Fluid kinematics

Ex. 1 Consider the following steady, two-dimensional velocity field:

$$\vec{V} = (u, v) = (a^2 - (b - cx)^2)\vec{i} + (-2cby + 2c^2xy)\vec{j}$$

Is there a stagnation point in this flow field? If so, where is it?

Analysis The velocity field is

$$\vec{V} = (u,v) = \left(a^2 - (b-cx)^2\right)\vec{i} + \left(-2cby + 2c^2xy\right)\vec{j} \qquad (1)$$

At a stagnation point, both u and v must equal zero. At any point (x,y) in the flow field, the velocity components u and v are obtained from Eq. 1,

Velocity components: $$u = a^2 - (b-cx)^2 \qquad v = -2cby + 2c^2xy \qquad (2)$$

Setting these to zero and solving simultaneously yields

Stagnation point: $$0 = a^2 - (b-cx)^2 \qquad x = \frac{b-a}{c} \qquad (3)$$
$$v = -2cby + 2c^2xy \qquad y = 0$$

So, **yes there is a stagnation point**; its location is ***x* = (*b* – *a*)/*c*, *y* = 0**.

Ex. 2 Consider steady, incompressible, two-dimensional flow through a converging duct

$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j}$$

where U_0 is the horizontal speed at $x = 0$. Note that this equation ignores viscous effects along the walls but is a reasonable approximation throughout the majority of the flow field. Calculate the material acceleration for fluid particles passing through this duct. Give your answer in two ways: (1) as acceleration components a_x and a_y and (2) as acceleration vector $\vec{a}$.

Analysis The velocity field is

$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

The acceleration field components are obtained from its definition (the material acceleration) in Cartesian coordinates,

$$\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 + (U_0 + bx)b + (-by)0 + 0 \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 + (U_0 + bx)0 + (-by)(-b) + 0 \end{aligned} \qquad (2)$$

where the unsteady terms are zero since this is a steady flow, and the terms with w are zero since the flow is two-dimensional. Eq. 2 simplifies to

Material acceleration components: $\boxed{a_x = b(U_0 + bx) \qquad a_y = b^2 y}$ (3)

In terms of a vector,

Material acceleration vector: $\boxed{\vec{a} = b(U_0 + bx)\vec{i} + b^2 y\vec{j}}$ (4)

Ex. 3 The velocity field for a flow is given by $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$ where $u = 3x$, $v = -2y$, $w = 2z$. Find the streamline that will pass through the point (1, 1, 0).

Solution For a given velocity field we are to calculate the streamline that will pass through a given point.

Assumptions **1** The flow is steady. **2** The flow is three-dimensional in the *x*-*y*-*z* plane.

Analysis

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{3x} = \frac{dy}{-2y} = \frac{dz}{2z}$$

For the first two pairs we have

$$\frac{\mathrm{dx}}{3\mathrm{x}} = \frac{dy}{-2y} \quad \text{or} \quad \frac{1}{3}\ln x = -\frac{1}{2}\ln y + \ln c_1$$

$$x^{1/3}y^{1/2} = c_1$$

For the point given $x = 1, \mathrm{y} = 1, \mathrm{z} = 0$

$$c_1 = 1^{1/3}.1^{1/2} = 1 \implies x^{1/3}.y^{1/2} = 1 \implies y = x^{-2/3}$$

on the other hand,

$$\frac{dz}{2z} = \frac{dx}{3x} \quad \text{or} \quad \frac{1}{2}\ln z - \frac{1}{3}\ln x = \ln c$$

$$\sqrt{z}/x^{1/3} = c \quad \text{or} \quad \frac{\mathrm{z}}{\mathrm{x}^{2/3}} = c \Rightarrow z = cx^{2/3}$$

A(1,1,0),

$0 = c.1^{2/3}$, $c = 0$ or $\mathrm{z} = 0$

Therefore the streamline is given by,

$$y = x^{-2/3}, z = 0$$

Ex. 4 A steady, incompressible, two-dimensional (in the *xy*-plane) velocity field is given by

$$\vec{V} = (0.523 - 1.88x + 3.94y)\vec{i} + (-2.44 + 1.26x + 1.88y)\vec{j}$$

Calculate the acceleration at the point $(x, y) = (-1.55, 2.07)$.

Solution For a given velocity field we are to calculate the acceleration.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *xy*-plane.

Analysis The velocity components are

Velocity components: $u = 0.523 - 1.88x + 3.94y \qquad v = -2.44 + 1.26x + 1.88y$ (1)

The acceleration field components are obtained from its definition (the material acceleration) in Cartesian coordinates,

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 + (0.523 - 1.88x + 3.94y)(-1.88) + (-2.44 + 1.26x + 1.88y)(3.94) + 0$$
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 + (0.523 - 1.88x + 3.94y)(1.26) + (-2.44 + 1.26x + 1.88y)(1.88) + 0 \qquad (2)$$

where the unsteady terms are zero since this is a steady flow, and the terms with w are zero since the flow is two-dimensional. Eq. 2 simplifies to

Acceleration components: $\boxed{a_x = -10.59684 + 8.4988x \qquad a_y = -3.92822 + 8.4988y}$ (3)

At the point $(x,y) = (-1.55, 2.07)$, the acceleration components of Eq. 3 are

Acceleration components at $(-1.55, 2.07)$: $a_x = -23.76998 \cong \mathbf{-23.8} \qquad a_y = 13.6643 \cong \mathbf{13.7}$

Ex. 5

For the velocity field of Ex. 2, generate an analytical expression for the flow streamlines.

Solution For a given velocity field we are to generate an equation for the streamlines.

Assumptions **1** The flow is steady. **2** The flow is two-dimensional in the *xy*-plane.

Analysis The steady, two-dimensional velocity field of Problem 4-16 is

Velocity field:
$$\vec{V} = (u,v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

For two-dimensional flow in the *xy*-plane, streamlines are given by

Streamlines in the xy-plane:
$$\left.\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \qquad (2)$$

We substitute the *u* and *v* components of Eq. 1 into Eq. 2 and rearrange to get

$$\frac{dy}{dx} = \frac{-by}{U_0 + bx}$$

We solve the above differential equation by separation of variables:

$$-\int \frac{dy}{by} = \int \frac{dx}{U_0 + bx}$$

Integration yields

$$-\frac{1}{b}\ln(by) = \frac{1}{b}\ln(U_0 + bx) + \frac{1}{b}\ln C_1 \qquad (3)$$

where we have set the constant of integration as the natural logarithm of some constant C_1, with a constant in front in order to simplify the algebra (notice that the factor of $1/b$ can be removed from each term in Eq. 3). When we recall that $\ln(ab) = \ln a + \ln b$, and that $-\ln a = \ln(1/a)$, Eq. 3 simplifies to

Equation for streamlines:
$$\boxed{y = \frac{C}{(U_0 + bx)}} \qquad (4)$$

The new constant C is related to C_1, and is introduced for simplicity.

Discussion Each value of constant C yields a unique streamline of the flow.

Ex. 6 Is this flow field in Ex. 2 rotational or irrotational?

Solution For a given velocity field, we are to determine whether the flow is rotational or irrotational.

Assumptions **1** The flow is steady. **2** The flow is incompressible. **3** The flow is two-dimensional in the *xy*-plane.

Analysis The velocity field is

$$\vec{V} = (u, v) = (U_0 + bx)\vec{i} - by\vec{j} \qquad (1)$$

By definition, the flow is rotational if the vorticity is non-zero. So, we calculate the vorticity. In a 2D flow in the *xy*-plane, the only non-zero component of vorticity is in the *z*-direction, i.e. ζ_z,

Vorticity component in the z-direction:

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - 0 = 0 \qquad (1)$$

Since the vorticity is zero, this flow is **irrotational**.