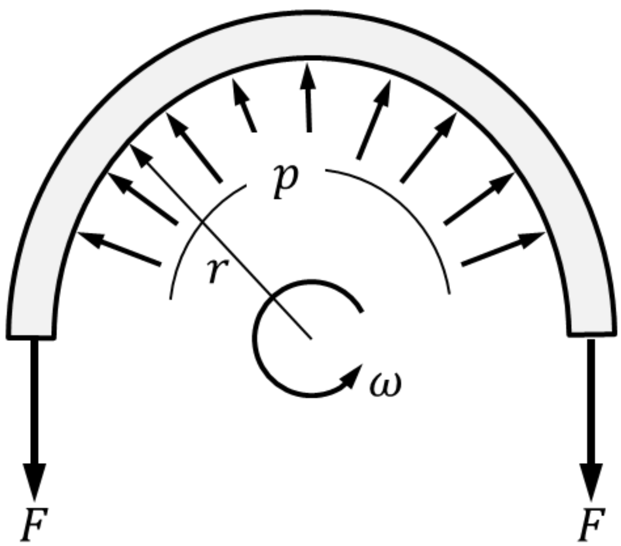
Advanced Structural Analysis
EGF316
6. Rotating Rings, Discs and Cylinders

6.1 Thin Rotating Ring or Cylinder

Let us consider a thin ring or cylinder as shown below. The cylinder is rotating with an angular velocity ω and is subjected to a radial pressure p (centrifugal force) caused by the centrifugal effect of its own rotating mass.

The centrifugal effect on a unit length of the circumference is:

$$p = m\omega^2 r$$



Considering equilibrium of half the ring shown above and assuming unit length:

$$2F = p \times 2r$$

$$F = pr = (m\omega^2 r)r = m\omega^2 r^2$$

Where F is the hoop tension force set up due to the rotation.

We will assume that the cylinder wall is thin so that the centrifugal effect is constant through the thickness of the wall. The hoop tension is transmitted through the entire circumference and is thus resisted by the complete cross-sectional area, giving:

$$hoop\ stress, \sigma_\theta = \frac{F}{A} = \frac{m\omega^2 r^2}{A}$$

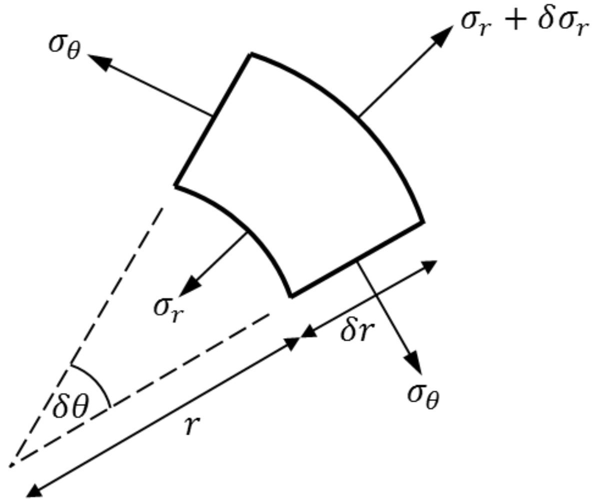
Where A is the cross-sectional area of the ring.

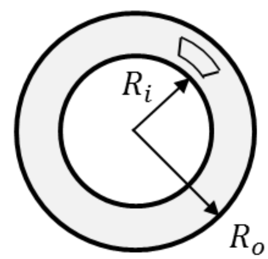
As we are considering a unit length, the ratio $\frac{m}{A}$ is the mass of the material per unit volume which is the density, ρ.

$$\sigma_\theta = \rho\omega^2 r^2 \qquad (6.1)$$

6.2 Rotating Disc

Consider the element of disc (of unit thickness) shown below:





The three principal stresses are:

$$radial, \sigma_r$$

$$hoop, \sigma_\theta \quad (or\ circumferetial\ or\ tangential)$$

$$longitudinal, \sigma_L \quad (or\ axial, normal\ to\ plane\ of\ paper)$$

At a radius r, and assuming unit thickness, the volume of the element is given by:

$$volume\ of\ element = r\delta\theta \times \delta r \times 1 = r\delta\theta\delta r$$

And the mass of the element is given by:

$$mass\ of\ element = density \times volume = \rho r\delta\theta\delta r$$

Therefore, the centrifugal force acting on the element is given by:

$$centrifugal\ force = m\omega^2 r = \ \rho r\delta\theta\delta r \times \omega^2 r = \rho r^2\omega^r\delta\theta\delta r$$

Considering radial equilibrium of the element, it can be shown (see text books) that:

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+v)}{8}\rho\omega^2 r^2 \qquad (6.2)$$

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{(1+3v)}{8}\rho\omega^2 r^2 \qquad (6.3)$$

Note: the longitudinal stress is neglected when considering discs which are by definition relatively thin.

6.3 Solid Circular Disc of Uniform Thickness

For a solid disc, the stress at the centre is given when $r = 0$. With r equal to zero, equations (6.2) and (6.3) would yield infinite stresses regardless of the speed of rotation unless B is also zero. Thus the only finite solution is given by:

$$B = 0 \quad \text{and hence} \quad \frac{B}{r^2} = 0$$

At the outside radius, R_o, the radial stresses must be zero since there are no external forces to provide the necessary balance of equilibrium if σ_r were not zero.

Therefore, from (6.2):

$$\sigma_r = A - \frac{(3+v)}{8}\rho\omega^2 {R_o}^2 = 0$$

Therefore:

$$A = \frac{(3+v)}{8}\rho\omega^2 {R_o}^2$$

Substituting this back into (6.2) and (6.3), the radial and hoop stresses at any radius, r, in a solid disc are given by:

$$\sigma_r = \frac{(3+v)}{8}\rho\omega^2 {R_o}^2 - \frac{(3+v)}{8}\rho\omega^2 r^2$$

$$\sigma_r = \frac{(3+v)}{8}\rho\omega^2 (R_o^2 - r^2) \qquad (6.4)$$

$$\sigma_\theta = \frac{(3+v)}{8}\rho\omega^2 {R_o}^2 - \frac{(1+3v)}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = \frac{\rho\omega^2}{8}[(3+v)R_o^2 - (1+3v)r^2] \qquad (6.5)$$

6.4 Circular Disc of Uniform Thickness with a Central Hole

The general equations for the stresses in a rotating hollow disc may be obtained in the same way as above, but by applying different boundary conditions to evaluate the constants A and B.

We have:

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+v)}{8}\rho\omega^2 r^2 \quad \text{and} \quad \sigma_\theta = A + \frac{B}{r^2} - \frac{(1+3v)}{8}\rho\omega^2 r^2$$

The radial stress is zero at both the inner and outer radius:

$$\sigma_r = 0 \quad \text{when} \quad r = R_i \quad \text{and when} \quad r = R_o$$

At $r = R_i$:

$$\sigma_r = A - \frac{B}{{R_i}^2} - \frac{(3+v)}{8}\rho\omega^2 {R_i}^2 = 0$$

At $r = R_o$:

$$\sigma_r = A - \frac{B}{{R_o}^2} - \frac{(3+v)}{8}\rho\omega^2 {R_o}^2 = 0$$

Equating and simplifying:

$$A - \frac{B}{{R_i}^2} - \frac{(3+v)}{8}\rho\omega^2 {R_i}^2 = A - \frac{B}{{R_o}^2} - \frac{(3+v)}{8}\rho\omega^2 {R_o}^2$$

$$\frac{B}{{R_i}^2} + \frac{(3+v)}{8}\rho\omega^2 {R_i}^2 = \frac{B}{{R_o}^2} + \frac{(3+v)}{8}\rho\omega^2 {R_o}^2$$

$$B\left(\frac{1}{{R_i}^2} - \frac{1}{{R_o}^2}\right) = \frac{(3+v)}{8}\rho\omega^2\left({R_o}^2 - {R_i}^2\right)$$

$$B\left(\frac{{R_o}^2 - {R_i}^2}{{R_i}^2 {R_o}^2}\right) = \frac{(3+v)}{8}\rho\omega^2\left({R_o}^2 - {R_i}^2\right)$$

$$B = \frac{(3+v)}{8}\rho\omega^2\left({R_o}^2 - {R_i}^2\right)\left(\frac{{R_i}^2 {R_o}^2}{{R_o}^2 - {R_i}^2}\right)$$

$$B = (3+v)\frac{\rho\omega^2 {R_i}^2 {R_o}^2}{8}$$

Therefore:

$$A = \frac{B}{{R_i}^2} + \frac{(3+v)}{8}\rho\omega^2 {R_i}^2 = (3+v)\frac{\rho\omega^2 {R_i}^2 {R_o}^2}{8}\frac{1}{{R_i}^2} + \frac{(3+v)}{8}\rho\omega^2 {R_i}^2$$

$$A = (3+v)\frac{\rho\omega^2\left({R_i}^2 + {R_o}^2\right)}{8}$$

Substituting this into (6.2) and (6.3) gives:

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+v)}{8}\rho\omega^2 r^2$$

$$\sigma_r = (3+v)\frac{\rho\omega^2\left({R_i}^2 + {R_o}^2\right)}{8} - (3+v)\frac{\rho\omega^2 {R_i}^2 {R_o}^2}{8}\frac{1}{r^2} - \frac{(3+v)}{8}\rho\omega^2 r^2$$

$$\sigma_r = (3+v)\frac{\rho\omega^2}{8}\left[{R_i}^2 + {R_o}^2 - \frac{{R_i}^2 {R_o}^2}{r^2} - r^2\right] \qquad (6.6)$$

And:

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{(1+3v)}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = (3+v)\frac{\rho\omega^2\left({R_i}^2 + {R_o}^2\right)}{8} + (3+v)\frac{\rho\omega^2 {R_i}^2 {R_o}^2}{8}\frac{1}{r^2} - \frac{(1+3v)}{8}\rho\omega^2 r^2$$

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+v)\left[{R_i}^2 + {R_o}^2 + \frac{{R_i}^2 {R_o}^2}{r^2}\right] - (1+3v)r^2\right] \qquad (6.7)$$

Example 1:

A steel ring of outer diameter 300mm and internal diameter 200mm is shrunk onto a solid steel shaft. The interface is such that the radial pressure between the mating surfaces remains above 30MN/m^2 at all times whilst the assembly rotates in practice. The circumferential stress on the inside surface of the ring must not exceed 240MN/m^2. Determine the maximum speed at which the assembly can rotate. You may assume that $\rho = 7500\text{kg/m}^3, v = 0.3, E = 210\text{GPa}$.

We know that:

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+v)}{8}\rho\omega^2 r^2$$

When $r = 0.15\text{m}, \sigma_r = 0$:

$$0 = A - \frac{B}{0.15^2} - \frac{(3+0.3)}{8}7500\omega^2 0.15^2 \qquad (1)$$

When $r = 0.1\text{m}, \sigma_r = -30$:

$$-30 \times 10^6 = A - \frac{B}{0.1^2} - \frac{(3+0.3)}{8}7500\omega^2 0.1^2 \qquad (2)$$

(1)-(2):

$$30 \times 10^6 = -\frac{B}{0.15^2} - \frac{(3+0.3)}{8}7500\omega^2 0.15^2 - \left(-\frac{B}{0.1^2} - \frac{(3+0.3)}{8}7500\omega^2 0.1^2\right)$$

$$30 \times 10^6 = -44.4B - 69.61\omega^2 + 100B + 30.94\omega^2$$

$$30 \times 10^6 = 55.56B - 38.67\omega^2$$

$$55.56B = 30 \times 10^6 + 38.67\omega^2$$

$$B = 0.54 \times 10^6 + 0.696\omega^2$$

Therefore:

$$A = \frac{B}{0.15^2} + \frac{(3 + 0.3)}{8} 7500\omega^2 0.15^2$$

$$A = \frac{0.54 \times 10^6 + 0.696\omega^2}{0.15^2} + \frac{(3 + 0.3)}{8} 7500\omega^2 0.15^2$$

$$A = 24 \times 10^6 + 30.676\omega^2 + 69.61\omega^2 = 24 \times 10^6 + 100.3\omega^2$$

From the question, we know that the circumferential (or hoop) stress on the inside surface of the ring must not exceed 240MN/m^2. We have:

$$\sigma_\theta = A + \frac{B}{r^2} - \frac{(1 + 3v)}{8} \rho\omega^2 r^2$$

Thus:

$$240 \times 10^6 = 24 \times 10^6 + 100.3\omega^2 + \frac{0.54 \times 10^6 + 0.696\omega^2}{0.1^2} - \frac{(1.9)}{8} 7500\omega^2 0.1^2$$

$$(240 \times 10^6) - (24 \times 10^6) - (54 \times 10^6) = \omega^2(100.3 + 69.6 - 17.8)$$

$$152.1\omega^2 = 162 \times 10^6$$

$$\omega = 1032\text{rad/s} = \frac{1032 \times 60}{2\pi} = 9855\text{rev/min}$$

6.5 Maximum Stresses

We have:

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+v)\left[{R_i}^2 + {R_o}^2 + \frac{{R_i}^2{R_o}^2}{r^2}\right] - (1+3v)r^2\right]$$

The maximum hoop stress, $\hat{\sigma}_\theta$, occurs at the inside radius when:

$$r = R_i$$

$$\hat{\sigma}_\theta = \frac{\rho\omega^2}{8}\left[(3+v)\left[{R_i}^2 + {R_o}^2 + \frac{{R_i}^2{R_o}^2}{{R_i}^2}\right] - (1+3v){R_i}^2\right]$$

$$\hat{\sigma}_\theta = \frac{\rho\omega^2}{8}\left[(3+v)\left[{R_i}^2 + {R_o}^2 + {R_o}^2\right] - (1+3v){R_i}^2\right]$$

$$\hat{\sigma}_\theta = \frac{\rho\omega^2}{8}\left[3{R_i}^2 + 3{R_o}^2 + 3{R_o}^2 + v{R_i}^2 + v{R_o}^2 + v{R_o}^2 - {R_i}^2 - 3v{R_i}^2\right]$$

$$\hat{\sigma}_\theta = \frac{\rho\omega^2}{8}\left[2{R_i}^2 + 6{R_o}^2 + 2v{R_o}^2 - 2v{R_i}^2\right] = \frac{\rho\omega^2}{4}\left[{R_i}^2 + 3{R_o}^2 + v{R_o}^2 - v{R_i}^2\right]$$

$$\hat{\sigma}_\theta = \frac{\rho\omega^2}{4}\left[(3+v){R_o}^2 + (1-v){R_i}^2\right] \qquad (6.8)$$

As the value of the inside radius approaches zero, the maximum hoop stress value approaches:

$$\hat{\sigma}_\theta = \frac{\rho\omega^2}{4}(3+v){R_o}^2 \qquad (6.9)$$

This is twice the value we obtain at the centre of solid disc rotating at the same speed. This implies that by drilling even a very small hole at the centre of a solid disc, the maximum hoop stress due to the rotation is doubled!

The maximum radial stress is found by considering equation (6.6):

$$\sigma_r = (3+v)\frac{\rho\omega^2}{8}\left[{R_i}^2 + {R_o}^2 - \frac{{R_i}^2{R_o}^2}{r^2} - r^2\right]$$

This will take a maximum value when:

$$\frac{d\sigma_r}{dr} = 0$$

So:

$$\frac{d}{dr}\left[{R_i}^2 + {R_o}^2 - \frac{{R_i}^2{R_o}^2}{r^2} - r^2\right] = 0$$

$$\frac{2{R_i}^2{R_o}^2}{r^3} - 2r = 0$$

$$r^4 = {R_i}^2{R_o}^2$$

$$r = \sqrt{R_i R_o}$$

Using this in (7.6):

$$\hat{\sigma}_r = (3+v)\frac{\rho\omega^2}{8}\left[{R_i}^2 + {R_o}^2 - \frac{{R_i}^2{R_o}^2}{\left(\sqrt{R_i R_o}\right)^2} - \left(\sqrt{R_i R_o}\right)^2\right]$$

$$\hat{\sigma}_r = (3+v)\frac{\rho\omega^2}{8}\left[{R_i}^2 + {R_o}^2 - \frac{{R_i}^2{R_o}^2}{R_i R_o} - R_i R_o\right]$$

$$\hat{\sigma}_r = (3+v)\frac{\rho\omega^2}{8}\left[{R_i}^2 + {R_o}^2 - 2R_i R_o\right]$$

$$\hat{\sigma}_r = (3+v)\frac{\rho\omega^2}{8}(R_o - R_i)^2 \qquad (6.10)$$