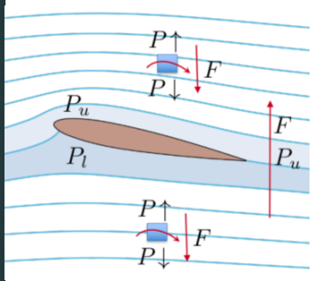
Swansea University
Prifysgol Abertawe

$$\mathrm{Re} = \frac{\rho V L}{\mu}$$

$$\mathrm{St} = \frac{f L}{V}$$

EGF320 - Fluid Flow



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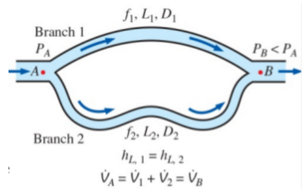
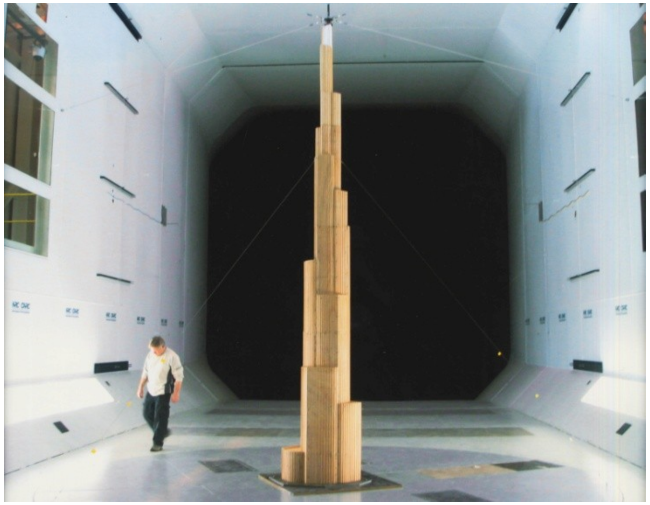


Dimensional Analysis

- “to determine the form of the dependence of one variable upon a range of other controlling parameters in the absence of an analytical solution...”
- Conducting the experiments for the whole range and recording the data is not a viable option because of the constraints on time and resources.
- Dimensional analysis offers a solution to this problem by allowing to group variables whose relationships may be determined experimentally.
- Dimensional analysis offers a **qualitative route**, determination of the quantitative relations still need experimental data.

Dimensional Analysis



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Dimensional Analysis

The three primary purposes of dimensional analysis are

- To predict trends in the relationship between parameters
- To obtain scaling laws so that prototype performance can be predicted from model performance
- To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results

Units and Dimensions

Dimension: A measure of a physical quantity (without numerical values).

Unit: A way to assign a number to that dimension

Some primary dimensions and their associated SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time†	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

All other non-primary variables can be written as a combination of the primary variables.

The law of dimensional homogeneity

Every additive term in an equation must have the same dimensions.

Example

$$\textit{Bernoulli's equation:} \qquad P + \frac{1}{2}\rho V^2 + \rho g z = C$$

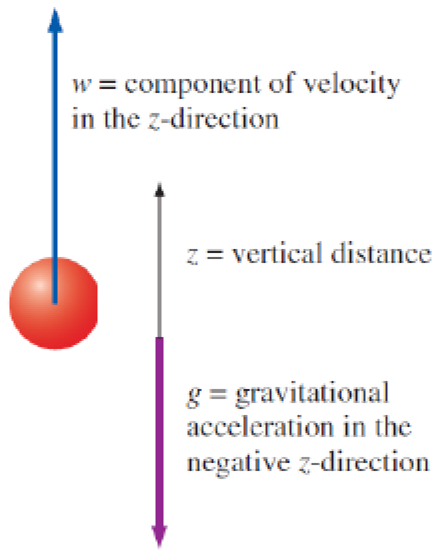
$$\{P\} = \{\text{Pressure}\} = \left\{\frac{\text{Force}}{\text{Area}}\right\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\,\frac{1}{\text{Length}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\left\{\frac{1}{2}\rho V^2\right\} = \left\{\frac{\text{Mass}}{\text{Volume}}\left(\frac{\text{Length}}{\text{Time}}\right)^2\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\{\rho g z\} = \left\{\frac{\text{Mass}}{\text{Volume}}\,\frac{\text{Length}}{\text{Time}^2}\,\text{Length}\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

Non-dimensionalisation

Example



Equation of motion $$\frac{d^2 z}{dt^2} = -g$$

Solution $$z = z_0 + w_0 t - \frac{1}{2} g t^2$$

Dimensional variables: z, t

Dimensional constants: g, z_0, w_0

All dimensions

$$\{z\} = \{L\} \quad \{t\} = \{t\} \quad \{z_0\} = \{L\} \quad \{w_0\} = \{L/t\} \quad \{g\} = \{L/t^2\}$$

Non dimensional variables $$z^* = \frac{z}{z_0} \quad \text{and} \quad t^* = \frac{t w_0}{z_0}$$

$$\frac{d^2 z}{dt^2} = \frac{d^2 z^* z_0}{d(z_0 t^*/w_0)^2} = \left(\frac{w_0^2}{z_0}\right) \frac{d^2 z^*}{dt^{*2}} = -g$$

$$\left(\frac{w_0^2}{z_0 g}\right) \frac{d^2 z^*}{dt^{*2}} = -1 \quad \text{or} \quad \mathrm{Fr}^2 \frac{d^2 z^*}{dt^{*2}} = -1$$

Froude Number

$$\mathrm{Fr} = \frac{w_0}{\sqrt{g z_0}}$$

Important in free surface flows too

The indicial method

Example 1: Force, **F**, acted depends upon the mass, **m**, and acceleration, **a**.

Find the expression for **F** in terms of **m** and **a**.

Buckingham Pi theorem

Method of repeating variables

How to *generate* the nondimensional parameters, i.e., the Π' s?

There are several methods that have been developed for this purpose, but the most popular (and simplest) method is the method of repeating variables.

The Method of Repeating Variables

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,

$$k = n - j$$

Step 4: Choose j *repeating parameters*.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

A concise summary of the six steps that comprise the *method of repeating variables.*

Detailed description of the six steps that comprise the *method of repeating variables*[*]	
Step 1	List the parameters (dimensional variables, nondimensional variables, and dimensional constants) and count them. Let n be the total number of parameters in the problem, including the dependent variable. Make sure that any listed independent parameter is indeed independent of the others, i.e., it cannot be expressed in terms of them. (E.g., don't include radius r and area $A = \pi r^2$, since r and A are *not* independent.)
Step 2	List the primary dimensions for each of the n parameters.
Step 3	Guess the **reduction** j. As a first guess, set j equal to the number of primary dimensions represented in the problem. The expected number of Π's (k) is equal to n minus j, according to the **Buckingham Pi theorem**, *The Buckingham Pi theorem:* $k = n - j$ (7–14) If at this step or during any subsequent step, the analysis does not work out, verify that you have included enough parameters in step 1. Otherwise, go back and *reduce j by one* and try again.
Step 4	Choose j **repeating parameters** that will be used to construct each Π. Since the repeating parameters have the potential to appear in each Π, be sure to choose them *wisely* (Table 7–3).
Step 5	Generate the Π's one at a time by grouping the j repeating parameters with one of the remaining parameters, forcing the product to be dimensionless. In this way, construct all k Π's. By convention the first Π, designated as Π_1, is the *dependent* Π (the one on the left side of the list). Manipulate the Π's as necessary to achieve established dimensionless groups (Table 7–5).
Step 6	Check that all the Π's are indeed dimensionless. Write the final functional relationship in the form of Eq. 7–11.

[*] This is a step-by-step method for finding the dimensionless Π groups when performing a dimensional analysis.

Buckingham Pi theorem

Example:

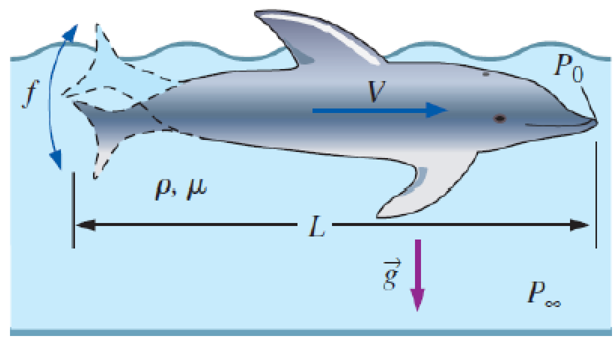
A force, **F**, is acted upon a body of length, **d**, and moving at a speed, **v**, by a fluid of density, ρ, and viscosity, μ.

Find the expression for F.

Nondimensional parameters

- A parameter without any dimension.
- Very important in comparative studies between real world models and prototypes.

Some important non-dimensional parameters



$\mathrm{Re} = \frac{\rho V L}{\mu}$ $\mathrm{Fr} = \frac{V}{\sqrt{gL}}$

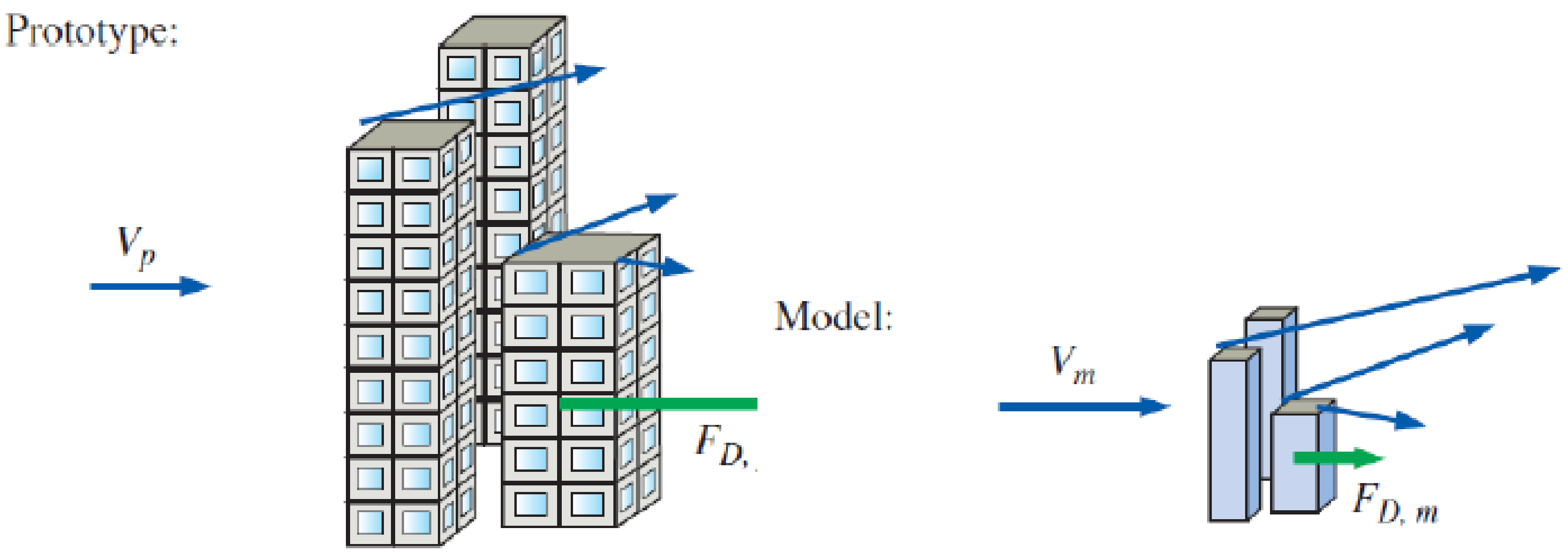
$\mathrm{St} = \frac{fL}{V}$ $\mathrm{Eu} = \frac{P_0 - P_\infty}{\rho V^2}$

Reynolds number, Froude number, Strouhal number, and Euler number

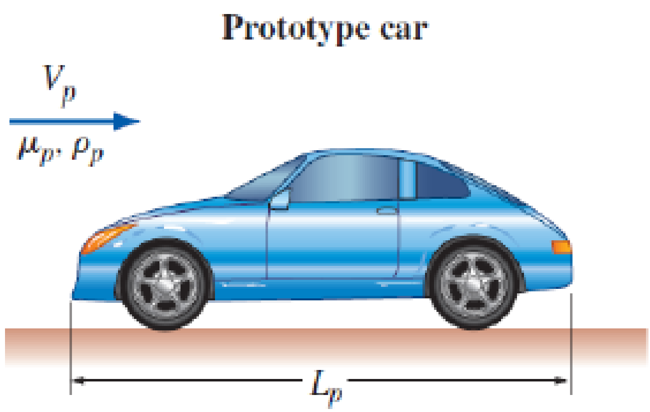
Similarity

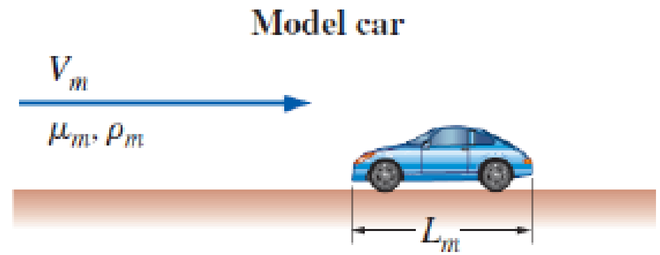
Complete similarity: three necessary conditions

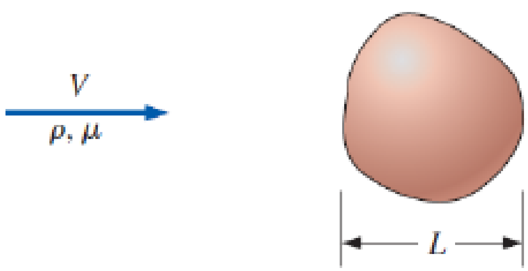
- Geometric similarity—the model must be the same shape as the prototype, but may be scaled by some constant scale factor.
- Kinematic similarity—the velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow.
- Dynamic similarity—When all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow (force-scale equivalence).



Aerodynamic drag on prototype car







$$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{VL}{\nu}$$

$$\Pi_1 = f(\Pi_2) \quad \text{where} \quad \Pi_1 = \frac{F_D}{\rho V^2 L^2} \quad \text{and} \quad \Pi_2 = \frac{\rho V L}{\mu}$$

Drag coefficient (non standard form)

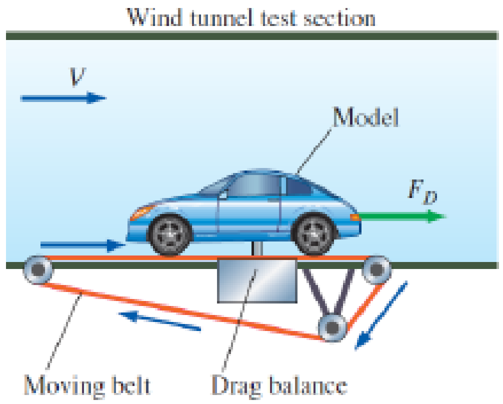
Reynolds number

If we match the independent Pi, the Reynolds number between the model and prototype then the dependent Pi, the drag coefficient, is also matched and hence the drag on the protoype can be predicted from measurement in the model.

The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics.

Example

The aerodynamic drag of a new sports car is to be predicted at a speed of 80.0 km/h at an air temperature of 25°C. Automotive engineers build a one-fifth scale model of the car to test in a wind tunnel. It is winter and the wind tunnel is located in an unheated building; the temperature of the wind tunnel air is only about 5°C. Determine how fast the engineers should run the wind tunnel in order to achieve similarity between the model and the prototype.



For air at atmospheric pressure and at T = 25°C, ρ = 1.184 kg/m^3 and $\mu = 1.849 \times 10^{-5}$ kg/m·s. Similarly, at T = 5°C, ρ = 1.269 kg/m^3 and $\mu = 1.754 \times 10^{-5}$ kg/m·s.

Analysis Since there is only one independent Π in this problem, the similarity equation (Eq. 7–12) holds if $\Pi_{2,\,m} = \Pi_{2,\,p}$, where Π_2 is given by Eq. 7–13, and we call it the Reynolds number. Thus, we write

$$\Pi_{2,\,m} = \mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \Pi_{2,\,p} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

which we solve for the unknown wind tunnel speed for the model tests, V_m,

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (80.0\ \text{km/h})\left(\frac{1.754 \times 10^{-5}\ \text{kg/m·s}}{1.849 \times 10^{-5}\ \text{kg/m·s}}\right)\left(\frac{1.184\ \text{kg/m}^3}{1.269\ \text{kg/m}^3}\right)(5) = \mathbf{354\ km/h}$$

Example (continued)

Suppose the engineers run the wind tunnel at 354 km/h to achieve similarity between the model and the prototype. The aerodynamic drag force on the model car is measured with a drag balance. Several drag readings are recorded, and the average drag force on the model is 94.3N. Predict the aerodynamic drag force on the prototype (at 80 km/h and 25°C).

SOLUTION Because of similarity, the model results are to be scaled up to predict the aerodynamic drag force on the prototype.

Analysis The similarity equation (Eq. 7–12) shows that since $\Pi_{2,\,m} = \Pi_{2,\,p}$, $\Pi_{1,\,m} = \Pi_{1,\,p}$, where Π_1 is given for this problem by Eq. 7–13. Thus, we write

$$\Pi_{1,m} = \frac{F_{D,m}}{\rho_m V_m^2 L_m^2} = \Pi_{1,p} = \frac{F_{D,p}}{\rho_p V_p^2 L_p^2}$$

which we solve for the unknown aerodynamic drag force on the prototype car, $F_{D,\,p}$,

$$F_{D,p} = F_{D,m}\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

$$= (94.3\text{ N})\left(\frac{1.184\text{ kg/m}^3}{1.269\text{ kg/m}^3}\right)\left(\frac{80.0\text{ km/h}}{354\text{ km/h}}\right)^2(5)^2 = \mathbf{112\text{ N}}$$

Special case: if the density and viscosity of the fluid in wind tunnel and prototype are identical, then drag force is also identical

If a water tunnel is used instead of a wind tunnel to test their one-fifth scale model, the water tunnel speed required to achieve similarity is

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (80.0\ \text{km/h})\left(\frac{1.002 \times 10^{-3}\ \text{kg/m}\cdot\text{s}}{1.849 \times 10^{-5}\ \text{kg/m}\cdot\text{s}}\right)\left(\frac{1.184\ \text{kg/m}^3}{998.0\ \text{kg/m}^3}\right)(5) = 25.7\ \text{km/h}$$

One advantage of a water tunnel is that the required water tunnel speed is much lower than that required for a wind tunnel using the same size model (354 km/h for air and 25.7 km/h for water) .

Similarity can be achieved even when the model fluid is different than the prototype fluid. Here a submarine model is tested in a wind tunnel.

Incomplete similarity

Example:

An air ship of 6m diameter and 30m length is to be studied in a wind tunnel. The airship speed to be investigated is at the decking end of its range, a maximum of 3 m/s. Determine the mean model wind tunnel air flow velocity if the model is made to a 1/30 scale, assuming the same sea level air pressure and temperature conditions for the model and the prototype.

Some important issues

When the velocities to be achieved are very high

- Use a bigger wind tunnel. Not always an option.
- Use a different fluid, for example, water can achieve high Reynolds numbers.
- Pressurise the wind tunnel and/or adjust the temperature to increase the Reynolds number capability of wind tunnel.
- Conduct the experiments close to the maximum capability of the wind tunnel and extrapolate.

Note: Nowadays computer simulations replace wind tunnel testing.