Season 3 Diagnostic

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2021

§ 1 Rules and Instructions

The MAST Diagnostic is a selection of problems the staff team has written that have appeared in various contests. Because the first lecture is so soon, I've abridged the diagnostic to 8 computational problems and 0 proof. (You're welcome.)

Major changes:

- The proof section is entirely removed; computational has been fixed to be 8 problems total, 2 per each subject. One problem is easy-medium and the other is medium-hard in difficulty. **However, there are still some things you must prove**: I will make it clear if I want a full and rigorous proof for a "find-all" style problem.[1] **Solution sketches, at a minimum, are still mandatory.** Convince me you know how to solve the problem on your own.[2]
- The points system has been removed.[3] Part of the reason is because the relative difficulty of each problem should be clear, given there's only two per subject.
- You must indicate if you remember a past problem.
- If you think you've accidentally broken any of these rules, please let me know. The difference between cheating and a mistake is **transparency**.
- If you've deliberately broken any of these rules, please also let me know. I believe that cheaters can become better people. Please prove me right.

§ 1.1 Asking for Help

If you've been stuck on a problem for a while and want a hint, you can ask me **or other people**. You may also consult static media. (This includes books, articles, and parts of the internet such as encyclopedias or other references. This **does not** include making a forum post on AoPS or StackExchange. You also may not look for the problem.) If you do ask other people, here are a couple of rules:

- Do not ask any MAST students/applicants/potential applicants.
- The person helping you should know that they are helping you on a diagnostic intended to gauge your ability. So they should not give you the entire solution and instead be giving helpful hints.

[1] Nevertheless, problems will still mostly be computational in nature.

[2] This used to be a saying I stole to indicate how much work you needed to do, but based on the number of cheaters last season, I actually mean it when I say it this time.

[3] It only ended up discouraging the people who had less of a chance to make it, which is the last thing I want to do. And people kept asking me "how much should I do" anyway, so it didn't save me any questions. (For the record, please don't ask how much you need to do: the answer is "as much as you can, to a reasonable extent.")

- ✦ Don't try to get the solution, try to get small hints. Just enough to get you moving forward on the problem.
- ✦ You **must** indicate to me that you have asked someone for help, and you must indicate to me how they have helped you. (Even if it wasn't actually helpful in the end.)

I want to stress that these are not suggestions or requests. These are hard rules that **must be followed** if you ask someone else for help. **Deliberately omitting any of these steps is academic dishonesty, and may get you banned from the program.** Some examples of people you can ask: past/current students, your math coach, your parents, college friends, people overqualified to be a student of the program,[4] et cetera.

§ 1.2 Seen Before

Because these are mostly problems that've appeared elsewhere, some applicants may have seen some of the problems. Policy change from last year: **if there is any way you may have been exposed to the problem before**, such as taking JMC and/or reviewing the solutions packet, **indicate this in your PDF submission**.[5]

Something that might be obvious but should be explicitly stated anyway: if you intend to apply, you should not look at any of the problems after you've seen this document. Nor should you frantically review JMC/NARML to "prepare" for this application. You know what's right and what's not, so please don't make yourself look stupid by claiming that something obviously unfair is "technically not cheating."

[4] If this describes you and you want to get involved with the program, shoot me a PM! I always appreciate more staff.

[5] Contrary to what might be popular belief, you actually have better chances if you have done MAC contests. Those of you who have followed my streams more closely know that **I am a nepotist**, and I think that if you care enough to take our mock contests, you deserve to make the program. Obviously I won't really factor it in either way, since your problemset, essays, and contest scores (to a lesser extent) are much more important to me.

§ 2 Problems

§ 2.1 Algebra

A1. What is the value of

$$\frac{(2019+2020)(2020+2021)(2021+2019)+2019\cdot 2020\cdot 2021}{2019\cdot 2020+2020\cdot 2021+2021\cdot 2019}?$$

A2. Find $\sum\limits_{a=1}^{\infty} \frac{32a}{16a^4+24a^2+25}$.

§ 2.2 Combo

C1. Steven has 4 lit candles and each candle is blown out with a probability $\frac{1}{2}$. After he finishes blowing, he randomly selects a possibly empty subset out of all the candles. What is the probability his subset has at least one lit candle?

C2. The mad scientist Kyouma is traveling on a number line from 1 to 2020, subject to the following rules:

- He starts at 1.
- Each move, he randomly and uniformly picks a number greater than his current number to go to.
- If he reaches 2020, he is instantly teleported back to 1.
- There is a time machine on 199.
- A foreign government is waiting to ambush him on 1729.

What is the probability that he gets to the time machine before being ambushed?

§ 2.3 Geometry

G1. Consider parallelogram $ABCD$ with $AB = 7$, $BC = 6$. Let the angle bisector of $\angle DAB$ intersect BC at X and CD at Y. Let the line through X parallel to BD intersect AD at Q. If $QY = 6$, find $\cos\angle DAB$.

G2. An equilateral triangle ACK is located inside a regular decagon $ABCDEFGHIJ$. If the area of the decagon is 1, find the area of $HIJAK$.

§ 2.4 Number Theory

N1. Compute the smallest positive integer n such that $9(n+3)$ divides $4n!+n+5$.

N2. Find all positive integers n such that

$$\frac{1!\cdot 2!\cdot 3!\cdots 2019!\cdot 2020!}{n!}$$

is a perfect square.