Multistage stochastic programs with the entropic risk measure

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Outline

Two-stage Stochastic Linear Programming

Classic two-stage stochastic linear programming:

$$\begin{aligned} \min_{x_1} \quad & c_1^T x_1 + \mathbb{E}\left[V_2(x_1,\omega)\right] \\ \text{s.t.} \quad & A_1 x_1 = b_1 \\ & x_1 \geq 0 \end{aligned} \tag{1}$$

where the second-stage problem is

$$\begin{aligned} V_2(x_1,\omega) = \quad \min_{x_2} \quad & c_2(\omega)^T x_2 \\ \text{s.t.} \quad & A_2(\omega) x_2 + B_2(\omega) x_1 = b_2(\omega) \\ & x_2 \geq 0 \end{aligned} \tag{2}$$

1. Making first-stage decision x_1
2. Uncertainty ω realized
3. Making second-stage decision x_2 given x_1 and ω

- Key: obtain first-stage decision x_1

Multi-stage Stochastic Programming

Classic multi-stage stochastic linear programming:

$$
\begin{aligned}
\min_{x_1} \quad & c_1^T x_1 + \mathbb{E}\left[V_2(x_1, \omega_2)\right] \\
\text{s.t.} \quad & A_1 x_1 = b_1 \\
& x_1 \geq 0
\end{aligned}
\tag{3}
$$

where the t-th-stage problem is recursively defined by

$$
\begin{aligned}
V_t(x_{t-1}, \omega_t) = \quad \min_{x_t} \quad & c_t(\omega_t)^T x_t + \mathbb{E}\left[V_{t+1}(x_t, \omega_t)\right] \\
\text{s.t.} \quad & A_t(\omega_t) x_t + B_t(\omega_t) x_{t-1} = b_t(\omega_t) \\
& x_t \geq 0
\end{aligned}
\tag{4}
$$

for $t = 2, ..., T$ where we assume $\mathbb{E}\left[V_{T+1}(\cdot, \cdot)\right] = 0$. We always assume finite sample space $\omega_t \in \Omega_t$ ($|\Omega_t|$ is finite) and existence of feasible and finite optimal solutions for all t-stage problems.

- Solution x_t is a optimal control policy $x_t(\omega_2, ..., \omega_t)$

Multi-stage Stochastic Programming

Classic multi-stage stochastic linear programming:

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\begin{aligned}
\min_{x_1} \quad & c_1^T x_1 + \mathbb{E}\left[V_2(x_1, \omega_2)\right] \\
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\begin{aligned}
V_t(x_{t-1}, \omega_t) = \quad \min_{x_t} \quad & c_t(\omega_t)^T x_t + \mathbb{E}\left[V_{t+1}(x_t, \omega_t)\right] \\
\text{s.t.} \quad & A_t(\omega_t) x_t + B_t(\omega_t) x_{t-1} = b_t(\omega_t) \\
& x_t \geq 0
\end{aligned}
\tag{4}
$$

for $t = 2, ..., T$ where we assume $\mathbb{E}\left[V_{T+1}(\cdot, \cdot)\right] = 0$. We always assume finite sample space $\omega_t \in \Omega_t$ ($|\Omega_t|$ is finite) and existence of feasible and finite optimal solutions for all t-stage problems.

- Solution x_t is a optimal control policy $x_t(\omega_2, ..., \omega_t)$
- Risk neutral: Expectation $\mathbb{E}\left[\cdot\right]$

Outline

Risk-averse Decision-making

- Does expectation make sense?

Example 1

Consider two games with random return:

1. X_1 is $\$0$ w.p. 1
2. X_2 is $-\$100$ w.p. 0.99, $\$9901$ w.p. 0.01

- If we choose to play only once, then most people prefer X_1

Risk-averse Decision-making

- Does expectation make sense?

Example 1

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1. X_1 is $\$0$ w.p. 1
2. X_2 is $-\$100$ w.p. 0.99, $\$9901$ w.p. 0.01

- If we choose to play only once, then most people prefer X_1

- People has risk attitude and most of them are risk-averse

Risk Measure[1]

- Coherent risk measure and convex risk measure

Definition 1

A functional $\mathbb{F} : \mathcal{X} \to \mathbb{R}$ is a convex risk measure if it has the following four properties:

- Monotonicity: $\mathbb{F}[Z_1] \leq \mathbb{F}[Z_2]$ if $Z_1 \leq Z_2$ a.s..
- Translation invariance: $\mathbb{F}[Z + t] = \mathbb{F}[Z] + t$ for all $t \in \mathbb{R}$.
- Convexity: $\mathbb{F}[\lambda Z_1 + (1 - \lambda)Z_2] \leq \lambda\mathbb{F}[Z_1] + (1 - \lambda)\mathbb{F}[Z_2]$ for all $\lambda \in [0, 1]$.

A convex risk measure is coherent if $\mathbb{F}$ also satisfies

- Positive homogeneity: $\mathbb{F}[kZ] = k\mathbb{F}[Z]$ for all $k \geq 0$.

- Example: Entropic risk measure with risk aversion parameter $\gamma > 0$

$$\mathbb{ENT}_\gamma[Z] = \frac{1}{\gamma} \ln \mathbb{E}[\exp(\gamma Z)] \tag{5}$$

is a convex risk measure but not coherent.

[1] R Tyrrell Rockafellar. "Coherent approaches to risk in optimization under uncertainty". In: *OR Tools and Applications: Glimpses of Future Technologies*. Informs, 2007, pp. 38–61.

Example: CVaR

- Conditional Value-at-Risk is a coherent risk measure defined by

$$\mathbb{CV@R}_\gamma [Z] = \inf_\zeta \zeta + \frac{1}{1-\gamma} \mathbb{E}\left[(Z-\zeta)_+\right] \tag{6}$$

  - If $\gamma = 0$, $\mathbb{CV@R}_\gamma [Z] = \mathbb{E}[Z]$
  - As $\gamma \to 1$, $\mathbb{CV@R}_\gamma [Z] \to \operatorname{ess\,sup} [Z]$

Example: CVaR

- Conditional Value-at-Risk is a coherent risk measure defined by

$$\mathbb{CV@R}_\gamma[Z] = \inf_\zeta \zeta + \frac{1}{1-\gamma}\mathbb{E}[(Z-\zeta)_+] \tag{6}$$

  - If $\gamma = 0$, $\mathbb{CV@R}_\gamma[Z] = \mathbb{E}[Z]$
  - As $\gamma \to 1$, $\mathbb{CV@R}_\gamma[Z] \to \operatorname{ess\,sup}[Z]$

- Value-at-Risk defined by

$$\mathbb{V@R}_\gamma[Z] = \inf\{\zeta : \mathbb{P}[Z \leq \zeta] \geq \gamma\} \triangleq F_Z^{-1}(\gamma) \tag{7}$$

is not a convex risk measure. Consider the example with three scenarios with equal probabilities:

ω	Z^1	Z^2	$\frac{1}{2}Z^1 + \frac{1}{2}Z^2$
1	300	0	150
2	0	0	0
3	0	300	150
$\text{VaR}_{0.6}$	0	0	150
$\mathbb{E}$	100	100	100

CVaR cont.

- If Z is continuous, then $\mathbb{CV@R}_\gamma[Z] = \mathbb{E}\left[Z \mid Z \geq F_Z^{-1}(\gamma)\right]$

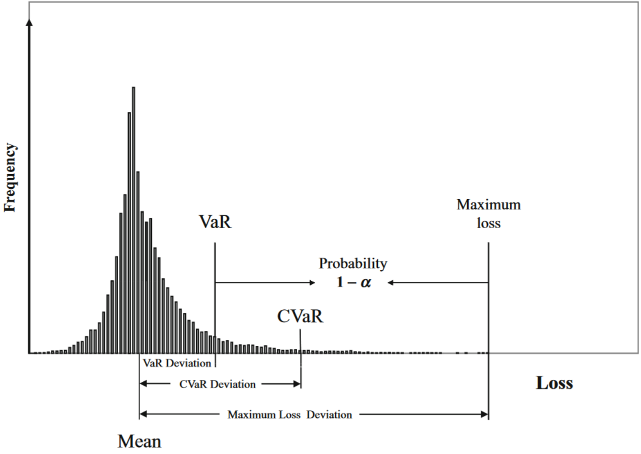


Figure 1: Illustration of CVaR_α and VaR_α

Risk-averse Decision-making

Example 2

Consider two games with random **loss**:

1. X_1 is $-\$9001$ w.p. 0.1, $\$1000$ w.p. 0.9
2. X_2 is $-\$1000$ w.p. 0.9, $\$9000$ w.p. 0.1

In terms of cost,

- $\mathbb{CV@R}_0[X_1] = -0.1$, $\mathbb{CV@R}_0[X_2] = 0$, prefer X_1
- $\mathbb{CV@R}_{0.2}[X_1] = 1000$, $\mathbb{CV@R}_{0.2}[X_2] = 250$, prefer X_2
- $\mathbb{CV@R}_{0.5}[X_1] = 1000$, $\mathbb{CV@R}_{0.5}[X_2] = 1000$, no preference
- $\mathbb{CV@R}_{0.8}[X_1] = 1000$, $\mathbb{CV@R}_{0.8}[X_2] = 4000$, prefer X_1
- $\mathbb{V@R}_{0.8}[X_1] = 1000$, $\mathbb{V@R}_{0.8}[X_2] = -1000$, prefer X_2

Risk Measure in Multi-stage Optimization

Given a sequence of correlated random variables $Z = \{Z_t\}_{t=1}^{T}$

- End-of-horizon approach

$$\text{End-of-Horizon-Risk}(Z) = \mathbb{F}\left[Z_1 + Z_2 + \cdots + Z_T\right] \tag{8}$$

- Nested approach

$$\text{Nested-Risk}(Z) = \mathbb{F}\left[Z_1 + \mathbb{F}\left[Z_2 + \mathbb{F}\left[\cdots + \mathbb{F}\left[Z_T\right] \cdots \mid Z_2\right] \mid Z_1\right]\right] \tag{9}$$

where the inner evaluations of risk measure are conditioned on the realizations of random variables in the outer layers of nesting.

Conditional Consistency

- Problem arises by replacing expectation by risk measure

Definition 2

A risk measure $\mathbb{F}$ is conditionally consistent if

$$\begin{aligned} & \mathbb{F}\left[X_1 + X_2\right] \leq \mathbb{F}\left[Y_1 + Y_2\right] \\ \iff \quad & \mathbb{F}\left[X_1 + \mathbb{F}\left[X_2 \mid X_1\right]\right] \leq \mathbb{F}\left[Y_1 + \mathbb{F}\left[Y_2 \mid Y_1\right]\right] \end{aligned} \tag{10}$$

for any two-dimensional random vectors (X_1, X_2), (Y_1, Y_2).

Examples: Expectation $\mathbb{E}\left[\cdot\right]$, worst-case $\operatorname{ess\,sup}\left[\cdot\right]$

An Example of Conditionally Inconsistency

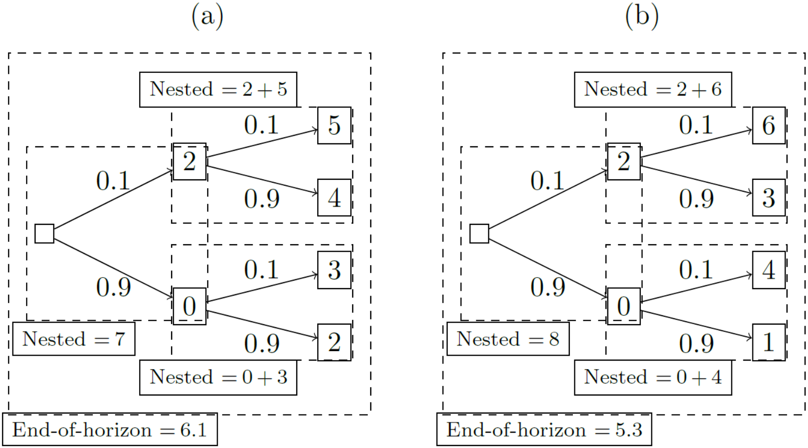


Figure 2: $\mathbb{CV@R}_{0.9}[\cdot]$ is not conditionally consistent

Conditional Consistency of Entropic Risk Measure

Theorem 1

Suppose $\mathbb{E}\left[e^{\gamma(X+Y)}\right] < +\infty$ *and* $\gamma > 0$, *then*

$$\mathbb{ENT}_\gamma\left[X+Y\right] = \mathbb{ENT}_\gamma\left[X + \mathbb{ENT}_\gamma\left[Y \mid X\right]\right]$$

$$\begin{aligned}
\mathbb{ENT}_\gamma\left[X + \mathbb{ENT}_\gamma\left[Y \mid X\right]\right] &= \frac{1}{\gamma}\ln \mathbb{E}\left[e^{\gamma(X+\frac{1}{\gamma}\ln \mathbb{E}[\exp(\gamma Y)|X])}\right] \\
&= \frac{1}{\gamma}\ln \mathbb{E}\left[e^{\gamma X}\mathbb{E}\left[e^{\gamma Y} \mid X\right]\right] \\
&= \frac{1}{\gamma}\ln \mathbb{E}\left[\mathbb{E}\left[e^{\gamma(X+Y)} \mid X\right]\right] \\
&= \mathbb{ENT}_\gamma\left[X+Y\right]
\end{aligned}$$

Corollary 1

Entropic risk measure $\mathbb{ENT}_\gamma\left[\cdot\right]$ *is conditionally consistent if the moment generating function exists.*

A General Result[2]

Theorem 2

*Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, \mathbb{P})$ be a standarad filtered probability space. The family $(\mathbb{F}_t)_{t\in\mathbb{N}_0}$ is a law invariant, time consistent, relevant dynamic risk measure **if and only if** there is $\gamma \in (-\infty, +\infty]$ such that:*

$$\mathbb{F}_t(Z) = \frac{1}{\gamma} \ln \mathbb{E}\left[\exp(\gamma Z) \mid \mathcal{F}_t\right], \ \forall t \in \mathbb{N}_0.$$

The limiting cases are $\mathbb{F}_t(Z) = \mathbb{E}\left[Z \mid \mathcal{F}_t\right]$ when $\gamma = 0$ and $\mathbb{F}_t(Z) = \operatorname{ess\,sup}\left[Z \mid \mathcal{F}_t\right]$ when $\gamma = \infty$.

Time consistency here means $\mathbb{F}\left[X+Y\right] = \mathbb{F}\left[X + \mathbb{F}\left[Y \mid X\right]\right]$, which is stronger than conditionally consistency.

- Open question: How large is the class of conditional consistent risk measure?

[2] Michael Kupper and Walter Schachermayer. "Representation results for law invariant time consistent functions". In: *Mathematics and Financial Economics* 2.3 (2009), pp. 189–210.

A Simple Illustration

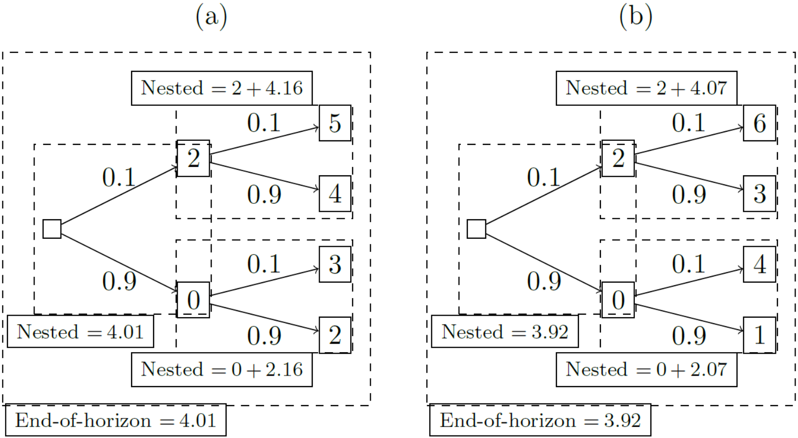


Figure 3: $\mathbb{ENT}_1$ is time consistent, hence conditionally consistent

Outline

Variational Formulation of $\mathbb{ENT}_\gamma[\cdot]$

Theorem 3

Let $\gamma > 0$, $\mathcal{P} = \left\{ q \in \mathbb{R}^{|\Omega|} : q \geq 0, \sum_{\omega\in\Omega} p_\omega = 1 \right\}$ *and* Z *has probability mass* $p \in \mathcal{P}$ *and* $p > 0$. *Then*

$$\begin{aligned} \mathbb{ENT}_\gamma[Z] &= \tfrac{1}{\gamma} \log \textstyle\sum_{\omega\in\Omega} p_\omega e^{\gamma z_\omega} \\ &= \max_{\boldsymbol{q}\in\mathcal{P}} \sum_{\omega\in\Omega} q_\omega z_\omega - \frac{1}{\gamma} \sum_{\omega\in\Omega} q_\omega \log\left(\frac{q_\omega}{p_\omega}\right) \end{aligned} \tag{11}$$

and the optimal probability $q^*_\omega = \frac{p_\omega e^{\gamma z_\omega}}{\sum_{\omega\in\Omega} p_\omega e^{\gamma z_\omega}}$. *Moreover,* $\mathbb{ENT}_\gamma[Z]$ *is increasing w.r.t.* γ. *As* $\gamma \to +\infty$, *we have* $\mathbb{ENT}_\gamma[Z] \to \operatorname{ess\,sup}[Z]$. *As* $\gamma \to 0$, *we have* $\mathbb{ENT}_\gamma[Z] \to \mathbb{E}[Z]$.

Variational Formulation of Risk Measure[3]

- Any convex risk measure has a variational formulation of

$$\mathbb{F}\left[Z\right] = \sup_{q \in \mathcal{P}} \left\{ \mathbb{E}_q[Z] - \alpha(q) \right\}$$

  - For $\mathbb{ENT}_\gamma\left[\cdot\right]$, the penalty $\alpha(q)$ is K-L divergence $\sum_{\omega \in \Omega} q_\omega \log\left(\frac{q_\omega}{p_\omega}\right)$

- Any coherent risk measure has has a variational formulation of

$$\mathbb{F}\left[Z\right] = \sup_{q \in \mathcal{M}(p)} \mathbb{E}_q[Z]$$

  - For $\mathbb{CV@R}_\gamma\left[\cdot\right]$, the risk set is

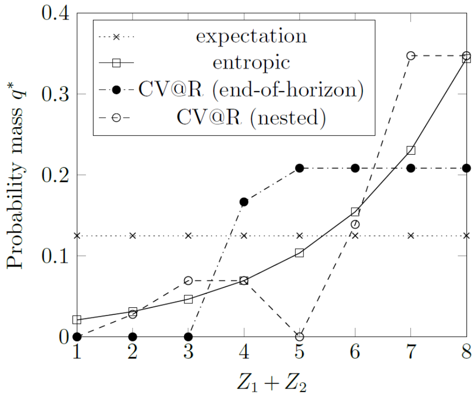
$$\mathcal{M}(p) = \left\{ q \in \mathcal{P} : q_\omega \le \frac{p_\omega}{1-\gamma} \right\}$$

[3] Andrzej Ruszczyński and Alexander Shapiro. "Optimization of convex risk functions". In: *Mathematics of operations research* 31.3 (2006), pp. 433–452.

Variational Interpretation of conditional inconsistency

Consider Z_1 uniform on $\{0, 4\}$ and Z_2 uniform on $\{1, 2, 3, 4\}$, so $Z_1 + Z_2$ is uniform on $\{0, 1, ..., 8\}$.

- The probability that attains the supremum in variational formulation is as follows.



A Conic Dual Representation of $\mathbb{ENT}_\gamma[\cdot]$

Theorem 4

The entropic risk measure $\mathbb{ENT}_\gamma[\cdot]$ *has a dual formulation*

$$\begin{aligned} \mathbb{ENT}_\gamma[Z] = \quad & \min_{\mu \in \mathbb{R}^{|\Omega|+1}} && \textstyle\sum_{\omega\in\Omega} p_\omega \mu_\omega + \mu_0 \\ & \quad \text{s.t.} && \left(-\tfrac{1}{\gamma}, \mu_0 - z_\omega, \mu_\omega\right) \in \mathcal{K}_{\exp}^* \quad \forall \omega \in \Omega \end{aligned} \tag{12}$$

where $\mathcal{K}_{\exp}^* = \left\{(u,v,w) \in \mathbb{R}^3 : -ue^{v/u} \le e^1 w, u < 0\right\}$ *is the dual of exponential cone* $\mathcal{K}_{\exp} = \left\{(x,y,z) \in \mathbb{R}^3 : ye^{x/y} \le z, y > 0\right\}$*.*

- Proof. Take the dual of the variational form (11). Strong duality holds since (11) is feasible and (12) is strictly feasible.
- The dual formulation is useful in obtaining tighter bound in computation later.

Outline

Two-stage Risk-averse Stochastic Programming

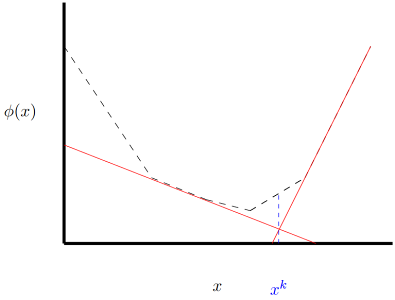
- Benders' decomposition (L-shaped method):
  - Idea: build piece-wise linear convex lower bound of recourse function $\mathbb{F}\left[V_2(x,\omega)\right]$ by adding cuts sequentially

Approximate

$$
\begin{aligned}
V_1 = \quad & \min_{x_1} \quad c_1^T x_1 + \mathbb{F}\left[V_2(x_1,\omega)\right] \\
& \text{s.t.} \quad A_1 x_1 = b_1 \\
& \qquad\quad x_1 \geq 0
\end{aligned}
$$

by the master problem

$$
\begin{aligned}
\text{Single-}V_1^k = \quad & \min_{x_1,\Theta} \quad c_1^T x_1 + \Theta \\
& \text{s.t.} \quad A_1 x_1 = b \\
& \qquad\quad x_1 \geq 0 \\
& \qquad\quad \Theta \geq -M \\
& \qquad\quad \Theta \geq \alpha_k + \beta_k^T x_1, k = 1, \ldots, K-1
\end{aligned}
$$



Theorem 5

Let $\omega \in \Omega$ be a random vector with finite support and with nominal probability mass $p > 0$. For a convex risk measure $\mathbb{F}$ with penalty $\alpha(q)$ in its variational form. Let $V(x,\omega)$ be convex w.r.t. x for all $\omega \in \Omega$. Let $\lambda(\tilde{x},\omega)$ be the subgradient of $V(x,\omega)$ w.r.t. x at $x = \tilde{x}$ for each $\omega \in \Omega$. Then $\sum_{\omega \in \Omega} q^*_\omega \lambda(\tilde{x},\omega)$ is a subgradient of $\mathbb{F}[V(x,\omega)]$ at $\tilde{x}$, where $q^* \in \arg\max_{q \in \mathcal{P}} \{\mathbb{E}_q[V(\tilde{x},\omega)] - \alpha(q)\}$.

Proof.

$$
\begin{aligned}
\mathbb{F}[V(x,\omega)] \quad &= \sup_{q \in \mathcal{P}} \{\mathbb{E}_q[V(x,\omega)] - \alpha(q)\} \\
&\geq \mathbb{E}_{q^*}[V(x,\omega)] - \alpha(q^*) \\
&= \textstyle\sum_{\omega \in \Omega} q^*_\omega V(x,\omega) - \alpha(q^*) \\
&\geq \textstyle\sum_{\omega \in \Omega} q^*_\omega (V(\tilde{x},\omega) + \lambda(\tilde{x},\omega)^T (x - \tilde{x})) - \alpha(q^*) \\
&= \textstyle\sum_{\omega \in \Omega} q^*_\omega V(\tilde{x},\omega) + \left(\sum_{\omega \in \Omega} q^*_\omega \lambda(\tilde{x},\omega)\right)^T (x - \tilde{x}) - \alpha(q^*) \\
&= \mathbb{F}[V(\tilde{x},\omega)] + \left(\textstyle\sum_{\omega \in \Omega} q^*_\omega \lambda(\tilde{x},\omega)\right)^T (x - \tilde{x})
\end{aligned}
$$

Single-cut Generation

Risk-averse cut generator at x_1^k

1. Given x_1^k at iteration k, for each $\omega \in \Omega$, solve

$$V_\omega^k = \begin{array}{ll} \min\limits_{x_2,\bar{x}} & c_2(\omega)^T x_2 \\ \text{s.t.} & \bar{x} = x_1^k \quad [\lambda] \\ & A_2(\omega)x_2 + B_2(\omega)\bar{x} = b_2(\omega) \\ & x_2 \geq 0 \end{array} \tag{13}$$

   to get dual solution λ_ω^k associated with constraint $\bar{x} = x_1^k$.

2. Set $q^k \in \arg\max_{q \in \mathcal{P}} \mathbb{E}_q[V_\omega^k] - \alpha(q)$
3. Set $\beta_k = \sum_{\omega \in \Omega} q_\omega^k \lambda_\omega^k$
4. Set $\alpha_k = \sum_{\omega \in \Omega} q_\omega^k V_\omega^k - \alpha(q^k) - \beta_k^T x_1^k$
5. Return the cut $\Theta \geq \alpha_k + \beta_k^T x_1$

Multi-cut Version

$$
\text{Multi-}V_1^k = \begin{array}{ll} \min\limits_{x_1,\Theta} & c_1^T x_1 + \Theta \\ \text{s.t.} & A_1 x_1 = b \\ & x_1 \geq 0 \\ & \Theta \geq -M \\ & \Theta \geq \sum_{\omega\in\Omega} q_\omega^k \theta_\omega - \alpha(q^k), k = 1, ..., K-1 \\ & \theta_\omega \geq V_\omega^k + \lambda_\omega^{kT}(x_1 - x_1^k), \forall\omega\in\Omega, k = 1, ..., K-1 \end{array}
$$

- Trade-off: Multi-cut typically requires fewer iterations than single-cut version but each iteration is more expensive
- Hybrid master problem is possible

Variant of Multi-cut Version

$$
\text{Conic-}V_1^k = \begin{array}{ll} \min\limits_{x_1,\Theta} & c_1^T x_1 + \sum_{\omega\in\Omega} p_\omega \mu_\omega + \mu_0 \\ \text{s.t.} & A_1 x_1 = b \\ & x_1 \geq 0 \\ & \theta_\omega \geq -M \\ & \theta_\omega \geq V_\omega^k + \lambda_\omega^{kT}(x_1 - x_1^k), \forall \omega \in \Omega, k = 1, ..., K-1 \\ & \left(-\frac{1}{\gamma}, \mu_0 - \theta_\omega, \mu_\omega\right) \in \mathcal{K}_{\exp}^*, \quad \forall \omega \in \Omega \end{array}
$$

- Compare with Multi-V_1^k, the difference is now we use

$$
\Theta \geq \max_{q\in\mathcal{P}} \left\{ \sum_{\omega\in\Omega} q_\omega \theta_\omega - \frac{1}{\gamma} \sum_{\omega\in\Omega} q_\omega \log\left(\frac{q_\omega}{p_\omega}\right) \right\}
$$

instead of

$$
\Theta \geq \max_{k=1,\dots K-1} \left\{ \sum_{\omega\in\Omega} q_\omega^k \theta_\omega - \frac{1}{\gamma} \sum_{\omega\in\Omega} q_\omega^k \log\left(\frac{q_\omega^k}{p_\omega}\right) \right\}
$$

Multi-stage Risk-averse Stochastic Programming

$$
\begin{aligned}
V_1^K = \quad \min_{x_1,\theta_2} \quad & c_1^T x_1 + \theta_2 \\
\text{s.t.} \quad & A_1 x_1 = b_1 \\
& x_1 \geq 0 \\
& \theta_2 \geq \alpha_{2,k} + \beta_{2,k}^T x_1 \\
& \quad k = 1, ..., K-1 \\
& \theta_2 \geq -M_2
\end{aligned} \tag{22}
$$

where $V_t^K(x_{t-1}, \omega)$ is

$$
\begin{aligned}
\min_{x_t \bar{x}_t, \theta_{t+1}} \quad & c_t^T x_t + \theta_{t+1} \\
\text{s.t.} \quad & \bar{x}_t = x_{t-1} \quad [\lambda] \\
& A_t x_t + B_t \bar{x}_t = b_t \\
& x_t \geq 0 \\
& \theta_{t+1} \geq \alpha_{t+1,k} + \beta_{t+1,k}^T x_t, \\
& \quad k = 1, ..., K-1 \\
& \theta_{t+1} \geq -M_{t+1}
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
V_1 = \quad \min_{x_1} \quad & c_1^T x_1 + \mathbb{F}\left[V_2(x_1, \omega_2)\right] \\
\text{s.t.} \quad & A_1 x_1 = b_1 \\
& x_1 \geq 0
\end{aligned}
$$

where $V_t(x_{t-1}, \omega_t)$ is

$$
\begin{aligned}
\min_{x_t, \bar{x}_t} \quad & c_t^T x_t + \mathbb{F}\left[V_{t+1}(x_t, \omega_{t+1})\right] \\
\text{s.t.} \quad & \bar{x}_t = x_{t-1} \quad [\lambda] \\
& A_t x_t + B_t \bar{x}_t = b_t \\
& x_t \geq 0
\end{aligned}
$$

for $t = 2, ..., T$ and $V_{T+1}(\cdot, \cdot) = 0$.

Algorithm 2: Stochastic dual dynamic programming algorithm with a convex risk measure.

Set $K = 1$

while *not converged* **do**

  `// Forward pass`

  solve master problem (22) and obtain solution x_1^K

  for $t = 2, \dots, T-1$ **do**

    sample ω_t from Ω_t

    solve master problem (23) given (x_{t-1}^K, ω_t) and obtain solution x_t^K

  end

  `// Backward pass`

  for $t = T, \dots, 2$ **do**

    for $\omega_t \in \Omega_t$ **do**

      solve (23) given (x_{t-1}^K, ω_t) to obtain $V_t^K(x_{t-1}^K, \omega_t)$ and an extreme point dual solution, λ

      set $V_{\omega_t}^K = V_t^K(x_{t-1}^K, \omega_t)$

      set $\lambda_{\omega_t}^K = \lambda$

    end

    set $q^K \in \arg\max_{q \in \mathcal{M}(p)} \left\{ \mathbb{E}_q[V_{\omega_t}^K] - \alpha(q) \right\}$

    set $\beta_{t,K} = \sum_{\omega_t \in \Omega_t} q_{\omega_t}^K \lambda_{\omega_t}^*$

    set $\alpha_{t,K} = \sum_{\omega_t \in \Omega_t} q_{\omega_t}^K V_{\omega_t}^K - \alpha(q^K) - \beta_{t,K}^\top x_{t-1}^K$

    Add the cut $\theta_t \geq \alpha_{t,K} + \beta_{t,K}^\top x_t$ to (23) for $t-1$, i.e., updating the model with value V_{t-1}^K to V_{t-1}^{K+1}

  end

  $K \leftarrow K + 1$

end

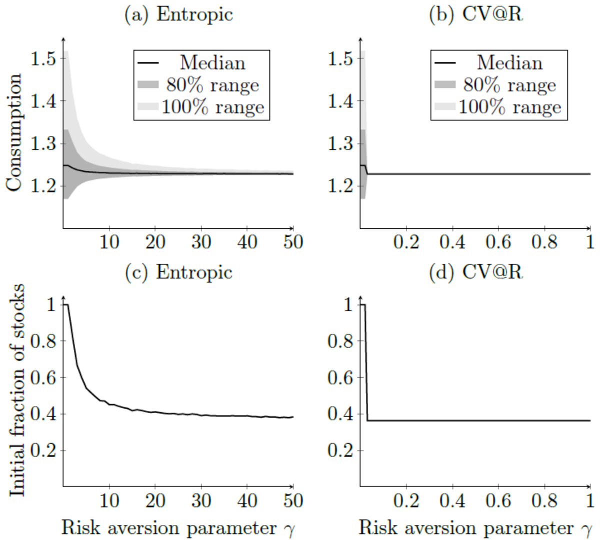
A Simple Example: Portfolio Management

- Number of stages: $T = 5$
- State variables (decision): x_t^s and x_t^b, the quantity of stocks and bonds held at the end of stage t
- Consumption variables (decision): u_t, the quantity of cash consumed in stage t
- Random variables: ω_t^s and ω_t^b, the random return of stocks and bonds realized at the beginning of stage t
  - $(\omega_1^s, \omega_1^b) = (1, 1)$ and $(\omega_t^s, \omega_t^b) = (1.11, 1.02)$ w.p. 0.2 and $(1.04, 1.06)$ w.p. 0.8.
  - (ω_t^s, ω_t^b) are independent across t
- Initial state $(x_0^s, x_0^b) = (0, 1)$.
- The goal is to maximize cumulative consumption $V_1((0, 1), (1, 1)))$.

$$
\begin{aligned}
V_t(x_{t-1}, \omega_t) = \quad \min_{u_t, x_t} \quad & -u_t + \mathbb{F}\left[V_{t+1}(x_t, \omega_{t+1})\right] \\
\text{s.t.} \quad & x_t^s + x_t^b + u_t = \omega_t^s x_{t-1}^s + \omega_t^b x_{t-1}^b \\
& x_t \geq 0 \\
& u_t \geq 0
\end{aligned}
$$

for $t = 1, ..., 5$ and $V_6(\cdot, \cdot) = 0$.

Results



Distribution of consumption and initial fraction of wealth placed in stocks, x_1^s, against the risk aversion parameter γ for the entropic risk measure (a) and (c) and nested $\mathbb{CV@R}$ risk measure (b) and (d).

Remarks on Numerical Examples

- Observations
  - In risk-neutral case, both methods invest all in stocks since the expected profit of stocks is higher:

$$0.2 \times 0.11 + 0.8 \times 0.04 = 0.54 > 0.52 = 0.2 \times 0.02 + 0.8 \times 0.06$$

  - In extremely risk-averse case (worst-case), both methods allocate $4/11$ to stocks and $7/11$ to bonds. A profit

$$\min\left\{\frac{4}{11} \times 0.11 + \frac{7}{11} \times 0.02, \frac{4}{11} \times 0.04 + \frac{7}{11} \times 0.06\right\} = \frac{\min\{0.58, 0.58\}}{11}$$

    is guaranteed regardless of uncertainty realization.
  - As risk aversion parameters increase, the solution of ENT gradually changes while the solution of CVaR changes sharply
- Implementation of stochastic dual dynamic programming (SDDP): `SDDP.jl`

Concluding Remarks

- Define conditional consistency and show ENT is conditionally consistent
  - No characterization of the class of conditionally consistent risk measure
- Extend L-shaped method and SDDP to multi-stage stochastic programming with ENT.
  - Computation are limited to toy examples, large-scale computational studies are needed
- The convergence result of SDDP directly follows from[4]
- Choice of risk aversion parameter γ is a problem

Thank You! Questions?

[4] Vincent Guigues. "Convergence analysis of sampling-based decomposition methods for risk-averse multistage stochastic convex programs". In: *SIAM Journal on Optimization* 26.4 (2016), pp. 2468–2494.