Some SDP Problems from Quantum Information

Chen Li

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The main references are [CS17] and [FF18]. More papers can be found in the references and citations of the two papers.

Notations

We denote the probability simplex

$$\Delta_n := \left\{p \in \mathbb{R}_+^n : 1^T p = 1\right\},$$

the set of density matrices

$$\mathrm{Spec}_n := \left\{X \in \mathbb{S}_+^n : \mathrm{Tr}\left(X\right) = 1\right\},$$

the von-Neumann entropy

$$\mathrm{H}\left(X\right) := -\mathrm{Tr}\left(X \log X\right),$$

the quantum relative entropy

$$D(X||Y) := \mathrm{Tr}\left(X(\log X - \log Y)\right)$$

1 Quantum channel capacity

The quantum channel capacity problem considered in [CS17] is

$$\sup_{p\geq 0, \sum_i p_i = 1, v_i \in S^n} \mathrm{H}\left(\sum_i p_i \mathcal{A}\left(v_i v_i^T\right)\right) - \sum_i p_i \mathrm{H}\left(\mathcal{A}\left(v_i v_i^T\right)\right) \tag{1}$$

where $\mathcal{A} : \mathbb{S}^n \longrightarrow \mathbb{S}^k$ is a completely positive linear map

$$\mathcal{A}(X) := \sum_j A_j X A_j^T, \text{ with } A_j \in \mathbb{R}^{k\times n} \text{ such that } \sum_j A_j A_j = I_k$$

and S^n is the unit sphere.

The problem (1) does not seem to be tractable. Typically, one can first consider some fixed $\{v_i\}_{i=1}^m$ and solve the problem

$$\max_{p\in\Delta_m} \mathrm{H}\left(\sum_{i=1}^m p_i \Phi_i\right) - \sum_{i=1}^m p_i \mathrm{H}\left(\Phi_i\right) \tag{2}$$

where $\Phi_i \triangleq \mathcal{A}(v_i v_i^T) \in \mathrm{Spec}_k$. Then one can add more v_i into the problem progressively to obtain a better lower bound of problem (1). In [FF18, CS17], they focus on solving problem (2), which is called *classical to quantum channel* capacity problem.

2 Quantum state tomography

Quantum state tomography (QST) is similar to a signal recovering problem. Given some measurement or observations, one aims to reconstruct a density matrix. There are many different optimization models associated with QST problem, I list a few as follows.

The maximum entropy model mentioned in [GLGR+13].

$$\begin{array}{ll} \max\limits_{X} & \mathrm{H}\left(X\right) \\ \text{s.t.} & \mathrm{Tr}\left(E_i X\right) = f_i \quad \forall i = 1, ..., m \\ & X \in \mathrm{Spec}_n \end{array} \tag{3}$$

QST is also related to compress sensing [GLF+10] in the sense that the objective can be chosen as

$$\begin{array}{ll} \min\limits_{X} & \|X\|_* \\ \text{s.t.} & \mathrm{Tr}\left(E_i X\right) = f_i \quad \forall i = 1, ..., m \\ & X \in \mathrm{Spec}_n \end{array} \tag{4}$$

where $\|\cdot\|_*$ is the nuclear norm.

Another maximum likelihood estimate (MLE) based optimization model presented in [GGRL16]

$$\begin{array}{ll} \max\limits_{X} & \displaystyle\sum_{j=1}^{m} r_j \log\left(\mathrm{Tr}\left(E_j X\right)\right) \\ \text{s.t.} & X \in \mathrm{Spec}_n \end{array} \tag{5}$$

There are other optimization models which are linear SDP or quadratic SDP problems in [GLGR+13, GGRL16].

3 Relative entropy of entanglement

Let **Sep** be the convex set of separable states on $A \otimes B$, i.e.,

$$\mathbf{Sep} := \mathrm{conv}\left\{\rho_A \otimes \rho_B : \rho_A \in \mathrm{Spec}_{n_A}, \rho_B \in \mathrm{Spec}_{n_B}\right\}$$

Given a state $Y \in \mathrm{Spec}_{n_A \times n_B}$, the relative entropy of entanglement [FF18] is the optimal value of the following optimization problem:

$$\min_{X \in \mathbf{Sep}} D(Y||X) \tag{6}$$

which is NP-hard. Typically, one can solve its partial positve transpose (PPT) relaxation:

$$\min_{X \in \mathbf{PPT}} D(Y||X) \tag{7}$$

where

$$\mathbf{PPT} := \left\{X \in \mathrm{Spec}_{n_A \times n_B} : (I \otimes T)(X) \succeq 0\right\}$$

and T is the transpose map. An explicit representation of partial transpose linear operator can be found at Peres–Horodecki criterion. It is worth mentioning that such relaxation is dual to sum-of-squares approximation of nonnegative polynomial, see QIP 2021 tutorial: Convex Optimization and Quantum Information.

4 Entanglement-assisted classical capacity

Besides the quantum channel capacity problem mentioned above, there are other different notions of capacity, see [GIN18, RISB20] for more details. We describe the entanglement-assisted classical capacity in [FF18] below.

$$\max_{X \in \mathrm{Spec}_{n_A}} \mathrm{H}\left(UXU^*\right) - \mathrm{H}\left(\mathrm{Tr}_B(UXU^*)\right) + \mathrm{H}\left(\mathrm{Tr}_E(UXU^*)\right) \tag{8}$$

where $UXU^* \in \mathrm{Spec}_{n_B \times n_E}$, U is given.

5 Kernel learning

Another problem involving quantum relative entropy is kernel learning [KSD09] where they consider two problems

$$\begin{array}{lll} \min\limits_{X} & D(X||Y) + \mathrm{Tr}\left(Y - X\right) & \\ \text{s.t.} & \mathrm{Tr}\left(A_i X\right) \leq b_i & \forall i = 1, ..., m \\ & \mathrm{rank}\left[X\right] \leq r & \\ & X \succeq 0 & \end{array}$$

and

$$\begin{array}{lll} \min\limits_{X} & \mathrm{Tr}\left(XY^{-1}\right) - \log\det(XY^{-1}) - n & \\ \text{s.t.} & \mathrm{Tr}\left(A_i X\right) \leq b_i & \forall i = 1, ..., m \\ & \mathrm{rank}\left[X\right] \leq r & \\ & X \succeq 0 & \end{array}$$

where the two objective functions are derived from Bregman divergence of von-Neumann entropy function and log determinant function. They focus on the cases of A_i's are rank-1. Then they adopt a dual coordinate descent type method to reduce per iteration computation complexity.

6 More problems

More SDP problems in quantum information theory can be found in [Wan18] Chapter 2.4.3.

References

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