Advanced Structural Analysis
**EGF316**
3. Section Properties and Bending

3.1 Loads in beams

When we analyse beams, we need to consider various types of loads acting on them, for example, axial forces, shear forces, bending moments and torques. Any beam structure subjected to any or all of these loads have associated stresses. In order to compute the stresses, we need to know the nature of the cross-section of the beam. Before going into the details of stress calculations for beams, let us recap some preliminaries of beam bending.

3.2 Beam bending

If the load is applied in such a way that the structure deforms perpendicular to the axis of the structure, then such a deformation behavior is called bending.

When a beam is loaded and subsequently bends, its longitudinal axis is deformed into a curve. One side is extended and in state of tension whereas the other side is shortened and in a state of compression, as illustrated in Figure 1.
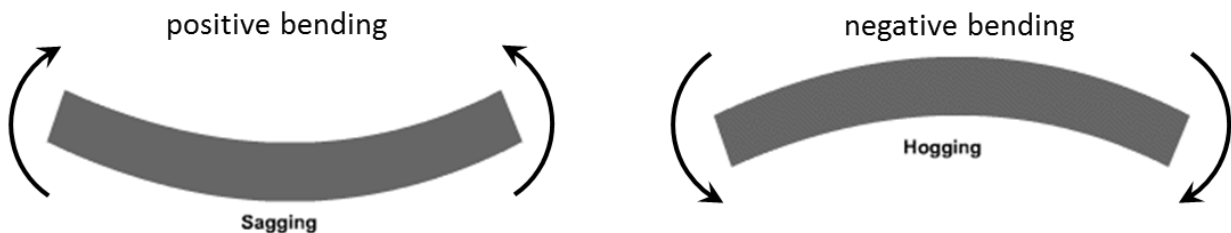
Figure 1: State of bending

If the loads cause the beam to sag, the upper surface of the beam is shorter that the lower surface and the opposite is true for hogging. Thus, the strains in the upper and lower portions of the beam are different, and knowing that stress is directly proportional to strain for linear elastic material, it follows that the stress varies through the depth of the beam.

Consider an appropriately supported beam aligned with the x-axis of the coordinate system, which is subjected to external loading such that it deflects in the y-direction as shown in Figure 2.
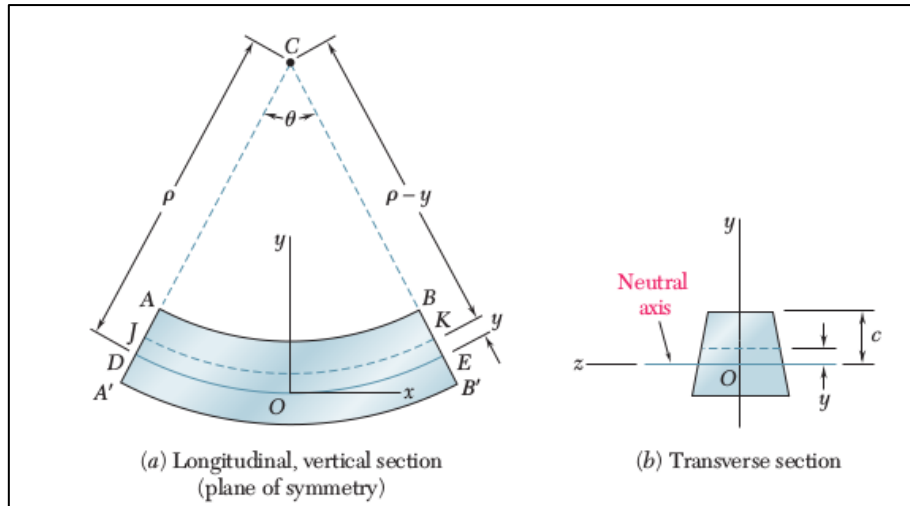
Figure 2: Bending of beam. Source: Mechanics of Materisl, Beer, Johnston, DeWolf, Mazurek

Stress analysis of the beam is based on some kinematical assumptions:

- The beam is slender (length >> thickness)
- Cross sections remain plane
- Cross sections remain normal to the beam axis (*Bernoulli hypothesis*)
- The beam material is linearly elastic (obeys Hooke's Law) and homogeneous
- Deformations and deflections are small

In the figure above, the direct stress varies from compression in the upper fibers of the beam to tension in the lower. Logically, the direct stress is zero in the fibers that do not undergo a change in length and we call the plane containing these fibers the *neutral plane*. The line of intersection of the neutral plane and any cross-section of the beam is termed the *neutral axis*.

The neutral axis denotes the material fiber parallel to the beam axis which, in pure bending, does not experience any elongation or compression.

Our challenge therefore is to be able to determine the variation of direct stress through the depth of the beam, calculate the values of stresses and also find the corresponding beam deflection.

The simple theory of elastic bending states that:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \qquad (3.1)$$

Where :
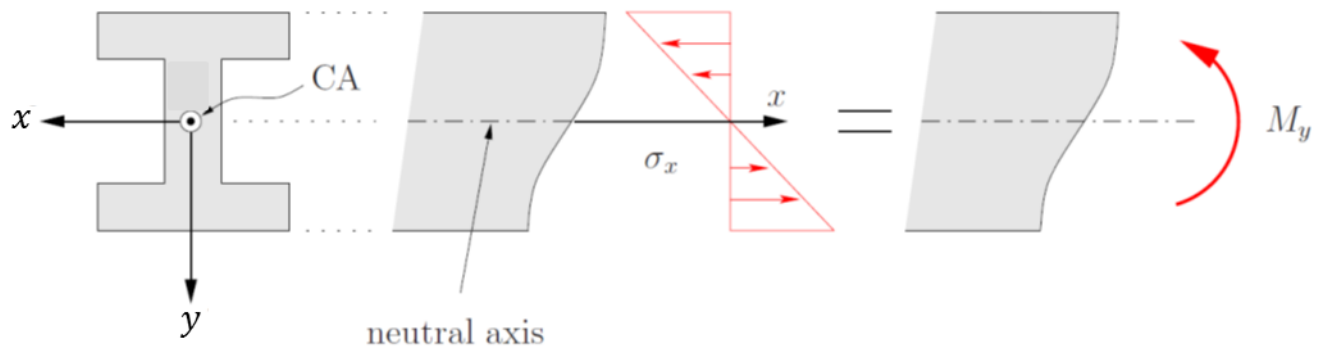M = applied bending moment (Nm or Nmm)
I = Second moment of area of cross-section (m^4 or mm^4)

σ = bending stress at 'y' (N/m^2 or MPa)

y = distance from neutral axis (m or mm)

E = Young's modulus for the material (N/m^2 or MPa)

R = Radius of curvature (m or mm)

Thus for a simply supported beam with central load subject to pure bending:

$$\sigma = \frac{My}{I} \qquad (3.2)$$

There is a linear variation in stress with distance from NA:

3.3 Recap on Some Simple Beam Types

There are different types of beams depending upon the boundary conditions. The three shown below are some of the simplest.

Cantilever Beam

A *cantilever beam* is built in or fixed at one end and the other end is free to move. When a load is applied to the cantilever, a reaction and resisting moment occur at the fixed end.
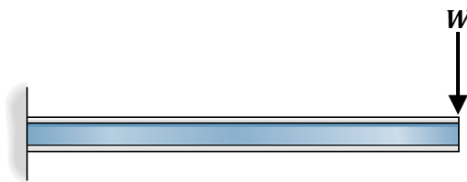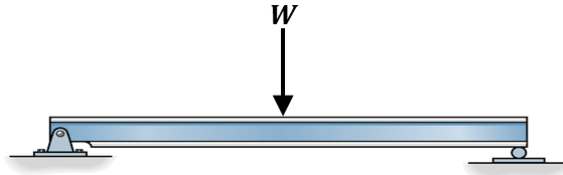

For a cantilever beam with end load it can be shown that:

$$\text{Maximum bending moment,} \qquad M_{max} = WL$$

$$\text{Maximum displacement,} \qquad \delta_{max} = \frac{WL^3}{3EI}$$

Simply Supported Beam

A *simply supported beam* is supported at its ends on rollers or smooth surfaces, or with one of these combined with a pin at the other end.
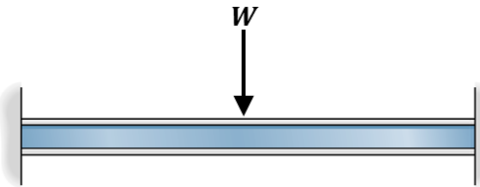
For a simply supported beam with a central load it can be shown that:

$$\text{Maximum bending moment,} \quad M_{max} = \frac{WL}{4}$$

$$\text{Maximum displacement,} \quad \delta_{max} = \frac{WL^3}{48EI}$$

Built in Beam

A *built-in beam* is built in or fixed at both ends.

For a built-in beam with central load it can be shown that:

$$\text{Maximum bending moment,} \quad M_{max} = \frac{WL}{8}$$

$$\text{Maximum displacement,} \quad \delta_{max} = \frac{WL^3}{192EI}$$

3.4 Computing stresses due to Bending

Steps in calculating the stresses in beams.

1.) As the bending moment changes along the length of a beam, we first need to choose a position along the beam at which to determine the stresses.
2.) We then need to get the cross-section at the chosen position and find the relevant sectional properties - namely the centroid and the second moment of areas.
3.) Then, we can calculate the corresponding stresses.

3.5 Centroid of Beam Cross Sections

The weight of a body is an example of a distributed force in that any body can be considered to be made up of a number of particles each with weight. It is convenient to replace all of these individual weight forces by a single weight force with a magnitude equal to the sum of the magnitudes of all of the constituent weight forces. This equivalent weight force acts at a particular point called the *centre of gravity*. When considering a section of a constant cross-section beam, we use the term *centroid* instead of centre of gravity.

The centroid is important in calculations as it tells us where the neutral axis lies. If we bend a beam, the neutral axis is the plane on which there is no strain. Some sections have multiple neutral axis and they all pass through the centroid.

Where would you anticipate the position of the centroid for the following shapes?
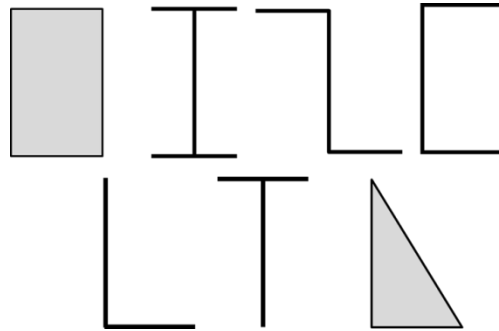
The first moment of area of a section is a measure of the distribution of mass relative to an axis.
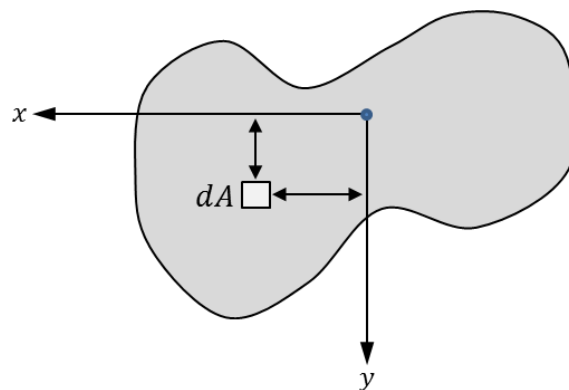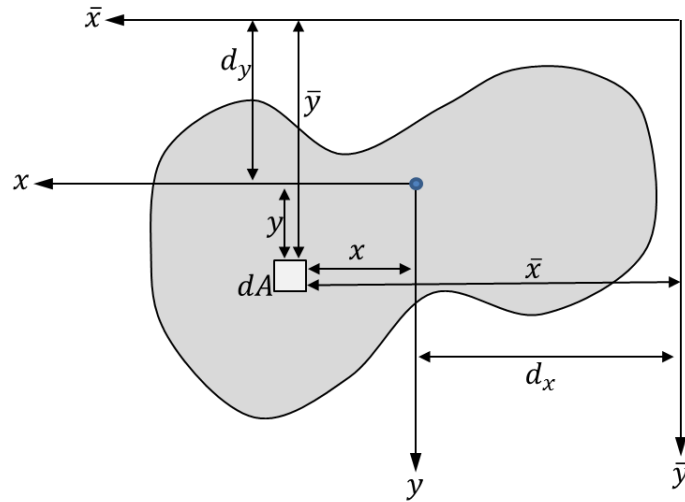
First moment of area A about the x-axis:

$$m_x = \int y dA$$

First moment of area A about the y-axis:

$$m_y = \int x dA$$

The centroid is the point at which the first moment of area goes to zero for any orthogonal axis system. The centroid of a section can be located as shown below:

$$\int x\,dA = 0$$

$$\int (\acute{x} - d_x)\,dA = 0$$

$$\int \acute{x}\,dA = \int d_x\,dA$$

Rearranging:

$$d_x = \frac{\int \acute{x}\,dA}{\int dA}; \quad \text{similarly,} \quad d_y = \frac{\int \acute{y}\,dA}{\int dA} \qquad (3.3)$$

For symmetrical homogeneous bodies, the centroid is located at the geometric centre. For composite sections, it can be obtained by considering the object to be made up of constituent parts each having weight acting through their own centroid. For objects containing holes or cutouts, the hole is treated as a negative mass.
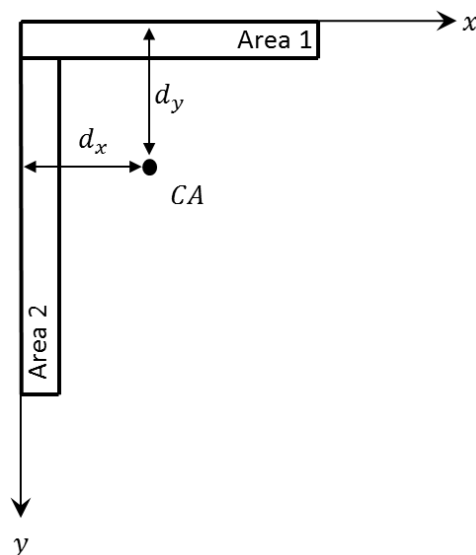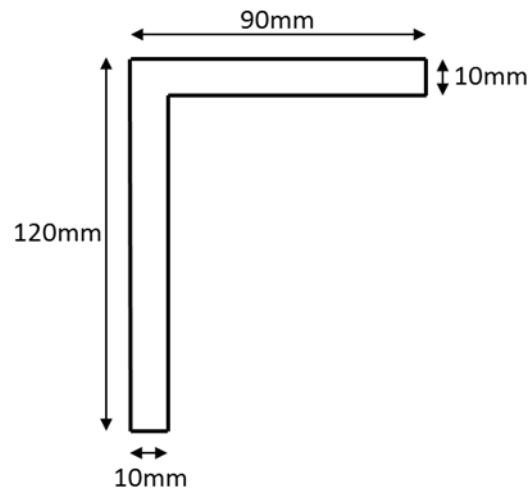
We find the centroid relative to any arbitrary orthogonal axis. Then once we know the position of the centroid, we can carry out subsequent calculations relative to this.

When considering discrete pieces of area, the integrals in the above equations can be replaced by a sum:

$$d_x = \frac{\sum_i \acute{x}_i A_i}{\sum_i A_i}; \quad \text{similarly,} \quad d_y = \frac{\sum_i \acute{y}_i A_i}{\sum_i A_i} \qquad (3.4)$$

Example 1

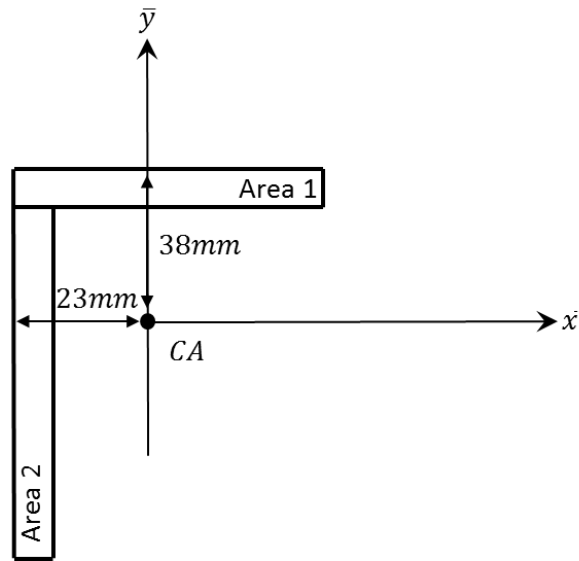Locate the centroid of the following shape:

Recall:

$$d_x = \frac{\sum_i \acute{x}_i A_i}{\sum_i A_i}$$

$$d_x = \frac{\acute{x}_1 A_1 + \acute{x}_2 A_2}{A_1 + A_2} = \frac{(45 \times 900) + (5 \times 1100)}{900 + 1100} = \frac{46000}{2000} = 23mm$$

$$d_y = \frac{\sum_i \acute{y}_i A_i}{\sum_i A_i}$$

7

$$d_y = \frac{\acute{y}_1 A_1 + \acute{y}_2 A_2}{A_1 + A_2} = \frac{(5 \times 900) + (65 \times 1100)}{(900 + 1100)} = \frac{76000}{2000} = 38mm$$

3.6 Second Moments of Area

The *second moment of area* (or moment of inertia) of a beam section is a measure of how far away the material is located from the neutral axis and therefore its resistance to bending. Thus the greater the second moment of area, the greater the bending moment needed to produce a given radius of curvature of the beam. In most cases, we would aim to maximize the second moments of area.

The second moments of area are given by:

$$I_{xx} = \int_A y^2 dA$$

$$I_{yy} = \int_A x^2 dA$$

$$I_{xy} = \int_A xy dA \qquad (3.5)$$

Again, if the areas are discrete, we can replace these integrals by a sum:

$$I_x = \sum y^2 A, I_y = \sum x^2 A, I_{xy} = \sum xy A \qquad (3.6)$$

The quantities above are geometric properties and can be evaluated for any cross section and must be taken relative to the centroid. Clearly, they depend on the origin and the orientation of the coordinate system.
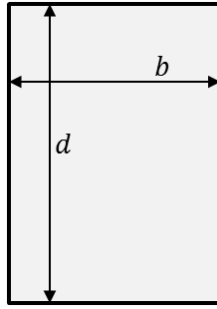
3.6.1 Some Useful Facts

For a circular section:

$$I = \frac{\pi D^4}{64}; \ J = \frac{\pi D^4}{32}$$

Where D is the diameter.

For a rectangular section:

$$I_{xx} = \frac{bd^3}{12}$$

$$I_{yy} = \frac{b^3d}{12}$$

$$I_{xy} = 0$$

Hence, for a square section (where $b = d$):

$$I_{xx} = I_{yy} = \frac{b^4}{12}$$

3.7 Parallel Axis Theorem

For the calculation of second moments of areas of complex sections, it is often convenient to perform the additive decompositions of the integrals above. Divide the area into a series of simpler shapes and the second moment of area for the entire shape is the sum of the second moment of areas of all of its parts *about a common axis* with origin at the centroid of the overall shape. Consider the elbow section below:
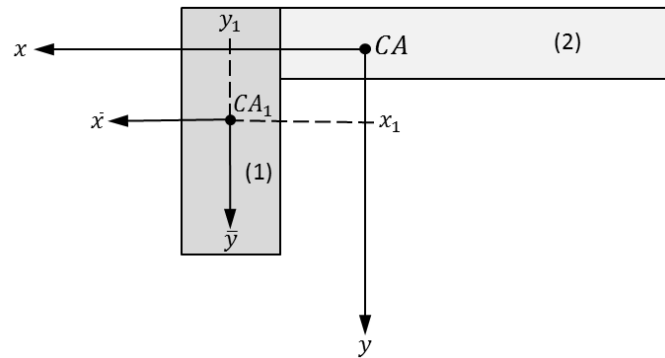
Firstly, the second moment of area of each rectangle needs to be calculated with respect to the 'global' coordinate system xy. A convenient strategy for this calculation consists of three steps:

1. Locate the centroid of the overall section
2. Determine the second moments of area $I_{xx_{ic}}, I_{yy_{ic}}, I_{xy_{ic}}$ of the subsections $i = 1,2,3, \dots n$ with respect to their own 'local' coordinate systems.
3. Account for the shift of the coordinate axis

We denote the coordinates of the centroid CA_1 of part 1 of the section with respect to the global system as x_1 and y_1. We can then write:

$$y^2 dA = \int_{A_1} \left(y_{1_c} + y_1\right)^2 dA$$

$$I_{xx_1} = \int_{A_1}$$

$$y_{1_c}^2 dA + \int_{A_1} 2 y_{1_c} y_1 dA + \int_{A_1} y_1^2 dA$$

$$I_{xx_1} = \int_{A_1}$$

$$I_{xx_1} = I_{xx_{1c}} + 2 y_1 \int_{A_1} y_{1_c} A + y_1^2 A_1$$

By definition, the $\acute{x}$- and $\acute{y}$-axes pass through the centroid CA_1 of subsection 1. Therefore, the remaining integral in the above disappears and we obtain:

11

$$I_{xx_1} = I_{xx_{1c}} + y_1{}^2 A_1$$

Similarly:

$$I_{xx_2} = I_{x_{2c}} + y_2{}^2 A_2$$

In the same way, we can show that in general:

$$I_{xx_i} = I_{xx_{ic}} + y_i{}^2 A_i$$

$$I_{yy_i} = I_{y_{ic}} + x_i{}^2 A_i$$

$$I_{xy_i} = I_{xy_{ic}} + x_i y_i A_i \quad (3.7)$$

It follows that the overall second moments of area of the compound section are given by:

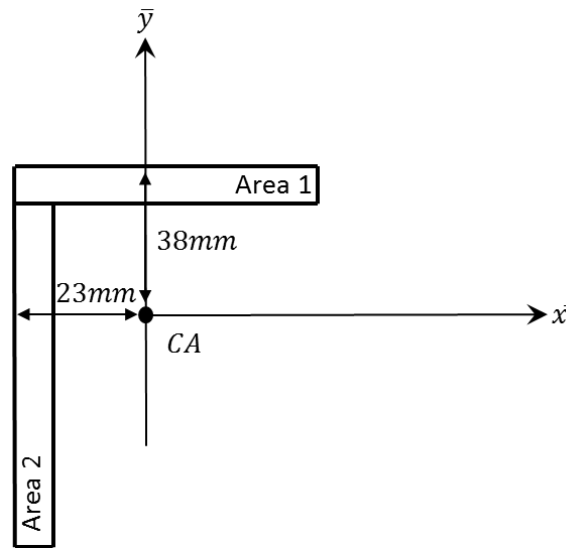$$I_{xx} = \sum I_{xx_{ic}} + y_i{}^2 A_i$$

$$I_{yy} = \sum I_{y_{ic}} + x_i{}^2 A_i$$

$$I_{xy} = \sum I_{xy_{ic}} + x_i y_i A_i \quad (3.8)$$

When applying these equations, take care as x_i and y_i can be positive or negative. These relations are known as the ***parallel axis theorem***.

Example 1 (continued):

Calculate I_{xx}, I_{yy} and I_{xy}

$$I_{xx} = \sum_{i=1}^{n} I_{xx}{}^{(i)}, I_{yy} = \sum_{i=1}^{n} I_{yy}{}^{(i)}, I_{xy} = \sum_{i=1}^{n} I_{xy}{}^{(i)}$$

And:

$$I_{xx}{}^{(i)} = I_{\acute{x}x}{}^{(i)} + y_i{}^2 A_i$$

$$I_{yy}{}^{(i)} = I_{\acute{y}y}{}^{(i)} + x_i{}^2 A_i$$

$$I_{xy}{}^{(i)} = I_{\acute{x}\acute{y}}{}^{(i)} + x_i y_i A_i$$

$$I_{xx} = I_{\acute{x}x}{}^{(1)} + y_1{}^2 A_1 + I_{\acute{x}x}{}^{(2)} + y_2{}^2 A_2$$

$$I_{xx} = \left(\frac{90 \times 10^3}{12}\right) + (33^2 \times 900) + \left(\frac{10 \times 110^3}{12}\right) + \{(-27)^2 \times 1100\}$$

$$I_{xx} = 7500 + 980100 + 1109166.7 + 801900 = 2.8987 \times 10^6 mm^4$$

$$I_{yy} = I_{\acute{y}y}{}^{(1)} + x_1{}^2 A_1 + I_{\acute{y}y}{}^{(2)} + x_2{}^2 A_2$$

$$I_{yy} = \left(\frac{10 \times 90^3}{12}\right) + (22^2 \times 900) + \left(\frac{110 \times 10^3}{12}\right) + \{(-18)^2 \times 1100\}$$

$$I_{yy} = 607500 + 435600 + 9166.7 + 356400 = 1.4087 \times 10^6 mm^4$$

$$I_{xy}^{(i)} = I_{\acute{x}\acute{y}}^{(i)} + x_i y_i A_i$$

$$I_{xy} = I_{\acute{x}\acute{y}}^{(1)} + x_1 y_1 A_1 + I_{\acute{x}\acute{y}}^{(2)} + x_2 y_2 A_2$$

$$I_{xy} = (33 \times 22 \times 900) + \{(-18)(-27) \times 1100\} = 1.188 \times 10^6 mm^4$$

Note: I_{xx} and I_{yy} are always positive, but I_{xy} can be positive or negative.

3.8 Principal Axis

There always exists an orientation of the coordinate system such that $I_{xy} = 0$. The associated coordinate axes are called the *principal directions* of the cross section.

A *principal axis* is one where bending about one axis does not result in any deflection (and hence stress/strain) perpendicular to that axis. There is no interaction between the two axes.

It follows that *every axis of symmetry is a principal axis*.

The principal axes for an open section are not so obvious. We need to calculate:

- The angle, θ, of the principal axes relative to the x and y axes
- The second moments of area about principal axes, I_u and I_v

It can be shown that:

$$tan2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})} \quad (3.9)$$

Where:
I_{xx} is the second moment of area about the x-axis
I_{xy} is the second moment of area about the y-axis
I_{xy} is the product moment of area about the x and y-axes
The angle θ is measured anticlockwise positive from the x-axis

And:

$$I_{u,v} = \frac{1}{2}(I_{xx} + I_{yy}) \pm \frac{1}{2}(I_{xx} - I_{yy})sec2\theta \, (cannot be used when \theta = 45°)(3.10)$$

Or:

$$I_{u,v} = \frac{1}{2}(I_{xx} + I_{yy}) \pm \frac{1}{2}\sqrt{(I_{xx} - I_{yy})^2 + 4I_{xy}^2}(3.11)$$

Example 1 (continued)

Calculate the second moments of area about principal axes, $I_{u,v}$ for the shape.

We need to find θ. Recall:

$$tan2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})}$$

$$tan2\theta = \frac{2(1.188 \times 10^6)}{(1.4087 \times 10^6 - 2.8987 \times 10^6)} = -1.5946$$

$$2\theta = tan^{-1}(-1.5946) = -57.91°$$

$$\theta = -28.95°$$

Finally we need to calculate I_u and I_v. Recall:

$$I_{u,v} = \frac{1}{2}(I_{xx} + I_{yy}) \pm \frac{1}{2}\sqrt{(I_{xx} - I_{yy})^2 + 4I_{xy}^2}$$

$$I_{u,v} = \frac{1}{2}\{(2.8987 \times 10^6) + (1.4087 \times 10^6)\}$$

$$\pm \frac{1}{2}\sqrt{(2.8987 \times 10^6 - 1.4087 \times 10^6)^2 + 4(1.188 \times 10^6)^2}$$
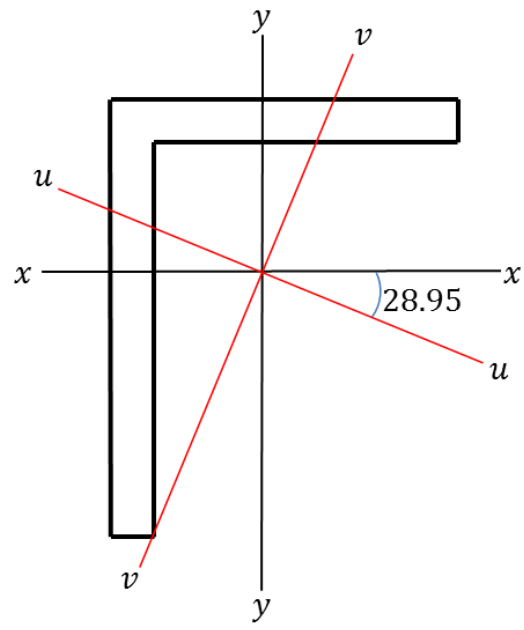
$$I_{u,v} = (2.1537 \times 10^6) \pm (1.4023 \times 10^6)$$

$$I_u = 3.556 \times 10^6 mm^4$$

$$I_v = 0.751 \times 10^6 mm^4$$

Note I_u and I_v are always positive.

28.95

Example 2:

A 50mm by 50mm square section steel cantilever beam is 1m long and supports an end load of 100N. Calculate the maximum bending stress and the maximum deflection in the beam. Assume a Young's modulus of 210GPa.

Solution:

$$\sigma = \frac{My}{I}; \ M_{max} = WL; \ \delta_{max} = \frac{WL^3}{3EI}$$

$$I = \frac{bd^3}{12} = \frac{50 \times 50^3}{12} = 520833.3$$

$$M_{max} = 100 \times 1000 = 1 \times 10^5 Nmm$$

$$\sigma_{max} = \frac{M_{max}y_{max}}{I} = \frac{(1 \times 10^5) \times 25}{520833.3} = 4.8MPa$$

$$\delta_{max} = \frac{100 \times 1000^3}{3 \times 210 \times 10^3 \times 520833.3} = 0.305mm$$

3.9 Unsymmetric (skew) Bending

Symmetric bending occurs in beams whose cross-sections have single or double lines of symmetry, or when the applied load is skew.

Our analysis so far has been limited to *symmetric* bending. This is when the x- and y-axes of the coordinate system have been assumed to coincide with the *principal directions* of the cross section. Every axis of symmetry of the section is a principal axis.
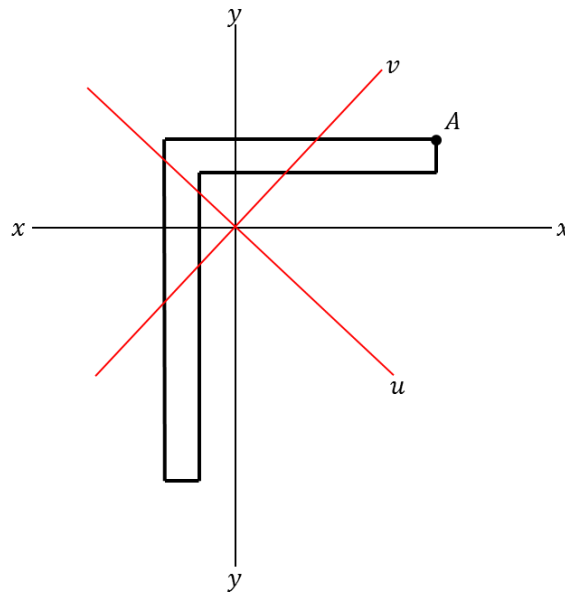
It can be shown that for unsymmetrical bending, at a point A as shown, the bending stress σ_A is given by:

$$\sigma_A = \frac{M_u}{I_u} v_A + \frac{M_v}{I_v} u_A \qquad (3.12)$$

Where M_u and M_v are the bending moments about the u- and v-axes, respectively.

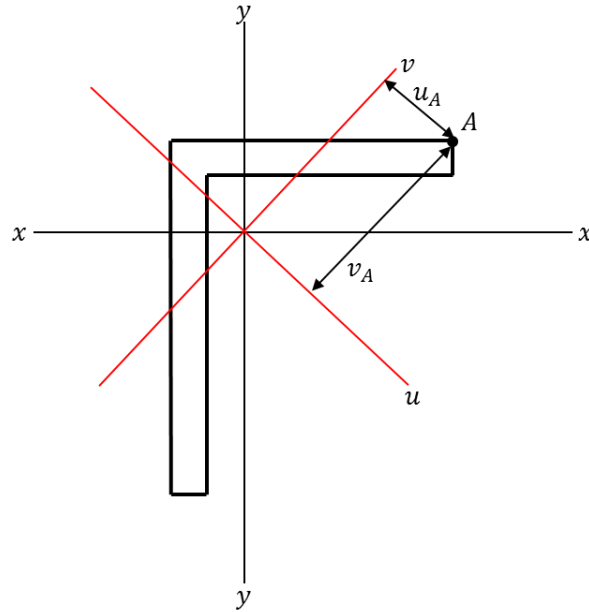$$M_u = M_x cos\theta; \quad M_v = M_x sin\theta \qquad (3.13)$$

Where M_x is the applied bending moment about the x-axis.

And where u_A and v_A are the coordinates of A about the u- and v-axes:

$$u_A = x_A cos\theta + y_A sin\theta; \quad v_A = y_A cos\theta - x_A sin\theta \qquad (3.14)$$

And x_A and y_A are the coordinates of A about the x- and y-axes.

We have previously defined that the neutral axis in the line along which the stresses due to bending are zero:
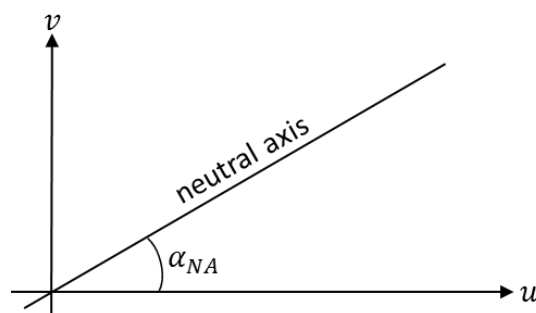
$$\sigma_A = \frac{M_u}{I_u} v_A + \frac{M_v}{I_v} u_A = 0$$

Therefore:

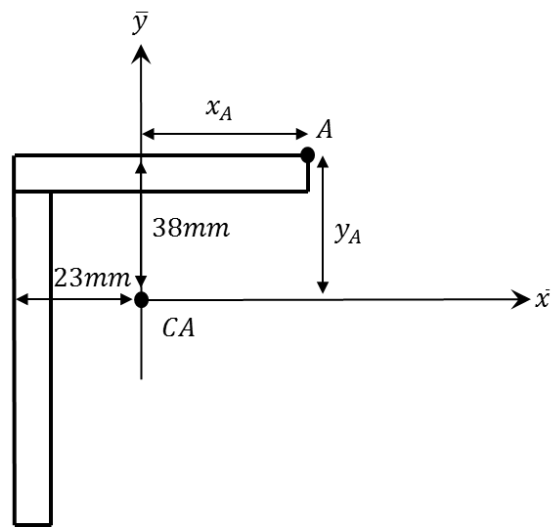$$\frac{M_u}{I_u} v_A = \frac{-M_v}{I_v} u_A$$

$$\frac{v_A}{u_A} = \frac{-M_v}{M_u} \frac{I_u}{I_v}$$

$$tan\alpha_{NA} = \frac{-M_v}{M_u} \frac{I_u}{I_v} \qquad\qquad (3.15)$$

Example 1 (continued)

Calculate the bending stress at Point A if $M_x = 1kNm$ and locate the neutral axis.

$$x_A = 90 - 23 = 67mm$$

$$y_A = 38mm$$

Recall:

$$u_A = x_A cos\theta + y_A sin\theta$$

$$v_A = y_A cos\theta - x_A sin\theta$$

$$u_A = 67cos(-28.954) + 38sin(-28.954)$$

$$u_A = 58.626 - 18.396 = 40.230mm$$

$$v_A = 38cos(-28.954) - 67sin(-28.954)$$

$$v_A = 33.250 + 32.435 = 65.685mm$$

Then:

$$M_u = M_x cos\theta = 1 \times cos(-28.954) = 0.875 \times 10^6 Nmm$$

$$M_v = M_x sin\theta = 1 \times sin(-28.954) = -0.484 \times 10^6 Nmm$$

Therefore:
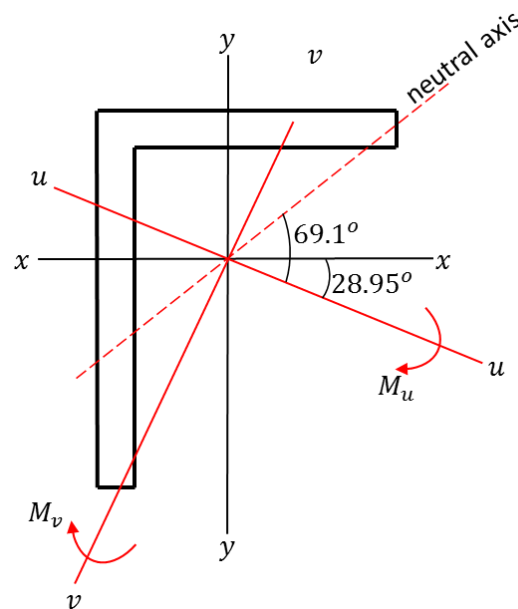
$$\sigma_A = \frac{M_u}{I_u} v_A + \frac{M_v}{I_v} u_A$$

$$\sigma_A = \left(\frac{0.875 \times 10^6}{3.556 \times 10^6} 65.685\right) + \left(\frac{-0.484 \times 10^6}{0.751 \times 10^6} 40.230\right)$$

$$\sigma_A = 16.163 - 25.927 = -9.764 MPa (i.e. compressive)$$

And to locate the neutral axis:

$$tan\alpha_{NA} = \frac{-M_v}{M_u} \frac{I_u}{I_v} = \left(\frac{0.484 \times 10^6}{0.875 \times 10^6}\right)\left(\frac{3.556 \times 10^6}{0.751 \times 10^6}\right) = 2.62$$

$$\alpha_{NA} = 69.1°$$

3.10 Shear Centre

We will not cover the *shear centre* in detail, but just to be aware of the meaning.

Depending on the location of the applied forces in the cross-section, the section will be subjected to a certain amount of torsion/twisting.

We define the **shear center** as that point in the cross-section through which the applied loads produce no twisting.

Where a cross-section has an axis of symmetry, the shear center must lie on this axis.

3.11 Torsion of Circular Sections

The twisting of a shaft about its longitudinal axis, due to an applied torque, is called *torsion*. When considering a circular shaft, the term pure torsion is used as the cross section of the shaft retains its shape. Here, we assume that circular sections remain circular and there is no change in diameter of the shaft. The relationship among various quantities is given by,

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \qquad (3.16)$$

where,

T = Applied torque (Nm or Nmm)

J = Polar moment of area of cross-section (m^4 or mm^4)

τ = shear stress at 'r' (N/m^2 or MPa)

r = radius (m or mm)

G = modulus of rigidity (N/m^2 or MPa) and is a function of the material given by $G = \frac{E}{2(1+v)}$

θ = Twist per unit length (radians/m or/mm)

L = Length of the shaft (m or mm)

The shear stress (τ) is a function of T, J and r which varies linearly with 'r' and does not depend on the material. There is a linear variation in shear stress with distance from centre:

$$\tau = \frac{Tr}{J} \qquad (3.17)$$
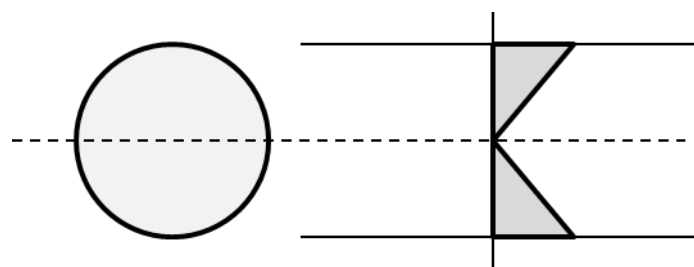
Example 3:

A 2m length of 20mm diameter steel bar is subjected to a torque of 5kNm. Calculate the maximum shear stress and the angle of twist. Assume a Young's modulus, Poisson's ratio and yield stress of 210 GPa, 0.3 and 300MPa respectively.

Solution:

For circular sections, we know that:

$$J = \frac{\pi d^4}{32}$$

$$J = \frac{\pi 20^4}{32} = 15707.96 mm^4$$

Recall:

$$\tau = \frac{Tr}{J}$$

$$\tau_{max} = \frac{Tr_{max}}{J} = \frac{5 \times 10^6 \times 10}{15707.96} = 3183 MPa$$

Recall:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

So:

$$\theta = \frac{TL}{GJ}$$

And:

$$G = \frac{E}{2(1+v)}$$

Therefore:

$$\theta = \frac{TL}{\frac{E}{2(1+v)}J} = \frac{2TL(1+v)}{EJ} = \frac{2 \times 5 \times 10^6 \times 1000(1+0.3) \times 2}{210 \times 10^3 \times 15707.96} = 7.88°$$