ECE641 Lab1 Report

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1 Introduction

N.A.

2 MAP Estimation with Gaussian Priors

2.1 Prior Model Formulation

2.1.1

We know that the PDF of the prior model is:

$$p(x) = \frac{1}{z(g,p,\sigma)} \exp\left\{ -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i - x_j|^p \right\}, \tag{1}$$

and it is obvious that:

$$\int p(x)dx = 1 \tag{2}$$

so,

$$\begin{aligned}
\int \frac{1}{z(g,p,\sigma)} \exp\left\{ -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i - x_j|^p \right\} dx = 1 \\
\frac{1}{z(g,p,\sigma)} \int \exp\left\{ -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i - x_j|^p \right\} dx = 1 \\
z(g,p,\sigma) = \int \exp\left\{ -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i - x_j|^p \right\} dx
\end{aligned} \tag{3}$$

Let $y_i = \frac{x_i}{\sigma}$, then $dy_i = \frac{1}{\sigma}dx_i$, hence $dx = \sigma^N dy$(Jacobian matrix), so:

$$\begin{aligned}
z(g,p,\sigma) &= \sigma^N \int \exp\left\{ -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|y_i - y_j|^p \right\} dy \\
&= \sigma^N z(g,p,1)
\end{aligned} \tag{4}$$

2.1.2

From (5):

$$p(x|\sigma) = \frac{1}{z(g,p,1)\sigma^N} \exp\left\{ -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i - x_j|^p \right\} \tag{5}$$

take log in both sides:

$$\log\{p(x|\sigma)\} = -\frac{1}{p\sigma^p} \sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i - x_j|^p - N\log\left\{\frac{\sigma}{z(g,p,1)}\right\} \tag{6}$$

calculate the derivatives of equation 6 w.r.t σ and make it as zero:

$$\begin{aligned}\frac{\partial \log\{p(x|\sigma)\}}{\partial\sigma} =&\sigma^{-p-1}\sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i-x_j|^p-\frac{N}{\sigma}=0\\ &\frac{1}{\sigma^p}(\sigma^{-p}\sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i-x_j|^p-N)=0\\ &\sigma^{-p}\sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i-x_j|^p-N=0\\ &\hat{\sigma^p}=\frac{1}{N}\sum_{\{i,j\}\in\mathcal{C}} g_{i,j}|x_i-x_j|^p\end{aligned}\tag{7}$$

2.1.3

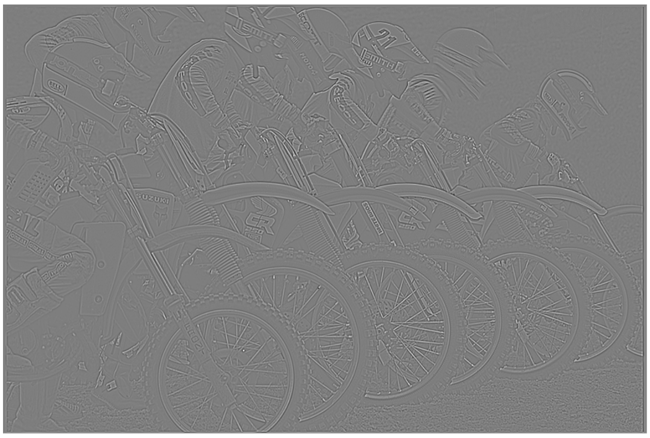

Figure 1: noncausal prediction error for the image img04.tif

2.1.4

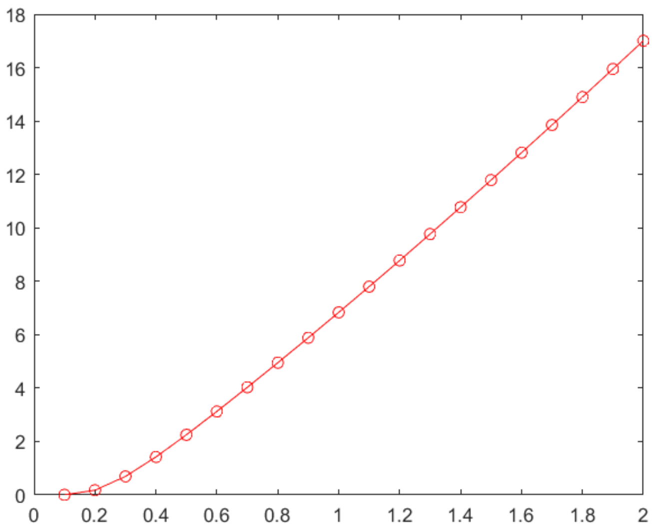

Figure 2: plot of the ML estimate of the scale parameter σ for values p in the range 0.1-2.0

2.2 MAP Restoration with Additive Noise and Gaussian Prior

2.2.1

The cost function can be represented by:

$$f(x) = \frac{1}{2}||y - x||_\Lambda^2 + \frac{1}{2}x^t Bx \tag{8}$$

where $\Lambda = \frac{1}{\sigma_W^2} I$ and $B_{i,j} = \frac{1}{\sigma_x^2}(\delta(i-j) - g_{i,j})$. So the Hessian matrix of the function is:

$$[H(x)]_{s,r} = \frac{\partial^2 f(x)}{\partial x_s \partial x_r} = \Lambda + B \tag{9}$$

which is positive definite, so it is strictly convex.

2.2.2

$$\begin{aligned}(\hat{x}, \hat{\sigma_W^2}) &= \underset{x,\sigma_W^2}{\operatorname{argmax}} \left\{ p_{x|y}(x|y, \sigma_W^2) \right\} \\ &= \underset{x,\sigma_W^2}{\operatorname{argmax}} \left\{ \log p(y|x, \sigma_W^2) + \log p(x) \right\}\end{aligned} \tag{10}$$

For σ_W^2, calculate derivative:

$$\frac{\partial \log p(x|y, \sigma_W^2)}{\partial \sigma_W} = \frac{\partial \log p(y|x, \sigma_W^2)}{\partial \sigma_W} = 0 \tag{11}$$

and we know that:

$$\log p(y|x, \sigma_W^2) = -\frac{1}{2\sigma_W^2} \sum_{i \in S} (y_i - x_i)^2 - \frac{N}{2} \log (2\pi\sigma_W^2) \tag{12}$$

so, substitute equation 12 to equation 11:

$$\frac{1}{2\sigma_W^4} \sum_{i \in S} (y_i - x_i)^2 - \frac{N}{2\sigma_W^2} = 0 \tag{13}$$

so,

$$\hat{\sigma_W^2} = \frac{1}{N} \sum_{i \in S} (y_i - x_i)^2 \tag{14}$$

for x:

$$\hat{x} = \underset{x}{\operatorname{argmax}} \left\{ \log p(y|x, \sigma_W^2) + \log p(x) \right\} \tag{15}$$

and we know that:

$$\log p(x) = -\frac{1}{2\sigma_x^2} \sum_{\{i,j\} \in C} g_{i,j}(x_i - x_j)^2 \tag{16}$$

Using the similar method as above, we can get:

$$\hat{x} = \underset{x}{\operatorname{argmin}} \left\{ \frac{N}{2} \log \left(\sum_{i \in S} (y_i - x_i)^2 \right) + \frac{1}{2\sigma_x^2} \sum_{\{i,j\} \in C} g_{i,j}(x_i - x_j)^2 \right\} \tag{17}$$

2.2.3

Yes, because it is changed towards the direction of gradient descent

2.2.4

No.

2.3 Iterative Coordinate Descent Optimization (ICD)

2.3.1

Coordinate descent is gradient descent with line search.

2.3.2

The Hessian matrix of the cost function is positive definite.

2.3.3

Adding i.i.d. Gaussian noise with mean zero and $\sigma_W^2 = 16^2$, the image is shown in Figure. 3.

Figure 3: The noisy image Y

2.3.4

The restored image with $\sigma_x^2 = 16.95$ and $\sigma_W^2 = 16$ is shown in Figure. 4.

Figure 4: MAP estimate of the image, $\sigma_x^2 = 16.95$

2.3.5

The cost decrease is shown in Figure. 5

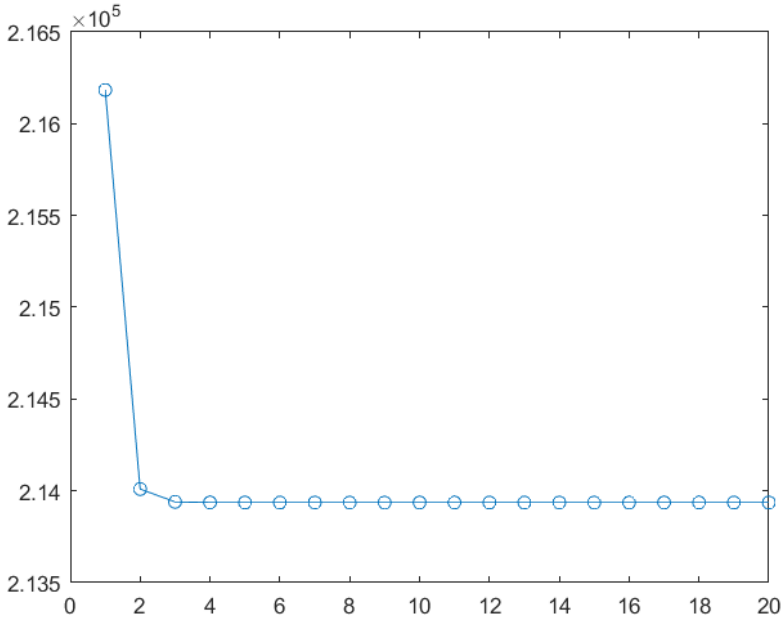

Figure 5: The cost of the noisy image restoration

2.3.6

For $\sigma_x^2 = 5 * \hat{\sigma_x^2}$, the restored image is shown in Figure. 6.

Figure 6: MAP estimate of the image, σ_x^2 = 5*16.95

For $\sigma_x^2 = (1/5) * \hat{\sigma_x^2}$, the restored image is shown in Figure. 7.

Figure 7: MAP estimate of the image, $\sigma_x^2 = 5*16.95$

2.4 MAP Restoration from Blurred/Noisy Image with Gaussian Prior

2.4.1

Adding i.i.d. Gaussian noise with mean zero and $\sigma_W^2 = 4^2$, the image is shown in Figure. 8.

Figure 8: The blurred and noisy image Y

2.4.2

The restored image is shown in Figure. 9

2.4.3

The cost is shown in Figure. 10

Figure 9: The restored image X

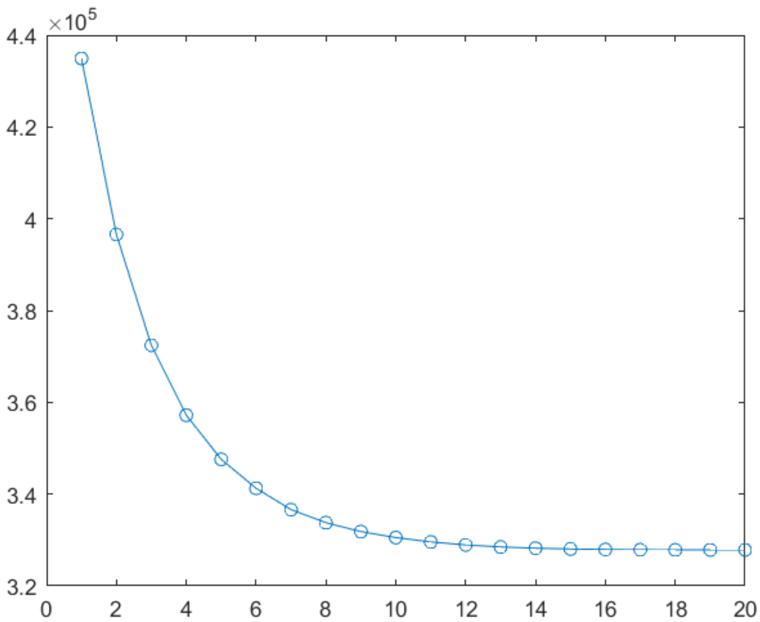

Figure 10: The cost of the noisy and blurred image restoration