**Advanced Structural Analysis**
EGF316
5. Compound Cylinders

5.1 Compound Cylinders

When a cylinder is subject to internal pressure, there is a significant variation in the hoop stress across the wall thickness. This implies that the material of the cylinder is not being used to maximum advantage. It is possible to obtain a more uniform hoop stress distribution by the process of shrinking one cylinder onto the outside of another.

The inner diameter of the outer cylinder is made slightly smaller than the outer diameter of the inner cylinder. Assembly is generally achieved by heating the outer cylinder until it can freely slide over the inner cylinder (it is possible to cool the inner cylinder). On cooling, the outer cylinder will contract and the inner cylinder is thus brought into a state of compression. The outer cylinder is conversely brought into a state of tension.

If this compound cylinder is subjected to an internal pressure, the hoop stress will be the algebraic sum of those due to the internal pressure and those due to the shrinkage. This yields a much smaller variation in hoop stress.


Internal pressure only   shrinkage only   combined shrinkage and internal pressure
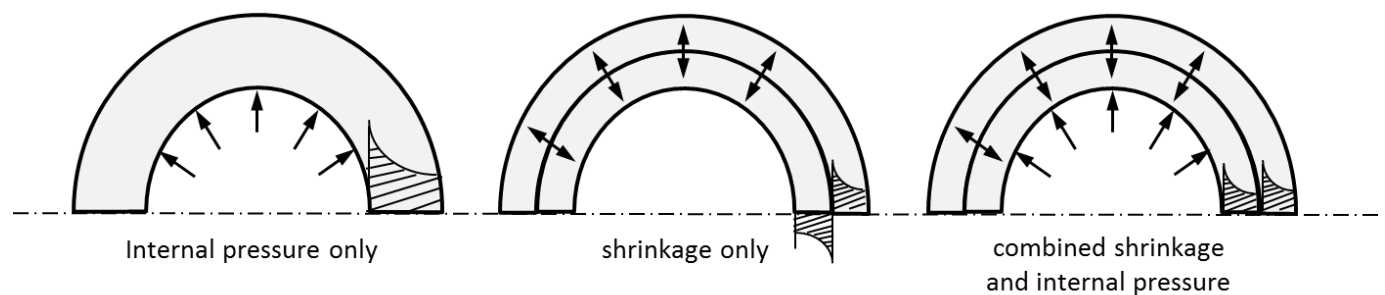
The implication of this is that a compound cylinder can withstand greater internal pressures.

5.1 Cylinders of the Same Material

Assuming that both cylinders are manufactured from the same material, the method of solution can be broken down into three separate effects:

Condition 1:   Shrinkage pressure on the inner cylinder only
Condition 2:   Shrinkage pressure on the outer cylinder only
Condition 3:   Internal pressure only on the compound cylinder

| shrinkage - internal cylinder | shrinkage - external cylinder | Internal pressure – compound cylinder |
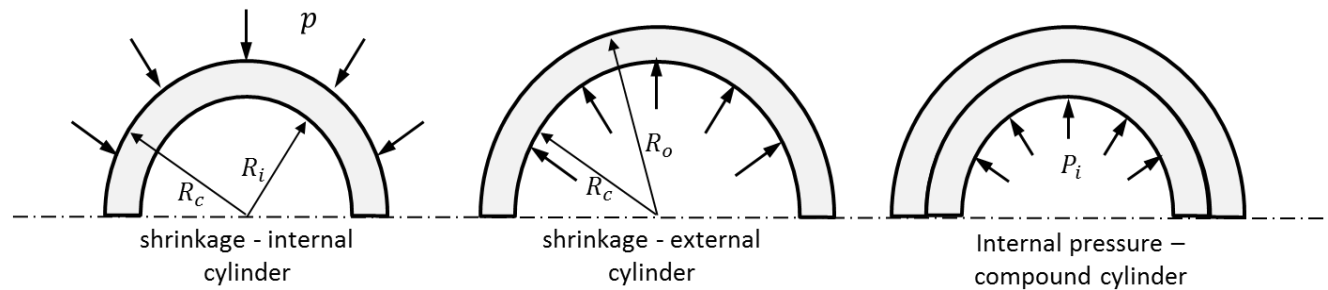
For each of the resulting load conditions, there are two known values of the radial stress which is sufficient to determine the associated Lame constants.

Condition 1:

$$At \quad r = R_i, \quad \sigma_r = 0$$

$$At \quad r = R_c, \quad \sigma_r = -p$$

where R_c is the interface radius. The stress is compressive since it tends to reduce wall thickness.

Condition 2:

$$At \quad r = R_o, \qquad \sigma_r = 0$$

$$At \quad r = R_c, \qquad \sigma_r = -p$$

Condition 3:

$$At \quad r = R_o, \qquad \sigma_r = 0$$

$$At \quad r = R_i, \qquad \sigma_r = -P_i$$

Therefore, for each condition, the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied. This means that the various stresses can simply be added up to compute the stresses in a compound cylinder subject to both shrinkage and internal pressure.

5.2 Shrinkage or Interference Allowance

When we design a compound cylinder, we need to relate the difference in diameter of the mating cylinders to the resultant stresses. The difference in diameter at the mating surfaces is called the *shrinkage* or *interference allowance* for compound cylinders produced by shrinking (normal process) or force fit procedure respectively.

Let us consider the compound cylinder shown below (the materials of the two cylinders are not necessarily the same):

Let:
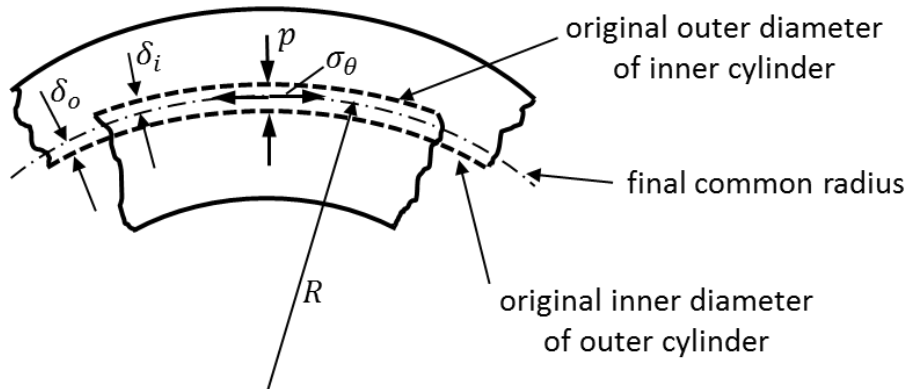
$$p = pressure\ setup\ at\ junction\ of\ cylinders\ due\ fit$$
$$\sigma_{\theta_i} = hoop\ stress\ on\ inner\ cylinder\ at\ junction (compressive)$$
$$\sigma_{\theta_o} = hoop\ stress\ on\ outer\ cylinder\ at\ junction (tensile)$$

Then if:

$$\delta_o = radial\ shift\ of\ outer\ cylinder$$
$$\delta_i = radial\ shift\ of\ inner\ cylinder$$

Since the circumferential strain is equal to the diametrical strain:

$$\varepsilon_\theta = \varepsilon_D (everywhere)$$

Then:

$$circumferential\ strain\ at\ radius\ R\ on\ outer\ cylinder = \frac{2\delta_o}{2R} = \frac{\delta_o}{R} = \varepsilon_{\theta_o}$$

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$$circumferential\ strain\ at\ radius\ R\ on\ inner\ cylinder = \frac{2\delta_i}{2R} = \frac{\delta_i}{R} = -\varepsilon_{\theta_i}$$

Note that ε_{θ_i} is –ve since it represents a decrease in diameter.

Thus:

$$shrinkage = \delta_o + \delta_i = R\varepsilon_{\theta_o} + R(-\varepsilon_{\theta_i}) = (\varepsilon_{\theta_o} - \varepsilon_{\theta_i})R$$

Assuming open ends ($\sigma_L = 0$):

$$\varepsilon_{\theta_o} = \frac{\sigma_{\theta_o} - v_1(-p)}{E_1}$$

And:

$$\varepsilon_{\theta_i} = \frac{\sigma_{\theta_i} - v_2(-p)}{E_2}$$

Since:

$$\sigma_{r_o} = \sigma_{r_i} = -p$$

Where E_1, v_1, E_2, v_2 relate to the two cylinders.

Therefore, the total shrinkage or interference (based on radius) is given by:

$$Total\ shrinkage(interference) = R\left[\frac{(\sigma_{\theta_o} + v_1 p)}{E_1} - \frac{(\sigma_{\theta_i} + v_2 p)}{E_2}\right] \qquad (5.1)$$

Where R is the initial nominal radius of the mating surfaces.

Generally, both cylinders are of the same material:

$$E_1 = E_2 = E \quad and \quad v_1 = v_2 = v$$

Giving the simplified equation:

$$Total\ shrinkage(interference) = \frac{R}{E}(\sigma_{\theta_o} - \sigma_{\theta_i}) \qquad (5.2)$$

Note:

i.   Since σ_{θ_i} is compressive, the sign will change
ii. Shrinkage allowance based on diameter is twice the values quoted above, with R replaced by D

Example:

A compound cylinder is formed by shrinking a tube of 250mm inside diameter and 300mm outside diameter onto another smaller tube of 200mm inside diameter and 250mm outside diameter (nominal dimensions). Both tubes are manufactured from steel. The interference pressure at the junction is 10MPa. If the compound cylinder is then subjected to an internal pressure of 80MPa, calculate the hoop stress distribution

Method: obtain results for shrinkage and internal pressure separately and then use superposition to combine.

Solution:

Shrinkage only, outer tube:
$$\text{At } r = 0.15m, \sigma_r = 0$$
$$\text{At } r = 0.125m, \sigma_r = -10MPa$$

We have:
$$\sigma_r = A - \frac{B}{R^2}$$

Giving:
$$0 = A - \frac{B}{0.15^2} \rightarrow A = \frac{B}{0.15^2}$$

And:
$$-10 = A - \frac{B}{0.125^2} = \frac{B}{0.15^2} - \frac{B}{0.125^2} = B\left(\frac{1}{0.15^2} - \frac{1}{0.125^2}\right) = -19.6B$$

$$B = 0.511$$

Thus:
$$A = \frac{0.51}{0.15^2} = 22.73$$

We know:
$$\sigma_\theta = A + \frac{B}{R^2}$$

When $r = 0.15m$:
$$\sigma_\theta = 22.73 + \frac{0.51}{0.15^2} = 45.4MPa$$

When $r = 0.125m$:
$$\sigma_\theta = 22.73 + \frac{0.51}{0.125^2} = 55.4MPa$$

<u>Shrinkage only, inner tube:</u>

When $r = 0.10m, \sigma_r = 0$

When $r = 0.125m, \sigma_r = -10MPa$

We have:

$$\sigma_r = A - \frac{B}{R^2}$$

Giving:

$$0 = A - \frac{B}{0.10^2} \rightarrow A = \frac{B}{0.10^2}$$

And:

$$-10 = A - \frac{B}{0.125^2} = \frac{B}{0.10^2} - \frac{B}{0.125^2} = B\left(\frac{1}{0.10^2} - \frac{1}{0.125^2}\right) = 36B$$

$$B = -0.278$$

Thus:

$$A = \frac{-0.278}{0.10^2} = -27.8$$

We know:

$$\sigma_\theta = A + \frac{B}{R^2}$$

When $r = 0.10m$:

$$\sigma_\theta = -27.8 - \frac{0.278}{0.10^2} = -55.6MPa$$

When $r = 0.125m$:

$$\sigma_\theta = -27.8 - \frac{0.278}{0.125^2} = -45.6MPa$$

<u>Internal pressure only on complete cylinder:</u>

When $r = 0.10m, \sigma_r = -80MPa$

When $r = 0.15m, \sigma_r = 0$

We have:

$$\sigma_r = A - \frac{B}{R^2}$$

Giving:

$$-80 = A - \frac{B}{0.10^2}$$

And:

$$0 = A - \frac{B}{0.15^2} \rightarrow A = \frac{B}{0.15^2}$$

Therefore:

$$-80 = \frac{B}{0.15^2} - \frac{B}{0.10^2} = B\left(\frac{1}{0.15^2} - \frac{1}{0.10^2}\right) = -55.6B$$

$$B = 1.44$$

Thus:

$$A = \frac{1.44}{0.15^2} = 64$$

We know:

$$\sigma_\theta = A + \frac{B}{R^2}$$

When $r = 0.10m$:

$$\sigma_\theta = 64 + \frac{1.44}{0.10^2} = 208MPa$$

When $r = 0.125m$:

$$\sigma_\theta = 64 + \frac{1.44}{0.125^2} = 156.2MPa$$

When $r = 0.15m$:

$$\sigma_\theta = 64 + \frac{1.44}{0.15^2} = 128MPa$$

Combined solution for shrinkage and internal pressure:

Inner tube:
When $r = 0.1, \sigma_\theta = -55.6 + 208 = 152.4MPa$
When $r = 0.125, \sigma_\theta = -45.6 + 156.2 = 110.6MPa$

Outer tube:
When $r = 0.125, \sigma_\theta = 55.4 + 156.2 = 211.6MPa$
When $r = 0.15, \sigma_\theta = 45.4 + 128 = 173.4MPa$

5.3 Shrink fitting of solid shafts

We have the Lame equations:

$$\sigma_r = A - \frac{B}{r^2} \quad and \quad \sigma_\theta = A + \frac{B}{r^2}$$

As $r \to 0$ (at the center of the solid shaft), σ_r, $\sigma_\theta \to \infty$! This is impossible.

Therefore, the only solution that can yield finite stress values is when $B = 0$. From the above equations, it follows that for all values of R:

$$\sigma_r = \sigma_\theta = A \qquad (5.3)$$

So at the outer surface of the shaft:

$$\sigma_r = -p \ (the \ shrinkage \ pressure)$$

Thus the hoop and radial stresses throughout a solid shaft are at all points constant and equal to the shrinkage or interference pressure and both are compressive.

The maximum shear stress given by $\frac{1}{2}(\sigma_1 - \sigma_3)$ is therefore zero throughout the shaft.

5.4 Force Fits

We have mentioned that compound cylinders can be manufactured by shrinking of force-fit methods. In the case of force-fits, the interference allowance is sufficiently small to allow the outer cylinder to be pressed (forced) over the inner cylinder with a large axial force.

Let the interference pressure set up at the common interface be p. The normal force, N, between the mating cylinders is then:

$$N = p \times 2\pi RL \quad (5.4)$$

Where:

$$R \ is \ the \ inter \ face \ radius$$
$$L \ is \ the \ contact \ length$$

The friction force, F, that must be overcome by the applied force is thus:

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$$F = \mu N \qquad (5.5)$$

Where μ is the coefficient of friction between the contact surfaces.

Therefore:

$$F = \mu(p2\pi RL) = 2\pi\mu pRL \qquad (5.6)$$

Is the applied force, F, is known, then the value of p can be determined. Alternatively, if the interference is known, p (and hence F) can be found.