# Solutions to Differentiation

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2020

## Contents

# 1 Exercises

## 1.1 The Fundamentals

**Exercise.** Find the equation of the line tangent to $x^4 + 3x^2$ at $(2, 28)$.

**Solution:** Note that $f'(x) = 3x^3 + 6x$, so $f'(2) = 36$. Thus the point-slope equation of the line is

$$y - 28 = 44(x - 2).$$

**Example.** Find $\lim_{x\to 0} \frac{\sin(2x)}{x+x^2}$.

**Solution 1 (Maclaurin Series):** Note that the Maclaurin Series of $\sin(2x)$ is $2x + O(x^3)$. Because the denominator has degree 2, and $x$ approaches 0, we don't care about $O(x^3)$. So the limit is equivalent to

$$\lim_{x\to 0} \frac{2x}{x + x^2} = \lim_{x\to 0} \frac{2}{1+x} = 2.$$

**Solution 2 (Factoring):** Note that this expression is equivalent to

$$\lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{1+x} = 1 \cdot \frac{2}{1} = 2.$$

**Solution 3 (L'Hopital's):** Note that 1 is the smallest number such that $g^{(1)}(x) \neq 0$, where $g(x) = x + x^2$, so

$$\lim_{x\to 0} \frac{\sin(2x)}{x + x^2} = \frac{2\cos(2\cdot 0)}{1 + 2\cdot 0} = 2.$$

## 1.2 Laws of Differentiation

**Exercise (AoPS Calculus, 3.6.3).** Find $\frac{dy}{dx}$ if $x^2 + y = \ln(y^2 - 1)$.

**Solution:** We implicitly differentiate. Note that

$$2x + \frac{dy}{dx} = \frac{1}{y^2 - 1} 2y \frac{dy}{dx}$$

$$2x = \frac{dy}{dx} \frac{2y - (y^2 - 1)}{y^2 - 1} = \frac{dy}{dx} \frac{-y^2 + 2y + 1}{y^2 - 1}$$

$$\frac{2x(y^2 - 1)}{-y^2 + 2y + 1} = \frac{dy}{dx}.$$

**Exercise (AoPS Calculus, 3.6.4).** Find the slope of the tangent line to the curve $x\sin(x + y) = y\cos(x - y)$ at the point $(0, \frac{\pi}{2})$.

**Solution:** We implicitly differentiate. First use the product rule and note that

$$\sin(x + y) + x(\cos(x + y))' = y'\cos(x - y) + y(\cos(x - y))'.$$

By the Chain Rule, this implies

$$\sin(x+y) + x\cos(x+y)(x+y)' = y'\cos(x-y) - y\sin(x-y)(x-y)'.$$

Now the linearity of derivatives implies

$$\sin(x+y) + x\cos(x+y)(1+y') = y'\cos(x-y) - y\sin(x-y)(1-y').$$

Plug in $(x,y) = (0, \frac{\pi}{2})$ to get

$$\begin{aligned}\sin(\frac{\pi}{2}) &= y'\cos(-\frac{\pi}{2}) - \frac{\pi}{2}\sin(-\frac{\pi}{2})(1-y') \\ 1 &= \frac{\pi}{2}(1-y') \\ \frac{\pi}{2}y' &= \frac{\pi}{2} - 1 \\ y' &= 1 - \frac{2}{\pi}.\end{aligned}$$

Thus the slope is $1 - \frac{2}{\pi}$.

## 1.3 Derivatives of Certain Functions

**Exercise (Periodic Derivatives).** If $f(x) = \sin x$, find $f'(x), f''(x), f'''(x)$, and $f''''(x)$. Do the same for $f(x) = \cos x$.

**Solution:** We already know from before that $f'(x) = \cos x$. Thus $f''(x) = -\sin x, f'''(x) = -\cos x$, and $f''''(x) = \sin x$.

A good way to think of this is that $f'(x) = \sin(x + \frac{\pi}{2})$. Then $f^{(n)}(x) = \sin(x + \frac{n\pi}{2})$, which intuitively explains why $f^{(n)}(x)$ has period 4.

**Exercise (Derivatives of Reciprocal Functions).** Given how the trigonometric derivatives for sin, cos, and tan were derived, determine and prove the derivatives of csc, sec, and cot.

**Solution:** The reciprocal rule solves the first two straightforwardly:

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = -\frac{\cos x}{\sin^2 x} = -\csc x\cot x(\sec x)' = \left(\frac{1}{\cos x}\right)' \qquad = \frac{\sin x}{\cos^2 x} = \sec x\tan x.$$

We could also use the reciprocal rule on $\cot x$, but it's more convenient to just use the quotient rule:

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\cos^2 x + \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} = \csc^2 x.$$

**Exercise.** Find the Maclaurin Series of $x\cos x$.

**Solution:** Note that the Maclaurin Series of $x$ is $x$, and the Maclaurin Series of $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$. Multiplying the two yields

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots.$$

**Exercise (Derivative of Inverse of Reciprocal Trigonometric Functions).** Find the derivative of $\operatorname{arccsc} x$, $\operatorname{arcsec} x$, and $\operatorname{arccot} x$.

**Solution:** We implicitly differentiate for all of these.

For the first one, let $f(x) = \operatorname{arccsc} x$, and note this implies $\csc y = x$. Then differentiating with respect to $x$ gives

$$\left(\frac{1}{\sin y}\right)' y' = 1$$
$$-\frac{\cos y}{\sin^2 y} y' = 1$$
$$y' = -\frac{\sin^2 y}{\cos y}.$$

Since $\sin y = \frac{1}{x}$ and $\cos y = \sqrt{1 - \frac{1}{x^2}}$,

$$y' = -\frac{1}{x^2\sqrt{1 - \frac{1}{x^2}}} = -\frac{1}{|x|\sqrt{x^2 - 1}}.$$ [1]

For the second one, let $f(x) = \operatorname{arcsec} x$, and note that this implies $\sec y = x$. Then differentiating with respect to $x$ gives

$$\left(\frac{1}{\cos x}\right) y' = 1$$
$$\frac{\sin y}{\cos^2 y} y' = 1$$
$$y' = \frac{\cos^2 y}{\sin y}.$$

Since $\cos y = \frac{1}{x}$ and $\sin y = \sqrt{1 - \frac{1}{x^2}}$,

$$y' = \frac{1}{x^2\sqrt{1 - \frac{1}{x^2}}} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$ [2]

For the last one, let $f(x) = \operatorname{arccot} x$, and note that this implies $\cot y = x$. Then differentiating with respect to $x$ gives

$$\left(\frac{\cos y}{\sin y}\right) y' = 1$$
$$\frac{-\sin^2 y - cos^2 y}{\sin y} y' = \frac{1}{\sin y} y' = 1$$
$$y' = \sin y.$$

Since $x = \cot y$,

$$y' = \frac{1}{\sqrt{x^2 + 1}}.$$

[1]The absolute value appears because $x^2 \geq 0$ for obvious reasons, and we need to preserve this even after factoring out an $x$.
[2]See above.

**Exercise.** Find the derivative of $f(x) = \log_a(g(x))$.

**Solution:** Note that $f(x) = \frac{\ln(g(x))}{\ln a}$, so

$$f'(x) = \frac{(\ln(g(x)))'}{\ln a} = \frac{\frac{1}{g(x)}g'(x)}{\ln a} = \frac{g'(x)}{g(x)\ln a}.$$

# 2 Problems

## 2.1 Unsourced

Prove that the derivative of $f(x) = e^{g(x)}$ is $e^{g(x)}g'(x)$.

### 2.1.1 Solution

This follows directly from the chain rule.

## 2.2 HMMT Calculus 2005/1

Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose that the tangent lines to the graph of $f$ at $x = a$ and $x = b$ are parallel. Find $f(1)$.

### 2.2.1 Solution

Note that $f'(x) = 3x^2 + a$, so $f'(a) = f'(b)$ implies $3a^2 + a = 3b^2 + a$, or that $a = -b$. Thus $f(1) = 1 + a + b = 1$.

## 2.3 HMMT Calculus 2010/1

Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

### 2.3.1 Solution

Note that $p(x)$ must have leading term $x^2$, because by the Power Rule $\deg(p(x) - p'(x)) = \deg(p(x))$, and furthermore the leading coefficients are the same. So we have

$$p(x) = x^2 + ax + bp'(x) \qquad\qquad = 2x + a$$

and we want $p(x) - p'(x) = x^2 + x(a-2) + (b-a) = x^2 + 2x + 1$, or

$$a - 2 = 2b - a \qquad\qquad = 1,$$

implying that $a = 4$ and $b = 5$. Therefore, $p(5) = 5^2 + 4 \cdot 5 + 5 = 50$.

## 2.4 HMMT Calculus 2010/3

Let $p$ be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeroes of $p'(x)$ are also zeros of $p(x)$. Find $p$. Note: monic means that the leading coefficient is 1.

### 2.4.1 Solution

There are either three distinct roots, two distinct roots, or one root. We look at all three cases.

If there are three distinct roots, then $p(x) = (x - r_1)(x - r_2)(x - r_3)$ and $p'(x) = 3x^2 - 2x(r_1 + r_2 + r_3) + (r_1r_2 + r_2r_3 + r_3r_1)$. We can verify that this function has zeroes and that none of them are $r_1, r_2, r_3$, since the roots are distinct.

If there are two distinct roots, there are two copies of one root and one copy of another. So $p(x) = (x - r_1)^2(x - r_2)$, and by the Product Rule,

$$p'(x) = 2(x - r_1)(x - r_2) + (x - r_1)^2 = (x - r_1)(3x - r_1 - 2r_2).$$

Since $r_1 \neq r_2$, $\frac{r_1+2r_2}{3}$ cannot be either $r_1$ or $r_2$, since it is a weighted mean.

If there is one distinct root, then $p(x) = (x-r)^3$. Note that $p'(x) = 3(x-r)^2$, and the only root is $x = r$, so this satisfies the condition. Since $p(0) = 1$, we must have $r = -1$, or $p(x) = (x+1)^3$.

## 2.5 Two-Term AM-GM

Determine the minimum value $f(x) = x + \frac{1}{x}$ can take over positive $x$.

### 2.5.1 Solution

Note that $f'(x) = 1 - \frac{1}{x^2}$. We claim that the minimum occurs at $x = 1$, and to prove it, note that $f'(x) < 0$ when $x < 1$ and $f'(x) > 0$ when $x > 1$. Thus $f(0) = 2$ is the minimum value it can take, and this minimum **is only achieved at** $x = 1$.

## 2.6 Unsourced

Find the derivative of $\frac{4^x}{4^x+1}$.

### 2.6.1 Solution

Let this function be $f(x)$. Note that $f(x) = 1 - \frac{1}{4^x+1}$, so $f'(x) = -\frac{d}{dx}(\frac{1}{4^x+1})$. By the Reciprocal Rule,

$$-\frac{d}{dx}(\frac{1}{4^x+1}) = \frac{\frac{d}{dx}(4^x+1)}{(4^x+1)^2} = \frac{4^x \ln 4}{(4^x+1)^2}.$$

## 2.7 Dennis Chen

Find the equation of the line tangent to

$$\tan x + \sin x = y \cos x - 1$$

at $(\frac{\pi}{4}, 2\sqrt{2}+1)$.

### 2.7.1 Solution

Differentiate both sides to get

$$\sec^2 x + \cos x = -y \sin x + y' \cos x.$$

Now plug in $(\frac{\pi}{4}, 2\sqrt{2}+1)$ to get

$$\begin{aligned} 2 + \frac{\sqrt{2}}{2} &= -2 - \frac{\sqrt{2}}{2} + y' \frac{\sqrt{2}}{2} \\ 4 + \sqrt{2} &= y' \frac{\sqrt{2}}{2} \\ 4\sqrt{2} + 2 &= y'. \end{aligned}$$

## 2.8 MIT OCW

Given that $f'(a)$ exists, show that $g(h) = \frac{f(a+h)-f(a)}{h}$ has a removable discontinuity at $h = 0$.

### 2.8.1 Solution

In order for g(h) to have a removable discontinuity at h = 0 it must follow two different rules. Firstly the $\lim_{h\to 0}$ for g(h) must exist. As the g(h) when evaluated for the limit leads to the equation $\lim_{h\to 0} g(h) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ it is therefore shown, through the limit definition of a limit that $\lim_{h\to 0} g(h) = f'(a)$. As we know, therefore, that $f'(a)$ exists at $a$ we therefore know that the $\lim_{h\to 0} g(h)$ exists. When paired with the structure of the equation, with $\frac{f(a+h)-f(a)}{h}$ demonstrating a rational equation and when evaluated for $\lim_{h\to 0}$ giving $\frac{0}{0}$ there is clearly a removable discontinuity. Therefore based on the structure of the equation and what g(h) represents it can be surmised that at h = 0, $g(h)$ has a removable discontinuity.

## 2.9 HMMT Calculus 2007/2

Determine the real number $a$ having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

### 2.9.1 Solution

Note that it is necessary (but not sufficient) for $f'(a) = 0$. Note that $f'(x) = 4x^3 - 3x^2 - 2x + a$, so

$$f'(a) = 4a^3 - 3a^2 - 2a + a = 4a^3 - 3a^2 - a = (a-1)a(4a+1).$$

Thus the possible values of $a$ are $-\frac{1}{4}, 0, 1$.

Note that we require $a = f(a)$, so the only case left to check is $a = 1$. For $a = 1$, we have

$$f'(x) = 4x^3 - 3x^2 - 2x - 1 = (x+1)(4(x+\frac{1}{8})^2 - \frac{17}{16}),$$

so the other roots of $f'(x)$ are $x = -\frac{1\pm\sqrt{17}}{8}$. Since both of these roots are less than 1, and the leading coefficient of $f'(x)$ is positive, $f(1-\epsilon) < 0$ and $f(1+\epsilon) > 0$ for small $\epsilon > 0$, which are necessary for a minimum to be achieved.

So the answer is just $a = 1$.

## 2.10 HMMT Calculus 2006/2

Compute $\lim\limits_{x\to 0} \frac{e^{x\cos x}-1-x}{\sin(x^2)}$.

### 2.10.1 Solution

We use L'Hopital's Rule to completely butcher this problem. Note that

$$\lim_{x\to 0} \frac{e^{x\cos x} - 1 - x}{\sin(x^2)} = \lim_{x\to 0} \frac{e^{x\cos x}(\cos x - x\sin x) - 1}{2x\cos(x^2)}{}^{3} = \lim_{x\to 0} \frac{1 - x\sin x - 1}{2x} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}.$$

## 2.11 MAST Diagnostic 2020/C10

Find the maximum value of $k$ such that $(x+1)^4 \geq kx^3$ for all $x$.

We solve this problem in two separate ways: one with calculus and another with AM-GM.

[3]We took the derivative of both sides of the fraction.

#### 2.11.1 Solution

[1 (Calculus)] Note that obviously $k \geq 0$, so $x \leq 0$ is not even a case worth considering since the left-hand side will be non-negative and the right-hand side will be non-positive.

This is equivalent to finding the minimum value of $f(x) = \frac{(x+1)^4}{x^3}$ over positive $x$. Note that the derivative of this function is, by the quotient/chain rules,

$$f'(x) = \frac{4(x+1)^3x^3 - 3(x+1)^4x^2}{x^6} = \frac{(x+1)^3(4x-3(x+1))}{x^4} = \frac{(x+1)^3(x-3)}{x^4}.$$

Note that $f'(x) > 0$ when $x > 3$ and $f'(x) < 0$ when $0 < x < 3$, so on the domain $(0, \infty)$, $f(x)$ is minimized when $x = 3$.

It is easy to verify that $x = 3$ gives $f(x) = \frac{64}{27}$, so that is our answer.

#### 2.11.2 Solution

[2 (AM-GM)] Note that by AM-GM, $(\frac{x}{3} + \frac{x}{3} + \frac{x}{3} + 1)^4 \geq 64 \cdot \frac{x^3}{27}$ with equality at $\frac{x}{3} = 1$, so our maximum is $k = \frac{64}{27}$.[4]

### 2.12 Extension of C10

Find the range of values $k$ such that $(x+1)^4 \geq kx^3$ for all $x$.

#### 2.12.1 Solution

As we can probably infer from the solution above, the problem behaves differently for $k \geq 0$ and $k \leq 0$, and each of these cases only care about $x \geq 0$ and $x \leq 0$, respectively.

We have already done $x \geq 0$ – in that case, $k \leq \frac{64}{27}$ will work. So we do $x \leq 0$.

The extension is really not hard; when $x = -1$ we have $0 \geq k \cdot -1^3$, so we must have $k \geq 0$. Thus the range is $k \in [0, \frac{64}{27}]$.

### 2.13 AMC 12B 2020/22

What is the maximum value of $\frac{(2^t - 3t)t}{4^t}$ for real values of $t$?

#### 2.13.1 Solution

Note this function is equivalent to

$$\frac{t}{2^t} - \frac{3t^2}{4^t}.$$

By the Quotient Rule, the derivative of this function is

$$\frac{2^t - t2^t \ln 2}{4^t} - \frac{6t4^t - 3t^2 4^t \ln 4}{16^t} =$$
$$\frac{1}{4^t}\left(2^t - t2^t \ln 2 - (6t - 3t^2 \ln 4)\right) =$$
$$\frac{(2^t - t2^t \ln 2) - (6t - 6t^2 \ln 2)}{4^t} =$$
$$\frac{(1 - t\ln 2)(2^t - 6t)}{4^t}.$$

[4]The case where $x$ is non-positive is not addressed in this solution, but it is completely trivial.

By definition, the function reaches its maximum when the derivative is 0. This means we either have $1 - t\ln 2 = 0$ or $2^t = 6t$; note that both have solutions. Plugging in the former gives $t = \frac{1}{\ln 2} = \log_2 e$, and the expression is equal to

$$\frac{e\log_2 e - 3(\log_2 e)^2}{e^2},$$

which is negative; we can obviously do better. Let's try $2^t = 6t$; then the function is equal to

$$\frac{(6t - 3t)t}{36t^2} = \frac{1}{12},$$

which is our maximum.

## 2.14 Leibniz Rule

Given two $n$th differentiable functions $f, g$, prove that

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

### 2.14.1 Solution

This is just algebraic manipulation with Taylor Series.

Note that the Taylor Series of $f(x)$ is $f(x+\epsilon) = f(x) + f'(x)\epsilon + \frac{f''(x)\epsilon^2}{2!} + \cdots$. A similar equation holds for $g(x+\epsilon)$. Take the product of the Taylor Series and note that the coefficient of the $\epsilon^n$ term can be expressed as

$$\sum_{k=0}^{n} \frac{f^{(k)}(x)\epsilon^k}{k!} \cdot \frac{g^{(n-k)}(x)\epsilon^{n-k}}{(n-k)!},$$

and since

$$\epsilon^n \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} \cdot \frac{g^{(n-k)}(x)}{(n-k)!} = \frac{(fg)^{(n)}(x)\epsilon^n}{n!},$$

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$