Optimization under Uncertainty using Exponential Cones

CHEN Li

Institute of Operations Research and Analytics
National University of Singapore

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Outline

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- Deterministic optimization

$$\max_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x}, \boldsymbol{z})$$

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    - Example: we do not know whether the coin is unbiased
- How to rank decisions under uncertainty?
  - Decision criteria

Decision-making under uncertainty

- Expected utility theory [VNM47]

$$\max_{\boldsymbol{x} \in \mathcal{X}} \mathbb{E}_{\mathbb{P}} \left[u(f(\boldsymbol{x}, \tilde{\boldsymbol{z}})) \right]$$

where the utility function $u : \mathbb{R} \to \mathbb{R}$ is increasing.

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  - Risk averse: $u(\cdot)$ is concave
- Maxmin expected utility [GS89]

$$\max_{\boldsymbol{x} \in \mathcal{X}} \inf_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[u(f(\boldsymbol{x}, \tilde{\boldsymbol{z}})) \right]$$

- Many others...

Computation challenges

How to solve optimization problems under uncertainty?

- Stochastic Optimization

$$\max_{\boldsymbol{x}\in\mathcal{X}} \mathbb{E}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$$

Fact: evaluating $\mathbb{E}_{\mathbb{P}}\left[\max\{\boldsymbol{a}^\top\tilde{\boldsymbol{z}}-b,0\}\right]$ is #P-hard for given $\boldsymbol{a}\in\mathbb{R}^n_+$, $b\in\mathbb{R}_+$, $\tilde{z}_i$'s are independent uniform on $[0,1]$.

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$$\max_{\boldsymbol{x} \in \mathcal{X}} \mathbb{E}_{\mathbb{P}} \left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \right]$$

  - Random approximation: Sample Average Approximation (SAA)

$$\max_{\boldsymbol{x} \in \mathcal{X}} \frac{1}{S} \sum_{s \in [S]} f(\boldsymbol{x}, \hat{\boldsymbol{z}}^s)$$

    - General: sampling oracle
    - Effective: statistics and optimization perspectives
    - Disadvantage: curse of dimensionality

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    - General: sampling oracle
    - Effective: statistics and optimization perspectives
    - Disadvantage: curse of dimensionality
  - Deterministic approximation: bound $\mathbb{E}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$
    - Ad hoc: utilize special structures of f and $\mathbb{P}$, e.g., moments
    - Advantages: scalability, handle decision-dependent uncertainty
    - Disadvantages: suboptimality

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$$\max_{\boldsymbol{x} \in \mathcal{X}} \mathbb{E}_{\mathbb{P}} \left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \right]$$

- Distributionally Robust Optimization (DRO)

$$\max_{\boldsymbol{x} \in \mathcal{X}} \inf_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}} \left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \right]$$

  - Tractability depends on underlying problem structures and choices of ambiguity sets
  - Mostly duality-based reformulation techniques: DRO $\rightarrow$ RO $\rightarrow$ convex (conic) optimization

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Why do we care tractability?

> *... can be treated as actual sources of "immunized against uncertainty" decisions only if these problems are computationally tractable; when that is not the case, these settings become more wishful thinking than actual decision-making tool.*
> *— Robust Optimization by Ben-Tal, El Ghaoui and Nemirovski*

Exponential Conic Optimization

- Exponential cone: 3-dimensional closed convex cone

$$\mathcal{K}_{\exp} := \mathsf{cl}\,\{(x_1, x_2, x_3) \mid x_1 \geq x_2 \exp(x_3/x_2), x_2 > 0\}$$

i.e. closure of epigraph of perspective of exponential function.

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  i.e. closure of epigraph of perspective of exponential function.

- Exponential cone programming (ECP):

$$\begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \in \mathcal{K}_{\exp} \times \cdots \times \mathcal{K}_{\exp} \\ & \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g} \end{array}$$

  - Generalization of linear programming (LP)
  - Great modeling power:
    - exponential cone $\succeq$ power cone $\succeq$ second-order cone
    - Many convex constraints involving exponential or logarithm functions are $\mathcal{K}_{\exp}$-representable, e.g.

$$t \leq -x \log x \Longleftrightarrow (1, x, t) \in \mathcal{K}_{\exp}$$

  - Computational advances: efficient interior point algorithm based solvers such as MOSEK

What is this thesis about?

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- Motivation of using MGFs:
  - Decomposability under independence of random variables
  - Close relation to exponential utility function and relative entropy based ambiguity set
  - The log-MGFs of many random variables are $\mathcal{K}_{\exp}$-representable.

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- Structure:
  - An ECP approach to electric vehicle (EV) charging management
  - **Robust CARA Optimization**
  - Extensions and future work

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Outline

A large-scale stochastic program

The EV charging scheduling problem is

$$\min_{\boldsymbol{x} \in \mathcal{X}} \mathbb{E}_{\mathbb{P}} \left[c(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \right] \tag{EV-SP}$$

where

$$c(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \triangleq \sum_{s \in [T]} e_s \sum_{v \in \mathcal{V}_s} x_{v,s} \tilde{z}_v + d \max_{t \in [T]} \left\{ \sum_{v \in \mathcal{V}_t} x_{v,t} \tilde{z}_v \right\}$$

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Computationally challenging for SAA due to large scale of the problem: more than 80,000 random variables and 700,000 decision variables!

ECP approximation in uncapacitated case

In uncapacitated case, $\tilde{\boldsymbol{z}} \sim \mathbb{P}$ are independent Poisson with rate $\boldsymbol{\lambda}$.

- Difficulty:

$$\sum_{s \in [T]} e_s f_s(\boldsymbol{x}, \boldsymbol{\lambda}) + d \boxed{\mathbb{E}_{\mathbb{P}} \left[\max_{t \in [T]} \{ f_t(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \} \right]}$$

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- Motivation of using MGF
  - Determines non-negative random variables uniquely
  - Utilize independence: $\mathbb{E}_{\mathbb{P}}\left[e^{\sum_v \theta_v \tilde{z}_v}\right] = \prod_v \mathbb{E}_{\mathbb{P}}\left[e^{\theta_v \tilde{z}_v}\right]$

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- Bound the order statistic:

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\begin{aligned}
& \mathbb{E}_{\mathbb{P}}\left[\max_{t\in[T]}\{f_t(\boldsymbol{x}, \tilde{\boldsymbol{z}})\}\right] \\
\leq \quad & \inf_{\mu>0} \mu \log \sum_{t\in[T]} \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{f_t(\boldsymbol{x}, \tilde{\boldsymbol{z}}) - f_t(\boldsymbol{x}, \boldsymbol{\lambda})}{\mu}\right)\right] + \max_{t\in[T]}\{f_t(\boldsymbol{x}, \boldsymbol{\lambda})\}
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- Exploit stochastic independence:

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&\mu \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{f_t(\boldsymbol{x},\tilde{\boldsymbol{z}}) - f_t(\boldsymbol{x},\boldsymbol{\lambda})}{\mu}\right)\right] \\
= \quad &\sum_{v\in\mathcal{V}_t} \mu \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{x_{v,t}\tilde{z}_v}{\mu}\right)\right] - f_t(\boldsymbol{x},\boldsymbol{\lambda})
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- Exponential cone representable log-MGF of Poisson variables:

$$\mu \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{x_{v,t}\tilde{z}_v}{\mu}\right)\right] = \lambda_v(\mu e^{x_{v,t}/\mu} - \mu) \text{ is } \mathcal{K}_{\exp}\text{-representable}$$

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  - Moments: truncated Poisson $\mathbb{P} \in \mathcal{F}^1$

$$\mathcal{F}^1 \triangleq \left\{ \mathbb{P} \;\middle|\; \begin{array}{l} \tilde{\boldsymbol{z}} \sim \mathbb{P} \\ \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(\boldsymbol{\theta}^\top \tilde{\boldsymbol{z}}\right)\right] \leq \sum_{v \in [V]} \lambda_v \left(e^{\theta_v} - 1\right) \quad \forall \boldsymbol{\theta} \geq \mathbf{0} \end{array} \right\}$$

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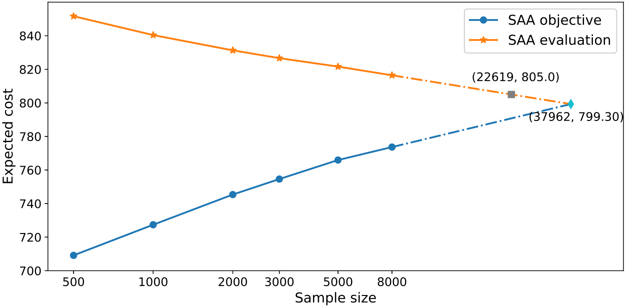
- ECP-C using $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ via infimal convolution

ECP vs SAA with different sample size

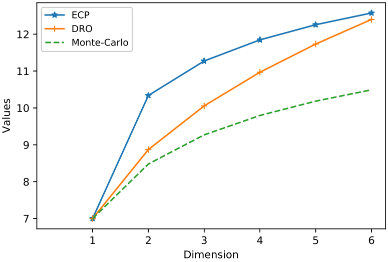
- Expected cost under SAA at different sample size given $C = 30$



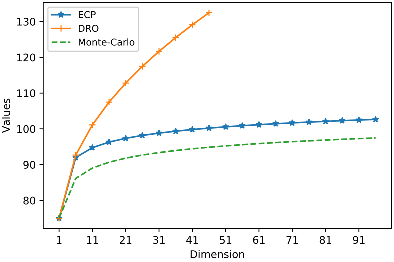
- SAA with 38,000 samples cannot be solved in 36 hours and is far from optimal upon termination
- ECP-C is near optimal: the optimality gap is at most $(805.00 - 773.67)/773.67 \times 100\% \approx 4.05\%$, even 0.71% when using the extrapolation results.

ECP vs DRO with mean-covariance information

Compare the upper bounds of $\mathbb{E}_{\mathbb{P}}\left[\max_{v\in[V]} \tilde{z}_v\right]$ where $\tilde{z}_v \sim \text{Poisson}(\lambda_v)$



(a) $\lambda_v = 7$



(b) $\lambda_v = 75$

Summary

- An ECP approach to a large-scale stochastic program
  - Bound the order statistic
  - Exploit stochastic independence
  - $\mathcal{K}_{\exp}$-representable upper bounds of log-MGF
- Scalability, tractability, superior performance.

Outline

Introduction

The Key Methodology of The ECP Approach to EV Charging

Robust CARA Optimization

Motivation

- Generalize the model of uncertainty and decision criteria
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- Generalize the objective function

Motivation

- Generalize the model of uncertainty and decision criteria
  - Risk neutral $\rightarrow$ risk averse
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- Generalize the objective function
  - EV-SP is a very special two-stage stochastic linear program with fixed recourse:

$$
\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{z}) = \quad & \min_{y} \quad y \\
& \text{s.t.} \quad y \geq \sum_{s \in [T]} e_s \sum_{v \in \mathcal{V}_s} x_{v,s} z_v + d \sum_{v \in \mathcal{V}_t} x_{v,t} z_v, \quad \forall t \in [T]
\end{aligned}
$$

    The dimension of recourse variable is one.
  - Extend to two-stage and multi-stage linear optimization problems

Constant absolute risk aversion (CARA)

- Exponential utility: $u_E(v) \triangleq 1 - e^{-v/\kappa} \to$ risk aversion
- CARA: **risk tolerance level** $-u_E'/u_E'' \equiv \kappa > 0$

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- Assessment of κ:
  - Roughly indifferent to accepting or rejecting a gamble involving a
    - 50-50 chance of winning κ or losing $\kappa/2$
    - 75-25 chance of winning κ or losing κ

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- Tractability in economic analysis
- Popularity: about five times more commonly adopted than all other types of utility functions combined [CC95]

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Robust optimization with CARA preference

A robust CARA optimization model:

$$\max_{\boldsymbol{x}\in\mathcal{X}} \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right] \tag{1}$$

where the **CARA certainty equivalent**

$$\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right] \triangleq u_E^{-1}\left(\mathbb{E}_{\mathbb{P}}\left[u_E(\tilde{v})\right]\right) = \begin{cases} \operatorname{ess\,inf}_{\mathbb{P}}\left[\tilde{v}\right] & \text{if } \kappa = 0 \\ \mathbb{E}_{\mathbb{P}}\left[\tilde{v}\right] & \text{if } \kappa = \infty \\ -\kappa \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\dfrac{\tilde{v}}{\kappa}\right)\right] & \text{if } \kappa \in (0,\infty) \end{cases}$$

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Properties of $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right]$; Extend to $\mathbb{C}^{\kappa}_{\mathcal{F}}\left[\tilde{v}\right] \triangleq \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right]$

1. $\lim_{\kappa\to\infty} \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right] = \mathbb{E}_{\mathbb{P}}\left[\tilde{v}\right]$, $\lim_{\kappa\downarrow 0} \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right] = \operatorname{ess\,inf}_{\mathbb{P}}\left[\tilde{v}\right]$.
2. $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right]$ is increasing in $\kappa > 0$
3. $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right]$ is jointly concave in $\tilde{v}$ and $\kappa > 0$.
4. $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v} + \tilde{\nu}\right] = \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right] + \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{\nu}\right]$ if $\tilde{v}$, $\tilde{\nu}$ are **independent**.
5. **Super-additivity**: $\mathbb{C}^{\kappa_1+\kappa_2}_{\mathbb{P}}\left[\tilde{v} + \tilde{\nu}\right] \geq \mathbb{C}^{\kappa_1}_{\mathbb{P}}\left[\tilde{v}\right] + \mathbb{C}^{\kappa_2}_{\mathbb{P}}\left[\tilde{\nu}\right]$ for any $\kappa_1, \kappa_2 \in \mathbb{R}_+$.

Model of uncertainty

Independent factors with ambiguous marginals

- $\tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_{I_z})$ has independent components
- $\tilde{z}_j \sim \mathbb{P}_j \in \mathcal{F}_j \subseteq \mathcal{P}_0([\underline{z}_j, \bar{z}_j])$ where $\underline{z}_j < \bar{z}_j$
- So $\tilde{\boldsymbol{z}} \sim \mathbb{P} \in \mathcal{F} \triangleq \times_{j \in [I_z]} \mathcal{F}_j \subseteq \mathcal{P}_0(\mathcal{Z})$ where $\mathcal{Z} \triangleq [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$

Model of uncertainty

Independent factors with ambiguous marginals

- $\tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_{I_z})$ has independent components
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- So $\tilde{\boldsymbol{z}} \sim \mathbb{P} \in \mathcal{F} \triangleq \times_{j\in[I_z]}\mathcal{F}_j \subseteq \mathcal{P}_0(\mathcal{Z})$ where $\mathcal{Z} \triangleq [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$

For an affine function

$$f(\boldsymbol{x}, \boldsymbol{z}) = a^0(\boldsymbol{x}) + \sum_{j\in[I_z]} a^j(\boldsymbol{x}) z_j,$$

we have

$$\mathbb{C}^{\kappa}_{\mathcal{F}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] = a^0(\boldsymbol{x}) + \sum_{j\in[I_z]} \phi_j(\kappa, a^j(\boldsymbol{x}))$$

where the function $\phi_j : [0, \infty] \times \mathbb{R} \to \mathbb{R}$,

$$\phi_j(\kappa, \lambda) \triangleq \inf_{\mathbb{P}_j\in\mathcal{F}_j} \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\lambda \tilde{z}_j\right].$$

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Tractability of CARA under independence

Example 1

Let $\tilde{z}_j \sim \mathsf{Unif}([0,1])$, $a^j(\boldsymbol{x}) \equiv a_j < 0$ and $a^0(\boldsymbol{x}) \equiv a_0 > 0$, then

$$
\begin{aligned}
\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] &= a_0 - \sum_{j\in[I_z]} \kappa \log \int_0^1 \exp\left(-\frac{a_j z_j}{\kappa}\right) \mathrm{d}z_j \\
&= a_0 - \sum_{j\in[I_z]} \kappa \log \left(\frac{\kappa - \kappa e^{-a_j/\kappa}}{a_j}\right).
\end{aligned}
$$

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$$\begin{aligned}\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] &= a_0 - \sum_{j \in [I_z]} \kappa \log \int_0^1 \exp\left(-\frac{a_j z_j}{\kappa}\right) \mathrm{d}z_j \\ &= a_0 - \sum_{j \in [I_z]} \kappa \log \left(\frac{\kappa - \kappa e^{-a_j/\kappa}}{a_j}\right).\end{aligned}$$

In contrast, evaluating an expected concave piecewise linear utility such as

$$\mathbb{E}_{\mathbb{P}}\left[\min\left\{a_0 + \sum_{j \in [I_z]} a_j \tilde{z}_j, 0\right\}\right]$$

is known to be #P-hard.

Tractability of CARA under independence

Example 2

Consider a simple mean-support ambiguity set:

$$\mathcal{G} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{I_z}) \;\middle|\; \begin{array}{l} \tilde{\boldsymbol{z}} \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{z}}] = \boldsymbol{\mu} \\ \mathbb{P}[\tilde{\boldsymbol{z}} \in [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]] = 1 \end{array} \right\}.$$

Note that evaluating $\mathbb{C}^{\kappa}_{\mathcal{G}}\left[\boldsymbol{a}^\top \tilde{\boldsymbol{z}}\right]$ requires

$$\sup_{\mathbb{P} \in \mathcal{G}} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\boldsymbol{a}^\top \tilde{\boldsymbol{z}}}{\kappa}\right)\right] = \begin{array}{rl} \inf\limits_{\alpha, \boldsymbol{\beta}} & \alpha + \boldsymbol{\beta}^\top \boldsymbol{\mu} \\ \text{s.t.} & \alpha \geq \sup\limits_{\boldsymbol{z} \in [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]} \exp\left(\dfrac{-\boldsymbol{a}^\top \boldsymbol{z}}{\kappa}\right) - \boldsymbol{\beta}^\top \boldsymbol{z}, \end{array}$$

which involves a convex maximization problem.

ECP formulations

Theorem 1

Let $g(\boldsymbol{x}, \kappa) = -\kappa \log \sum_{i \in [I]} p_i e^{-x_i/\kappa}$ *with* $\kappa > 0$ *and* $p_i > 0$ *for all* $i \in [I]$, *then the closure of* $\{(\boldsymbol{x}, \kappa, y) : y \le g(\boldsymbol{x}, \kappa), \kappa > 0\}$ *can be represented by*

$$\left\{ (\boldsymbol{x}, \kappa, y) \;\middle|\; \exists\, \boldsymbol{q} \in \mathbb{R}^I \;:\; \sum_{i \in [I]} p_i q_i \le \kappa,\; (q_i, \kappa, y - x_i) \in \mathcal{K}_{\exp} \quad \forall i \in [I] \right\}.$$

ECP formulations

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Recall that for an affine function,

$$\mathbb{C}_{\mathcal{F}}^{\kappa}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] = a^0(\boldsymbol{x}) + \sum_{j \in [I_z]} \phi_j(\kappa, a^j(\boldsymbol{x})).$$

ECP formulations

- $\phi(\kappa,\lambda) \triangleq \mathbb{C}^{\kappa}_{\mathcal{G}}\left[\lambda\tilde{z}\right]$ is $\mathcal{K}_{\exp}$-representable for many ambiguity sets. [NS07]

Table 1 **Equivalent representations of $\phi(\kappa,\lambda)$**

Ambiguity set $\mathcal{F}$	$\phi(\kappa,\lambda)$
$\{\mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\}$	$-\lvert\lambda\rvert$
$\left\{\begin{array}{l}\mathbb{P}\text{ is symmetric}\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$-\kappa\log\left(\frac{e^{\lambda/\kappa}+e^{-\lambda/\kappa}}{2}\right)$
$\left\{\begin{array}{l}\mathbb{P}\text{ is unimodal w.r.t. }0\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$-\kappa\log\int_0^1 e^{s\lvert\lambda\rvert/\kappa}\mathrm{d}s$
$\left\{\begin{array}{l}\mathbb{P}\text{ is symmetric,}\\ \text{unimodal w.r.t. }0\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$\lambda-\kappa\log\int_0^1 e^{-2\lambda s/\kappa}\mathrm{d}s$
$\left\{\begin{array}{l}\mathbb{E}_{\mathbb{P}}\left[\tilde{z}\right]\in[\underline{\mu},\overline{\mu}]\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$\min\left\{\begin{array}{l}-\kappa\log\left(\frac{(1+\underline{\mu})e^{-\lambda/\kappa}+(1-\underline{\mu})e^{\lambda/\kappa}}{2}\right),\\ -\kappa\log\left(\frac{(1+\overline{\mu})e^{-\lambda/\kappa}+(1-\overline{\mu})e^{\lambda/\kappa}}{2}\right)\end{array}\right\}$
$\left\{\begin{array}{l}\mathbb{E}_{\mathbb{P}}\left[\tilde{z}\right]=\mu\\ \mathbb{E}_{\mathbb{P}}\left[\lvert\tilde{z}-\mu\rvert\right]\le\delta\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$-\kappa\log\left(\frac{\delta}{2(\mu+1)}e^{\lambda/\kappa}+\frac{\delta}{2(1-\mu)}e^{-\lambda/\kappa}+\left(1-\frac{\delta}{2(\mu+1)}-\frac{\delta}{2(1-\mu)}\right)e^{-\mu\lambda/\kappa}\right)$
$\left\{\begin{array}{l}\mathbb{E}_{\mathbb{P}}\left[\tilde{z}\right]=\mu\\ \mathbb{E}_{\mathbb{P}}\left[\lvert\tilde{z}\rvert^2\right]\le\sigma^2\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$\min\left\{\begin{array}{l}-\kappa\log\left(\frac{(1-\mu)^2\exp\left(\frac{-(\mu-\sigma^2)\lambda}{(1-\mu)\kappa}\right)+(\sigma^2-\mu^2)\exp(-\lambda/\kappa)}{1-2\mu+\sigma^2}\right),\\ -\kappa\log\left(\frac{(1+\mu)^2\exp\left(\frac{-(\mu+\sigma^2)\lambda}{(1-\mu)\kappa}\right)+(\sigma^2-\mu^2)\exp(\lambda/\kappa)}{1+2\mu+\sigma^2}\right)\end{array}\right\}$
$\left\{\begin{array}{l}\mathbb{P}\text{ is symmetric,}\\ \mathbb{E}_{\mathbb{P}}\left[\lvert\tilde{z}\rvert^2\right]\le\sigma^2\\ \mathbb{P}\left[\tilde{z}\in[-1,1]\right]=1\end{array}\right\}$	$-\kappa\log\left(\frac{\sigma^2(e^{\lambda/\kappa}+e^{-\lambda/\kappa})}{2}+1-\sigma^2\right)$

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ECP formulations

- $\phi(\kappa, \lambda) \triangleq \mathbb{C}^{\kappa}_{\mathcal{G}}\left[\lambda \tilde{z}\right]$ is $\mathcal{K}_{\exp}$-representable for many ambiguity sets. [NS07]
- $\mathbb{C}^{\kappa}_{\mathcal{F}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] = a^0(\boldsymbol{x}) + \sum_{j \in [I_z]} \phi_j(\kappa, a^j(\boldsymbol{x}))$ is $\mathcal{K}_{\exp}$-representable if $a^0(\boldsymbol{x})$, $a^j(\boldsymbol{x})$ are affine.

Concave piecewise affine functions

- However, in practice the payoff functions $f(\boldsymbol{x}, \boldsymbol{z})$ are usually nonlinear in $\boldsymbol{z}$.

Concave piecewise affine functions

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- Consider a concave piecewise affine payoff function

$$f(\boldsymbol{x}, \boldsymbol{z}) = \min_{i \in \mathcal{I}} \left\{ a_i^0(\boldsymbol{x}) + \sum_{j \in [I_z]} a_i^j(\boldsymbol{x}) z_j \right\}.$$

- Recall evaluating $\mathbb{C}_{\mathbb{P}}^{\kappa}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right]$ under known distribution can be #P-hard.

Concave piecewise affine functions

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- Recall evaluating $\mathbb{C}_{\mathbb{P}}^{\kappa}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right]$ under known distribution can be #P-hard.
- Tractable approximations are needed.

A tractable lower bound

Let $\alpha_i = a_i^0(\boldsymbol{x})$, $\beta_i^j = a_i^j(\boldsymbol{x})$,

$$
\begin{aligned}
&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i + \boldsymbol{\beta}_i^\top \tilde{\boldsymbol{z}}\right\}\right] \\
=&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}} + \min_{i\in\mathcal{I}}\{\alpha_i + (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \tilde{\boldsymbol{z}}\}\right]
\end{aligned}
$$

A tractable lower bound

$$
\begin{aligned}
& \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i + \boldsymbol{\beta}_i^\top \tilde{\boldsymbol{z}}\right\}\right] \\
=& \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}} + \min_{i\in\mathcal{I}}\{\alpha_i + (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \tilde{\boldsymbol{z}}\}\right] \\
\geq & \sup_{\kappa_0+\kappa_1=\kappa, \boldsymbol{\kappa}\geq \mathbf{0}} \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}}\right] + \mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\min_{i\in\mathcal{I}}\{\alpha_i + (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \tilde{\boldsymbol{z}}\}\right]
\end{aligned}
$$

(Super-additivity of $\mathbb{C}_{\mathcal{F}}^{\kappa}[\tilde{v}]$ w.r.t. $(\tilde{v}, \kappa)$ with $\kappa \geq 0$)

A tractable lower bound

$$
\begin{aligned}
&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i+\boldsymbol{\beta}_i^\top\tilde{\boldsymbol{z}}\right\}\right]\\
=&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}+\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}\right]\\
\geq&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\boldsymbol{0}}\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]+\mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}\right]\\
=&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\boldsymbol{0}}\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\max_{i\in\mathcal{I}}\{-\alpha_i-(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}}{\kappa_1}\right)\right]
\end{aligned}
$$

A tractable lower bound

$$
\begin{aligned}
&\mathbb{C}^{\kappa}_{\mathcal{F}}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i+\boldsymbol{\beta}_i^\top\tilde{\boldsymbol{z}}\right\}\right] \\
=&\mathbb{C}^{\kappa}_{\mathcal{F}}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}+\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}\right] \\
\geq&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}^{\kappa_0}_{\mathcal{F}}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]+\mathbb{C}^{\kappa_1}_{\mathcal{F}}\left[\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}\right] \\
=&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}^{\kappa_0}_{\mathcal{F}}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\max_{i\in\mathcal{I}}\{-\alpha_i-(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}}{\kappa_1}\right)\right] \\
\geq&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}^{\kappa_0}_{\mathcal{F}}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sum_{i\in\mathcal{I}}\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{-\alpha_i+(\boldsymbol{\gamma}-\boldsymbol{\beta}_i)^\top\tilde{\boldsymbol{z}}}{\kappa_1}\right)\right]
\end{aligned}
$$

(Bound max by sum)

A tractable lower bound

$$
\begin{aligned}
&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i+\boldsymbol{\beta}_i^\top\tilde{\boldsymbol{z}}\right\}\right]\\
=&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}+\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}\right]\\
\geq&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]+\mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}\right]\\
=&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\max_{i\in\mathcal{I}}\{-\alpha_i-(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\}}{\kappa_1}\right)\right]\\
\geq&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sum_{i\in\mathcal{I}}\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{-\alpha_i+(\boldsymbol{\gamma}-\boldsymbol{\beta}_i)^\top\tilde{\boldsymbol{z}}}{\kappa_1}\right)\right]\\
=&\sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}}\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sum_{i\in\mathcal{I}}\exp\left(-\frac{\mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^\top\tilde{\boldsymbol{z}}\right]}{\kappa_1}\right)
\end{aligned}
$$

ECP approximations

All in all, $\mathbb{C}_{\mathcal{F}}^{\kappa}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right]$ has an ECP lower bound

$$
\begin{aligned}
\Lambda(\kappa, \boldsymbol{x}) \triangleq \max\ & r_0 + \rho \\
\text{s.t.}\ & \kappa_0 + \kappa_1 = \kappa \\
& \sum_{i \in \mathcal{I}} q_i \leq \kappa_1 \\
& (q_i, \kappa_1, \rho - r_i) \in \mathcal{K}_{\exp} \qquad \forall i \in \mathcal{I} \\
& \sum_{j \in [I_z]} \phi_j(\kappa_0, \gamma^j) \geq r_0 \\
& \alpha_i + \sum_{j \in [I_z]} \phi_j(\kappa_1, \beta_i^j - \gamma^j) \geq r_i \qquad \forall i \in \mathcal{I} \\
& \boldsymbol{\gamma} \in \mathbb{R}^{I_z}, \boldsymbol{r} \in \mathbb{R}^{1+N}, \boldsymbol{\kappa} \in \mathbb{R}_+^2, \rho \in \mathbb{R}, \boldsymbol{q} \in \mathbb{R}^N
\end{aligned}
$$

if $a_i^0(\boldsymbol{x})$ and $a_i^j(\boldsymbol{x})$ are affine.

Properties of the approximation

Theorem 2

For any $\boldsymbol{x} \in \mathcal{X}$, $\Lambda(\kappa, \boldsymbol{x})$ *is nondecreasing in* $\kappa \in [0, \infty]$ *and satisfies* $\Lambda(\kappa, \boldsymbol{x}) \geq \inf_{\boldsymbol{z} \in \mathcal{Z}} f(\boldsymbol{x}, \boldsymbol{z})$. *Moreover,* $\mathbb{C}^{\kappa}_{\mathcal{F}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] = \Lambda(\kappa, \boldsymbol{x})$ *if there exists some* $i^* \in \mathcal{I}$ *such that*

$$a^0_{i^*}(\boldsymbol{x}) + \sum_{j \in [I_z]} a^j_{i^*}(\boldsymbol{x}) z_j \leq a^0_i(\boldsymbol{x}) + \sum_{j \in [I_z]} a^j_i(\boldsymbol{x}) z_j \qquad \forall \boldsymbol{z} \in \mathcal{Z}, i \in \mathcal{I}.$$

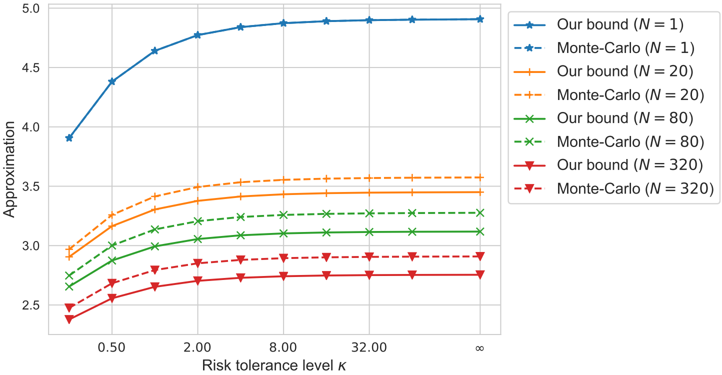
Cases of good approximation:

- Low risk tolerance κ
- Low coefficient of variations among the payoff components

Comparison with Monte-Carlo approximation

Approximate $\mathbb{C}_{\mathbb{P}}^{\kappa}\left[\min_{i\in[N]}\left\{\boldsymbol{a}_i^\top\tilde{\boldsymbol{z}}\right\}\right]$ where $\tilde{z}_j\sim\mathrm{Unif}([0,1])$

Figure: Comparison of our bound and Monte-Carlo approximation (10^6 samples) for $\kappa\geq 0.25$



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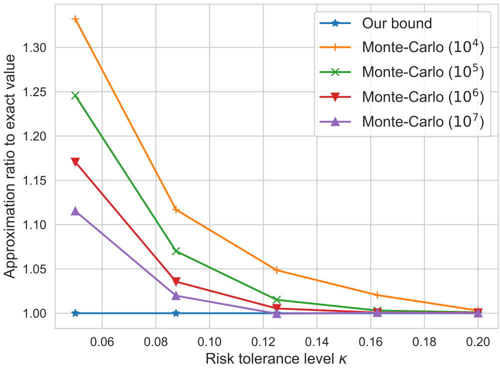
Approximate $\mathbb{C}_{\mathbb{P}}^{\kappa}\left[\min_{i\in[N]}\left\{\boldsymbol{a}_i^\top \tilde{\boldsymbol{z}}\right\}\right]$ where $\tilde{z}_j \sim \text{Unif}([0,1])$

Figure: Ratio of Monte-Carlo approximation to our bound for $\kappa \leq 0.2$ at $N = 1$ where our bound is exact while Monte-Carlo is upward biased



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Two-stage optimization

Consider the two-stage problem with fixed recourse

$$
\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{z}) = \quad \max_{\boldsymbol{y}} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\
\text{s.t.} \quad & \boldsymbol{b}_i^\top \boldsymbol{y} \le a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall i \in \mathcal{I}, \\
& \boldsymbol{y} \in \mathbb{R}^{I_y}
\end{aligned}
$$

where $\boldsymbol{y}$ is the recourse decision adaptive to uncertainty realization.

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- Assume no here-and-now variable in objective w.l.o.g.

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- Assume no here-and-now variable in objective w.l.o.g.
- Assume $\boldsymbol{b}_i \neq \boldsymbol{0}$ w.l.o.g.

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$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{z}) = \quad \max_{\boldsymbol{y}} \quad & \boldsymbol{c}^\top \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{b}_i^\top \boldsymbol{y} \le a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall i \in \mathcal{I}, \\ & \boldsymbol{y} \in \mathbb{R}^{I_y} \end{aligned}$$

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- Assume inequality constraints only w.l.o.g.

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& \qquad\quad \boldsymbol{y} \in \mathbb{R}^{I_y}
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- Assume no here-and-now variable in objective w.l.o.g.
- Assume $\boldsymbol{b}_i \neq \mathbf{0}$ w.l.o.g.
- Assume inequality constraints only w.l.o.g.

Note: Generalize the concave piecewise affine payoff function where $I_y = 1$, $\boldsymbol{c} = 1$, and $\boldsymbol{b}_i = 1$, $i \in \mathcal{I}$.

Multi-deflected linear decision rule

- A **multi-deflected linear decision rule (MLDR)**

$$\hat{\boldsymbol{y}}(\boldsymbol{z}) \triangleq \bar{\boldsymbol{y}}(\boldsymbol{z}) + \sum_{\ell \in [m]} \boldsymbol{y}_*^{\ell} \left(\max_{i \in \mathcal{I}_\ell^o} \left\{ \frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|} \right\} \right)^+ .$$

where

$$\begin{aligned} \bar{\boldsymbol{y}}(\boldsymbol{z}) &\triangleq \boldsymbol{y}^0 + \boldsymbol{Y}\boldsymbol{z} \\ h_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &\triangleq \boldsymbol{b}_i^\top \boldsymbol{y} - a_i^0(\boldsymbol{x}) - \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall i \in \mathcal{I}. \end{aligned}$$

Multi-deflected linear decision rule

- A **multi-deflected linear decision rule (MLDR)**

$$\hat{\boldsymbol{y}}(\boldsymbol{z}) \triangleq \bar{\boldsymbol{y}}(\boldsymbol{z}) + \sum_{\ell \in [m]} \boldsymbol{y}_*^{\ell} \left(\max_{i \in \mathcal{I}_\ell^o} \left\{ \frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|} \right\} \right)^+ .$$

where $\boldsymbol{y}_*^{\ell}$ and $\mathcal{I}_\ell^o$ are chosen as follows:

- Solve for each $i \in \mathcal{I}$,

$$\begin{array}{lll} \max\limits_{\boldsymbol{y} \in \mathbb{R}^{I_y}} & \boldsymbol{c}^\top \boldsymbol{y} & \\ & \boldsymbol{b}_k^\top \boldsymbol{y} \leq 0 & \forall k \in \mathcal{I} \backslash \{i\} \\ & \boldsymbol{b}_i^\top \boldsymbol{y} = -\|\boldsymbol{b}_i\| & \end{array}$$

  and denote $\boldsymbol{y}_*^{i}$ as its optimal solution if feasible.

- Partition feasible index set

$$\mathcal{I}^o = \bigcup_{\ell \in [m]} \mathcal{I}_\ell^o$$

  such that $\boldsymbol{y}_*^{i_1} = \boldsymbol{y}_*^{i_2}$ if and only if i_1 and i_2 are in the same $\mathcal{I}_\ell^o$.

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- Better than previous deflected LDR in [CSSZ08]
- Can replicate optimal decision rule under complete recourse with $I_y = 1$
- Obtain ECP lower bound based on approximation for concave piecewise affine functions.

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A multi-period consumption model

Given $\boldsymbol{\xi}_t \triangleq (\boldsymbol{z}_1, ..., \boldsymbol{z}_t)$,

$$\begin{array}{lll} \max\limits_{\substack{\boldsymbol{x}\in\mathcal{X}, \\ \boldsymbol{y}_1,\ldots,\boldsymbol{y}_T}} & \mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}\left[\boldsymbol{c}_1^\top \boldsymbol{y}_1(\tilde{\boldsymbol{\xi}}_1), \ldots, \boldsymbol{c}_T^\top \boldsymbol{y}_T(\tilde{\boldsymbol{\xi}}_T)\right] & \\ \text{s.t.} & \sum\limits_{\tau\in[t]} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_\tau(\boldsymbol{\xi}_\tau) \le a_{t,i}^0(\boldsymbol{x}) + \boldsymbol{a}_{t,i}^\top(\boldsymbol{x})\boldsymbol{\xi}_t & \forall t\in[T],\ i\in\mathcal{I}_t,\ \boldsymbol{z}\in\mathcal{Z} \\ & \boldsymbol{y}_t \in \mathcal{R}^{I_{\xi_t}, I_{y_t}} & \forall t\in[T]. \end{array}$$

where the **multi-period ambiguity-averse CARA certainty equivalent**

$$\mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] \triangleq \begin{cases} \min\limits_{t\in[T]:\theta_t>0} \left\{ \inf\limits_{\mathbb{P}\in\mathcal{F}} \operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}_t] \right\} & \text{if } \kappa = 0 \\ \sum\limits_{t\in[T]} \theta_t \inf\limits_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[\tilde{v}_t] & \text{if } \kappa = \infty \\ -\kappa \log\left(\sum\limits_{t\in[T]} \theta_t \sup\limits_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\dfrac{\tilde{v}_t}{\kappa}\right)\right]\right) & \text{if } \kappa \in (0,\infty). \end{cases}$$

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- Multi-period MLDR $\Longrightarrow$ Tractable ECP approximation

Multi-period inventory management

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- In each period $t = 1, ..., T$,

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  - Determine consumption level f_t and obtain utility $1 - e^{-f_t/\kappa}$
  - Inventory update:
  $$x_{t+1} = y_t - \tilde{z}_t$$
  - Wealth update:
  $$w_{t+1} = (w_t + q_t - f_t)(1 + \beta)$$
  where the income
  $$q_t = p_t\tilde{z}_t - h(y_t - \tilde{z}_t)^+ - b(\tilde{z}_t - y_t)^+ - c_t(y_t - \tilde{z}_t)$$
- Assume $w_{T+1} = 0$.

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Multi-period inventory management

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{f},\boldsymbol{w},\boldsymbol{q}} \quad & \mathbb{E}_{\mathbb{P}}\left[\sum_{t\in[T]} \theta_t\left(1 - e^{-f_t(\tilde{\boldsymbol{\xi}}_t)/\kappa}\right)\right] \\
\text{s.t.} \quad & f_t(\tilde{\boldsymbol{\xi}}_t) = w_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \frac{w_{t+1}(\tilde{\boldsymbol{\xi}}_t)}{1+\beta} + q_t(\tilde{\boldsymbol{\xi}}_t) && \forall t \in [T] \\
& q_t(\tilde{\boldsymbol{\xi}}_t) \le p_t\tilde{z}_t - h(y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tilde{z}_t) - c_t(y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - x_t(\tilde{\boldsymbol{\xi}}_{t-1})) && \forall t \in [T] \\
& q_t(\tilde{\boldsymbol{\xi}}_t) \le p_t\tilde{z}_t - b(\tilde{z}_t - y_t(\tilde{\boldsymbol{\xi}}_{t-1})) - c_t(y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - x_t(\tilde{\boldsymbol{\xi}}_{t-1})) && \forall t \in [T] \\
& y_t(\tilde{\boldsymbol{\xi}}_{t-1}) \ge x_t(\tilde{\boldsymbol{\xi}}_{t-1}) && \forall t \in [T] \\
& x_{t+1}(\tilde{\boldsymbol{\xi}}_t) = y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tilde{z}_t && \forall t \in [T-1] \\
& w_{T+1}(\tilde{\boldsymbol{\xi}}_T) = 0
\end{aligned}
$$

- Approach 1: dynamic programming (DP)
- Approach 2: Fourier-Motzkin elimination + multi-period MLDR

Numerical experiments

- Similar parameter setting as in [CSSLS07]:
  $h = 6$, $b = 3$, $c_t = 1$, $p_t = 8$, $\beta = 0.1$ for all $t \in [T]$

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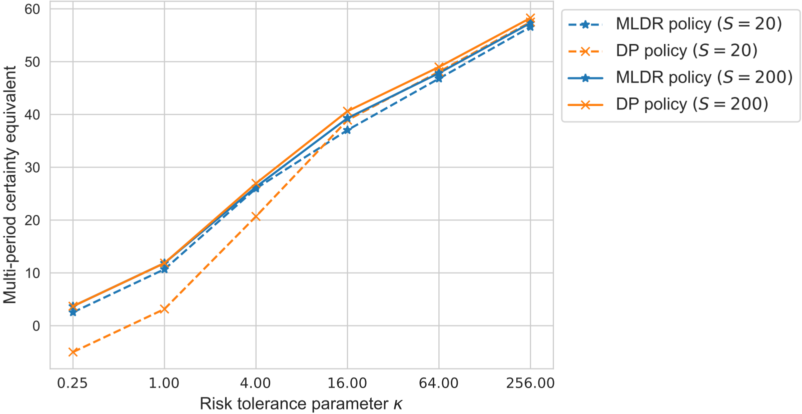
- The results are averaged over 50 random instances.

Numerical experiments

Figure: Multi-peirod CARA certainty equivalent under different risk tolerance parameters



Summary

- A robust decision model with CARA preference

$$\max_{\boldsymbol{x} \in \mathcal{X}} \mathbb{C}^{\kappa}_{\mathcal{F}} \left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \right]$$

- Tractable ECP approximations for a hierarchy of payoff functions:
  - Affine perturbations
  - Concave piecewise affine perturbations
  - Two-stage optimization with fixed recourse
- Extend to a multi-period consumption model
- Robust performance in data-driven setting when risk tolerance is low.

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Extensions and future work

- Other entropy related decision criteria such as entropic value-at-risk, entropic risk measure.
- Faster computation for large-scale ECP (with integer constraints)
- Handle correlated uncertain factors in robust CARA optimization
- Stronger probability bound of functions of (partially) independent random variables

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