EGF316 – Advanced Structual Analysis

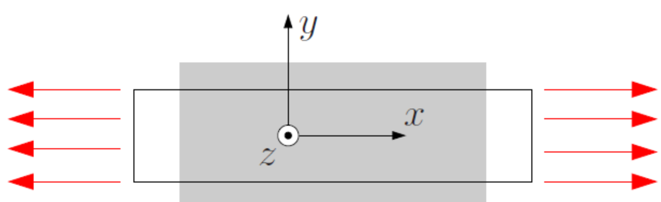
2. Stress-Strain Relationships

2.1 Introduction

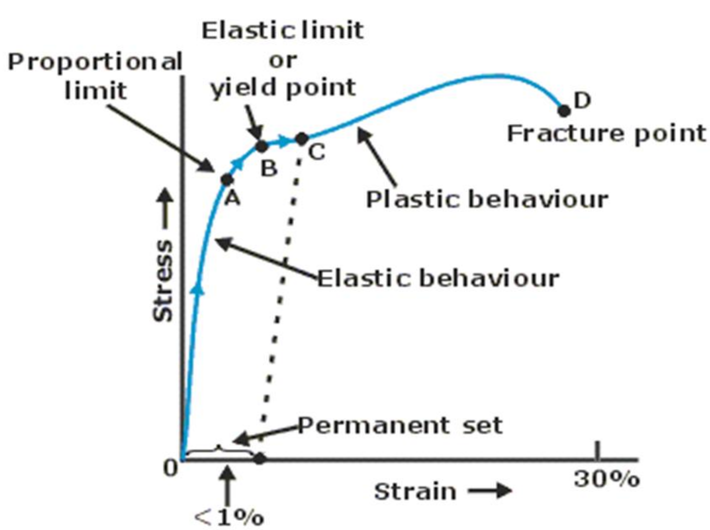
Our subsequent analysis will be limited to the elastic behavior of materials. A material is behaving elastically if it returns to its original dimensions on removal of the applied external load.

If we want to be able to compare the strengths (and thus the suitability for a given application) of various materials, we need to carry out some kind of standard test to establish their relative properties.

A common test is the standard tensile test during which a test specimen is subjected to axial tension - a gradually increasing tensile load until failure occurs.



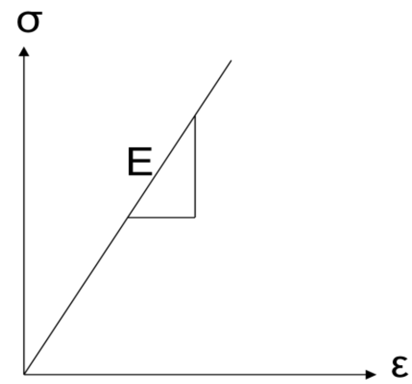
The test machine measures the changes in length of the bar with increased loading and plots a graph of load against extension, or load against strain. A typical stress strain curve for a ductile material is shown below:



This graph represents a one-dimensional state of stress (could be σ_{xx}, σ_{yy} or σ_{zz}). The gradient of the linear part of the curve represents the Young's Modulus.

So for small values of ε_x, the relation between the axial stress σ_x and the strain ε_x is accurately represented by the linear equation:

$$\sigma_x = E\varepsilon_x \qquad (2.1)$$



Where E is a constant known as the *Young's Modulus* and represents the stiffness of a material. Equation (2.1) is the simplest representation of *Hooke's Law* or *linear elasticity*.

$$E = constant = \frac{stress}{strain} = \frac{\sigma}{\varepsilon} = \frac{P}{A} \bigg/ \frac{\delta L}{L} = \frac{PL}{A\delta L}$$

If a material is stiff, it will only undergo small deflections when subjected to a large stress. A material with low stiffness will undergo a large deflection when subjected to a large stress.

We will assume Young's Modulus to be the same in tension and compression.

A typical value for steel would be around $200 \times 10^9\ N/m^2$. In most engineering applications, the strains are very small and do not often exceed 0.003 or 0.3%.

Note: a material which has a uniform structure throughout is called a *homogeneous* material (the converse if *non-homogeneous* or *inhomogeneous*). If a material exhibits uniform properties throughout in all directions, it is said to be *isotropic* (the converse is *non-isotropic* or *anisotropic*). An *orthotropic* material has different properties in different planes, such as wood.

2.3 Poisson's Ratio

When subjected to a direct stress, there will also be an associated contraction of the cross-section. This contraction is assumed to be proportional to the axial strain.

Consider a bar subjected to a load along the x-axis. From Hooke's Law, we have:

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}}$$

The normal stresses on the faces perpendicular to both the y- and z-axis are zero:

$$\sigma_y = \sigma_z = 0$$

However, the corresponding strains, $\varepsilon_{yy}\ and\ \varepsilon_{zz}$ are NOT zero.

Under the action of the load, the bar will increase in length by an increment δL giving rise to a longitudinal strain in the bar given by:

$$\varepsilon_{xx} = \frac{\delta L}{L}$$

At the same time, the bar will contract laterally. In 3-dimensions, its breadth and depth will both reduce. The associated lateral strains will be equal and will be in the opposite sense to the longitudinal strain. Assuming that the material is kept within its elastic range, the ratio of longitudinal and lateral strains will remain constant and is termed Poisson's Ratio:

$$v = \frac{lateral\ strain}{longitudinal\ strain} = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}} = constant \qquad (2.2)$$

Typically, for metals, Poisson's Ratio lies somewhere between 0.28 and 0.32. A value of 0.3 is very often used in analysis.

2.4 Direct Stress-Direct Strain Relationships

Therefore, for a one-dimensional state-of–stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \quad or \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} \quad or \quad \varepsilon_{zz} = \frac{\sigma_{zz}}{E} \qquad (2.3)$$

And:

$$\sigma_{xx} = E\varepsilon_{xx} \quad or \quad \sigma_{yy} = E\varepsilon_{yy} \quad or \quad \sigma_{zz} = E\varepsilon_{zz} \qquad (2.4)$$

For a 3-dimensional state of stress it can be shown that:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - v\frac{\sigma_{yy}}{E} - v\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - v\frac{\sigma_{xx}}{E} - v\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - v\frac{\sigma_{xx}}{E} - v\frac{\sigma_{yy}}{E} \qquad (2.5)$$

And:

$$\sigma_{xx} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{xx} + v\varepsilon_{yy} + v\varepsilon_{zz}\right]$$

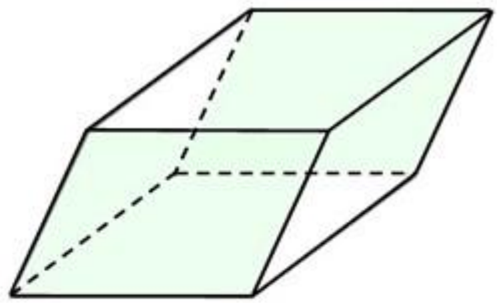
$$\sigma_{yy} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{yy} + v\varepsilon_{xx} + v\varepsilon_{zz}\right]$$

$$\sigma_{zz} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{zz} + v\varepsilon_{xx} + v\varepsilon_{yy}\right] \qquad (2.6)$$

These relations are the generalized Hooke's Law for multiaxial loading of a homogenous isotropic material.

2.5 Shear Stress - Shear Strain Relationships

We have derived relations between normal stresses and normal strains in a homogenous isotropic material. We assumed that no shearing stresses were involved. In the more generalized case there will be shearing stresses present, $\tau_{xy(yx)}, \tau_{yz(zy)}, \tau_{zx(xz)}$. These shear stresses have no direct effect on the normal strains. They tend to deform a cubic element of material into an oblique parallelepiped.



As for normal stresses and strains, the initial portion of a shear stress-strain diagram would be a straight line. For values of shear stress within the proportional limit, we can write for a one-dimensional state-of–stress:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

This is Hooke's Law for shear stress and strain of a homogeneous isotropic material.

Where G is the *modulus of rigidity* or *shear modulus* of the material given by:

$$G = \frac{E}{2(1+v)} \qquad (2.7)$$

For a generalised three-dimensional state-of–stress:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G} \qquad (2.8)$$

Thus:

$$\gamma_{xy} = \frac{2(1+v)\tau_{xy}}{E}, \quad \gamma_{yz} = \frac{2(1+v)\tau_{yz}}{E}, \quad \gamma_{zx} = \frac{2(1+v)\tau_{zx}}{E} \qquad (2.9)$$

2.6 Two-Dimensional State of Stress

We need to make an assumption about the third direction – we have two alternatives:

- Plane Stress Assumption
- Plane Strain Assumption

Plane Stress Assumption:

We assume that there are no stresses in the third direction. This is used for 'thin' plates with in-plane loads. The geometry of the body is such that one dimension is much smaller than the others. The loads are applied uniformly over the thickness of the plate and act in the plane of the plate. This simple stress condition represents a large number of real-life situations.

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

Therefore:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - v\frac{\sigma_{yy}}{E} - v\frac{\sigma_{zz}}{E} \quad \rightarrow \quad \varepsilon_{xx} = \frac{\sigma_{xx}}{E} - v\frac{\sigma_{yy}}{E} = \frac{1}{E}(\sigma_{xx} - v\sigma_{yy})$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - v\frac{\sigma_{xx}}{E} - v\frac{\sigma_{zz}}{E} \quad \rightarrow \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - v\frac{\sigma_{xx}}{E} = \frac{1}{E}(\sigma_{yy} - v\sigma_{xx})$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - v\frac{\sigma_{xx}}{E} - v\frac{\sigma_{yy}}{E} \quad \rightarrow \quad \varepsilon_{zz} = -v\frac{\sigma_{xx}}{E} - v\frac{\sigma_{yy}}{E} = \frac{v}{E}(-\sigma_{xx} - \sigma_{yy})$$

$$E\varepsilon_{xx} = \sigma_{xx} - v\sigma_{yy}, \quad E\varepsilon_{yy} = \sigma_{yy} - v\sigma_{xx}$$

Solving these simultaneously:

$$E\varepsilon_{xx} = \sigma_{xx} - v\left(E\varepsilon_{yy} + v\sigma_{xx}\right) = \sigma_{xx} - vE\varepsilon_{yy} - v^2\sigma_{xx} = \sigma_{xx}(1 - v^2) - vE\varepsilon_{yy}$$

$$\sigma_{xx}(1 - v^2) = E\varepsilon_{xx} + vE\varepsilon_{yy} = E(\varepsilon_{xx} - v\varepsilon_{yy})$$

$$\sigma_{xx} = \frac{E}{(1-v^2)}\left(\varepsilon_{xx} + v\varepsilon_{yy}\right) \qquad (2.10)$$

And similarly:

$$\sigma_{yy} = \frac{E}{(1-v^2)}\left(\varepsilon_{yy} + v\varepsilon_{xx}\right) \qquad (2.11)$$

$$\tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy} \qquad (2.12)$$

Example 1:

The principal strains at a point on a loaded thin plate are found to be 320×10^{-6} and 200×10^{-6}. What are the principal stresses if the modulus of elasticity is 200GPa and the Poisson's ratio is 0.3?

$$\sigma_1 = \frac{E}{(1-v^2)}(\varepsilon_1 + v\varepsilon_2) = \frac{200 \times 10^9}{(1-0.3^2)}[320 \times 10^{-6} + 0.3(200 \times 10^{-6})] = 83.5MPa$$

$$\sigma_2 = \frac{E}{(1-v^2)}(\varepsilon_2 + v\varepsilon_1) = \frac{200 \times 10^9}{(1-0.3^2)}[200 \times 10^{-6} + 0.3(320 \times 10^{-6})] = 65.1MPa$$

Plane Strain Assumption

We assume that there are no strains in the third direction. This is used for 'thick' plates with in-plane loads. This state of stress describes a situation in which the dimension of the structure in one direction (say the z-direction) is very large in comparison with the dimensions of the structure in the other two dimensions. The applied forces in the x-y plane do not vary in the z-direction (the loads are uniformly distributed with respect to the larger dimension, and act perpendicular to it).

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{v}{E}\sigma_{xx} - \frac{v}{E}\sigma_{yy} \quad \rightarrow \quad \sigma_{zz} = v(\sigma_{xx} + \sigma_{yy})$$

Giving:

$$\sigma_{xx} = \frac{E}{(1+v)(1-2v)}\big((1-v)\varepsilon_{xx} + v\varepsilon_{yy}\big) \qquad (2.13a)$$

$$\sigma_{yy} = \frac{E}{(1+v)(1-2v)}\big((1-v)\varepsilon_{yy} + v\varepsilon_{xx}\big) \qquad (2.13b)$$

$$\sigma_{zz} = \frac{E}{(1+v)(1-2v)}\big(v\varepsilon_{xx} + v\varepsilon_{yy}\big) \qquad (2.13c)$$

$$\tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy} \qquad (2.14)$$

Example 2:

A strain gauge rosette is attached to a thin flat steel plate. When loaded in-plane, the following strain values are measured:

$$\varepsilon_A = 500 \times 10^{-6}$$
$$\varepsilon_B = -400 \times 10^{-6}$$
$$\varepsilon_C = -200 \times 10^{-6}$$

Gauge A lies along the x axis of the structure and Gauge B and C are at $+45^{\circ}$ and -45° to Gauge A, respectively. Determine the values of:

(i) The in-plane strains and stresses
(ii) The in-plane principal strains
(iii) The principal stresses and the maximum shear stress

For steel, assume $E = 200GPa, v = 0.3$ and $\sigma_{yield} = 300MPa$.

(i)

$$\varepsilon_A = \varepsilon_{xx} = 500$$

$$\varepsilon_B = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_B = 500\cos^2 45 + \varepsilon_{yy}\sin^2 45 + \gamma_{xy}\sin 45\cos 45$$

$$\varepsilon_B = \frac{500}{2} + \frac{\varepsilon_{yy}}{2} + \frac{\gamma_{xy}}{2} = -400 \times 10^{-6}$$

$$-800 = 500 + \varepsilon_{yy} + \gamma_{xy} \qquad (1)$$

$$\varepsilon_C = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_C = 500\cos^2(-45) + \varepsilon_{yy}\sin^2(-45) + \gamma_{xy}\sin(-45)\cos(-45)$$

$$\varepsilon_C = \frac{500}{2} + \frac{\varepsilon_{yy}}{2} - \frac{\gamma_{xy}}{2} = -200 \times 10^{-6}$$

$$-400 = 500 + \varepsilon_{yy} - \gamma_{xy} \qquad (2)$$

(1)+(2):

$$-1200 = 1000 + 2\varepsilon_{yy}$$

$$-1200 - 1000 = 2\varepsilon_{yy}$$

$$\varepsilon_{yy} = -1100$$

Therefore:

$$\gamma_{xy} = 500 + \varepsilon_{yy} + 400 = 900 + (-1100) = -200$$

We know:

$$\sigma_{xx} = \frac{E}{(1-v^2)}(\varepsilon_{xx} + v\varepsilon_{yy}), \quad \sigma_{yy} = \frac{E}{(1-v^2)}(\varepsilon_{yy} + v\varepsilon_{xx}), \quad \tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy}$$

$$\sigma_{xx} = \frac{E}{(1-v^2)}\left(\varepsilon_{xx} + v\varepsilon_{yy}\right) = \frac{200 \times 10^9}{(1-0.3^2)}[500 \times 10^{-6} + 0.3(-1100 \times 10^{-6})]$$

$$\sigma_{xx} = 37.4MPa$$

$$\sigma_{yy} = \frac{E}{(1-v^2)}\left(\varepsilon_{yy} + v\varepsilon_{xx}\right) = \frac{200 \times 10^9}{(1-0.3^2)}[-1100 \times 10^{-6} + 0.3(500 \times 10^{-6})]$$

$$\sigma_{yy} = -208.8MPa$$

$$\tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy} = \frac{200 \times 10^9}{2(1+0.3)}(-200 \times 10^{-6})$$

$$\tau_{xy} = 15.4MPa$$

(ii):

$$\varepsilon_{max,min} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{1,2} = \frac{500 + (-1100)}{2} \pm \sqrt{\left(\frac{500 - (-1100)}{2}\right)^2 + \left(\frac{-200}{2}\right)^2}$$

$$\varepsilon_{1,2} = -300 \pm \sqrt{800^2 + 100^2} = -300 \pm \sqrt{650000}$$

$$\varepsilon_1 = 506.2 \times 10^{-6}$$

$$\varepsilon_2 = -1106.2 \times 10^{-6}$$

(iii):

$$\sigma_{max,min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + {\tau_{xy}}^2}$$

$$\sigma_{1,2} = \frac{37.4 + (-208.8)}{2} \pm \sqrt{\left(\frac{37.4 - (-208.8)}{2}\right)^2 + 15.4^2}$$

$$\sigma_{1,2} = -85.7 \pm \sqrt{15153.6 + 237.2} = -85.7 \pm \sqrt{15390.8}$$

$$\sigma_1 = 38.3MPa$$

$$\sigma_2 = -209.7MPa$$

Or:

$$\sigma_1 = \frac{E}{(1 - v^2)}(\varepsilon_1 + v\varepsilon_2) = \frac{200 \times 10^9}{(1 - 0.3^2)}[506.2 \times 10^{-6} + 0.3(-1106.2 \times 10^{-6})]$$

$$\sigma_1 = 38.3MPa$$

$$\sigma_2 = \frac{E}{(1 - v^2)}(\varepsilon_2 + v\varepsilon_1) = \frac{200 \times 10^9}{(1 - 0.3^2)}[-1106.2 \times 10^{-6} + 0.3(506.2 \times 10^{-6})]$$

$$\sigma_2 = -209.7MPa$$