MULTIPRODUCT MONOPOLY BUNDLING [1]

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Abstract

A monopolist must choose how to bundle her goods. We make progress toward characterizing the optimal bundling strategy of an expected profit-maximizing monopolist with several goods to sell. When consumer values are stochastically independent, any optimal menu of bundles must be "connected". Every two goods must be connected by a path, each step of which is between two goods that are in a bundle together.

Introduction

How should a monopolist bundle? When there are just two goods, McAfee, McMillan, and Whinston (1989) provides an answer. If consumer values for two non-complementary goods are stochastically independent, they should always be bundled. That is to say, a monopolist may increase profits in expectation by offering the menu of bundles $\{1, 2, (1, 2)\}$ at prices $\{p_1^*, p_2^*, p_{(1,2)}^*\}$ than by offering the menu $\{1, 2\}$ at prices $\{p_1^{'}, p_2^{'}\}$. This paper addresses the case of several goods. We capitalize on the perturbation intuition at the heart of McAfee, McMillan, and Whinston's result to prove that certain sorts of bundling are sub-optimal when consumer values are independent. We show that any optimal menu of bundles must be "connected". Every two goods must be connected by a path, each step of which is between two goods that are in a bundle together.

The case for bundling two goods rests essentially upon a perturbation intuition. Suppose that two unbundled goods are offered at expected profit-maximizing prices $\{p_1^{'}, p_2^{'}\}$. By the first-order condition, if we raise $p_2 : p_2^{'} \to p_2^{'} + \varepsilon$, our gains from those who continue to purchase good 2 will be counterbalanced by our losses from those who cease to buy good 2. If we include a bundle at the redundant price $p_1^{'} + p_2^{'}$ and *then* raise p_2, however, the dynamics of consumer reaction to the price perturbation will be different. By "adding up across lines" the relative profit effects of consumer behavior in this as opposed to the initial scenario with no bundle, we can understand in a visual way why the bundle's presence makes the monopolist better off.

- Consumers with low values for good 1, $v_1 < p_1^{'} - \varepsilon$, will not change their behavior.

- Consumers with borderline values for good 1, $v_1 \in [p_1^{'} - \varepsilon, p_1^{'}]$, may change their behavior. A) All with high values for good 2, $v_2 \in (p_2^{'} + \varepsilon, \infty)$, will purchase the bundle rather than continuing to purchase good 2. These types yield a first-order relative gain. B) Some with borderline values for good 2, $v_2 \in [p_2', p_2' + \varepsilon]$, will choose to purchase the bundle rather than ceasing to purchase good 2. These types yield a second-order relative gain. *In total, these types yield a first-order relative gain.*

- Consumers with high values for good 1, $v_1 \in (p_1', \infty)$, will change their behavior and acquire exactly what they did before the price perturbation. *By independence and the first-order condition, these types yield no first-order relative gain or loss.*

Overall, we have a first-order relative gain; bundling is better. Independence allows us to conclude that the tradeoff from increasing p_2 due to consumers along each line in the type space, $\{(\hat{v}_1, v_2) : \hat{v}_1 > p_1'\}$, is precisely the same as the aggregate tradeoff from increasing p_2. And this aggregate tradeoff is zero by the first-order condition. In this paper, we extend and apply this appealing perturbation intuition to a setting of several goods.

Let $N = \{1, 2, \ldots, n\}$ be a set of unique[1] non-complementary goods with zero production costs[2] and $\mathfrak{B} = \mathbf{P}(N)$ the set of subsets of N; each subset $\mathfrak{U} \subset \mathfrak{B}$ is a menu of bundles. To ascertain a partial expected profit-ordering on the set of possible menus, we ask the question, "When can a monopolist increase expected profit by adding another bundle?" Suppose that the bundles of an original menu $\mathfrak{U}$ are optimally priced. By the first-order condition, any perturbation of prices will have no first-order effect on profit. Once we add a bundle B, however, the consumer may substitute between bundles in new ways; in the new setting, the same perturbation of prices can have a first-order effect. To determine whether this is the case, it suffices to characterize

[1] Our analysis does not require uniqueness but that individual consumers do not acquire multiple copies of the same good. This restriction constitutes a loss of generality in that our analysis does not also apply to the expanded (and infinite) space of all bundles with multiple copies of the various goods.

[2] We can accomodate our analysis to scenarios of positive costs by replacing consumer values $(v_1, \ldots, v_N)$ with net consumer values, $(v_1 - c_1, \ldots, v_N - c_N)$, which will be independent given the independence of $v_1, \ldots, v_N$.

all consumer types whose behavior in response to the price perturbation differs depending on whether B is offered and to aggregate these relative effects on profit. If they constitute a net first-order gain, then the monopolist is strictly better off adding B to $\mathfrak{U}$.

Section 1 simplifies the problem by aggregating all first-order relative effects into just a few classes. Section 2 offers a proof of the main result. Section 3, finally, illustrates with an example the limits of perturbation analysis to characterize optimal bundling, even given independence.

1 Simplifying the Problem

Consumer types who adjust in the same way to the price perturbation with and without a new bundle can be grouped into equivalence classes. We will denote by $\{X_1 \to X_2 \Rrightarrow X_1 \to X_3\}$ the consumer class which purchases the goods in X_1 at the original optimal prices, switches to purchase the goods in X_2 after the price perturbation when the new bundle is not offered, and switches to X_3 after the price perturbation with the new bundle. For any set X_i of goods, there is a cheapest way to acquire them. All consumers will acquire these goods in this way, at the cheapest possible price $p^*_{X_i}$.

The aggregate relative effect of a consumer class is first-order or second-order and so on depending on the magnitude of the relative effect of each of the members of that class and the probability that a randomly drawn consumer belongs to that class. For example, in the two-good case when we add the bundle at the price $p_{(1,2)} = p^*_1 + p^*_2$ and perturb $p_2 : p^*_2 \to p^*_2 + \delta$,

- $\{1, 2 \to 1, 2 \Rrightarrow 1, 2 \to (1, 2)\}$ constitutes a first-order relative loss since each of its consumer types pays marginally less with the bundle;
- $\{1, 2 \to 1 \Rrightarrow 1, 2 \to (1, 2)\}$ constitutes a first-order relative gain since its corresponding region in the type-space is constrained to satisfy $v_2 - p^*_2 \in [0, \delta]$;

- $\{2 \to 2 \Rrightarrow 2 \to (1,2)\}$ constitutes a first-order relative gain since its region is constrained to satisfy $p_1^* - v_1^* \in [0, \delta]$; and
- $\{2 \to \emptyset \Rrightarrow 2 \to (1,2)\}$ constitutes a second-order relative gain since its region must satisfy two constraints:

$$\begin{aligned}
&v_2 - p_2^* \in [0, \delta] \\
&(v_2 - p_2^*) - (v_1 + v_2 - p_{(1,2)}) = p_1^* - v_1 \in [0, \delta]
\end{aligned}$$

In general, if the cheapest price of X_1 is raised, $p_{X_1} : p_{X_1}^* \to p_{X_1}^* + \delta$, the region corresponding to $\{X_1 \to X_2 \Rrightarrow X_1 \to X_3\}$ must satisfy the constraints

$$\begin{aligned}
&\Big(\sum_{g \in X_1} v_g - p_{X_1}^*\Big) - \Big(\sum_{g \in X_2} v_g - p_{X_2}^*\Big) \in [0, \delta] \\
&\Big(\sum_{g \in X_1} v_g - p_{X_1}^*\Big) - \Big(\sum_{g \in X_3} v_g - p_{X_3}^*\Big) \in [0, \delta].
\end{aligned}$$

Whenever these constraints are linearly independent,[3] the relative effect of $\{X_1 \to X_2 \Rrightarrow X_1 \to X_3\}$ consumers is at most second-order. As long as the sum of all first-order relative effects is non-zero, we may ignore these second-order effects. Thus, we may greatly simplify our analysis of the (numerous) substitution effects.

Suppose that there exists $C_1 \subseteq N : B \in \mathfrak{U}$ and $B \cap C_1 \neq \emptyset \Rrightarrow B \subseteq C_1$. We will consider the effect of adding the bundle $(C_1 \cup B_2)$, the union of C_1 with a bundle $B_2 \in \mathfrak{U}$ disjoint from C_1, when we raise the price of $B_2, p_{B_2} : p_{B_2}^* \to p_{B_2}^* + \delta$. In this case, we may enumerate all relevant consumer classes with a first-order relative effect:

- **I**: $\{C_1, B_2 \to C_1, B_2 \Rrightarrow C_1, B_2 \to (C_1 \cup B_2)\}$: relative loss;

[3]Regard X_i as a vector in $\mathbf{R}^n$ whose kth coordinate is 1 if good k belongs to a bundle in X_i and 0 otherwise. "The constraints are linearly independent" means that $X_1 - X_2$ and $X_1 - X_3$ are linearly independent when viewed in this way.

- **II**:$\{C_1, B_2 \to C_1 \Rrightarrow C_1, B_2 \to (C_1 \cup B_2)\}$: relative gain;

- **III**: $\{C_1, B_2, Y_2 \to C_1, Z_2 \Rrightarrow C_1, B_2, Y_2 \to (C_1 \cup B_2), Y_2\}$, where $Y_2 \subseteq N \setminus (C_1 \cup B_2)$ and $Z_2 \cap B_2 \neq \emptyset$: possibly relative gain or loss;[4]

- **IV**: $\{Y_1, B_2 \to Y_1, B_2 \Rrightarrow Y_1, B_2 \to (C_1 \cup B_2)\}$, where $Y_1 \subsetneq C_1$: relative gain;

Claim 1: *Any consumer class with a first-order relative effect on monopolist profit when the monopolist raises $p_{B_2} : p^*_{B_2} \to p^*_{B_2} + \delta$ with and without the bundle $(C_1 \cup B_2)$ is of type* **I**, **II**, **III**, or **IV**.

Proof: First, the consumer class $\{X_1 \to X_2 \Rrightarrow X_1 \to X_3\}$ can not have a first-order effect when the constraints imposed by the two substitutions $X_1 \to X_2$ and $X_1 \to X_3$ are linearly independent. And we get linear dependence here only when one of the constraints is empty or both are equal, i.e. if and only if $X_1 = X_2, X_1 = X_3$, or $X_2 = X_3$. Second, the only consumers with any relative effect on profits (the "relevant" consumers) are those who purchase B_2 before the price perturbation when $(C_1 \cup B_2)$ is not available and who purchase $(C_1 \cup B_2)$ after the price perturbation when it is available.

Since relevant consumers must switch to the bundle $(C_1 \cup B_2)$ in the second scenario, $X_1 = X_3$ only when $C_1 \cup B_2 \subseteq X_1$. Among these consumers, those for whom $C_1 \cup B_2 \subseteq X_2$ are type **I**; those who originally drop B_2 without switching to any bundles that overlap B_2 are type **II**; and those who originally drop B_2 and switch to some bundles that overlap B_2 are type **III**.

Similarly, $X_2 = X_3$ only when $C_1 \cup B_2 \subseteq X_2$. In this case, however, $C_1 \cup B_2 \subseteq X_1$ since only the price of B_2 increases. (Any consumer who

[4]Types **I**, **II**, and **IV** are each aggregations of several consumer classes – they may or may not purchase a variety of bundles disjoint from both C_1 and B_2. In types **III**, we explicitly include every class of consumers who originally switch to some bundles which overlap B_2 and therefore may adjust their entire portfolio of purchases in $N \setminus C_1$ in a complicated way. Some of these type **III** classes constitute a relative gain, others a relative loss.

continues to purchase B_2 when p_{B_2} is raised will not wish to alter his other purchases.) Thus, $X_2 = X_3$ for type **I** consumers only.

Finally, for the same reason, the relevant consumers for whom $X_1 = X_2$ are precisely those who continue to purchase B_2 after the price increase. Those who also continue to purchase all goods in C_1 are type **I**, while those who continue to purchase not all of the goods in C_1 are type **IV**.

■

2 Connected Bundle Menus

A menu of bundles is **connected** if every two goods can be connected by a **bundle path,** each step of which is between two goods that are in a bundle together.

Claim 2: *If a menu of bundles $\mathfrak{U}$ is not connected, then there exist $C_1, C_2 : C_1 \cup C_2 = N, C_1 \cap C_2 = \emptyset$, and $B \in \mathfrak{U} \Rrightarrow B \subseteq C_1$ or $B \subseteq C_2$.*

Proof: If $\mathfrak{U}$ is not connected, then by definition there exist two goods g_1, g_2 which can not be connected by a bundle path. Let Γ_1 be the set of goods which are included in a bundle with g_1. Define Γ_{i+1} recursively to be the set of goods which are included in a bundle with some good in Γ_i. (Each good is included with itself, so that $\Gamma_i \subseteq \Gamma_{i+1}$.) Since N is finite, Γ_∞ is well-defined. Since $\mathfrak{U}$ is not connected, $\Gamma_\infty \neq N$.

Suppose that there exists $B \in \mathfrak{U}$: $B \cap \Gamma_\infty, B \cap (N \setminus \Gamma_\infty) \neq 0$. In this case, $\Gamma_\infty \supsetneq \Gamma_\infty \cup (B \cap (N \setminus \Gamma_\infty))$, a contradiction. We may therefore set $C_1 = \Gamma_\infty, C_2 = N \setminus \Gamma_\infty$ to satisfy Claim 2.

■

Theorem: *Any expected profit-maximizing menu of bundles must be connected.*

PROOF: Suppose not. Let C_1, C_2 be the sets guaranteed by CLAIM 2 and $B_2 \in \mathfrak{U}$ a bundle disjoint from C_1. (B_2 may be a degenerate bundle, any good in C_2.)

Consider the relative effects of including the bundle $(C_1 \cup B_2)$ when the price of B_2 is marginally increased. Since all original bundles are either disjoint from or contained in C_1, no consumer will adjust his purchases of goods in C_1 when p_{B_2} rises. We may therefore characterize consumers by a portfolio substitution $B_2, Y_2 \to Z_2$, where $Y_2, Z_2 \subseteq C_2$ are disjoint from C_1.

By the first-order condition, when the price of B_2 is raised without the bundle $(C_1 \cup B_2)$, consumers will make a variety of changes in their portfolio of purchases whose profit effects will exactly offset each other first-order in sum. By independence, when p_{B_2} is raised without the bundle $(C_1 \cup B_2)$, the effects of all consumer types exactly offset on each B_2-hyperplane.[5] Types **I**, **II**, and **III** are precisely the consumers who originally purchase B_2 and all of the goods in C_1. Restricting ourselves to those B_2-hyperplanes on which consumers wish to purchase all of the goods in C_1,[6] we may therefore conclude that the net original effect of type **I**, **II**, and **III** consumers due to the perturbation before the bundle is added is zero. But since $p_{(C_1 \cup B_2)} = p^*_{C_1} + p^*_{B_2}$, the revenue from types **I**, **II**, and **III** after the perturbation with the bundle is the same as before the perturbation without the bundle. Hence, the net *relative* effect of these types is also zero.

All type **IV** consumers have a positive first-order relative effect, so we are done. ■

The partial ordering induced by this result is incomplete. In the case of three goods, we can only eliminate

[5]Define a "B_2-hyperplane" to be a $|B_2|$-dimensional subspace of the type-space on which consumers' valuations of all of the goods in B_2 are fixed.

[6]If it is impossible for consumers to acquire all of the goods in C_1, then consumers of type **I**, **II**, and **III** do not exist. Trivially, we reach the same conclusion that the first-order relative effects of these types sum to zero.

$$\{1,2,3\},\{1,2,3,(1,2)\}\{1,2,3,(1,3)\},\{1,2,3,(2,3)\}$$

as unpreferred. Nonetheless, we can conclude in general that every good should be offered in some sort of bundle. If a good g is not offered in a bundle, then we may set $C_1 = g$ so that $\mathfrak{U} \cup (g \cup B)$ is a strictly better menu for the monopolist to offer than $\mathfrak{U}$, for any $B \in \mathfrak{U}$.

3 The General Difficulty

Consider an example in which there are four goods[7] and $\mathfrak{U} = \{1,2,3,4,(1,2),(3,4)\}$. We examine the relative effect of including (2,3) when we increase $p_2 : p_2^* \to p_2^* + \delta$.

The relevant consumer classes are

- **I**: $\{2,3 \to 2,3 \Rrightarrow 2,3 \to (2,3)\}$: relative loss;
- **II**:$\{2,3 \to 3 \Rrightarrow 2,3 \to (2,3)\}$: relative gain;
- **III**: $\{2,3 \to (1,2),3 \Rrightarrow 2,3 \to (2,3)\}$: relative loss;
- **IV**: $\{2 \to 2 \Rrightarrow 2 \to (2,3)\}$: relative gain;
- **V**: $\{2,(3,4) \to 2,(3,4) \Rrightarrow 2,(3,4) \to (2,3)\}$: relative loss.

Types **I**, **II**, **III**, and **IV** in this scenario are analogous to the four consumer types of CLAIM 1. A fifth type is necessary since some relevant consumers may originally purchase some bundles that contain/overlap good 3. (No bundles contained or overlapped C_1 by construction, so this type did not

[7]The reader may be curious what we can not prove in the case of three goods. The most sticky comparison to make is between menus {1,2,3,(1,2),(2,3),(1,2,3)} and {1,2,3,(1,2),(1,3),(2,3),(1,2,3)} (we add the bundle (1,3) and raise p_1). The problem here is that we can not represent all consumers' initial adjustments to the price perturbation as a portfolio substitution $Z_1 \to Z_2$ disjoint from good 3. Thus, we can no longer pin down the net relative effect of **I**, **II**, and **III** types by "adding up along v_3-planes" as we did in the proof of the theorem.

exist in the setting of Claim 1. In the language of Claim 1, type **V** is another – and the only other – sort of consumer for which $X_1 = X_2$.)

When p_2 is raised in the absence of (2,3), the first-order loss from $2 \to \emptyset$ types is exactly offset by the gain from $2 \to 2$ and $2 \to (1,2)$ types. There are no complicated substitutions because the menu $\mathfrak{U}$ has no overlapping bundles. Indeed, we can characterize consumers by a portfolio substitution $Z_1 \to Z_2$ disjoint from goods 3 and 4. Hence, by independence the original effects of the price perturbation exactly offset on each v_3v_4-plane. Restricting attention to those v_3v_4-planes on which the consumer chooses to purchase good 3 (and not the bundle (3,4)), we see that the original effects of types **I**, **II**, and **III** exactly offset. (**I**, **II**, and **III** aggregate precisely the consumers on these v_3v_4-planes which purchase good 2.) Hence, their net relative effect is zero.

We are left to compare the positive relative effect of type **IV** with the negative relative effect of type **V** consumers. Either effect may dominate the other, depending upon the magnitude of each (a $p_3^* - \delta$ relative gain due to type **IV** versus a $p_{3,4}^* - p_3^* + \delta$ relative loss due to type **V**) and the distribution of consumer values. We can tailor an independent distribution so that the addition of the bundle $(2,3)$ to the menu $\{1, 2, 3, 4, (1,2), (3,4)\}$ is locally a loser.

Conclusion

At least in the special case of independent consumer values, a straightforward geometrical argument suffices to make significant progress to characterize optimal non-stochastic[8] multiproduct monopoly bundling. Independence allows us to "add up relative effects across hyperplanes" and thereby prove that any

[8] Rochet (1994) shows that the assumption of independence does not rule out the possibility that stochastic bundle prices may be best. McAfee and McMillan's (1988) "generalized single-crossing property" does imply, however, that optimal bundling will be non-stochastic.

optimal menu of bundles must be connected. Even in this specialized setting, however, it is impossible to prove that "more bundling is always better" or to pinpoint the expected profit-maximizing menu of bundles with just the tools of price perturbation analysis.

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