EGF316 – Advanced Structual Analysis 1

1. Stress and Strain

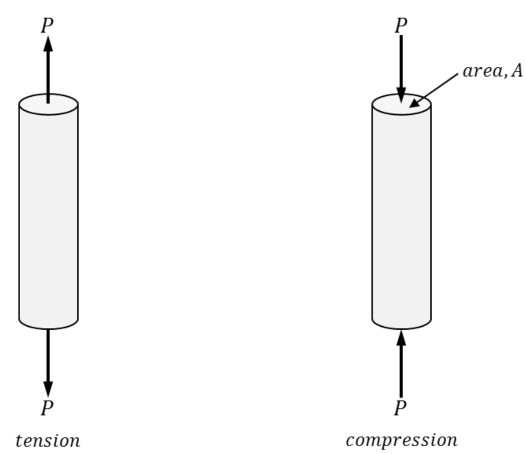
1.1 Introduction

Stress is important as it is the root cause of the majority of engineering failures, and it causes strain and deformation. Stress analysis describes a technique whereby the stress distribution within a component or structure due to external loading is determined on order that the strength/integrity of the structure can be assessed. Stress analysis is an essential element of the DESIGN process.
In all engineering structures, individual components will be subject to external forces due to the service conditions and/or the environment in which the structure operates.

For a component of a structure to be in equilibrium, there must be no overall/resultant external forces. The external forces can nevertheless subject the component to a load which will cause it to deform. This load must be reacted by internal forces which are set up in the material due to the application of the external load

If a cylinder is subject to a direct pull along its axis, it is said to be subjected to ***tension.*** This is considered positive and will cause stretching.

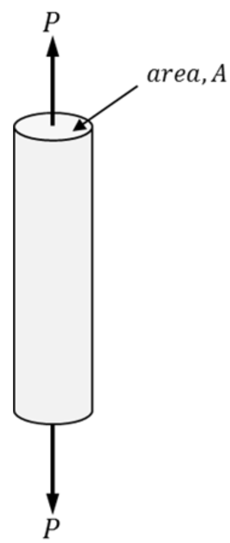
If a cylinder is subject to a direct push along its axis, it is said to be subjected to ***compression*.** This is considered negative and will cause contraction.



There are numerous ways in which load can be applied to a member/structure. Some of the most obvious are static (non-fluctuating) loads caused by the effect of gravity, live loads caused for example buy traffic travelling over a bridge, impact (or shock) loads caused by sudden blows and fatigue (or fluctuating) loads caused by for example the wind on a building.

1.2 Stress

Let us consider the most basic situation where we have a bar of cross-sectional area *A* subjected to a tensile load *P* and think about how we would be calculate the stress in the bar.



We know that:

$$direct\ stress, \sigma = \frac{force}{area} = \frac{P}{A}\left(\frac{N}{m^2}\right)$$

Note:

$$1\,N/m^2 = 1\,Pa, \quad 1{,}000{,}000\,N/m^2 = 1MPa = 1\,N/mm^2$$

Stress has the same units as pressure. Atmospheric pressure is approximately 1 bar:

$$1\ bar = 10^5 Pa = 0.1MPa$$

We have made some assumptions in the above equation. We have assumed that the force is evenly distributed across the bar, so that the stress is constant anywhere in the cross-sectional area. This is a reasonable assumption for this constant cross-section bar, and detailed analysis would confirm this.

This is not always the case. Let us consider a slightly more complicated set-up as shown below:

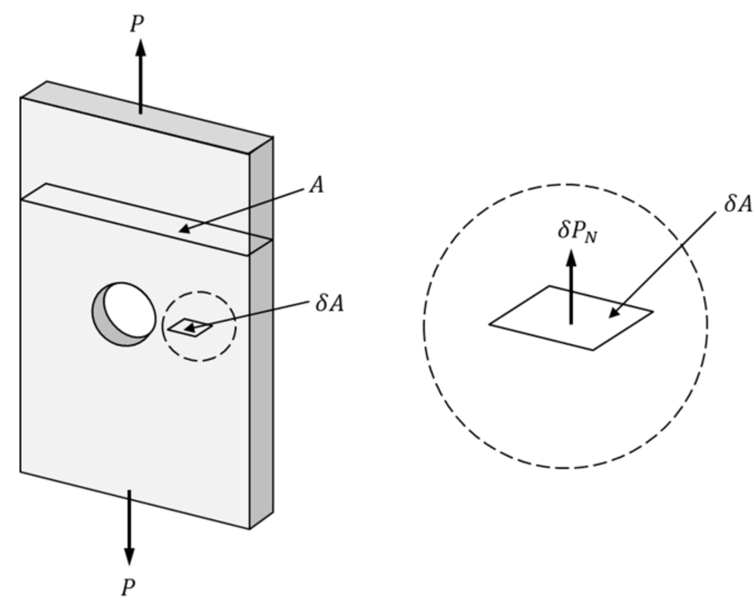
This bar has a hole, so we can no longer apply the above equation. Therefore we need to adapt the above equation to make it more general to allow us to calculate the stress at any given point.

We know that If we have a given amount of stress acting on a given area, we can calculate the stress. Looking far away from the hole, we simply divide the force by the area. If we look close to the hole, the assumption that the load is evenly distributed no longer holds true. We need to consider a small element of area δA and small part of the force which we will call δP_N to represent the fact that we are only considering the portion of the force acting normal to the small area considered.

We can now work out the stress at a point if we keep shrinking this element of area and measuring the force on it. The ratio of the force to area as we shrink this area down to zero will give us the stress.

In direct stress the cross-sectional area is *perpendicular* to the loading direction.

Direct stress is generally defined as:



$$Direct\ stress, \sigma = \lim_{\delta A \to 0} \frac{\delta P_N}{\delta A} \qquad (1.1)$$

This only considers the normal stress which occurs when we pull directly on something. There are of course other stresses such as shear stresses due to twisting and shearing action. The simplest example to get pure shear stress is a hollow section as shown below where the wall thickness is very small compared to the diameter. If we twist the section by applying a torque then we get shear stress.

One layer of material will tend to try to slide over the other. If this system is to remain in equilibrium and the layers of material do not move relative to one another, then an internal resisting shear stress must be set up within the material.

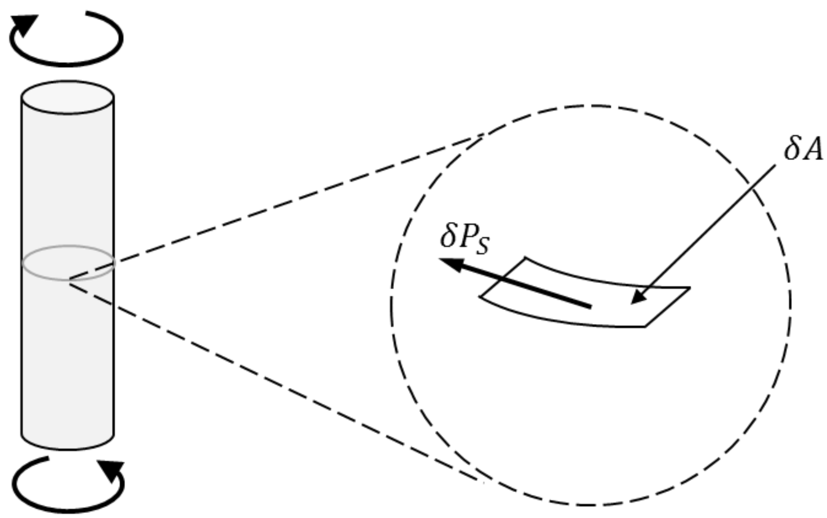
This shear stress will always be *tangential* to the areas on which it acts, the cross-sectional areas is *parallel* to the loading. The area considered is the *area resisting s*hear.

We adopt the same reasoning as for direct stress. We consider a small area and look at the shear force acting perpendicular to the surface.

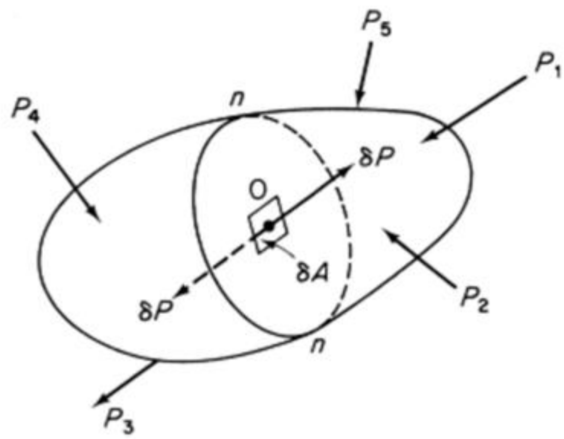
We can take any point within a body and keep shrinking down the area considered until we get the normal stress and the shear stress at that point.

Shear Stress is generally defined as:

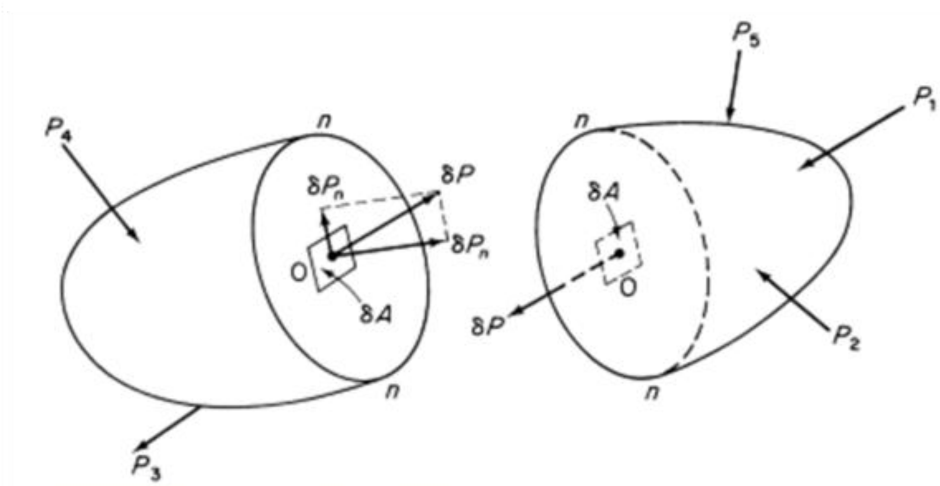
$$Shear\ stress, \tau = \lim_{\delta A \to 0} \frac{\delta P_S}{\delta A} \qquad (1.2)$$



If we now consider a very general body:



If this body is cut in half we can consider a small element of area δA and consider the force acting on it.



This force isn't necessarily normal to our element of area and could act in any direction based on the external loading conditions. The force can be split into a component normal to the surface δP_N and a component tangential to the surface δP_S and these are our *normal* forces and *shear* forces. Again, if we shrink this area down, we can calculate the normal and shear stresses at that point in the material.

1.3 Properties of Stress

In engineering, we are familiar with scalar and vector quantities. A vector quantity needs a magnitude and direction to be fully defined. Stress had a further property. We measure a vector on a surface so we need some properties of the surface also. We end up with two vectors: one that defines the direction normal to the surface on which we are measuring the force and one that defines the force. Stress is classed as a *tensor* quantity and has the notation:

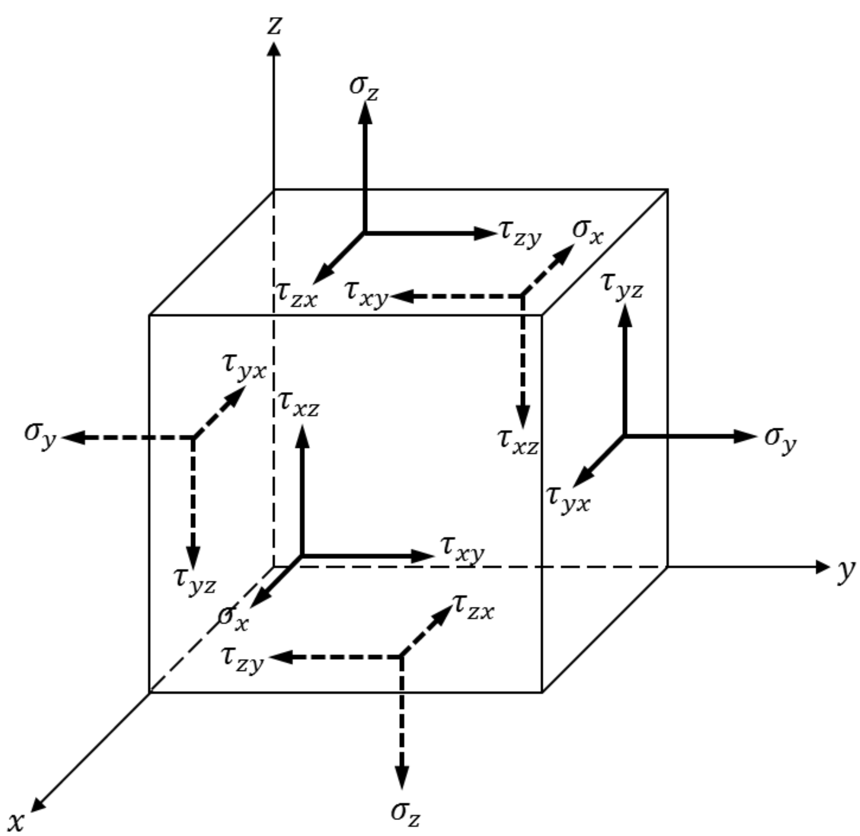
$$\sigma_{ij} \qquad \text{or} \qquad \tau_{ij}$$

Where:

$$i = \text{plane (plane of constant i), face on which stesss acts (for direct stresses i = j)}$$

$$j = \text{direction (for shear stresses i} \neq \text{j)}$$

Consider an element of material in equilibrium:



At any point in a component or structure under load, the stress state can be represented by nine quantities.

$$\sigma_{xx}, \sigma_{yy}, \sigma_{zz} \quad (normal\ stresses)$$

For simplicity, we can reduce normal stress notation to one subscript:

$$\sigma_x, \sigma_y, \sigma_z$$

And:

$$\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}. \tau_{zx}, \tau_{zy} \quad (shear\ stresses)$$

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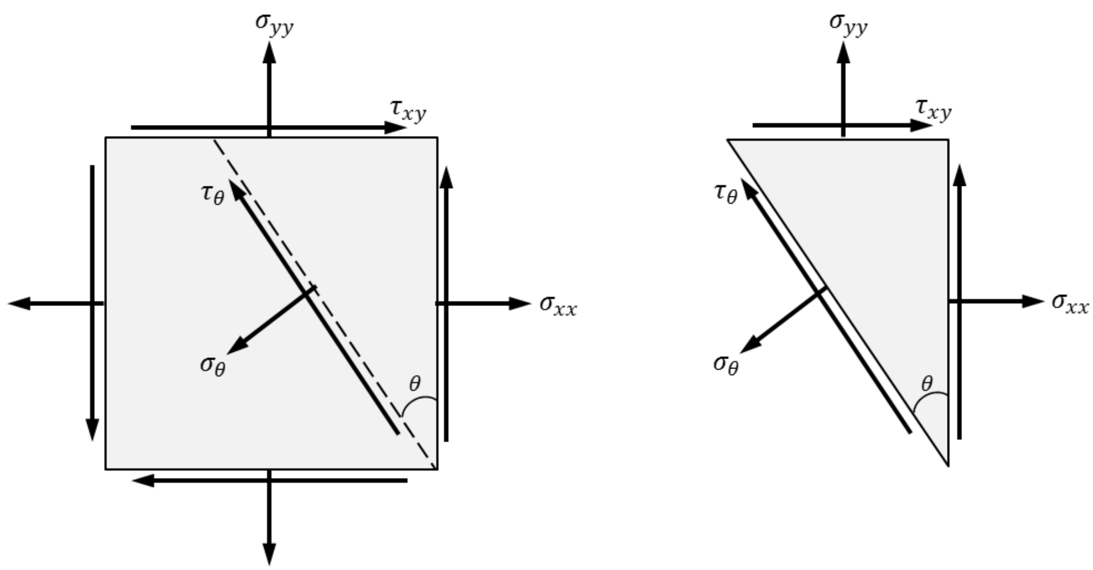
1.4 Plane Stress

Many structural components are made from thin metal sheets. This means that the stresses acting across the thickness of the sheets are very small and can be neglected in order to simplify our analysis.

Assume for example that the z-axis is in the direction of the thickness, then the stress state reduces to a two-dimensional stress state in the xy-plane where:

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0$$

Then the stress state is two-dimensional in the xy-plane. Consider the triangular element shown below:



The triangular element is in equilibrium therefore the sum of the resolved components of the forces in any given direction must be zero. The cross sectional area of the inclined plane is A.

Considering forces perpendicular to the hypotenuse plane of the triangular element we can show that:

$$\sigma_\theta A - (\sigma_{xx} A \cos\theta)\cos\theta - (\sigma_{yy} A \sin\theta)\sin\theta - (\tau_{xy} A \cos\theta)\sin\theta - \left(\tau_{xy} A \sin\theta\right)\cos\theta = 0$$

Which simplifies to:

$$\sigma_\theta = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

Considering forces parallel to the hypotenuse plane of the triangular element we can show that:

$$\tau_\theta A - (\sigma_{xx} A \cos\theta)\sin\theta + (\sigma_{yy} A \sin\theta)\cos\theta + (\tau_{xy} A \cos\theta)\cos\theta - (\tau_{xy} A \sin\theta)\sin\theta$$

Which simplifies to:

$$\tau_\theta = \sigma_{xx}\cos\theta\sin\theta - \sigma_{yy}\sin\theta\cos\theta - \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

And since:

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta), \quad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

Equations become:

$$\sigma_\theta = \sigma_{xx}\frac{1}{2}(\cos 2\theta + 1) + \sigma_{yy}\frac{1}{2}(1 - \cos 2\theta) + 2\tau_{xy}\frac{1}{2}\sin 2\theta$$

$$\sigma_\theta = \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) + \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1.3)$$

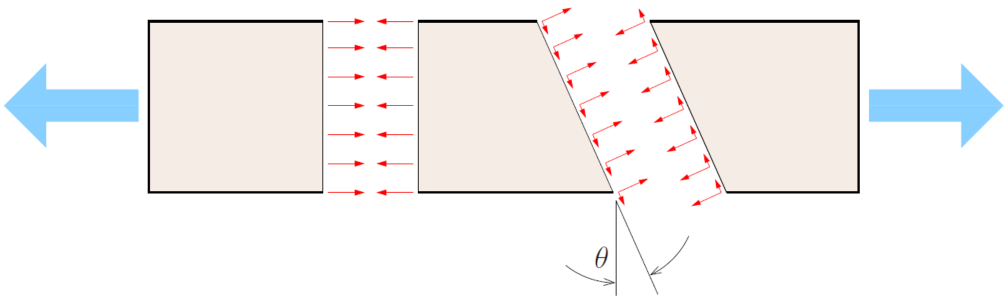
$$\tau_\theta = \sigma_{xx}\frac{1}{2}\sin 2\theta - \sigma_{yy}\frac{1}{2}\sin 2\theta - \tau_{xy}\left[\frac{1}{2}(\cos 2\theta + 1) - \frac{1}{2}(1 - \cos 2\theta)\right]$$

$$\tau_\theta = \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta - \tau_{xy}\cos 2\theta \qquad (1.4)$$

The same calculations can be done for a three dimensional state of stress, but they are much harder. We will now go on to consider the minimum and maximum stresses that occur on any plane within the material.

1.5 Principal Stresses

The stress coefficients change as the coordinate system is rotated. For each possible stress state, there is one specific orientation of the coordinate axes, such that the shear stresses disappear and the direct stresses adopt maximum and minimum values. The associated directions are denoted the ***principal directions*** of the stress state under consideration. The corresponding normal stresses are known as ***principal stresses***.



In other words, principal stresses occur on planes of zero shear stress. For the general 2D stress system, the shear stress τ_θ is zero when from equation (1.4):

$$\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta - \tau_{xy}\cos 2\theta = 0$$

This is when:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad (1.5)$$

Solving equation (1.5) gives two values of 2θ differing by 180° and hence two values of θ differing by 90°. These principal stresses lie on mutually perpendicular *principal planes* and the values of the normal stresses for those planes are *principal stresses*.

For σ_θ to adopt a minimum or maximum value, we know that:

$$\frac{d\sigma_\theta}{d\theta} = 0$$

$$\frac{d\sigma_\theta}{d\theta} = \frac{d}{d\theta}\left[\frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) + \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta + \tau_{xy}\sin 2\theta\right] = 0$$

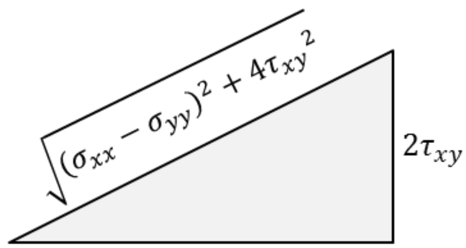
$$-\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Thus the minimum or maximum is when:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad (1.6)$$

Which is the same as (1.5). Hence the principal stresses are also maximum and minimum normal stresses.

Considering the triangle below:



It can be seen that:

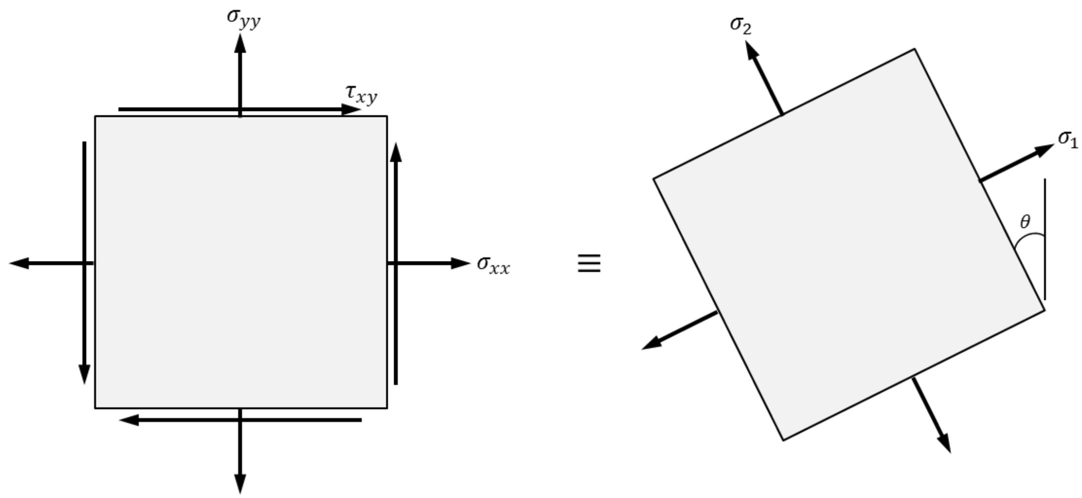
$$\sin 2\theta = \pm \frac{2\tau_{xy}}{\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\tau_{xy}^2}} \quad \text{and} \quad \cos 2\theta = \pm \frac{\sigma_{xx}-\sigma_{yy}}{\sqrt{\left(\sigma_{xx}-\sigma_{yy}\right)^2+4\tau_{xy}^2}}$$

Using this in equation (1.3) the principal stresses can thus be shown to be:

$$\sigma_{max,min} = \sigma_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2+\tau_{xy}^2} \qquad (1.7)$$

These are the principal stresses of the system and occur on mutually perpendicular planes, termed *principal planes*.

We have shown that the complex stress system can be reduced to an equivalent system of principal stresses.



At any point in a component or structure under load, there can be up to three principal stresses:

$$\sigma_1 \quad \text{(largest)}$$
$$\sigma_2$$
$$\sigma_3 \quad \text{(smallest)}$$

For a 1D state of stress, $\sigma_2 = \sigma_3 = 0$.
For a 2D state of stress, $\sigma_2 \text{ or } \sigma_3 = 0$ (depends on which book you use)

1.6 Maximum Shear Stress

The shear stress is given in equation (1.4) as:

$$\tau_\theta = \frac{1}{2}\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Differentiating to obtain the maximum gives:

$$\frac{d\tau_\theta}{d\theta} = \left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta + 2\tau_{xy}\sin 2\theta$$

$$tan2\theta = -\frac{\left(\sigma_{xx} - \sigma_{yy}\right)}{2\tau_{xy}} \qquad (1.8)$$

The maximum shear stress is therefore when $2\theta = 90°$ ie: on planes at $45°$ to the principal planes.

In the same way as previously we can consider a triangular element and it follows that:

$$sin2\theta = \frac{-(\sigma_{xx} - \sigma_{yy})}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4{\tau_{xy}}^2}}, \qquad cos2\theta = \frac{2\tau_{xy}}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4{\tau_{xy}}^2}}$$

$$sin2(\theta + \pi/2) = \frac{(\sigma_{xx} - \sigma_{yy})}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4{\tau_{xy}}^2}}, \qquad cos2(\theta + \pi/2) = \frac{-2\tau_{xy}}{\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4{\tau_{xy}}^2}}$$

Using these in (1.6):

$$\tau_{max,min} = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + {\tau_{xy}}^2} \qquad (1.9)$$

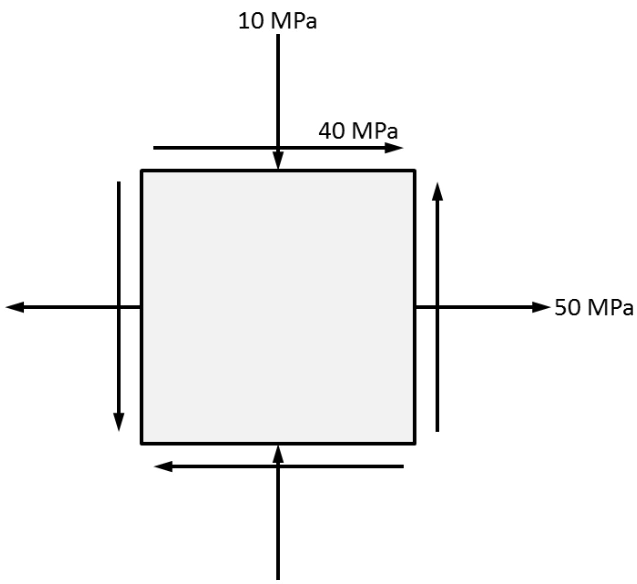
Comparing (1.9) with (1.7), we have:

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) \qquad (1.10)$$

Example

For the state of plane stress shown below, determine:

i) The principal planes
ii) The principal stresses
iii) The maximum shearing stress



(i) We have:

$$\sigma_x = 50MPa, \quad \sigma_y = -10MPa, \quad \tau_{xy} = 40MPa$$

Using:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2(40)}{50 - (-10)} = 1.3$$

$$\theta = 26.6^{\circ}$$

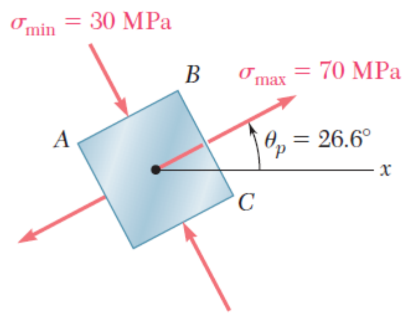
The principal planes are hence at 26.6° and $(90^{\circ} + 26.6^{\circ}) = 116.6^{\circ}$.

(ii) Using:

$$\sigma_{max,min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{max,min} = \frac{50 + (-10)}{2} \pm \sqrt{\left(\frac{50 - (-10)}{2}\right)^2 + 40^2}$$

$$\sigma_{max,min} = 20 \pm 50$$
$$\sigma_{max} = 70MPa$$
$$\sigma_{min} = -30MPa$$



As a check:

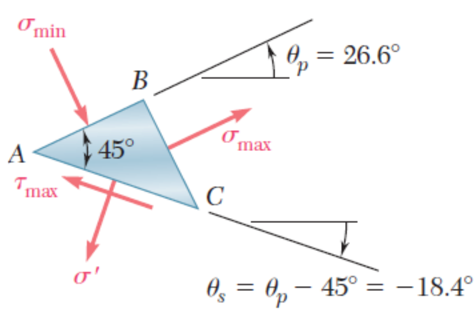
$$\sigma_{max} + \sigma_{min} = 70 + (-30) = 40MPa = \sigma_x + \sigma_y$$

(iii) We have:

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + 4\tau_{xy}{}^2}$$

$$\tau_{max} = \frac{1}{2}\sqrt{(50 - (-10))^2 + 4(40)^2} = 50MPa$$

We have said that the maximum shear stress is on planes at 45° to the principal planes.



1.7 Displacements and Strain

The internal and external forces considered earlier cause linear and angular displacements in a deformable body, such displacements generally being defined in terms of strains. When we consider the elasticity of materials, we need to understand not only stresses, but strains also.

Direct stresses give rise to direct or longitudinal strains and relate to a change in length. Shear stresses give rise to shear strains which relate to a change in angle.

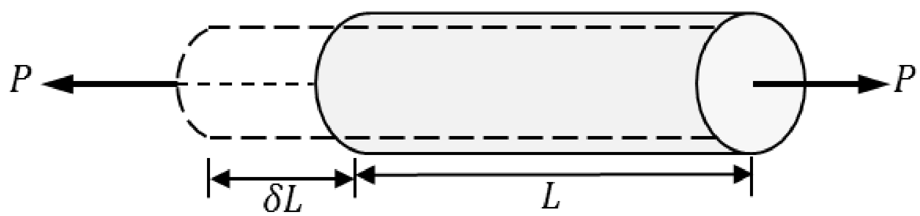
$$\varepsilon \text{ is used to represent direct strains}$$

$$\gamma \text{ is used to represent shear strains}$$

As with stress, strain can be measured at any point. When we try to measure direct strain, we take a length of material, often called the gauge length, apply a load and divide the resulting extension by the original length. In a more complicated shape (such as the bar with a hole that we considered for stress), we need to take the limit as L tends towards zero.

Instead of looking a small change in length of a large piece, we look at a smaller change in length on a smaller piece, then a tiny change in length on a tiny piece.

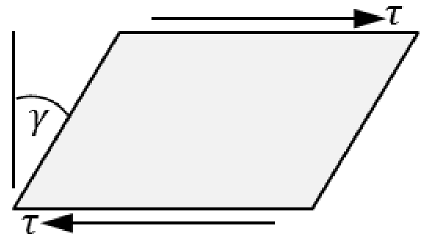
Direct strain is generally defined as:



$$\varepsilon = \lim_{L \to 0} \frac{\delta L}{L} \qquad (1.11)$$

Strain is a measure of the deformation of the material and is non-dimensional. Since elastic strain values are very small, microstrain is often used.

Consider the block shown below. It can be seen that on application of shear force, the block will undergo a change in shape. The angle of deformation γ is called the shear strain.



Within the elastic range, the shear strain is proportional to the shear stress which is causing it:

$$\frac{shear\ stress}{shear\ strain} = \frac{\tau}{\gamma} = G \qquad (1.12)$$

Where G is called the *modulus of rigidity* or the *shear modulus* of the material.

At any point in a component or structure under load, the state of strain can be represented by six scalar quantities.

$$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz} \qquad (direct\ strains)$$

And:

$$\gamma_{xy}\,(\gamma_{yx}),\ \gamma_{yz}\,(\gamma_{zy}),\ \gamma_{xz}\,(\gamma_{zx}) \qquad (shear\ strains)$$

1.8 Plane Strain

For thick sections, in the case where:

$$\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$$

Then the strain state is two-dimensional in the xy-plane.

It can be shown that:

$$\varepsilon_\theta = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad (1.13)$$

And since:

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1), \quad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta), \quad \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\varepsilon_\theta = \varepsilon_{xx}\frac{1}{2}(\cos 2\theta + 1) + \varepsilon_{yy}\frac{1}{2}(1 - \cos 2\theta) + \gamma_{xy}\frac{1}{2}\sin 2\theta$$

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) + \frac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy})\cos\ 2\theta + \gamma_{xy}\frac{1}{2}\sin 2\theta \qquad (1.14)$$

It is evident that this takes the same form as the equation for direct stress on an inclined plane with ε_{xx} and ε_{yy} instead of σ_{xx} and σ_{yy} and $\frac{1}{2}\gamma_{xy}$ instead of τ_{xy}. Thus *the shear stress is replaced by half the shear strain*.

Similarly, it can be shown that for shear strain:

$$\gamma_\theta = \frac{1}{2}\left(\varepsilon_{xx} - \varepsilon_{yy}\right)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta \qquad (1.15)$$

Again, the same calculations can be done for a 3-dimensional state of strain, but they are much harder.

1.9 Principal Strains

As for principal stresses, principal strains occur on planes of zero shear strain and can be shown to be given by:

$$\varepsilon_{max,min} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (1.16)$$

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \qquad (1.17)$$

At any point in a component or structure under load, there can be up to three principal strains:

$$\varepsilon_1 \quad \text{(largest)}$$
$$\varepsilon_2$$
$$\varepsilon_3 \quad \text{(smallest)}$$

For a 1D state of stress, $\varepsilon_2 = \varepsilon_3 = 0$.
For a 2D state of stress, $\varepsilon_2 \text{ or } \varepsilon_3 = 0$ (depends on which book you use)

Note: Tensile stresses and strains are considered to be positive in sense producing an increase in length (stretching). Compressive stresses and strains are considered negative in sense producing a decrease in length (contraction).