Advanced Structural Analysis

EGF316

7. Theories of Failure and Stress Concentration Effects

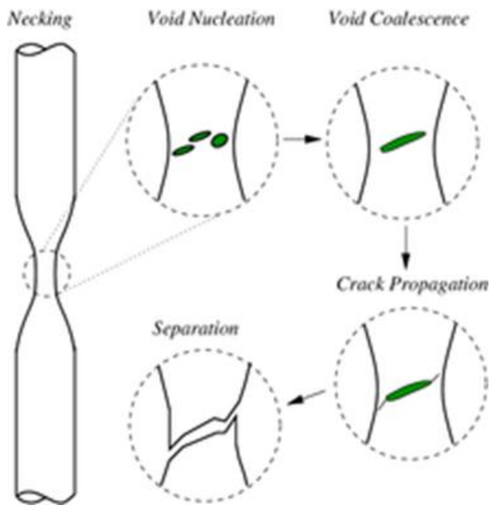
7.1 Failure Mechanisms

In this module, we will consider ways in which design can predict and aviod failure. It is therefore logical to consider the extensive number of different ways in which components can fail.

- Mechanical overload (or underdesign)
- Ductile fracture
- Brittle fracture
- Elastic yielding (due to applied force and/or temperature)
- Fatigue (high cycle, low cycles, thermal, corrosion, fretting….)
- Creep
- Corrosion (chemical, galvanic, cavitation, pitting…)
- Impact
- Instability (buckling)
- Wear (adhesive, abrasive, corrosive…)
- Vibration
- Environmental (thermal shock, radiation damage, lubrication failure)
- Contact (spalling, pitting, seizure)

7.2 Ductile Failure

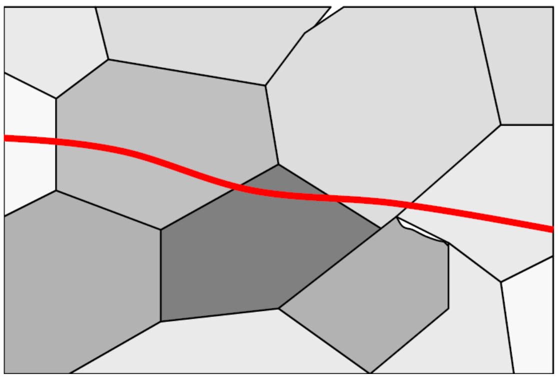
When elastically strained, a ductile material will plastically deform and necking will be observed. Small cavities will form in the internal cross sectional area and will grow and join together upon further application of stress. These cavities will continue to link together and the crack will propagate parallel to its major axis until eventual fracture by shear deformation at an angle of about 45^{o} to the applied stress. In many cases it is possible to detect ductile crack growth prior to, and thus preventing, catastrophic failure.



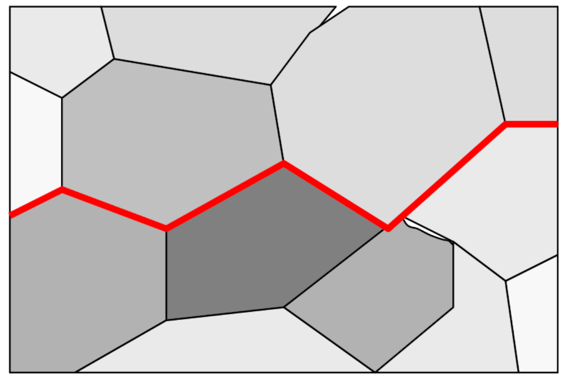
7.3 Brittle Fracture

Only very limited plastic flow occurs during brittle fracture. A crack, once initiated, can propagate very rapidly without warning. The crack will spread perpendicular to the applied stress and is classed as unstable. Usually, the crack develops by cleavage whereby atomic bonds are broken.

The crack can propagate through the grains (transgranular fracture) or along the grain boundaries (intergranular facture). Intergranular propagation is usually only seen in a few specific circumstances such as in the presence of creep, stress or corrosion – aggressive environments.



Transgranular Fracture



Intergranular Fracture

The main differences between ductile and brittle fracture are shown in the table below:

	Ductile	Brittle
Fracture Stress	Greater than yield strength	Lower than yield strength
Energy Absorption	High	Low
Nature of Fracture	Necking, rough fracture surface, linking up of cavities	No necking, shiny granular surface, cleavage or intergranular
Type of Material	Metals	Ceramics, glasses
Crack Propagation	Slow	Fast
Nature of Failure	Plastic deformation warning, less catastrophic	Little deformation, more catastrophic

It is important to note that the working temperature of a material will have an effect on its ability to absorb energy prior to fracture. For example, steel has a high toughness and behaves in a ductile manner at room temperature, but will exhibit brittle behaviour at lower temperatures. Some materials have what is termed as a ductile-brittle transition temperature. It is imperative to take into account the likely working temperature during the design stage.

7.4 Failure Theories

How do we test the strength of a material? Firstly it is useful to qualify to what type of strength we are referring:

- Tensile
- Compressive
- Shear

In addition to this, as many materials are ductile, we also need to consider the following strengths:

- Yield
- Ultimate tensile strength

In a simple tensile test, a dog bone specimen is often used with the weakest part in the middle. A force is applied and the material will begin to yield at a given section. We measure the force and extension and transform these to stress and strain (force to stress by dividing by cross-sectional area, extension to strain by dividing by original length). The resultant stress (y-axis) strain (x-axis) plot allows us to determine the UTS and yield stress.

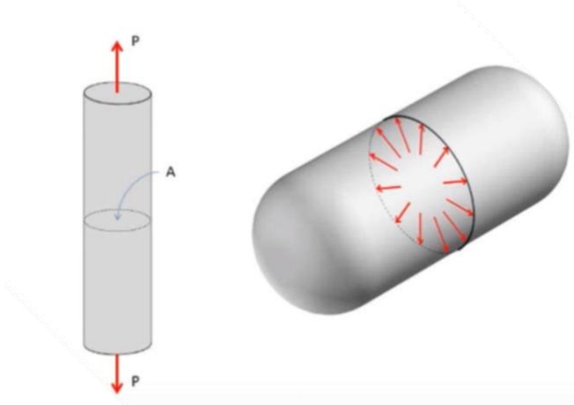
In the simple case of a member subjected to uniaxial stress, it is common to base failure prediction for ductile materials upon the yield stress.

In a one-dimensional stress state, there is one principal stress, σ_1 and yielding will start when:

$$\sigma_1 = \sigma_{yield}$$

We have seen that stress has multiple components. In the above, we are only measuring the stress in one direction. In practice materials aren't often loaded in such a simplistic manner, the strengths of materials under complex stress systems are not generally known but such stress systems are the most frequently encountered in reality. So we need a mechanism to allow us to predict the failure (yielding or rupture) for loading cases when there is more than one stress component.

Let us consider a simple example, the comparison between a simple tensile bar test and a pressure vessel with an internal pressure.



We need to develop ways to use a yield stress calculated based on one stress component to compare to a more complex real-life situation. This is where failure theories and failure criterion come into play. They allow us to determine allowable working stresses and thus avoid failure. Such theories aim to predict, from the behaviors of materials in a simple tensile test, when elastic failure will occur under *any* condition of applied stress.

Predicting the yielding of ductile materials or the fracture of brittle materials depends on a *Theory of Failure.*

A good failure theory requires:

- Observation of a large number of test results in different load conditions (empirical knowledge)
- A theory explaining the microscopic mechanisms involved which must be consistent with the observations

In this way, we can have a high level of confidence that not only does a given theory of failure work for a loading condition that we have observed, but that it will also apply to other conditions that have not been exhaustively tested.

Various theoretical criteria have been proposed to obtain adequate correlation between the estimated component life and the actual life achieved in service. We will consider two of the most commonly used theories for *ductile* materials.

Our chosen failure theory needs to be appropriate to the type of failure that will occur – for example we couldn't apply the same failure theory to glass and aluminum. Ductile failure characterized by yielding tends to occur by shearing of material so we need a failure theory which is based around shear stress.

Numerous theories of failure exist in the literature developed over the last 200 years. Some of the popular ones are:

a) Rankine or Maximum Principal stress theory
b) Saint venant or Maximum Principal strain theory
c) Tresca or Maximum shear stress theory
d) Von Mises & Hencky or Shear strain energy theory
e) Haigh or Total strain energy per unit volume theory
f) Mohr-Coulomb failure theory – cohesive-frictional solids
g) Drucker-Prager failure theory – pressure dependent solids
h) Cam-Clay failure theory - soils

In this course, we discuss (a), (b), (c) and (d).

7.4.1) Rankine or Maximum Principal stress theory

According to this theory, the failure occurs when the maximum principal stress is greater than the yield strength of the material.

$$\max\,(\sigma_{1,}\sigma_{2,}\sigma_3) \leq \sigma_{yield}$$

7.4.2 Saint venant or Maximum Principal strain theory

According to this theory, failure occurs when the maximum principal strain exceeds the strain at the yield point.

$$\max\,(\varepsilon_{1,}\varepsilon_{2,}\varepsilon_3) \leq \varepsilon_{yield}$$

7.4.2 Tresca or Maximum Shear Stress Theory

This theory considers that failure (yielding) will occur when the maximum shear stress in the complex stress state becomes equal to the material's limiting shear strength in a simple tensile test.

We are essentially extrapolating the results from a simple tensile test to give us information about failure under a more complex stress situation.

Since the maximum shear stress is equal to half the greatest difference between two principal stresses:

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

And since the maximum shear in simple tension is equal to half the tensile stress at yield:

$$\tau_{max} = \tau_{yield} = \frac{1}{2}\sigma_{yield}$$

Equating these gives:

$$\frac{1}{2}\sigma_{yield} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$\sigma_y = \sigma_1 - \sigma_3$$

Where the value of σ_3 is algebraically the smallest value (taking into account the sign and the fact that one stress may be zero).

E.g. if $\sigma_1 = 50Mpa$ and $\sigma_3 = -20MPa$ then $\sigma_1 - \sigma_3 = 70MPa$.

Remember that in 2D analysis where the two principal actresses are 50MPa and 15MPa, the other principal stress is zero thus:

$$\sigma_1 - \sigma_3 = 50 - 0 = 50MPa \quad \text{NOT} \quad 50 - 15 = 35MPa$$

This criterion has been shown to give reasonably accurate prediction of failure, especially for ductile materials.

7.4.4 Von Mises Criterion (1913) - Maximum Shear Strain Energy Theory

The strain energy of a stressed component can be divided into two components, volumetric strain energy (associated with volume change but no distortion) and shear strain energy (associated distortion of the stressed elements). This theory states that failure will occur when the maximum shear energy component in the complex stress system is equal to that at the yield point in a simple tensile test.

Defining a von Mises equivalent stress σ_{eq} where:

For Principal Stresses:

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}$$

For components of stress:

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2 + 6({\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{xz}}^2)}$$

It can be shown that yielding occurs when:

$$\sigma_{eq} \leq \sigma_{yield}$$

When $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = \tau_{xz} = 0$ i.e. there is one direct stress, σ_{xx} and one shear stress, τ_{xy} only (such as in bending and torsion)

$$\sigma_{eq} = \sqrt{(\sigma_{xx})^2 + 3({\tau_{xy}}^2)}$$

This theory is widely regarded as the most reliable basis for design and has received significant practical verification, particularly for ductile materials.

Example 1:

If $\sigma_1 = 200MPa$ and $\sigma_2 = 100MPa$, calculate the limiting value of σ_3 to avoid yielding in accordance with Tresca and Von Mises. The yield stress $\sigma_y = 300MPa$.

Tresca:

$$\sigma_1 - \sigma_3 = \sigma_y$$

$$\sigma_3 = \sigma_1 - \sigma_y = 200 - 300$$

$$\sigma_3 = -100MPa$$

Von Mises:

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}$$

$$300 = \frac{1}{\sqrt{2}}\sqrt{(200 - 100)^2 + (100 - \sigma_3)^2 + (200 - \sigma_3)^2}$$

$$300^2 = \frac{1}{2}\{(200 - 100)^2 + (100 - \sigma_3)^2 + (200 - \sigma_3)^2\}$$

$$180{,}000 = 10{,}000 + 10{,}000 - 200\sigma_3 + {\sigma_3}^2 + 40{,}000 - 400\sigma_3 + {\sigma_3}^2$$

$$2{\sigma_3}^2 - 600\sigma_3 - 120{,}000 = 0$$

$$\sigma_3{}^2 - 300\sigma_3 - 60{,}000 = 0$$

$$\sigma_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-300) \pm \sqrt{(-300)^2 - 4(1)(-60{,}000)}}{2(1)}$$

$$\sigma_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{300 \pm \sqrt{90{,}000 + 240{,}000}}{2} = \frac{300 \pm \sqrt{330{,}000}}{2}$$

$$\sigma_3 == \frac{300 \pm 574.5}{2} = \cancel{437.2MPa} \quad \text{or} \quad -137.2MPa$$

7.5 Safety Factor (SF) or Factor of Safety (FoS):

In order to avoid failure of components, it is common practice in mechanical engineering design to apply a safety factor (SF). SF is always greater than 1. Thus, a component is actually designed to withstand a higher load than it is intended to. However, choosing the proper value of safety factor is a very difficult task, as a higher value requires additional material, thus increasing the cost.

By introducing a safety factor into the above criteria, we get,

Tresca:

$$\sigma_1 - \sigma_3 \leq \frac{\sigma_y}{SF}$$

Von Mises:

$$\sigma_{eq} \leq \frac{\sigma_{yield}}{SF}$$

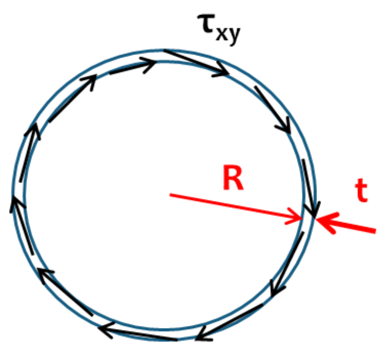
The value used for the SF depends on the confidence in the accuracy of the stress values.

Example 2:

A thin mild steel tube has a mean diameter of 100mm and a thickness of 3mm. Determine the maximum torque that the tube can transmit according to the Tresca and von Mises criteria and using a Safety Factor of 2.25.

Assume a constant shear stress (thin tube) and a yield stress of 230 MPa.

Approximate cross-sectional area = $2\pi Rt$
Calculate principal stresses first



$$T = Force \times distance = shear\ stress \times area \times distance\ (r)$$

$$T = shear\ stress \times area \times radius = \tau \times 2\pi Rt \times R$$

$$\tau = \frac{T}{2\pi R^2 t}$$

Using:

$$\sigma_1 = \frac{(\sigma_{xx} + \sigma_{yy})}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_3 = \frac{(\sigma_{xx} + \sigma_{yy})}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}{}^2}$$

We have $\sigma_{xx} = \sigma_{yy} = 0$, therefore:

$$\sigma_1 = \tau \quad and \quad \sigma_3 = -\tau$$

Tresca:

$$\sigma_1 - \sigma_3 \leq \frac{\sigma_y}{SF}$$

$$\tau - (-\tau) = 2\tau = \frac{\sigma_y}{2.25}$$

$$2\frac{T}{2\pi R^2 t} = \frac{\sigma_y}{2.25}$$

$$T = \frac{\pi R^2 t \sigma_y}{2.25} = \frac{\pi(50)^2 3 \times 230}{2.25} = 2408554Nmm$$

Von Mises:

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}$$

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{(\tau - 0)^2 + (0 - (-\tau))^2 + (\tau - (-\tau))^2} = \frac{1}{\sqrt{2}}\sqrt{\tau^2 + \tau^2 + 4\tau^2}$$

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\sqrt{6\tau^2} = \tau\sqrt{3}$$

$$\sigma_{eq} \leq \frac{\sigma_{yield}}{SF} = \frac{230}{2.25} = 102.2$$

We have:

$$\tau = \frac{T}{2\pi R^2 t} = \frac{\sigma_{eq}}{\sqrt{3}} = \frac{102.2}{\sqrt{3}}$$

$$T = \frac{(102.2)2\pi R^2 t}{\sqrt{3}} = \frac{(102.2)2\pi 50^2 3}{\sqrt{3}} = 2781159Nmm$$

7.6 Stress Concentrations

Mechanically and/or thermally loaded components have stress distributions set up in them which maintain equilibrium with externally applied loads.

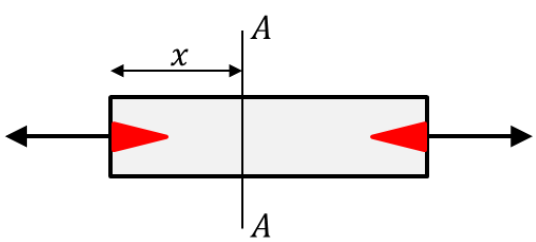
A uniform cross-section bar subjected to an axial tensile or compressive load is assumed to experience uniform stress across the section. However, in practice the presence of any sudden change of section can cause the local stress to rise significantly and rapidly over a short distance.

The stress gradient ($d\sigma/dx$) in one area of a loaded component can be orders of magnitude greater than in other areas of the same component. For example, in the vicinity of the point of application of a concentrated load where the maximum stress can be much higher than the average or nominal value.

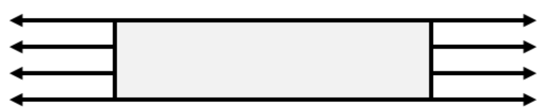
This situation is called a *stress concentration*. Stress concentrations are also produced at geometric discontinuities in a component such as holes, notches, keyways, fillets and material flaws. The effect of a stress concentration is local, but it can result in failure of material in both static and dynamic loading conditions.

Also St. Venant's Principle states that the actual distribution of the load over the surface of its application will not affect the distribution of stress or strain on sections of the body which are at an appreciable distance (relative to the dimensions) away from the load.

e.g. for a rod in simple tension



(a) concentrated load (W)

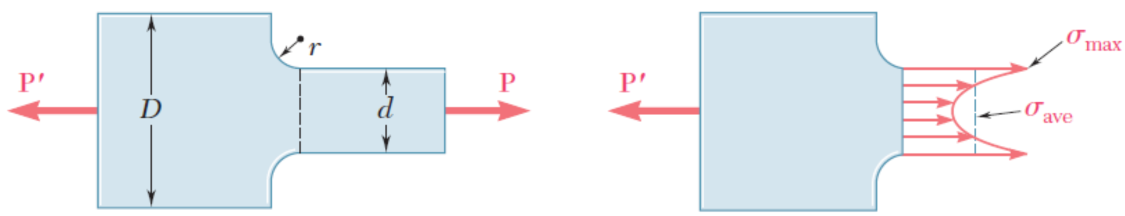

(b) distributed load (or pressure)

The two loading conditions are statically equivalent and at the Section AA the stress is reasonably uniform and equal to $load/area$ so long as 'x' is greater than 3 times the diameter.

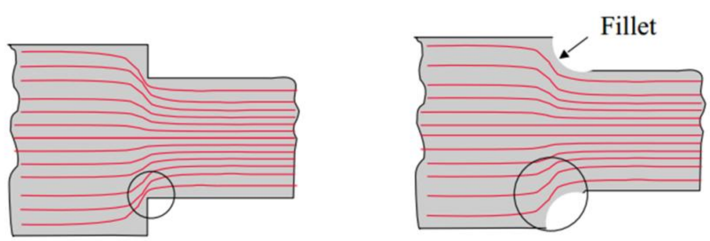
7.6.1 Geometric Discontinuities

Abrupt changes in geometry due to a hole, fillet, keyway etc., give rise to stress concentrations, as already stated. These high stresses are often the root cause of failure.

The presence of a stress concentration can be explained clearly using a stepped bar as an example. At the step there is a discontinuity in stress, due to the change in section.



There is a significant increase in the stretching of the flow lines in the region of the discontinuity; hence the strain and stress are higher in this region than along the rest of the lines.



The larger the fillet radius at the step, the smoother the transition and hence the less noticeable is the stress concentration. Therefore, the smaller the fillet radius the greater is the stress concentration.

Note: Stress concentrations play a significant part in the failure of components due to fatigue (and to an extent brittle fracture). The fatigue life of a component, in terms of the number of cycles to failure, is a function of the stress range being experienced by the component. Fatigue calculations which ignore the presence of a stress concentration will therefore over-predict the life of that component since the stress range in the region of the stress concentration will be significantly greater than the nominal value.

Stresses in concentrated locations can be computed using

1) Analytical methods

   - Often limited to simple geometries and loading conditions

2) Photoelastic methods - laboratory experiments

   - Widely used method before the invention of numerical methods
   - Limited to small components

3) Empirical data – Stress concentration factors

   - Precomputed tables and graphs.
   - "Peterson's Stress Concentration Factors" by Pilkey & Pilkey contains many results for a large number of component configurations and loading conditions.
   - Concentrated stresses can be computed using the empirical formulae.

4) Numerical methods

   - usually Finite Element Analysis.

7.6.2. Stress concentration factor (SCF)

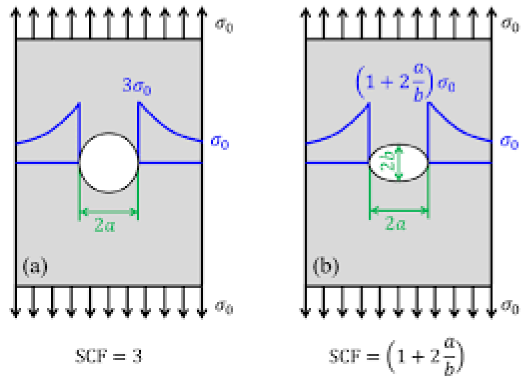
The stress concentration due to a geometric discontinuity is a function of the shape and dimensions of the discontinuity and is expressed in terms of an *elastic stress concentration factor*, K_t. These values of stress concentration factors help in calculating the values of concentrated shapes for the simple geometries and loading conditions.

SCF is defined as the ratio of the maximum stress occurring near the discontinuity to the nominal stress at the section in the absence of a stress concentration.

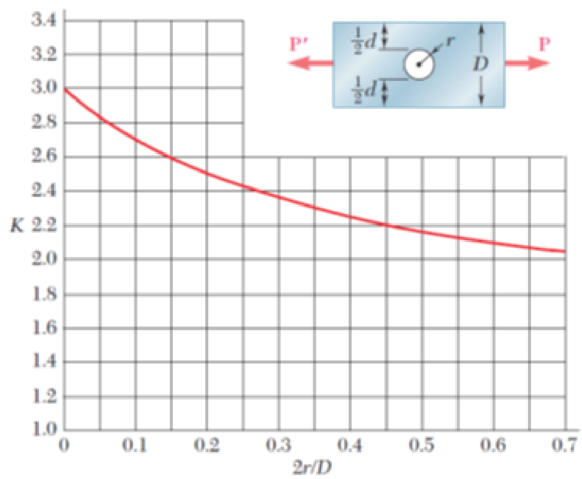
$$K_t = \frac{\sigma_{max}}{\sigma_{nom}}$$

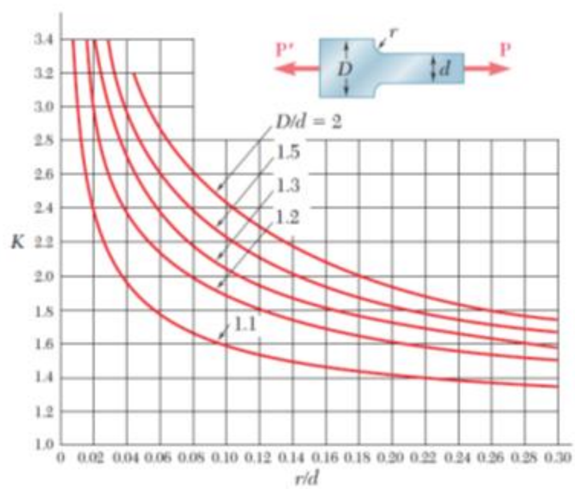
This factor is constant within the elastic range of the material; and changes when the material becomes plastic.

A typical example of a stress concentration is the classical hole-in-a-plate problem. A hole machined in a plate produces both an increase in the mean stress and also a peak in stress at the surface of the hole. This peak stress can be up to three times the mean value.



Some typical examples are shown below for a flat bar with holes, and a flat bar with fillets:





7.6.3 Static Load Design

This simplifies the work of a designer. All they need to do in order to determine the maximum stress occurring near a discontinuity in a given member subjected to an axial load, *P* is calculate the average or nominal stress $\sigma_{nom} = P/A$ in the critical section. This result is then multiplied by the appropriate stress concentration factor, K_t.

$$\sigma_{max} = K_t \sigma_{nom}$$

One of two possible design models can then be applied.

Elastic only design model when all the stresses must be elastic and the nominal stress must be such that the maximum stress does not exceed yield. A factor of safety is often employed.

Limited plasticity where local yielding is allowed in the region of the stress concentration and hence the nominal stress can be significantly closer to yield within some factor of safety.

7.6.3 Fatigue Design

The effect of a notch (due to a hole, fillet etc.) on the fatigue strength of a component depends on the material and the notch geometry and is generally less than the effect that would be predicted by the straight application of an elastic stress concentration factor. The *notch sensitivity* relates fatigue notch factor (used to determine fatigue life) with the elastic stress concentration factor:

$$K_f = q(K_t - 1) + 1$$

Where:

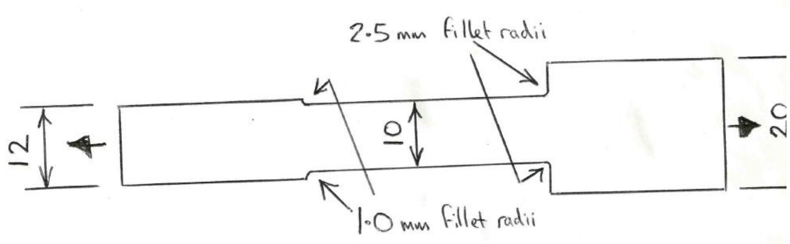
q is the notch sensitivity
K_f is the fatigue notch factor
K_t is the elastic stress concentration factor

Notch sensitivity data is available (see Pilkey for references to q data).

K_f is then used to determine the elastic stress range used for fatigue life calculations.

<u>Example 3:</u>

Determine the stresses in the fillets of the component shown below due to an axial load of 100N. The thickness of the part is 1mm.



$$\sigma_{nom} = \frac{100}{10 \times 1} = 10MPa$$

Considering the 1mm fillets:

$$\frac{D}{d} = \frac{12}{10} = 1.2$$

$$\frac{r}{d} = \frac{1}{10} = 0.1$$

$$K_t = 1.9$$

Therefore:

$$\sigma_{max} = K_t \sigma_{nom} = 1.9 \times 10 = 19MPa$$

Considering the 2.5mm fillets:

$$\frac{D}{d} = \frac{20}{10} = 2.0$$

$$\frac{r}{d} = \frac{2.5}{10} = 0.25$$

$$K_t = 1.83$$

Therefore:

$$\sigma_{max} = K_t \sigma_{nom} = 1.83 \times 10 = 18.3MPa$$