$$

$$
\begin{aligned}
&* \exp\left[rC(t+1:T,u) + A(t+1:T,u) - \frac{1}{2}\ln(1-2\alpha_1 B(t+1:T,u))\right]\\
&* \exp\left[\left(\beta_1 B(t+1:T,u) + \left(\lambda - \frac{1}{2}\right)C(t+1:T,u) - \frac{\lambda^2}{2}\right.\right.\\
&\left.\left. + \frac{\frac{1}{2}(\lambda - C(t+1:T,u))^2}{1-2\alpha_1 B(t+1:T,u)}\right)h_{t+1}\right]. \qquad \text{(A.5)}
\end{aligned}
$$

Appendix B. Derivation of Conditional m.g.f. in CHJ Model for Y_T

As in HN model, we suppose that the relation (A.1) holds a time $t+1$ and therefore, by iteration property of the conditional expected value, we compute the conditional m.g.f a time t:

$$
\begin{aligned}
\varphi_t(u) = \exp&\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t) + A(t+1,T,u)\right]\\
&* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})]\\
&* E_t\left[\exp\left(\varepsilon_{t+1}(C(t+1:T,u)\eta + \alpha_1 B(t+1,T,u)) + \frac{B(t+1,T,u)\gamma h_{t+1}^2}{\varepsilon_{t+1}}\right)\right]
\end{aligned}
\qquad \text{(B.1)}
$$

the expected value in above formula can be calculated using the generalized moment generating function of Inverse Gaussian with δ degree of freedom:

$$
E\left[\exp\left(\theta y + \frac{\phi}{y}\right)\right] = \frac{\delta}{\sqrt{(\delta^2 - 2\phi)}}\exp(\delta - \sqrt{(\delta^2-2\phi)(1-2\theta)}). \qquad \text{(B.2)}
$$

Defining

$$
I_t(u) := E_t\left[\exp\left(\varepsilon_{t+1}(C(t+1:T,u)\eta + \alpha_1 B(t+1,T,u)) + \frac{B(t+1,T,u)\gamma h_{t+1}^2}{\varepsilon_{t+1}}\right)\right] \qquad \text{(B.3)}
$$

and remembering that $\delta_t = \frac{h_t}{\eta^2}$ we get

$$
I_t(u) = \frac{\exp\left[\frac{h_{t+1}}{\eta^2} - h_{t+1}\sqrt{\begin{array}{l}[\frac{1}{\eta^4} - 2B(t+1,T,u)\gamma][1-2(C(t+1:T,u)\eta\\ \qquad +\alpha_1 B(t+1,T,u))]\end{array}}\right]}{\eta^2\sqrt{\frac{1}{\eta^4} - 2B(t+1,T,u)\gamma}} \qquad \text{(B.4)}
$$

from which the recursive relation can be recovered.

Appendix C. Derivation of Conditional m.g.f. in BM Model for Y_T

Following the same approach proposed in HN model, we obtain for the conditional moment generating function:

$$\begin{aligned}\varphi_t(u) = \exp&\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t) + A(t+1,T,u)\right]\\ &* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})]\\ &* E_t[\exp(\varepsilon_{t+1}(C(t+1:T,u) + \alpha_1 B(t+1,T,u)))]\end{aligned} \tag{C.1}$$

hence, using the moment generating function of Gamma variable, we get

$$\begin{aligned}\varphi_t(u) = \exp&\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t)\right]\\ &* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + A(t+1,T,u)\\ &+ B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})]\left(1 + \frac{\theta + \alpha_1 B(t+1,T,\theta)}{\sqrt{a}}\right)^{-ah_{t+1}}.\end{aligned} \tag{C.2}$$

Appendix D. Derivation of Conditional m.g.f. in M Model for Y_T

For this model, the first step is given by:

$$\begin{aligned}\varphi_t(u) = \exp&\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t) + A(t+1,T,u)\right]\\ &* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})]\\ &* E_t\left[\exp\left(\frac{(\alpha_1 B(t+1,T,u) - C(t+1:T,u))}{2\sqrt{\alpha a(1-\alpha)(b)^{(\alpha-2)/\alpha}}}Z_{t+1}\right)\right]\end{aligned} \tag{D.1}$$

recalling that the moment generating function of Tempered Stable distribution with parameters (α, a, b) is given by:

$$E[\exp(\theta X)] = \exp[ab[1 - (1 - 2\theta b^{-1/\alpha})^{\alpha}]] \tag{D.2}$$

we get

$$\begin{aligned}\varphi_t(u) = \exp&\left[\frac{u}{T+1}\sum_{h=0}^{t-1}\ln(S_h) + \left(\frac{u}{T+1} + C(t+1:T,u)\right)\ln(S_t) + A(t+1,T,u)\right]\\ &* \exp[C(t+1:T,u)(X_t + r + \lambda h_{t+1}) + B(t+1,T,u)(\alpha_0 + \beta_1 h_{t+1})]\\ &* \exp\left[ah_{t+1}b\left[1 - \left(1 - \frac{(\alpha_1 B(t+1,T,u) - C(t+1:T,u))}{\sqrt{\alpha b a(1-\alpha)}}\right)^{\alpha}\right]\right]\end{aligned} \tag{D.3}$$

from which the recursive system for the coeffecients in (A.1).

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