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Pricing of Product Bundles

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Gaussian Demand and Commodity Bundling

In order to obtain comparative results for alternative pricing strategies, the distribution of reservation prices in the Adams-Yellen framework is assumed to be bivariate normal. Implications of Gaussian demand are explored. Pure bundling is shown to operate by reducing buyer diversity, thus facilitating the capture of consumers' surplus. It generally makes buyers worse off than unbundled sales; it is more profitable if average reservation prices are high enough. Mixed bundling combines advantages of both pure bundling and unbundled sales, and it is generally strictly more profitable than either. Bundling, which treats goods symmetrically, seems most attractive under symmetric reservation price distributions.

I. Introduction

In an important and widely cited essay, Adams and Yellen (1976) examine the use of package selling as a price discrimination device.[1] They consider a monopolist producing two goods with constant unit costs and facing buyers with diverse tastes. The marginal utility of a second unit of either good is assumed to be zero for all buyers. The two goods are independent in demand for all buyers, so that any buyer's reservation price for the first unit of either good is independent of the market price of the other. Thus the maximum amount any buyer would pay for a bundle consisting of one unit of each good is the sum of the two reservation prices, and buyers are completely described by the values of those two prices.[2] Resale markets are assumed away.

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1. The first clear recognition of this possibility seems to have been by Stigler (1968), but also see Burstein (1960).

2. Paroush and Peles (1981) relax the assumption that consumers purchase at most one unit of each good, but they consider only two types of buyers, both with linear demand functions. (They retain the assumption that demands are independent for all buyers.) Telser (1979) and Phillips (1981) use different frameworks to analyze bundling policies. Spence (1980) explores the relation between bundling across commodities, as here, and bundling multiple units of the same

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Adams and Yellen consider three different sales strategies: *unbundled sales* (the two goods are priced and sold separately), *pure bundling* (only a bundle consisting of one unit of each good is sold), and *mixed bundling* (both the bundle and the two goods are offered). (They refer to unbundled sales as "a pure components strategy.")

Mainly through the use of examples, Adams and Yellen (1976) examine the possible implications of switches among these strategies for seller profit and net welfare (profit plus consumers' surplus). Unfortunately, they do not derive general conditions under which bundling increases either profit or welfare, and they do not suggest any general principles or insights that could develop the reader's understanding of such issues.[3] Thus, they observe merely that "whether one [pricing strategy] generates more profits than another depends on the prevailing level of costs and on the distribution of customers in reservation price space" (p. 488). Similarly, at the conclusion of their normative analysis, they note only that "prohibition of bundling without more might make society worse off" and that "the deadweight loss associated with bundling might also exceed the corresponding loss associated with simple monopoly pricing" (p. 495). It is not at all apparent, however, that much more is implied by the general assumptions that Adams and Yellen employ.

The present essay takes what seems to be a logical next step under these circumstances. The Adams-Yellen model is specialized by imposing restrictions on the admissible patterns of buyers' tastes or, equivalently, on the distribution of reservation price pairs in the population. This restriction to a class of examples enables us to say a good deal about the effects of bundling on seller profit, consumers' surplus, and net welfare. While my detailed findings are, of course, necessarily valid only for the class of examples considered, they serve both to enhance intuition about bundling and its consequences and to suggest patterns that might hold more generally.

Specifically, this essay adds to the assumptions of Adams and Yellen (1976) outlined above the additional restriction that buyers' reservation price pairs follow a bivariate normal distribution.[4] The frequency with which normal distributions arise in the social sciences makes the Gaussian family a plausible choice to describe the distribution of tastes in a

commodity in nonlinear pricing. For applications of the basic Adams-Yellen framework employed here, see Adams and Yellen (1977) and Schmalensee (1982).

3. Adams and Yellen (1976) do develop a general relation between profits under pure bundling and under mixed bundling. This relation is discussed below. In addition, the Adams-Yellen analysis of the difficulties of using standard tools to measure the social cost of a bundling monopoly leads to some general conclusions.

4. Adams and Yellen (1976, p. 488, n. 15) indicate that they performed some simulations for this case. However, they report only that "for every characterization of tastes we studied, bundling in *some* form was preferred to pure components pricing for *some* cost conditions" (emphasis added).

population of buyers. In addition, the bivariate normal has a small number of easily interpreted parameters. A final attraction of the Gaussian case is that the distributions of reservation prices for each good separately and for the bundle (composed of one unit of each good) are all normal. This greatly facilitates comparison of unbundled sales and pure bundling.

There are, however, two difficulties with the assumption of normality. First, even under a policy of unbundled sales, profit-maximizing solutions do not exist in closed form. Though analytical methods can carry us a long way, full exploration of the Gaussian case seems to require a good deal of numerical analysis. In the interests of clarity and brevity, assertions in what follows that are supported only by computer analysis and not by general proofs are often indicated by the italicized adverb *apparently*. Second, it follows from the analysis of Adams and Yellen (1976, p. 483) that for all reservation price distributions that have density everywhere in the positive quadrant (as the nondegenerate bivariate normal does), mixed bundling is strictly more profitable than pure bundling. (It can obviously never be strictly less profitable than pure bundling for any distribution.) But in the Gaussian case it is quite difficult to obtain optimal mixed bundling policies numerically, and we have accordingly not performed a complete analysis of this strategy.

Section II analyzes unbundled sales under Gaussian demand. Some general results are obtained that may be of independent interest because Gaussian demand has not received much study. These results are used extensively in later sections. Section III compares pure bundling and unbundled sales. Even though pure bundling cannot be profit maximizing in this model, as noted above, it may in fact be optimal when there are setup costs associated with offering additional product varieties or with maintaining more complex pricing arrangements. It may also be more difficult to prevent resale under mixed bundling than under pure bundling. Moreover, this comparison may provide insights useful in the analysis of other situations, involving non-Gaussian demands, in which pure bundling is optimal. Finally, since mixed bundling differs from unbundled sales by the addition of a bundle to the firm's offerings, an analysis of pure bundling enhances understanding of the operation of mixed bundling policies. In Section IV the profitability of mixed bundling is compared to that of unbundled sales. Section V summarizes the general results and implications of this analysis.

II. Unbundled Sales

Let $Q^1(P^1, P^2)$ and $Q^2(P^1, P^2)$ be the demand functions for the monopoly's two products when both are priced and sold separately. The assumption of demand independence implies that the demand for either good under unbundled sales is independent of the price of the other, so that these demand functions may be written as $Q^1(P^1)$ and $Q^2(P^2)$. Because buyers are interested in at most one unit of either good, if we pass to a continuum model and drop superscripts to reduce clutter, either demand function may be written as

$$Q(P) = \int_P^\infty g(x)dx, \tag{1}$$

where $g(x)$ gives the density of buyers at reservation price x. Since unit costs are constant, we can normalize by setting $Q(-\infty) = 1$ with no loss of generality. In the Gaussian case, $g(x)$ is then a univariate normal density function.

If the mean and the standard deviation of $g(x)$ are μ and σ, respectively, $(x - \mu)/\sigma$ follows the standard (mean $= 0$, standard deviation $= 1$) normal distribution. Let $f(t)$ be the standard normal density, $[(2\pi)^{-1/2} \exp(-t^2/2)]$, and define

$$F(x) = \int_x^\infty f(t)dt. \tag{2}$$

The function F is one minus the standard normal distribution function; it is everywhere strictly decreasing. It follows from the discussion above that in the Gaussian case, demand for a single good under unbundled sales is given by

$$Q(P) = F[(P - \mu)/\sigma]. \tag{3}$$

One can show that for this demand function, elasticity is increasing in P and decreasing in μ. Higher values of σ make demand less elastic if $P \geqslant \mu$ and (by continuity) for some values of $P < \mu$. The demand curve given by (3) is strictly concave for $P < \mu$ and strictly convex for $P > \mu$. While curves of this shape are not commonly encountered in textbooks, the Gaussian distribution from which (3) follows seems much more plausible in this context than the uniform distribution, for instance, that one would need to posit in order to justify more familiar looking linear demand curves. Note also that *any* reservation price distribution with a nonzero mode implies a demand function that goes from concavity to convexity as price rises past the modal value.

A number of readily derived properties of the functions f and F that hold for all values of x are employed in what follows:

$$F_x(x) = -f(x), \tag{4a}$$

$$F_{xx}(x) = -f_x(x) = xf(x), \tag{4b}$$

$$\int_x^\infty t\, f(t)dt = f(x), \tag{4c}$$

$$f(x) > xF(x). \tag{4d}$$

(Equation [4c] directly implies [4d] and is obtained by integration by parts.) Here and in all that follows, subscripts indicate differentiation.

Let C be the constant unit cost of producing the good being considered, and define the following new variables:

$$z = (P - C)/\sigma \tag{5a}$$

and

$$\alpha = (\mu - C)/\sigma. \tag{5b}$$

Under competition z would equal zero and, from (3), sales would equal $F(-\alpha)$. Thus α is a scaling variable, reflecting the strength of demand relative to cost. One can think of z as a normalized or standardized markup.

Using (3), (5a), and (5b), profit from a single good under unbundled sales, Π, can be written as follows:

$$\Pi = (P - C)F[(P - \mu)/\sigma] = \sigma[zF(z - \alpha)] = \sigma\{\pi\}. \tag{6a}$$

Note that $\pi\,(z, \alpha)$ is equal to total profit divided by σ and that choosing z to maximize π serves to maximize Π as well. Proceeding similarly, consumers' surplus, S, and net welfare, W, are as follows:

$$S = \int_P^\infty (x - P)g(x)dx = \sigma f[(P - \mu)/\sigma] - (P - \mu)F[(P - \mu)/\sigma] \tag{6b}$$

$$= \sigma\{s\} = \sigma\{f(z - \alpha) - (z - \alpha)F(z - \alpha)\},$$

$$W = \sigma\{s + \pi\} = \sigma\{f(z - \alpha) + \alpha F(z - \alpha)\} = \sigma\{w\}. \tag{6c}$$

Let asterisks indicate evaluation at the profit-maximizing point, so that $z^* = z^*(\alpha)$ is the profit-maximizing value of z for given α. I now want to show that there exists a unique z^* for every α. Differentiating (6a), the first-order condition for profit maximization is

$$F(z^* - \alpha) - z^*f(z^* - \alpha) = 0, \tag{7}$$

and the corresponding second-order condition is

$$z^*(z^* - \alpha) - 2 < 0. \tag{8}$$

For any α, condition (8) is satisfied only for z^* values in a single interval that includes the origin: The profit function is thus *not* globally concave in the single-good Gaussian case. Since $\partial\pi/\partial\alpha$ is declining in this interval and only there, if there exists a solution to the first-order condition (7) satisfying (8), it is unique. To show existence, first note that the left-hand side of (7) is positive for any finite α if $z^* = 0$. Using (4d), this expression can be seen to be less than zero if $z^* = [\alpha + (\alpha^2 + 4)^{1/2}]/2$. There is thus a solution to (7) in this range at which, from (4d) and (7),

$$z^*(z^* - \alpha) - 1 < 0. \tag{9}$$

But satisfaction of (9) implies satisfaction of (8), and we have shown both existence and uniqueness of $z^*(\alpha)$.

Differentiation of (7) and use of (8) and (9) establishes that z^* is an increasing function of α:

$$0 < z^*_\alpha(\alpha) = [z^*(z^* - \alpha) - 1]/[z^*(z^* - \alpha) - 2] < 1. \quad (10)$$

A plot of $z^*(\alpha)$, obtained by solving (7) numerically, is given in figure 1. It follows from (10) that $\tilde{\alpha}$, defined by $z^*(\tilde{\alpha}) = \tilde{\alpha}$, is unique, as figure 1 suggests.[5]

Going back to (5a) and (5b), one can use (10) to show that the profit-maximizing price, P^*, increases with both C and μ, as one might expect. The derivative of P^* with respect to σ has the sign of $(z^* - \alpha z^*_\alpha)$. This expression is clearly positive for negative α, and $\alpha < 3.69$ is *apparently* necessary and sufficient for it to be positive.[6]

Let $\pi^*(\alpha) = \pi[z^*(\alpha), \alpha]$, and define $s^*(\alpha)$ and $w^*(\alpha)$ similarly. Then differentiation of (6) and use of (7) and (10) yield

$$0 < \pi^*_\alpha(\alpha) = F(z^* - \alpha) < 1, \quad (11a)$$

$$0 < s^*_\alpha(\alpha) = F(z^* - \alpha)(1 - z^*_\alpha) < 1, \quad (11b)$$

$$0 < w^*_\alpha(\alpha) = F(z^* - \alpha)(2 - z^*_\alpha) < 2. \quad (11c)$$

Differentiation of (11a) establishes that $\pi^*_{\alpha\alpha}$ is everywhere positive, so that π^* is globally strictly convex. Differentiation and simulation show that w^* is *apparently* strictly convex if and only if $\alpha < 8.18$, while s^* is *apparently* strictly convex if and only if $\alpha < 1.54$.[7] Inequalities (11) can be used to show directly that Π, S, and W are increasing functions of μ and decreasing functions of C, as one might suspect. The dependence of these quantities on σ is more complex and is of central importance in what follows.

Differentiation of (6a) and use of (11a) yields

$$\Pi^*_\sigma = \pi^* - \alpha\pi^*_\alpha = (z^* - \alpha)F(z^* - \alpha). \quad (12a)$$

Thus for $\alpha < \tilde{\alpha}$ (implying $z^* > \alpha$), increases in σ increase total profit, while for larger values of α, profit is reduced when σ increases. To see why this happens, note that changes in σ affect both the level and the elasticity of demand. If α is positive, so that $\mu > C$, increases in σ shift some buyers' reservation prices below C and thus lower the level of demand. On the other hand, as we noted in the paragraph below equation (10), above, increases in σ *apparently* raise P^* by lowering de-

5. Figure 1 also suggests that $z^*_{\alpha\alpha}$ is positive. This quantity has the sign of $[z^*(z^* - \alpha)^2 + \alpha]$, which is clearly positive for $\alpha \geqslant 0$ and is *apparently* positive at least for $\alpha \geqslant -3$.

6. Assertions of this sort, which make global statements about all values of α, are supported by numerical function evaluations for an array of α values whose upper and lower bonds are set by the inability of our routines to handle normal variables above 18 in absolute value. Thus, even if some of these assertions are not true for some values outside this range, it is unlikely that such values would ever be encountered.

7. The term $w^*_{\alpha\alpha}$ can be shown to have the sign of $(6 + 4\alpha z^* - 5z^{*2})$, while $s^*_{\alpha\alpha}$ has the sign of $(2 - z^{*4} + 2\alpha z^{*3} - \alpha^2 z^{*2} - z^{*2})$.

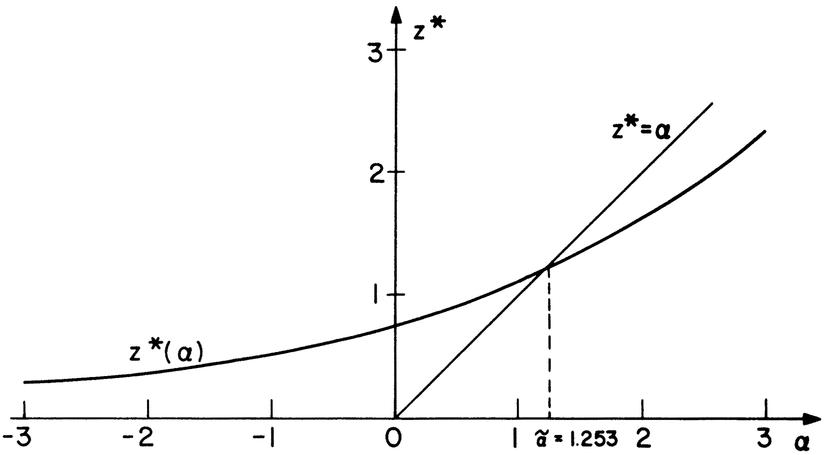


FIG. 1

mand elasticity at the profit-maximizing point as long as $\alpha < 3.69$. Thus for $\alpha \leqslant 0$, increases in σ increase profit by raising the level of demand and lowering elasticity, while for $\alpha \geqslant 3.69$, profit is decreased because demand is lowered and made more elastic. At $\alpha = \tilde{\alpha} = 1.253$, an increase in σ lowers demand elasticity, but this is exactly offset (in profit terms) by the associated fall in the level of demand.

Differentiation of (6b) and use of (7) and (11b) does not yield an expression as simple as in (12a):

$$S_\sigma^* = \{-f(z^* - \alpha)/[(z^* - \alpha) - 2]\}\{[z^*(z^* - \alpha) - 1]^2 + [1 - z^{*2}]\}. \tag{12b}$$

The first of the two right-hand terms is always positive by (8), while the second is positive as long as $z^{*2} \leqslant 1$. From (9), $z^*(0)^2 < 1$, and $z_\alpha^* > 0$, so that S_α^* is positive for all $\alpha \leqslant 0$. Simulation shows that S_α^* is *apparently* positive for all $\alpha > 0$ as well. As σ rises, buyers become more diverse, and it is more difficult for a monopolist not practicing price discrimination to convert consumers' surplus into profit.

Finally, differentiation of (6c) and use of (7) and (11c) yields

$$W_\sigma^* = \{-f(z^* - \alpha)/[z^*(z^* - \alpha) - 2]\}\{2 - z^{*2}\}. \tag{12c}$$

Increases in σ raise net welfare as long as $z^* < \sqrt{2}$. Solving $z^*(\hat{\alpha}) = \sqrt{2}$ numerically, we find $\hat{\alpha} = 1.561$, so that W_σ^* is positive if and only if $\alpha < \hat{\alpha} = 1.561$. It should be clear that W^* must be positive for $\alpha < \tilde{\alpha}$, since both Π_σ^* and S_σ^* are (*apparently*) positive there.

III. Pure Bundling

Let us now use superscripts to identify goods, as before, so that C^1 is the unit cost of good 1, σ^2 is the demand standard deviation of good 2, and so on. Let ρ be the correlation coefficient of the joint reservation price distribution. Under bivariate normality, it is straightforward to show that the distribution of reservation prices for the bundle (consisting of one unit of each good) is normal with mean $\mu^B = (\mu^1 + \mu^2)$ and standard deviation $\sigma^B = \delta(\sigma^1 + \sigma^2)$, where δ is defined by

$$\delta = [1 - 2(1 - \rho)\theta(1 - \theta)]^{1/2}, \tag{13a}$$

$$\theta = \sigma^1/(\sigma^1 + \sigma^2). \tag{13b}$$

The important function $\delta(\rho, \theta)$ satisfies $0 \leq \delta \leq 1$, $\delta(1, \theta) = 1$ for all θ, $\delta_\rho > 0$, and δ is minimized for any ρ by $\theta = 1/2$. The following definitions are also useful:

$$d = \alpha^1 - \alpha^2, \tag{13c}$$

$$\bar{\alpha} = \theta\alpha^1 + (1 - \theta)\alpha^2, \tag{13d}$$

$$\alpha^B = (\mu^B - C^B)/\sigma^B = \bar{\alpha}/\delta, \tag{13e}$$

where $C^B = (C^1 + C^2)$ is the unit cost of producing the bundle.

Total profits under optimal pure bundling are simply $[\sigma^B\pi^*(\alpha^B)]$, while unbundled selling would yield $[\sigma^1\pi^*(\alpha^1) + \sigma^2\pi^*(\alpha^2)]$. Using definitions (13) to substitute for the α's and σ's, it is straightforward to show that pure bundling is more profitable than unbundled sales if and only if the following quantity is positive:

$$D\Pi = \delta\pi^*\left(\frac{\bar{\alpha}}{\delta}\right) - \theta\pi^*[\bar{\alpha} + (1 - \theta)d] - (1 - \theta)\pi^*(\bar{\alpha} - \theta d). \tag{14a}$$

Analogous reasoning produces the following test quantities for consumers' surplus and net welfare, respectively:

$$DS = \delta s^*\left(\frac{\bar{\alpha}}{\delta}\right) - \theta s^*[\bar{\alpha} + (1 - \theta)d] - (1 - \theta)s^*(\bar{\alpha} - \theta d), \tag{14b}$$

$$DW = \delta w^*\left(\frac{\bar{\alpha}}{\delta}\right) - \theta w^*[\bar{\alpha} + (1 - \theta)d] - (1 - \theta)w^*(\bar{\alpha} - \theta d). \tag{14c}$$

Pure bundling increases consumers' surplus (net welfare) if and only if DS (DW) is positive.

In general, one needs values of seven parameters to describe any particular case of this model fully: μ^1, σ^1, μ^2, σ^2, ρ, C^1, and C^2. Conditions (14) show that only four parameters need to be known to evaluate the desirability of a switch from unbundled selling to pure bundling: $\bar{\alpha}$, ρ, d, and θ. It turns out to be most instructive to consider first the symmetric case, in which $d = 0$ and $\theta = 1/2$. (These conditions ensure that the marginal distributions of $[P - C]$ are the same for both goods.) I show that pure bundling operates in this case by reducing the effective dispersion of the distribution of reservation prices. If symmetry does not hold, pure bundling is less likely to be profit or welfare enhancing.

A. The Symmetric Case

Because $\theta = ½$, we have $\delta = \sqrt{(1 + \rho)/2}$, so that δ goes from 0 to 1 as ρ goes from -1 to $+1$. In the symmetric case, equations (14) reduce immediately to

$$D\Pi = \delta\pi^*\left(\frac{\bar{\alpha}}{\delta}\right) - \pi^*(\bar{\alpha}), \tag{15a}$$

$$DS = \delta s^*\left(\frac{\bar{\alpha}}{\delta}\right) - s^*(\bar{\alpha}), \tag{15b}$$

$$DW = \delta w^*\left(\frac{\bar{\alpha}}{\delta}\right) - w^*(\bar{\alpha}). \tag{15c}$$

If $\rho = 1$, then $\delta = 1$, and $D\Pi = DS = DW = 0$. Pure bundling and unbundled sales are then publicly and privately equivalent, since every buyer consumes either both goods or neither one under both strategies. This provides a boundary condition on these three expressions for all values of $\bar{\alpha}$.

Let us first see when pure bundling is more profitable than unbundled sales. As ρ goes to -1 ($\delta \rightarrow 0$), σ^B goes to zero, since in the limit all buyers have reservation price μ^B for the bundle. If $\mu^B < C^B$, so that $\bar{\alpha} < 0$, the bundle clearly cannot be sold at a profit, even though unbundled sales will yield positive profits. If $\bar{\alpha}$ is positive, however, the bundle can be sold profitably to all buyers as long as it is priced between C^B and μ^B. Maximum profits under pure bundling are thus $(\mu^B - C^B) = (\mu^1 - C^1) + (\mu^2 - C^2) = 2\sigma\bar{\alpha}$, where $\sigma = \sigma^1 = \sigma^2$. Total profit under unbundled sales is just $2\sigma\pi^*(\bar{\alpha})$, so that pure bundling is more profitable than unbundled sales when $\rho = -1$ if and only if $[\bar{\alpha} - \pi^*(\bar{\alpha})]$ is positive. *Apparently* this quantity is positive for $\bar{\alpha} > \underline{\tilde{\alpha}} = .227$. (It follows from [11a] that there can be at most one solution to $\underline{\tilde{\alpha}} = \pi^*[\underline{\tilde{\alpha}}]$.)

Differentiating (15a) and recalling (12a), we have

$$D\Pi_\delta = \pi^*\left(\frac{\bar{\alpha}}{\delta}\right) - \left(\frac{\bar{\alpha}}{\delta}\right)\pi^*_\alpha\left(\frac{\bar{\alpha}}{\delta}\right) = \Pi^*_\sigma\Big|_{\alpha = \bar{\alpha}/\delta}. \tag{16a}$$

Now consider figure 2. From the discussion below (12a), it follows that $D\Pi_\delta < 0$ for all δ if $\bar{\alpha} > \tilde{\alpha}$. Since $D\Pi = 0$ when $\delta = 1$, for any $\bar{\alpha}$, a curve like I must apply whenever $\bar{\alpha} > \tilde{\alpha}$. Pure bundling is then more profitable than unbundled sales for all $\rho < 1$. Using the strict convexity of π^*, one can show that $D\Pi$ is strictly convex in δ. Since if $\bar{\alpha} < \underline{\tilde{\alpha}}$, $D\Pi < 0$ for $\delta = 0$, a curve like III thus must apply in this range, and pure bundling is always less profitable than unbundled sales for $\bar{\alpha} < \underline{\tilde{\alpha}}$. If $\bar{\alpha}$ is between $\underline{\tilde{\alpha}}$ and $\tilde{\alpha}$, curve II must be the relevant one. (This function attains its minimum at $\delta = \bar{\alpha}/\tilde{\alpha}$, from [16a] and the discussion of [12a].) By strict convexity, there is a unique $\delta^\pi(\bar{\alpha})$ between zero and one for $\bar{\alpha}$ in this range, such that pure bundling is more profitable than unbundled sales if and only if $\delta < \delta^\pi(\bar{\alpha})$. (From the discussion above, $\delta^\pi[\underline{\tilde{\alpha}}] = 0$ and $\delta^\pi(\tilde{\alpha}) = 1$, and one can show analytically that δ^π is strictly increasing at intermediate points.) Solving for δ^π numerically and transforming

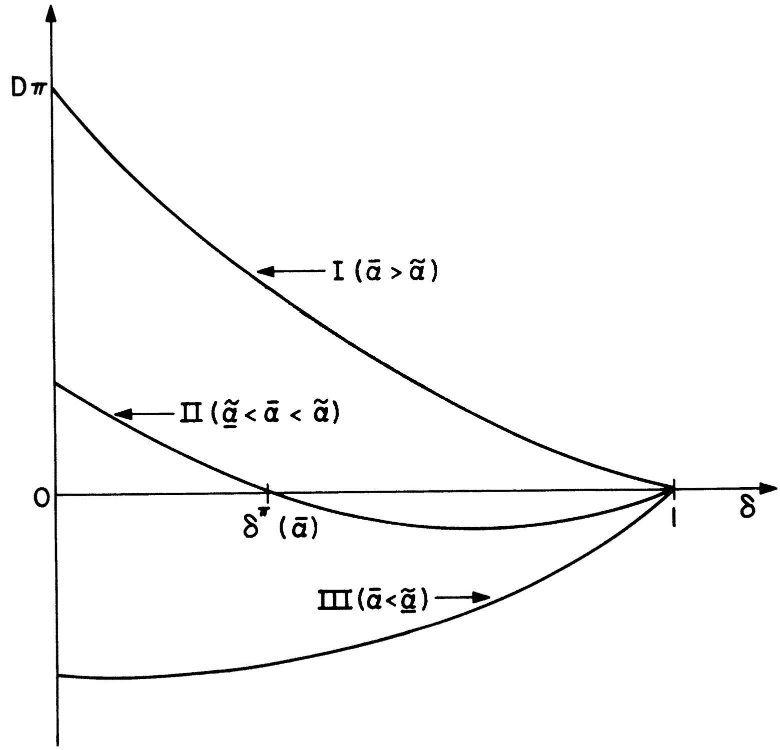


FIG. 2

back to ρ, we obtain the curve labeled $\rho^{\pi}(\underset{\sim}{\alpha})$ in figure 3. Pure bundling is *apparently* more profitable than unbundled sales at all $(\bar{\alpha}, \rho)$ points below this curve.

The numerical examples in Stigler (1968) and in Adams and Yellen (1976) might suggest that the profitability of pure bundling requires a negative correlation in the population of buyers between reservation prices for the two goods. This is clearly *not* the case, however: Pure bundling is always more profitable than unbundled sales in the symmetric Gaussian case for any positive ρ if $\bar{\alpha}$ is large enough. A more useful interpretation of bundling is suggested by our use of (12a) in the analysis above. Pure bundling acts to reduce buyer diversity, in that as long as $\rho \neq 1$, the standard deviation of reservation prices for the bundle is always less than the sum of the standard deviations for the two goods of which it is composed. (Since σ is measured in dollars, it is the relevant index of dispersion or buyer heterogeneity in this context, as equations [6] indicate.) If the level of demand for the two goods, as measured by $\bar{\alpha}$, is high enough, a reduction in heterogeneity always increases profits by permitting more efficient extraction of consumers' surplus. The correlation among reservation prices matters only because the lower is ρ, the greater the reduction in dispersion produced by bundling.

Differentiating (15b) and recalling (12b), we have as above

$$DS_\delta = S^*_\sigma\Big|_{\alpha = \bar{\alpha}/\delta}. \tag{16b}$$

Since we found above that S^*_σ is *apparently* everywhere positive, it follows that pure bundling *apparently* always lowers consumers' surplus as long as $\rho \neq 1$. (If one redrew fig. 2 to show DS, $DS_\delta > 0$ everywhere means that a curve like III applies for all $\bar{\alpha}$.) It is thus *apparently* the case that the reduction in effective buyer heterogeneity induced by pure bundling leaves buyers in aggregate worse off, even if mean α is so low that seller profit is not increased.

Finally, let us consider DW, the change in net welfare brought about by a shift from unbundled sales to pure bundling. When $\rho = -1$ ($\delta = 0$), pure bundling extracts all consumers' surplus so that $W = \Pi$. Reasoning as above, pure bundling produces a larger value of W when $\rho = -1$ if and only if $[\bar{\alpha} - w^*(\underline{\alpha})]$ is positive. *Apparently* $\underline{\hat{\alpha}} > \hat{\alpha} = .575$ is necessary and sufficient for this quantity to be positive.[8] Differentiation of (15c) yields, as above,

$$DW_\delta = W^*_\sigma\Big|_{\alpha = \bar{\alpha}/\delta} \tag{16c}$$

Since $W^*_\sigma < 0$ for $\alpha > \hat{\alpha}$, it follows exactly as in the discussion below (16a) that pure bundling leads to higher net welfare when $\rho \neq 1$ and $\bar{\alpha} > \hat{\alpha}$.[9] Exploring the interval $[\tilde{\underline{\alpha}}, \hat{\alpha}]$ numerically, we obtain the curve labeled $\rho^W(\bar{\alpha})$ in figure 3. Pure bundling is *apparently* more efficient than unbundled sales, in the sense of producing a higher value of W, at all $(\bar{\alpha}, \rho)$ points below this curve. If a move from unbundled sales to pure bundling is welfare enhancing, it is *apparently* also profit enhancing, but the converse is *apparently* not correct.[10] If $\bar{\alpha}$ is not especially large, the drop in S *apparently* always caused by a move to pure bundling can outweigh the associated increase in Π.

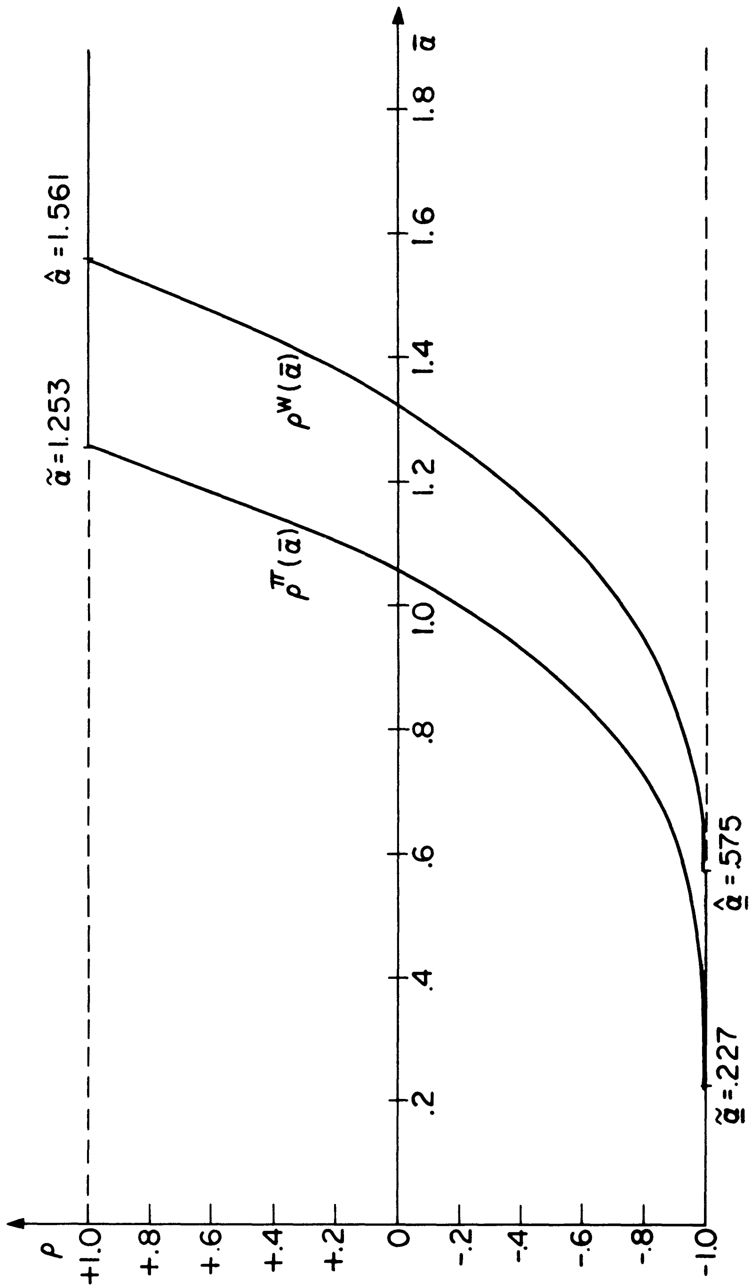
B. Departures from Symmetry

Pure bundling treats the two goods symmetrically, while unbundled sales do not involve this constraint. It is thus not surprising that the

8. Since $w^*(\bar{\alpha}) > \pi^*(\bar{\alpha})$ for all $\bar{\alpha}$ and both functions are increasing, it must be that $\underline{\hat{\alpha}} > \tilde{\underline{\alpha}}$. As we show below, if $\rho \neq 1$ pure bundling always raises W, if $\bar{\alpha} > \hat{\alpha}$. Thus any solutions to $\underline{\hat{\alpha}} = w(\underline{\hat{\alpha}})$ must lie in the open interval $(\tilde{\underline{\alpha}}, \hat{\alpha})$, and the solution given in the text is *apparently* the only one there.

9. In particular, this result does not depend on whether or not w^* is convex. As long as DW is everywhere decreasing in δ and equals zero when $\delta = 1$, it must be everywhere positive; see curve I in fig. 2.

10. When $\rho = \rho^W$, $DW = D\Pi + DS = 0$. Since DS is *apparently* everywhere negative, $D\Pi$ must be positive along the ρ^W locus, and the latter therefore *apparently* cannot intersect the ρ^π locus. Using the convexity of π^*, one can show that $D\Pi$ is increasing in $\bar{\alpha}$ for $\rho \neq +1$, so that the ρ^W locus must lie everywhere to the right of the ρ^π locus if they do not intersect. This is confirmed by the computations underlying fig. 3.



ρ
+1.0
+.8
+.6
+.4
+.2
0
-.2
-.4
-.6
-.8
-1.0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
ᾱ
α̃ = 1.253
α̂ = 1.561
ρ^W(ᾱ)
ρ^π(ᾱ)
α̰̃ = .227
α̰̂ = .575

FIG. 3

symmetric case is the one most favorable to pure bundling, in terms of both profit and net welfare.

Let us first assume that $d = 0$ but $\theta \neq \frac{1}{2}$. Then equations (14) reduce to equations (15) exactly as above, but the δ corresponding to any given ρ is larger by the discussion below (13a). This means that pure bundling produces a smaller reduction in diversity. It then follows from the analysis above that DS is still *apparently* negative everywhere but that $D\Pi$ and DW are positive only for a subset of the set of $(\bar{\alpha}, \rho)$ values for which they were positive in the symmetric case. Since $\delta = 1$ when $\rho = 1$ for any θ, pure bundling is more profitable (efficient) than unbundled sales if $\bar{\alpha} > \tilde{\alpha}$ $(\bar{\alpha} > \hat{\alpha})$ for any θ as long as $d = 0$. For smaller values of $\bar{\alpha}$, however, lower values of ρ are generally required for pure bundling to increase profits (or net welfare) when $\theta \neq \frac{1}{2}$ than in the symmetric case.

If $d \neq 0$, then for any θ not equal to zero or one, strict convexity of π^* implies

$$\begin{aligned} &\theta\pi^*[\bar{\alpha} + (1 - \theta)d] + (1 - \theta)\pi^*[\bar{\alpha} - \theta d] \\ &> \pi^*\{\theta[\bar{\alpha} + (1 - \theta)d] + (1 - \theta)[\bar{\alpha} - \theta d]\} = \pi^*(\bar{\alpha}). \end{aligned} \tag{17}$$

(We ignore the uninteresting degenerate cases $\theta = 0$ and $\theta = 1$ here and below.) Comparing (14a) and (15a), this means that for any values of the other parameters, $D\Pi$ is larger when $d = 0$ than when the goods' α's are unequal. In particular, when $d \neq 0$, pure bundling always *lowers* profits when $\rho = 1$. Nonzero values of d thus tend to reduce the set of values of the other parameters for which pure bundling is more profitable than unbundled sales. As long as α^1 and α^2 are less than 8.18, w^* is *apparently* strictly convex over the relevant range, and all the results of this paragraph thus apply to net welfare as well as to profit.

Numerical analysis of nonsymmetric cases is quite straightforward. The procedures employed in the preceding subsection need only slight modification to be used for any values of d and θ. In particular, figure 2 remains valid except that the boundary value at $\delta = 1$ is generally negative.[11] This sort of computation did not seem likely to yield any new insights, however, so none was performed.

IV. Mixed Bundling

Under a policy of mixed bundling, goods 1 and 2 are offered for sale at prices P^1 and P^2, respectively, and a bundle of one unit of each is sold

11. From this and $\delta_\rho^\pi > 0$ it follows that if pure bundling is more profitable than unbundled sales for $\rho = \rho_0$, then regardless of the values of the other parameters it is also more profitable for all $\rho < \rho_0$. Intuitively, this is because smaller ρ's *always* enable bundling to achieve greater reduction in effective buyer diversity.

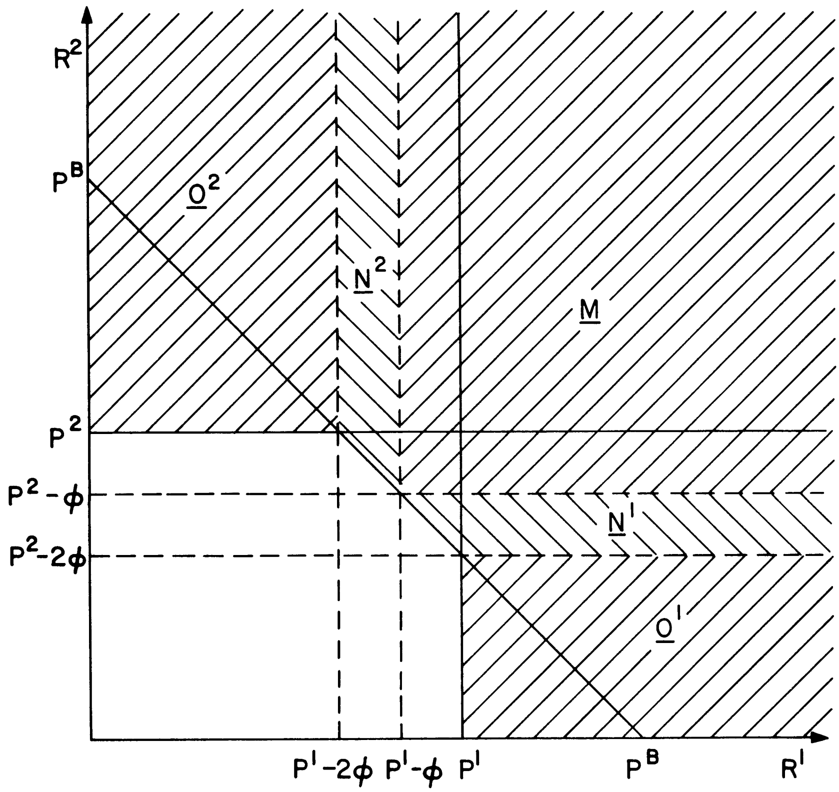


FIG. 4

for $P^B = P^1 + P^2 - 2\phi$, where $\phi > 0$. (If $\phi \leqslant 0$, nobody would ever buy the bundle, and this strategy would reduce to unbundled sales.) Let $g(R^1,R^2)$ be the (bivariate normal) density function of reservation price pairs in the population of buyers. Then, referring to figure 4, total profit under mixed bundling is given by the following expression (see Adams and Yellen 1976, p. 480):

$$\begin{aligned}\Pi(P^1, P^2, \phi) = (P^1 + P^2 - 2\phi - C^1 - C^2)(M + N^1 + N^2) \\ + (P^1 - C^1)0^1 + (P^2 - C^2)0^2,\end{aligned} \tag{18}$$

where the areas appearing in this equation and in figure 4 are defined as follows:

$$\underline{M} = \int_{P^1-\phi}^{\infty}\int_{P^2-\phi}^{\infty} g(x, y)\, dx\, dy, \tag{19a}$$

$$\underline{N}^1 = \int_{P^2-2\phi}^{P^2-\phi} \int_{P^1+P^2-2\phi-y}^{\infty} g(x, y)\, dx\, dy,$$

$$\underline{N}^2 = \int_{P^1-2\phi}^{P^1-\phi} \int_{P^1+P^2-2\phi-x}^{\infty} g(x, y)\, dy\, dx, \tag{19b}$$

$$\underline{0}^1 = \int_{-\infty}^{P^2-2\phi} \int_{P^1}^{\infty} g(x, y)\, dx\, dy,$$

$$\underline{0}^2 = \int_{-\infty}^{P^1-2\phi} \int_{P^1}^{\infty} g(x, y)\, dy\, dx. \tag{19c}$$

Iterative solution of the three first-order conditions for maximization of Π would involve several numerical integrations at each step. I do not attempt computation or characterization of such solutions here. Instead, I develop and apply a sufficient condition for mixed bundling to be strictly more profitable than unbundled sales.

If P^1 and P^2 are chosen to maximize profits from unbundled sales, these values maximize $\Pi(P^1, P^2, \phi)$ when $\phi = 0$. If at such a point $\Pi_\phi > 0$, it follows that *some* mixed bundling policy is strictly more profitable than unbundled sales. (If Π were globally concave in these prices, this would also be a necessary condition, but we found in Sec. II that profit functions do not exhibit global concavity in the Gaussian case.) Differentiating (19) and evaluating the derivatives at $\phi = 0$, one obtains after considerable algebra

$$\begin{aligned} t = \left(\frac{1}{2}\right)\Pi_\phi(P^1, P^2, 0) = (P^1 - C^1) \int_{P^2}^{\infty} g(P^1, y)dy \\ + (P^2 - C^2) \int_{P^1}^{\infty} g(x, P^2)dx \\ - \int_{P^1}^{\infty} \int_{P^2}^{\infty} g(x, y)\, dx\, dy. \end{aligned} \tag{20}$$

The right-hand side of (20) has the form of most monopoly first-order conditions: The revenue loss on inframarginal sales is subtracted from the profit gain on marginal sales. Setting the partial derivatives of Π with respect to its first two arguments equal to zero at $\phi = 0$, one obtains condition (7) for each good. If $t > 0$ when these two conditions are satisfied, mixed bundling is more profitable than unbundled sales.

In order to take (20) into more usable form, let $f(u, v)$ be the bivariate normal density function of two standard normal deviates with correlation coefficient ρ, let $f(u)$ be the standard normal (marginal) density, as before, and let $f(u|v)$ be the corresponding conditional density. (Recall that $f[u|v]$ is normal with mean ρv and variance $[1 - \rho^2]$.) Let $r^i(\alpha^i) = z^{i*}(\alpha^i) - \alpha^i$, for $i = 1, 2$. Then standard manipulations and condition (7) yield

$$\begin{aligned}(P^1 - C^1)\int_{P^2}^{\infty} g(P^1, y)\,dy &= z^1\int_{r^2}^{\infty} f(r^1, v)\,dv = z^1 f(r^1)\int_{r^2}^{\infty} f(v|r^1)\,dv \\ &= z^1 f(r^1)F[(r^2 - \rho r^1)/(1 - \rho^2)^{\frac{1}{2}}] \qquad (21a)\\ &= F(r^1)F[(r^2 - \rho r^1)/(1 - \rho^2)^{\frac{1}{2}}].\end{aligned}$$

Similarly, the second and third terms on the right of (20) can be rewritten as follows:

$$(P^2 - C^2)\int_{P^1}^{\infty} g(x, P^2)\,dx = F(r^2)F[(r^1 - \rho r^2)/(1 - \rho^2)^{\frac{1}{2}}], \qquad (21b)$$

$$\int_{P^1}^{\infty}\int_{P^2}^{\infty} g(x, y)\,dx\,dy = \int_{r^1}^{\infty}\int_{r^2}^{\infty} f(u, v)\,du\,dv. \qquad (21c)$$

Let us define the following:

$$\bar{r} = \frac{r^1 + r^2}{2}, \qquad (22a)$$

$$s = \frac{r^1 - r^2}{2}, \qquad (22b)$$

$$\omega = \sqrt{(1 - \rho)/(1 + \rho)}. \qquad (22c)$$

Using (21) and (22), equation (20) can be rewritten as follows:

$$\begin{aligned}t = {} & F(\bar{r} + s)F\left(\omega\bar{r} - \frac{s}{\omega}\right) + F(\bar{r} - s)F\left(\omega\bar{r} + \frac{s}{\omega}\right)\\ & - \int_{\bar{r}+s}^{\infty}\int_{\bar{r}-s}^{\infty} f(u, v)\,du\,dv.\end{aligned} \qquad (23)$$

A. The Symmetric Case

In the symmetric case, $s = 0$, so that equation (23) simplifies to

$$t = 2F(\bar{r})F(\omega\bar{r}) - I(\bar{r}), \qquad (23a)$$

where we define

$$I(\bar{r}) = \int_{\bar{r}}^{\infty}\int_{\bar{r}}^{\infty} f(u, v)\,du\,dv. \qquad (23b)$$

(Note that we do not need to impose any second condition analogous to $\theta = ½$ in order to obtain symmetry here.) When $\rho = 1$, $I(\bar{r}) = F(\bar{r})$ and $\omega = 0$, so that $t = 0$ as one would expect. Mixed bundling can have no more effect in the case of perfect positive correlation among reservation prices than pure bundling.

If $\rho \neq 1$, we can easily show that t is positive for $\bar{\alpha} \geqslant \tilde{\alpha}$. Clearly $I(\bar{r}) < F(\bar{r})$ when $\rho \neq 1$, so that

$$t > F(\bar{r})[2F(\omega\bar{r}) - 1].$$

The quantity in brackets is nonnegative if and only if $\bar{r} \leqslant 0$, and this is equivalent to the condition $\bar{\alpha} \geqslant \tilde{\alpha}$. Thus t is positive when this last condition is satisfied, as asserted above. This is hardly a surprise, of course. Adams and Yellen (1976, p. 483) showed that mixed bundling is more profitable than pure bundling in this model for all distributions that, like the normal, have probability density everywhere in the positive quadrant. But we established in Section III that pure bundling is always at least as profitable as unbundled sales when $\bar{\alpha} \geqslant \tilde{\alpha}$ if $\rho \neq 1$.

Now let us establish that for any $\bar{\alpha}$, $t > 0$ if $\rho \leqslant 0$. If $\rho = 0$, $I(\bar{r}) = F(\bar{r})^2$, and $\omega = 1$, so that $t = F(\bar{r})^2 > 0$. This shows immediately how much more powerful mixed bundling is than pure bundling. For pure bundling to dominate unbundled sales when $\rho = 0$, figure 3 indicates that $\bar{\alpha}$ must be at least 1.05. To deal with $\rho < 0$, rewrite $I(\bar{r})$ as follows:

$$I(\bar{r}) = \int_{\bar{r}}^{\infty}\left\{\int_{\bar{r}}^{\infty} f(u|v)\,du\right\}f(v)\,dv = \int_{\bar{r}}^{\infty} F[(\bar{r} - \rho v)/(1 - \rho^2)^{\frac{1}{2}}]f(v)\,dv. \tag{24}$$

If $\rho < 0$ and $v > \bar{r}$, $F[(\bar{r} - \rho v)/(1 - \rho^2)^{\frac{1}{2}}] < F[\omega\bar{r}]$, since F is strictly decreasing. Then from (23) and (24), $I(\bar{r}) < F(\bar{r})F(\omega\bar{r})$, so that for $\rho < 0$, $t > F(\bar{r})F(\omega\bar{r}) > 0$.

We have shown that if $\rho \neq 1$, mixed bundling is more profitable than unbundled sales if either $\bar{\alpha} \geqslant \tilde{\alpha}$ or $\rho \leqslant 0$. It is *apparently* the case that t is positive for all $\rho < 1$ as long as $\bar{\alpha} > -2.8$, and it may be positive for all values of $\bar{\alpha}$.[12] It is thus apparently true that if $\rho \neq 1$, mixed bundling is more profitable than unbundled sales in the symmetric case if either $\bar{\alpha} > -2.8$ or $\rho \leqslant 0$. Mixed bundling is thus a very powerful price discrimination device in the Gaussian symmetric case. The advantage of pure bundling is its ability to reduce effective buyer heterogeneity, while the advantage of unbundled sales is its ability to collect a high price for each good from some buyers who care very little for the other. Mixed bundling can make use of both these advantages by selling the bundle to a group of buyers with accordingly reduced effective heterogeneity, while charging high markups to those on the fringes of the taste distribution who are mainly interested in only one of the two goods (see fig. 4).

12. A few negative values of t were encountered for $\bar{\alpha} < -2.8$, but the routine used to evaluate $I(\bar{r})$ yields errors in this region that are of the same order of magnitude as the computed values of t. It is thus quite possible that t is everywhere positive, but I do not now see a straightforward way of pursuing this point further.

B. *Departures from Symmetry*

Mixed bundling differs from unbundled sales only in that a bundle of one unit of each good is offered for sale under the former policy but not under the latter. Since the two goods are treated symmetrically in the bundle, it is perhaps not surprising that mixed bundling seems to perform best relative to unbundled sales in the symmetric case. Differentiation of (23) shows that $t_s = 0$ at $s = 0$. Differentiating again, one obtains

$$t_{ss}\Big|_{s=0} = [4f(\bar{r})/\omega][(\omega\bar{r})F(\omega\bar{r}) - f(\omega\bar{r})] + [2f(\omega\bar{r})/\omega][\bar{r}F(\bar{r}) - f(\bar{r})] < 0,$$

using (4d). Thus t attains a local maximum at $s = 0$ for all values of $\bar{r}$ and ω.[13] Of course, since $t > 0$ is only sufficient for the superiority of mixed bundling, and no special importance attaches to the magnitude of t, this result merely suggests that the symmetric case is most favorable to mixed bundling; it does not prove it.

V. Conclusions

By requiring the distribution of reservation price pairs in the Adams-Yellen (1976) model to be bivariate normal, we have been able to obtain a number of interesting and suggestive results about the comparative properties of commodity bundling and unbundled sales.

A number of basic properties of monopoly pricing under constant costs and Gaussian demand are derived in Section II. Even though the profit function is not globally concave, a unique profit-maximizing price always exists, and it depends in the expected ways on unit cost and the mean of the reservation price distribution. The more complex effects of changes in the standard deviation are explored.

The results of Section II are used in Section III to compare pure bundling with unbundled sales. In the symmetric case, it is shown explicitly that pure bundling operates by reducing the effective dispersion in buyers' tastes. This happens simply because as long as reservation prices are not perfectly correlated (i.e., $\rho \neq 1$), the standard deviation of reservation prices for the bundle is less than the sum of the standard deviations for the two component goods. The greater is the average willingness to pay, as measured by the normalized difference between mean reservation price and cost, $\bar{\alpha}$, the more likely it is that such a reduction in diversity will enhance profits by permitting more

13. It is perhaps also worth noting that for any s, as $\rho \rightarrow 1$ (so that $\omega \rightarrow 0$), $t \rightarrow 0$ for all values of $\bar{r}$. Let $s > 0$ without loss of generality. Then as $\omega \rightarrow 0$, the first term on the right of (24) approaches $F(\bar{r} + s)$, the second term goes to zero, and the third approaches $-F(\bar{r} + s)$.

efficient capture of consumers' surplus. On the other hand, pure bundling *apparently* always makes buyers worse off in the symmetric case for the same reason. If $\bar{\alpha}$ is large enough, the increase in profit caused by pure bundling is *apparently* larger than the fall in consumers' surplus, so that pure bundling increases net welfare. Since pure bundling requires symmetric treatment of the two goods while unbundled sales does not, it is not surprising that we find that pure bundling is most likely to be profit or welfare enhancing in the symmetric case for any values of ρ and $\bar{\alpha}$.

Because mixed bundling is a much more complex strategy than either unbundled sales or pure bundling, the analysis of mixed bundling in Section IV yields fewer results than are obtained for pure bundling. In the symmetric case, mixed bundling is always more profitable than unbundled sales if $\rho \leq 0$ or if $\bar{\alpha}$ is large enough, and it is *apparently* more profitable for positive $\rho \neq 1$ at least as long as $\bar{\alpha} > -2.8$. Mixed bundling combines the advantages of pure bundling and unbundled sales. This policy enables the seller to reduce effective heterogeneity among those buyers with high reservation prices for both goods, while still selling at a high markup to those buyers willing to pay a high price for only one of the goods. At least in the Gaussian case, this makes mixed bundling a very powerful price discrimination device. It appears likely that mixed bundling, like pure bundling, is most attractive in the symmetric case, all else equal, but I do not have a rigorous proof of this.

While all of the derivations here rely on the assumption of Gaussian demand, that case is of some interest in its own right. Moreover, the methods developed here can be applied to other cases, and the key finding that bundling permits more efficient extraction of surplus by reducing effective buyer heterogeneity seems likely to hold more generally.

Following Adams and Yellin (1976), this analysis has dealt exclusively with a monopolist selling two products of which consumers purchase at most one unit and for which demands are independent. Only the restriction to two products would seem to be innocuous. That is, it is not clear how one should model seller rivalry here or how the assumption of independent zero/one demands can best be relaxed.[14] And it must be acknowledged that the Adams-Yellin case is mainly of interest as a (relatively) tractable stepping-stone to cases of more direct empirical relevance.

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14. See n. 2, above.

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