Preface

It is very easy to teach geometry wrong. There are plenty of high school geometry books written for profit that do exactly this. Rarely are underlying connections pointed out, or are thought-provoking problems required. More often than not, those who do not enjoy math associate it with brutal calculations with ugly numbers. Rounding to the nearest hundredth or to the nearest inch, and messing up the calculations has been a constant source of frustration for those in typical math classes.

But the art of math is filled with beauty. In reality, most ratios are simple. A surprising fact is that in a triangle, $\overline{HG} = 2\overline{GO}.$ Most configurations will have beautiful numbers involved, and each problem's numbers, if any, have been chosen with care. Rarely will a problem require unreasonable computation, because understanding the concept will be enough to solve it.

While these theorems may seem memorization based at first, a deeper understanding of each theorem and enough practice with them on significant, thought-provoking problems will be more than enough to have them ingrained in your head. Putting formulas on flashcards, writing down a formula hundreds of times, and so on are not efficient methods because the best way to learn to solve problems is to do them! However, if a problem does not provoke thought, it will not stick around in your brain. Only when our brains are sufficiently stimulated will they be able to remember what stimulated them in the first place.

Geometry can be taught wrong in so many ways, but there are so many ways to teach it right. Only through a variety of perspectives will you understand a concept deeper. This is one of the many resources out there, and other good resources should be used as well.

With that said, let us begin Exploring Euclidean Geometry!

<u>Using the Book</u>

A couple of notes on the style of this book and usage.

All exercises are either classic or written by yours truly. I acknowledge that contest problems are probably the ideal way to train. As part of this acknowledgement, I will release an exercise packet exclusively with contest and classic problems eventually.

Exercises are also very roughly ordered. Since the book's problems were not all written in order, rough inserts are present. I do not believe this impacts the value of them that much, as the beginning is usually filled with the introductory problems anyway. Once the introductory problems are over, the reader can probably easily judge the difficulty of the rest on their own.

A couple of conventional notations have not been followed. For example, $\overline{AB}$ usually denotes the segment, while AB denotes the length. This convention has been reversed. If a convention is not followed, it will be obvious when you see it. Usually this boils down to personal preference or a lack of knowledge about the convention until it was too late to change it.

Dedication

To Mr. Lomas.

You can trick yourself into creating something so grand that you would never have dared to plan such a thing.

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The Fundamentals

The Axioms of Geometry

You probably already intuitively know what a point, line, ray, line segment, angle, and a plane are. However, formalizing the axioms of geometry will reveal much more about geometry than an intuitive understanding, just like formalizing divisibility reveals that 0 divides 0. These axioms are important, so make sure that you fully understand what each of them are saying.

Axiom 1

A point is a location in a space. It is 0 dimensional.

Axiom 2

A line segment is a straight path between two points. It has 1 dimension.

Note that two points have a unique line segment connecting them.

Axiom 3

A ray AB is the line segment AB extended infinitely past point B.

Remember that this implies that rays AB and BA are distinct.

Axiom 4

A line is the extension of a line segment infinitely in both directions.

Axiom 5

An angle is formed by two rays BA and BC that share a common endpoint. The angle is the smallest amount that ray BA needs to be rotated to form ray BC. (This means it can be either rotated clockwise or counterclockwise.)

Axiom 6

A plane is a flat and infinitely extending surface. It has 2 dimensions.

Make sure you understand these fundamental axioms. The entire book will be built upon them. Unlike other geometry books, we will not have example problems to refresh your knowledge; you probably already know this information. Other prerequisite information includes how many degrees there are in an angle, the properties of the angles formed by lines, classifications of triangles, and the like.

Definitions and Properties

Let us first define a few equivalence properties. Even though these are not categorized under geometry, they are crucial for our study of geometry.

The transitive property states that if $a = b$ and $a = c$, $b = c$.

The zero product property states that for given numbers (real or imaginary) $A_1, A_2 ... A_n$ such that $A_1 A_2 ... A_n = 0$, then at least one of $A_1, A_2 ... A_n$ is equivalent to 0.

There are quite a few properties of equality. They are all based off the assumption $a = b$. This implies $a + x = b + x$, $ax = bx$, $a^x = b^x$, and $x^a = x^b$ (if x is not 0 or a and b are not 0 in the last case). Similar properties are true for similar operations. These can easily be deduced by common sense, but do remember not all of these properties are reversible. For example, $x^a = x^b$ does not necessarily mean $a = b$, particularly when x is $-1, 0,$ or 1.

Unlike the other properties, the properties of equality will be used very often, but when they are used, it will almost never be stated. This means that you will have to know when we are using it, and you should take great care that you fully understand what this property states and when it is used before continuing onward with the book.

Now, we shall define some properties of lines.

The midpoint of line AB is the point X such that X lies on AB and $\overline{AX} = \overline{BX}$. There is one unique midpoint for every line.

Points A, B, C are collinear if all the points A, B, C can be connected by a single line. In general, points $A_1, A_2 ... A_n$ are collinear if they all lie on the same line.

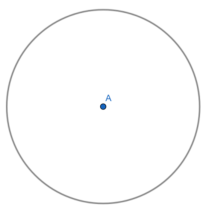
Lines AB, CD, EF are concurrent lines if there is a common point that lies on all of the lines AB, CD, EF. In general, lines $A_1B_1, A_2B_2 ... A_nB_n$ are concurrent if there is a point that lies on all of these lines.

Lines AB and CD are coplanar if they lie on the same plane.

Lines AB and CD are parallel if no point lies on AB and CD, and if AB and CD are coplanar. This is denoted as $AB \parallel CD$.

Have lines AB and CD intersect at X. Lines AB and CD are perpendicular if $\angle AXC = 90^{\circ}$. This is denoted as $AB \perp CD$.

Then, we shall define some shapes. We begin by defining a circle as the locus of points a constant distance, known as the radius, away from a point, known as the center.



Then, we define triangle ABC as the plane bounded by the lines AB, BC, CA. We can then define quadrilateral $ABCD$ as the plane bounded by the lines AB, BC, CD, DA. In general, for $n - gon$ $A_1A_2...A_n$, we define it as the plane bounded by the lines $A_1A_2, A_2A_3...A_{n-1}A_n, A_nA_1$.

Here are some problems to reinforce your understanding of the material. (These are mostly just semantics.)

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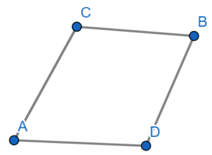
1. Draw, and label, quadrilateral $ACBD$.

2. Is the center of a circle part of the circle? Explain why or why not.

3. Draw, and label, heptagon $AOPSFTW$.

1. Draw, and label, quadrilateral $ACBD.$

Solution: We draw lines $AC,$ $CB,$ $BD,$ and DA to form our quadrilateral.

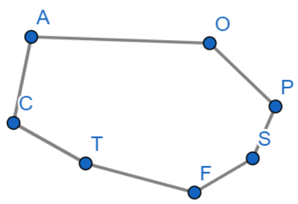


2. Is the center of a circle part of the circle? Explain why or why not?

Solution: The center of a circle is not part of it. This is because it has a distance of 0 from the center of the circle, whereas the circle has a radius of $x.$ Note that x cannot be 0 because that would not give us a circle and only give us a point.

3. Draw, and label, heptagon $AOPSFTW.$

Solution: We draw lines $AO,$ $OP,$ $PS,$ $SF,$ $FT,$ $TW,$ and WA to form our heptagon.



Then, let us define properties concerning relations between figures.

We define $\triangle ABC$ and $\triangle DEF$ to be congruent if $\overline{AB} = \overline{DE},$ $\overline{BC} = \overline{EF},$ $\overline{CA} = \overline{FD},$ $\angle A = \angle D,$ $\angle B = \angle E,$ and $\angle C = \angle F.$ This is denoted as $\triangle ABC \cong \triangle DEF.$ Note that $\triangle ABC \cong DEF$ does not necessarily mean $\triangle ABC \cong \triangle EFD.$

In general, n-gons $A_1A_2...A_n$ and $B_1B_2...B_n$ are congruent if $\overline{A_iA_{i+1}} = \overline{B_iB_{i+1}}$ for $0 < i < n,$ $\overline{A_nA_1} = \overline{B_nB_1},$ and $\angle A_j = \angle B_j$ for all $0 < j < n+1.$ While this book will denote this as $A_1A_2...A_n \cong B_1B_2...B_n,$ this is not a very common notation and to the best of my knowledge, this notation is unique to *Exploring Euclidean Geometry*.

Additionally, $\triangle ABC$ is similar to $\triangle DEF$ if there is some constant x such that $\overline{AB} = x\overline{DE}$, $\overline{BC} = x\overline{EF}$, $\overline{CA} = x\overline{FD}$, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. This is denoted as $\triangle ABC \sim \triangle DEF$. Similar to congruence, note that $\triangle ABC \sim \triangle DEF$ does not necessarily mean $\triangle ABC \sim \triangle EFD$.

In general, n-gons $A_1A_2...A_n$ and $B_1B_2...B_n$ are similar if there exists a constant x such that $\overline{A_iA_{i+1}} = x\overline{B_iB_{i+1}}$ for $0 < i < n$, $\overline{A_nA_1} = x\overline{B_nB_1}$, and $\angle A_j = \angle B_j$ for all $0 < j < n + 1$. This will be denoted as $A_1A_2...A_n \sim B_1B_2...B_n$.

Below are a few problems related to congruence of polygons.

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1. Consider $\triangle ABC$ and $\triangle DEF$. If $\triangle ABC \cong \triangle DEF$ and $\triangle ABC \cong \triangle EFD$, what are the values of $\angle A, \angle B, \angle C$?

2. Generalizing, if $A_1A_2...A_n \cong B_1B_2...B_n$ and $A_1A_2...A_n \cong B_2B_3...B_nB_1$, prove $A_1A_2...A_n$ is a regular polygon.

1. Consider $\triangle ABC$ and $\triangle DEF$. If $\triangle ABC \cong \triangle DEF$ and $\triangle ABC \cong \triangle EFD$, what are the values of $\angle A, \angle B, \angle C$?

Solution: Note that triangles are unique based on sides. Then, note that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, $\angle A = \angle E$, $\angle B = \angle F$, and $\angle C = \angle D$. By the transitive property, $\angle A = \angle B = \angle C$ (try figuring out yourself how $\angle A$ is connected to these other two angles), which means $\angle A = 60^\circ$, $\angle B = 60^\circ$, and $\angle C = 60^\circ$.

2. Generalizing, if $A_1A_2...A_n \cong B_1B_2...B_n$ and $A_1A_2...A_n \cong B_2B_3...B_nB_1$, prove $A_1A_2...A_n$ is a regular polygon.

Solution: Have $\overline{A_1A_2} = a_1, \overline{A_2A_3} = a_2 ... \overline{A_nA_1} = a_n$, and have $\overline{B_1B_2} = b_1, \overline{B_2B_3} = b_2 ... \overline{B_nB_1} = b_n$. Then note that the two givens imply $a_1 = b_1, a_2 = b_2 ... a_n = b_n$ and $a_1 = b_2, a_2 = b_3 ... a_n = b_1$. By the transitive property, $a_1 = a_2 = b_2$, $a_2 = a_3 = b_3 ...$ and so on, until $a_n = a_1 = b_1$. Another application of the transitive property gives us $a_1 = a_2 = ... = a_n$.

Doing something similar for the angles, note that $\angle A_1 = \angle B_1, \angle A_2 = \angle B_2 ... \angle A_n = \angle B_n$ and $\angle A_1 = \angle B_2, \angle A_2 = \angle B_3 ... \angle A_n = \angle B_1$. Applying the transitive property in an identical manner yields $\angle A_1 = \angle A_2 = \angle A_3 = ... = \angle A_n$.

These are the properties of regular polygons, so $A_1A_2A_3...A_n$ is a regular polygon.

Logic In Mathematics

In this book, we'll be using logic a lot, especially to prove theorems. We prove things by using a set of assumptions that cannot be proven called *axioms*. (As a rule of thumb, we limit the amount of axioms we have; if we can have something proven, it will be proven. If it cannot be proven but is essential to the study of a field, then it will be made an axiom.) In the first section, we have already defined our axioms. Before we study any proofs, let's introduce some notation and some terminology.

A proposition P is said to imply a result R if the truth of P means that R is also true. This is notated as $P \rightarrow R.$

The negation of a proposition P is the proposition $-P$ such that no matter what circumstances, exactly one of the propositions P and $-P$ are true. This means that if P is true then $-P$ is false, if P is false then $-P$ is true, if $-P$ is true then P is false, and if $-P$ is false then P is true. This also means that $-(-P) = P$ (the two propositions are identical).

The statement P is true if and only if R is true means either P and R are both true or neither of them are. This means the truthfulness of one leads to the truthfulness of the other, and the untruthfulness of one leads to the untruthfulness of the other. This is notated as $P \leftrightarrow R.$ This also implies $-P \leftrightarrow -R.$

Now, let us take a look at some proof techniques and some logical statements.

The Principle of Mathematical Induction

To prove something by induction, we must prove that there is a base case b such that $p(n)$ is true. We then want to prove that if $P(n)$ is true, then $P(n+1)$ is true for all $n.$ This works because if our base case $P(b)$ is true, then $P(b+1)$ is true. Since $P(b+1)$ is true, $P(b+2)$ is, and so on.

An example of a proof by mathematical induction is the formula for the nth triangular number. If $P(n)$ denotes the nth triangular number, $P(n) = \frac{n(n+1)}{2}.$ Try to prove this yourself using induction; the solution will be below.

Note that plugging in $n = 1$ obviously makes $P(n)$ true, as $1 = \frac{1(1+1)}{2}$. Then note that if $P(n)$ is true, then $P(n+1) = P(n) + n + 1 = \frac{n(n+1)}{2} + (n+1) = \frac{(n+2)(n+1)}{2}$. By induction, we are done.

The proof for this identity is all and well, but where did we derive this from? Well, there is a geometric way to find the value of $P(n)$. Note that when drawing lines between $n+1$ points, there are two ways to count the amount of lines. First, note that drawing a line between a point and the other n yields n lines. Then $n-1$ points can be drawn, and so on. This means we have $1 + 2 + ... + n$ lines. Then note that you can choose two points to make a line, and this implies that we have $\binom{n+1}{2} = \frac{n(n+1)}{2}$ lines, so $1 + 2 + ... + n = \frac{n(n+1)}{2}$, which is how we derive the formula.

Below are a few exercises based on induction.

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1. Prove that $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

2. Prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$.

3. Prove that a square can be split into n smaller non-overlapping squares for all $n \geq 6$.

4. Prove that $5^{2n-1} + 7^{2n-1}$ is always divisible by 6 for all positive n.

1. Prove that $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: Our base case is $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$. If $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, then $1^2 + 2^2 + ... + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$. Note that this implies $\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$, and algebraic manipulation on the left hand side gives us $(n+1)(\frac{n(2n+1)}{6}) + (n+1)(\frac{6n+6}{6}) = (n+1)(\frac{2n^2+7n+6}{6}) = (n+1)(\frac{(n+2)(2n+3)}{6}) = \frac{(n+1)(n+2)(2n+3)}{6}$, as desired. By the principle of induction, we are done.

2. Prove that $1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$.

Solution: Our base case is $1^3 = 1^2$. If $1^3 + 2^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^2$, then $1^3 + 2^3 + ... + (n+1)^3 = (1 + 2 + 3 + ... + n)^2 + (n+1)^3$. Applying algebraic manipulations give us the following.

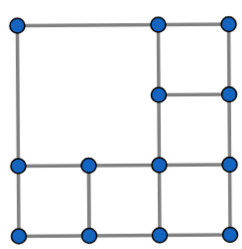
$$(1 + 2 + ... + n + 1)^2 - (1 + 2 + ... + n)^2 =$$
$$(n+1)(2 + 4 ... + 2n + n + 1) =$$
$$(n+1)(n(n+1) + n + 1) = (n+1)^3$$

By induction, we are done.

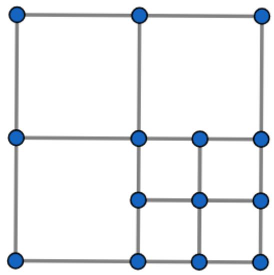
3. Prove that a square can be split into n smaller non-overlapping squares for all $n \geq 6$.

Solution: Note that if a square can be split into n pieces it can be split into $n+3$ pieces (splitting a square into four smaller squares is obviously possible).

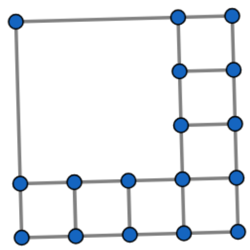
Here is the diagram for 6 squares.



Here is the diagram for 7 squares.



Here is the diagram for 8 squares.



Applying the inductive process finishes the problem.

4. Prove that $5^{2n-1}+7^{2n-1}$ is always divisible by 6 for all positive n.

Solution: Plugging in $n=1$, we see that $5^{2-1}+7^{2-1}=12$ which is obviously divisible by 6. Assuming $5^{2n-1}+7^{2n-1}$ is divisible by 6, if and only if $5^{2(n+1)-1}+7^{2(n+1)-1}$ is divisible by 6, then $5^{2(n+1)-1}+7^{2(n+1)-1}-5^{2n-1}+7^{2n-1}$ is divisible by 6. Factoring, this yields $24(5^{n-1})+48(7^{n-1})$ which is obviously divisible by 6, and we are done.

Proof By Contradiction

To prove P is true, it suffices to prove that $-P \rightarrow R$ and $-P \rightarrow -R$ for any two results $R, -R.$ This is because the truthfulness of $-P$ leads to a contradiction, implying that $-P$ is false. By the definition of an inverse, $-P$ being false implies P is true.

Some well-known facts can be proven by contradiction; for example, the proof of the irrationality of $\sqrt{2}$ or the proof of the existence of infinite primes.

The problems will be formally stated below; their solutions will follow. These problems are approachable, so give them a try before looking at the solutions. However, it is fine if these problems are not doable as first, as they are intended as examples.

Prove $\sqrt{2}$ is irrational.

Assume $\sqrt{2}$ is rational and can be expressed in simplest form as $\frac{n}{m}.$ Then $\frac{n^2}{m^2} = 2,$ implying that $n^2 = 2m^2.$ Since $2m^2$ has a factor of $2,$ n^2 must as well. Since n is integer, n must have a factor of $2.$ Have $n = 2k$ for some other integer $k.$ Substituting, we see that $4k^2 = 2m^2,$ which implies $2k^2 = m^2.$ By the same argument above, m has a factor of two. Since both n and m have a factor of two, $\frac{m}{n}$ is not in simplest form, leading to a contradiction. By contradiction, $\sqrt{2}$ is not rational, which means it must be irrational.

Prove there are infinitely many primes.

Assume there are finitely many primes. Have them be $p_1, p_2 ... p_n.$ Note that $p_1 p_2 ... p_n - 1$ is not divisible by $p_1, p_2 ... p_n$ which implies that $p_1 p_2 ... p_n - 1$ is divisible by some other prime (possibly $p_1 p_2 ... p_n - 1$) as $p_1 p_2 ... p_n - 1$ must have a prime factorization. This leads to a contradiction as we assume there are no other primes, but we see there must be other primes. By contradiction, there are not finitely many primes. Therefore, there are infinitely many primes.

A few more exercises in contradiction will be presented below.

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1. Prove $\sqrt{3}$ is irrational.

2. Prove $\sqrt[3]{4}$ is irrational.

3. Prove $\sqrt{k}$ is either irrational or integer for positive integer values of k.

4. Prove $\sqrt[n]{k}$ is either irrational or integer for positive integer values of k.

5. Prove there are infinitely many primes of the form $3n+2$.

6. Prove there are infinite primes of the form $4n+3$.

7. Prove there are infinite primes of the form $6n+5$.

1. Prove $\sqrt{3}$ is irrational.

Solution: Assume $\sqrt{3}$ is rational and can be expressed in simplest form as $\frac{m}{n}$. Then $\frac{n^2}{m^2} = 3$, implying that $n^2 = 3m^2$. Since $3m^2$ has a factor of 3, n^2 must as well. Since n is integer, n must have a factor of 3. Have $n = 3k$ for some other integer k. Substituting, we see that $9k^2 = 3m^2$, which implies $3k^2 = m^2$. By the same argument above, m has a factor of three. Since both n and m have a factor of three, $\frac{m}{n}$ is not in simplest form, leading to a contradiction. By contradiction, $\sqrt{3}$ is not rational, which means it must be irrational.

2. Prove $\sqrt[3]{4}$ is irrational.

Solution: Assume $\sqrt[3]{4}$ is rational and can be expressed in simplest form as $\frac{m}{n}$. Then $\frac{n^3}{m^3} = 4$, implying that $n^3 = 4m^3$. Since $4m^3$ has a factor of 4, n^3 must as well. Since n is integer, n must have a factor of 2. Have $n = 2k$ for some other integer k. Substituting, we see that $8k^3 = 4m^3$, which implies $2k^3 = m^3$. By a similar argument as above, m has a factor of two. Since both n and m have a factor of two, $\frac{m}{n}$ is not in simplest form, leading to a contradiction. By contradiction, $\sqrt[3]{4}$ is not rational, which means it must be irrational.

3. Prove $\sqrt{k}$ is either irrational or integer for positive integer values of k.

Solution: Have $k = ab^2$, such that a and b are integers, and that b is maximized. Note that $\sqrt{ab^2} = b\sqrt{a}$, meaning b is clearly rational. Note that for $b\sqrt{a}$ to be rational, $\sqrt{a}$ must be rational.

Assume that $\sqrt{a}$ is rational and can be expressed in simplest form as $\frac{m}{n}$. This implies $\frac{m^2}{n^2} = a$, or $m^2 = an^2$. Since the prime factors of a can only have an exponent of 1 (any factors with an exponent greater than 1 would not have b maximized), we note that $a|m^2$ implies $a|m$. Have $m = aj$ for some integer j. Substituting, we see that $a^2j^2 = an^2$, which implies $aj^2 = n^2$. By the same argument above, $a|n$. Since n, m share a common factor, $\frac{n}{m}$ is not in simplest form, and by contradiction, $\sqrt{a}$ is irrational… if and only if

a is not $1.$ If $a = 1,$ then sharing a common factor of 1 leads to no contradiction. This means that perfect square values of k are rational, and since b is integer, perfect squares are also integer. Having covered all cases, we are done.

4. Prove $\sqrt[n]{k}$ is either irrational or integer for positive integer values of $k.$

Solution: Have $k = ab^n,$ such that a and b are integers, and that b is maximized. Note that $\sqrt[n]{ab^n} = b\sqrt[n]{a},$ meaning b is clearly rational. For $b\sqrt[n]{a}$ to be rational, $\sqrt[n]{a}$ must be rational.

Assume that $\sqrt[n]{a}$ is rational and can be expressed in simplest form as $\frac{x}{y}.$ This implies $\frac{x^n}{y^n} = a,$ or $x^n = ay^n.$ Have the prime factorization of a be $p_1^{e_1}p_2^{e_2}...p_c^{e_c}.$ Note that $e_1, e_2...e_c < n,$ otherwise b is not maximized. Then note that $p_1^{e_1}p_2^{e_2}...p_c^{e_c}|y^n,$ implies $p_1p_2...p_c|y,$ as y must be integer. Have $y = p_1p_2...p_cj$ for some integer $j.$ Substituting, we see that $(p_1p_2...p_c)^nj^n = ax^n,$ or $(p_1p_2...p_c)^nj^n = p_1^{e_1}p_2^{e_2}...p_c^{e_c}x^n,$ which implies $p_1^{n-e_1}p_2^{n-e_2}...p_c^{n-e_c}a = x^n.$ Since this works for any arbitrary $e_1, e_2...e_n,$ our argument can be applied to this expression to attain $p_1p_2...p_c|x.$ Since x, y share a common factor, $\frac{x}{y}$ is not in simplest form, and by contradiction, $\sqrt[n]{a}$ is irrational… if and only if a has no prime factors. If a has no prime factors, then $a = 1,$ and sharing a common factor of 1 leads to no contradiction. This means that perfect nth power values of k are rational, and since b is integer, perfect nth powers are also integer. Having covered all cases, we are done.

5. Prove there are infinitely many primes of the form $3n + 2.$

Solution: Assume there are finitely many primes of this form. Have $p_1, p_2...p_n$ be our finitely many odd primes of form $3n + 2.$ Then note that $3p_1p_2...p_n + 2$ is not divisible by any of the primes $p_1, p_2...p_n,$ and that it is not divisible by $3.$ Note then that some odd prime must divide $3p_1p_2...p_n + 2,$ as $3p_1p_2...p_n + 2 > 1$ and is odd. We cannot have all of these primes be of the form $3k + 1,$ because multiplying numbers with a remainder of 1 yields a remainder of $1,$ and we desire a remainder of $2.$ Thus, $3k + 2|3p_1p_2...p_n + 2$ for some $k.$ This leads to a contradiction, as we assumed there were no other primes in the form of $3k + 2.$ By contradiction, there are not finitely many primes of the form $3n + 2,$ implying that there are infinitely many primes of the form $3n + 2.$

6. Prove there are infinite primes of the form $4n + 3.$

Solution: Assume there are finite primes of this form. Have them be $p_1, p_2 ... p_n$. Then note that $4p_1p_2...p_n + 3$ is not divisible by any of these primes. Since $4p_1p_2...p_n + 3 \equiv 3\ (mod\ 4)$, all of its prime factors must be congruent to 1 or 3 $(mod\ 4)$. We must have at least one prime factor congruent to $3\ (mod\ 4)$, meaning there is some prime q in the form $4n + 3$ that divides $4p_1p_2...p_n + 3$. However, we also have that no prime of the form $4n + 3$ divides $4p_1p_2...p_n + 3$, which is a contradiction.

7. Prove there are infinite primes of the form $6n + 5$.

Solution: Assume there are finite primes of this form. Have them be $p_1, p_2 ... p_n$. Then note that $6p_1p_2...p_n + 5$ is not divisible by any of these primes. Since $6p_1p_2...p_n + 5 \equiv 5\ (mod\ 6)$, all of its prime factors must be congruent to 1 or 5 $(mod\ 6)$. We must have at least one prime factor congruent to $5\ (mod\ 6)$, meaning there is some prime q in the form $6n + 5$ that divides $6p_1p_2...p_n + 5$. However, we also have that no prime of the form $6n + 5$ divides $6p_1p_2...p_n + 5$, which is a contradiction.

Playing with Circles

Circles and Angles

Let us begin by defining a chord, secant, tangent, and arc.

A chord is a line segment formed by two distinct points on a circle.

A secant is a line that intersects a circle twice.

A tangent is a line that intersects a circle once.

The measure of $arc(AB)$ of circle with center I is the measure of $\angle AIB.$ Unless specified, this means the minor arc, or the smaller arc.

With the following in mind, we may start looking at a few properties of circles, chords, secants, tangents, and arcs.

The Inscribed Angle Theorem (1.1)

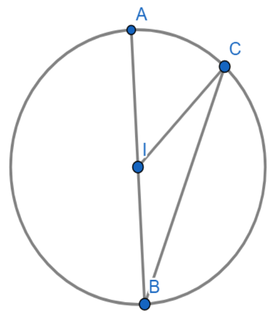
The Inscribed Angle Theorem states that given two chords AB and $BC,$ the measure of $\angle ABC$ is half of $arc(AC).$ In other words, $\angle ABC = \frac{1}{2}arc(AC).$

This is the foundation for which all of our other properties will be built upon. As thus, to fully understand our other properties, we must first understand the inscribed angle theorem.

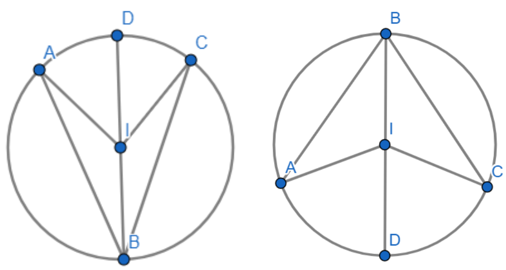
Theorem 1.1's Proof

We shall have three Cases as following.

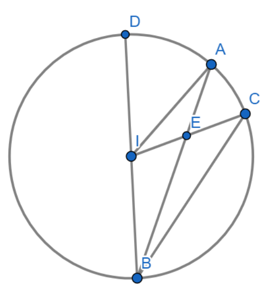
Let Case 1 be the case such that $\angle ABC$ and $\angle AIC$ intersect only at points $A, C.$ If we draw $CI,$ we see that BIC is an isosceles triangle because BI and BC are both radii. This implies $180^\circ - \angle AIC + 2\angle ABC = 180^\circ,$ or $\angle ABC = \frac{1}{2}\angle AIC,$ which completes Case 1.



Let Case 2 be the case such that $\angle AIC$ intersects $\angle ABC$ at points A, C. Let us then construct diameter BD. This implies that $\angle AIB + \angle CIB = \angle AIC$ or $\angle AID + CID = \angle AIC$, depending on the value of $\angle AIB + \angle CIB$. (Without loss of generality, let us use the $\angle AIB + \angle CIB$ case.) This also implies that $\angle ABI + \angle CBI = \angle ABC$. We may apply Case 1 to find that $\angle ABD = \frac{1}{2}\angle AID$ and that $\angle CBD = \frac{1}{2}\angle CID$. Summing these two cases up gives us $\angle ABD + \angle CBD = \frac{1}{2}\angle AID + \frac{1}{2}\angle CID$, and substituting yields $\angle ABC = \frac{1}{2}\angle AIC$ as desired.



Let Case 3 be the case such that $\angle AIC$ and $\angle ABC$ intersect at A, C, and some other point E. After constructing diameter BD, note that by Case 1, $\angle ABD = \frac{1}{2}\angle AID$, and $\frac{1}{2}\angle AID + \frac{1}{2}\angle AEI = \angle ABD + \angle CBE$. Substituting yields $\frac{1}{2}\angle AID + \frac{1}{2}\angle AEI = \frac{1}{2}\angle AID + \angle CBE$, and rearranging yields $\frac{1}{2}\angle AID = \angle CBE$, as desired.



Make sure you understand this proof. Observe that Case 1 is the main idea of this entire proof. A suggested exercise for any confused reader is to mark the angles and rewrite the proof in their own words; that is how the I initially understood the proof.

This idea is the basis for all of the properties below in this chapter -- almost everything else will be proved using subtended arcs and inscribed angles.

Angle of Intersecting Secants/Tangents Theorem (1.2)

Have points A, C be on a circle. The Angle of Intersecting Secants/Tangents Theorem states that given two secants or tangents AB and BC that intersect the circle at points the D, E, $\angle ABC = \frac{1}{2}arc(AC) - \frac{1}{2}arc(DE)$.

Angle of Intersecting Chords Theorem (1.3)

The Angle of Intersecting Chords Theorem states that given two chords AB and CD that intersect inside the circle at E, that $\angle AEC = \frac{1}{2}arc(AC) + \frac{1}{2}arc(BD)$.

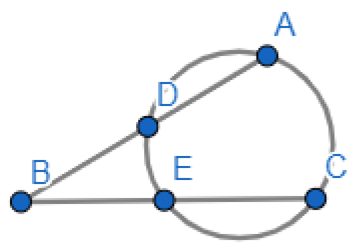
Angle of A Tangent and Chord Theorem (1.4)

Consider chord AX of a circle and consider the tangent of the circle passing through X. Then let B be a point on the tangent through X such that $\angle AXB$ is acute. Then, $\angle AXB = \frac{1}{2}arc(AX)$.

Try to prove these on your own. Note that we proved Theorem 1.1 using straight angles and the fact that a triangle's angles sum up to 180 degrees. Draw some angles that subtend some arcs, and you'll find the solution. The proofs will be below for you to check your work, or if you are completely lost, they will be here to catch you up.

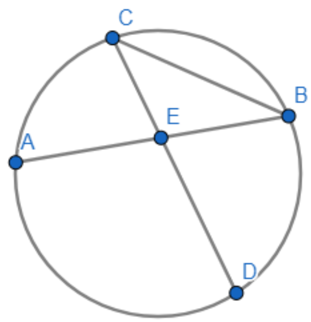
Theorem 1.2's Proof

Draw line AE. Then note that $\angle AEC = \frac{1}{2}arc(AC)$, which implies $\angle AEB = 180^{\circ} - \frac{1}{2}arc(AC)$, due to straight angles. Then note that $\angle BAE = \frac{1}{2}arc(DE)$, and because ABE is a triangle, $\angle AEB + \angle BAE + \angle ABE = 180^{\circ}$. Substituting yields $180^{\circ} = 180^{\circ} - \frac{1}{2}arc(AC) + \frac{1}{2}arc(DE) + \angle ABE$, implying that $\angle ABE = \frac{1}{2}arc(AC) + \frac{1}{2}arc(DE)$. Since $\angle ABE$ is the same as $\angle ABC$, we are done.



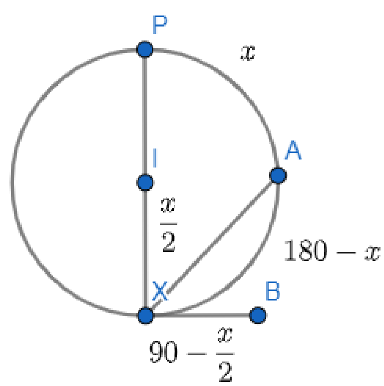
Theorem 1.3's Proof

Note that $\angle AEC = 180 - \angle BEC$. Then note that $\angle BEC + \angle CBE + \angle BCE = 180^\circ$. Since $\angle BCE = \angle BCD$ and $\angle BCD = \frac{1}{2}arc(BD)$, $\angle BCE = \frac{1}{2}arc(BD)$. Then note that since $\angle CBE = \angle CBA$ and $\angle CBA = \frac{1}{2}arc(AC)$, $\angle CBE = \frac{1}{2}arc(BD)$. This implies that $\angle BEC = 180 - \frac{1}{2}arc(AC) - \frac{1}{2}arc(BD)$, and that $\angle AEC = 180 - (180 - \frac{1}{2}arc(AC) - \frac{1}{2}arc(BD))$. Simplifying, we see that $\angle AEC = \frac{1}{2}arc(AC) + \frac{1}{2}arc(BD)$, as desired.



Theorem 1.4's Proof

Let XP be a diameter of the circle. It is a property of tangents that $\angle PXB = 90^\circ$. Denote $arc(PA)$ as x. Then by the Inscribed Angle Theorem (1.1), $\angle PXA = \frac{1}{2}arc(PA) = \frac{x}{2}$. It is common sense that $arc(AX) = 180^\circ - arc(PA) = 180^\circ - x$, and it is also common sense that $\angle AXB = \angle PXB - \angle PXA = 90^\circ - \frac{x}{2}$. This implies $\angle AXB = \frac{1}{2}arc(PA)$, as desired.



I implied earlier that the proofs were based on triangles. Where we have angles, we look for triangles. If you do not understand these proofs, do not worry; that is to be expected. Don't sit around and hope for it to come to you; draw circles and points inside or outside of the circle. Look for angles; chances are, the first thing you will do is stumble upon the proofs for these identities. If not, you may have made some new mathematical problems; that is not a loss either.

Now, we will move on to cyclic quadrilaterals, which are quadrilaterals whose vertices all lie on a single circle. Let's look at some properties of convex cyclic quadrilaterals.

Opposite Angles of Cyclic Quadrilaterals Theorem (2.1)
Given cyclic quadrilateral $ABCD$, $\angle A + \angle C = \angle B + \angle D = 180^\circ$.

Diagonal Angles of Cyclic Quadrilaterals Theorem (2.2)
Given cyclic quadrilateral $ABCD$, the following four properties are true.

$$\angle ABD = \angle ACD$$
$$\angle BCA = \angle BDA$$
$$\angle BAC = \angle BDC$$
$$\angle CAD = \angle CBD$$

Theorem 2.1's Proof
Note that the arcs $\angle A$ and $\angle C$ combine to form a circle. This means that the angles add to $\frac{1}{2} \cdot 360 = 180.$ The same can be done for $\angle B$ and $\angle D.$

Theorem 2.2's Proof
Note that $\angle ABD = \angle ACD$ because they subtend the same angle. The same can be done for the other three properties.

Note that these cyclic quadrilateral identities are based on Theorem 1.1. After you fully understand these identities, here are a few problems involving circles, arcs, and quadrilaterals. The solutions for them will be below the problems.

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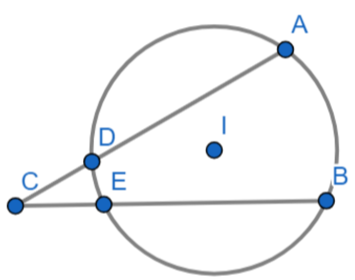
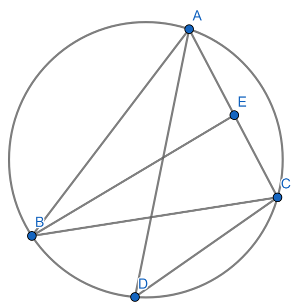
1. If $\angle ABC = 60^\circ$ and $\angle CAB = 70^\circ$, find $arc(AB) - arc(DE).$



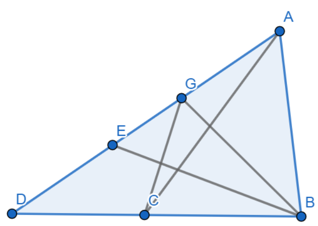
2. Given that A, B, C, and D are all on the circumference of the same circle, that BE is the angle bisector of $\angle ABC$, that $\angle AEB = \angle CEB$, and that $\angle ADC = 50^\circ$, find $\angle BAC$.



3. Given points A, B, C, D, E such that BE is the angle bisector of $\angle ABC$, $\angle AEB = \angle CEB$, $\angle BAC + \angle BDC = \angle ABD + \angle ACD$, and $\angle ADC = 48^\circ$, find $\angle BCA$.

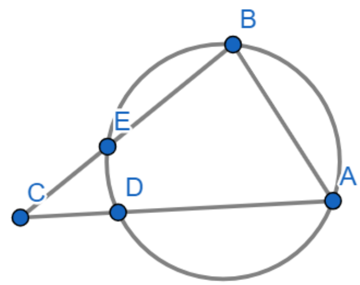
4. Consider any cyclic pentagon (a pentagon that can be inscribed within a circle) $ABCDE$. Then prove that, no matter what, $ABCP$ is not cyclic, where P is the center of the circle.

5. Given that $m\measuredangle BAC = m\measuredangle BGC = 40^\circ$, $m\measuredangle ABG = 80^\circ$, $m\measuredangle GEB = 2m\measuredangle DBE$, and $m\measuredangle DBE = m\measuredangle GBE$, find $m\measuredangle ADB$.



6. Consider $\triangle ABC$ with point D on BC. Let M, N be the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

1. If $\angle ABC = 60^\circ$ and $\angle CAB = 70^\circ$, find $arc(AB) - arc(DE)$.



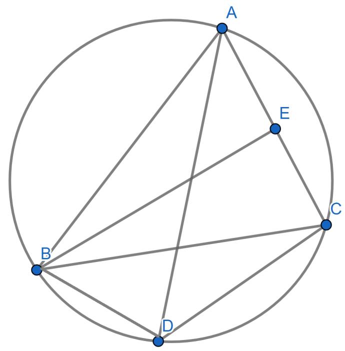
Solution: Since the angles of a triangle have a total measure of 180°, $\angle C = 50^\circ$. Then, by the Angle of Intersecting Secants/Tangents Theorem (1.2), $\angle C = 50^\circ = \frac{1}{2}arc(AB) - \frac{1}{2}arc(DE)$. As thus, $100^\circ = arc(AB) - arc(DE)$.

2. Given that A, B, C, and D are all on the circumference of the same circle, that BE is the angle bisector of $\angle ABC$, that $\angle AEB = \angle CEB$, and that $\angle ADC = 50^\circ$, find $\angle BAC$.

Solution: Since $\angle ADC$ and $\angle ABC$ subtend the same arc, they have the same degree measure, implying $\angle ADC = \angle ABC = 50^\circ$. Note that $\angle AEB = \angle CEB$ implies $\angle AEB = \angle CEB = 90^\circ$ because $\angle AEB$ and $\angle CEB$ form a straight angle. Since BE is an angle bisector, this also implies $\angle BEA = 25^\circ$. Since a triangle has 180 total degrees, $\angle AEB + \angle EBA + \angle BAC = 90^\circ + 25^\circ + \angle BAC = 180^\circ$, implying $\angle BAC = 65^\circ$, and we are done.

3. Given points A, B, C, D, E such that BE is the angle bisector of $\angle ABC$, $\angle AEB = \angle CEB$, $\angle BAC + \angle BDC = \angle ABD + \angle ACD$, and $\angle ADC = 48^\circ$, find $\angle BCA$.

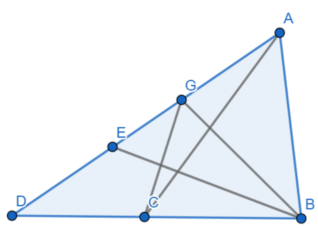
Solution: Note that $\angle BAC + \angle BAD = \angle ABD + \angle ACD$ and $\angle BAC + \angle BDC + \angle ABD + \angle ACD = 360^\circ$, which is true because a quadrilateral has angles with a total measure of 360 degrees. This implies that $\angle BAC + \angle BDC = \angle ABD + \angle ACD = 180^\circ$, which means $ABDC$ is cyclic. Then note that since $\angle ADC$ and $\angle ABC$ subtend the same arc, they have the same degree measure, implying $\angle ADC = \angle ABC = 48^\circ$. We are also given that $\angle AEB + \angle CEB = 180^\circ$ and $\angle AEB = \angle CEB$, which implies $\angle AEB + \angle CEB = 90^\circ$, or that BE is an altitude. Since BE is an altitude and angle bisector, triangle ABC is isosceles with $\overline{BA} = \overline{BC}$. Therefore, $\angle BCA = \angle BAC = 66^\circ$, which is our answer.



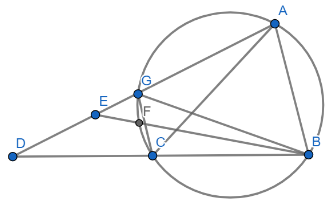
4. Consider any cyclic pentagon (a pentagon that can be inscribed within a circle) $ABCDE$. Then prove that, no matter what, $ABCP$ is not cyclic, where P is the center of the circle.

Solution: Assume that there is such a point P that $ABCP$ is cyclic. Note that by the Inscribed Angle Theorem (1.1), $\angle ABC = \frac{1}{2}arc(AEDC)$ and $\angle APC = arc(AC)$. Since $arc(AEDC) + arc(AC) = 360$ because the two arcs make a circle, this implies $\angle ABC + \angle APC = 180^\circ + \frac{1}{2}arc(AC)$. To prevent $ABCP$ from being a degenerate quadrilateral, $arc(AC) > 0^\circ$, so $180^\circ + \frac{1}{2}arc(AC) > 180^\circ$. However, since $ABCP$ is cyclic, $\angle ABC + \angle APC = 180^\circ$, leading to a contradiction.

5. Given that $m\measuredangle BAC = m\measuredangle BGC = 40^\circ$, $m\measuredangle ABG = 80^\circ$, $m\measuredangle GEB = 2m\measuredangle DBE$, and $m\measuredangle DBE = m\measuredangle GBE$, find $m\measuredangle ADB$.

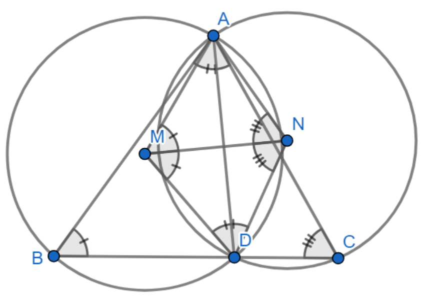


Solution: Because $\angle BAC = \angle BGC$, $ABCG$ is a cyclic quadrilateral. Let EB intersect the circumcircle of $ABCG$ again at F Note that since $\angle ABG = 80^\circ$ and $\angle BAC = 40^\circ$, $arc(AG) = 160^\circ$ and $arc(BC) = 80^\circ$, so $arc(AB) + arc(GF) + arc(FC) = 120^\circ$. Let $arc(AB) = a$, $arc(GF) = b$, and $arc(FC) = c$. Note that $\angle GEB = 2\angle DBE$ implies $a - b = 2c$ by the Angle of Intersecting Secants/Tangents Theorem (1.2) and the Inscribed Angle Theorem (1.1), respectively. Similarly, $b = c$, and $arc(AB) + arc(GF) + arc(FC) = 120^\circ$ implies $a + b + c = 120^\circ$. Solving, we see that $a = 72^\circ$, $b = 24^\circ$, and $c = 24^\circ$. By the Angle of Intersecting Secants/Tangents Theorem (1.2), $\angle ADB = \frac{1}{2}(a - b - c) = 12^\circ$, which is our answer.



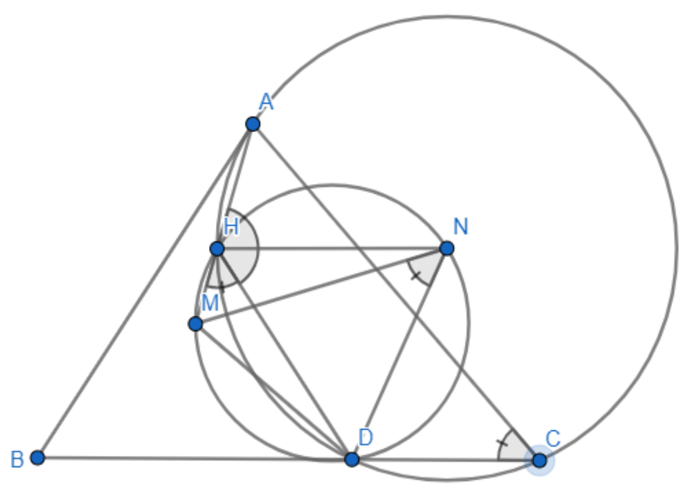
6. Consider $\triangle ABC$ with point D on BC. Let M, N be the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

Solution: Note that by the definition of a circumcenter, $\overline{AM} = \overline{DM}$, and since MN is a perpendicular bisector of isosceles $\triangle AMD$, it follows MN bisects $\angle AMD$, implying $\angle AMN = \angle DMN$. Similarly, $\angle ANM = \angle DNM$. Thus, by the Inscribed Angle Theorem (1.1), $\angle ABD = \frac{1}{2}\angle AMD = \angle AMN$, implying $\triangle ABC \sim \triangle AMN$ and $\triangle AMN \cong \triangle DMN$.

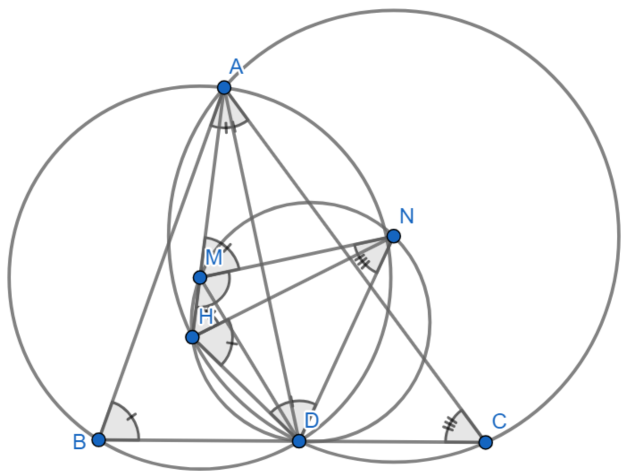


By the definition of H, either $HMDN$ or $MHDN$ are cyclic.

Note that in the first case, $\angle AHD + \angle ACD = 180^\circ$ and $\angle DNM = \angle DHM$ by the Inscribed Angle Theorem (1.1). Then we note that since $\triangle DMN \sim \triangle ABC$ as we have proven before, $\angle DNM = \angle DCA$. Substituting yields $\angle AHD + \angle DHM = 180^\circ$, as desired.



In the second case, $\angle HMN + \angle HDN = 180^\circ$. Since $NMHD$ is cyclic, $\angle MHD + \angle DNM = 180^\circ$, which can also be expressed as $\angle MHN + \angle NHD + \angle DNM$. By the Inscribed Angle Theorem (1.1), we note that $\angle MHN = \angle MDN$, and since $\triangle DMN \sim \triangle ABC$, $\angle MDN = \angle BAC$ and $\angle MND = \angle ACB$. This implies that $\angle BAC + \angle NHD + \angle ACB = 180^\circ$, or that $\angle NHD = \angle ABC$. Then we note that $\overline{HN} = \overline{HD}$ as they are both radii of the circumcircle of $\triangle ACD$, implying that $\angle NHD = \angle NDH$ since $\triangle HND$ is isosceles. Since $\triangle ABC \sim \triangle AMN$, $\angle AMN = \angle ABC = \angle NHD = \angle NDH$, implying that $\angle HMN + \angle AMN = \angle HMN + \angle HDN = 180^\circ$, proving that A, M, H are collinear, as desired.



Circles and Lines

We have already defined what a chord, secant, and tangent in the first section of this chapter. We shall now define a special chord, known as the diameter.

A diameter of a circle is a chord that passes through the center of the circle.

An important property of a diameter is that it is the longest chord of a circle. We shall formalize and prove this statement when we have the tools to do so. For now, let's prove a few prerequisites.

Diameter Through A Point Theorem (3.1)

Given a point K inside of a circle, there is one unique diameter that goes through $K.$

Try to prove this on your own. Remember that one unique line passes through two points; Theorem 3.1 should not be hard to prove if you know how to utilize this axiom.

Theorem 3.1's Proof

Note that the point K and our radius, which we will have as $R,$ forms a unique line because two points form a unique lines. The portion of KR that is contained by the circle will form the diameter.

Another prerequisite for this is the Power of a Point Theorem. This theorem has three parts, which we will state separately. Unlike Theorem 3.1, this is useful in many situations. Make sure you understand what these theorems are stating, and how to prove them. If you want to prove this on your own, try drawing extra lines to make similar triangles.

Power of a Point With Chords Theorem (3.2.1)

Consider chords AB and CD that intersect at $E.$ Then, $\overline{AE} \cdot \overline{BE} = \overline{CE} \cdot \overline{DE}.$

Power of a Point With a Tangent And a Secant Theorem (3.2.2)

Consider tangent line AB such that A is on the circle and secant BD that intersects the circle again at point $C,$ with D being on the circle. Then, $\overline{AB}^2 = \overline{BC} \cdot \overline{BD}.$

Power of a Point With Two Secants Theorem (3.2.3)

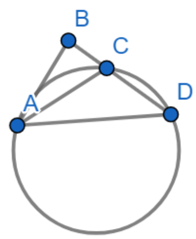
Consider secants AC and CE where points A, C are on the circle. Have them intersect the circle again at B and $D,$ respectively. Then, $\overline{CB} \cdot \overline{CA} = \overline{CD} \cdot \overline{CE}.$

Theorem 3.2.1's Proof

Drawing AD and BC gives us that $\triangle ABC \sim \triangle CBE$, which implies $\overline{DE} = x\overline{BE}$ and $\overline{AE} = x\overline{CE}$. Substituting, we see that $\overline{AE} \cdot \overline{BE} = \overline{CE} \cdot \overline{DE}$ implies $x\overline{CE} \cdot \overline{BE} = \overline{CE} \cdot x\overline{BE}$, which is obviously true.

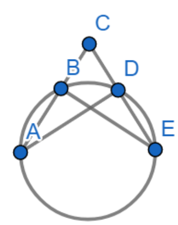
Theorem 3.2.2's Proof

Drawing AC and AD, we note that $\triangle ABC \sim \triangle DBA$. This implies $\overline{AB} = x\overline{DB}$ and $\overline{BC} = x\overline{AB}$. Substituting, we see that $\overline{AB}^2 = \overline{BC} \cdot \overline{BD}$ implies $\overline{AB} \cdot x\overline{DB} = x\overline{AB} \cdot \overline{DB}$, which is obviously true.



Theorem 3.2.3's Proof

Drawing AD and BE, we see that $\triangle ACD \sim \triangle ECB$. This implies $\overline{AC} = x\overline{EC}$ and $\overline{CD} = x\overline{CB}$. Then, substituting this into $\overline{CB} \cdot \overline{CA} = \overline{CD} \cdot \overline{CE}$, we see this implies $\overline{CB} \cdot x\overline{EC} = x\overline{CB} \cdot \overline{EC}$, which is obviously true.



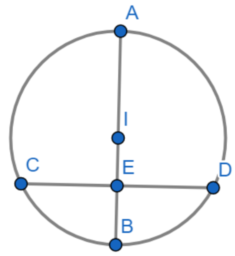
If you are wondering why we know these triangles are similar, think about the Inscribed Angle (1.1) Theorem and everything based off it. With these tools, we may now prove that the longest chord of a circle is the diameter. Let us formalize and prove this.

Longest Chord Theorem (3.3)

The longest chord of a circle is the diameter.

Theorem 3.3's Proof

Let us assume that there is some other chord DE longer than the diameter whose midpoint is C. Then, we may uniquely construct a diameter AB that passes through point C, and note that by power of a point, $\overline{AC} \cdot \overline{BC} = \overline{DC} \cdot \overline{EC}$. Have $\overline{AC} = x$, $\overline{BC} = y$, and $\overline{DC} = \overline{EC} = n$. Then note by AM-GM, $\frac{x+y}{2} \geq \sqrt{xy}$. Based on our results from power of a point, $\frac{x+y}{2} \geq n$. If the radius is r, this implies $r \geq n$. The equality condition implies that $x = y$, which is only possible when DE passes through the center. This is a contradiction, so the diameters of a circle alone are the longest chords of a circle.



The Power of a Point Theorem in particular is fairly powerful. Here are a few problems involving power of a point.

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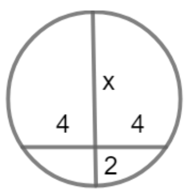
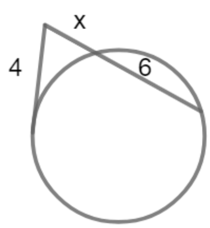
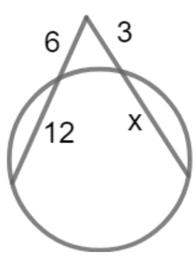
1. Find x.



2. Find x.

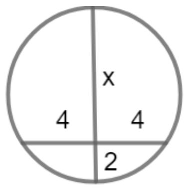


3. Find x.



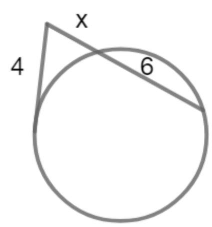
4. Consider chord AB of length 8 inside a circle of radius $5.$ Prove that only one line DE has a length of 2 such that D is on the arc AB and E is on the line AB.

1. Find $x.$



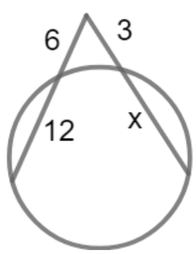
Solution: By Power of a Point, $4 \cdot 4 = x \cdot 2.$ This implies $x = 8.$

2. Find $x.$



Solution: By Power of a Point, $4 \cdot 4 = x \cdot (x + 6).$ This implies $x = 2.$

3. Find $x.$



Solution: By Power of a Point, $6 \cdot (6 + 12) = 3 \cdot (3 + x).$ This implies $x = 33.$

4. Consider chord AB of length 8 inside a circle of radius $5.$ Prove that only one line DE has a length of 2 such that D is on the arc AB and E is on the line $AB.$

Solution: Let the diameter be $FD.$ By Power of a Point, $\overline{AE} \cdot \overline{BE} = \overline{DF} \cdot \overline{DE}.$ Note that $\overline{DF} = 10 - \overline{DE} = 8,$ so $\overline{AE} \cdot \overline{BE} = \overline{DF} \cdot \overline{DE} = 16.$ Since $\overline{AE} + \overline{BE} = 8,$ $\overline{AE} = \overline{BE} = 4.$ This means there is a unique point E that FD must pass through, and by Theorem 3.1, FD is a unique diameter, meaning DE is unique, and we are done.

Now, we shall consider a special property of tangents, which is known as the Two Tangents Theorem. But first, we will need to prove that a tangent line must be perpendicular to the radius that intersects it.

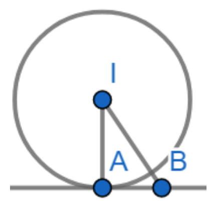
Radius-Tangent Perpendicularity Theorem (3.4)

Have a circle with center I. Prove that any line tangent to the circle is perpendicular to the radius that intersects it.

Try to notice a contradiction that occurs if this is false.

Theorem 3.4's Proof

We assume there is some circle and some tangent such that this is not true. Have the foot of the perpendicular from the center to the tangent be A, and have the tangent point be B. Assume A and B are different points. Since $\overline{IB} < \overline{IA}$, $\angle IBA > 90^\circ$. But then $\triangle IAB$ has angles that have a total degree measure of more than 180°, which is a contradiction. Therefore, A and B must be the same point, and we are done.



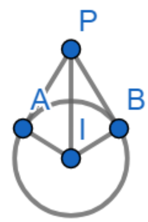
Two Tangent Theorem (3.5)

Consider tangents PA and PB such that P is not on the circle and A, B are points on the circle. Then, $\overline{PA} = \overline{PB}$.

Try to use triangle congruence conditions to prove this yourself. Ask yourself why we proved Theorem 3.4 earlier.

Theorem 3.5's Proof

Have the center be I. Then note that by Theorem 3.4, $\angle PAI = \angle PBI = 90^\circ$. By the Pythagorean Theorem, $\overline{PA} = \sqrt{\overline{PI}^2 + \overline{AI}^2}$ and $\overline{PB} = \sqrt{\overline{PI}^2 + \overline{BI}^2}$. Since $\overline{AI} = \overline{BI}$, $\overline{PA} = \overline{PB}$, and we are done.

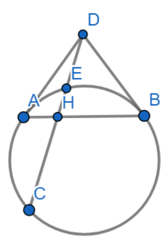


Below are a few more problems involving all of the concepts discussed above; that includes Power of a Point as well as the Two Tangent Theorem, which is really just a special case of Power of a Point.

---

1. Consider points A, B, I such that $\overline{AI} = \overline{BI}$. Given a point X such that $\angle IAX = \angle IBX = 90^\circ$, find $\overline{AX} - \overline{BX}$.

2. Given that $\overline{AD} = 4$, $\overline{DC} = 8$, $\overline{AH} = 1$, and $\overline{EH} = 1$, find the area of $\triangle ABD$.



3. Consider $\triangle ABC$ with inradius r such that $\overline{AB} = 9$, $\overline{BC} = 12$, and $\overline{AC} = \overline{AB} + \overline{BC} - 2r$. Find $[ABC]$.

4. Consider $\overline{AB} = x$ and circle N centered at B with radius r such that $r < x$. Find the length of the tangent from A to N.

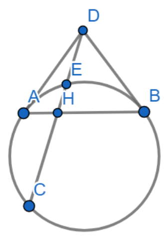
5. Consider a circle centered at O and chord AB. Let P be a point on segment AB such that $\overline{AP} = 2$ and $\overline{BP} = 8$. If $\angle APO = 150^\circ$, what is the area of the circle?

6. Consider $\triangle ABC$ such that $\angle C = 90^\circ$. Let P be the foot of the altitude from C to AB, and let X and Y be the feet of the altitudes from P to AC and BC respectively. Prove that $AXYB$ is cyclic.

1. Consider points A, B, I such that $\overline{AI} = \overline{BI}$. Given a point X such that $\angle IAX = \angle IBX = 90^\circ$, find $\overline{AX} - \overline{BX}$.

Solution: Draw a circle with center I and radii AI and BI. Theorem 3.4 implies that AX and BX are tangent to the circle. The Two Tangent Theorem then implies $\overline{AX} = \overline{BX}$, so $\overline{AX} - \overline{BX} = 0$.

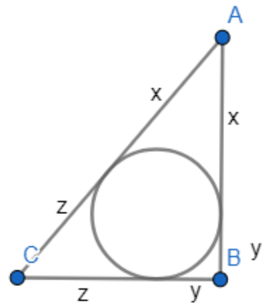
2. Given that $\overline{AD} = 4$, $\overline{DC} = 8$, $\overline{AH} = 1$, and $\overline{EH} = 1$, find the area of $\triangle ABD$.



Solution: By The Two Tangents Theorem (3.5), $\overline{AD} = \overline{BD} = 4$. Power of a Point (3.2.2) implies $\overline{AD}^2 = \overline{DE} \cdot \overline{DC}$, or $16 = \overline{DE} \cdot \overline{DC}$. Since $\overline{DE} + \overline{DC} = 8$, $\overline{DE} = 2$ and $\overline{EC} = 6$. Then note that by Power of a Point (3.2.1), $\overline{EH} \cdot \overline{HC} = \overline{AH} \cdot \overline{HB}$. This implies $1 \cdot 5 = 1 \cdot \overline{HB}$, or $\overline{HB} = 6$. Then, we have a triangle with side lengths $4, 4, 6$. By the Pythagorean Theorem, the altitude to side AB is $\sqrt{4^2 - 3^2} = \sqrt{7}$, meaning that the area of $\triangle ABD$ is $3\sqrt{7}$.

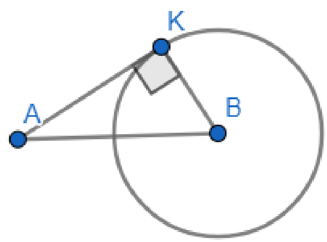
3. Consider $\triangle ABC$ with inradius r such that $\overline{AB} = 9$, $\overline{BC} = 12$, and $\overline{AC} = \overline{AB} + \overline{BC} - 2r$. Find $[ABC]$.

Solution: By the Two Tangent Theorem, $x + z = x + 2y + z - 2r$, implying $y = r$. This implies that since drawing radii perpendicular to the sides makes a square, $\angle ABC = 90^\circ$, implying $[ABC] = \frac{9 \cdot 12}{2} = 54$.



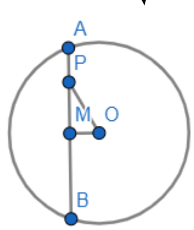
4. Consider $\overline{AB} = x$ and circle N centered at B with radius r such that $r < x$. If line AK intersects N exactly once, find $\overline{AK}$ in terms of x, r.

Solution: By Theorem 3.4, $\angle AKB = 90^\circ$. Note that $\overline{KB} = r$, $\overline{AB} = x$, and by the Pythagorean Theorem, $\overline{AK} = \sqrt{\overline{AK}^2 - \overline{KB}^2} = \sqrt{x^2 - r^2}$.



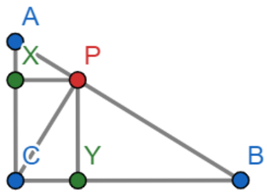
5. Consider a circle centered at O and chord AB. Let P be a point on segment AB such that $\overline{AP} = 2$ and $\overline{BP} = 8$. If $\angle APO = 150^\circ$, what is the area of the circle?

Solution: Let M be the midpoint of AB. Notice that by definition, $\overline{PM} = 3$ and $\angle MPO = 30^\circ$. Then as $\triangle POM$ is a $30 - 60 - 90$ triangle, $\overline{MO} = \sqrt{3}$. Since $r = \overline{AO} = \sqrt{\overline{AM}^2 + \overline{MO}^2} = \sqrt{25 + 3} = \sqrt{28}$, the area of the circle is 28π.



6. Consider $\triangle ABC$ such that $\angle C = 90^\circ$. Let P be the foot of the altitude from C to AB, and let X and Y be the feet of the altitudes from P to AC and BC respectively. Prove that $AXYB$ is cyclic.

Solution: By Power of a Point (3.2), if $\overline{CX} \cdot \overline{CA} = \overline{CY} \cdot \overline{YB}$, then $AXYB$ is cyclic. Notice that $\triangle ACP \sim \triangle PCX$. Thus $\frac{\overline{CP}}{\overline{AC}} = \frac{\overline{CX}}{\overline{PC}}$, implying $\overline{CX} = \frac{\overline{PC}^2}{\overline{CA}}$. Similarly, $\overline{CY} = \frac{\overline{PC}^2}{\overline{CB}}$. Thus, $\overline{CX} \cdot \overline{CA} = \overline{PC}^2 = \overline{CY} \cdot \overline{YB}$, as desired.



Radical Axes

With the Power of a Point Theorem comes a natural extension known as radical axes.

First, we define the *power* of point P with respect to circle ω centered at O with radius r as $\overline{PO}^2 - r^2$. We denote this as $\pi(P,\omega) = \overline{PO}^2 - r^2$.

---

1. If circle ω with center O has radius 3 and $\overline{PO} = 5$, find $\pi(P,\omega)$.

2. Consider a point P with power 36 respective to a circle with center O. If $\overline{PO} = 10$, find the radius of the circle.

3. When is the power of a point negative, zero, or positive?

4. For P outside of ω, prove that $\pi(P,\omega)$ is the square of the length of the tangent from P to ω.

---

1. If circle ω with center O has radius 3 and $\overline{PO} = 5,$ find $\pi(P, \omega).$

Solution: By definition, $\pi(P, \omega) = 5^2 - 3^2 = 16.$

2. Consider a point P with power 36 respective to a circle with center $O.$ If $\overline{PO} = 10,$ find the radius of the circle.

Solution: By definition, $\pi(P, \omega) = 10^2 - r^2.$ But it is given that $\pi(P, \omega) = 36,$ so $10^2 - r^2 = 36 \rightarrow r = 8.$

3. When is the power of a point negative, zero, or positive?

Solution: When $\overline{PO} < r$ (i.e. when P is in the interior of the circle), $\pi(P, \omega) < 0.$ When $\overline{PO} = r$ (i.e. P is on the circle), $\pi(P, \omega) = 0.$ When $\overline{PO} > r$ (i.e. P is outside the circle), $\pi(P, \omega) > 0.$

4. For P outside of $\omega,$ prove that $\pi(P, \omega)$ is the square of the length of the tangent from P to $\omega.$

Solution: Let ω have a center of O and let the tangent from P to ω intersect ω at $T.$ Notice by Theorem 3.4, $\overline{PT}^2 = \overline{PO}^2 - \overline{TO}^2.$ But $\pi(P, \omega) = \overline{PO}^2 - \overline{TO}^2,$ so equality holds.

Now, we'll prove some theorems that hold for powers of points with respect to multiple circles.

First, we define a *radical axis* of two circles ω_1, ω_2 as the locus of points P such that $\pi(P, \omega_1) = \pi(P, \omega_2).$ Notice that not every circle has one; any two concentric but not congruent circles will not have a radical axis. We will ignore these exceptions, however.

Radical Axis Theorem (4.1)

The radical axis of circles ω_1, ω_2 with centers O_1, O_2 is a line perpendicular to $O_1O_2.$

Radical Axis Theorem's Proof

Lemma: For any $P,$ let the foot of the perpendicular from P to O_1O_2 be $Q.$ Then

$$\pi(P, \omega_1) - \pi(Q, \omega_1) = \pi(P, \omega_2) - \pi(Q, \omega_2).$$

Proof: Let ω_1, ω_2 have radii of r_1, r_2. Then by the definition of the power of a point,
$\pi(P, \omega_1) - \pi(Q, \omega_1) = \overline{PO_1}^2 - r_1^2 - (\overline{QO_1}^2 - r_1^2) = \overline{PO_1}^2 - \overline{QO_1}^2 = \overline{PQ}^2$. Similarly,
$\pi(P, \omega_2) - \pi(Q, \omega_2) = \overline{PO_2}^2 - r_2^2 - (\overline{QO_2}^2 - r_2^2) = \overline{PO_2}^2 - \overline{QO_2}^2 = \overline{PQ}^2$.

Lemma: There is exactly one Q on segment O_1O_2 such that $\pi(Q, \omega_1) = \pi(Q, \omega_2)$.

Proof: Notice that the "graphs" of $\pi(Q, \omega_1)$ and $\pi(Q, \omega_2)$ are continuous and are monotonic. If we start from O_1 and slide to O_2, the graph of $\pi(Q, \omega_1)$ will go from 0 to x for some positive x, and the graph of $\pi(Q, \omega_2)$ will go from y to 0 for some positive y, so they must intersect.

Then the combination of these two lemmas finishes the proof.

Radical Center Theorem (4.2)

The pairwise radical axes of $\omega_1, \omega_2, \omega_3$ are concurrent.

Try to prove this yourself, as the proof is fairly easy.

Theorem 4.2's Proof

Let the radical axes of ω_1, ω_2 and ω_2, ω_3 intersect at P. Then $\pi(P, \omega_1) = \pi(P, \omega_2)$ and $\pi(P, \omega_2) = \pi(P, \omega_3)$, so $\pi(P, \omega_1) = \pi(P, \omega_3)$. Thus, P lies on the radical axis of ω_1, ω_3.

Radical Axis of Intersecting Circles (4.3)

Given circles ω_1, ω_2 that intersect at X, Y, their radical axis is XY.

Radical Axis of Intersecting Circles Proof

Notice that $\pi(X, \omega_1) = \pi(X, \omega_2) = 0, \pi(Y, \omega_1) = \pi(Y, \omega_2) = 0$. Then the Radical Axis Theorem (4.1) finishes the proof.

Radical Axis of Tangent Circles (4.4)

Given circles ω_1, ω_2 that are tangent at P, their radical axis is the common tangent to both circles at P.

Radical Axis of Tangent Circles Proof

Let the centers of ω_1, ω_2 be O_1, O_2. Notice that since O_1, O_2, P are collinear, the radical axis is the line perpendicular to O_1O_2 through P by the Radical Axis Theorem (4.1), which is the tangent through P.

1. Consider two circles ω_1, ω_2 that intersect at X, Y. Let AB be a line segment tangent to ω_1, ω_2 with A, B on ω_1, ω_2, respectively. Prove that XY bisects AB.

2. Consider four concyclic points A, B, C, D. Then let AD, BD, CD intersect BC, CA, AB at X, Y, Z, respectively. Let the circumcircle of $\triangle ABC$ have center O and let the circumcircle of $\triangle XYZ$ have center I. Then let these two circles intersect at M, N, and let BC intersect MN at J. Then draw the two tangents from J to the two circumcircles that only intersect one circle. Let the point of tangency to the circumcircle of $\triangle ABC$ be K and let the point of tangency to the circumcircle of $\triangle XYZ$ be L. Then let the circumcenter of $\triangle JKL$ be H, and let the circumcircle of $\triangle JKL$ intersect the circumcircles of $\triangle ABC$, $\triangle XYZ$ at E, F, respectively. Then let the midpoint of EK be Q and the midpoint of FL be R. Prove H, O, Q and H, I, R are collinear.

(Assume that all mentioned points exist.)

3. Consider two circles ω_1, ω_2 that intersect at X, Y. Then pick M on XY such that M is not in either circle. Let A, B be on ω_1, ω_2 such that AM, BM are tangent to ω_1, ω_2, respectively. Prove $AM = BM$.

4. Consider two externally tangent circles ω_1, ω_2. Let them have common external tangents AC, BD such that A, B are on ω_1 and C, D are on ω_2. Let AC intersect BD at P, and let the common internal tangent intersect AC and BD at X and Y. If $\frac{[PCD]}{[PAB]} = \frac{1}{25}$, find $\frac{[PCD]}{[PXY]}$.

1. Consider two circles ω_1, ω_2 that intersect at X, Y. Let AB be a line segment tangent to ω_1, ω_2 with A, B on ω_1, ω_2, respectively. Prove that XY bisects AB.

Solution: Let XY intersect AB at P. Notice that $\pi(P, \omega_1) = \pi(P, \omega_2)$, so the tangent from P to ω_1, ω_2 have the same length. But notice these tangents are PA, PB, so $\overline{PA} = \overline{PB}$, as desired.

2. Consider four concyclic points A, B, C, D. Then let AD, BD, CD intersect BC, CA, AB at X, Y, Z, respectively. Let the circumcircle of $\triangle ABC$ have center O and let the circumcircle of $\triangle XYZ$ have center I. Then let these two circles intersect at M, N, and let BC intersect MN at J. Then draw the two tangents from J to the two circumcircles that only intersect one circle. Let the point of tangency to the circumcircle of $\triangle ABC$ be K and let the point of tangency to the circumcircle of $\triangle XYZ$ be L. Then let the circumcenter of $\triangle JKL$ be H, and let the circumcircle of $\triangle JKL$ intersect the circumcircles of $\triangle ABC$, $\triangle XYZ$ at E, F, respectively. Then let the midpoint of EK be Q and the midpoint of FL be R. Prove H, O, Q and H, I, R are collinear.

(Assume that all mentioned points exist.)

Solution: For convenience, let $\omega_1, \omega_2, \omega_3$ be the circumcircles of $ABCD, \triangle XYZ, \triangle JKL$. By the Radical Axis Theorem (4.1), $HO \perp EK$ as EK is the radical axis of ω_1, ω_3, and similarly, $HI \perp FL$, implying Q is on HO and R is on HI.

3. Consider two circles ω_1, ω_2 that intersect at X, Y. Then pick M on XY such that M is not in either circle. Let A, B be on ω_1, ω_2 such that AM, BM are tangent to ω_1, ω_2, respectively. Prove $AM = BM$.

Solution: There are two ways to prove this.

The first is using the definition of the power of a point. The power of a point is the length of the tangent if the point is outside the circle, which trivially finishes the problem.

The second method is to use the Power of a Point Theorem (3.2). Notice that $\overline{AM}^2 = \overline{MX} \cdot \overline{MY} = \overline{BM}^2$, and a simple square root proves the rest.

4. Consider two externally tangent circles ω_1, ω_2. Let them have common external tangents AC, BD such that A, B are on ω_1 and C, D are on ω_2. Let AC intersect BD at P, and let the common internal tangent intersect AC and BD at X and Y. If $\frac{[PCD]}{[PAB]} = \frac{1}{25}$, find $\frac{[PCD]}{[PXY]}$.

Solution: Notice that $AX = XC$ and $BY = YD$, by a well known theorem. (This was problem 1, for reference.) Then it is obvious that $\frac{PC}{PA} = \frac{1}{5}$, so $PC = x, CX = 2x, XA = 2x$. Then $\frac{PC}{PX} = \frac{x}{x+2x} = \frac{1}{3}$, so $\frac{[PCD]}{[PXY]} = \frac{1}{9}$.

Triangles

Area of a Triangle

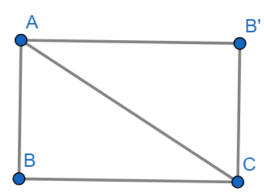
There are numerous ways to find the area of a triangle. Perhaps the most famous is $\frac{bh}{2}$. We shall introduce some common synthetic methods of finding the area of a triangle. (Coordinate methods will be explored later.) We will denote the area of $\triangle ABC$ as $[ABC]$. We will set off to prove $\frac{bh}{2}$ and some of the theorems that are associated with it, but first we shall prove a prerequisite theorem. This is merely a specific version of $\frac{bh}{2}$ that can be applied generally to prove the rest of $\frac{bh}{2}$.

Area of a Right Triangle (5.1)

Given right triangle $\triangle ABC$ such that $\angle ABC = 90^\circ$, $[ABC] = \frac{1}{2} \cdot \overline{AB} \cdot \overline{BC}$.

Theorem 5.1's Proof

Reflecting B about the midpoint $\overline{AC}$ gives us two identical triangles which form a rectangle $ABCB'$. Note that $[ABC] = \frac{1}{2}[ABCB']$, and $[ABCB'] = \overline{AB} \cdot \overline{BC}$, so $[ABC] = \frac{1}{2} \cdot \overline{AB} \cdot \overline{BC}$.



Now that we have proved $\frac{bh}{2}$ for right triangles, we can use this to generalize $\frac{bh}{2}$ for acute and obtuse triangles. I implore you to try this proof yourself; it is a fairly standard application of right triangle areas, cancelling out and substituting, and casework.

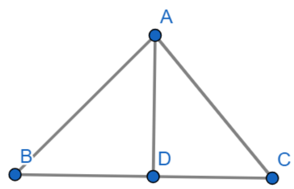
$[ABC] = \frac{bh}{2}$ *(5.2)*

Consider triangle $\triangle ABC$, and have $\overline{BC} = b$. Then drop an altitude from A to BC, and have its length be h. Then, $[ABC] = \frac{bh}{2}$.

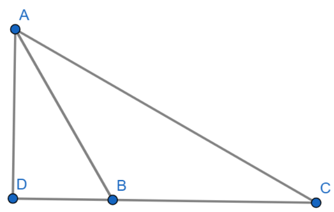
Try to use the formula for the area of a right triangle to prove $\frac{bh}{2}$. This theorem also works for obtuse triangles; all you have to do is extend side BC if the altitude is outside the triangle. In this case, the height is the length of the segment formed by A and the altitude's intersection point at BC.

Theorem 5.2's Proof

Have the altitude intersect BC at D. If the altitude intersects BC inside the triangle, $[ABC] = [ABD] + [ACD]$, and that $\overline{BD} + \overline{CD} = b$. By Theorem 5.1, $[ABD] = \frac{1}{2} \cdot \overline{AD} \cdot \overline{BD}$ and $[ACD] = \frac{1}{2} \cdot \overline{AD} \cdot \overline{CD}$. This implies $[ABD] + [ACD] = \frac{1}{2} \cdot \overline{AD} \cdot (\overline{BD} + \overline{CD}) = \frac{1}{2} \cdot h \cdot b = [ABC]$, as desired.



If the altitude intersects BC outside the triangle, then by Theorem 5.1, $[ABC] = [ACD] - [ABD] = \frac{1}{2}\overline{DC} \cdot h - \frac{1}{2}\overline{DB} \cdot h = \frac{1}{2}\overline{BC} \cdot h = \frac{bh}{2}$ as desired.



What follows below is essentially a rewriting of Theorem 5.2. However, this technique expands the range of perspectives which you have when dealing with area, which is very helpful.

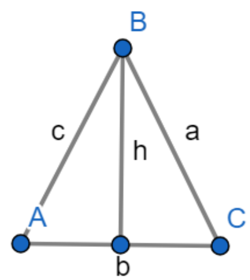
$[ABC] = \frac{1}{2}ab \cdot \sin(C)$ ***(5.3)***

Given two side lengths a, b and the angle C between them, $[ABC] = \frac{1}{2}ab \cdot \sin(C)$, if $\triangle ABC$ is acute.

Try to use Theorem 5.2 to help you prove this. Drawing the altitude will be extremely useful. Remember that $\sin(C)$ represents a ratio!

Theorem 5.3's Proof

Note that by $\frac{bh}{2}$ (5.2), $[ABC] = \frac{1}{2}bh$. Then note $\sin(C) = \frac{h}{a}$. This implies that $\frac{1}{2}ab \cdot \sin(C) = \frac{1}{2}ab \cdot \frac{h}{a} = \frac{1}{2}bh$, and by the transitive property, $\frac{1}{2}ab \cdot \sin(C) = [ABC]$, as desired.



What follows below are two formulas involving the inradius and circumradius of a triangle. If you do not know what incircles and circumcircles are, you may skip ahead and go to the section for a more detailed explanation on it. (In fact, I encourage you to work this book out of order.) In short, incircles are inscribed within a triangle and triangles are inscribed within circumcircles. However, incircles and circumcircles are not hard to understand, so you will probably understand what these theorems are stating.

$[ABC] = rs$ *(5.4)*

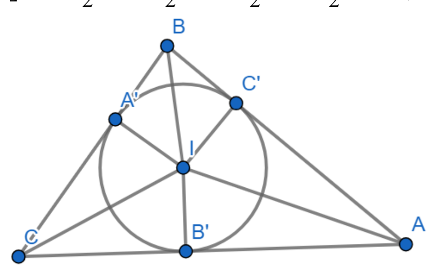
The area of a triangle is the product of the semiperimeter and the inradius of the triangle.

$[ABC] = \frac{abc}{4R}$ *(5.5)*

The area of a triangle is the product of a fourth of all of its sides divided by its circumradius.

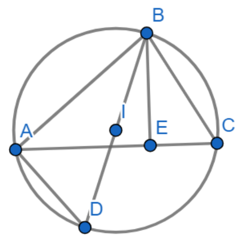
Theorem 5.4's Proof

Have the center of the incircle be I. Have the points the circle intersects $\triangle ABC$ be A', B', and C', such that A' does not share an edge with A, B' does not share an edge with B, and C' does not share an edge with C. Then note that by Theorem 3.4, the radii are perpendicular to the sides. Note that $[ABC] = [IAB] + [IAC] + [IBC]$. By $\frac{bh}{2}$ (5.2), $[IAB] + [IAC] + [IBC] = r\frac{\overline{AB}}{2} + r\frac{\overline{AC}}{2} + r\frac{\overline{BC}}{2} = r\frac{p}{2} = rs$, which implies $[ABC] = rs$.



Theorem 5.5's Proof

Have $\overline{AB} = c$, $\overline{AC} = b$, $\overline{CB} = a$, $\overline{BE} = h$, and $\overline{BI} = R$. (The first three are the side lengths of the triangle, and the last two are the height and circumradius, respectively.) Then note that $\angle BAD = 90^\circ$ because it subtends an arc that makes up half of a circle. Since $\angle ADB$ and $\angle ACB$ subtend the same arc, $\angle ADB = \angle ACB$. This implies that $\triangle BAD \sim \triangle BEC$, so $\frac{\overline{BA}}{BD} = \frac{\overline{BE}}{BC}$, which implies $\frac{c}{2R} = \frac{h}{a}$. By $\frac{bh}{2}$ (5.2), $\frac{bh}{2} = [ABC]$, so $h = \frac{2[ABC]}{b}$. Substituting, we see that $\frac{c}{2R} = \frac{2[ABC]}{ab}$, and rearranging yields $\frac{abc}{4R} = [ABC]$.



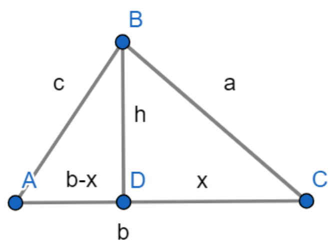
Very rarely in competition problems will you be required to find the area using these two formulas. More likely is the case that you will be required to find the inradius or circumradius as an intermediate step to a problem. Next, we shall introduce a clever example of using the Pythagorean Theorem and $\frac{bh}{2}$ to find the area of a triangle based off of its side lengths. This is a famous formula known as Heron's Formula, and it is a way to derive the altitudes or area of triangles.

Heron's Formula (5.6)

Given $\triangle ABC$ with side lengths a, b, c, $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$, where s denotes the semiperimeter.

Theorem 5.6's Proof

Without loss of generality, let $\overline{AC} = b$ be the longest side of the triangle. Then, we may use $\frac{bh}{2}$. Dropping an altitude from B to $\overline{AC}$, we see that $h^2 + x^2 = a^2$ and $h^2 + (b-x)^2 = c^2$. Note that we want to find $\frac{bh}{2}$, implying that we want to find h and substitute it in. We may subtract the first equation from the second, yielding $b^2 - 2bx = c^2 - a^2$. This implies $2bx = a^2 + b^2 - c^2$, or $x = \frac{a^2+b^2-c^2}{2b}$. We substitute this into $h^2 = a^2 - x^2$ and we see $h^2 = a^2 - (\frac{a^2+b^2-c^2}{2b})^2$. Using difference of squares, $h^2 = \frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{4b^2}$. This can be further factored into $h^2 = \frac{((a+b)^2-c^2)(c^2-(a-b)^2)}{4b^2}$, and applying difference of squares again, we see that $h^2 = \frac{(a+b+c)(a+b-c)(c-a+b)(c+a-b)}{4b^2} = \frac{2s(2s-2c)(2s-2a)(2s-2b)}{4b^2} = \frac{4s(s-a)(s-b)(s-c)}{b^2}$, implying that $h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{b}$. By $\frac{bh}{2}$ (5.2), $[ABC] = \frac{bh}{2} = \frac{b}{2} \cdot \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{b} = \sqrt{s(s-a)(s-b)(s-c)}$.



That might be difficult to follow at first. I would recommend rereading the proof a few times, trying to rewrite it with different variables, then testing yourself to see if you remember the proof.

Now that we have introduced some formulas for finding the area of a triangle, below are a few problems revolving around them.

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1. If you draw three radii of a circle and lines perpendicular to these radii that only intersect the circle once, what relation does the polygon made by the intersections of the lines and the circle have?

2. Given a triangle with side lengths $3, 4, 5,$ find the radius of its incircle.

3. Consider acute $\triangle ABC$ with $\overline{AB} = 6,$ $\overline{AC} = 8,$ and $\overline{BC} = 9.$ Find the sum of all of its altitudes.

4. Consider $\triangle ABC$ with integer side lengths a, b, c such that $\overline{AB} = c,$ $\overline{AC} = b,$ and $\overline{BC} = a.$ The inradius is $2,$ and $ab \cdot \sin(C) = 48.$ Find the side lengths of the triangle.

5. A triangle has side lengths 4 and $8,$ and it has an area of $3\sqrt{15}.$ Find the length of the third side.

6. A circle with area 9π is inscribed within $\triangle ABC,$ and a circle with area 72.25π intersects all of the vertices of $\triangle ABC.$ Provided that $\triangle ABC$ has an area of $60,$ find its side lengths.

7. A semicircle is inscribed within a right triangle with an area of 30 such that its diameter lies on a leg of the triangle and its area is maximized. Provided that the hypotenuse of the triangle is $13,$ find the area of the semicircle.

8. Prove that for parallelogram $ABCD$ with the lengths of AB and BC fixed, that $[ABCD]$ is maximized when $ABCD$ is a rectangle.

9. Given that $[ABC]=x$ and $abc=y$ for $\triangle ABC,$ find $\sin(A)\sin(B)\sin(C)$ in terms of $x,y.$

10. Let P,A,C and P,B,D be collinear. Then prove that $[PCD]=[PAB]\cdot\frac{\overline{PC}\cdot\overline{PD}}{\overline{PA}\cdot\overline{PB}}.$

11. If a triangle has side lengths of 10 and $11,$ find its maximal area.

12. Consider rectangle $ABCD$ such that $\overline{AB}=2$ and $\overline{BC}=1.$ Let X,Y trisect $AB.$ Then let DX and DY intersect AC at P and Q respectively. What is the area of quadrilateral $XYQP$?

13. Consider trapezoid $ABCD$ with bases AB and $CD.$ If AB and CD intersect at $P,$ prove the sum of the areas of $\triangle ABP$ and $\triangle CDP$ exceeds half the area of trapezoid $ABCD.$

14. What positive value of x maximizes $(21+x)(1+x)(x-1)(21-x)$?

---

1. If you draw three radii of a circle and lines perpendicular to these radii that only intersect the circle once, what relation does the polygon made by the intersections of the lines and the circle have?

Solution: By Theorem 3.4, these lines are tangent. As such, the circle is contained within the lines, which makes it an incircle.

2. Given a triangle with side lengths $3, 4, 5,$ find the radius of its incircle.

Solution: This triangle is clearly a right triangle, which means it has an area of $6,$ by Theorem 4.1. Then, by $[ABC] = rs$ (5.4), $6 = r(\frac{3+4+5}{2}),$ implying $r = 1.$

3. Consider $\triangle ABC$ with $\overline{AB} = 8,$ $\overline{AC} = 12,$ and $\overline{BC} = 18.$ Find the sum of all of its altitudes.

Solution: By Heron's Formula, the area is $\sqrt{19(19-18)(19-12)(19-8)} = \sqrt{19(1)(7)(11)} = \sqrt{1463}.$ For convenience, we shall substitute this area as $[ABC].$ Then note that by $\frac{bh}{2}$ (5.2), $h = \frac{2[ABC]}{b}.$ This implies that the sums of our altitudes is $\frac{2[ABC]}{8} + \frac{2[ABC]}{12} + \frac{2[ABC]}{18} = \frac{[ABC]}{4} + \frac{[ABC]}{6} + \frac{[ABC]}{9} = \frac{19[ABC]}{36} = \frac{19\sqrt{1463}}{36},$ which is our answer.

4. Consider $\triangle ABC$ with integer side lengths a, b, c such that $\overline{AB} = c,$ $\overline{AC} = b,$ and $\overline{BC} = a.$ The inradius is $2,$ and $ab \cdot \sin(C) = 48.$ Find the side lengths of the triangle.

Solution: Note that $[ABC] = \frac{1}{2}ab \cdot \sin(C) = \frac{1}{2} \cdot 48 = 24,$ and since $[ABC] = 24 = rs = 2s,$ the perimeter of the triangle is 24, which implies $a + b + c = 24.$ Then, we note that we want to solve for a, b, c that satisfies the following system of equations.

$$a + b + c = 24$$
$$\sqrt{12(12-a)(12-b)(12-c)} = 24$$

Squaring both sides of the second equation, we see that $24^2 = 12(12-a)(12-b)(12-c).$ We could expand, but that seems unnecessary and painful. Instead, have $a' = 12 - a,$ $b' = 12 - b,$ and $c' = 12 - c.$ This then implies the following system of equations.

$a' + b' + c' = 12$
$12a'b'c' = 24^2$
$a'b'c' = 48$

Looking at the factors of $48,$ we see that the ones that sum to 12 are $2, 4, 6.$ (Otherwise, the largest factor would be too large, and that means they would sum to more than 12 and the system would not be satisfied.) This means that $a' = 2,$ $b' = 4,$ and $c' = 6,$ in no particular order. Then, this implies $a = 10,$ $b = 8,$ and $c = 6,$ which are the side lengths we were looking for.

5. A triangle has side lengths 4 and $8,$ and it has an area of $3\sqrt{15}.$ Find the possible lengths of the third side.

Solution: Use the Law of Cosines. By $[ABC] = \frac{1}{2}ab \cdot \sin(C)$ (5.3), note that $\sin(C) = \frac{3\sqrt{15}}{16},$ implying $\cos(C) = \pm\frac{11}{16}.$ Using the Law of Cosines, we have $c = 6, c = 2\sqrt{31}.$ (This solution is better.)

6. A circle with area 9π is inscribed within $\triangle ABC,$ and a circle with area 72.25π intersects all of the vertices of $\triangle ABC.$ Provided that $\triangle ABC$ has an area of $60,$ find its side lengths.

Solution: This implies that $r = 3$ and $R = 8.5.$ Substituting in, we note that $[ABC] = 60 = 3 \cdot \frac{a+b+c}{2} = \frac{abc}{34}.$ Also, by Heron's Formula, $\sqrt{s(s-a)(s-b)(s-c)}.$

This gives us the following system of equations:
$40 = a + b + c$
$2040 = abc$
$\frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} = 3600$

In particular, the third equation implies $\frac{40}{2} \cdot \frac{40-2a}{2} \cdot \frac{40-2b}{2} \cdot \frac{40-2c}{2} = 3600.$ Multiplying both sides by $\frac{1}{20}$ gives us $(20-a)(20-b)(20-c) = 180.$ Expanding gives us $20^3 - 20^2a - 20^2b - 20^2c + 20ab + 20ac + 20bc - abc = 180.$ Let's substitute again. The following gives us $20^3 - 20^3 \cdot 2 + 20ab + 20ac + 20bc - 2040 = 20(ab + ac + bc) - 2040 - 20^3 = 180.$ Rearranging and dividing both sides by $20,$ we see that $ab + ac + bc = 511.$

Now, our new system of equations follow:
$40 = a + b + c$
$511 = ab + ac + bc$
$2040 = abc$

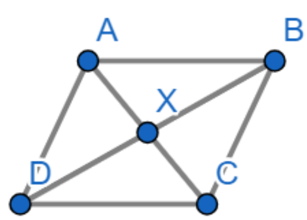
Basic algebraic manipulations give us $a = 8,\ b = 15,$ and $c = 17.$

7. A semicircle is inscribed within a right triangle with an area of 30 such that its diameter lies on a leg of the triangle and its area is maximized. Provided that the hypotenuse of the triangle is $13,$ find the area of the semicircle.

Solution: Have the legs be a, b and the hypotenuse be $c.$ Without loss of generality, have $a < b.$ Then note that to maximize the area of the semicircle, which is what we desire, the semicircle is on leg $b.$ Reflecting the triangle and the semicircle to make a full circle inscribed within an isosceles triangle, we see that our isosceles triangle has an area of $60.$ This implies $60 = r(13 + a).$ By $\frac{bh}{2},\ ab = 60$ and $a^2 + b^2 = 169.$ Note then that this means $a + b = 17,$ which gives us $a = 5$ and $b = 12.$ This means that $60 = r(13 + 5),$ or $r = \frac{10}{3}.$ As thus, our semicircle has area $\frac{50}{9}\pi.$

8. Prove that for parallelogram $ABCD$ with the lengths of AB and BC fixed, that $[ABCD]$ is maximized when $ABCD$ is a rectangle.

Solution: Draw diagonals AC and BD and have them intersect at $X.$ Note that $[ABCD] = [ABC] + [ADC],$ and by $[ABC] = \frac{1}{2}ab \cdot \sin(C)$ (4.3), $[ABCD] = [ABC] + [ADC] = \frac{1}{2}\overline{AB} \cdot \overline{BC} \cdot \sin(B) + \frac{1}{2}\overline{CD} \cdot \overline{DA} \cdot \sin(D).$ Note that by the definition of a parallelogram, $\overline{AB} = \overline{CD},\ \overline{BC} = \overline{DA},$ and $\sin(B) = \sin(D).$ Substituting yields $[ABCD] = \overline{AB} \cdot \overline{BC} \cdot \sin(B),$ and since $\overline{AB}$ and $\overline{BC}$ are fixed, we maximize $\sin(B)$ by having $\angle B = 90^\circ,$ implying $[ABCD]$ is maximized when $ABCD$ is a rectangle.



9. Given that $[ABC] = x$ and $abc = y$ for $\triangle ABC,$ find $\sin(A)\sin(B)\sin(C)$ in terms of $x, y.$

Solution: Note that by $\frac{1}{2}ab\sin(C) = [ABC]$ (4.3) and cyclic variants,

$\frac{1}{2}ab\sin(C) = [ABC]$
$\frac{1}{2}bc\sin(A) = [ABC]$
$\frac{1}{2}ca\sin(B) = [ABC].$

Multiplying these together gives us $\frac{1}{8}a^2b^2c^2\sin(A)\sin(B)\sin(C) = [ABC]^3.$ Substituting x, y yields $\frac{1}{8}y^2\sin(A)\sin(B)\sin(C) = x^3,$ and rearrangement gives $\sin(A)\sin(B)\sin(C) = \frac{8x^3}{y^2}.$

10. Let P, A, C and P, B, D be collinear. Then prove that $[PCD] = [PAB] \cdot \frac{\overline{PC}\cdot\overline{PD}}{\overline{PA}\cdot\overline{PB}}.$

Solution: It is obvious that $\angle PAB = \angle PCD.$ Let $\angle PAB = \angle PCD = \theta,$ and let $\overline{PA} = a, \overline{PB} = b, \overline{PC} = x, \overline{PD} = y.$ Notice that by $[ABC] = \frac{1}{2}ab \cdot \sin(C)$ (4.3), $[PAB] = \frac{1}{2}ab \cdot \sin(\theta)$ and $[PCD] = \frac{1}{2}xy \cdot \sin(\theta).$ Rearranging yields $\sin(\theta) = \frac{2[PAB]}{ab} = \frac{2[PCD]}{xy}.$ Using the transitive theorem on the latter two and rearranging yields $[PCD] = [PAB] \cdot \frac{xy}{ab},$ as desired.

11. If a triangle has side lengths of 10 and $11,$ find its maximal area.

Solution: By Theorem 4.3, the area of our triangle is $\frac{1}{2} \cdot 10 \cdot 11 \cdot \sin(\theta) = 55\sin(\theta),$ where θ is the angle between the sides of $10, 11.$ Obviously the area is maximized when $\theta = 90^\circ$ or $\sin(\theta) = 1,$ yielding a maximal area of $55.$

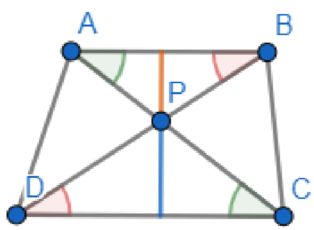
12. Consider rectangle $ABCD$ such that $\overline{AB} = 2$ and $\overline{BC} = 1.$ Let X, Y trisect $AB.$ Then let DX and DY intersect AC at P and Q respectively. What is the area of quadrilateral $XYQP$?

Solution: Without loss of generality, let X be between A and $Y.$ Then notice we want to find $[AQY] - [APX].$ Since $\triangle APX$ is similar to $\triangle CPD$ with a ratio of $1:3,$ $[APX] = \frac{1}{2}(\frac{2}{3} \cdot \frac{1}{4}),$ and since $\triangle AQY$ is similar to $\triangle CQD$ with a ratio of $2:3,$ $[AQY] = \frac{1}{2}(\frac{4}{3} \cdot \frac{2}{5}).$ So $[XYQP] = [AQY] - [APX] = \frac{11}{60}.$

13. Consider trapezoid $ABCD$ with bases AB and $CD.$ If AB and CD intersect at $P,$ prove the sum of the areas of $\triangle ABP$ and $\triangle CDP$ exceeds half the area of trapezoid $ABCD.$

Solution: Let $\overline{AB} = b_1$, $\overline{CD} = b_2$, and let the height of the trapezoid be h. By similar triangles, the height from AB to P is $h\frac{b_1}{b_1+b_2}$, and the height from CD to P is $h\frac{b_2}{b_1+b_2}$. By $\frac{bh}{2}$ (5.2), $[ABP] = \frac{h}{2} \cdot \frac{b_1^2}{b_1+b_2}$ and $[CDP] = \frac{h}{2} \cdot \frac{b_2^2}{b_1+b_2}$.

Notice that $[ABCD] = h\frac{b_1+b_2}{2}$, so we want to prove that $[ABP] + [CDP] \geq \frac{[ABCD]}{2}$, or $\frac{h(b_1^2+b_2^2)}{2(b_1+b_2)} \geq h\frac{b_1+b_2}{4}$. Multiplying both sides by $\frac{4(b_1+b_2)}{h}$ yields $2(b_1^2 + b_2^2) \geq (b_1 + b_2)^2$. Subtracting $(b_1 + b_2)^2$ from both sides yields $b_1^2 + b_2^2 - 2b_1b_2 = (b_1 - b_2)^2 \geq 0$, with equality at $b_1 = b_2$. Since all steps are reversible, we are done.



14. What positive value of x maximizes $(21 + x)(1 + x)(x - 1)(21 - x)$?

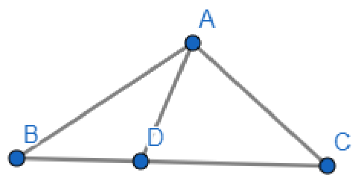
Solution: Notice that if $\triangle ABC$ has side lengths $20, 22, 2x$, then by Heron's Formula (5.6), $[ABC]^2 = (21 + x)(1 + x)(x - 1)(21 - x)$. By $\frac{1}{2}ab \cdot \sin(C)$ (5.3), the area is maximized when the angle between the sides of length 20 and 22 is 90°. Thus,
$2x = \sqrt{20^2 + 22^2} = 2\sqrt{221}$, so $x = \sqrt{221}$.

Concurrency and Collinearity

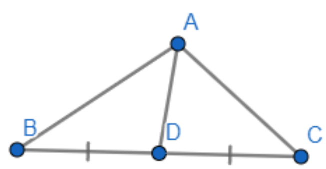
As we have seen in the previous chapters, lines are closely connected to triangles. (This should be true, especially because triangles are made from three lines.) We will explore the relations of triangles and lines. We shall define a cevian as a line such that one of the endpoints is a vertice of the triangle and the other endpoint is on the opposite side of the triangle. (When we say "opposite side," we are referring to the side that does not contain the vertice it is opposite to.)

There are a few special cevians. The three cevians that we shall be discussing in-depth are angle bisectors, medians, and altitudes.

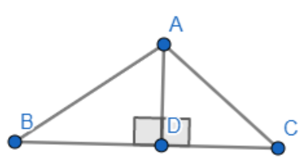
First, let us set up our cevians. Consider $\triangle ABC$ with cevian $AD.$ This implies that D lies on $BC.$



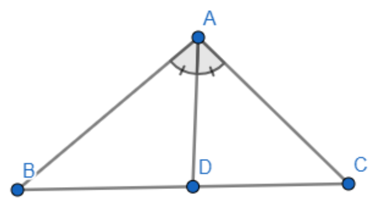
A *median* is a cevian such that $\overline{BD} = \overline{CD}.$



An *altitude* is a cevian such that $\angle ADB = \angle ADC = 90^\circ.$



An *angle bisector* is a cevian such that $\angle BAD = \angle CAD.$ This definition is useful at times, but we will use the definition that the angle bisector of $\angle CAB$ is the locus of points equidistant from AB and CA at other times.



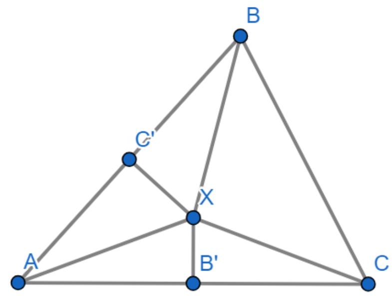
Let us consider some properties of medians, altitudes and angle bisectors. We shall first prove that the perpendicular bisectors of a triangle are concurrent (there exists a point that lies on all three perpendicular bisectors). However, we need to first define a perpendicular bisector before proving anything. Then, we will prove that they are all concurrent.

A *perpendicular bisector* of AB is the line that is perpendicular to line AB and intersects the midpoint of AB. But this definition is quite useless, so instead we'll use the definition that the perpendicular bisector of AB is *the locus of points equidistant from* A, B.

Concurrency of Perpendicular Bisectors Theorem (6.1)
In any $\triangle ABC$, perpendicular bisectors $A'D$, $B'E$, and $C'F$ are concurrent.

Without loss of generality, have A' be on BC, B' on AC, and C' on AB. Our naming convention uses opposite sides (the side that a vertice is not part of). When two perpendicular bisectors intersect at a point, what is true of the segments formed by the intersection point and the vertices? (We are specifically looking for equivalence.) What do our other equivalences imply? Remember that the perpendicular bisector of AB is also the locus of points that is equidistant from points A, B.

Theorem 6.1's Proof
Any two non-parallel lines are concurrent. Since AC and BC are not parallel, it follows that $B'E$ and $C'F$ are concurrent. Let then intersect at X. Then, we see that $\overline{AX} = \overline{CX}$ and $\overline{AX} = \overline{BX}$. By the transitive property, $\overline{BX} = \overline{CX}$. Since the perpendicular bisector of BC is also the locus of points equidistant from B and C, X lies on $A'D$, $B'E$, and $C'F$, which means that these three lines are concurrent.



In addition, this yields the useful result that for $\triangle ABC$ with circumcenter O, $\overline{AO} = \overline{BO} = \overline{CO}$. **Take note of this.**

This is an example of a time where the formal definition of concurrent we used (the one involving the existence of a point on all three lines) is much more helpful than the intuitive understanding of having all three lines in a point. We would rather say that they are concurrent by its definition than say that $A'D$ passes through the point, all three lines pass through the point, and the like.

The point of concurrency is called the circumcenter of the triangle. (Try to reason out why it is called the circumcenter. We will explain this in the next section of the book.) The circumcenter does not always lie inside the triangle; it only lies inside the triangle if it is acute, it lies on the triangle if it is a right triangle, and it is outside the triangle if it is obtuse.

Now, we shall prove that some other sets of cevians are concurrent. Try to prove these on your own, and make sure to take advantage of the fact that the perpendicular bisectors of a triangle are concurrent.

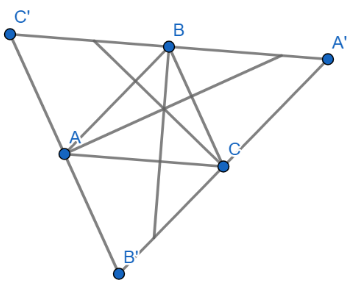
Concurrency of Altitudes Theorem (6.2)
In any $\triangle ABC$, altitudes AD, BE, and CF are concurrent.

Concurrency of Medians Theorem (6.3)
In any $\triangle ABC$, medians AD, BE, and CF are concurrent.

Concurrency of Angle Bisectors Theorem (6.4)
In any $\triangle ABC$, angle bisectors AD, BE, and CF are concurrent.

Theorem 6.2's Proof

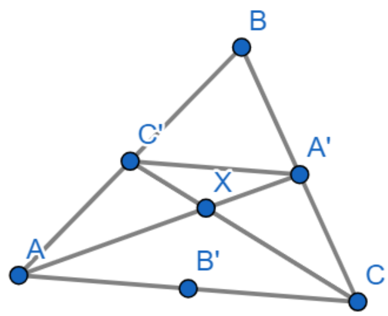
Draw a line through A parallel to BC, a line through B parallel to AC, and a line through C parallel to AB. Without loss of generality, have them intersect at A', B', C' such that A' is the point not on the line parallel to BC, B' is the point not on the line parallel to AC, and C' is the point not on the line parallel to AB. Note that $\overline{C'B} = \overline{AC}$ and $\overline{A'B} = \overline{AC}$ because $C'ACB$ and $A'CAB$ are parallelograms, respectively. By the transitive property, this means that $\overline{A'B} = \overline{C'B}$, and doing the same process for the other sides gives us $\overline{A'C} = \overline{B'C}$ and $\overline{B'A} = \overline{C'A}$. This means that the altitudes of $\triangle ABC$, when extended, are the perpendicular bisectors of $\triangle A'B'C'$. By Theorem 5.1, the perpendicular bisectors of $\triangle A'B'C'$ are concurrent, so the altitudes of $\triangle ABC$ are as well, since they are the same lines.



The point of concurrency of the altitudes is known as the orthocenter of a triangle. The orthocenter does not necessarily lie inside the triangle.

Theorem 6.3's Proof

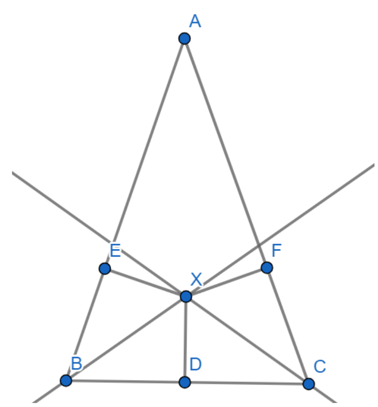
Since C' and A' are midpoints, $\overline{AB} = 2\overline{C'B}$ and $\overline{CB} = 2\overline{A'B}$. Note that by SAS similarity, $\triangle ABC \sim \triangle C'BA'$. This implies that $AC \parallel A'C'$ since the corresponding angles are the same by similarity. Then this implies $\angle XA'C' = \angle XAC$ and $\angle XC'A' = \angle XCA$, by the properties of a transversal. By ASA similarity, $\triangle AXC \sim \triangle A'XC'$, implying that $\overline{AX} = 2\overline{A'X}$, or $\overline{AX} = \frac{2}{3}\overline{AA'}$. Repeating this for all pairs of two sides, we see that all intersection points of the medians must divide the medians into two segments of ratio $2:1$. However, X already satisfies this for AA' and CC', so the only point of intersection for BB' with the other two lines is X. This means AA', BB', and CC' are concurrent, as desired.



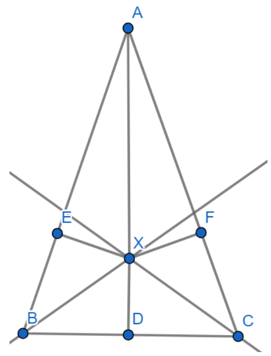
This point of intersection is called the centroid. The reason why will be hinted at in an exercise after the next theorem. The proof for the next theorem is very similar to the proof for the one before it; the key idea is drawing two cevians, finding some equal angles and sides, and proving that the third cevian must be concurrent with the previous two. The centroid lies inside the triangle.

Theorem 6.4's Proof

First draw angle bisectors BX and CX. Then, from the intersection point X, draw perpendiculars to AB, AC, and BC, and have them intersect at E, F, and D, respectively.



Then note by ASA congruence, $\triangle BEX \cong \triangle BDX$ and $\triangle CFX \cong \triangle CDX$. This implies $\overline{XE} = \overline{XF} = \overline{XD}$. Drawing AX, we see that since $\triangle AEX$ and $\triangle AFX$ are right triangles with an equal hypotenuse and an equal leg, $\triangle AEX \cong \triangle AFX$, implying that $\angle EAX = \angle FAX$. This means AX is also an angle bisector, which implies that the angle bisectors are concurrent.



Note that the reason we know two angles are equivalent to each other is because $\angle BEX = \angle BDX = 90^{\circ}$ because these angles are formed by perpendiculars and $\angle XEB = \angle XDB$ because $\angle B$ is bisected to form these two lines. The same can be said for $\angle CDX$ and $\angle CFX$.

Aside from Theorem 6.2, all of the other cevian theorems used the fact that two lines had to be concurrent, and proved that the third must be as well based on the information we obtained. Below are a set of problems involving cevians; the solutions will be right below the problems.

---

1. Prove that the point of concurrency of the angle bisectors of a triangle is always inside the triangle.

2. Prove that if the incenter and circumcenter of a triangle are the same point, the triangle must be equilateral.

3. Prove that in $\triangle ABC$ with medians AA', BB', CC' and centroid X, that $[AB'X] = [AC'X] = [BA'X] = [BC'X] = [CA'X] = [CB'X]$. (Remember that $[ABC]$ denotes the area of $\triangle ABC$ in this book.)

4. Consider $\triangle ABC$ with $\overline{AB} = 5$, $\overline{BC} = 12$, and $\overline{AC} = 13$. Angle bisector AD and median $\overline{AE}$ is drawn such that B, C, D, E are collinear. Find $[ADE]$.

5. Consider $\triangle ABC$ with circumcenter O. If $\overline{AO} = 20, \overline{BC} = 32$, find $[BOC]$.

6. Prove that $\triangle ABC$ with circumcenter O cannot simultaneously satisfy $BO = 100$, the distance from O to AB is 95, the distance from O to BC is 2, and the distance from O to CA is 97.

7. Consider $\triangle ABC$ with circumcenter O such that the distance from O to AB is $5,$ the distance from O to BC is $2,$ and the distance from O to AB is $7,$ find the minimum integer length of AO such that the triangle exists.

---

1. Prove that the point of concurrency of the angle bisectors of a triangle is always inside the triangle.

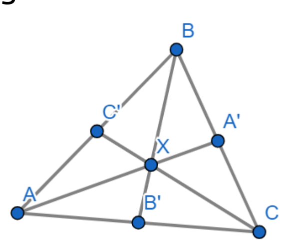
Solution: The point of concurrency of the angle bisectors is the incenter of the triangle. Since none of the incircle can be outside the triangle, the incenter must be inside the triangle.

2. Prove that if the incenter and circumcenter of a triangle are the same point, the triangle must be equilateral.

Solution: For the incenter and circumcenter to be the same, the medians and perpendicular bisectors must be the same. If cevian AA' is a median and a perpendicular bisector, $\triangle ABC$ must be isosceles with $\overline{AB} = \overline{AC}$. Similarly, if cevian BB' is a median and a perpendicular bisector, $\triangle ABC$ must be isosceles with $\overline{BC} = \overline{AB}$. By the transitive property, $\overline{AB} = \overline{AC} = \overline{BC}$, which means $\triangle ABC$ is equilateral.

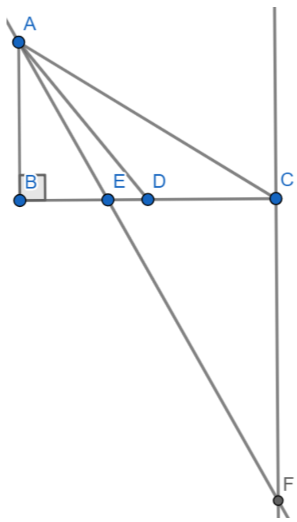
3. Prove that in $\triangle ABC$ with medians AA', BB', CC' and centroid X, that $[AB'X] = [AC'X] = [BA'X] = [BC'X] = [CA'X] = [CB'X]$. (Remember that $[ABC]$ denotes the area of $\triangle ABC$ in this book.)

Solution: By $\frac{bh}{2}$ (5.2), $[AC'X] = [BC'X]$, $[AB'X] = [CB'X]$, and $[BA'X] = [CA'X]$. Have $a = [BA'X] = [CA'X]$, $b = [AB'X] = [CB'X]$, and $c = [AC'X] = [BC'X]$. (Here, we have defined the variables such that they represent the areas of the triangles who have that variable primed as a vertice.) Then note by $\frac{bh}{2}$, $2a + b = 2c + b$ and $2b + a = 2c + a$, implying $a = c$ and $b = c$. By the transitive property, $a = b = c$, and menial substitutions give us what we desire.



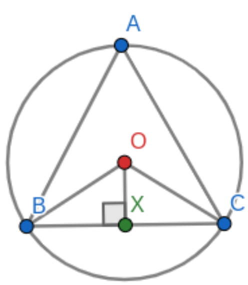
4. Consider $\triangle ABC$ with $\overline{AB} = 5$, $\overline{BC} = 12$, and $\overline{AC} = 13$. Angle bisector AD and median $\overline{AE}$ is drawn such that B, C, D, E are collinear. Find $[ADE]$.

Solution: We can use $[ABC] = \frac{bh}{2}$ (5.2) to find $[ADE] = \frac{5}{2} \cdot \overline{DE}$. Note that B, C, D, E being collinear implies $\overline{BD} = 6$, by the definition of a median. Here it gets a little trickier. We extend angle bisector AE and draw line CF parallel to AB such that they meet at F. Looking at vertical angles, we note that $\angle AEB = \angle FEC$, and seeing the transversal formed by AF gives us $\angle BAE = \angle CFE$. This implies $\triangle BAE \sim \triangle CFE$, or $\frac{\overline{BE}}{5} = \frac{\overline{CE}}{\overline{CF}}$. Since AE is an angle bisector, $\angle BAE = \angle CAE$, and by the transitive property, $\angle CAE = \angle CFE$. This implies that $\triangle ACF$ is isosceles, giving us $\overline{CF} = \overline{AC} = 13$, which implies $\frac{\overline{BE}}{5} = \frac{\overline{CE}}{13}$. Additionally, $\overline{BE} + \overline{CE} = 12$, or $\overline{CE} + \frac{5\overline{CE}}{13} = \frac{18\overline{CE}}{13} = 12$. Note that this implies $\overline{CE} = \frac{12 \cdot 13}{18} = \frac{26}{3}$, which also implies $\overline{ED} = \frac{26}{3} - 6 = \frac{8}{3}$. Applying $\frac{bh}{2}$ (5.2), we see that $[ADE] = \frac{5 \cdot \frac{8}{3}}{2} = \frac{20}{3}$, which is the answer we desired.



5. Consider $\triangle ABC$ with circumcenter O. If $\overline{AO} = 20, \overline{BC} = 32$, find $[BOC]$.

Solution: Let the midpoint of BC be X. Then $\overline{OX}^2 = \overline{AO}^2 - \overline{BX}^2 = 20^2 - 16^2 = 12^2$, implying $\overline{OX} = 12$. Thus, $[BOC] = \overline{OX} \cdot \overline{BX} = 12 \cdot 16 = 192$.



6. Prove that $\triangle ABC$ with circumcenter O cannot simultaneously satisfy $BO = 100,$ the distance from O to AB is $95,$ the distance from O to BC is $2,$ and the distance from O to CA is $97.$

Solution: Suppose for the sake of contradiction this is possible. Let the feet of the altitudes from O to AB, BC, CA be $X, Y, Z,$ respectively. By Pythagorean's Theorem, $\sqrt{975} = \overline{AX}, \sqrt{9996} = \overline{BY}, \sqrt{591} = \overline{CZ}.$ Then $2\sqrt{975} = \overline{AB}, 2\sqrt{9996} = \overline{BC}, 2\sqrt{591} = \overline{CA}.$ But notice $\overline{BC} > \overline{AB} + \overline{CA},$ failing the Triangle Inequality. Thus this is not a triangle.

7. Consider $\triangle ABC$ with circumcenter O such that the distance from O to AB is $5,$ the distance from O to BC is $2,$ and the distance from O to AB is $7,$ find the minimum integer length of AO such that the triangle exists.

Solution: Let $\overline{OA} = x.$ Then applying Pythagorean's yields that $\overline{AB} = 2\sqrt{x^2 - 25}, \overline{BC} = 2\sqrt{x^2 - 4}, \overline{CA} = 2\sqrt{x^2 - 49}.$ We want to find the minimum integer x such that $2\sqrt{x^2 - 4} < 2\sqrt{x^2 - 25} + 2\sqrt{x^2 - 49}.$ This value of x is $8,$ which is our answer.

The fourth problem leads very well into one of the theorems we will discuss in the next chapter, known as the Angle Bisector Theorem. However, we will first prove a general theorem for concurrency known as Ceva's Theorem. This is useful because problem writers may use it to hide cevian concurrency in problems that use it, and recognizing it will help. (Also, mass points.)

Ceva's Theorem (6.5)

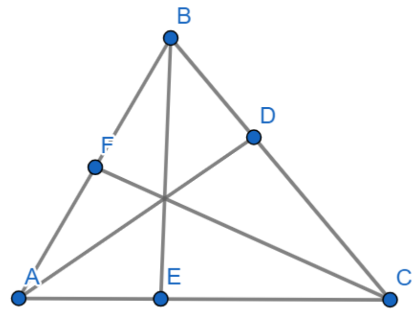
In $\triangle ABC$ with cevians $AD, BE, CF,$ the cevians are concurrent if and only if

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1.$$

Think about using $\frac{bh}{2}$ (5.2) to prove this. Areas are very important, and if you want 1 on the right hand side, the ratios must be equated to areas which cancel out. A good way to remember the statement for this theorem, specifically not switching the numerator and denominator, is what I like to call the "travelling technique." We travel from A to E, forming AE, then travel from E to C, forming EC. Our first ratio is $\frac{AE}{EC}$. We then travel from C to D, forming CD, then travel from D to B, forming DB. Our next ratio is $\frac{CD}{DB}$. Finally, we travel from B to F, forming BF, then travel from F to A, forming FA. Our final ratio is $\frac{BF}{FA}$. Multiplying these ratios, we have $\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1$. This is identical to the statement we made above, only that we took the reciprocal.

Theorem 6.5's Proof

Have the point of concurrency be X. Note that by $\frac{bh}{2}$ (5.2), $\frac{[AXD]}{[CXD]} = \frac{\frac{hAD}{2}}{\frac{hCD}{2}} = \frac{AD}{CD}$, and $\frac{[ABD]}{[CBD]} = \frac{\frac{hAD}{2}}{\frac{hCD}{2}} = \frac{AD}{CD}$. (The h can be cancelled out because the height is consistent.) We then note that this implies $\frac{[ABD]}{[CBD]} = \frac{[ABD]-[AXD]}{[CBD]-[CXD]}$ because $\frac{[ABD]}{[CBD]} = \frac{[AXD]}{[CXD]}$. Note that $[ABD] - [AXD] = [ABX]$ because $[ABX]$ is the rest of $\triangle ABD$, and $[CBD] - [CXD] = [CBX]$ by the same reasoning. This implies that $\frac{[ABX]}{[CBX]} = \frac{AD}{CD}$. Similarly, $\frac{BF}{AF} = \frac{[CBX]}{[ACX]}$, and $\frac{EC}{BE} = \frac{[ACX]}{[ABX]}$. Multiplying, we see that $\frac{AD}{CD} \cdot \frac{AF}{FB} \cdot \frac{EC}{BE} = \frac{[ABX]}{[CBX]}$.



Another useful technique to prove collinearity of three points on the sides of a triangle is Menelaus's Theorem. Again, problems that involve or are solved by concurrency may be hid using this theorem.

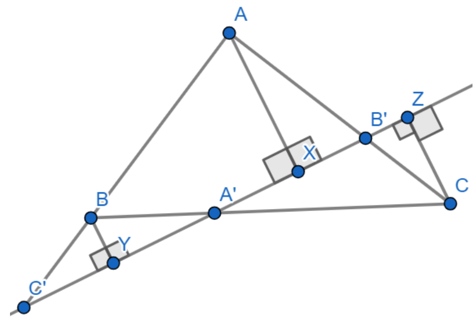
Menelaus's Theorem (6.6)

Given $\triangle ABC$ with A', B', C' on lines BC, AC, AB respectively (note that these lines can be extended if necessary), A', B', C' are collinear if and only if $\frac{\overline{AC'}}{\overline{BC'}} \cdot \frac{\overline{BA'}}{\overline{CA'}} \cdot \frac{\overline{CB'}}{\overline{AB'}} = 1.$

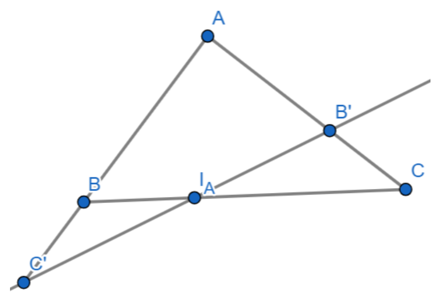
Like Ceva's Theorem (6.5), there is a convenient way to memorize this one. Note that the first letters are $A, B, B, C, C, A.$ This means that the first letter of these lines are cycling. Then note that in each fraction, the prime of the unincluded letter will be the second letter of both of the segments. For example, A, B becomes AC', BC' because C was not included in the list $A, B.$

Theorem 6.6's Proof

We first prove that if A', B', C' are collinear, that $\frac{\overline{AC'}}{\overline{BC'}} \cdot \frac{\overline{BA'}}{\overline{CA'}} \cdot \frac{\overline{CB'}}{\overline{AB'}} = 1.$ Without loss of generality, force C' to be on the extension of AB but not $AB.$ If A', B', C' are collinear then $\triangle AXC' \sim \triangle BYC'$, $\triangle BYA' \sim \triangle CZA'$, and $\triangle CZB' \sim \triangle AXB'$ where X, Y, Z are the feet of the perpendiculars from A, B, C to line $C'A'B'$ respectively, implying $\frac{\overline{AC'}}{\overline{BC'}} = \frac{\overline{AX}}{\overline{BY}}$, $\frac{\overline{BA'}}{\overline{CA'}} = \frac{\overline{BY}}{\overline{CZ}}$, and $\frac{\overline{CB'}}{\overline{AB'}} = \frac{\overline{CZ}}{\overline{AX}}$. Multiplication of these three equalities yields $\frac{\overline{AC'}}{\overline{BC'}} \cdot \frac{\overline{BA'}}{\overline{CA'}} \cdot \frac{\overline{CB'}}{\overline{AB'}} = 1,$ as desired.



Then we prove the converse. Have line $C'B'$ intersect line BC at I_A. Since $\frac{\overline{AC'}}{\overline{BC'}} \cdot \frac{\overline{BA'}}{\overline{CA'}} \cdot \frac{\overline{CB'}}{\overline{AB'}} = 1$ and, by the first proof, $\frac{\overline{AC'}}{\overline{BC'}} \cdot \frac{\overline{BI_A}}{\overline{CI_A}} \cdot \frac{\overline{CB'}}{\overline{AB'}} = 1$, we note that $\frac{\overline{BA'}}{\overline{CA'}} = \frac{\overline{BI_A}}{\overline{CI_A}}$. We can use directed segments (which we will elaborate on in Analytic Geometry) to prove that A' is I_A, implying that C', A', B' are collinear, as desired.



An interesting fact is that cevians are named after Giovanni Ceva, the person who discovered Ceva's Theorem (6.5). Below are a few problems based on his theorem and the lines named after him, which are cevians.

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1. In $\triangle ABC$ with cevians AD, BE, CF such that $\overline{BD} = 4, \overline{DC} = 6, \overline{AE} = 4, \overline{EC} = 6, \overline{AF} = \overline{BF} = 5,$ find whether AD, BE, CF are collinear or not.

2. Given concurrent cevians AD, BE, CF in $\triangle ABC$ such that $\overline{BD} = 12, \overline{DC} = 6, \overline{EC} = 8, \overline{AE} = 15, \overline{BF} = 15,$ find $[ABC]$.

3. Is there a triangle ABC with concurrent cevians AD, BE, CF such that $\overline{BD} = 12, \overline{DC} = 6, \overline{EC} = 8, \overline{AE} = 15, \overline{BF} = 18?$

4. Use Menelaus's Theorem to prove the concurrency of angle bisectors of a triangle.

5. Prove that given $\triangle ABC$ and point P on the circumcircle of $\triangle ABC,$ that the feet of the perpendiculars from P to AB, AC, BC are collinear.

6. Let the incircle of $\triangle ABC$ touch BC, CA, AB at $X, Y, Z,$ respectively. Show that AX, BY, CZ are concurrent.

7. Use the Angle Bisector Proportionality Theorem (7.1.1) and Ceva's Theorem (6.5) to prove the concurrency of angle bisectors (6.4). (To see Theorem 7.1.1, which you probably already know, go to the next section.)

8. Consider $\triangle ABC$ with cevians $AD, BE, CF.$ Denote $\angle CAD = \alpha_1, \angle DAB = \alpha_2, \angle ABE = \beta_1, \angle CBE = \beta_2, \angle BCF = \gamma_1, \angle ECF = \gamma_2.$ Prove that AD, BE, CF concur if and only if $\frac{\sin(\alpha_1)}{\sin(\alpha_2)} \cdot \frac{\sin(\beta_1)}{\sin(\beta_2)} \cdot \frac{\sin(\gamma_1)}{\sin(\gamma_2)} = 1.$ (This is known as the trigonometric form of Ceva.)

9. Consider $\triangle ABC$ with circumcenter $O.$ Let the feet of the altitudes from O to BC, CA, AB be $P, Q, R.$ Then let AO, BO, CO intersect BC, CA, AB at $D, E, F,$ respectively. If X, Y, Z are the reflections of D about $P,$ E about $Q,$ and F about $R,$ prove that AX, BY, CZ concur.

10. Consider $\triangle ABC,$ and let its incircle intersect AB, BC, CA at $P, Q, R,$ respectively. Let line l be the line passing through Q perpendicular to $AQ,$ and let l intersect PR at $X.$ If K is the midpoint of $AX,$ prove that B, C, X are collinear.

---

1. In $\triangle ABC$ with cevians AD, BE, CF such that $\overline{BD} = 4, \overline{DC} = 6, \overline{AE} = 4, \overline{EC} = 6, \overline{AF} = \overline{BF} = 5,$ find whether AD, BE, CF are collinear or not.

Solution: Even though at first glance we may try to apply Ceva's Theorem (6.5) by multiplying $\frac{4}{6} \cdot \frac{4}{6} \cdot \frac{5}{5}$ and seeing that this is not equal to $1,$ this is not what we desire. The order of the lines was mixed up to trick you, so remember the "travelling technique" and note that we want to find $\frac{\overline{BD}}{DC} \cdot \frac{\overline{EC}}{AE} \cdot \frac{\overline{AF}}{BF} = \frac{4}{6} \cdot \frac{6}{4} \cdot \frac{5}{5} = 1.$ As thus, these three cevians are concurrent.

2. Given concurrent cevians AD, BE, CF in $\triangle ABC$ such that $\overline{BD} = 12, \overline{DC} = 6, \overline{EC} = 8, \overline{AE} = 15, \overline{BF} = 16,$ find $[ABC].$

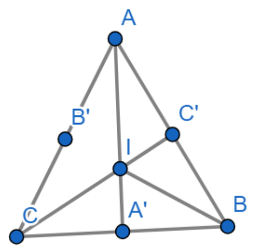
Solution: By Ceva's Theorem (6.5), $\frac{\overline{BD}}{DC} \cdot \frac{\overline{EC}}{AE} \cdot \frac{\overline{AF}}{BF} = \frac{12}{6} \cdot \frac{8}{15} \cdot \frac{\overline{AF}}{16} = 1.$ This implies $\overline{AF} = 15,$ which leads to $\overline{BC} = 18, \overline{AC} = 14, \overline{AB} = 31.$ Applying Heron's Formula (5.6), we see that $[ABC] = \sqrt{\frac{63}{2}(\frac{35}{2})(\frac{27}{2})(\frac{1}{2})} = \frac{\sqrt{3^2 \cdot 7 \cdot 5 \cdot 3^3}}{4} = \frac{9\sqrt{105}}{4},$ which is our answer.

3. Is there a triangle ABC with concurrent cevians AD, BE, CF such that $\overline{BD} = 12, \overline{DC} = 6, \overline{EC} = 8, \overline{AE} = 15, \overline{BF} = 18?$

Solution: If there is such a triangle, $\frac{\overline{BD}}{DC} \cdot \frac{\overline{EC}}{AE} \cdot \frac{\overline{AF}}{BF} = \frac{12}{6} \cdot \frac{8}{15} \cdot \frac{\overline{AF}}{18} = 1$ by Ceva's Theorem (6.5). This implies $\overline{AF} = \frac{135}{8},$ which leads to $\overline{BC} = 18, \overline{AC} = 14, \overline{AB} = 34 + \frac{7}{8}.$ This does not satisfy the triangle inequality, so no such triangle exists.

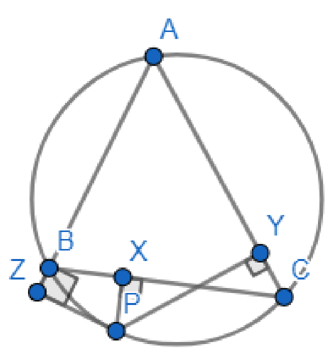
4. Use Menelaus's Theorem to prove the concurrency of angle bisectors of a triangle.

Solution: Note that we want to use Menelaus's Theorem (6.6) when we have proven that $\frac{\overline{AC'}}{BC'} \cdot \frac{\overline{BC}}{A'C} \cdot \frac{\overline{A'I}}{AI} = 1$ where we apply it on $\triangle A'BA,$ with points C', I, C on $AB, AA', A'B,$ respectively. Note that by the Angle Bisector Proportionality Theorem (7.1.1), $\frac{\overline{AC'}}{BC'} = \frac{\overline{AC}}{BC}$ and $\frac{\overline{A'I}}{AI} = \frac{\overline{BA'}}{BA},$ using $\triangle ACB$ and $\triangle ABA'$ with bisectors CC' and $BI,$ respectively. Substituting yields $\frac{\overline{AC'}}{BC'} \cdot \frac{\overline{BC}}{A'C} \cdot \frac{\overline{A'I}}{AI} = \frac{\overline{AC}}{BC} \cdot \frac{\overline{BC}}{A'C} \cdot \frac{\overline{BA'}}{BA} = \frac{\overline{AC} \cdot \overline{BA'}}{A'C \cdot BA},$ which is equivalent to 1 when the Angle Bisector Proportionality Theorem (7.1.1) is applied on $\triangle ABC$ with angle bisector AA', which gives us $\frac{\overline{AC}}{\overline{A'C}} = \frac{\overline{BA}}{\overline{BA'}}$, implying $\frac{\overline{AC} \cdot \overline{BA'}}{\overline{A'C} \cdot \overline{BA}} = 1$ as desired.



5. Prove that given $\triangle ABC$ and point P on the circumcircle of $\triangle ABC$, that the feet of the perpendiculars from P to AB, AC, BC are collinear.

Solution: Let the feet of the perpendiculars from P to AB, AC, BC be Z, Y, X, respectively. Then note that we can prove that $\angle ZXB = \angle YXC$ because vertical angles are equal when it is one straight line passing through the other. Note that right angles imply cyclic quadrilaterals. We use cyclic $ABPC$ and note that $\angle BPC = 180 - \angle A$. Then note that cyclic $AZPY$ gives us $\angle ZPY = \angle 180 - \angle A = \angle BPC$. Since $\angle ZPY$ and $\angle BPC$ share $\angle BPY$, we have $\angle ZPB = \angle YPC$. By cyclic $ZBXP$ and $XYCP$, respectively, $\angle ZXB = \angle ZPB$ and $\angle YPC = \angle YXC$. By the transitive property, $\angle ZXB = \angle YXC$, as desired.



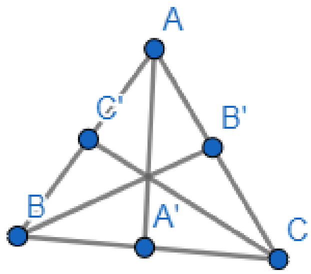
Note that this line is known as the Simson line of P with $\triangle ABC$.

6. Let the incircle of $\triangle ABC$ touch BC, CA, AB at X, Y, Z, respectively. Show that AX, BY, CZ are concurrent.

Solution: By the Two Tangents Theorem (3.5), $\overline{AY} = \overline{AZ}, \overline{ZB} = \overline{BX}, \overline{XC} = \overline{CY}$. By Ceva's Theorem (6.5), since $\frac{\overline{AZ}}{\overline{ZB}} \cdot \frac{\overline{BX}}{\overline{XC}} \cdot \frac{\overline{CY}}{\overline{AY}} = 1$, these cevians are concurrent. (The point they concur at is known as the Gergonne Point, and $\triangle XYZ$ has it as its symmedian point.)

7. Use the Angle Bisector Proportionality Theorem (7.1.1) and Ceva's Theorem (6.5) to prove the concurrency of angle bisectors (6.4). (To see Theorem 7.1.1, which you probably already know, go to the next section.)

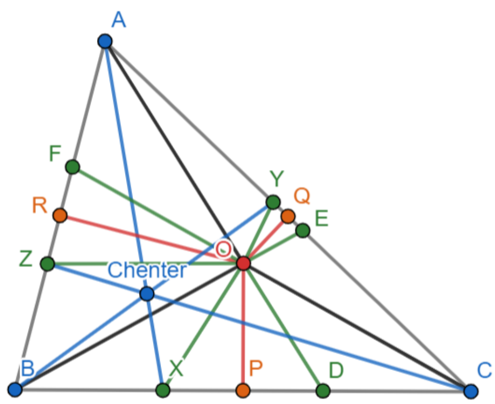
Solution: Note that by the Angle Bisector Proportionality Theorem (7.1.1), $\frac{\overline{AC}}{\overline{AC'}} = \frac{\overline{BC}}{\overline{BC'}}$, $\frac{\overline{BA}}{\overline{BA'}} = \frac{\overline{CA}}{\overline{CA'}}$, and $\frac{\overline{AB}}{\overline{AB'}} = \frac{\overline{CB}}{\overline{CB'}}$. We note that by Ceva's Theorem (6.5), we want to prove that $\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = 1.$ Our results from the Angle Bisector Proportionality Theorem (7.1.1) imply that $\frac{\overline{AC'}}{\overline{C'B}} = \frac{\overline{AC}}{\overline{BC}}$, $\frac{\overline{BA'}}{\overline{A'C}} = \frac{\overline{BA}}{\overline{CA}}$, and $\frac{\overline{CB'}}{\overline{B'A}} = \frac{\overline{CB}}{\overline{AB}}$. This substitution implies $\frac{\overline{AC'}}{\overline{C'B}} \cdot \frac{\overline{BA'}}{\overline{A'C}} \cdot \frac{\overline{CB'}}{\overline{B'A}} = \frac{\overline{AC}}{\overline{BC}} \cdot \frac{\overline{BA}}{\overline{CA}} \cdot \frac{\overline{CB}}{\overline{AB}} = 1,$ as desired.



8. Consider $\triangle ABC$ with cevians AD, BE, CF. Denote $\angle CAD = \alpha_1, \angle DAB = \alpha_2, \angle ABE = \beta_1, \angle CBE = \beta_2, \angle BCF = \gamma_1, \angle ECF = \gamma_2$. Prove that AD, BE, CF concur if and only if $\frac{\sin(\alpha_1)}{\sin(\alpha_2)} \cdot \frac{\sin(\beta_1)}{\sin(\beta_2)} \cdot \frac{\sin(\gamma_1)}{\sin(\gamma_2)} = 1.$ (This is known as the trigonometric form of Ceva.)

9. Consider $\triangle ABC$ with circumcenter O. Let the feet of the altitudes from O to BC, CA, AB be P, Q, R. Then let AO, BO, CO intersect BC, CA, AB at D, E, F, respectively. If X, Y, Z are the reflections of D about P, E about Q, and F about R, prove that AX, BY, CZ concur.

Solution: Notice P, Q, R are midpoints of BC, CA, AB. By the definition of reflection, $AF = BZ, AZ = BF, BD = CX, BX = CD, CE = AY, CY = AE$. By Ceva's (5.5), $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$ Substituting, $\frac{\overline{BZ}}{\overline{AZ}} \cdot \frac{\overline{CX}}{\overline{BX}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$ Applying Ceva (5.5) again proves that AX, BY, CZ concur.


A
F
Y
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Lengths in a Triangle

We shall take another look at the fourth problem in the previous section. The latter part asks basically asks for the lengths of the sections of BC split by angle bisector AD. Let's generalize. We want to find a formula for the lengths of the segments an angle bisector splits the side into. Applying a similar process to the one used for the fourth problem gets us our result.

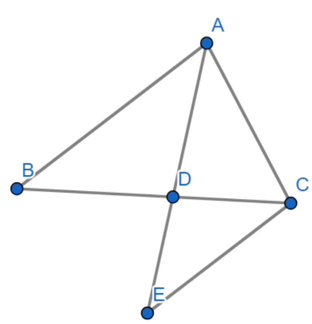
The Angle Bisector Proportionality Theorem (7.1.1)

Given $\triangle ABC$ with angle bisector AD, $\frac{\overline{AB}}{BD} = \frac{\overline{AC}}{CD}$.

Even though we first state the theorem, it is important to note that these theorems are a result of the proofs, not the other way around. We do not pull a theorem out of our butts and try to prove it. We play around with constructions and notice something that can be generalized to get our theorems. If you want to prove this yourself, look at the second part of the fourth problem's solution. You should be able to do so; give it a try!

Theorem 7.1.1's Proof

Note that since $BA \parallel CE$, $\triangle ABD \sim \triangle ECD$. This implies $\frac{\overline{AB}}{BD} = \frac{\overline{CE}}{CD}$. Note that since AD is an angle bisector, $\angle BAD = \angle CAD$, and since AD is a transversal to parallel lines BA and CE, $\angle BAD = \angle AEC$. By the transitive property, $\angle CAD = \angle AEC$, implying that $\triangle ACE$ is isosceles with $\overline{AC} = \overline{CE}$. We substitute this into $\frac{\overline{AB}}{BD} = \frac{\overline{CE}}{CD}$ and obtain $\frac{\overline{AB}}{BD} = \frac{\overline{AC}}{CD}$ as desired.



The Angle Bisector Length Theorem (7.1.2)

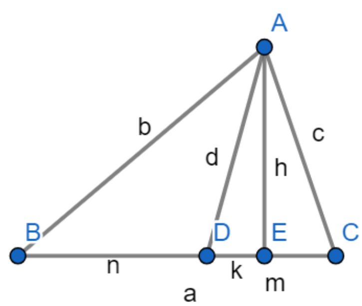
In $\triangle ABC$ with angle bisector AD, $\overline{AB} \cdot \overline{AC} - \overline{BD} \cdot \overline{BC} = \overline{AD}^2$.

Theorem 7.1.2's Proof

Assume our triangle is acute. Let's first get an equation that we can use to solve for $\overline{AD}$. Without loss of generality, have $\overline{AB} > \overline{AC}$. Then have $\overline{AB} = b$, $\overline{AC} = c$, $\overline{BD} = n$, $\overline{DE} = k$, $\overline{DC} = m$, $\overline{BC} = a$, $\overline{AD} = d$, and $\overline{AE} = h$. (Refer to the diagram instead of this line of text for the values; this is done for formality, and the diagram is labeled for convenience.) By the Pythagorean Theorem, the following three equations are true.

$$k^2 + h^2 = d^2$$
$$(k-m)^2 + h^2 = c^2$$
$$(k+n)^2 + h^2 = b^2$$

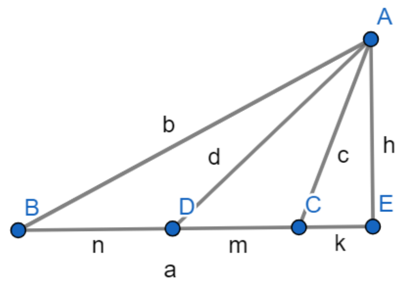
Let's expand the second and third equations. We see that expanding these equations yields $k^2 - 2mk + m^2 + h^2 = c^2$ and $k^2 + 2nk + n^2 + h^2 = b^2$. Multiplying the former by n and the latter by m gives us $nk^2 - 2mnk + nm^2 + nh^2 = nc^2$ and $mk^2 + 2mnk + mn^2 + mh^2 = mb^2$. Adding these two equations together gives us $(n+m)k^2 + (n+m)(nm) + (n+m)h^2 = cnc + bmb$. Substituting $n + m = a$ and our results from applying the Pythagorean Theorem, we see $man + dad = bmb + cnc$.



Now assume the triangle is right or obtuse. Dropping an altitude, have it have length h and intersect BC at E. Then have $\overline{CE} = k$. By the Pythagorean Theorem, the following equations are true.

$$k^2 + h^2 = c^2$$
$$(k+m)^2 + h^2 = d^2$$
$$(k+a)^2 + h^2 = b^2$$

Expanding the second and third equations give us $k^2 + 2km + m^2 + h^2 = d^2$ and $k^2 + 2ka + a^2 + h^2 = b^2$. Multiplying the first equation by a and the second by $-m$ yields $ak^2 + 2kam + am^2 + ah^2 = ad^2$ and $-mk^2 - 2kam - a^2m - mh^2 = -mb^2$. Adding these two equations together yields $(a - m)(k^2 + h^2) + (am)(m - a) = dad - bmb.$ Note that $a - m = n$ and $k^2 + h^2 = c^2.$ Applying these substitutions yield $cnc - man = dad - bmb.$ Rearranging gives us $man + dad = bmb + cnc,$ as desired.



We would like to prove that $\overline{AB} \cdot \overline{AC} - \overline{BD} \cdot \overline{BC} = \overline{AD}^2$, or $bc - nm = d^2.$ Taking our result above, we see that $d^2 = \frac{bmb+cnc}{a} - mn.$ Substituting $a = n + m$ into the equation gives us $d^2 = \frac{b^2m+c^2n}{m+n} - mn.$ Applying the Angle Bisector Proportionality Theorem (7.1.1), we see that $n = \frac{bm}{c}$ and $m = \frac{cn}{b}.$ Applying this substitution into $d^2 = \frac{b^2m+c^2n}{m+n} - mn$ gives us $d^2 = \frac{b^2m+c^2n}{\frac{bm}{c}+\frac{cn}{b}} - mn = \frac{b^2m+c^2n}{\frac{b^2m}{bc}+\frac{c^2n}{bc}} - mn = bc - mn.$ Substituting the lengths of the lines for the variables gives us $\overline{AB} \cdot \overline{AC} - \overline{BD} \cdot \overline{BC} = \overline{AD}^2$.

The lemma above that states $man + dad = bmb + cnc$ is very useful. It is known as Stewart's Theorem, and we shall formalize it. Whenever we have a problem involving lengths of a cevian, Stewart's Theorem will be involved in some way, whether it be directly or as part of the proof for a theorem that will be used to prove it.

Stewart's Theorem (7.2)

In $\triangle ABC$ with cevian AD and variables a, b, c, m, n, d representing $\overline{BC}, \overline{AB}, \overline{AC}, \overline{DC}, \overline{BD}, \overline{AD}$ respectively, $man + dad = bmb + cnc.$

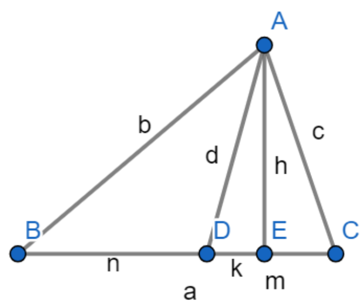
The proof is quite straightforward. I implore you to do this on your own instead of relying on the proof below. For the sake of completeness, it will be below, but please first look at the proof for the Angle Bisector Length Theorem (7.1.2) and try to find the proof for Stewart's there yourself. I recommend you try to write it using different variables; while I say this a lot, it is because this exercise works. It forces you to understand the relations between certain values, instead of memorizing them.

Theorem 7.2's Proof

Assume our triangle is acute. First, drop an altitude from A to BC that intersects BC at K. Without loss of generality, have $\overline{AB} > \overline{AC}$. Then have $\overline{DE} = k$ and $\overline{AE} = h$. (Refer to the diagram instead of this line of text for the values; this is done for formality, and the diagram is labeled for convenience.) By the Pythagorean Theorem, the following three equations are true.

$$k^2 + h^2 = d^2$$
$$(k - m)^2 + h^2 = c^2$$
$$(k + n)^2 + h^2 = b^2$$

Let's expand the second and third equations. We see that expanding these equations yields $k^2 - 2mk + m^2 + h^2 = c^2$ and $k^2 + 2nk + n^2 + h^2 = b^2$. Multiplying the former by n and the latter by m gives us $nk^2 - 2mnk + nm^2 + nh^2 = nc^2$ and $mk^2 + 2mnk + mn^2 + mh^2 = mb^2$. Adding these two equations together gives us $(n+m)k^2 + (n+m)(nm) + (n+m)h^2 = cnc + bmb$. Substituting $n + m = a$ and our results from applying the Pythagorean Theorem, we see $man + dad = bmb + cnc$.

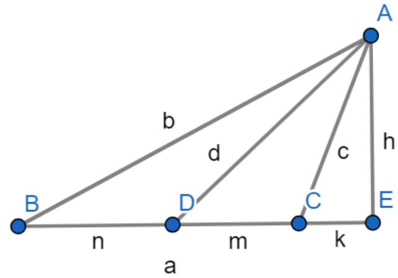


Now assume the triangle is right or obtuse. Dropping an altitude, have it have length h and intersect BC at E. Then have $\overline{CE} = k$. By the Pythagorean Theorem, the following equations are true.

$$k^2 + h^2 = c^2$$
$$(k + m)^2 + h^2 = d^2$$
$$(k + a)^2 + h^2 = b^2$$

Expanding the second and third equations give us $k^2 + 2km + m^2 + h^2 = d^2$ and $k^2 + 2ka + a^2 + h^2 = b^2$. Multiplying the first equation by a and the second by $-m$ yields $ak^2 + 2kam + am^2 + ah^2 = ad^2$ and $-mk^2 - 2kam - a^2m - mh^2 = -mb^2$. Adding these two equations together yields $(a-m)(k^2 + h^2) + (am)(m - a) = dad - bmb$. Note that $a - m = n$

and $k^2 + h^2 = c^2$. Applying these substitutions yield $cnc - man = dad - bmb$. Rearranging gives us $man + dad = bmb + cnc$, as desired.



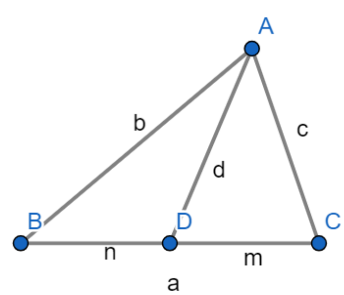
We shall now prove a formula for the length of a median. This formula is useful, and demonstrating the proof using Stewart's will make it all the more derivable in the middle of a contest. This is another useful formula because it is faster, and we can do more with less given information. However, this is not an essential formula because the same can be achieved with Stewart's.

Apollonius' Theorem (7.3)

Given $\triangle ABC$ with median AD, $\overline{AB}^2 + \overline{AC}^2 = 2(\overline{AD}^2 + \overline{BD}^2)$.

Theorem 7.3's Proof

By Stewart's Theorem, $man + dad = bmb + cnc$. Note that Apollonius' Theorem states $\overline{AB}^2 + \overline{AC}^2 = 2(\overline{AD}^2 + \overline{BD}^2)$, which can be translated into $b^2 + c^2 = 2d^2 + 2n^2$. Note that since $m = n$, the following implies $b^2 + c^2 = 2d^2 + 2mn$. Then note $2n = 2m = a$, and applying this substitution yields $b^2 + c^2 = 2d^2 + an$. Multiplying both sides by m and substituting yields $bmb + cnc = dad + man$, as desired.



No similar formula shall be provided for altitudes for the reason that the length of an altitude can easily be deduced from $\frac{bh}{2}$. Below are a few problems involving finding the lengths of certain cevians.

1. Consider $\triangle ABC$ such that $\overline{AB} = 3, \overline{AC} = 5.$ Angle bisector AD exists such that $\overline{AD}^2 = \frac{3[ABC]}{2 \cdot \sin(A)}.$ Find $\overline{DB}.$

2. Try to find a direct formula for the length of a median, as that may come handy in the next problem (though it certainly may not look that way).

3. Let $n = \frac{2}{3}\sqrt{\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{4}} + \frac{1}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}} + \frac{x}{2}$ and let $m = \frac{2}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}} + \frac{1}{3}\sqrt{\frac{x^2}{2} + \frac{z^2}{2} - \frac{y^2}{4}} + \frac{y}{2}.$ Prove $\frac{n}{m}(n - \frac{4}{3}\sqrt{\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{4}})(n - \frac{2}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}}) = (m - \frac{4}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}})(m - \frac{2}{3}\sqrt{\frac{x^2}{2} + \frac{z^2}{2} - \frac{y^2}{4}})\frac{(m-y)}{(n-x)}$ for all x, y, z such that $x, y, z < \frac{1}{2}(x + y + z).$

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1. Consider $\triangle ABC$ such that $\overline{AB} = 3, \overline{AC} = 5.$ Angle bisector AD exists such that $\overline{AD}^2 = \frac{3[ABC]}{2 \cdot \sin(A)}.$ Find $\overline{DB}.$

Solution: Recall that by $\frac{1}{2}ab \cdot sin(C)$ (5.3), $[ABC] = \frac{1}{2} \cdot \overline{AC} \cdot \overline{AB} \cdot sin(A).$ Applying this substitution yields $\overline{AD}^2 = \frac{3 \cdot \frac{1}{2} \cdot \overline{AC} \cdot \overline{AB} \cdot sin(A)}{2sin(A)} = \frac{45}{4}.$ The Angle Bisector Length Theorem (7.1.2) then implies $3 \cdot 5 - \overline{DB} \cdot \overline{DC} = (\frac{45}{4})^2$ or $\overline{DB} \cdot \overline{DC} = \frac{2025}{16} - 16 = \frac{1785}{16}.$ By the Angle Bisector Proportionality Theorem (7.1.1), $\overline{DC} = \frac{5}{3}\overline{DB}.$ The answer we desire comes from $\frac{5}{3}\overline{DB} \cdot \overline{DB} = \frac{1785}{16},$ or $\overline{DB} = \frac{3\sqrt{119}}{4}.$

2. Try to find a direct formula for the length of a median based on the side lengths of the triangle, as that may come handy in the next problem (though it certainly may not look that way).

Solution: Have $\triangle ABC$ and let our median be $AD.$ By Apollonius' Theorem (7.3), $\overline{AB}^2 + \overline{AC}^2 = 2(\overline{AD}^2 + \overline{BD}^2).$ Rearranging yields $\overline{AB}^2 + \overline{AC}^2 - 2\overline{BD}^2 = 2\overline{AD}^2.$ Dividing by 2 and taking the square root yields $\sqrt{\frac{\overline{AB}^2}{2} + \frac{\overline{AC}^2}{2} - \overline{BD}^2} = \overline{AD}.$ Substituting $\overline{BD} = \frac{1}{2}\overline{BC}$ gives us our formula of $\sqrt{\frac{\overline{AB}^2}{2} + \frac{\overline{AC}^2}{2} - \frac{\overline{BC}^2}{4}} = \overline{AD}.$

3. Let $n = \frac{2}{3}\sqrt{\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{4}} + \frac{1}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}} + \frac{x}{2}$ and let $m = \frac{2}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}} + \frac{1}{3}\sqrt{\frac{x^2}{2} + \frac{z^2}{2} - \frac{y^2}{4}} + \frac{y}{2}.$ Prove $\frac{n}{m}(n - \frac{4}{3}\sqrt{\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{4}})(n - \frac{2}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}}) = (m - \frac{4}{3}\sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}})(m - \frac{2}{3}\sqrt{\frac{x^2}{2} + \frac{z^2}{2} - \frac{y^2}{4}})\frac{(m-y)}{(n-x)}$ for all x, y, z such that $x, y, z < \frac{1}{2}(x + y + z).$

Solution: Messy algebra bash is possible, but it's not what we're looking for. For the sake of readability, we shall have $x' = \sqrt{\frac{y^2}{2} + \frac{z^2}{2} - \frac{x^2}{4}}, y' = \sqrt{\frac{x^2}{2} + \frac{z^2}{2} - \frac{y^2}{4}}, z' = \sqrt{\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{4}}.$ Substituting, we see that what we want to prove is equivalent to $n(n - x)(n - \frac{4}{3}z')(n - \frac{2}{3}x') = m(m - y)(m - \frac{4}{3}x')(m - \frac{2}{3}y').$ The expressions $\frac{2}{3}x'$ and $\frac{1}{3}x'$ (and their symmetric equivalents) are very suspicious. This is because they are forms of Apollonius' Theorem (7.3).

Consider a triangle with side lengths $x, y, z.$ The medians intersect at the centroid of a triangle. Since the centroid divides the median into segments of ratio $1:2,$ we attain our $\frac{2}{3}$ and $\frac{1}{3}$ coefficients. Then note that the medians split the triangle into six smaller triangles of equal area. By Heron’s Formula, $\sqrt{\frac{n}{2}(\frac{n-x}{2})(\frac{n-\frac{4}{3}z'}{2})(\frac{n-\frac{2}{3}x'}{2})} = \sqrt{\frac{m}{2}(\frac{(m-y)}{2})(\frac{m-\frac{4}{3}x'}{2})(\frac{m-\frac{2}{3}y'}{2})}.$ What we desire follows directly after multiplying both sides by $\sqrt{16}$ and squaring.

Remind yourself of what x', y', z' as you follow the solution. It may be hard to follow, so try to rewrite the solution using your own sets of variables to replace $m, n, x, y, z.$ (This is the same trick I commonly advocate for understanding proofs for theorems.) The last condition $(x, y, z < \frac{1}{2}[x+y+z])$ ensures that such a triangle exists.

Circles and Triangles

Circles and triangles are the fundamental shapes of Euclidean geometry. The reason for this is because every polygon can be split into triangles, and the only other shapes in Euclidean geometry are either a circle or transformations of a circle (think stretching and shrinking).

The first two circles we shall define are the incircle and circumcircle. We will also explore other circles, particularly those that inscribe triangles formed by the endpoints of cevians.

The *incircle* of a triangle is the unique circle that is inscribed within the triangle. (This means the circle intersects the circle at all of its sides exactly once.) The *incenter* of a triangle is the center of the incircle.

The *circumcircle* of a triangle is the unique circle circumscribing the triangle. (This means the circle intersects the triangle at all of its vertices, and only its vertices.) The *circumcenter* of a triangle is the center of the circumcircle.

We want to be able to construct incircles and circumcircles with precision and exactness. This means that we want a way to determine the incenter or circumcenter of a triangle consistently; drawing the inradius which we obtain from $[ABC] = rs$ (5.4) and drawing an circumradius by connecting the circumcenter to a vertice is trivial. To do that, we need to consider the properties of the inradius and circumradius. Try to figure out how to construct the inradius and exradius; we mentioned which points were the inradius and circumradius before. The theorems below will reveal the answer.

Construction of an Incircle (8.1)

The incenter of a triangle is formed by the point of concurrency of the angle bisectors.

Construction of a Circumcircle (8.2)

The circumcenter of a triangle is formed by the point of concurrency of the perpendicular bisectors.

The answers have been revealed. Can you prove that this construction *always works* on your own? The proofs are below for you if you are confused or want to check your work.

Theorem 8.1's Proof

Remembering the proof for the concurrency of angle bisectors (6.4), note that drawing perpendiculars from the incircle to the triangle yields three segments of the same length. Drawing tangents, these perpendiculars must be radii of a circle (this is elaborated on in an earlier exercise). The perpendiculars have the same length and touch the triangle, so we are done.

Theorem 8.2's Proof

Note that the perpendicular bisector of a line is the locus of points that is equidistant from said line. This implies our circumcenter is equidistant from the three vertices. By the definition of the circumcircle, we are done.

Let's consider cyclic quadrilaterals now. The circumcenter of the cyclic quadrilateral can be constructed in a similar fashion. Remember that it must be equidistant from all four of the vertices of the quadrilateral. Some properties of cyclic quadrilaterals will be presented as theorems below; however, I recommend you try to find them on your own first.

Circumcenter of a Cyclic Quadrilateral (8.3)

A cyclic quadrilateral's circumcenter is the point of concurrency of the perpendicular bisectors of all the sides of the quadrilateral. If a quadrilateral's perpendicular bisectors are not concurrent, then it is not cyclic.

We need to prove that the perpendicular bisectors are concurrent, and that the point they are concurrent on is in fact equidistant from all the points. We also need to prove the opposite; there is no point that is equidistant if the perpendicular bisectors are not concurrent.

Theorem 8.3's Proof

Have our quadrilateral be $ABCD.$ Note that the perpendicular bisector of a line is the set of points equidistant from its endpoints.

If the perpendicular bisectors of the sides of $ABCD$ are concurrent at a point $X,$ then the following four equations are implied.

$$\overline{AX} = \overline{BX}$$
$$\overline{BX} = \overline{CX}$$
$$\overline{CX} = \overline{DX}$$

$$\overline{DX} = \overline{AX}$$

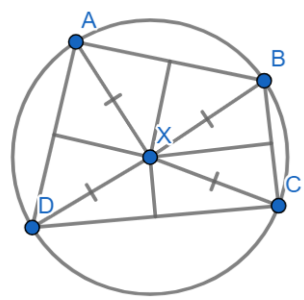
Applying the transitive property yields $\overline{AX} = \overline{BX} = \overline{CX} = \overline{DX}$. Since there is a point in the quadrilateral equidistant from all of its vertices, it is cyclic.

Conversely, if a quadrilateral is cyclic, then the perpendicular bisectors of the quadrilateral must be concurrent. Note that the circumcenter must be equidistant from all of the vertices, implying $\overline{AX} = \overline{BX} = \overline{CX} = \overline{DX}$.

Note that $\overline{AX} = \overline{BX}$ implies X lies on the perpendicular bisector of AB.
Note that $\overline{BX} = \overline{CX}$ implies X lies on the perpendicular bisector of BC.
Note that $\overline{CX} = \overline{DX}$ implies X lies on the perpendicular bisector of CD.
Note that $\overline{DX} = \overline{AX}$ implies X lies on the perpendicular bisector of DA.

By the definition of concurrency, AB, BC, CD, DA are concurrent because point X lies on all four of these lines.

This sets up an if and only if situation, so its inverse must also be true, as desired.



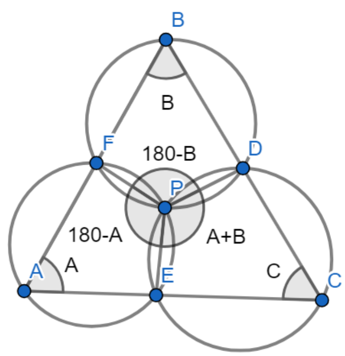
Let's explore more about circles and triangles. In particular, cevians have interesting properties. We could prove that altitudes, medians, and angle bisectors form concurrent circles with their feet and with the vertices of the triangle. (This is another example of when the definition of concurrency helps us out; we do not have to make other restrictions or exceptions such as, “three circles can be concurrent and intersect more than once.” This can also be generalized to planes and three dimensions!) However, there is something much more general which we could do. Consider the theorem below; it is an example of when we can draw two shapes (in this case, circles) and note that the third one also must pass through the common intersection point.

Miquel's Theorem (8.4)

Consider cevians AD, BE, CF in $\triangle ABC$. The circumcircles of $\triangle AEF$, $\triangle BDF$, $\triangle CDE$ are all concurrent.

Theorem 8.4's Proof

Have the circumcircles of $\triangle AEF$ and $\triangle BDF$ intersect inside the triangle at P. Then note that since $AEPF$ and $BDPF$ are cyclic, $\angle EPF = 180^\circ - \angle A$ and $\angle DPF = 180^\circ - \angle B$. This implies that $\angle DPE = \angle A + \angle B$, and since $\angle DPE + \angle ACB = \angle A + \angle B + \angle C = 180^\circ$, $CDPE$ is cyclic as well, implying that P lies on the circumcircle of $\triangle CDE$. Since P lies on all three circumcircles, the three circumcircles are concurrent, as desired.



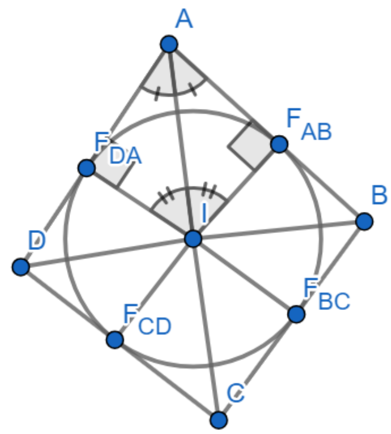
The opposite of a cyclic quadrilateral, known as a tangential quadrilateral, is a quadrilateral that can have a circle inscribed within it. We shall prove a few properties of those as well.

Incenter of a Tangential Quadrilateral (8.5)

If and only if the angle bisectors of a quadrilateral are concurrent, then the quadrilateral is tangential.

Theorem 8.5's Proof

Have quadrilateral $ABCD$ such that the angle bisectors of $ABCD$ are concurrent. As thus, they all pass through a point I. Have the perpendiculars from I to AB, BC, CD, DA be $F_{AB}, F_{BC}, F_{CD}, F_{DA}$ respectively. Drawing AI, BI, CI, DI gives us $\triangle AF_{DA}I \cong \triangle AF_{AB}I$ by SAS congruence. (This is illustrated in the diagram.) Similarly, $\triangle BF_{AB}I \cong \triangle BF_{BC}I$, $\triangle CF_{BC}I \cong \triangle CF_{CD}I$, $\triangle DF_{CD}I \cong \triangle DF_{DA}I$. This implies that $\overline{IF_{DA}} = \overline{IF_{AB}}, \overline{IF_{AB}} = \overline{IF_{BC}}, \overline{IF_{BC}} = \overline{IF_{CD}}, \overline{IF_{CD}} = \overline{IF_{DA}}$. By the transitive property, $\overline{IF_{AB}} = \overline{IF_{BC}} = \overline{IF_{CD}} = \overline{IF_{DA}}$. Since there is a point I whose perpendiculars to the sides are equivalent (recall Theorem 3.4), the quadrilateral is tangential, as desired.



If the quadrilateral does not have concurrent angle bisectors, then there exists no point $I,$ and there is no point whose perpendiculars to the side are the same length.

Here are two formulas for cyclic quadrilaterals. Even though you may not know trigonometry, using the methods for similarity and area of a triangle will get you some progress on the proofs.

Ptolemy's Theorem (8.6)

Given a cyclic quadrilateral with sides of lengths a, b, c, d and diagonals of lengths $p, q,$ $ac + bd = pq.$

Brahmagupta's Formula (8.7)

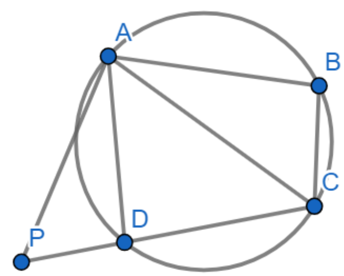
Given a cyclic quadrilateral with sides of lengths $a, b, c, d,$ the area of said quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}.$

Brahmagupta's Formula, based on the information given by this book, can be proved as a result of Ptolemy's. First, try to prove Ptolemy's, then, whether you are successful or not, try to use Ptolemy's in conjunction with Heron's (5.6) to prove Brahmagupta's. Similar triangles are involved in both proofs.

Theorem 8.6's Proof

Have our cyclic quadrilateral be $ABCD.$ Have point P on ray CD such that $\angle BAD = \angle CAP.$ Since $ABCD$ is cyclic, $\angle B + \angle D = 180^\circ,$ and since $\angle ADP$ is supplementary to $\angle D,$ $\angle ADP + \angle D = 180^\circ,$ implying $\angle B = \angle ADP.$ Then note $\triangle ABC \sim \triangle ADP$ because two of the angles are the same; it is given that $\angle BAD = \angle CAP,$ implying that $\angle BAC = \angle DAP,$ and we have $\angle ABC = \angle ADP$ as well.

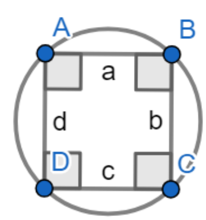
This implies that $\frac{\overline{AB}}{AD} = \frac{\overline{BC}}{DP}$, or that $\overline{DP} = \frac{\overline{AD}\cdot\overline{BC}}{AB}$. The Inscribed Angle Theorem (1.1) implies that $\angle ABD = \angle ACD$ since these two angles subtend the same arc. When used in conjunction with the fact that $\angle BAC = \angle DAP$, we get that $\triangle BAD \sim \triangle CAP$, which gives us $CP = \frac{\overline{AC}\cdot\overline{BD}}{\overline{AB}}$. We note that $CP = CD + DP$, and substituting our previous two results for CP and CD yields $\frac{\overline{AC}\cdot\overline{BD}}{\overline{AB}} = \overline{CD} + \frac{\overline{AD}\cdot\overline{BC}}{\overline{AB}}$. Multiplying both sides by $\overline{AB}$ gives us $\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC}$, and substituting line and diagonal lengths for arbitrary values p, q, a, b, c, d gives us $ac + bd = pq$, as desired.



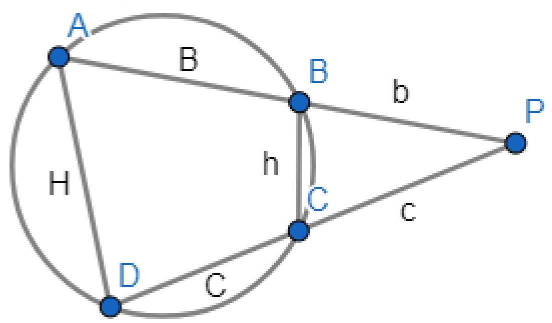
Theorem 8.7's Proof

Have our cyclic quadrilateral be $ABCD$. We have two cases; the first is if the quadrilateral is a parallelogram, and the second is if it is not.

If the quadrilateral is a parallelogram, it must be a rectangle, because the opposite angles must sum to 180°. In this case, Brahmagupta's Theorem is obviously true; some algebraic manipulation will get us there. Note that $s = a + b$, and that $a = c$ and $b = d$; substitution gives us $[ABCD] = \sqrt{(s-a)(s-b)(s-c)(s-d)} = (s-a)(s-b) = ab$.



If the quadrilateral is not a parallelogram, then we can extend its sides such that they meet. Without loss of generality, have rays BA and CD meet at P. (If these rays do not meet, then rays AB and DC will meet, and if they are parallel, the other pair of sides cannot be, because we already have covered that case.) Then note that $\triangle PBC \sim \triangle PDA$, and that the ratio of similarity is $\frac{\overline{BC}}{DA}$. Have $\overline{PB} = b$, $\overline{PC} = c$, $\overline{BC} = h$, $\overline{DA} = H$, $\overline{BA} = B$, $\overline{CD} = C$, S be the semiperimeter of $\triangle PDA$, and have s be the semiperimeter of $\triangle PBC$. By Heron's Theorem (4.6), $[ABCD] = \sqrt{S(S-b-B)(S-c-C)(S-H)} - \sqrt{s(s-b)(s-c)(s-h)}$. By our ratio of similarity, $[ABCD] = \frac{H^2-h^2}{H^2}\sqrt{S(S-b-B)(S-c-C)(S-h-H)}$, $B+b = \frac{H}{h}b$, and $C+c = \frac{H}{h}c$. Substituting yields $[ABCD] = \frac{H^2-h^2}{H^2}\sqrt{S(S-\frac{H}{h}b)(S-\frac{H}{h}c)(S-H)}$. This implies we want to find b and c in terms of B, C, H, h. Note that by similarity, $\frac{B+b}{H} = \frac{c}{h}$, and $\frac{C+c}{H} = \frac{b}{h}$. This implies that $b = \frac{cH}{h} - B$ and $c = \frac{bH}{b} - C$. (Make sure you understand this; this is a crucial step!) By more similarity, $bh + Bh = cH$, or $bh + Bh = (\frac{bH}{b} - C)H$. Algebraic manipulations yield $b = \frac{h(Bh+CH)}{H^2-h^2}$ and $c = \frac{h(Ch+BH)}{H^2-h^2}$. More substitution (have the semiperimeter of $ABCD$ be S_Q for clarity) gives us $[ABCD] = \frac{H^2-h^2}{H^2}\sqrt{\frac{H^4}{(H-h)^2(H+h)^2}\cdot(S_Q-B)(S_Q-C)(S_Q-h)(S_Q-H)}$. Taking out $\frac{H^4}{(H-h)^2(H+h)^2}$ gives us $\frac{H^2}{H^2-h^2}$, and substituting this into the formula gives us $[ABCD] = \frac{H^2-h^2}{H^2}\cdot\frac{H^2}{H^2-h^2}\sqrt{(S_Q-B)(S_Q-C)(S_Q-h)(S_Q-H)}$, as desired.



We will now introduce a lemma that ties together the two circle centers of a triangle (the incenter and the excenter, hence the name Incenter-Excenter Lemma). It is a relatively well-known theorem, and it will pop up occasionally in this book as well. (See the Inversion section of the book for a great exercise involving the Incenter-Excenter Lemma.)

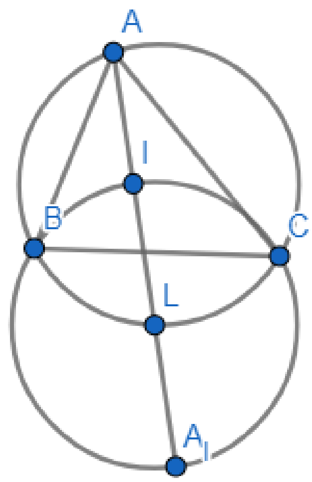
Incenter-Excenter Lemma (8.8)

Consider $\triangle ABC$ with incenter I, let I_A be the A excenter of $\triangle ABC$, and have L be the midpoint of $arc(BC)$, where $arc(BC)$ is an arc of the circumcircle of $\triangle ABC$. We claim that L is the center of a circle that intersects I, B, C, I_A.

The A excenter of $\triangle ABC$ is the center of the circle tangent to the extensions of rays AB, AC and line segment BC.

Theorem 8.8's Proof

Note that A, I, L, I_A are collinear since L lies on the angle bisector of $\angle A$. Then note that we want to prove $\overline{LI} = \overline{LB} = \overline{LC} = \overline{LA_I}$. We can show $\overline{LI} = \overline{LB}$, with the other case being symmetrical. Note that $\angle LBI = \angle LBC + \angle CBI$. Since $\angle LBC$ and $\angle LAC$ subtend the same arc, $\angle LBC = \angle LAC$, which implies $\angle LBI = \angle LAC + \angle CBI = \frac{1}{2}\angle A + \frac{1}{2}\angle B$. Similarly, $\angle BIL = \angle BAI + \angle ABI = \frac{1}{2}\angle A + \frac{1}{2}\angle B$, implying that $\triangle BIL$ is isosceles with $\overline{LB} = \overline{LI}$. Symmetrical solving gives us $\overline{LI} = \overline{LB} = \overline{LC}$. Then note that since BL and $A_I L$ are transversals, $\angle LBA_I = \angle LA_I B$, implying $\overline{LI} = \overline{LB} = \overline{LC} = \overline{LA_I}$, and we are done.



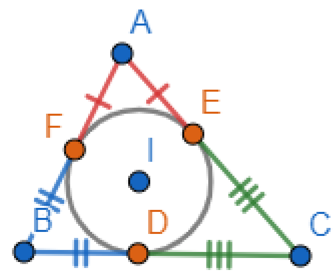
After introducing the incenter and incircle in conjunction with the excenter and excircles, we present two nice length lemma based off of tangents for the incircle and excircle.

Incenter Length Lemma (8.9)

Let the incircle of $\triangle ABC$ be tangent to BC at D. Then $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$.

Theorem 8.9's Proof

Let the incircle be tangent to CA, AB at E, F, respectively. Notice that $\overline{AB} + \overline{CD} = \overline{AF} + \overline{FB} + \overline{CD}$, and $\overline{AC} + \overline{BD} = \overline{AE} + \overline{BD} + \overline{EC}$. But notice that by the Two Tangent Theorem (3.5), $\overline{AF} = \overline{AE}$, $\overline{FB} = \overline{BD}$, and $\overline{CD} = \overline{EC}$. Thus, $\overline{AB} + \overline{CD} = \overline{AF} + \overline{FB} + \overline{CD} = \overline{AC} + \overline{BD} = \overline{AE} + \overline{BD} + \overline{EC}$, implying $\overline{AB} + \overline{CD} = \overline{AC} + \overline{BD}$, as desired.



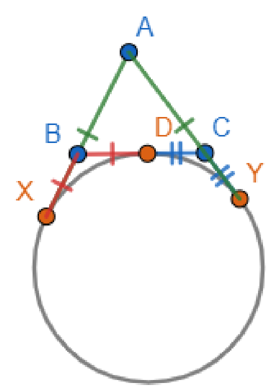
Excenter Length Lemma (8.10)

Let the A excircle of $\triangle ABC$ be tangent to BC at D. Then $\overline{AB} + \overline{BD} = \overline{AC} + \overline{CD} = s,$ where s is the semiperimeter of $\triangle ABC$.

Theorem 8.10's Proof

Let the excircle be tangent to AB, AC at X, Y, respectively.

Notice that by the Two Tangent Theorem (3.5), $\overline{BD} = \overline{BX}$ and $\overline{CD} = \overline{CY}$. This implies $\overline{AB} + \overline{BD} = \overline{AB} + \overline{BX} = \overline{AX},$ and $\overline{AC} + \overline{CD} = \overline{AC} + \overline{CY} = \overline{AY}$. But notice that another application of the Two Tangent Theorem (3.5) yields $\overline{AX} = \overline{AY}$, implying that $\overline{AB} + \overline{BD} = \overline{AC} + \overline{CD}$. As $\overline{AB} + \overline{AC} + \overline{BD} + \overline{CD} = 2s,$ we have proven that $\overline{AB} + \overline{BD} = \overline{AC} + \overline{CD} = s,$ as desired.



This lemma implies that $\overline{BD} = s - c$ and $\overline{CD} = s - b$. (Can you see why?)

Finally, we introduce the Gergonne and Nagel points, triangle centers that are not as widely known.

Gergonne Point (8.11)

Let the incircle of $\triangle ABC$ intersect BC, CA, AB at D, E, F. Then AD, BE, CF concur at a point known as the Gergonne Point.

Theorem 8.11's Proof

By the Two Tangents Theorem (3.5), $\overline{AY} = \overline{AZ}$, $\overline{ZB} = \overline{BX}$, and $\overline{XC} = \overline{CY}$. By Ceva's Theorem (6.5), since $\frac{\overline{AZ}}{\overline{ZB}} \cdot \frac{\overline{BX}}{\overline{XC}} \cdot \frac{\overline{CY}}{\overline{AY}} = 1$, these cevians are concurrent.

Proving that this triangle center is called the Gergonne Point is left as an exercise for a search engine.

Nagel Point (8.12)

Let the A excircle of $\triangle ABC$ be tangent to BC at D, the B excircle be tangent to CA at E, and the C excircle be tangent to AB at F. Then AD, BE, CF concur at a point known as the Nagel Point.

Theorem 8.12's Proof

Notice that $\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} \cdot \frac{s-b}{s-a} = 1$. Thus, by Ceva's Theorem (6.5), they concur.

Most of these theorems are not very important on their own, but they show a method of thinking that will be important. Below are a few problems related to the theorems we showed and the proofs for them.

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1. Given a cyclic polygon, find a general construction for its circumcenter.

2. Given a tangential polygon, find a general construction for its incenter.

3. A cyclic quadrilateral has sides $5, 7, 8, 10,$ in that order. Find the product of the lengths of its diagonals.

4. A cyclic quadrilateral has side lengths $4, 5, 6, 7.$ Find its area.

5. Consider cyclic quadrilateral with sides $20, 20, 52, x$ in that order, with diagonals whose lengths multiply to $40 \cdot 63.$ Find its perimeter.

6. Consider tangential quadrilateral $ABCD.$ If $\overline{AB} = 6,$ and $\overline{CD} = 8,$ find the perimeter of $ABCD.$

7. Consider arbitrary $\triangle ABC$. Construct the excenters of $\triangle ABC$. (You may only use a straightedge and compass for this problem.)

8. Prove that for a cyclic quadrilateral with a fixed perimeter, that its area is maximized when the side lengths are equal.

9. Consider cyclic quadrilateral $ABCD$ with point X on BC. Have line AX intersect BD at Y such that $\overline{DY} = 3\overline{YB}$. Have the line that intersects B and is perpendicular to BD and the extension of AX intersect at I. If $CI \parallel BD$ and $[XYB] = 2$, find $[ABCD]$.

10. Consider $\triangle ABC$ such that $\overline{AB} = 8, \overline{BC} = 5, \overline{CA} = 7$. Let AB, CA be tangent to the incircle at T_C, T_B, respectively. Find $[AT_BT_C]$.

11. Prove that the incenter and circumcenter of a triangle are the same point if and only if the triangle is equilateral.

12. Consider $\triangle ABC$ with D on segment BC, E on segment CA, and F on segment AB. Let the circumcircles of $\triangle FBD$ and $\triangle DCE$ intersect at $P \neq D$. If $\angle A = 50^\circ, \angle B = 35^\circ$, find $\angle DPE$.

13. Let A, B, C be points such that $\angle ABC = 90^\circ$ and $\overline{AB} = \overline{BC} = 5$. Then consider a circle of radius 2 tangent to segments AB and BC. Let X, Y be points on the circle such that AX and BY are tangent to the circle. If AX and CY intersect at P, find $[PXY]$.

14. Consider circle O with diameter AB. Let T be on the circle such that $TA < TB$. Let the tangent line through T intersect AB at X and intersect the tangent line through B at Y. Let M be the midpoint of YB, and let XM intersect circle O at P and Q. If $\overline{XP} = \overline{MQ}$, find AT.

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1. Given a cyclic polygon, find a general construction for its circumcenter.

Solution: By similar reasoning to the construction of a cyclic quadrilateral's circumcenter, drawing the perpendicular bisectors of all the sides of the polygon, they are concurrent at the circumcenter of the polygon.

2. Given a tangential polygon, find a general construction for its incenter.

Solution: Drawing the angle bisectors of all the angles, they are concurrent at the incenter of the polygon.

3. A cyclic quadrilateral has sides $5, 7, 8, 10,$ in that order. Find the product of the lengths of its diagonals.

Solution: By Ptolemy's Theorem (8.6), the product of the diagonals is $5 \cdot 8 + 7 \cdot 10 = 130.$

4. A cyclic quadrilateral has side lengths $4, 5, 6, 7.$ Find its area.

Solution: Note that the semiperimeter is $11.$ By Brahmagupta's Theorem (8.7), the area of the quadrilateral is $\sqrt{(11-4)(11-5)(11-6)(11-7)} = 2\sqrt{210}.$

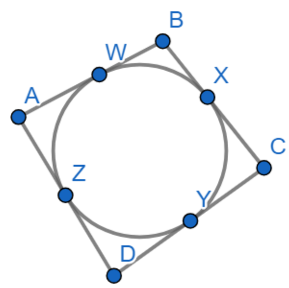
5. Consider cyclic quadrilateral with sides $20, 20, 52, x$ in that order, with diagonals whose lengths multiply to $40 \cdot 63.$ Find its perimeter.

Solution: Note that by Ptolemy's Theorem (8.6), $20 \cdot 52 + 20x = 40 \cdot 63 = 20 \cdot 126.$ Dividing both sides through by 20 yields $52 + x = 126,$ or $x = 74.$ Then note that the perimeter is $166,$ which is what we desired.

6. Consider tangential quadrilateral $ABCD.$ If $\overline{AB} = 6,$ and $\overline{CD} = 8,$ find the perimeter of $ABCD.$

Solution: Let the incircle of $ABCD$ be $O.$ Have AB, BC, CD, DA intersect O at $W, X, Y, Z,$ respectively. Note that by the Two Tangent Theorem (3.5), $\overline{AZ} = \overline{AW}, \overline{BW} = \overline{BX}, \overline{CX} = \overline{CY}, \overline{DY} = \overline{DZ}.$ We are trying to find

$\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA} = 2(\overline{AZ} + \overline{DZ} + \overline{BX} + \overline{CX}) = 2(\overline{AB} + \overline{CD}) = 2 \cdot 14 = 28.$ As thus, the perimeter of tangential quadrilateral $ABCD$ is $28.$



7. Consider arbitrary $\triangle ABC.$ Construct the excenters of $\triangle ABC.$ (You may only use a straightedge and compass for this problem.)

Solution: Without loss of generality, we will just construct the A excenter of $\triangle ABC.$ We notice that we want I_A equidistant from the *extensions* of $AB, AC,$ so we draw the perpendicular of the angle bisector of $\angle B, \angle C,$ which turns out to bisect the supplementary angles next to them. Then, we draw the angle bisector of $\angle A,$ and they intersect at $I_A.$

Let the distance from I_A to AB, BC, CA be $z, y, x,$ respectively. This works because by the definition of an angle bisector, $x = y, y = z, z = x \to x = y = z,$ as desired. Also, the A excenter falls out of $\triangle ABC$ and ends up on the other side of $BC,$ as desired.

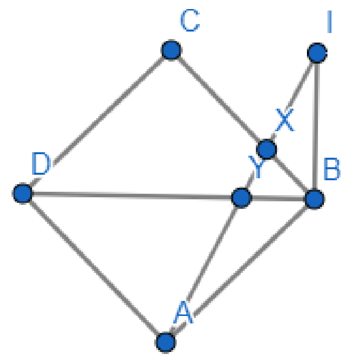
(A construction using the Incenter-Excenter Lemma (8.8), however, may be more accurate.)

8. Prove that for a cyclic quadrilateral with a fixed perimeter, that its area is maximized when the side lengths are equal.

Solution: Note that by Brahmagupta's Theorem (8.7), the area of the quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}.$ Then note that s is fixed as the perimeter is fixed. Have $s - a = a',$ $s - b = b',$ $s - c = c',$ and $s - d = d'.$ Then note $a' + b' + c' + d'$ is fixed. By AM-GM, $(\frac{2s}{4})^2 \geq \sqrt{a'b'c'd'},$ with equality occurring when $a' = b' = c' = d',$ and we are done.

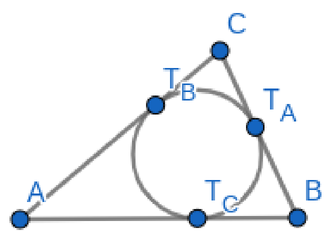
9. Consider cyclic quadrilateral $ABCD$ with point X on $BC.$ Have line AX intersect BD at Y such that $\overline{DY} = 3\overline{YB}.$ Have the line that intersects B and is perpendicular to BD and the extension of AX intersect at $I.$ If $CI \parallel BD$ and $[XYB] = 2,$ find $[ABCD].$

Solution: Reversing this process on the other side (let I' be the reflection of I across C in this solution) gives us $\overline{IC} = \overline{I'C}$. Reflecting across BD, this implies $ABCD$ is a square. (It must be a square, otherwise the quadrilateral with equal side lengths will not be cyclic.) Note that this implies $AB \parallel CD$, and by AAA similarity, $\triangle XYB \sim \triangle ADY$. This implies that the altitude of $\triangle XYB$ is $\frac{1}{3}$ of the altitude of $\triangle ADY$ (which is the altitude of $\triangle BCD$ as well) and the base of $\triangle XYB$, which is YB, is $\frac{1}{4}$ the length of DB. This implies that $[BCD] = 12[XYB] = 24$, which leads to $[ABCD] = 2[BCD] = 48$.



10. Consider $\triangle ABC$ such that $\overline{AB} = 8, \overline{BC} = 5, \overline{CA} = 7$. Let AB, CA be tangent to the incircle at T_C, T_B, respectively. Find $[AT_BT_C]$.

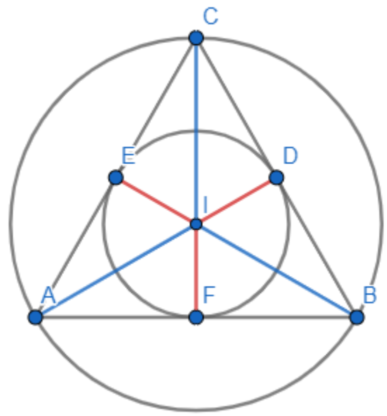
Solution: Add extra point T_A where the incircle touches BC. By the Two Tangent Theorem (3.5), $\overline{AT_B} = \overline{AT_C}, \overline{BT_C} = \overline{BT_A}, \overline{CT_A} = \overline{CT_B}$, and $\overline{AT_C} + \overline{BT_C} = 8, \overline{BT_A} + \overline{CT_A} = 5, \overline{CT_B} + \overline{AT_B} = 7$ based on our given lengths. Thus, $\overline{AT_B} = \overline{AT_C} = 5$.



Then we use Heron's Formula (5.6) and note that $[ABC] = 10\sqrt{3}$. By $\frac{1}{2}ab \cdot \sin(C)$ (5.3), $\frac{[AT_BT_C]}{[ABC]} = \frac{0.5\sin(A)\cdot\overline{AT_B}\cdot\overline{AT_C}}{0.5\sin(A)\cdot\overline{AB}\cdot\overline{AC}} = \frac{5\cdot5}{8\cdot7} = \frac{25}{56}$. Plugging in $[ABC]$ yields $\frac{[AT_BT_C]}{10\sqrt{3}} = \frac{25}{56} \rightarrow [AT_BT_C] = \frac{125\sqrt{3}}{28}$.

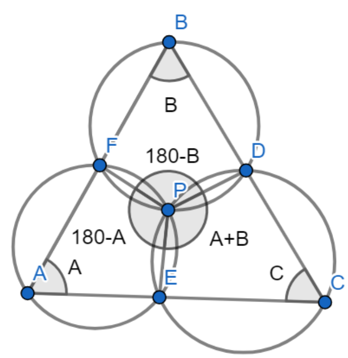
11. Prove that the incenter and circumcenter of a triangle are the same point if and only if the triangle is equilateral.

Solution: An equilateral triangle trivially implies that the incenter and circumcenter are the same point. If the incenter and circumcenter are the same point, then let AD, BE, CF be angle bisectors, and let the incenter be I. Notice that by HL congruence, $\overline{AF} = \overline{FB} = \overline{BD} = \overline{DC} = \overline{CE} = \overline{EA} \rightarrow \overline{AB} = \overline{BC} = \overline{CA}$, as desired.



12. Consider $\triangle ABC$ with D on segment BC, E on segment CA, and F on segment AB. Let the circumcircles of $\triangle FBD$ and $\triangle DCE$ intersect at $P \neq D$. If $\angle A = 50^\circ, \angle B = 35^\circ$, find $\angle DPE$.

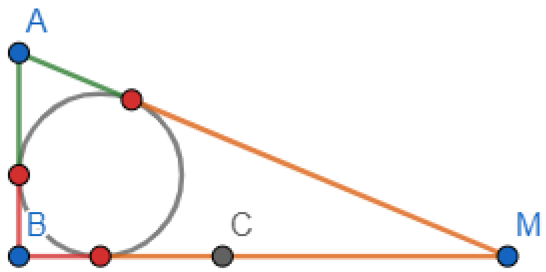
Solution: By Miquel's Theorem (8.4), $AEOF$ is a cyclic quadrilateral. Then notice that $\angle DPE = \angle A + \angle B = 85^\circ$.



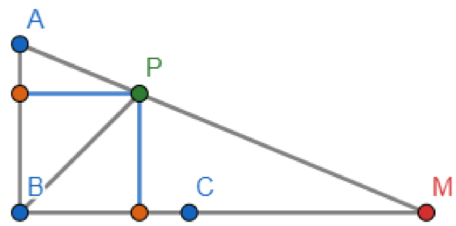
13. Let A, B, C be points such that $\angle ABC = 90^\circ$ and $\overline{AB} = \overline{BC} = 5$. Then consider a circle of radius 2 tangent to segments AB and BC. Let X, Y be points on the circle such that AX and BY are tangent to the circle. If AX and CY intersect at P, find $[PXY]$.

Solution: This solution involves a light use of coordinates.

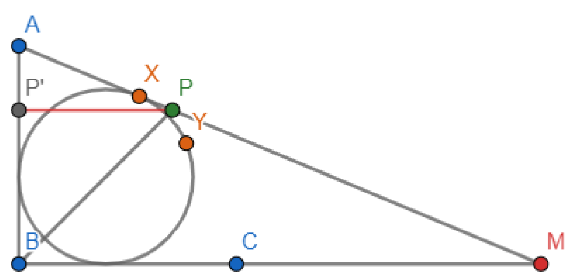
Let the tangent from A to the circle meet at M. By the Two Tangent Theorem (3.4), $\overline{AM} = \overline{AB} + \overline{BC} - 2r = 5 + \overline{BM} - 4 = 1 + \overline{BM}$. By the Pythagorean Theorem, $5^2 + \overline{BM}^2 = (\overline{BM} + 1)^2$. Notice that $\overline{BM} = 12$ via the Pythagorean triple.



By symmetry, P must lie on the angle bisector of $\angle ABC$, and by the conditions of the problem, P must lie on AM. Since the side length of a square inscribed in a triangle is $\frac{bh}{b+h}$ where b is a base and h is its corresponding height, $\overline{BP} = \sqrt{2} \cdot \frac{12 \cdot 5}{12+5} = \frac{60\sqrt{2}}{17}$.

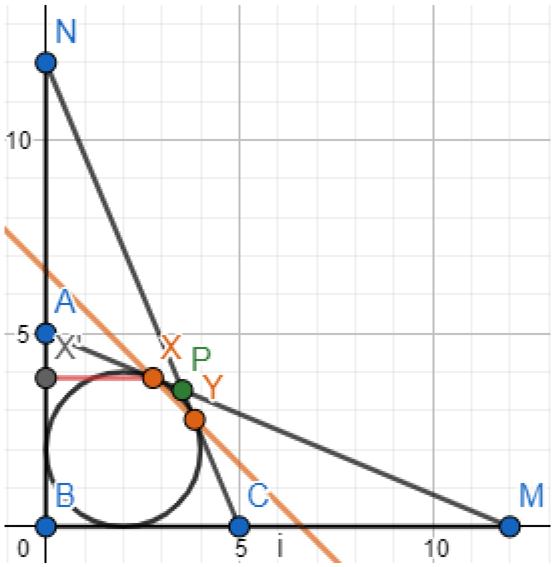


Now all we need to do is find $\overline{XP}$ and find the altitude from P to XY. Finding $\overline{XP}$ is equivalent to finding $\overline{AP} - \overline{AX}$. By the Two Tangent Theorem (3.4), $\overline{AX} = 3$. Since $\overline{PP'} = \frac{60}{17}$, by similarity, $\overline{AP} = \frac{65}{17}$, so $\overline{XP} = \frac{65}{17} - 3 = \frac{14}{17}$.



Now for the light use of coordinates. If $B = (0,0)$ and AB and BC are the y and x axes respectively, then $X = (3 \cdot \frac{12}{13}, 5 - 3 \cdot \frac{5}{13}) = (\frac{36}{13}, \frac{50}{13})$. It is also obvious that the slope of XY is -1, so X lies on $x + y = \frac{86}{13}$. Thus, the distance from B to XY is $\frac{43\sqrt{2}}{13}$. It is obvious that the distance from P to XY is $\overline{PB} - \frac{43\sqrt{2}}{13} = \frac{60\sqrt{2}}{17} - \frac{43\sqrt{2}}{13} = \frac{49\sqrt{2}}{221}$. Since

$\overline{PX} = \frac{14}{17}$, $\overline{XY} = 2\sqrt{PX^2 - (\frac{49\sqrt{2}}{221})^2} = 2\sqrt{(\frac{14}{17})^2 - (\frac{49\sqrt{2}}{221})^2} = \frac{14}{17}\sqrt{2^2 - (\frac{7\sqrt{2}}{13})^2} = \frac{14}{17}\sqrt{\frac{578}{169}}$.
Simplifying yields $\overline{XY} = \frac{14\sqrt{2}}{13}$. Then $\triangle PXY = \frac{49\sqrt{2}}{221} \cdot \frac{14\sqrt{2}}{13} \cdot \frac{1}{2} = \frac{686}{2873}$, which is our answer.



Now, we will discuss concyclic points. If there exists a point X such that $\overline{A_1X} = \overline{A_2X} = ... = \overline{A_nX},$ then $A_1, A_2 ... A_n$ are concyclic. Two points are trivially always concyclic, as are three (think diameter of a circle and circumcircle). However, four points are not always concyclic, and it is not so trivial either. For four points to be concyclic, all of these conditions have to be true. One of them being true implies all of them are true, and one of them being false implies all of them are false.

Without loss of generality, have the four points be A, X, B, Y in that order. Have AB, XY intersect at P; then $\overline{AP} \cdot \overline{BP} = \overline{XP} \cdot \overline{YP}.$

The quadrilateral formed by connecting the four points is cyclic. If it is convex and does not self-intersect, opposite angles sum up to $180^{\circ}.$

Without loss of generality, have the four points be A, B, C, D in that order. Then $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} = \overline{AC} \cdot \overline{BD}.$

Make two line segments with any of the points. Draw their perpendicular bisectors, find the point of concurrency, and check if it is equidistant from all four points. If so, then the four points are concyclic; otherwise, they are not.

Below are some problems involving concyclic points.

---

1. Consider self-intersecting cyclic quadrilateral $ABCD,$ such that BC and DA intersect at point $P.$ If $\angle BAP + \angle CDP = 140^{\circ},$ find $\angle BPD.$

2. Prove that given points A, B, C, D such that $\overline{AB} = \overline{BC}$ and $\overline{CD} = \overline{DA},$ $[ABCD] = \overline{AB} \cdot \overline{CD}.$

3. Prove that given concyclic points $A, B, C, D,$ that the perpendicular bisectors of AB and CD intersect at the center of the circle that contains all four points.

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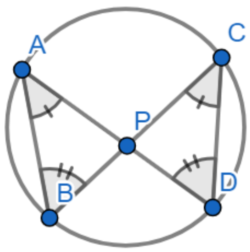
1. Consider self-intersecting cyclic quadrilateral $ABCD$, such that BC and DA intersect at point P. If $\angle BAP + \angle CDP = 140^\circ$, find $\angle BPD$.

Solution: Note that since $\angle CDP$ and $\angle ABP$ subtend the same arc, $\angle BAP + \angle CDP = \angle BAP + \angle ABP = 140^\circ$. Since the measures of the angles of a triangle sum up to 180°, $\angle APB = 40^\circ$, and since $\angle BPD$ is a supplement of $\angle APB$, $\angle BPD = 180^\circ - 40^\circ = 140^\circ$.



2. Prove that given points A, B, C, D such that $\overline{AB} = \overline{BC}$ and $\overline{CD} = \overline{DA}$, $[ABCD] = \overline{AB} \cdot \overline{CD}$.

Solution: Note that $ABCD$ is a kite. This means that it is a cyclic quadrilateral, implying that $[ABCD] = (s - \overline{AB})(s - \overline{CD}) = \overline{AB} \cdot \overline{CD}$.

Alternatively, note that the diagonals are perpendicular. By Ptolemy's Theorem (8.6), $\overline{AD} \cdot \overline{BC} = 2 \cdot \overline{AB} \cdot \overline{CD}$. Then, note that $[ABCD] = \frac{1}{2} \cdot \overline{AD} \cdot \overline{BC} = \overline{AB} \cdot \overline{CD}$, as desired.

3. Prove that given concyclic points A, B, C, D, that the perpendicular bisectors of AB and CD intersect at the center of the circle that contains all four points.

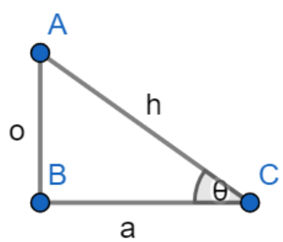
Solution: If AB and CD do not intersect then $ABCD$ is a cyclic quadrilateral whose center can be constructed as the point of concurrency of the perpendicular bisectors. If they do intersect, then note that any chord that is a perpendicular bisector of a circle is also a diameter. Two non-identical diameters are concurrent the center of the circle, so we are done.

Trigonometry

Sine, Cosine, and Tangent

Trigonometry is about expressing the relations of the ratios of the sides and angles of triangles. As thus, an introduction to Trigonometry must include the basics: Sine, Cosine, and Tangent. For now, we will only consider values between 0° and 90° exclusive, and we will not deal with radians just yet.

To consider the Sine, Cosine, and Tangent functions (which will be abbreviated as sin, cos, tan from this point), we must first consider a right triangle.



Let the side opposite to θ be o (opposite means that the angle is not formed by that side), let the adjacent side to θ be a (adjacent means the side that forms the angle but is not the longest), and let the hypotenuse be h.

Then, we define $\sin(\theta) = \frac{o}{h}$, $\cos(\theta) = \frac{a}{h}$, $\tan(\theta) = \frac{o}{a}$. A good memorization mnemonic for this is "Soh, Cah, Toa." The first letter represents the first letter of the trigonometric formulas, and the ratio is the middle letter divided by the last letter. Nevertheless, this mnemonic is about as useless as "A man and his dad put the bomb in the sink" if the concept that said mnemonic explains is not understood.

Here are a few problems involving the trigonometric formulas $\sin, \cos, \tan$. For the rest of this book, non-central trigonometric formulas will be presented as exercises, and central trigonometric formulas (for example, take the Law of Sines) will be presented as theorems, much the same way as it was done before.

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1. Prove that $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$.

2. Consider a right triangle such that $\sin(\theta) = \frac{3}{5}$. Find $\cos(\theta)$.

3. Prove that $\sin(\theta) = \cos(90 - \theta)$.

4. Prove that $\sin^2(\theta) + \cos^2(\theta) = 1$. (In trigonometry, $\sin^2(\theta) = (\sin(\theta))^2$, not $\sin(\sin(\theta))$. The same is true for cosine.)

5. A right triangle with an angle θ such that $\sin(\theta) = \frac{5}{13}$ has a hypotenuse of 117. Find its area.

6. Prove that $\tan^2(\theta) + \sin^2(\theta) = \tan^2(\theta) \cdot (2 - \sin^2(\theta))$.

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1. Prove that $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$.

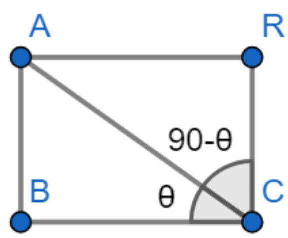
Solution: Note that $\sin(\theta) = \frac{o}{h}$ and $\cos(\theta) = \frac{a}{h}$. Substituting gives us $\frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{o}{h}}{\frac{a}{h}} = \frac{o}{a} = \tan(\theta)$, as desired.

2. Consider a right triangle such that $\sin(\theta) = \frac{3}{5}$. Find $\cos(\theta)$.

Solution: Note that this implies $o = 3x$ and $h = 5x$. We want to find $\frac{a}{h}$. By the Pythagorean Theorem, $3x^2 + a^2 = 5x^2$, implying $a^2 = 16x^2$. Since a is positive, note that $a = 4x$. This implies $\frac{a}{h} = \frac{4x}{5x} = \frac{4}{5}$, so $\cos(\theta) = \frac{4}{5}$.

3. Prove that $\sin(\theta) = \cos(90 - \theta)$.

Solution: Reflecting the right triangle across its hypotenuse yields a rectangle. Note that $\sin(\theta) = \frac{\overline{AB}}{AC}$ and $\cos(90 - \theta) = \frac{\overline{RC}}{AC}$. Since $ABCR$ is a rectangle, $\overline{AB} = \overline{RC}$, implying that $\sin(\theta) = \cos(90 - \theta)$, as desired.



Similarly, $\cos(\theta) = \sin(90 - \theta)$, $\sin(\theta) = \cos(\theta - 90)$, and $\cos(\theta) = -\sin(\theta - 90)$ are all true. This combines to give us $\sin(\theta) = \cos(\theta - 90) = \cos(90 - \theta)$ and $\cos(\theta) = \sin(90 - \theta) = -\sin(\theta - 90)$.

4. Prove that $\sin^2(\theta) + \cos^2(\theta) = 1$. (In trigonometry, $\sin^2(\theta) = (\sin(\theta))^2$, not $\sin(\sin(\theta))$. The same is true for cosine.)

Solution: Substituting $\sin(\theta) = \frac{o}{h}$ and $\cos(\theta) = \frac{a}{h}$ yields $\sin^2(\theta) + \cos^2(\theta) = \frac{o^2+a^2}{h^2}$. By the Pythagorean Theorem (remember o, a are legs of a right triangle and h is the hypotenuse), we see that $o^2 + a^2 = h^2$. Applying this substitution yields $\sin^2(\theta) + \cos^2(\theta) = 1$, as desired.

5. A right triangle with an angle θ such that $\sin(\theta) = \frac{5}{13}$ has a hypotenuse of $117.$ Find its area.

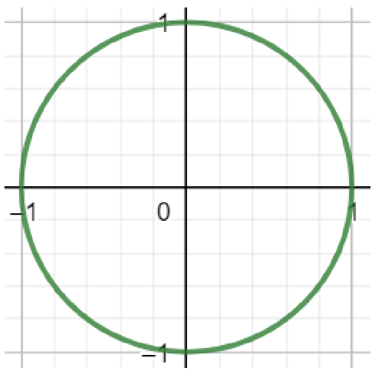
Solution: Note that this implies $o = \frac{5}{13} \cdot 117 = 45.$ By the Pythagorean Theorem, $o^2 + a^2 = h^2,$ or $45^2 + a^2 = 117^2.$ This implies $a = 108.$ By $\frac{bh}{2}$ (4.2), the area of our right triangle is $\frac{45 \cdot 108}{2} = 2430.$

6. Prove that $\tan^2(\theta) + \sin^2(\theta) = \tan^2(\theta) \cdot (2 - \sin^2(\theta)).$

Solution: Note that $\cos^2(\theta) = 1 - \sin^2(\theta).$ Applying this substitution yields $\tan^2(\theta) \cdot (1 + \cos^2(\theta)) = \tan^2(\theta) + \tan^2(\theta) \cdot \cos^2(\theta).$ Note that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)},$ so $\tan^2(\theta) \cdot \cos^2(\theta) = \sin^2(\theta),$ and substituting implies $\tan^2(\theta) + \sin^2(\theta) = \tan^2(\theta) \cdot (2 - \sin^2(\theta)),$ as desired.

The uses of trigonometry are limited if we are only allowed to use $\sin, \cos, \tan$ for degrees between 0 and $90.$ We want to generalize for any degree, even those greater than $360.$ However, rotating something by x degrees is the same as rotating it by $x+360$ degrees, so we keep a periodicity of 360 degrees. Unit circle trigonometry is a little bit complicated, so we're going to define sine, cosine, and tangent differently. Note that we only need to consider values of θ between 0 and $360,$ due to the periodicity of our functions. However, our definition will be general enough to cover any degree value, *even negative values.*

First, consider a unit circle (a circle with radius 1) centered at the origin. The angles will be formed by the x axis and by a radius. Let the origin be $O,$ and let $(1,0)$ be P.



To find the trigonometric function of any degree value, we must rotate OP counterclockwise by θ degrees. (If θ is negative, rotate it clockwise by $|\theta|$ degrees.) Let the image of P rotated around O be P', and let the coordinates of P' be $(x,y).$

We define $\cos(\theta) = x.$

We define $\sin(\theta) = y.$

We define $\tan(\theta) = \frac{y}{x}.$

Now that's all well and good, but we need some context as to the reason we define $\sin, \cos, \tan$ this way. This gives us a continuous function for the three formulas. Consider the graph of $\sin$ for the values of 0 through $90,$ as an example. Obviously, the absolute value of the shortest leg divided by the longest leg cannot exceed $1,$ and as $\lim_{\theta \to 0} \sin(\theta) = 0,$ it makes sense that $\sin(0) = 0.$ Similarly, as $\lim_{\theta \to 90} \sin(\theta) = 1,$ we can define $\sin(90) = 1.$ Then, our definition makes the functions continuous. This is one of the reasons we define unit circle trigonometry as such. (This is not something that can be taught at you; build some intuition and see why unit circle trigonometry is defined this way, and why it is so convenient for it to be.)

Now, let us use our extended knowledge of trigonometry to prove a few important theorems known as the laws of trigonometry. This is all based on the soh-cah-toa definition of sine, cosine, and tangent, and a few extensions. Drawing altitudes will help you get to the solution.

The Law of Sines (9.1)

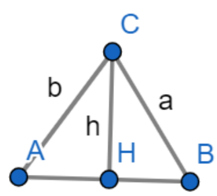
In $\triangle ABC$ with angles $\angle A, \angle B, \angle C$ with corresponding opposite sides BC, AC, AB whose lengths shall be denoted as a, b, c respectively, $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$.

Theorem 9.1's Proof

Without loss of generality, we only need to prove three things. We desire to prove $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ for acute triangles and and we desire to prove the same with obtuse triangles. (We can ignore right triangles since that is a trivial case by the definition of the sine function.) We will do this by splitting our proof into two cases.

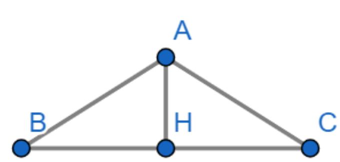
Case 1: Acute Triangle

Note that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ implies $a \cdot \sin(B) = b \cdot \sin(A)$. Substituting $\sin(A) = \frac{h}{b}$ and $\sin(B) = \frac{h}{a}$ gives us $a \cdot \frac{h}{a} = b \cdot \frac{h}{b}$, which is obviously true.



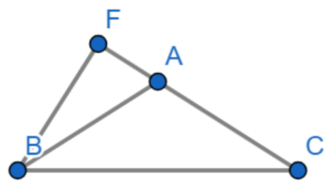
Case 2: Obtuse Triangle

Note that two altitudes will intersect the extensions of the sides of the triangle outside the triangle. Without loss of generality, let the altitude from A to BC intersect BC inside $\triangle ABC$.



The proof for $\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ is identical to the one for acute triangles.

We can just prove $\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$ without loss of generality.



Let $\overline{FB} = h$. Then, we see that $\sin(x) = \sin(180 - x)$. This means that $\sin(A) = \frac{h}{c}$ (by our supplement rule) and $\sin(C) = \frac{h}{a}$. Substituting gives us $\frac{a}{\frac{h}{c}} = \frac{c}{\frac{h}{a}}$, which is obviously true, and we are done.

We will introduce an extension of the Law of Sines including the radius of the circumcircle of the triangle. The Inscribed Angle Theorem (1.1) will be used to prove this.

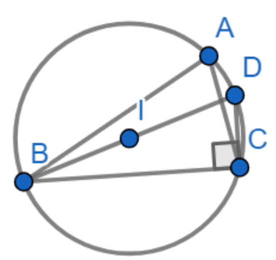
The Extended Law of Sines (9.2)

In $\triangle ABC$ with circumradius R and angles $\angle A, \angle B, \angle C$ with corresponding opposite sides BC, AC, AB whose lengths shall be denoted as a, b, c respectively,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R.$$

Theorem 9.2's Proof

Without loss of generality, let us just prove $\frac{a}{\sin(A)} = 2R$. Draw the circumcircle of $\triangle ABC$. Choose a point D on the circumcircle of $\triangle ABC$ such that B is diametrically opposite to it. Note that by the Inscribed Angle Theorem (1.1), $\angle BAC = \angle BDC$ and $\angle BCD = 90^\circ$, so $\sin(A) = \sin(D)$. Note that $\frac{a}{\sin(D)} = \frac{a}{\frac{a}{2R}} = 2R$, and we are done.



The Law of Cosines (9.3)

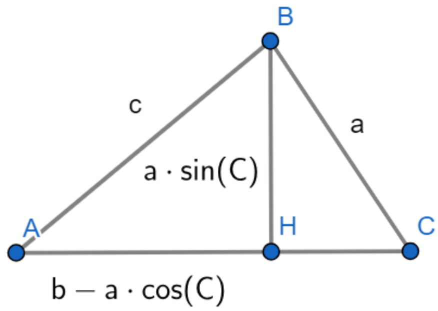
In $\triangle ABC$ with angles $\angle A, \angle B, \angle C$ with corresponding opposite sides BC, AC, AB whose lengths shall be denoted as a, b, c respectively, $c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$.

Equivalently, $a^2 = b^2 + c^2 - 2bc \cdot \cos(A)$ and $b^2 = a^2 + b^2 - 2ab \cdot \cos(B)$.

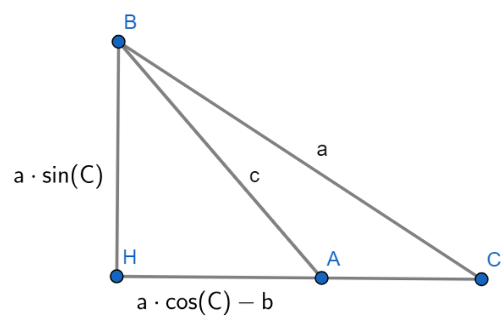
Theorem 9.3's Proof

We have two cases; either the altitude of B is inside of $\triangle ABC$ or it is outside of $\triangle ABC$. (This is trivial for right triangles due to the Pythagorean Theorem.)

If the altitude is inside the triangle, then note that $\overline{BH} = a \cdot \sin(C)$, because $\sin(C) = \frac{h}{a}$, and note that $\overline{AH} = b - a \cdot \cos(C)$ because $b - a \cdot \cos(C) = b - a \cdot \frac{\overline{CH}}{a} = b - \overline{CH}$, and $b - \overline{CH} = \overline{AH}$. By the Pythagorean Theorem, we see that $c^2 = (a \cdot \sin(C))^2 + (b - a \cdot \cos(C))^2 = a^2\sin^2(C) + b^2 - 2ab \cdot \cos(C) + a^2\cos^2(C)$. Since $\sin^2(C) + \cos^2(C) = 1$, factoring gives us $c^2 = a^2(\sin^2(C) + \cos^2(C)) + b^2 - 2ab \cdot \cos(C)$. Substituting yields $c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$, as desired.



If the altitude falls outside the triangle, then note that $\overline{BH} = a \cdot \sin(C)$ by the same reasoning as the case where the altitude lies inside the triangle, and note that $\overline{HA} = a \cdot \cos(C) - b$ because $a \cdot \cos(C) = a \cdot \frac{\overline{HC}}{a} = \overline{HC}$. By the Pythagorean Theorem, $(a \cdot \sin(C))^2 + (a \cdot \cos(C) - b)^2 = c^2$. Expanding gives us $a^2\sin^2(C) + a^2\cos^2(C) - 2ab \cdot \cos(C) + b^2 = c^2$. Since this is the same expression we simplified for the earlier case, we already know that this implies $a^2 + b^2 - 2ab \cdot \cos(C) = c^2$, and we are done.



Now that we have introduced the Law of Cosines, we will introduce the Law of Tangents, another form of the Law of Sines. However, we will need to first prove a few lemmas; this will give you a taste for trigonometric identities which will come later.

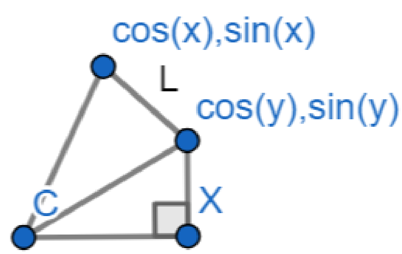
Lemma 1

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x)$$

Lemma 1's Proof

We first will prove the second equation.

By the Law of Cosines, $L^2 = 2 - 2 \cdot \cos(x-y)$, and by the distance formula, $L^2 = (\cos(x) - \cos(y))^2 + (\sin(x) - \sin(y))^2 = 2 - 2\cos(x)\cos(y) - 2\sin(x)\sin(y)$. (We can substitute $\sin^2(x) + \cos^2(x) = 1$ and the symmetrical case for y to get the 2 in the equation.) By the transitive property, $2 - 2 \cdot \cos(A-B) = 2 - 2\cos(x)\cos(y) - 2\sin(x)\sin(y)$, which implies $\cos(A-B) = \cos(x)\cos(y) + \sin(x)\sin(y)$, as desired.



We now will prove the first equation.

Substituting y for $-y'$ (this y' is really an arbitrary term, and y' is used in lieu of y for explanation purposes) gives us $\cos(x+y') = \cos(x)\cos(-y') - \sin(x)\sin(-y')$. Since cosine is an even function and sine is an odd function (this can be learned by analyzing the graphs of the two functions; we will elaborate on another chapter), we see that $\cos(x+y') = \cos(x)\cos(y) + \sin(x)\sin(-y')$. Since y' is arbitrary, we can substitute y, and we get our desired equation.

We now prove the fourth equation.

Note that $\cos(x) = \sin(90-x)$ and $\sin(90-x) = \cos(x)$ when these trigonometric functions are in degrees. Substituting x for $90 - x'$ in the first equation, we get $\cos(90 - x' + y) = \cos(90 - x')\cos(y) - \sin(90 - x')\sin(y)$. Substituting our translations gives us $\sin(x' - y) = \sin(x')\cos(y) - \sin(y)\cos(x')$, as desired.

Finally, we prove the third equation.

Substituting $-y = y'$ gives us $\sin(x + y') = \sin(x)\cos(-y') - \sin(-y')\cos(x')$, which becomes $\sin(x + y') = \sin(x)\cos(y') + \sin(y')\cos(x)$, as desired.

Lemma 2

$$\sin(x) + \sin(y) = 2 \cdot \sin(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2})$$
$$\sin(x) - \sin(y) = 2 \cdot \sin(\tfrac{x-y}{2})\cos(\tfrac{x+y}{2})$$

Lemma 2's Proof

Let $x = a + b$ and $y = a - b$. Then this implies $x + y = 2a$ and $x - y = 2b$.

Substituting gives us $\sin(a + b) + \sin(a - b) = 2 \cdot \sin(a)\cos(a)$ for the first equation. By Lemma 1, $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, and $\sin(a - b) = \sin(a)\cos(b) - \sin(b)\cos(a)$. Substituting then yields $\sin(a + b) + \sin(a - b) = 2 \cdot \sin(a)\cos(b)$. This implies that $\sin(x) + \sin(y) = 2 \cdot \sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$, as desired.

For the second equation, substituting gives us $\sin(a + b) - \sin(a - b) = 2 \cdot \sin(b)\cos(a)$. By Lemma 1, $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, and $\sin(a - b) = \sin(a)\cos(b) - \sin(b)\cos(a)$, implying that $\sin(a + b) - \sin(a - b) = 2\sin(b)\cos(a)$. Substituting our initial definitions of a and b back in yield $\sin(x) - \sin(y) = 2 \cdot \sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$, as desired.

Note that Lemma 2 is a direct result of Lemma 1.

The Law of Tangents (9.4)

In $\triangle ABC$ with angles $\angle A, \angle B, \angle C$ with corresponding opposite sides BC, AC, AB whose lengths shall be denoted as a, b, c respectively, $\frac{a-b}{a+b} = \frac{\tan(\frac{1}{2}(A-B))}{\tan(\frac{1}{2}(A+B))}$.

Theorem 9.4's Proof

This will utilize the Extended Law of Sines (9.2). By the Extended Law of Sines (9.2), $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = 2R$. This implies $a = 2R \cdot \sin(A)$ and $b = 2R \cdot \sin(B)$. Substituting, we see that $\frac{a-b}{a+b} = \frac{2R\cdot\sin(A)-2R\cdot\sin(B)}{2R\cdot\sin(A)+2R\cdot\sin(B)} = \frac{\sin(A)-\sin(B)}{\sin(A)+\sin(B)}$. Note that by Lemma 2, $\sin(A) - \sin(B) = 2 \cdot \sin(\frac{A-B}{2})\cos(\frac{A+B}{2})$, and $\sin(A) + \sin(B) = 2 \cdot \sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$. Substituting yields $\frac{a-b}{a+b} = \frac{2\cdot\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})}{2\cdot\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})}$. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, $\tan(\frac{A-B}{2}) = \frac{\sin(\frac{A-B}{2})}{\cos(\frac{A-B}{2})}$ and

$\tan(\frac{A+B}{2}) = \frac{\sin(\frac{A+B}{2})}{\cos(\frac{A+B}{2})}$. Substituting yields $\frac{a-b}{a+b} = \frac{2\cdot\tan(\frac{A-B}{2})}{2\cdot\tan(\frac{A+B}{2})}$, and simplifying yields $\frac{a-b}{a+b} = \frac{\tan(\frac{1}{2}(A-B))}{\tan(\frac{1}{2}(A+B))}$, as desired.

Now that we have introduced the unit circle definitions of sine, cosine, and tangent, and have proven the Laws of Trigonometry, we will introduce some problems.

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1. Find the exact value of $\sin(75)$.

2. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively. If $\frac{\tan(\frac{1}{2}[A-B])}{\tan(\frac{1}{2}[A+B])} = \frac{1}{5}$, find $\frac{a}{b}$.

3. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively. If $a = 4,\ b = 2\sqrt{6}$, and $c = 2\sqrt{3} + 2$, find $\angle A, \angle B, \angle C$.

4. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively. If $a = 6,\ b = 4$, and $\angle C = 120^\circ$, find $[ABC]$.

5. Consider $\triangle ABC$ with $\overline{BC} = 5$. Then have $\triangle DEF$ with $\overline{EF} = 10$. If the circumcircle of $\triangle DEF$ has an area four times the area of $\triangle ABC$, then the two values of $\angle D$ are x, y such that $x > y$. If $\frac{x}{y} = 3$, find the area of the circumradius of $\triangle ABC$.

6. If $\sin(x) = \frac{4}{5}$, find $\tan(45 - x)$. (Assume that $0 < x < 90$ for this problem.)

7. Prove the Pythagorean Inequality, which states that for $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as a, b, c such that without loss of generality, $a < b < c$, that the following three statements are true:
$a^2 + b^2 > c^2$ if and only if $\triangle ABC$ is acute.
$a^2 + b^2 = c^2$ if and only if $\triangle ABC$ is right.
$a^2 + b^2 < c^2$ if and only if $\triangle ABC$ is obtuse.

8. Use the Law of Cosines to prove Heron's Formula and Stewart's Theorem. (There are more identities that can be proved using the Law of Cosines in this book; try to find them!)

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1. Find the exact value of $\sin(75)$.

Solution: Note this is equivalent to $\cos(15)$. We can use our knowledge of 45-45-90 and 30-60-90 triangles to solve this. Since $15 = 45 - 30$, we can use Lemma 1 to state that $\sin(75) = \cos(45 - 30) = \cos(45)\cos(30) + \sin(45)\sin(30) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$, as desired.

2. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as a, b, c, respectively. If $\frac{\tan(\frac{1}{2}[A-B])}{\tan(\frac{1}{2}[A+B])} = \frac{1}{5}$, find $\frac{a}{b}$.

Solution: Note that $\frac{a-b}{a+b} = \frac{1}{5}$, implying $5a - 5b = a + b$, or $6a = 4b$, which means that $\frac{a}{b} = \frac{1}{3}$.

3. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as a, b, c, respectively. If $a = 4$, $b = 2\sqrt{6}$, and $c = 2\sqrt{3} + 2$, find $\angle A, \angle B, \angle C$.

Solution: Note that $a : b : c$ is equivalent to $\frac{\sqrt{2}}{2} : \frac{\sqrt{3}}{2} : \frac{\sqrt{6}+\sqrt{2}}{4}$. These values seem a bit suspicious, and for good reason; they are $\sin(45)$, $\sin(60)$, and $\sin(75)$, respectively. As thus, we have $\angle A = 45^\circ$, $\angle B = 60^\circ$, and $\angle C = 75^\circ$.

4. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as a, b, c, respectively. If $a = 6$, $b = 4$, and $\angle C = 120^\circ$, find $[ABC]$.

Solution: The $\frac{1}{2}ab\sin C = [ABC]$ theorem (4.3) kills this. Clearly $\frac{1}{2} \cdot 6 \cdot 4 \cdot \sin(120) = 6\sqrt{3}$.

We show another alternate method to demonstrate an example of the Law of Cosines. By the Law of Cosines, $c^2 = 6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cdot \cos(120) = 36 + 16 + 24 = 76$, implying $c = 2\sqrt{19}$. Heron's Formula would be messy, so let's use $[ABC] = \frac{abc}{4R}$ (4.5) instead. Note that by the Extended Law of Sines (9.2), $2R = \frac{2\sqrt{19}}{\sin(120)} = \frac{4\sqrt{57}}{3}$. Plugging this in yields $[ABC] = \frac{6 \cdot 4 \cdot 2\sqrt{19}}{\frac{8\sqrt{57}}{3}} = 6\sqrt{3}$.

5. Consider $\triangle ABC$ with $\overline{BC} = 5.$ Then have $\triangle DEF$ with $\overline{EF} = 10.$ If the circumcircle of $\triangle DEF$ has an area four times the area of $\triangle ABC,$ then the two possible values of $\angle D$ are x, y such that $x > y.$ If $\frac{x}{y} = 3,$ find the area of the circumcircle of $\triangle ABC.$

Solution: The problem implies $4(\frac{\frac{5}{\sin(A)}}{2})^2 = (\frac{\frac{10}{\sin(D)}}{2})^2,$ implying $\sin(A) = \sin(D).$ Note that $\sin(y) = \sin(180 - y),$ so $x = 180 - y$ as $y < 90.$ Then note that $\frac{180-y}{y} = 3,$ implying that $y = 45,$ as y must be positive. Then this implies $\sin(A) = \frac{\sqrt{2}}{2},$ and by the circumradius formula, $2R = \frac{5}{\frac{\sqrt{2}}{2}} = 5\sqrt{2},$ or $R = \frac{5\sqrt{2}}{2}.$ This means the area of the circumcircle of $\triangle ABC$ is $(\frac{5\sqrt{2}}{2})^2 = \frac{25}{2},$ which is our answer.

6. If $\sin(x) = \frac{4}{5},$ find $\tan(45 - x).$ (Assume that $0 < x < 90$ for this problem.)

Solution: Let's draw a 3-4-5 right triangle. By the Law of Tangents (8.4), $\frac{4-3}{4+3} = \frac{\tan(\frac{1}{2}[x-(90-x)])}{\tan(\frac{1}{2}[x+90-x])}.$ Simplifying gives us $\frac{1}{7} = \frac{\tan(x-45)}{\tan(45)}.$ Note that $\tan(45) = 1,$ implying $\tan(x - 45) = \frac{1}{7}.$ However, since we want to find $\tan(45 - x),$ we just note that $\tan(x)$ is an odd function, yielding $\tan(45 - x) = \frac{-1}{7}.$

7. Prove the Pythagorean Inequality, which states that for $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as a, b, c such that without loss of generality, $a < b < c,$ that the following three statements are true:

$a^2 + b^2 > c^2$ if and only if $\triangle ABC$ is acute.

$a^2 + b^2 = c^2$ if and only if $\triangle ABC$ is right.

$a^2 + b^2 < c^2$ if and only if $\triangle ABC$ is obtuse.

Solution: By the Law of Sines, $\angle A < \angle B < \angle C.$ Applying the Law of Cosines, $a^2 + b^2 - 2ab \cdot \cos(C) = c^2,$ implying $a^2 + b^2 = c^2 + 2ab \cdot \cos(C).$

If $\angle C < 90^\circ,$ then $2ab \cdot \cos(C) > 0,$ meaning that $a^2 + b^2 > c^2.$

If $\angle C = 90^\circ,$ then the Pythagorean Theorem applies.

If $180^\circ > \angle C > 90^\circ,$ then $2ab \cdot \cos(C) < 0,$ implying $a^2 + b^2 < c^2.$

8. Use the Law of Cosines to prove Heron's Formula and Stewart's Theorem. (There are more identities that can be proved using the Law of Cosines in this book; try to find them!)

Solution: Consider $\triangle ABC,$ and for Stewart's Theorem, consider D on $\overline{BC}.$

For Heron's Formula, apply the Law of Cosines to get $\cos(C) = \frac{a^2+b^2-c^2}{2ab}.$ Then we use the $\sqrt{1-\cos^2(\theta)} = \sin(\theta)$ identity and we get $\sin(\theta) = \sqrt{\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2}}.$ By $[ABC] = \frac{1}{2}ab \cdot \sin(C)$ (4.3), we achieve $[ABC] = \frac{1}{2} \cdot \sqrt{\frac{4a^2b^2-(a^2+b^2-c^2)^2}{4}} = \frac{1}{4} \cdot \sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}.$ Our formula then can easily be proven by some substitution.

For Stewart's Theorem, note that by the Law of Cosines, $\cos(ADB) = \frac{c^2-d^2-m^2}{-2dm} = -\frac{b^2-d^2-n^2}{-2dn} = -\cos(ADC).$ Multiplying both sides by $-2dnm$ results in $c^2n - d^2n - nm^2 = -b^2m + d^2m + n^2m.$ Rearranging yields $b^2m + c^2n = d^2n + d^2m + n^2m + nm^2.$ Since $n + m = a,$ this then implies $bmb + cnc = dad + man,$ as desired.

Reciprocals and Inverses

What would happen if we took the reciprocal of Sine, Cosine, or Tangent? And what would you need to take the sine, cosine, or tangent of to attain a certain value? In the last section, we covered our basic trigonometric functions: Sine, Cosine, and Tangent. In this section, we'll cover the reciprocals and inverses of our functions.

First, we will cover reciprocals. To do this, let us consider the Cosecant, Secant, and Cotangent functions, which will be abbreviated as csc, sec, and cot, respectively.

We define $\csc(\theta) = \frac{1}{\sin(\theta)}$, $\sec(\theta) = \frac{1}{\cos(\theta)}$, $\cot(\theta) = \frac{1}{\tan(\theta)}$. This time, instead of introducing a tacky memorization mnemonic, we shall use a memorization technique I like to call the "Co-reciprocal technique." Each trigonometric function sine, cosine, and tangent are matched up with cosecant, secant, and cotangent such that each pair has one function beginning with "co," and the other function doesn't begin with "co." It is quite obvious cotangent goes with tangent, and since cosecant has a "co" and sine doesn't, cosecant goes with sine. (It works the other way too; since cosine has a "co" and secant doesn't, secant goes with cosine.) Eventually, you'll probably outgrow this memorization technique, but it's quite useful for acclimating yourself to these functions.

These functions are quite special. The graphs of them are interesting; we'll go more in-depth in another section, but a few important things to note are that the graphs of $\csc(\theta)$ and $\sec(\theta)$ are horizontal translations of each other, much like $\sin(\theta)$ and $\cos(\theta)$, and that $\cot(\theta)$ is a series of reflections of $\tan(\theta)$ in certain regions.

Remember that cosecant, secant, and cotangent functions are reciprocals of the sine, cosine, and tangent functions, respectively. Above all, remember that in certain cases, $\sin(\theta) = \frac{o}{h}$, $\cos(\theta) = \frac{a}{h}$, and $\tan(\theta) = \frac{o}{a}$. (This works especially well for squares of trigonometric functions, where the end result will always be positive.) These problems will require algebraic manipulation, and substitutions using o, a, h are perfectly fine, especially when problems involves squares of trigonometric functions.

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1. Prove that $(\csc(\theta) - 1)(\csc(\theta) + 1)(\sec(\theta) - 1)(\sec(\theta) + 1) = 1$, for all θ such that $\csc(\theta)$ and $\sec(\theta)$ are defined.

2. Prove that $\tan^2(\theta) + 1 = \sec^2(\theta)$.

3. Prove that $\cot^2(\theta) + 1 = \csc^2(\theta)$.

4. Given triangle $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively, find the circumradius of $\triangle ABC$ if $a \cdot \csc(A) = 8$.

5. Prove that $\tan(\theta) = \cot(-\theta + 90)$.

6. Find the minimum value $\csc^2(\theta) + \sec^2(\theta)$ can take.

7. Write versions of the Extended Law of Sines (9.2) and the Law of Cosines (9.3) in terms of Cosecant and Secant, respectively.

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1. Prove that $(\csc(\theta) - 1)(\csc(\theta) + 1)(\sec(\theta) - 1)(\sec(\theta) + 1) = 1,$ for all θ such that $\csc(\theta)$ and $\sec(\theta)$ are defined.

Solution: Note that this multiplies out to $(\csc^2(\theta) - 1)(\sec^2(\theta) - 1) = 1.$ This implies $\csc^2(\theta) \cdot \sec^2(\theta) = \csc^2(\theta) + \sec^2(\theta).$

2. Prove that $\tan^2(\theta) + 1 = \sec^2(\theta).$

Solution: Note that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and that $\sec(\theta) = \frac{1}{\cos(\theta)}.$ Substituting yields $\frac{\sin^2(\theta)}{\cos^2(\theta)} + 1 = \frac{1}{\cos^2(\theta)},$ which further implies $\frac{\sin^2(\theta)+\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta).}$ Since $\sin^2(\theta) + \cos^2(\theta) = 1,$ we get $\frac{1}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)},$ which is obviously true.

3. Prove that $\cot^2(\theta) + 1 = \csc^2(\theta).$

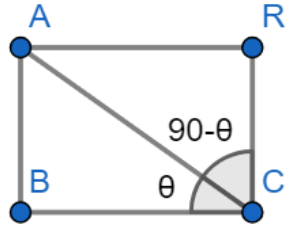
Solution: Applying a similar process as the problem before, we get $\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}.$ The proof follows from the $\sin^2(\theta) + \cos^2(\theta) = 1$ identity.

4. Given triangle $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively, find the circumradius of $\triangle ABC$ if $a \cdot \csc(A) = 8.$

Solution: Note that $\csc(A) = \frac{1}{\sin(A)}.$ Substituting yields $\frac{a}{\sin(A)} = 8.$ By the Extended Law of Sines (9.2), $\frac{a}{\sin(A)} = 2R,$ and applying the transitive property yields $R = 4.$

5. Prove that $\tan(\theta) = \cot(90 - \theta)$ for all θ such that $\tan(\theta)$ and $\cot(\theta)$ are defined.

Solution: For values of θ between 0 and $90,$ this diagram of a rectangle suffices. Note that $\tan(\theta) = \frac{\overline{AB}}{BC}$ and $\cot(90 - \theta) = \frac{\overline{RC}}{AR},$ and since $ABCR$ is a rectangle, $\tan(\theta) = \cot(90 - \theta).$



Now we attempt to prove this for all values of θ. By induction, all we need to do to prove is that if it works for θ, it works for $\theta + 90$. (This is because the cases for multiples of 90 are either trivial or undefined.) Substituting in $\theta + 90$ gives us $\tan(\theta + 90) = \cot(-\theta)$. Now note that $\sin(\theta + 90) = \sin(90 - \theta)$, and $\cos(\theta + 90) = -\cos(90 - \theta)$. Using our knowledge of even and odd functions, this implies $\tan(90 - \theta) = \cot(\theta)$, which is true by the same argument as the initial identity, so we are done.

6. Find the minimum value $\csc^2(\theta) + \sec^2(\theta)$ can take.

Solution: Substituting in sine and cosine yields $\frac{1}{\sin^2(\theta)} + \frac{1}{\cos^2(\theta)}$, and this value is equivalent to $\frac{\sin^2(\theta)+\cos^2(\theta)}{\sin^2(\theta)\cdot\cos^2(\theta)}$. Since we know $\sin^2(\theta) + \cos^2(\theta) = 1$, this leaves us with $\frac{1}{\sin^2(\theta)\cdot\cos^2(\theta)}$, implying that we wish to maximize $\sin^2(\theta) \cdot \cos^2(\theta)$. Note that this means we want to maximize $\sin(\theta) \cdot \cos(\theta)$. By AM-GM, we note that $\frac{\sin^2(\theta)+\cos^2(\theta)}{2} \geq \sin(\theta) \cdot \cos(\theta)$, or $\frac{1}{2} \geq \sin(\theta) \cdot \cos(\theta)$, with equality occuring at $\theta = 45 + 360x$. This means that the minimum value $\csc^2(\theta) + \sec^2(\theta)$ can take is 4.

7. Write versions of the Extended Law of Sines (9.2) and the Law of Cosines (9.3) in terms of Cosecant and Secant, respectively.

Solution: The first and second are quite easy due to the definitions of cosecant and secant as the reciprocals of sine and cosine, respectively.

The Extended Law of Cosecants claims that $a \cdot \csc(A) = b \cdot \csc(B) = c \cdot \csc(C) = 2R$.

The Law of Secants claims that $a^2 + b^2 - \frac{2ab}{\sec(C)} = c^2$.

(There's a reason we haven't mentioned anything about cotangents; that is because there is legitimately something known as the Law of Cotangents that is not a reformulation of the Law of Tangents.)

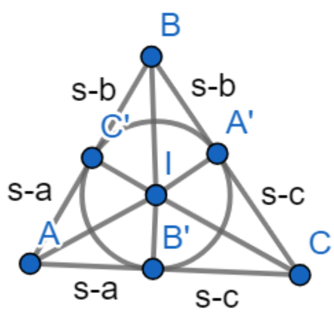
In the solution to Problem 7 I mentioned a Law of Cotangents. Before we get into inverse functions, we'll cover the Law of Cotangents. Quite a few things can be proved using it, such as The Law of Tangents. Let's take a look.

The Law of Cotangents (9.5)

In $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ and the inradius denoted as a, b, c, r respectively, and with the semiperimeter s being equal to $\frac{a+b+c}{2}$, $\frac{\cot(A)}{s-a} = \frac{\cot(B)}{s-b} = \frac{\cot(C)}{s-c} = \frac{1}{r}$.

Theorem 9.5's Proof

Have the incenter of $\triangle ABC$ be I, and draw perpendiculars from I to AB, BC, AC whose feet are C', A', B', respectively. Then note that A', B', C' are all on the incircle of $\triangle ABC$, and $\angle IA'B = \angle IA'C = \angle IB'A = \angle IB'C = \angle IC'A = \angle IC'B = 90^\circ$ by Theorem 3.4. By the Two Tangent Theorem (3.5), note that $\overline{B'A} = \overline{C'A}$, $\overline{A'B} = \overline{C'B}$, and $\overline{AC'} = \overline{BC'}$. Additionally, note that $\overline{BA'} + \overline{CA'} = a$, $\overline{AB'} + \overline{CB'} = b$, and $\overline{AC'} + \overline{BC'} = c$. This implies $\overline{AB'} = \overline{AC'} = s - a$, $\overline{BA'} = \overline{BC'} = s - b$, and $\overline{CA'} = \overline{CB'} = s - c$.



Note that $\cot(IAB') = \cot(\frac{A}{2}) = \frac{s-a}{r}$, and similarly, $\cot(\frac{B}{2}) = \frac{s-b}{r}$ and $\cot(\frac{C}{2}) = \frac{s-c}{r}$. Rearranging and using the transitive property finishes the proof.

Now that we have proved the Law of Cotangents, we will introduce a few problems revolving around the Law of Cotangents (and maybe some of the other trigonometry material we've introduced) before we introduce inverses.

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1. Prove that in $\triangle ABC$, $\cot(\frac{A}{2}) + \cot(\frac{B}{2}) + \cot(\frac{C}{2}) = \cot(\frac{A}{2}) \cdot \cot(\frac{B}{2}) \cdot \cot(\frac{C}{2})$.

2. In $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$, and its inradius denoted as a, b, c, r respectively, prove that $r \cdot \cot(B + \frac{C}{2}) = \frac{a-b}{2} + \frac{bc-ac}{2(a+b)}$ where $\cot(B + \frac{C}{2})$ is defined.

3. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB},$ and its inradius denoted as a, b, c, r respectively. Use the formula in the problem above to find $\cot(B + \frac{C}{2})$ where $a = 5, \ b = 7, \ c = 8.$

4. Prove that in $\triangle ABC$ with inradius $r, \ [ABC] = r^2(\cot(\frac{A}{2}) + \cot(\frac{B}{2}) + \cot(\frac{C}{2})).$

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1. Prove that in $\triangle ABC$, $\cot(\frac{A}{2}) + \cot(\frac{B}{2}) + \cot(\frac{C}{2}) = \cot(\frac{A}{2}) \cdot \cot(\frac{B}{2}) \cdot \cot(\frac{C}{2})$.

Solution: Applying the Law of Cotangents gives us $\frac{s}{r} = \frac{(s-a)(s-b)(s-c)}{r^3}$. This implies $r^2s^2 = s(s-a)(s-b)(s-c)$, and applying $[ABC] = rs$ (5.4) and Heron's Formula (5.6) give us $[ABC]^2 = [ABC]^2$, which is obviously true.

2. In $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$, and its inradius denoted as a, b, c, r respectively, prove that $r \cdot \cot(B + \frac{C}{2}) = \frac{a-b}{2} + \frac{bc-ac}{2(a+b)}$ where $\cot(B + \frac{C}{2})$ is defined.

Solution: Applying the Law of Tangents (9.4) and using the $\tan(\theta) = \cot(-\theta + 90)$ identity gives us $\frac{\cot(B+\frac{C}{2})}{\cot(\frac{C}{2})} = \frac{a-b}{a+b}$. By the Law of Cotangents (9.5), $\frac{a-b}{a+b} = \frac{\cot(B+\frac{C}{2})}{\frac{s-c}{r}} = \frac{r \cdot \cot(B+\frac{C}{2})}{s-c}$. This implies that $(a-b)(s-c) = (a+b)(r \cdot \cot(B + \frac{C}{2})$. Substituting in $s = \frac{a+b+c}{2}$ gives us $(a-b)(\frac{a+b-c}{2}) = \frac{a^2+ab-ac-ab-b^2+bc}{2} = \frac{a^2-b^2-ac+bc}{2} = (a+b)(r \cdot \cot(B + \frac{C}{2})$. Dividing both sides by $a+b$ yields $r \cdot \cot(B + \frac{C}{2}) = \frac{(a-b)(a+b)}{2(a+b)} + \frac{bc-ac}{2(a+b)} = \frac{a-b}{2} + \frac{bc-ac}{2(a+b)}$.

3. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$, and its inradius denoted as a, b, c, r respectively. Use the formula in the problem above to find $\cot(B + \frac{C}{2})$ where $a = 5$, $b = 7$, $c = 8$.

Solution: Using the formula above, $r \cdot \cot(B + \frac{C}{2}) = \frac{7-5}{2} + \frac{7\cdot 8-5\cdot 8}{2(5+7)} = -\frac{1}{3}$. Then we use Heron's Formula (5.6) and we find the area to be $10\sqrt{3}$. By $[ABC] = rs$ (5.4), $10\sqrt{3} = 10r$, implying $r = \sqrt{3}$. This implies that $\cot(B + \frac{C}{2}) = -\frac{\sqrt{3}}{9}$.

4. Prove that in $\triangle ABC$ with inradius r, $[ABC] = r^2(\cot(\frac{A}{2}) + \cot(\frac{B}{2}) + \cot(\frac{C}{2}))$.

Solution: Applying the Law of Cotangents (9.5) yields $[ABC] = \frac{r^2(3s-a-b-c)}{r} = rs$, which is true by $[ABC] = rs$ (5.4).

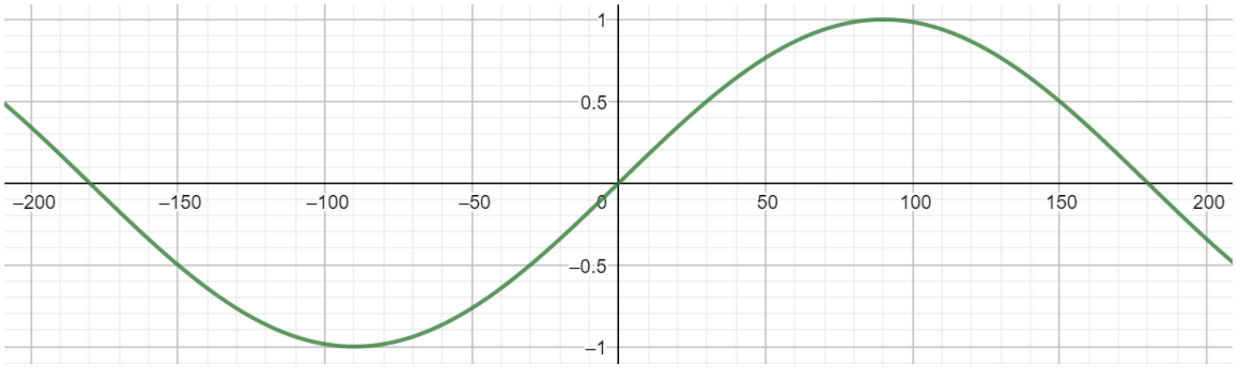
Now we'll consider inverse functions. Just like algebra started with the shift from $x+y=?$ to $x+?=y,$ inverses are born of the shift from $\sin(x)=?$ to $\sin(?)=x.$ Basically, with inverses, we want to find what value we need to take a sine, cosine, or tangent of to get another value.

We will denote our inverse functions by adding an exponent of $-1.$ For example, $\sin^{-1}$ is the inverse of $\sin.$ This may be denoted as $\arcsin$ in other texts; we will use $\sin^{-1}$ instead. Note that $\sin^{-1}$ does not denote $\csc,$ because $\sin(\csc(\theta))$ is not necessarily $\sin(\theta).$ Instead, we define our inverse functions as follows.

$$\sin(\sin^{-1}(x))=x$$
$$\cos(\cos^{-1}(x))=x$$
$$\tan(\tan^{-1}(x))=x$$
$$\csc(\csc^{-1}(x))=x$$
$$\sec(\sec^{-1}(x))=x$$
$$\cot(\cot^{-1}(x))=x$$

At first glance, everything seems fine. For example, if we wanted to find $\sin^{-1}(1),$ we could just note that $\sin(90)=\sin(\sin^{-1}(1))=1.$ However, note that trigonometric functions have a periodicity of $360^\circ,$ which means that $\sin^{-1}(1)=90+360x.$ To make our inverse functions only have one possible value, which we will call our *principal value*, we must consider a non-arbitrary set of degrees such that all possible values of the trigonometric functions are taken. To make it not arbitrary, let's make sure we include the acute angles which can be formed in a right triangle. This means that our principal value for all functions includes 0° through $90^\circ.$ We will denote that we want to find the principal value of an inverse by capitalizing its first letter.

Let's take Sin^{-1} as an example. We note that we want all values -1 through 1 to be covered in a continuous manner. We also want to include 0° through $90^\circ.$ This means that we cannot go on further than 90° or we would have repeats, so we have to go back. When we go back to $-90^\circ,$ we have covered all the values that $\sin(\theta)$ can take, so we have $-90^\circ \le \mathrm{Sin}^{-1}(\mathrm{x}) \le 90^\circ$ as our principal values for $\mathrm{Sin}^{-1}(\mathrm{x})$. (For your convenience, below is a graph of $\sin(x)$ in degrees.)



Similarly, $0^{\circ} \leq \mathrm{Cos}^{-1}(\mathrm{x}) \leq 180^{\circ}$, and $-90^{\circ} \leq \mathrm{Tan}^{-1}(\mathrm{x}) \leq 90^{\circ}$. These ranges hold for the inverses of the reciprocal functions; $-90^{\circ} \leq \mathrm{Csc}^{-1}(\mathrm{x}) \leq 90^{\circ}$, $0^{\circ} \leq \mathrm{Sec}^{-1}(\mathrm{x}) \leq 180^{\circ}$, and $-90^{\circ} \leq \mathrm{Cot}^{-1}(\mathrm{x}) \leq 90^{\circ}$, though these functions will not be used as often, because $\mathrm{Sin}^{-1}(\mathrm{x}) = \mathrm{Csc}^{-1}(\frac{1}{x})$, and so on. (Understand why this is the case.)

Now, we will present some problems based on inverse trigonometric functions.

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1. Find $\mathrm{Sin}^{-1}(\frac{1}{2})$.

2. Given that $\mathrm{Sin}^{-1}(\mathrm{x}) = 69^{\circ}$, find $\sin^{-1}(x)$.

3. Find the values of x such that $\mathrm{Sin}^{-1}(\mathrm{x})$ is defined.

4. Do the same for the inverses of the other five trigonometric functions.

5. Prove that $\mathrm{Cos}^{-1}(\mathrm{x}) = \mathrm{Sec}^{-1}(\mathrm{x})$, and prove that $\tan^{-1}(x) = \cot^{-1}(\frac{1}{x})$.

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1. Find $\mathrm{Sin}^{-1}(\frac{1}{2})$.

Solution: Note that $\sin^{-1}(x) = 30 + 360x$. The value that lies between -90° and 90° is 30°, which is the value we are looking for. This implies that $\mathrm{Sin}^{-1}(\frac{1}{2}) = 30^{\circ}$.

2. Given that $\mathrm{Sin}^{-1}(\mathrm{x}) = 69^{\circ}$, find $\sin^{-1}(x)$.

Solution: Since $\sin(\theta)$ is a periodic function with a period of 360°, $\sin^{-1}(x) = 69 + 360x$.

3. Find the values of x such that $\mathrm{Sin}^{-1}(\mathrm{x})$ is defined.

Solution: This is basically finding the values of x that can be taken through $\sin(\theta)$, which are $-1 \leq x \leq 1$.

4. Do the same for the inverses of the other five trigonometric functions.

Solution: For $\mathrm{Cos}^{-1}(\mathrm{x})$, the values are $-1 \leq x \leq 1$ because the possible values of $\cos(\theta)$ are $-1 \leq \cos(\theta) \leq 1$. By similar reasoning, $\mathrm{Tan}^{-1}(\mathrm{x})$ is defined for all values of x. $\mathrm{Csc}^{-1}(\mathrm{x})$ is defined for $x \leq -1$ or $x \geq 1$, the same is true for $\mathrm{Sec}^{-1}(\mathrm{x})$, and $\mathrm{Cot}^{-1}(\mathrm{x})$ is defined over all values of x.

5. Prove that $\mathrm{Cos}^{-1}(\mathrm{x}) = \mathrm{Sec}^{-1}(\frac{1}{x})$, and prove that $\tan^{-1}(x) = \cot^{-1}(\frac{1}{x})$.

Solution: Note that $\cos(\mathrm{Cos}^{-1}(\mathrm{x})) = x$, and that $\sec(\mathrm{Sec}^{-1}(\frac{1}{x})) = \frac{1}{x}$. By the definition of $\cos$ and $\sec$, $\cos(\theta) = \frac{1}{\sec(\theta)}$, and this result implies that $\mathrm{Cos}^{-1}(\mathrm{x}) = \mathrm{Sec}^{-1}(\frac{1}{x})$.

Similarly, $\tan(\tan^{-1}(x)) = x$, and $\cot(\cot^{-1}(\frac{1}{x})) = \frac{1}{x}$. By the definitions of $\tan$ and $\cot$, $\tan(\theta) = \frac{1}{\cot(\theta)}$, and this result implies that $\tan^{-1}(x) = \cot^{-1}(\frac{1}{x})$.

Trigonometric Identities

This is by far the hardest section of this entire chapter, and possibly the entire book so far. My approach is to prove every concept as we go, using concepts from previous parts of the chapter, but that probably won't work for everyone, particularly as this is such a difficult section. For ease of reference and as a summary to what we'll be doing in this chapter, I'll cover all of the identities without any proof, prove them, then show what we used to prove each identity. All of these identities will be in degrees.

Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

$$\tan^2(\theta) + 1 = \sec^2(\theta).$$

$$\cot^2(\theta) + 1 = \csc^2(\theta).$$

These are called Pythagorean Identities because they are all proved using the basic definition of sine, cosine, and the like, and by using the Pythagorean Theorem.

Odd/Even Functions

$$\sin(\theta) = -\sin(-\theta).$$

$$\cos(\theta) = \cos(-\theta).$$

$$\tan(\theta) = -\tan(-\theta).$$

Sine and tangent are referred to as odd functions, and cosine is an even function. We will cover this more deeply when we talk about graphing. Try to derive which reciprocal functions (cosecant, secant, and cotangent) are odd and which are even.

Cofunction Identities

$\sin(\theta) = \cos(90 - \theta),$ and $\cos(\theta) = \sin(90 - \theta).$

$\tan(\theta) = \cot(90 - \theta),$ and $\cot(\theta) = \tan(90 - \theta).$

These are called cofunction identities because they denote a relationship between pairs of trigonometric functions. Try to derive similar identities for cosecant and secant.

Periodicity Identities

$$\sin(\theta) = \sin(\theta + 360x)$$

$$\cos(\theta) = \cos(\theta + 360x)$$

$$\tan(\theta) = \tan(\theta + 180x)$$

Periodicity means repeating, and these identities demonstrate the intervals at which these functions repeat at. This is true for all *integer* values of x. Similar periodicity functions can be derived for the reciprocal functions.

Sum/Difference Identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y).$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
$$\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x).$$
$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\cdot\tan(y)}.$$
$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\cdot\tan(y)}.$$
$$\cot(x+y) = \frac{\cot(x)\cdot\cot(y)-1}{\cot(x)+\cot(y)}.$$
$$\cot(x-y) = \frac{\cot(x)\cdot\cot(y)+1}{\cot(y)-\cot(x)}.$$

These are known as sum/difference identities because we are taking the trigonometric function of a sum and expressing it in terms of the individual parts.

Double Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x).$$
$$\cos(2x) = \cos^2(x) - \sin^2(x).$$
$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}.$$
$$\cot(2x) = \frac{1-\cot^2(x)}{2\cot(x)}.$$

These are known as double angle identities because we are taking the trigonometric function of the double of a value and expressing it in terms of said value. The reciprocal functions have quite trivial double angles, except for cotangent.

Half Angle Identities

$$\sin(\tfrac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{2}}.$$
$$\cos(\tfrac{x}{2}) = \pm\sqrt{\frac{1+\cos(x)}{2}}.$$
$$\tan(\tfrac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}} = \frac{\sin(x)}{1+\cos(x)} = \frac{1-\cos(x)}{\sin(x)}.$$

These are known as half angle functions because we are taking the trigonometric function of the half of a value and expressing it in terms of said value. For the functions expressed in terms of square roots, the sign depends on which quadrant $\frac{x}{2}$ so happens to lie in.

Sum to Product Identities

$$\sin(x) + \sin(y) = 2\sin(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2}).$$
$$\sin(x) - \sin(y) = 2\sin(\tfrac{x-y}{2})\cos(\tfrac{x+y}{2}).$$
$$\cos(x) + \cos(y) = 2\cos(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2}).$$
$$\cos(x) - \cos(y) = -2\sin(\tfrac{x+y}{2})\sin(\tfrac{x-y}{2}).$$
$$\tan(x) + \tan(y) = \frac{\sin(x+y)}{\cos(x)\cos(y)}.$$
$$\tan(x) - \tan(y) = \frac{\sin(x-y)}{\cos(x)\cos(y)}.$$

These are called sum to product identities because you take the sum of two trigonometric functions and put it in the terms of the products of trigonometric functions of related degrees. Deriving functions for the reciprocal functions should be trivial.

Product to Sum Identities

$$\sin(x)\sin(y) = \tfrac{1}{2}(\cos(x-y) - \cos(x+y)).$$
$$\sin(x)\cos(y) = \tfrac{1}{2}(\sin(x-y) + \sin(x+y)).$$
$$\cos(x)\cos(y) = \tfrac{1}{2}(\cos(x-y) + \cos(x+y)).$$
$$\tan(x)\tan(y) = \frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}.$$

These are called product to sum identities because you take the products of two trigonometric functions and put it in the terms of the sums of trigonometric functions of related degrees. Deriving functions for the reciprocal functions should be trivial.

Mollweide's Formulas

Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ expressed as $a, b, c,$ respectively. Then,

$$\frac{a+b}{c} = \frac{\cos(\frac{A-B}{2})}{\sin(C)},$$

and $\frac{a-b}{c} = \frac{\sin(\frac{A-B}{2})}{\cos(C)}.$

Interestingly enough, these formulas use all six parts of the triangle; the three sides, and the three angles.

These are our three pages of trigonometric functions alone, not to mention the proofs needed for them. We've already proved our Pythagorean, Cofunction, and Periodicity Identities. We should start by proving what we proved before; we'll prove our Sum/Difference Identities, which is an extension of *Lemma 1* of "Sine, Cosine, and Tangent." For convenience, we shall put the proof below.

Sum/Difference Identities (10.1)

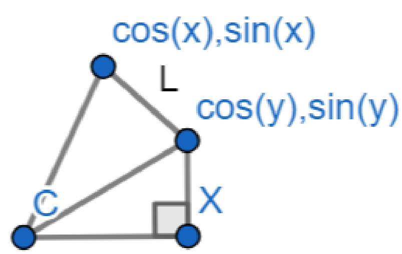
For any values x, y,
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y).$$
$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x).$$
$$\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x).$$
$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\cdot\tan(y)}.$$
$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\cdot\tan(y)}.$$
$$\cot(x+y) = \frac{\cot(x)\cdot\cot(y)-1}{\cot(x)+\cot(y)}.$$
$$\cot(x-y) = \frac{\cot(x)\cdot\cot(y)+1}{\cot(y)-\cot(x)}.$$

The proof for the second equation is in "Sine, Cosine, and Tangent." You can use the second equation to prove the rest. The proofs for the tangent and cotangent identities have not been included, and they will be included here.

Theorem 10.1's Proof

We first will prove the second equation.

By the Law of Cosines, $L^2 = 2 - 2\cdot\cos(x-y)$, and by the distance formula, $L^2 = (\cos(x) - \cos(y))^2 + (\sin(x) - \sin(y))^2 = 2 - 2\cos(x)\cos(y) - 2\sin(x)\sin(y)$. (We can substitute $\sin^2(x) + \cos^2(x) = 1$ and the symmetrical case for y to get the 2 in the equation.) By the transitive property, $2 - 2\cdot\cos(A-B) = 2 - 2\cos(x)\cos(y) - 2\sin(x)\sin(y)$, which implies $\cos(A-B) = \cos(x)\cos(y) + \sin(x)\sin(y)$, as desired.



We now will prove the first equation.

Substituting y for $-y'$ (this y' is really an arbitrary term, and y' is used in lieu of y for explanation purposes) gives us $\cos(x+y') = \cos(x)\cos(-y') + \sin(x)\sin(-y')$. Since cosine is an even function and sine is an odd function (this can be learned by analyzing the graphs of the two functions; we will elaborate on another chapter), we see that $\cos(x+y') = \cos(x)\cos(y) - \sin(x)\sin(-y')$. Since y' is arbitrary, we can substitute y, and we get our desired equation.

We now prove the fourth equation.

Note that $\cos(x) = \sin(90 - x)$ and $\sin(90 - x) = \cos(x)$ when these trigonometric functions are in degrees. Substituting x for $90 - x'$ in the first equation, we get $\cos(90 - x' + y) = \cos(90 - x')\cos(y) - \sin(90 - x')\sin(y)$. Substituting our translations gives us $\sin(x' - y) = \sin(x')\cos(y) - \sin(y)\cos(x')$, as desired.

We now prove the third equation.

Substituting $-y = y'$ gives us $\sin(x + y') = \sin(x)\cos(-y') - \sin(-y')\cos(x')$, which becomes $\sin(x + y') = \sin(x)\cos(y') + \sin(y')\cos(x)$, as desired.

We now prove the fifth equation.

Note that $\tan(x + y) = \frac{\sin(x+y)}{\cos(x+y)}$. By the first and third equations, $\frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y)+\sin(y)\cos(x)}{\cos(x)\cos(y)-\sin(x)\sin(y)}$. Dividing the numerator and denominator of the fraction by $\cos(x)\cos(y)$ yields $\tan(x + y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\tan(x)+\tan(y)}{1-\tan(x)\cdot\tan(y)}$, as desired.

We now prove the sixth equation.

Substitute in $-y' = y$ into the fifth equation and we see that $\tan(x - y') = \frac{\tan(x)-\tan(y)}{1+\tan(x)\cdot\tan(y)}$, as desired.

We now prove the seventh equation.

Take the fifth equation and take its reciprocal. This gives us $\cot(x + y) = \frac{1-\tan(x)\cdot\tan(y)}{\tan(x)+\tan(y)}$. Multiply both sides of the fraction by $\cot(x)\cot(y)$ and we get $\cot(x + y) = \frac{\cot(x)\cdot\cot(y)-1}{\cot(x)+\cot(y)}$, as desired.

Finally, we prove the eight equation.

Take the sixth equation and take its reciprocal. This gives us $\cot(x - y) = \frac{1+\tan(x)\cdot\tan(y)}{\tan(x)-\tan(y)}$. Multiply both sides of the fraction by $\cot(x)\cot(y)$ and we get $\cot(x - y) = \frac{\cot(x)\cdot\cot(y)+1}{\cot(y)-\cot(x)}$, as desired.

The only function you need to have by heart are the $\sin(x + y)$ and $\cos(x + y)$ ones, for convenience. (Taking cofunction identities would not be fun, after all.) Even though the proofs for the rest of the identities in this entire section are based off of the proof for $\cos(x - y)$, the most important identities to understand are the ones based off of it. (This is because it will give you insight into how these identities are used.)

Double Angle Identities (10.2)

$$\sin(2x) = 2\sin(x)\cos(x).$$
$$\cos(2x) = \cos^2(x) - \sin^2(x).$$
$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}.$$
$$\cot(2x) = \frac{\cot^2(x)-1}{2\cot(x)}.$$

These should be easy to prove, as they are just direct applications of the sum/difference formulas (10.1).

Theorem 10.2's Proof

We first will prove the first equation.

Applying the sum formula for sines gives us

$\sin(2x) = \sin(x+x) = \sin(x)\cos(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x),$ as desired.

Now we prove the second equation.

Applying the sum formula for sines gives us

$$\cos(2x) = \cos(x+x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2(x) - \sin^2(x).$$

Now we prove the third equation.

Note that $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$. By the first and second equation, $\tan(2x) = \frac{2\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)}$. Dividing both sides of the fraction by $\cos^2(x)$ gives us $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$, as desired.

Finally, we prove the last equation.

Note that $\cot(2x) = \frac{1}{\tan(2x)}$. This implies $\cot(2x) = \frac{1-\tan^2(x)}{2\tan(x)}$. Multiplying both sides of the fraction by $\cot^2(x)$ yields $\cot(2x) = \frac{\cot^2(x)-1}{2\cot(x)}$, as desired.

Half Angle Identities (10.3)

$$\sin(\tfrac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{2}}.$$
$$\cos(\tfrac{x}{2}) = \pm\sqrt{\frac{1+\cos(x)}{2}}.$$
$$\tan(\tfrac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}} = \frac{\sin(x)}{1+\cos(x)} = \frac{1-\cos(x)}{\sin(x)}.$$

Applying double angle to $\frac{x}{2}$ will give the desired results.

Theorem 10.3's Proof

We first will prove the first equation.

Applying double angle to $\cos(\frac{x}{2} \cdot 2)$ yields $\cos(x) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})$. Note that by the $\sin^2(\theta) + \cos^2(\theta) = 1$ identity, $\cos(x) = 1 - 2\sin^2(\frac{x}{2})$ and $\cos(x) = 2\cos^2(\frac{x}{2}) - 1$. Rearranging for $\cos(x) = 1 - 2\sin^2(\frac{x}{2})$ gives us $\sin^2(\frac{x}{2}) = \frac{1-\cos(x)}{2}$, and taking the square root (and accounting for sign) gives us $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos(x)}{2}}$, as desired.

We now prove the second equation.

We have already proved $\cos(x) = 2\cos^2(\frac{x}{2}) - 1$. Rearranging yields $\cos(\frac{x}{2}) = \sqrt{\frac{1+\cos(x)}{2}}$, as desired.

Finally, we prove the third equation.

The first part is easy; substitute $\tan(\frac{x}{2}) = \frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})} = \frac{\sqrt{1-\cos(x)}}{\sqrt{1+\cos(x)}}$, as desired.

For the second part, multiply both sides of the fraction by $\sqrt{1+\cos(x)}$ to achieve $\tan(\frac{x}{2}) = \frac{\sqrt{1-\cos^2(x)}}{1+\cos(x)} = \frac{\sqrt{\sin^2(x)}}{1+\cos(x)} = \frac{\sin(x)}{1+\cos(x)}$, as desired.

For the last part, we can prove $\frac{\sin(x)}{1+\cos(x)} = \frac{1-\cos(x)}{\sin(x)}$ by cross multiplying and getting $\sin^2(x) = 1 - \cos^2(x)$, which proves the last part of our identity.

Sum to Product Identities (10.4)

$$\sin(x) + \sin(y) = 2\sin(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2}).$$
$$\sin(x) - \sin(y) = 2\sin(\tfrac{x-y}{2})\cos(\tfrac{x+y}{2}).$$
$$\cos(x) + \cos(y) = 2\cos(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2}).$$
$$\cos(x) - \cos(y) = -2\sin(\tfrac{x+y}{2})\sin(\tfrac{x-y}{2}).$$
$$\tan(x) + \tan(y) = \frac{\sin(x+y)}{\cos(x)\cos(y)}.$$
$$\tan(x) - \tan(y) = \frac{\sin(x-y)}{\cos(x)\cos(y)}.$$
$$\cot(x) + \cot(y) = \frac{\sin(x+y)}{\sin(x)\sin(y)}.$$
$$\cot(x) - \cot(y) = \frac{\sin(x-y)}{\sin(x)\sin(y)}$$

We want to simplify the right hand side into the left hand side with our sum and difference identities.

Theorem 10.4's Proof

We shall first prove the first equation.

By the Sum/Difference Identities (10.1), $\sin(\frac{x+y}{2} + \frac{x-y}{2}) = \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$. Then note that $\sin(\frac{x+y}{2} - \frac{x-y}{2}) = \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) - \cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$. We can add these two equalities together to get $\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$, as desired.

We now prove the second equation.
By the Sum/Difference Identities (10.1), $\sin(\frac{x+y}{2} + \frac{x-y}{2}) = \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$. Then note that $\sin(\frac{x+y}{2} - \frac{x-y}{2}) = \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) - \cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$. We can subtract the second equality from the first to get $\sin(x) - \sin(y) = 2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$, as desired.

We now prove the third equation.
By the Sum/Difference Identities (10.1), $\cos(\frac{x+y}{2} + \frac{x-y}{2}) = \cos(\frac{x+y}{2})\cos(\frac{x-y}{2}) - \sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$ and $\cos(\frac{x+y}{2} - \frac{x-y}{2}) = \cos(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$. Adding these equations up, we get $\cos(x) + \cos(y) = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$, as desired.

We now prove the fourth equation.
By the Sum/Difference Identities (10.1), $\cos(\frac{x+y}{2} + \frac{x-y}{2}) = \cos(\frac{x+y}{2})\cos(\frac{x-y}{2}) - \sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$ and $\cos(\frac{x+y}{2} - \frac{x-y}{2}) = \cos(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$. Subtracting the second equality from the first gives us $\cos(x) - \cos(y) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$, as desired.

We now prove the fifth equation.
Note that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ implies $\tan(x) + \tan(y) = \frac{\sin(x)}{\cos(x)} + \frac{\sin(y)}{\cos(y)} = \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cdot\cos(y)}$. Note that by the Sum/Difference Identities (10.1), $\tan(x) + \tan(y) = \frac{\sin(x+y)}{\cos(x)\cdot\cos(y)}$, as desired.

We now prove the sixth equation.
A trivial plugin of $-x$ and knowledge that $\sin(\theta)$ is an odd function while $\cos(\theta)$ is an even function gives us $\tan(x) - \tan(y) = \frac{\sin(x)\cos(y)-\cos(x)\sin(y)}{\cos(x)\cdot\cos(y)} = \frac{\sin(x-y)}{\cos(x)\cdot\cos(y)}$, as desired.

We now prove the seventh equation.
Substituting $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ and using the Sum/Difference Identities (10.1) implies that $\cot(x) + \cot(y) = \frac{\cos(x)}{\sin(x)} + \frac{\cos(y)}{\sin(y)} = \frac{\cos(x)\sin(y)+\sin(x)\cos(y)}{\sin(x)\cdot\sin(y)} = \frac{\sin(x+y)}{\sin(x)\cdot\sin(y)}$, as desired.

Finally, we prove the last equation.
Knowledge of even and odd functions gives us
$\cot(x) - \cot(y) = \frac{\cos(x)\sin(y) - \sin(x)\cos(y)}{\sin(x)\cdot\sin(y)} = \frac{\sin(x-y)}{\sin(x)\cdot\sin(y)}$, as desired.

Note that for the first four equations, we didn't apply the Sum/Difference Identities (10.1) to x and y. Instead, we applied them to $\frac{x+y}{2}$ and $\frac{x-y}{2}$. For the last four equations, we applied them to x and y directly.

Product to Sum Identities (10.5)

$$\sin(x)\sin(y) = \tfrac{1}{2}(\cos(x-y) - \cos(x+y)).$$
$$\sin(x)\cos(y) = \tfrac{1}{2}(\sin(x-y) + \sin(x+y)).$$
$$\cos(x)\cos(y) = \tfrac{1}{2}(\cos(x-y) + \cos(x+y)).$$
$$\tan(x)\tan(y) = \frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}.$$

Just as we used Sum/Difference Identities (10.1) to prove our Sum to Product Identities (10.4), we should expect to use them to prove this theorem as well.

Theorem 10.5's Proof

First, we shall prove the first equation.
By the Sum/Difference Identities (10.1),
$\frac{1}{2}(\cos(x-y) - \cos(x+y)) = \frac{1}{2}(\cos(x)\cos(y) + \sin(x)\sin(y) - [\cos(x)\cos(y) - \sin(x)\sin(y)])$,
which simplifies to $\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$, as desired.

Now, we shall prove the second equation.
By the Sum/Difference Identities (10.1),
$\frac{1}{2}(\sin(x-y) + \sin(x+y)) = \frac{1}{2}(\sin(x)\cos(y) - \cos(x)\sin(y) + \sin(x)\cos(y) - \cos(x)\sin(y))$, which simplifies to $\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \cos(x+y))$, as desired.

Now, we shall prove the third equation.
By the Sum/Difference Identities (10.1),
$\frac{1}{2}(\cos(x-y) + \cos(x+y)) = \frac{1}{2}(\cos(x)\cos(y) + \sin(x)\sin(y) + \cos(x)\cos(y) - \sin(x)\sin(y))$, which simplifies to $\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$, as desired.

Finally, we prove the last equation.
Multiplying both sides of $\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}$ by $\tan(x)\tan(y)$ gives us
$\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)} = \frac{\tan(x)\tan(y)(\tan(x)+\tan(y))}{\tan(x)+\tan(y)} = \tan(x)\tan(y)$, as desired.

Most of these identities are proved backwards; our mess turns into our neat formulas. Nevertheless, problem writers will use these formulas both ways, so make sure you can derive these proofs in either way.

Mollweide's Formulas (10.6)

Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ expressed as $a, b, c,$ respectively. Then,

$$\frac{a+b}{c} = \frac{\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})},$$

and $\frac{a-b}{c} = \frac{\sin(\frac{A-B}{2})}{\cos(\frac{C}{2})}$.

Theorem 10.6's Proof

Note that by the Extended Law of Sines (8.2), $a = 2R\sin(A),\ b = 2R\sin(B),$ and $c = 2R\sin(C).$ This implies that $\frac{a+b}{c} = \frac{\sin(A)+\sin(B)}{\sin(C)}$, and $\frac{a-b}{c} = \frac{\sin(A)-\sin(B)}{\sin(C)}$. By the Sum to Product Identities (10.4), $\frac{a+b}{c} = \frac{2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})}{\sin(C)}$ and $\frac{a-b}{c} = \frac{2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})}{\sin(C)}$. Then, note that by the Double Angle Identities (10.2), $\frac{a+b}{c} = \frac{2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})}{2\sin(\frac{C}{2})\cos(\frac{C}{2})}$ and $\frac{a-b}{c} = \frac{2\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})}{2\sin(\frac{C}{2})\cos(\frac{C}{2})}$. Note that by the Cofunction Identities, $\cos(\frac{C}{2}) = \sin(\frac{A+B}{2})$, and $\sin(\frac{C}{2}) = \cos(\frac{A+B}{2})$. Applying these substitutions yields $\frac{a+b}{c} = \frac{\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})}{\sin(\frac{A+B}{2})\sin(\frac{C}{2})} = \frac{\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})}$ and $\frac{a-b}{c} = \frac{\sin(\frac{A-B}{2})\cos(\frac{A+B}{2})}{\cos(\frac{C}{2})\cos(\frac{A+B}{2})} = \frac{\sin(\frac{A-B}{2})}{\cos(\frac{C}{2})}$, as desired.

The key motivation to the proof is to express a, b, c in terms of trigonometric functions. The most convenient would be the Law of Sines, which is why we chose it.

This might be a confusing section. To make it easier to digest, we'll go over how each of these identities were derived. The Sum/Difference Identities (10.1) were proved by using the Law of Cosines (8.3) to prove the second equation, and using odd/even and cofunction identities to derive the rest. The Double Angle Identities (10.2) and the Half Angle Identities (10.3) were derived by using the Sum/Difference Identities (10.1) on $\sin(x+x)$ and $\sin(\frac{x}{2}+\frac{x}{2})$, respectively. The Sum to Product Identities (10.4) were proved by applying the Sum/Difference Identities (10.1) on $\sin(\frac{x+y}{2}+\frac{x-y}{2})$ and $\sin(\frac{x+y}{2}-\frac{x-y}{2})$. The Product to Sum Identities (10.5) were proved by applying the Sum/Difference Identities (10.1) on $\frac{1}{2}(\cos(x)-\cos(y))$ and the like. Finally, Mollweide's identities were proved using the Extended Law of Sines (9.2) and the Sum/Difference Identities (10.1). The main takeaway from this is that the most important identities are the Sum/Difference Identities (10.1); with those, and knowledge of the outline of the proofs for the other identities, you could easily derive the rest with the Sum/Difference Identities (10.1).

Now that we've proved our trigonometric identities and reviewed their motivations, we will provide a few problems that can be solved using them. Don't hesitate to look up the solutions for a few of these problems; as long as you are actively trying to understand them, you will be building intuition to solve these.

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1. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively. If $\angle A = 45^\circ$ and $\angle B = 15^\circ$, find $\frac{a}{c}$.

2. Prove that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$.

3. Find $\frac{\tan(15)+\tan(45)}{\cot(15)+\cot(45)}$.

4. Use Mollweide's Formulas to prove the Law of Sines.

5. Find $\tan(0) + \tan(1) + ... + \tan(179)$.

6. Find $\csc(1)\sec(1) + \csc(2)\sec(2) + ... + \csc(359)\sec(359)$.

7. If $\tan^{-1}(x) + \tan^{-1}(y)$ cannot be expressed as $\tan^{-1}(z)$ for some z, find xy.

8. Given that $\tan(x) + \tan(y) = 7$ and $\tan(x + y) = -\frac{7}{9}$, find $\tan(x) - \tan(y)$, provided that $\tan(x) > \tan(y)$.

9. If $\cot(x) = 3$ and $\cot(x - y) + \cot(x + y) = 6$, find $\tan(y)$.

10. Prove that $\sin(x) + \cos(x) = \pm\sqrt{1 + \sin(2x)}$.

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1. Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively. If $\angle A = 45^\circ$ and $\angle B = 15^\circ,$ find $\frac{a}{c}.$

Solution: Note that this implies $\angle C = 120^\circ.$ By the Law of Sines, $\frac{a}{\sin(45)} = \frac{c}{\sin(120)},$ implying $\frac{a}{c} = \frac{\sin(45)}{\sin(120)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{6}}{3}.$

2. Prove that $\cos(3x) = 4\cos^3(x) - 3\cos(x).$

Solution: By the Sum/Difference Identities (10.1), $\cos(x + 2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x) = \cos(x)(\cos^2(x) - \sin^2(x)) - 2\sin^2(x)\cos(x).$ Expanding yields $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x).$ Note that $3\sin^2(x) = 3 - 3\cos^2(x).$ Substitution yields $\cos(3x) = \cos^3(x) - (3 - 3\cos^2(x))(\cos(x)) = 4\cos^3(x) - 3\cos(x),$ as desired.

3. Find $\frac{\tan(15)+\tan(45)}{\cot(15)+\cot(45)}.$

Solution: By the Product to Sum Identities (10.5), $\frac{\tan(15)+\tan(45)}{\cot(15)+\cot(45)} = \tan(15)\tan(45).$ By the Sum to Product Identities (10.4), $\tan(15)\tan(45) = \frac{\sin(15)\sin(45)}{\cos(15)\cos(45)} = \frac{\cos(-30)-\cos(60)}{\cos(-30)+\cos(60)}.$ Note that $\cos(-30) = \frac{\sqrt{3}}{2}$ and $\cos(60) = \frac{1}{2}.$ This implies $\frac{\tan(15)+\tan(45)}{\cot(15)+\cot(45)} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = 2 - \sqrt{3}.$

4. Use Mollweide's Formulas to prove the Law of Sines.

Solution: Consider $\triangle ABC$ with $\overline{BC}, \overline{AC}, \overline{AB}$ denoted as $a, b, c,$ respectively. Note that by Mollweide's Formulas, $\frac{a+b}{c} = \frac{\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})}$ and $\frac{a-b}{c} = \frac{\sin(\frac{A-B}{2})}{\cos(\frac{C}{2})}.$ This implies that $\frac{a}{c} = \frac{1}{2}(\frac{\cos(\frac{A-B}{2})}{\sin(\frac{C}{2})} + \frac{\sin(\frac{A-B}{2})}{\cos(\frac{C}{2})}) = \frac{\cos(\frac{A-B}{2})\cos(\frac{C}{2})+\sin(\frac{A-B}{2})\sin(\frac{C}{2})}{2\sin(\frac{C}{2})\cos(\frac{C}{2})}.$ By the Sum/Difference Identities (10.1), $2\sin(\frac{C}{2})\cos(\frac{C}{2}) = \sin(C),$ and $\cos(\frac{A-B}{2})\cos(\frac{C}{2}) + \sin(\frac{A-B}{2})\sin(\frac{C}{2}) = \cos(\frac{A-B-C}{2}),$ implying that $\frac{a}{c} = \frac{\cos(\frac{A-B-C}{2})}{\sin(C)}.$ By the Odd/Even Identities, $\cos(\frac{A-B-C}{2}) = \cos(\frac{B+C-A}{2}),$ and by the Cofunction Identities, $\cos(\frac{B+C-A}{2}) = \sin(A),$ implying $\frac{a}{c} = \frac{\sin(A)}{\sin(C)},$ as desired.

5. Find $\tan(0) + \tan(1) + ... + \tan(179).$

Solution: Note that by the Even/Odd Identities, $\tan(90) + ... + \tan(179) = -\tan(-90) - ... - \tan(-179)$, and by the Periodicity Identities, $-\tan(-90) - ... - \tan(-179) = -\tan(90) - ... - \tan(1)$. Substitution yields $\tan(0) + \tan(1) + ... + \tan(179) = \tan(0) - \tan(0) + \tan(1) - \tan(1)... + \tan(90) - \tan 90) = 0$.

6. Find $\csc(1)\sec(1) + \csc(2)\sec(2) + ... + \csc(359)\sec(359)$.

Solution: Note that by the Periodicity Identities, $\csc(181)\sec(181) + ... + \csc(359)\sec(359) = \csc(-179)\sec(-179) + ... + \csc(-1)\sec(-1)$. By the Odd/Even Identities, this is equal to $-\csc(1)\sec(1) - ... - \csc(179)\sec(179)$. Substitution yields $\csc(1)\sec(1) + \csc(2)\sec(2) + ... + \csc(359)\sec(359) = 0$.

7. If $\tan^{-1}(x) + \tan^{-1}(y)$ cannot be expressed as $\tan^{-1}(z)$ for some z, find xy.

Solution: Let us assume that there is some z that $\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}(z)$. Taking the tangent of both sides yields $\tan(\tan^{-1}(x) + \tan^{-1}(y)) = z$ and the Sum/Difference Identities (10.1) imply that $z = \frac{\tan(\tan^{-1}(x)) + \tan(\tan^{-1}(y))}{1 - \tan(\tan^{-1}(x))\tan(\tan^{-1}(y))} = \frac{x+y}{1-xy}$. The only value of xy that leaves z undefined is $xy = 1$, which is our answer.

8. Given that $\tan(x) + \tan(y) = 7$ and $\tan(x+y) = -\frac{7}{9}$, find $\tan(x) - \tan(y)$, provided that $\tan(x) > \tan(y)$.

Solution: Note that by the Sum/Difference Identities (10.1), $\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} = \frac{7}{1-\tan(x)\tan(y)} = -\frac{7}{9}$, implying $\tan(x)\tan(y) = 10$. Note that $(\tan(x) + \tan(y))^2 = \tan^2(x) + 2\tan(x)\tan(y) + \tan^2(y) = 49$, which implies that $\tan^2(x) + \tan^2(y) = 29$. This implies that $\tan(x) = 5$ and $\tan(y) = 2$, because $\tan(x) > \tan(y)$. The Sum/Difference Identities (10.1) also imply that $\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} = \frac{3}{11}$. (Note that $\tan(x) > \tan(y)$, so this fraction is positive.)

9. If $\cot(x) = 3$ and $\cot(x-y) + \cot(x+y) = -6$, find $\tan^2(y)$.

Solution: Note that by the Sum/Difference Identities (10.1), $\cot(x-y) + \cot(x+y) = \frac{\cot(x)\cot(y)+1}{\cot(y)-\cot(x)} + \frac{\cot(x)\cot(y)-1}{\cot(y)+\cot(x)} = \frac{2\cot(x)\cot^2(y)+2\cot(x)}{\cot^2(y)-\cot^2(x)} = -6$. Substitution yields

$\frac{2\cdot 3\cdot \cot^2(y)+2\cdot 3}{\cot^2(y)-9} = -6,$ or $\frac{\cot^2(y)+1}{\cot^2(y)-9} = -1.$ This implies that $\cot^2(y) = 4,$ or $\tan^2(y) = \frac{1}{4},$ which is our answer.

10. Prove that $\sin(x) + \cos(x) = \pm\sqrt{1 + \sin(2x)}.$

Solution: Squaring both sides gives us $\sin^2(x) + \cos^2(x) + 2\sin(x)\cos(x) = 1 + \sin(2x).$ By the Pythagorean Identity and the Sum/Difference Identities (10.1), these two are equal.

Graphing Trigonometric Functions

How would trigonometric functions look graphed out? Well, using degrees would be really inconvenient, because you'd have to move from 0 to 90 on the x axis to get from 0 to 1. Before we get into graphing our functions, we shall introduce radians.

Radians are a measure of angles based on distance traveled in a circle. Considering a circle, we note that the total distance (the circumference) is $2\pi r,$ where r denotes the radius. We want radians to be consistent with degrees, so we have to find a way to get rid of $r.$ We can do this by dividing $2\pi r$ by $r.$ So, for a given arc that is $d\pi$ units long, the amount of radians it measures is $\frac{d\pi}{r}.$ Since a circle is 360 degrees, or 2π radians, we note that π radians is equivalent to 180 degrees. Now we can extend radians to be used in trigonometry. There are a few reasons we would use radians; one is for graphing to be easier, and others are Fourier Series, Taylor Sums, and Analytic Geometry, among others.

In this section, we will be discussing amplitude, period, frequency, phase shifts, and stretches/shrinks. Let our general function be $f(x) = a[\sin(n[x - p]) + c],$ where a represents the vertical stretch/shrink, n represents the horizontal dilation factor, p represents a horizontal translation, and c represents the vertical translation.

Let $\max_{f(x)}$ denote the maximum value $f(x)$ can take and let $\min_{f(x)}$ denote the minimum value $f(x)$ can take. The **amplitude** of $f(x)$ is $\frac{1}{2}(\max_{f(x)} - \min_{f(x)}).$ For example, the amplitude of $f(x) = \sin(x)$ is $\frac{1}{2}(1 - (-1)) = 1.$ In general, the amplitude of $f(x) = a[\sin(n[x - p]) + c]$ is $a.$ (We will prove this assertion along with others as exercises once we have finished introducing these terms.)

The **period** of a function $f(x)$ is the length of the smallest x interval that needs to be drawn before the graph of $f(x)$ repeats. For example, the period of $f(x) = \sin(x)$ is $2\pi.$ In general, the period of $f(x) = a\sin[(n[x - p]) + c]$ is $n.$

The **frequency** of a function $f(x)$ is how often its graph repeats. For example, the frequency of $\sin(x)$ is 1 per $2\pi,$ which is expressed as $\frac{1}{2\pi}.$ In general, the frequency of $f(x) = a\sin(n[x - p]) + c$ is $\frac{1}{n}.$ (Note that the frequency of $f(x)$ is the reciprocal of its period.)

The **phase shift** of a function $f(x) = a[\sin(n[x - p]) + c]$ represents its horizontal translation compared to its parent function $a[\sin(nx) + c]$. Usually, phase shifts to the right are denoted positively, and phase shifts to the left are denoted negatively. This is because as a graph is shifted to the right, its x value increases, while if it is shifted to the left, its x value decreases. For example, the phase shift of $f(x) = \sin(x + \pi)$ is $-\pi$, and the phase shift of $f(x) = a[\sin(n[x - p]) + c]$ is p. (Do not forget that p cannot be distributed out; it must be multiplied by n. If $f(x) = a[\sin(nx - p') + c]$, then the phase shift is $\frac{p'}{n}$, instead of n.)

To **vertically stretch/shrink** $f(x) = \sin(n[x - p]) + c$, multiply it by a factor of a and have it become $a \cdot f(x) = a\sin[(n[x - p]) + c]$. This means that all points $(x, f(x))$ are translated to $(x, a \cdot f(x))$, which is the definition of a vertical stretch/shrink.

To **horizontally stretch/shrink** $f(x) = a[\sin(x - p) + c]$, multiply x by n (leave p alone) to get $f_n(x) = a[\sin(nx - p) + c]$. This is because for it to be a horizontal stretch/shrink, $f(x) = f_n(x)$ when $x = 0$, and $a[\sin(-p) + c] = a[\sin(-p) + c]$, not $a[\sin(-np) + c]$.

All of these terms apply for cosine, and all of these terms except for amplitude apply for tangent, cotangent, secant, and cosecant. Remember that phase shift and vertical/horizontal stretches and shrinks are based off of a parent function. (Don't worry; we won't randomly have our parent function be $\sin(2x)$, and it usually can be assumed to be $\sin(x)$ unless we are dealing with a problem in terms of cosine, or other trigonometric functions.) Below are a few problems based on graphing trigonometric functions; all of these will be in radians from now on.

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1. Given parent function $\sin(x)$, find the phase shift of $\cos(x)$.

2. Find the period of $\sin(\frac{3}{2}x)$.

3. Find the frequency and period of $\sin(2x) + \cos(3x)$.

4. Find the amplitude of $2\sin(x) + 3\cos(x)$.

5. Prove that $f(x) = a\sin(nx + p) + c$ has an amplitude of a, a phase shift of $-\frac{p}{n}$, a period of $\frac{2\pi}{n}$, and a frequency of $\frac{n}{2\pi}$.

6. Generalizing for Problem 3, find the period of $f(x) = \sin(x\frac{m}{n}) + \cos(x\frac{a}{b})$ in terms of m, n, a, b where m, n, a, b are integers, $\gcd(m, n) = 1$, $\gcd(a, b) = 1$, and $n, b > 1$.

7. Generalizing for Problem 4, find the amplitude of $f(x) = m\sin(x) + n\cos(x)$.

8. Find, with proof, the period of $\tan(x)$.

9. Find the amplitude of $f(x) = \sin(x) + \cos(2x)$.

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1. Given parent function $\sin(x)$, find the phase shift of $\cos(x)$.

Solution: Note that by the Cofunction Identities, $\sin(x + \frac{\pi}{2}) = \cos(-x)$. By the Odd/Even Identities, $\sin(x + \frac{\pi}{2}) = \cos(x)$. For any point $(x, \sin(x))$ on $f(x) = \sin(x)$, there is a point $(x - \frac{\pi}{2}, g(x))$ where $g(x) = \cos(x) = \sin(x + \frac{\pi}{2})$. As thus, the phase shift of $\cos(x)$ given parent function $\sin(x)$ is $-\frac{\pi}{2}$.

2. Find the period of $\sin(\frac{3}{2}x)$.

Solution: Note that $\frac{3}{2}x$ repeats every 2π radians. So we want $\frac{3}{2}x = 2\pi$, implying $x = \frac{4\pi}{3}$, which is the period of $\sin(\frac{3}{2}x)$.

3. Find the frequency and period of $\sin(4x) + 2\cos(3x)$.

Solution: Note that for $f(x) = \sin(4x) + 2\cos(3x)$ to go through a period, $\sin(4x)$ and $2\cos(3x)$ must both go through an integral amount of periods. Note that the smallest number for which this is true is 2π, so the period of $f(x) = \sin(4x) + 2\cos(3x)$ is 2π.

4. Find the amplitude of $2\sin(x) + 3\cos(x)$.

Solution: Note that there is no value for x such that $2\sin(x) + 3\cos(x) = 5$. This is because there is no value of x such that $\sin(x) = \cos(x) = 1$. This implies that we'd like to write this as a single trigonometric function. Consider that by the Sum/Difference Identities (10.1), $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$. Then note that we want to find $\sin(x) \cdot 2 + \cos(x) \cdot 3$. Clearly, 2 replaces $\cos(y)$ and 3 replaces $\sin(y)$. Yet $\cos(y) = 2$ and $\sin(y) = 3$ are impossible. However, we can factor something out and scale down the equation such that $\sin(y)$ and $\cos(y)$ satisfy the famous Pythagorean Equality $\sin^2(x) + \cos^2(x) = 1$. Consider $f(x) = 2\sin(x) + 3\cos(x)$ and $c \cdot f(x) = 2c\sin(x) + 3c\cos(x)$ such that $2c = \cos(y)$ and $3c = \sin(y)$ satisfy $(2c)^2 + (3c)^2 = \cos^2(y) + \sin^2(y) = 1$, which implies $c^2 = \frac{\sqrt{13}}{13}$. Note that the amplitude of $\frac{\sqrt{13}}{13}f(x) = \sin(x)\cos(y) + \cos(x)\sin(y)$ where $\cos(y) = \frac{2\sqrt{13}}{13}$ and $\sin(y) = \frac{3\sqrt{13}}{13}$ is 1, because $\sin(x)\cos(y) + \cos(x)\sin(y) = \sin(x + y)$, whose amplitude is obviously 1. Then, $f(x) = \sqrt{13}\sin(x + y)$, implying the amplitude of $f(x) = 2\sin(x) + 3\cos(x)$ is $\sqrt{13}$.

5. Prove that $f(x) = a\sin(nx + p) + c$ has an amplitude of a, a phase shift of $-\frac{p}{n}$, a period of $\frac{2\pi}{n}$, and a frequency of $\frac{n}{2\pi}$.

Solution: For the amplitude, let's look at the maximum and minimum of $f(x)$ in terms of a, c. Note that $\min_{f(x)} = c - a$ and $\max_{f(x)} = c + a$, implying the amplitude of $f(x)$ is $\frac{1}{2}(c + a - (c - a)) = a$.

For the phase shift, note that $f(x) = a\sin(n[x - \frac{-p}{n}]) + c$, implying that compared to parent function $f_p(x) = a\sin(nx) + c$ with points $(f_p(x), a\sin(nx) + c)$, we have points $(f(x + \frac{p}{n}), a\sin(nx) + c)$, which is a horizontal shift of $\frac{-p}{n}$, implying the phase shift is $\frac{-p}{n}$.

For the period/frequency, note that $f(x) = a\sin(nx + p) + c$, the function $\sin(nx + p)$ repeats once every difference of 2π. A difference of 2π is spanned by nx every $\frac{2\pi}{n}$ traveled on the x axis, implying the period is $\frac{2\pi}{n}$ and the frequency is $\frac{n}{2\pi}$.

6. Generalizing for Problem 3, find the period of $f(x) = \sin(x\frac{m}{n}) + \cos(x\frac{a}{b})$ in terms of m, n, a, b where m, n, a, b are integers, $\gcd(m, n) = 1$, $\gcd(a, b) = 1$, and $n, b > 1$.

Solution: Note that the smallest x to make an integral amount of periods for $\sin(x\frac{m}{n})$ is $2\pi\frac{n}{m}$ and the smallest x to make an integral amount of periods for $\cos(x\frac{a}{b})$ is $2\pi\frac{b}{a}$. Note that we want to find the smallest x such that $x\frac{m}{n}$ and $x\frac{a}{b}$ are "multiples" of 2π, implying that we can substitute $x = 2\pi y$. This means that we desire $y\frac{m}{n}$ and $y\frac{a}{b}$ are integers, so a solution for y is $y = lcm(b, n)$. Note that we then can divide y by $\gcd(m, a)$ because that would still leave $y\frac{m}{n}$ and $y\frac{a}{b}$ integral, while ensuring $\gcd(y\frac{m}{n}, y\frac{a}{b}) = 1$ when $y = \frac{lcm(b,n)}{\gcd(m,a)}$. This implies that x, which is our period, is equivalent to $2\pi \cdot \frac{lcm(b,n)}{\gcd(m,a)}$.

7. Generalizing for Problem 4, find the amplitude of $f(x) = m\sin(x) + n\cos(x)$.

Solution: The problem goes similarly to Problem 4, only with variables m, n instead of constants. We have $f(x) = m\sin(x) + n\cos(x)$ which we scale down to $c \cdot f(x) = cm\sin(x) + cn\cos(x)$ such that $(cm)^2 + (cn)^2 = 1$. This implies that $m^2 + n^2 = \frac{1}{c^2}$, or $\frac{1}{c} = \sqrt{m^2 + n^2}$. Since the amplitude of $c \cdot f(x)$ is obviously 1, the amplitude of $f(x)$ is $1 \cdot \frac{1}{c} = \sqrt{m^2 + n^2}$, which is our answer.

8. Find, with proof, the period of $\tan(x)$.

Solution: We claim the period of $\tan(x)$ is π. Note that $\sin(x) = -\sin(x+\pi)$ and $\cos(x) = -\cos(x+\pi)$, because considering a unit circle, π is half of the length of the circumference, which means $(\cos(x), \sin(x))$ and $(\cos(x+\pi), \sin(x+\pi))$ are diametrically opposite. Since the center is the origin, the point diametrically opposite to $(\cos(x), \sin(x))$ is $(-\cos(x), -\sin(x))$, implying $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{-\sin(x)}{-\cos(x)} = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \tan(x+\pi)$, as desired.

Then note that for the period k of $\tan(x)$ to be less than π, that $\tan(-\frac{\pi}{2}) = \tan(k-\frac{\pi}{2})$. Yet we know that when $-\frac{\pi}{2} < x < \frac{\pi}{2}$, that $\tan(x)$ encompasses all values $-1 \leq \tan(x) \leq 1$ exactly once, the period of $\tan(x)$ must be greater than or equal to π.

9. Find the amplitude of $f(x) = \sin(x) + \cos(2x)$.

Solution: Note that by the Double Angle Identities (10.2), $f(x) = \sin(x) + 1 - 2\sin^2(x)$. This is a quadratic in $\sin(x)$, and completing the square yields $f(x) = -2(\sin(x) - \frac{1}{4})^2 + \frac{9}{8}$. Note that $\max_{f(x)}$ is clearly $\frac{9}{8}$, while the minimum is achieved at $\sin(x) = -1$, which gives us $\min_{f(x)} = -2$. This implies that the amplitude is $\frac{1}{2}(\frac{9}{8} + 2) = \frac{25}{16}$.

<u>Analytic Geometry</u>

Cartesian Coordinates

Analytic Geometry is the study of geometry within a coordinate plane. To commence our study, we begin with Cartesian Coordinates, the most basic of them. Cartesian Coordinates describe a pair of values with a set distance from the x and y axes. Cartesian Coordinates can be used in the real and complex plane, though we will first focus on the basics of Cartesian Coordinates, which involve the real plane. We will be defining 1 dimensional, 2 dimensional, and 3 dimensional coordinates, though we will be focusing on the xy plane in this section.

First, though, we shall define the distance from a point to a line, and a point to a plane. In a plane with point P and line $X,$ the length of the shortest path from P to X is the distance from P to $X.$ This happens to be the perpendicular from P to $X.$ In space with point P and plane $N,$ the length of the shortest path from P to N is the distance from P to $N.$ This happens to be the perpendicular from P to $N.$ (This we will define later.) The distance of a line from a plane, assuming they are parallel (otherwise they intersect) is the distance of an arbitrary point on said line to said plane.

For $(x, y),$ you go x units to the right of the origin (where right is perpendicular to the y axis) and you go y units to the up of the origin (where up is perpendicular to the x axis). Note that negative values for x, y imply leftwards and downwards movement, respectively. This can be generalized to higher dimensions with x axes $n_1, n_2 \ldots n_x,$ where for ordered $(n_1, n_2 \ldots n_x),$ n_k denotes going n_k units to the right of the origin, where right is perpendicular to all $n_1, n_2 \ldots n_x$ except for $n_k.$

Now, we shall synthetically prove that the distance of a point to a line is the length of the perpendicular, and extend this argument to a plane.

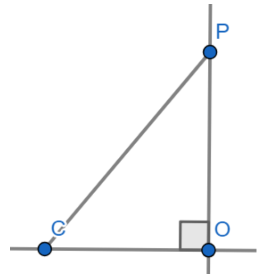
Distance of a Point and a Line (11.1)

The shortest path from point P to line X is the perpendicular from P to $X.$

Theorem 11.1's Proof

Assume that there is a shorter path. Let said path be from point P to point $C.$ Now let the perpendicular from P to X intersect X at $O.$ By the Pythagorean Theorem,

$\overline{PC}^2 = \overline{PO}^2 + \overline{OC}^2$. Yet we already assume that $\overline{PC} < \overline{PO}$. This implies that $\overline{OC}^2 < 0$, which is clearly ridiculous for real $\overline{OC}$. Thus no shorter path exists.



Now, as a prerequisite to our next theorem, we will define a perpendicular from point P to plane N as a line X such that any line passing through the intersection point of N and X contained within plane N is perpendicular to X. We also define a perpendicular pair of planes N, M such that there is a line X in plane M such that X is perpendicular to N.

Distance of a Point and Plane (11.2)

The shortest path from point P to plane N is the perpendicular from P to N.

Theorem 11.2's Proof

Let us assume there is a shorter path created by line X with points P, Q. There are two cases of lines that are not perpendicular. The first case is that the plane M containing X is perpendicular to N. In this case, by Theorem 11.1, we already know that the shortest distance is the perpendicular. The second case is if our line is not contained by a perpendicular plane. Then we can draw a cube with P, Q diagonally opposite each other. Note that the diagonal of a cube is always longer than its sides, so there is no shorter line X.

Now our Cartesian Coordinates are properly defined as the directed distance with perpendiculars. We look at the xy plane. Note that while our Cartesian Coordinates can be negative (consider $(-1,-1)$, $(-1,1)$, and $(1,-1)$ as examples), distance cannot be. We will have uses for both the Cartesian Coordinates and the distance, so let us define the magnitude of (x, y) in the real plane as $\sqrt{x^2+y^2}$. Similarly, the magnitude of (x) is $\sqrt{x^2}$ and the magnitude of (x, y, z) is $\sqrt{x^2+y^2+z^2}$. (This is also known as absolute value.)

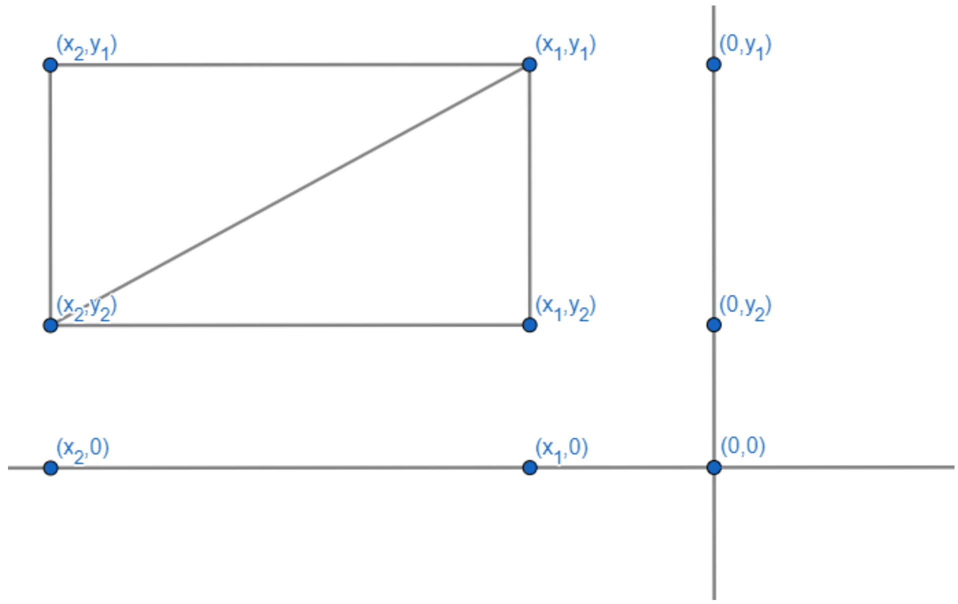
Now, with our coordinate system and magnitude properly defined, we'll prove a few basic facts about distance.

The Distance Formula (11.3)

Given (x_1, y_1) and (x_2, y_2), the distance of the two points is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Theorem 11.3's Proof

Plotting out points $(x_1, 0), (x_2, 0), (0, y_1), (0, y_2)$ and drawing a rectangle such that (x_1, y_1) and (x_2, y_2) are diagonally opposite corners, we note that the other two corners are (x_1, y_2) and (x_2, y_1). Since lines $(x_1, y_1), (x_1 y_2)$ and $(x_1, y_1), (x_2, y_1)$ are parallel to $(0, y_1), (0, y_2)$ and $(x_1, 0), (x_2, 0)$ respectively, our rectangle has dimensions $|x_1 - x_2|, |y_1 - y_2|$. By the Pythagorean Theorem, the length of the diagonal, or the distance of $(x_1, y_1), (x_2, y_2)$, is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, as desired.



You may already know that the graph of a line is $y = mx + b$, and the like for higher degree single variable polynomials. Instead, we look at the graph of a circle. Let's first look at the standard graph of a unit circle, with the origin as the center. Recall that a circle is the locus of points equidistant from a given point known as the center. This implies that the graph of the circle is the locus of points (x, y) such that the magnitude of (x, y) is 1. Recall that this implies $\sqrt{x^2 + y^2} = 1$, or $x^2 + y^2 = 1^2$. Generalizing, the graph of a circle centered at the origin with radius r has graph $\sqrt{x^2 + y^2} = r$, which in its more well-known form is $x^2 + y^2 = r^2$. Then, to translate the center to (h, v), we set the equation to $(x - h)^2 + (y - v)^2 = r^2$. To shrink or stretch the x factor by a, our equation becomes $a^2(x - h)^2 + (y - v)^2 = r^2$. Similarly, we can do the same to y and get our final equation of an ellipse as $a^2(x-h)^2+b^2(y-v)^2=r^2$, or the standard $\frac{(x-h)^2}{r_1^2}+\frac{(y-v)^2}{r_2^2}=1$, such that $(x-h)$ is shrunk by a factor of $\frac{1}{r_1}$, implying it is stretched by a factor of r_1, and similarly, $(y-v)$ is stretched by a factor of r_2.

Now, we recall the graph of polynomial $\sum_{n=0}^{d} c_n x^n = c_d x^d + ... + c_0$. (Note that a polynomial has a finite amount of terms, and only has positive exponents for all terms when in simplest form.) We want to find the zeros of $\sum_{n=0}^{d} c_n x^n$, or the intersections of $\sum_{n=0}^{d} c_n x^n = y$ and $0 = y$. Remember that the intersections of $y = f(x)$ and $y = g(x)$ are $(x, f(x))$ for all x such that $f(x) = g(x)$. (This is due to the transitive property.) Note that $\sum_{n=0}^{d} c_n x^n = 0$ can have up to n roots $r_1, r_2 ... r_d$, and some of them may be imaginary. This is because linear equations have only one solution, and $\sum_{n=0}^{d} c_n x^n$ can be expressed as $c_d \prod_{n=1}^{d}(x-r_n)$, and it is possible for all r_n to be distinct. The idea that a single variable polynomial with degree n has at least 1 zero and has at most n zeros is the Fundamental Theorem of Algebra.

In summary, the main ideas of Cartesian Coordinates are the Fundamental Theorem of Algebra, considering intersections as a system of equations, circles and their transformations, the shortest distance between two points being a line, and algebraic expressions.

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1. Find, with proof, the largest number of times a quadratic and a circle can intersect.

2. Prove that two lines are parallel if and only if they share the same slope, with different y intercepts.

3. If we freely rotate the point $(5, 8)$ around the point $(9, 5)$ and the point $(6, 17)$ around the point $(4, 17)$, what is the minimum distance these two rotated points could have from each other?

4. What about the maximum distance they could have from each other?

5. If $x^2+8x+y^2-10y=23$, find the sum of the maximum and minimum values of x^2+y^2.

6. Find the equation of the line, in any form, such that any point on that line makes an isosceles triangle in conjunction with points $(5, 2)$ and $(7, 4)$.

1. Find, with proof, the largest number of times a quadratic and a circle can intersect.

Solution: Note that the graph of a quadratic is $ax^2 + bx + c = y$ for constant a, b, c and the graph of a circle is $x^2 + y^2 = r^2,$ or $y^2 = r^2 - x^2,$ for constant $r.$ Then note that squaring both sides of the quadratic yields $ax^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2 = y^2.$ By the transitive property, the intersections of the two graphs are characterized at x such that $ax^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2 = r^2 - x^2.$ Since a, b, c, r are constant, this is a single variable polynomial, and by the Fundamental Theorem of Algebra, it has at most 4 solutions. Thus, the quadratic and circle can intersect at most 4 times.

2. Prove that two lines are parallel if and only if they share the same slope, with different y intercepts.

Solution: We first prove that two lines with the same slope are parallel. Note that we can express them as $f(x) = mx + b$ and $g(x) = mx + c$ with $b \neq c.$ This implies that their intersection point is the point where $b = c,$ which is no point. So there is no intersection, implying the two lines are parallel.

Then we prove that two parallel lines have the same slope but different y intercepts. Note that two lines with different slopes cannot be parallel, because the solution for $f(x) = mx + b$ and $g(x) = nx + c$ with $m \neq n$ is $mx + b = nx + c,$ implying $mx - nx = c - b$ or $x = \frac{c-b}{m-n},$ which means there is an interception point. It is obvious that they cannot have the same slope and same y intercept, else they are the same line, so parallel lines must have the same slope and have different y intercepts.

3. If we freely rotate the point $(5, 8)$ around the point $(9, 5)$ and the point $(6, 17)$ around the point $(4, 17),$ what is the minimum distance these two rotated points could have from each other?

Solution: Note that these are circles with centers at $(9, 5)$ and $(4, 17)$ with radii 5 and $2,$ respectively. Since the shortest distance between two points is a line, we note that any other path can be characterized as a quadrilateral. The line between $(4, 17)$ and $(9, 5)$ is shorter than the other three lengths, and the lengths of the radii can be subtracted, leaving us with the knowledge that the shortest path lies on the line between $(4, 17)$ and $(9, 5).$ This line has a length of $13,$ and subtracting the lengths of the radii yields a minimum distance of $6,$ as desired.

4. What about the maximum distance they could have from each other?

Solution: Similarly, we want to make the direct distance equivalent as going from the point on the circle we pick to the center to the other center and to the point we pick on the other circle. This happens if the points and the centers are in a straight line. Otherwise, by the quadrilateral inequality, they will be shorter. Thus, our answer is $13 + 6 = 19,$ when we add in the lengths of the radii.

5. If $x^2 + 8x + y^2 - 10y = 23,$ find the sum of the maximum and minimum values of $x^2 + y^2.$

Solution: $x^2 + 8x + y^2 - 10y = 23$ implies $(x+4)^2 + (y-5)^2 = 64.$ This is a circle, and we want to draw $x^2 + y^2 = r^2$ such that it is maximized and minimized. Let us call the circle formed by $(x+4)^2 + (y-5)^2 = 64$ circle C, and let us call the circle formed by $x^2 + y^2 = r^2$ circle M. For the radius of M to be maximized (and subsequently, $x^2 + y^2$ to be maximized), we want C internally tangent to M, and for the radius of M to be minimized, we want M internally tangent to C. We note that for there to only be one intersection point, said intersection point must pass through a line that bisects both circles. This means that said line must pass through the centers of both M and C (because a chord is only a diameter if it passes through the center), quickly giving us our equation of $y = \frac{-5x}{4}.$ Substitution yields $(x+4)^2 + (\frac{-5x}{4} - 5)^2 = 64,$ which can be simplified into $(x+4)^2 + \frac{25}{16}(x+4)^2 = 64.$ This implies that $(x+4)^2 = \frac{64 \cdot 16}{41},$ or $x + 4 = \frac{\pm 32\sqrt{41}}{41},$ or $x = -4 \pm \frac{32\sqrt{41}}{41},$ which also implies $y = 5 \mp \frac{40\sqrt{41}}{41}.$ Keeping with the knowledge that $y = \frac{-5x}{4}$ gives us our maximum and minimum as $x^2 + y^2 = (-4 - \frac{32\sqrt{41}}{41})^2 + (5 + \frac{40\sqrt{41}}{41})^2$ and $x^2 + y^2 = (-4 + \frac{32\sqrt{41}}{41})^2 + (5 - \frac{40\sqrt{41}}{41})^2.$ Summing these all up, we note that $(-4 - \frac{32\sqrt{41}}{41})^2 + (5 + \frac{40\sqrt{41}}{41})^2 + (-4 + \frac{32\sqrt{41}}{41})^2 + (5 - \frac{40\sqrt{41}}{41})^2 = 210,$ which is our answer.

6. Find the equation of the line, in any form, such that any point on that line makes an isosceles triangle in conjunction with points $(5, 2)$ and $(7, 4).$

Solution: Note that this implies every point on this line is equidistant from those two points, implying this is a perpendicular bisector. This bisector, by the fact that the slope of a perpendicular is always the negative of the reciprocal of the original line, has a slope of -1. It passes through the midpoint, which is $(6, 3)$, so the equation of the line is $x + y = 9$.

We previously discussed synthetic methods to find the area of a triangle. Now, we will prove two formulas for the area of a polygon using Cartesian Coordinates in the real plane; the Shoelace Theorem and Pick's Theorem.

Shoelace Theorem (12.1)

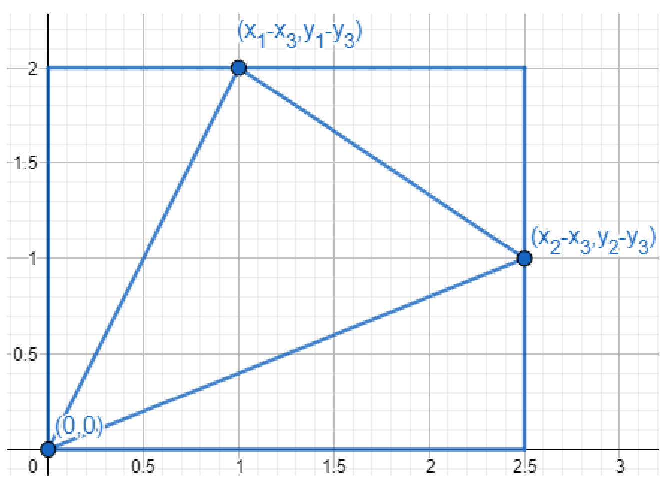
Given a polygon with coordinates $(x_1, y_1), (x_2, y_2) ... (x_n, y_n)$ listed in a clockwise or counterclockwise order, its area is $\frac{1}{2}|x_1y_2 + x_2y_3 + ... + x_ny_1 - x_1y_2 - x_2y_3 - ... - x_ny_1|$.

Theorem 12.1's Proof

We proceed by induction. First, we prove the formula for a triangle. Since translating will not affect the area of the triangle, we translate our points $(x_1, y_1), (x_2, y_2), (x_3y_3)$ to $(x_1 - x_3, y_1 - y_3), (x_2 - x_3, y_2 - y_3), (0, 0)$, respectively. This yields two cases:

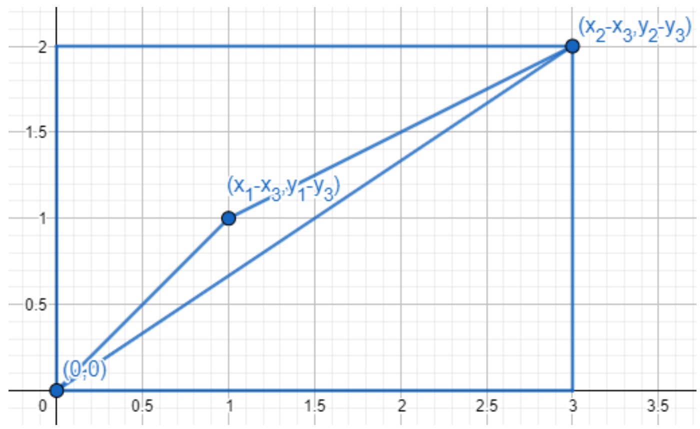
Case 1: All the points of the triangle are on the perimeter of the rectangle.

Then we draw a rectangle around our triangle, whose coordinates are clearly $(0, 0), (x_2 - x_3, 0), (x_2 - x_3, y_1 - y_3), (0, y_1 - y_3)$, when listed counterclockwise. We clearly know that the area of the rectangle is $(x_2 - x_3)(y_1 - y_3)$, and that the area of our three periphery triangles are $\frac{1}{2}(y_1 - y_3)(x_1 - x_3)$, $\frac{1}{2}(y_2 - y_3)(x_2 - x_3)$, $\frac{1}{2}(y_1 - y_2)(x_2 - x_1)$, listed counterclockwise. Then, subtracting gives us our area of $\frac{1}{2}|(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1)|$. (Sign can easily be accounted for with different quadrants. This also works if two of the vertices are situated as corners.)



Case 2: One vertice lies inside the rectangle.

Note that this means the other vertices are corners. Clearly the area of the rectangle is $(x_2 - x_3)(y_2 - y_3)$. Now we ignore the rectangle and only focus on the area of the right triangle containing our original triangle, and subtract the remaining area. Clearly we start out with an area of $\frac{1}{2}(x_2 - x_3)(y_2 - y_3)$. The remaining area can be split into two triangles, by connecting $(x_1 - x_3, y_1 - y_3)$ to $(x_2 - x_3, 0)$ or $(0, y_2 - y_3)$, depending on which side the point is on. Regardless, the area of these two triangles is $\frac{1}{2}(x_2 - x_3)(y_1 - y_3)$ and $\frac{1}{2}(x_1 - x_3, y_2 - y_3)$. We then subtract from our initial expression and we get $\frac{1}{2}|(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1)|$, as desired. (Again, sign is easy to deal with.)



Then we induct for $(x_1, y_1)...(x_{n+1}, y_{n+1})$. We work with $(x_1, y_1), (x_2, y_2)...(x_n, y_n)$ and $(x_1, y), (x_n, y_n), (x_{n+1}, y_{n+1})$, and we get their areas as $\frac{1}{2}|x_1y_2 + ... + x_ny_1 - x_1y_2 - x_2 - ... - x_ny_1|$ and $\frac{1}{2}|x_1y_n + x_ny_{n+1} + x_{n+1}y_1 - y_1x_n - y_nx_{n+1} - y_{n+1}x_1|$. Since these were traced in the same direction, their signs are the same, and summing these up gives us our total area of $\frac{1}{2}|x_1y_2 + x_2y_3 + ... + x_ny_1 - x_1y_2 - x_2y_3 - ... - x_ny_1|$, as desired.

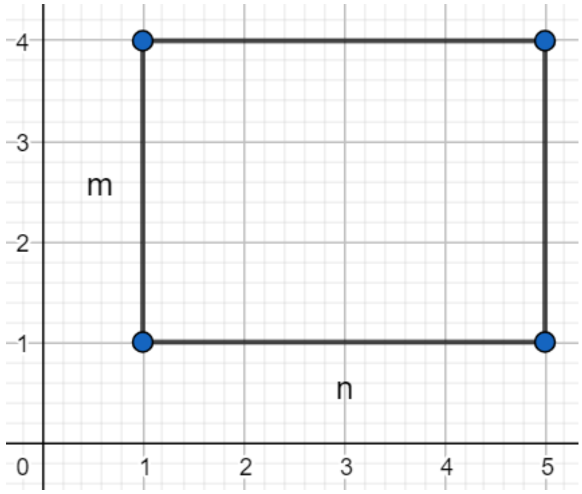
Work through these manipulations on your own to understand the process that was used to prove the Shoelace Theorem.

Pick's Theorem (12.2)

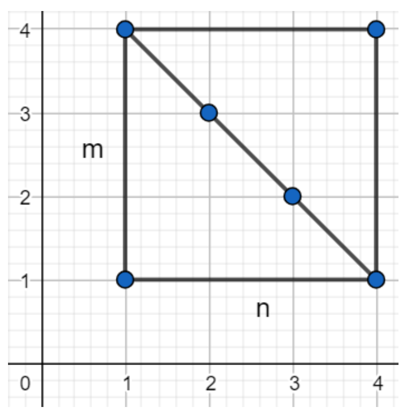
Given a non-self intersecting polygon with lattice coordinates, its area is $i + \frac{b}{2} - 1$ where i denotes the amount of lattice points in the interior of our polygon and b denotes the amount of lattice points on the boundary of our polygon.

Theorem 12.2's Proof

We start by proving this works for all triangles. To do this, we first prove that this is true for lattice rectangles parallel to the axes. For general $m \times n$ rectangles we have an area of mn, $2m + 2n$ boundary points, and $(m-1)(n-1)$ interior points. Pick's Theorem states $(m-1)(n-1) + \frac{2m+2n}{1} - 1 = mn$, which holds in this case.



Then we work with right triangles. Think of right triangles as half of a rectangle. Let there be d points on the diagonal. Then the area of our right triangle is $mn,$ there are $m+n-1+d$ boundary points, and the amount of interior points can be found by subtracting the $d-2$ points previously in the interior (remember that two points are instead corners of the rectangle and never counted) and dividing by $2.$ Thus, Pick's Theorem states $\frac{m+n-1+d}{2}+\frac{(m-1)(n-1)-(d-2)}{2}-1=\frac{mn-m-n+1m+n-1+d+2}{2}-1=\frac{mn}{2},$ which is true.

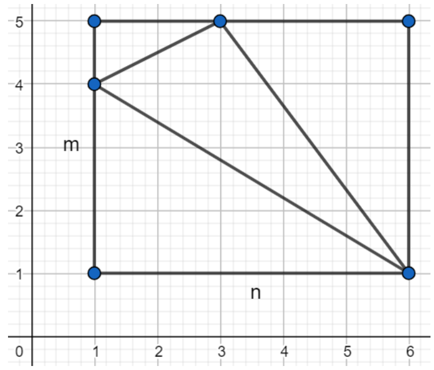


Then we prove this is true for every triangle, as we desired. Let our desired triangle be $T,$ and call the three other triangles $A,B,C,$ in any order. Then let I_A be the amount of interior points A possesses, B_A represent the amount of boundary points A contains, and similar definitions for $I_B, I_C, B_B, B_C,$ and for I_T, I_R, B_T, B_R as well.

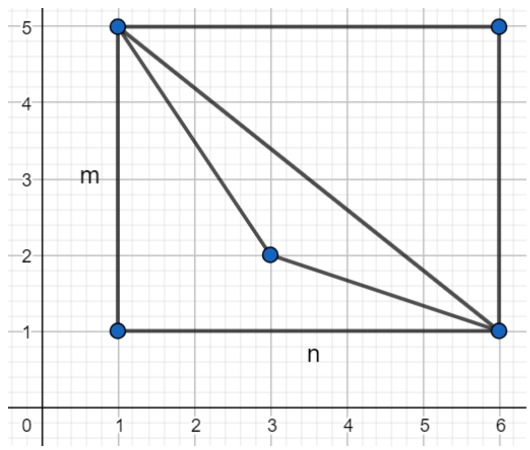
Case 1: All three vertices lie on the rectangle.

We note that $[A]=I_A+\frac{B_A}{2}-1,$ $[B]=I_B+\frac{B_B}{2}-1,$ $[C]=I_C+\frac{B_C}{2}-1,$ and $[R]=I_R+\frac{B_R}{2}-1,$ where $[R]$ denotes the total area of the rectangle. This then implies $[R]-[A]-[B]-[C]=[T]$ is the area we are solving for.

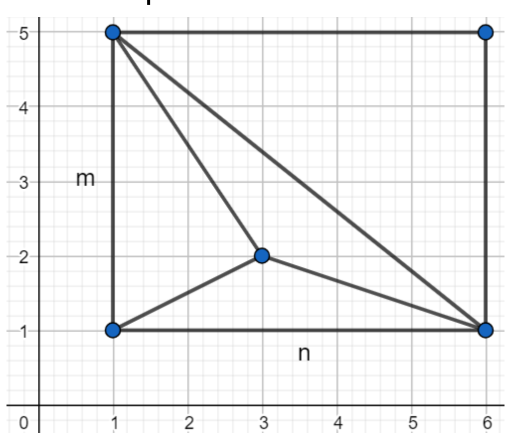
Then we note that $B_A + B_B + B_C = B_R + B_T$ and $I_R = I_A + I_B + I_C + I_T + B_T - 3$. Substituting gives us $[T] = I_R + \frac{B_R}{2} - 1 - (I_A + \frac{B_A}{2} - 1 + I_B + \frac{B_B}{2} - 1 + I_C + \frac{B_C}{2} - 1)$, which becomes $I_R + \frac{B_R}{2} - 1 - (\frac{B_R+B_T}{2} + I_R - I_T - B_T)$. This then is simplified into $I_T + \frac{B_T}{2} - 1 = [T]$, as desired.



Case 2: Only two vertices are on the rectangle.



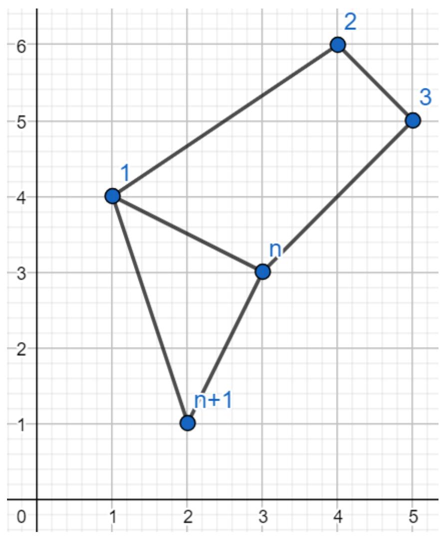
Connect the other point to a vertice of the rectangle.



Then the proof continues similarly. (Note that while two triangles share part of their boundary, it is counted individually for each triangle.) Let the line we constructed have

M boundary points. Then we note that $B_A + B_B + B_C = B_R + B_T + M$ and $I_R = I_A + I_B + I_C + I_T + B_T - 3 - M$. Then the substitution is identical, as the M terms cancel out.

Finally, we induct. Assuming all n sided polygon follow Pick's Theorem, we can prove this for any $n+1$ sided polygon by appending a triangle onto the side such that the extra vertice does not create a self-intersecting polygon. Let our polygon be P and our triangle be T, and let the side they share intersect M lattice points. We note that $[P]+[T] = P_I + T_I + \frac{P_B+T_B}{2} - 2$, and we note that the area of our total polygon is $P_I + T_I + (M-2) + \frac{(P_B-[M-2])+(T_B-[M-2]-2)}{2} - 1$. Note that this is equal to $P_1 + T_1 + (M-2) + \frac{P_B+T_B}{2} - \frac{2(M-2)}{2} - \frac{2}{2} - 1 = [P]+[T]$, as desired.



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1. Find the area of the triangle whose vertices lie on $(3,5), (4,9), (-4,-6)$.

2. Find the area of the polygon whose vertices lie on $(-1,1), (1,1), (1,-1), (-4,-4)$.

3. If the area of the polygon made by points $(5,3), (3,8), (4,6), (x,y)$ is 4, find the equation of the two lines that encompass all possible points (x,y).

4. Let $A=(5,4)$, $B=(6,-2)$, and $C=(-3,5)$. There are three distinct points X, Y, Z such that the quadruplets of points (A,B,C,X), (A,B,C,Y), and (A,B,C,Z) all form parallelograms. What is the area of $\triangle XYZ$?

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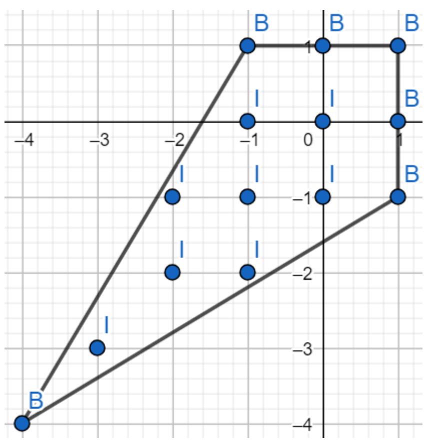
1. Find the area of the triangle whose vertices lie on $(3, 5), (4, 9), (-4, -6)$.

Solution: By the Shoelace Theorem (12.1), the area of our triangle is $\frac{1}{2}|3 \cdot 9 + 4 \cdot (-6) + (-4) \cdot 5 - 5 \cdot 4 - 9 \cdot (-4) - (-6) \cdot 3| = \frac{1}{2}|17| = \frac{17}{2}$.

2. Find the area of the polygon whose vertices lie on $(-1, 1), (1, 1), (1, -1), (-4, -4)$.

Solution: By the Shoelace Theorem (12.1), our area is $\frac{1}{2}|-1 \cdot 1 + 1 \cdot -1 + 1 \cdot (-4) + (-1) \cdot (-4) - 1 \cdot 1 - 1 \cdot 1 - 1 \cdot 4 - (-1) \cdot (-4)| = \frac{1}{2}|-12| = 6$.

Alternatively, by Pick's Theorem (12 .2), our area is $\frac{6}{2} + 8 - 1 = 10$.



3. If the area of the polygon made by points $(5, 3), (3, 8), (4, 6), (x, y)$ is 4, find the equation of the two lines that encompass all possible points (x, y).

Solution: Be careful about counting everything clockwise/counterclockwise consistently. Note that $(5, 3), (4, 6), (3, 8), (x, y)$ is the order that is either clockwise or counterclockwise; it doesn't particularly matter. By the Shoelace Theorem (11.1), $4 = \frac{1}{2}|5 \cdot 6 + 4 \cdot 8 + 3y + 3x - 3 \cdot 4 - 6 \cdot 3 - 8x - 5y| = \frac{1}{2}|32 - 5x - 2y|$. This implies that $8 = |32 - 5x - 2y|$. Either $8 = 32 - 5x - 2y$ or $8 = 5x + 2y - 32$, which can be simplified into $24 = 5x + 2y$ or $40 = 5x + 2y$, which is what we desired.

4. Let $A = (5, 4)$, $B = (6, -2)$, and $C = (-3, 5)$. There are three distinct points X, Y, Z such that the quadruplets of points (A, B, C, X), (A, B, C, Y), and (A, B, C, Z) all form parallelograms. What is the area of $\triangle XYZ$?

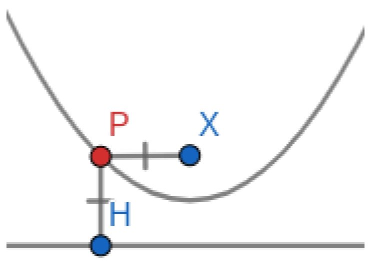
Solution: Notice that $\triangle ABC$ is the medial triangle of $\triangle XYZ$. Thus, $[XYZ] = 4[ABC]$.
By Shoelace,
$[ABC] = \frac{1}{2} \cdot |5 \cdot (-2) + 6 \cdot 5 + (-3) \cdot 4) - (4 \cdot 6 + (-2) \cdot (-3) + 5 \cdot 5)| = \frac{1}{2}|8 - 55| = \frac{47}{2}$. Thus, $[XYZ] = \frac{47}{2} \cdot 4 = 94$.

Conic Sections

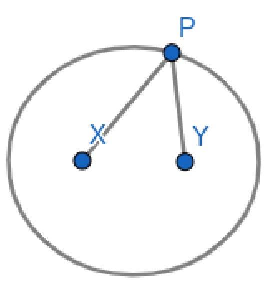
There are four definitions of a conic section. The first is the all-too-familiar $ax^2 + bxy + cy^2 + dx + ey + f = 0$ definition, which can be very useful for rotation. The next is the locus definition, which is the fundamental definition of a conic. Finally, there is the directrix definition. Then there is the double-cone definition, which we will ignore due to its lack of use in problem-solving.

Let's begin with the locus definition.

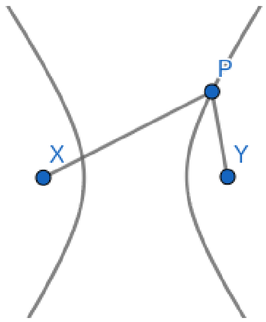
We define a *parabola* as the locus of points P such that for some point X and some line l, $\overline{PX} = d(P, l)$. The point X is known as the *focus*, and the line l is known as the *directrix*.



We define an *ellipse* as the locus of points P such that for two points X, Y, $\overline{PX} + \overline{PY} = c$ for some constant c. The points X, Y are known as the *foci*.



Similarly, we define a *hyperbola* as the locus of points P such that for two points X, Y, $|\overline{PX} - \overline{PY}| = c$ for some constant c. The points X, Y are known as the *foci* as well.



The algebraic definition of a conic is a second-degree equation in x, y. We will introduce the standard form of a parabola, an ellipse, and a hyperbola, along with a method on how to see whether a general conic is a parabola, ellipse, or a hyperbola.

Finally, the directrix definition states that every conic can be defined with two parameters; the distance from a focus to a directrix and an *eccentricity*. Given a focus P and a directrix l, the locus of points such that $\overline{PX} = d(P, l) \cdot \epsilon$ is the conic, where ϵ denotes the eccentricity.

First, we will investigate certain special properties of an ellipse. Let the ellipse have foci P, Q, and let the midpoint of PQ be O. Then the *major* radius of the ellipse is maximum length of OX, where X is a point on the ellipse. Conversely, the *minor* radius of the ellipse is the minimum length of OX.

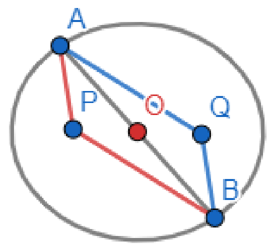
Major Radius of an Ellipse (13.1)

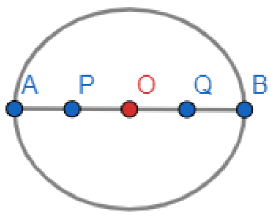
Consider an ellipse with foci P, Q such that any point X on the ellipse satisfies $\overline{PX} + \overline{QX} = c$. Then the major radius of the ellipse is $\frac{c}{2}$.

Theorem 13.1's Proof

We prove that the major diameter of the ellipse has length c. We must prove that this is the maximum. Also, we define O as the midpoint of PQ.

Let some line passing through O intersect the ellipse at A, B. Clearly we want to maximize $\overline{AB}$. Notice that by the Triangle Inequality, $\overline{AB} \leq \overline{AP} + \overline{PB} = c$, and $\overline{AB} \leq \overline{AQ} + \overline{QB} = c$. For both inequalities, equality is achieved when A, P, O, Q, B are collinear. Then $\overline{AB} \leq c$. As the major radius is $\overline{AO} = \frac{1}{2}\overline{AB}$, we have $\overline{AO} \leq \frac{c}{2}$, as desired.





Minor Radius of an Ellipse (13.2)

Consider an ellipse with foci P, Q such that any point X on the ellipse satisfies $\overline{PX} + \overline{QX} = c$ and such that $\overline{PQ} = a$. Then the minor radius of the ellipse is $\frac{\sqrt{c^2-a^2}}{2}$.

Theorem 13.2's Proof

Let $\overline{OX} = b$ and let $\angle POX = \theta$. Then it is clear that $\angle QOX = 180 - \theta$. By the Law of Cosines (9.3) and by the definition of an ellipse,

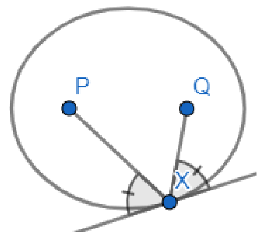
$\overline{PX} + \overline{PY} = \sqrt{\frac{a^2}{4} + b^2 - ab\cos(\theta)} + \sqrt{\frac{a^2}{4} + b^2 + ab\cos(\theta)} = c$. But notice that $\sqrt{\frac{a^2}{4} + b^2 - ab\cos(\theta)} + \sqrt{\frac{a^2}{4} + b^2 + ab\cos(\theta)} = c \leq 2\sqrt{\frac{a^2}{4} + b^2}$. Dividing both sides by 2 and squaring yields $\frac{c^2}{4} \leq \frac{a^2}{4} + b^2 \to \frac{c^2-a^2}{4} \leq b^2 \to \frac{\sqrt{c^2-a^2}}{2} \leq b$. Equality occurs when $\theta = 90^\circ$.

Equal Tangent Angles of an Ellipse (13.3)

Consider an ellipse with foci P, Q and point X on the ellipse. If α denotes the acute angle formed by PX and the tangent to the ellipse through X and β denotes the acute angle formed by QX and the tangent to the ellipse through X, $\alpha = \beta$.

Theorem 13.3's Proof

Let $\overline{PX} + \overline{QX} = c$ for some constant c, and let the tangent line be l Then for every point N on l, $\overline{PN} + \overline{QN} \geq c$, by the definition of a tangent line. Then notice that by *Running to the River*, the point that optimizes this is the intersection of PQ' (where Q' is the image of Q after a reflection about the tangent line) and l. Then it is a property of similar triangles and reflections that $\alpha = \beta$, as desired.



Perpendicular to Tangent Point Bisects Generating Angle (13.4)

Consider an ellipse with foci P, Q and point X on the ellipse. Then the line through X perpendicular to the tangent at X bisects $\angle PXQ$.

Theorem 13.4's Proof

Trivial by a combination of Theorem 13.3 and angle addition postulate.

Equation of an Ellipse (13.5)

The equation of an ellipse with center (h, k) and minor/major axes parallel to the axes of the coordinate system with the horizontal axis having length a and vertical axis having length b is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

Theorem 13.5's Proof

Notice that the graph of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ translated by (h, k). Thus we only have to prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ describes an ellipse with axes a, b. Without loss of generality, let $a > b$. Then the foci are $(-\sqrt{a^2 - b^2}, 0)$ and $(\sqrt{a^2 - b^2}, 0)$. This implies that $\sqrt{(x - \sqrt{a^2 - b^2})^2 + y^2} + \sqrt{(x + \sqrt{a^2 - b^2})^2 + y^2} = 2a$. For simplicity, let $c = \sqrt{a^2 - b^2}$. Expanding yields $\sqrt{x^2 + y^2 + c^2 - 2cx} + \sqrt{x^2 + y^2 + c^2 + 2cx} = 2a$. Squaring both sides yields $2(x^2 + y^2 + c^2) + 2\sqrt{(x^2 + y^2 + c^2)^2 - 4x^2c^2} = 4a^2$. Rearranging into $2\sqrt{(x^2 + y^2 + c^2)^2 - 4x^2c^2} = 4a^2 - 2(x^2 + y^2 + c^2)$ and squaring both sides yields $(2x^2 + 2y^2 + 2c^2)^2 - 16x^2c^2 = (4a^2 - 2x^2 - 2y^2 - 2c^2)^2$. Difference of squares yields $-16x^2c^2 = (4a^2)(4a^2 - 4x^2 - 4y^2 - 4c^2)$. Dividing by a common factor of 16 yields $-x^2c^2 = a^2(a^2 - x^2 - y^2 - c^2)$. Plugging in $c^2 = a^2 - b^2$ yields $x^2b^2 - x^2a^2 = a^2(b^2 - x^2 - y^2)$. Adding a^2x^2 to both sides yields $x^2b^2 = a^2b^2 - a^2y^2 \rightarrow x^2b^2 + y^2a^2 = a^2b^2$. Dividing both sides by a^2b^2 yields $\frac{a^2}{x^2} + \frac{b^2}{y^2}$, as desired.

Expanding the equation of an ellipse with axes not parallel to the coordinate axes is more computationally tedious. Usually, though, if the ellipse has foci $P = (x_1, y_1), Q = (x_2, y_2)$ and $\overline{PX} + \overline{QX} = c,$ then $\sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = c$ will suffice.

This equation of an ellipse isn't actually terribly useful on its own (though a large number of problems involving ellipses will usually use the coordinate plane), but it does reveal something important. An ellipse is simply a stretching of a circle, which is especially useful for projective transformations, and for area problems.

We'll also define the *latus rectum* of an ellipse as the line segment passing through a focus of an ellipse parallel to the minor radius with both endpoints on the ellipse.

Length of a Latus Rectum (13.6)

Let the equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ with $a \geq b.$ Then the length of the latus rectum is $\frac{2b^2}{a}.$

Theorem 13.6's Proof

Notice that the foci of the ellipse are $(\pm\sqrt{a^2 - b^2}, 0).$ Without loss of generality, we let the latus rectum be through $(\sqrt{a^2 - b^2}, 0).$ Then by the equation of the ellipse, $\frac{a^2 - b^2}{a^2} + \frac{y^2}{b^2} = 1,$ implying $y = \pm\frac{b^2}{a}.$ This implies that the latus rectum intersects the ellipse at $(\sqrt{a^2 - b^2}, \frac{-b^2}{a})$ and $(\sqrt{a^2 - b^2}, \frac{b^2}{a}).$ It should be obvious that the distance between the two points is $\frac{2b^2}{a},$ as desired.

Standard Form of a Parabola (14.1)

The standard form of a parabola with a directrix parallel to the x axis is $y - k = \frac{1}{4a}(x - h)^2.$ Then the vertex is $(h, k),$ the focus is $(h, k + a),$ and the directrix is $y = k - a.$

Theorem 14.1's Proof

Let the vertex be $(h, k),$ the focus be $X = (h, k + a),$ and the directrix be $y = k - a.$ Let some point on the parabola be $P = (x, y).$ Then, $\overline{XP} = \sqrt{(x - h)^2 + (y - k - a)^2},$ and the distance from P to the directrix is $y - k + a.$ Thus we have $\sqrt{(x - h)^2 + (y - k - a)^2} = y - k + a.$ Squaring both sides yields

$(x-h)^2+(y-k-a)^2=(y-k+a)^2$, which by difference of squares implies $(x-h)^2=(2a)(2y-2k)$, or $\frac{1}{4a}(x-h)^2=y-k$, as desired.

Notice that this implies there is a *line of symmetry* in every parabola. Its equation is $x=h$. Taking coordinates out of the picture, the line of symmetry of a parabola is the line through the focus perpendicular to the directrix.

We define the *latus rectum* of a parabola as the line segment through the focus parallel to the directrix with both endpoints on the parabola.

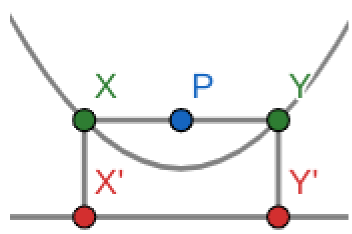
Length of the Latus Rectum (14.2)

If the distance from the focus and directrix of a parabola is $2a$, the length of the latus rectum is $4a$.

We choose $2a$ because a represents the distance from the focus to the vertex.

Theorem 14.2's Proof

Let the latus rectum intersect the parabola at X and Y, and let the feet of the perpendiculars from X and Y to the directrix be X' and Y'. Since the latus rectum is parallel to the directrix, $\overline{XX'}=\overline{YY'}=2a$. By the definition of a parabola, $\overline{XP}=\overline{XX'}=2a$ and $\overline{YP}=\overline{YY'}=2a$. Then $\overline{XY}=\overline{XP}+\overline{YP}=2a+2a=4a$, as desired.



We define the *vertices* of a hyperbola as the points where the line connecting the foci of the hyperbola meet the hyperbola. Additionally, the *center* of a hyperbola is the midpoint of its foci.

Standard Form of a Hyperbola (15.1)

Consider a hyperbola with center (h,k), vertices $(h\pm a,k)$, and foci $(h\pm c,k)$. Then the equation of the hyperbola is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, where $b^2=c^2-a^2$.

Theorem 15.1's Proof

Clearly, the h and k in the equation are just transformations - we will ignore them for now. If the vertices are $(\pm a, 0)$ and the foci are $(\pm c, 0)$, then our equation is $|\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}|=2a.$ Since the absolute value is annoying, we square both sides of the equation to get $(x-c)^2+y^2+(x+c)^2+y^2-2\sqrt{[(x-c)^2+y^2][(x+c)^2+y^2]}=4a^2.$ Cleaning this equation up gives us $2x^2+2y^2+2c^2-2\sqrt{x^4-2x^2c^2+c^4+2x^2y^2+y^4+2c^2y^2}=4a^2.$ We rearrange and get $\sqrt{x^4-2x^2c^2+c^4+2x^2y^2+y^4+2c^2y^2}=2a^2-c^2-x^2-y^2.$ Since $c^2=a^2+b^2,$ we get $\sqrt{x^4-2x^2c^2+c^4+2x^2y^2+y^4+2c^2y^2}=a^2-b^2-x^2-y^2.$ Squaring both sides yields $x^4-2x^2c^2+c^4+2x^2y^2+y^4+2c^2y^2=a^4-2a^2b^2+b^4+x^4+y^4+2x^2y^2-2a^2x^2-2a^2y^2+2b^2x^2+2b^2y^2.$ Cancelling like terms yields $c^4-2x^2c^2+2y^2c^2=a^4-2a^2b^2+b^4-2a^2x^2+2b^2x^2-2a^2y^2+2b^2y^2.$ Since $c^2=a^2+b^2,$ we can substitute to get $(a^2+b^2)^2-2x^2(a^2+b^2)+2y^2(a^2+b^2)=(a^2-b^2)^2+2x^2(b^2-a^2)-2y^2(b^2-a^2).$ Rearranging yields $4a^2b^2=4x^2b^2-4y^2a^2.$ Dividing both sides by $4a^2b^2$ yields $1=\frac{x^2}{a^2}-\frac{y^2}{b^2},$ as desired.

If the foci are vertical, the equation is $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ (which should seem very similar).

Asymptotes of a Hyperbola (15.2)

The asymptotes (lines that approach the hyperbola but never intersect it) of the hyperbola are $y-k=\pm\frac{b}{a}(x-h).$

Theorem 15.2's Proof

Once again, the k and h are just transformations and we can ignore them.

Plugging in $y=\pm\frac{b}{a}x$ into $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ gives us $0=1,$ which can never happen. Thus, the asymptote never intersects the hyperbola. However, we have $\frac{x^2}{a^2}=\frac{y^2}{b^2}+1\rightarrow\frac{x}{a}=\pm\sqrt{\frac{y^2}{b^2}+1}.$ In the "limit case," the graph approaches $\frac{x}{a}=\pm\frac{y}{b}\rightarrow y=\pm\frac{b}{a}x,$ as desired.

The *latus rectum* of a hyperbola is the line segment through a focus of the hyperbola with both endpoints on the hyperbola that is perpendicular to the line containing the foci.

Length of Latus Rectum (15.3)

The length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

Theorem 15.3's Proof

Notice the foci are $(\pm\sqrt{a^2+b^2}, 0)$. Without loss of generality, let the latus rectum pass through $(\sqrt{a^2+b^2}, 0)$. Then by the equation of the hyperbola, $\frac{a^2+b^2}{a^2} - \frac{y^2}{b^2} = 1$, implying $y = \pm\frac{b^2}{a}$. It is obvious that the distance between $(\sqrt{a^2+b^2}, -\frac{b^2}{a})$ and $(\sqrt{a^2+b^2}, \frac{b^2}{a})$ is $\frac{2b^2}{a}$.

Now we'll discuss two general conic identification methods. We will first show the discriminant method for the algebraic conic, and we will then show the eccentricity method (and how to find the eccentricity of a conic).

Conic by Discriminant (16.1)

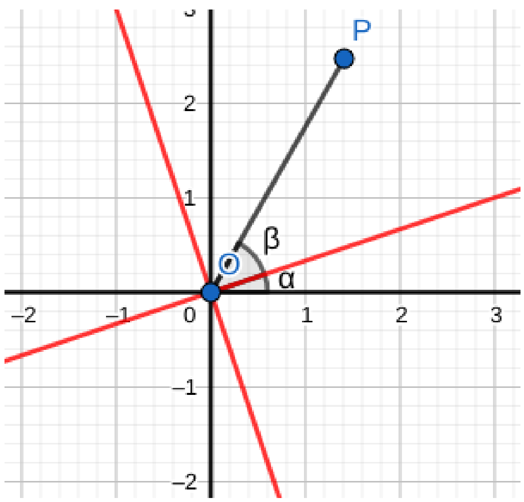
Consider conic $ax^2 + bxy + c^2 + dx + ey + f = 0$. Then let $\Delta f(x,y) = b^2 - 4ac$. If $\Delta f(x,y) < 0$, the conic is an ellipse. If $\Delta f(x,y) = 0$, the conic is a parabola. Finally, if $\Delta f(x,y) > 0$, the conic is a hyperbola.

Before we prove this theorem, we should discuss the idea of *rotating the axes*. This is motivated by the xy term basically rotating a conic. (As an example, consider the graph of $xy = 1$, which is a hyperbola.)

To get rid of the xy term, you can rotate the axes to transform the equation of the conic. Let the new axes be rotated counterclockwise by α. Then, for any point P with polar coordinates $(r, \alpha+\beta)$ in terms of the old coordinate system, A has coordinates (r, β) in terms of the new coordinates. Expanding the polar coordinates, we have $P' = (r\cos(\beta), r\sin(\beta))$. (This is useful because we can directly substitute our equations for x, y without changing the validity of the conic equation!)

This implies that $x = r\cos(\alpha+\beta) = r\cos(\alpha)\cos(\beta) - r\sin(\alpha)\sin(\beta) = x'\cos(\alpha) - y'\sin(\alpha)$ and $y = r\sin(\alpha)\cos(\beta) + r\cos(\alpha)\sin(\beta) = y'\cos(\alpha) + x'\sin(\alpha)$, by the Sum/Difference Identities (10.1).

Just to make it even easier to look at,
$x = x'\cos(\alpha) - y'\sin(\alpha)$.
$y = y'\cos(\alpha) + x'\sin(\alpha)$.



Theorem 16.1's Proof

We first prove that $b^2 - 4ac$ is invariant, no matter how the axes are rotated. Notice that $ax^2 + bxy + cy^2 + \ldots$ is transformed to

$$a(x'\cos(\alpha) - y'\sin(\alpha))^2 + b(x\cos(\alpha) - y\sin(\alpha))(x\sin(\alpha) + y\cos(\alpha)) + c(x\sin(\alpha) + y\cos(\alpha))^2 + \ldots$$

(We omit the d, e, f terms since their degree will not be high enough to affect $a, b, c.$)

Expanding, we get our transformed conic as

$$a(x^2\cos^2\alpha + y^2\sin^2\alpha - 2xy\sin\alpha\cos\alpha) +$$
$$b(x^2\sin\alpha\cos\alpha - y^2\sin\alpha\cos\alpha + xy\cos^2\alpha - xy\sin^2\alpha)$$
$$+ c(x^2\sin^2\alpha + y^2\cos^2\alpha + 2xy\sin\alpha\cos\alpha).$$

Notice that a' is the coefficient of the new x^2 term, b' is the coefficient of the new xy term, and c' is the coefficient of the new y^2 term. We then see that

$$a' = a\cos^2\alpha + b\sin\alpha\cos\alpha + c\sin^2\alpha$$
$$b' = -2a\sin\alpha\cos\alpha + xy\cos^2\alpha - xy\sin^2\alpha + 2c\sin\alpha\cos\alpha$$
$$c' = a\sin^2\alpha - b\sin\alpha\cos\alpha + c\cos^2\alpha.$$

It can then be verified with a computer program that $b'^2 - 4a'c' = b^2 - 4ac.$

Then rotate the axes such that $b' = 0.$ Then it should be obvious that if $\Delta f'(x, y) < 0,$ you have an ellipse, if $\Delta f'(x, y) = 0$ you have a parabola, and if $\Delta f'(x, y) > 0$ you have a hyperbola.

If you want to rotate the conic to get rid of the xy term, notice that we desire $b' = 0 = -2a \sin\alpha\cos\alpha + b\cos^2\alpha - b\sin^2\alpha + 2c\sin\alpha\cos\alpha$. Then notice $0 = (c-a)(\sin 2\alpha) + b\cos 2\alpha \rightarrow \frac{b}{a-c} = \tan 2\alpha$. So $\alpha = \frac{\arctan(\frac{b}{a-c})}{2}$.

Eccentricity of Conics (16.2)

Let the eccentricity of a conic be ϵ.

If $\epsilon < 1$, then the conic is an ellipse. (If $\epsilon = 0$, the conic is a circle. This is valid because of the projective plane.)

If $\epsilon = 1$, the conic is a parabola.

If $\epsilon > 1$, the conic is a hyperbola.

For an ellipse and a hyperbola, let the distance between the vertices be a and the distance between the foci be c. (For an ellipse, the vertices are the endpoints of the major axis.) Then $\epsilon = \frac{c}{a}$.

Theorem 16.2's Proof

Without loss of generality, let the directrix be $y = 0$ and let the focus be $(0,1)$. (Changing the focus only changes the "perspective" and leaves the "shape" of the conic intact.)

Then the equation of the conic is $\epsilon y = \sqrt{x^2 + (y-1)^2}$. Squaring both sides yields $\epsilon^2 y^2 = x^2 + y^2 - 2y + 1 \rightarrow x^2 + y^2 - \epsilon^2 y^2 - 2y + 1 = 0$. By Theorem 16.1, if $\epsilon < 1$, if is an ellipse, if $\epsilon = 1$, the conic is a parabola, and if $\epsilon > 1$, the conic is a hyperbola.

For the proof of the ellipse, write the standard form as $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.

We rewrite $x^2 + y^2 - \epsilon^2 y^2 - 2y + 1 = 0$ as $x^2 + (1-\epsilon^2)(y + \frac{1}{1-\epsilon^2})^2 - \frac{1}{1-\epsilon^2} + 1 = 0$. This rearranges to $x^2 + (1-\epsilon^2)(y + \frac{1}{1-\epsilon^2})^2 = \frac{\epsilon^2}{1-\epsilon^2}$. Notice that $a = \frac{\epsilon}{1-\epsilon^2}$ and $b = \frac{\epsilon}{\sqrt{1-\epsilon^2}}$. Since $a^2 - b^2 = \frac{\epsilon^2}{(1-\epsilon^2)^2} - \frac{\epsilon^2}{(1-\epsilon^2)} = \frac{\epsilon^2 - \epsilon^2(1-\epsilon^2)}{(1-\epsilon^2)^2} = \frac{\epsilon^4}{(1-\epsilon^2)^2} = (\frac{\epsilon^2}{1-\epsilon^2})^2$, $c = \frac{\epsilon^2}{1-\epsilon^2}$. Then $\frac{c}{a} = \epsilon$, as desired.

For the proof of the hyperbola, write the standard form as $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

We rewrite $x^2+y^2-\epsilon^2y^2-2y+1=0$ as $(\epsilon^2-1)(y+\frac{1}{1-\epsilon})^2-x^2=\frac{\epsilon^2}{\epsilon^2-1}$. (Notice that this is identical to the equation for the ellipse.) Notice $a=\frac{\epsilon}{1-\epsilon^2}$ and $b=\frac{\epsilon}{\sqrt{1-\epsilon^2}}$. Since $a^2+b^2=\frac{\epsilon^2}{(1-\epsilon^2)^2}+\frac{\epsilon^2}{(1-\epsilon^2)}=\frac{\epsilon^2+\epsilon^2(1-\epsilon^2)}{(1-\epsilon^2)^2}=\frac{\epsilon^4}{(\epsilon^2-1)^2}=(\frac{\epsilon^2}{1-\epsilon^2})^2$, so $c=\frac{\epsilon^2}{1-\epsilon^2}$. Then $\frac{c}{a}=\epsilon$, as desired.

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1. Consider an ellipse with major radius 10 and minor radius 5. Inscribe a quadrilateral $ABCD$ inside this ellipse. What is the maximum area of $ABCD$?

2. Two lovers Bob and Alice are surrounded by a ring of magical fairy dust. As part of his marriage proposal, Bob wants to deliver some magical fairy dust to Alice. He notices that he is 6 meters away from Alice, but no matter which part of the ring of magical fairy dust he goes to, he must always travel 10 meters to collect fairy dust and deliver it to Alice. What is the area of the region enclosed by the fairy dust?

3. Consider an ellipse ω with foci P,Q and some arbitrary point X. If O is the midpoint of PQ, and Y is the point on ω that minimizes $\overline{XY}$, for which X are X,Y,O collinear?

4. Consider a parabola with directrix $x=y$ and focus $(-1,1)$. Let $O=(0,0)$ and let A,B be points on the parabola such that $\triangle OAB$ is equilateral. Find the slope of AB.

5. Consider an ellipse with equation $\frac{x^2}{a^2}+y^2=1$, with $a\geq 1$. If a latus rectum of the ellipse intersects the ellipse at P,Q, and $O=(0,0)$, what is the value of a that makes $\triangle OPQ$ an equilateral triangle?

6. Consider a hyperbola with equation $\frac{x^2}{a^2}-y^2=1$, with $a\geq 1$. If a latus rectum of the hyperbola intersects the hyperbola at P,Q, and $O=(0,0)$, what is the value of a that makes $\triangle OPQ$ an equilateral triangle?

7. Consider a parabola with equation $y=ax^2$, with $a>0$. If a latus rectum of the parabola intersects the parabola at P,Q, and $O=(0,0)$, what is the value of a that such that $[OPQ]=18$?

8. Prove that the graph of $y^2=x^2+axy$ for some constant a is always two perpendicular lines passing through the origin.

9. Consider conic $ax^2 + bxy + cy^2 + dx + ey + f = 0.$ If $a > 0$ and $c < 0,$ prove that this conic is always a hyperbola.

1. Consider an ellipse with major radius 10 and minor radius $5.$ Inscribe a quadrilateral $ABCD$ inside this ellipse. What is the maximum area of $ABCD$?

Solution: As an ellipse can be achieved after stretching a circle, we consider the maximal area of a quadrilateral inscribed within a circle of radius $5,$ and multiply by 2 afterwards.

We claim that the maximal area is achieved when $ABCD$ is a square.

Let the points on the circle be in the order $A, B, C, D.$ Then

$$[ABCD] =$$
$$[AOB] + [BOC] + [COD] + [DOA] =$$
$$\tfrac{1}{2}r^2\sin(\theta_1) + \tfrac{1}{2}r^2\sin(\theta_2) + \tfrac{1}{2}r^2\sin(\theta_3) + \tfrac{1}{2}r^2\sin(\theta_4) =$$
$$\tfrac{1}{2}\cdot 5^2(\sin(\theta_1) + \sin(\theta_2) + \sin(\theta_3) + \sin(\theta_4)).$$

Individually, $\sin(\theta_1), \sin(\theta_2), \sin(\theta_3), \sin(\theta_4)$ are maximized when $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 90^\circ.$ Conveniently, this is possible. Since this describes a square, a square is the maximum. Thus, the maximum area is $5^2 \cdot 2 = 50.$ Multiplying by 2 to account for the stretch of the ellipse, our answer is $100.$

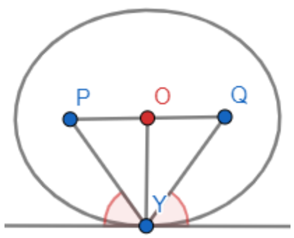
2. Two lovers Bob and Alice are surrounded by a ring of magical fairy dust. As part of his marriage proposal, Bob wants to deliver some magical fairy dust to Alice. He notices that he is 6 meters away from Alice, but no matter which part of the ring of magical fairy dust he goes to, he must always travel 10 meters to collect fairy dust and deliver it to Alice. What is the area of the region enclosed by the fairy dust?

Solution: Let P be any point on the ring, and let B, A be the positions of Bob and Alice. We notice that $\overline{BA} = 6$ and $\overline{BP} + \overline{AP} = 10$ for all $P.$ Notice that this is the definition of an ellipse. Then the major axis is $5,$ and the minor axis is $\sqrt{5^2 - (\frac{6}{2})^2} = \sqrt{16} = 4.$ Thus, the area of the ellipse is $5 \cdot 4 = 20.$

3. Consider an ellipse ω with foci P, Q and some arbitrary point $X.$ If O is the midpoint of $PQ,$ and Y is the point on ω that minimizes $\overline{XY},$ for which X are X, Y, O collinear?

Solution: This implies that the circle with center X containing point Y is tangent to ellipse ω. This also implies that OY must be perpendicular to the common internal tangent. We can "work backwards" once we've gotten this intuition; we want to find all possible Y and look for X from there.

We want the tangent to ω through Y to be perpendicular. Let the acute angle formed by AY and the tangent line be α and let the acute angle formed by BY and the tangent line be β. By Theorem 13.3, $\alpha = \beta$. This implies that if we want two angles formed by OY and the tangent to be equal, $\alpha + \angle AYO = \beta + \angle BYO \rightarrow \angle AYO = \angle BYO$. This means either $\overline{AY} = \overline{BY}$ (implying Y lies on the minor axis) or A, B, Y are collinear (implying Y lies on the major axis). Thus, X must lie on an axis of the ellipse or the extension of an axis.



4. Consider a parabola with directrix $x = y$ and focus $(-1, 1)$. Let $O = (0, 0)$ and let A, B be points on the parabola such that $\triangle OAB$ is equilateral. Find the slope of AB.

Solution: Let the focus be P and the directrix be l. Notice that $OP \perp l$ and $AB \perp OP$, so $l \parallel AB$. Thus, the slope of AB is 1.

5. Consider an ellipse with equation $\frac{x^2}{a^2} + y^2 = 1$, with $a \geq 1$. If a latus rectum of the ellipse intersects the ellipse at P, Q, and $O = (0, 0)$, what is the value of a that makes $\triangle OPQ$ an equilateral triangle?

Solution: Without loss of generality, let the focus that the latus rectum passes through be $(\sqrt{a^2 - 1}, 0)$. The latus rectum intersects the ellipse at $(\sqrt{a^2 - 1}, \frac{-1}{a})$ and $(\sqrt{a^2 - 1}, \frac{1}{a})$. Notice that the length of the latus rectum is $\frac{2}{a}$ (see Theorem 13.6), and by the Distance Formula (11.3), $\overline{OP} = \sqrt{a^2 - 1 + \frac{1}{a^2}}$. We desire $\sqrt{a^2 - 1 + \frac{1}{a^2}} = \frac{2}{a}$. Squaring gives $a^2 - 1 + \frac{1}{a^2} = \frac{4}{a^2} \rightarrow a^2 - 1 - \frac{3}{a^2} = 0 \rightarrow a^4 - a^2 - 3 = 0$. By the Quadratic Formula, $a^2 = \frac{1+\sqrt{13}}{2}$ (since a is real, a^2 must be positive). Then $a = \sqrt{\frac{1+\sqrt{13}}{2}}$, which is our answer.

6. Consider a hyperbola with equation $\frac{x^2}{a^2} - y^2 = 1,$ with $a \geq 1.$ If a latus rectum of the hyperbola intersects the hyperbola at $P, Q,$ and $O = (0, 0),$ what is the value of a that makes $\triangle OPQ$ an equilateral triangle?

Solution: Let the focus that the latus rectum passes through be $(\sqrt{a^2+1}, 0).$ The latus rectum intersects the hyperbola at $(\sqrt{a^2+1}, \frac{-1}{a})$ and $(\sqrt{a^2+1}, \frac{1}{a}).$ Notice the length of the latus rectum is $\frac{2}{a}$ (see Theorem 15.3), and by the Distance Formula (11.3), $\overline{OP} = \sqrt{a^2+1+\frac{1}{a^2}}.$ We desire $\sqrt{a^2+1+\frac{1}{a^2}} = \frac{2}{a}.$ Squaring gives $a^2+1+\frac{1}{a^2} = \frac{4}{a^2} \rightarrow a^2+1-\frac{3}{a^2} = 0 \rightarrow a^4 - a^2 - 3 = 0.$ By the Pythagorean Theorem, $a^2 = \frac{-1+\sqrt{13}}{2}$ (notice a^2 must be positive). Then $a = \sqrt{\frac{-1+\sqrt{13}}{2}}.$

7. Consider a parabola with equation $y = ax^2,$ with $a > 0.$ If a latus rectum of the parabola intersects the parabola at $P, Q,$ and $O = (0, 0),$ what is the value of a that such that $[OPQ] = 18?$

Solution: By Theorem 14.1, the equation can be rewritten as $y = \frac{1}{4a'}x^2.$ The vertex is $(0, 0),$ the focus is $(0, a'),$ and the directrix is $y = -a'.$ We only care about the focus right now; the focus passes through the parabola at $(\pm 2a', a').$ By Theorem 14.2, the length of the latus rectum is $4a'.$ By $\frac{bh}{2}$ (5.2), $[OPQ] = 2a'^2 = 18.$ Thus, $a' = 3.$ Since $a = \frac{1}{4a'} = \frac{1}{12},$ which is our answer.

8. Prove that the graph of $y^2 = x^2 + axy$ for some constant a is always two perpendicular lines passing through the origin.

Solution: It is obvious that $(0, 0)$ is part of this "conic." Then notice that the xy term just rotates the conic, so proving the rest for $a = 0$ suffices. When $a = 0,$ the result is obvious, so we are done.

9. Consider conic $ax^2 + bxy + cy^2 + dx + ey + f = 0.$ If $a > 0$ and $c < 0,$ prove that this conic is always a hyperbola.

Solution: This is equivalent to proving $b^2 - 4ac > 0.$ Notice $b^2 - 4ac \geq -4ac > 0$ since $|a|, |c| > 0$ and a, c have opposite signs.

The Complex Plane

With a Cartesian system in the reals, the point (a, b) represented reals $a, b.$ For complex $z = a + bi,$ we plot z on the complex plane as $(a, b),$ where the x axis represents reals and the y axis represents imaginaries. Thus, we can denote the real and imaginary counterparts of z separately. Let $\Re(z)$ denote the real counterpart of $z,$ and $\Im(z)$ denote the imaginary counterpart of $z.$ It is important to note that $\Im(z)$ is real, as it doesn't include the coefficient i of $a + bi.$ As thus, the point z can be universally expressed as $z = (\Re(z), \Im(z)).$ $\Re(z)$ and $\Im(z)$ may also be denoted as $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ in other texts. (However, we prefer using $a + bi,$ and have defined $\Re(z)$ and $\Im(z)$ just in case we need it.)

In the previous section, we defined magnitude as the distance from $0,$ which can also be expressed as the origin. However, the uses of magnitude mostly shine when studying complex numbers. We take $z = a + bi$ and plot $(a, b),$ and by the distance formula, its distance from the origin, or $(0, 0),$ is $\sqrt{a^2 + b^2}.$ We denote this as $|z| = \sqrt{[\Re(z)]^2 + [\Im(z)]^2}$ or $|a + bi| = \sqrt{a^2 + b^2},$ where $|z|$ denotes the magnitude of $z.$

We then define polar coordinates. Note that $(\frac{a}{\sqrt{a^2+b^2}})^2 + (\frac{b}{\sqrt{a^2+b^2}})^2 = 1.$ This means that we can express $\frac{a}{|z|}$ and $\frac{b}{|z|}$ as $\cos(\theta)$ and $\sin(\theta),$ respectively, because $\sin^2(\theta) + \cos^2(\theta) = 1.$ Let the magnitude of z be $r,$ denoted as $|z| = r.$ In that case, $z = r(\frac{a}{r}, \frac{b}{r}),$ in the Cartesian complex plane. We note that $(\frac{a}{r}, \frac{b}{r})$ lies on the unit circle, and that for some θ we can represent $(\frac{a}{r}, \frac{b}{r})$ as $(\cos(\theta), \sin(\theta)),$ which is defined as the point achieved by rotating $(1, 0)$ around $(0, 0).$ Then we scale it up and note that (a, b) lies on the circle $x^2 + y^2 = \sqrt{a^2 + b^2},$ and that furthermore (a, b) can be represented as $(a\cos(\theta), b\sin(\theta)).$ Each value of θ represents a unique point for set $(a, b),$ which means that we only need the magnitude and the angle to find an angle. Thus, we define polar coordinates (r, θ) as rotating $(r, 0)$ around $(0, 0)$ by θ degrees or radians, depending on context (though we will mostly be using radians at this point). We call r the magnitude and we call θ the argument. Thus, we can represent $z = a + bi$ as $(r, \theta),$ or $z = r\cos(\theta) + ir\sin(\theta),$ for some $\theta.$

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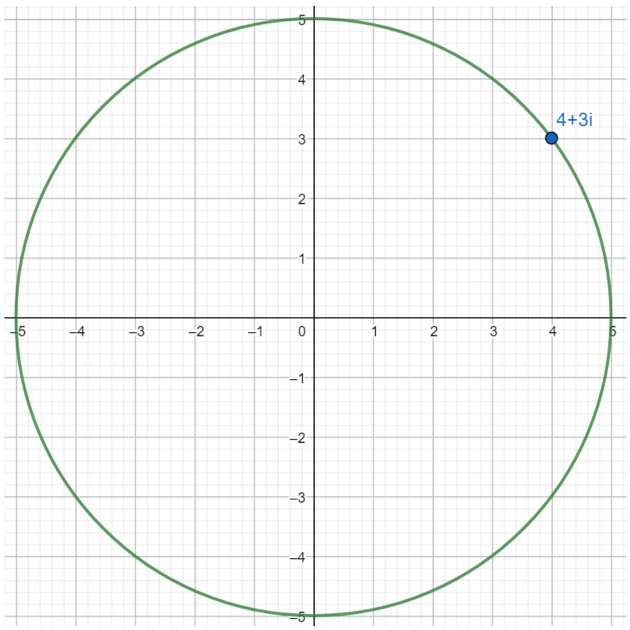
1. Graph $3i + 4$ and graph the circle centered at the origin that contains it.

2. Find the magnitude and argument of $\frac{\sqrt{3}}{3} + i$.

1. Graph $3i + 4$ and graph the circle centered at the origin that contains it.

Solution: The real numbers lie on the x axis and the imaginaries on the y axis. Thus, our point is $(4, 3)$. Then note that the magnitude of $3i + 4$ is 5, implying that the radius of our circle is 5.

We end up with the following.



2. Find the magnitude and argument of $\frac{\sqrt{3}}{3} + i$.

Solution: Note that $|\frac{\sqrt{3}}{3} + i| = \sqrt{(\frac{\sqrt{3}}{3})^2 + (1)^2} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$. Then we note that $\frac{2\sqrt{3}}{3}\cos(\theta) + i\frac{2\sqrt{3}}{3}\sin(\theta) = \frac{\sqrt{3}}{3} + i$. Coefficient matching yields $\cos(\theta) = \frac{1}{2}$ and $\sin(\theta) = \frac{\sqrt{3}}{2}$. Then we note that all θ that satisfy this are in the form $\theta = \frac{\pi}{6} + 2\pi x$ for any integer x, which means that it is technically correct to say that the argument of θ is $\frac{\pi}{6} + 2\pi x$. However, we generally want to give a value of the argument from 0 to 2π.

We note that we can express all z as $r(\cos(\theta) + i\sin(\theta))$. While adding complex numbers and subtracting them is easy without changing z into polar form, multiplication and division is not so easy when we introduce more terms. This means that we need a good way to multiply and divide complex numbers (which can be covered with multiplication alone), and polar coordinates seems the right way to do this.

First, let us define the complex conjugate of z, denoted $\overline{z}$, as $\Re(z) - i\Im(z)$. Noting that $z = \Re(z) + i\Im(z)$, we note that for $z = a + bi$, $\overline{z} = a - bi$. We have a few identities revolving complex conjugates; let us prove one of them now.

$|z|^2 = z \cdot \overline{z}$ *(17.1)*

Given complex z, the square of the magnitude of z is the product of z and its conjugate $\overline{z}$.

Theorem 17.1's Proof

Letting $z = a + bi$, we note $\overline{z} = a - bi$. Then we note that $(a + bi)(a - bi) = a^2 + abi - abi - (bi)^2 = a^2 + b^2$. By the definition of magnitude, $|z| = \sqrt{a^2 + b^2}$, and thus, $|z|^2 = a^2 + b^2$. By the transitive property, $|z|^2 = z \cdot \overline{z}$, as desired.

This method of proof is all and well, but for larger amounts of multiplication, such as exponentiation, we would rather use a quicker method. First, note that $|z| = r$, when z can be expressed in polar coordinates as (r, θ). In polar coordinates, $|z|$ is expressed as $(r, 0)$. Then we note that expressing z as (r, θ) implies $z = r(\cos(\theta) + i\sin(\theta))$. We want to express $\overline{z} = r(\cos(\theta) - i\sin(\theta))$ in polar coordinates. Noting that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we can express $\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$. Multiplying $z \cdot \overline{z} = r^2(\cos(\theta) + i\sin(\theta))(\cos(-\theta) + i\sin(-\theta))$ yields $z \cdot \overline{z} = r^2(\cos(\theta)\cos(-\theta) - \sin(\theta)\sin(-\theta) + i[\sin(\theta)\cos(-\theta) + \cos(\theta)\sin(-\theta)])$. By the Sum/Difference Identities (10.1), $\cos(\theta)\cos(-\theta) - \sin(\theta)\sin(-\theta) + i[\sin(\theta)\cos(-\theta) + \cos(\theta)\sin(-\theta)]$ is equivalent to $\cos(\theta + (-\theta)) + i\sin(\theta + (-\theta)) = 1$. Substituting yields $z \cdot \overline{z} = r^2 = |z|^2$, as desired.

We can generalize. If we desire to multiply two complex numbers together, we can express them in polar coordinates. This makes finding the product of multiple complex numbers easy, as we can repeat the process more.

Multiplication of Complex Numbers with Polar Coordinates (17.2)

Given $z_1, z_2 ... z_n$ expressed as $(r_1, \theta_1), (r_2, \theta_2) ... (r_n, \theta_n)$ in polar coordinates, the product $z_1 z_2 ... z_n$ can be expressed as $(r_1 r_2 ... r_n, \theta_1 + \theta_2 + ... + \theta_n)$ in polar coordinates.

Theorem 17.2's Proof

To prove this, we induct. First, we prove that this holds true for $n = 2.$ Note that $z_1 z_2 = r_1 r_2 (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)),$ which can be expanded into $r_1 r_2 (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i[\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)]).$ By the Sum/Difference Identities (9.1), this is equivalent to $r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)),$ which in polar coordinates is $(r_1 r_2, \theta_1 + \theta_2),$ as desired.

Then, we assume this is true for n and prove it for $n + 1.$ Note that this means $z_1 z_2 ... z_{n+1} = r_1 ... r_n r_{n+1} (\cos(\theta_1 + ... + \theta_n) + i\sin(\theta_1 + ... + \theta_n))(\cos(\theta_{n+1})\sin(\theta_{n+1})).$ Note that this is the case for two terms, so our proof is done.

The key idea is that the magnitudes multiply and the angles add. With this geometric representation of complex numbers, we can do a lot of things. We can easily find $z^n,$ or look for the roots of unity. (We will define the roots of unity later.) Speaking of $z^n,$ we will prove De Moivre's Theorem, which gives us z^n in a derivative of polar form.

De Moivre's Theorem (17.3)

Given $z = r(\cos(\theta) + i\sin(\theta)),$ $z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$

Theorem 17.3's Proof

This really is just a direct result of Theorem 17.2; we note that z in polar coordinates is $(r, \theta),$ and that z^n can be expressed as $(r^n, n\theta).$ Thus, we express this as $z^n = r^n(\cos(n\theta) + i\sin(n\theta)),$ as desired.

De Moivre's (17.3) can be expressed in polar coordinates as $(r, \theta)^n = (r^n, n\theta).$ This works because this is how we have defined polar coordinates, and this is one of the properties that result from it.

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1. Consider $\triangle XA_0A_1$ with $\overline{XA_0} = 3,$ $\overline{A_0A_1} = 4,$ and $\overline{XA_1} = 5.$ Then for all $n,$ construct $\triangle XA_nA_{n+1}$ by making it *directly similar* to $\triangle XA_0A_1.$ (Directly similar implies that they have the same orientation). Prove that for no pair of distinct integers $p, q,$ X, A_p, A_q are collinear.

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1. Consider $\triangle XA_0A_1$ with $\overline{XA_0} = 3,$ $\overline{A_0A_1} = 4,$ and $\overline{XA_1} = 5.$ Then for all $n,$ construct $\triangle XA_nA_{n+1}$ by making it *directly similar* to $\triangle XA_0A_1.$ (Directly similar implies that they have the same orientation). Prove that for no pair of distinct integers $p, q,$ X, A_p, A_q are collinear.

Let X be the origin and let $A_0 = 3$ in the complex plane. Then $A_n = 3 \cdot (\frac{5}{3})^n \cdot (\cos(n\theta) + i\sin(n\theta)),$ where $\theta = \sin^{-1}(\frac{4}{5}).$ Notice that for p, q to be collinear, $\cos(p\theta) + i\sin(p\theta) = \cos(q\theta) + i\sin(p\theta).$ But this is impossible, as there is no rational number k such that $k\theta = 2\pi.$

The nth roots of unity are the n numbers ω_i such that $\omega_i^n = 1$ for all $0 \le i < n$. Many interesting things can be done with the roots of unity, including but not limited to linking them with combinatorics and with number theory, particularly with the Binomial Theorem and divisibility. An important thing to remember is that the nth roots of unity are expressible as $(1, \theta)^n = (1, n\theta)$. Thus, the roots of unity are of the form $(1, k\frac{2\pi}{n})$ for some k. Thus, we can define the nth roots of unity as $\omega, \omega^2 ... \omega^n$ where $\omega = (1, \frac{2\pi}{n})$, $\omega^2 = (1, 2\frac{2\pi}{n})$, and so on. (There is no point to defining ω^{n+1} as ω is equivalent to it.) We will prove an important fact and an interesting one; we will be proving the sum and the product of the roots of unity.

Sum of Roots of Unity (17.4)

Let $\omega, \omega^2 ... 1$ be the nth roots of unity. Then, $\omega + \omega^2 + ... + \omega^{n-1} + 1 = 0$.

Theorem 17.4's Proof

Note that this is equivalent to $\frac{\omega^n - 1}{\omega - 1}$. Since $\omega^n = 1$, this value is 0.

There is also a geometrical argument for this using the fact that the roots of unity are equally spaced on the unit circle.

Product of Roots of Unity (17.5)

Let $\omega, \omega^2 ... 1$ be the nth roots of unity. Then $\omega \cdot \omega^2 \cdot ... \cdot \omega^{n-1} \cdot 1 = (-1)^{n+1}$.

Theorem 17.5's Proof

Expressing this in polar coordinates gives us $(1, \frac{2\pi}{n}) \cdot (1, 2\frac{2\pi}{n}) \cdot ... \cdot (1, n\frac{2\pi}{n})$. By Theorem 17.2, this is equivalent to $(1, (1 + 2 + ... + n)\frac{2\pi}{n}) = (1, (n+1)\pi)$. This is equivalent to $\cos((n+1)\pi) + i\sin((n+1)\pi) = \cos((n+1)\pi)$. We note that if $2|n+1$, then $\cos((n+1)\pi) = 1$, and otherwise, $\cos((n+1)\pi) = -1$. Thus, $\omega \cdot \omega^2 \cdot ... \cdot \omega^{n-1} \cdot 1 = (-1)^{n+1}$, as desired.

A lot of interesting things can be done with this. For one, we can use the roots of unity filter to evaluate problems such as $\binom{14}{1} + \binom{14}{4} + \binom{14}{7} + \binom{14}{10} + \binom{14}{13}$. Otherwise, we can multiply roots by it to find real or even integer polynomials with a root $\frac{5}{2} + i\frac{\sqrt{3}}{2}$, or to prove that a polynomial of some form is divisible by another, among other crazy things.

Let's use these problems to take a look at some of the things roots of unity can do. Since most of these problems will be harder, multiple of a type will be provided. This means that in each category, you can choose a problem to use as an example to provide the motivation to solve the other ones.

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1. Find $\sin(\frac{\pi}{3}) + \sin(\frac{2\pi}{3}) + \sin(\pi) + \cos(\pi) + \cos(\frac{2\pi}{3}) + \cos(\frac{\pi}{3})$.

2. Simplify $\sin(\frac{2\pi}{n}) + \sin(2\frac{2\pi}{n}) + ... + \sin(n\frac{2\pi}{n})$.

3. Create a polynomial with integer coefficients that has a root of $3 + \sqrt[4]{2}$.

4. Create a polynomial with integer coefficients with a root of $\sqrt{3} + \sqrt[4]{2}$.

5. Prove that $31|5^{31} + 5^{17} + 1$.

6. Prove that $x^2 + x + 1|x^{31} + x^{17} + 1$.

7. Prove that for positive integers $a, b, c,$ $x^2 + x + 1|x^{3a} + x^{3b+1} + x^{3c+2}$.

8. Find $\binom{12}{0} + \binom{12}{3} + \binom{12}{6} + \binom{12}{9} + \binom{12}{12}$.

9. Find $\binom{12}{1} + \binom{12}{4} + \binom{12}{7} + \binom{12}{10}$.

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1. Find $\sin(\frac{\pi}{3}) + \sin(\frac{2\pi}{3}) + \sin(\pi) + \cos(\pi) + \cos(\frac{2\pi}{3}) + \cos(\frac{\pi}{3})$.

Solution: Note that this is equivalent to $\sin(\frac{\pi}{3}) + \sin(\frac{2\pi}{3}) + \sin(\pi) + \sin(\frac{4\pi}{3}) + \sin(\frac{5\pi}{3}) + \sin(2\pi)$. We recognize this as the imaginary counterpart of the sum of the $6th$ roots of unity. By Theorem 17.4, this sum is 0, implying its imaginary part is 0, or that $\sin(\frac{\pi}{3}) + \sin(\frac{2\pi}{3}) + \sin(\pi) + \sin(\frac{4\pi}{3}) + \sin(\frac{5\pi}{3}) + \sin(2\pi) = 0$.

2. Simplify $\sin(\frac{2\pi}{n}) + \sin(2\frac{2\pi}{n}) + ... + \sin(n\frac{2\pi}{n})$.

Solution: Similarly to before, we coefficient match with the imaginary counterpart of the sum of the nth roots of unity, which yields an answer of 0.

3. Create a polynomial with integer coefficients that has a root of $3 + \sqrt[4]{2}$.

Solution: This implies that our polynomial is divisible by $x - (3 + \sqrt[4]{2})$. While this doesn't seem like a useful fact at first, this implies that we can multiply $(x - [3 + \sqrt[4]{2}])$ by anything and our polynomial still has a root of $3 + \sqrt[4]{2}$. Remembering that the product of the fourth roots of unity is -1 and that their sum is 0, we try multiplying $(x - [3 + \sqrt[4]{2}])(x - [3 + \omega\sqrt[4]{2}])(x - [3 + \omega^2\sqrt[4]{2}])(x - [3 + \omega^3\sqrt[4]{2}])$, where ω is one of the fourth roots of unity that isn't equal to 1.

We then note the nice thing that happens is that our irrational terms will cancel themselves out. Expanding, we see that this polynomial has an x^4 coefficient of 1 (which should be immediately obvious), an x^3 coefficient of $-(3 + \sqrt[4]{2}) - (3 + \omega\sqrt[4]{2}) - (3 + \omega^2\sqrt[4]{2}) - (3 + \omega^3\sqrt[4]{2})$, an x^2 coefficient of $(3 + \sqrt[4]{2})(3 + \omega\sqrt[4]{2}) + (3 + \sqrt[4]{2})(3 + \omega^2\sqrt[4]{2}) + (3 + \sqrt[4]{2})(3 + \omega^3\sqrt[4]{2}) + (3 + \omega\sqrt[4]{2})(3 + \omega^2\sqrt[4]{2}) + (3 + \omega\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2}) + (3 + \omega^2\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2})$, an x coefficient of $-(3 + \sqrt[4]{2})(3 + \omega\sqrt[4]{2})(3 + \omega^2\sqrt[4]{2}) - (3 + \sqrt[4]{2})(3 + \omega\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2}) - -(3 + \sqrt[4]{2})(3 + \omega^2\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2}) - (3 + \omega\sqrt[4]{2})(3 + \omega^2\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2})$, and a constant term of $(3 + \sqrt[4]{2})(3 + \omega\sqrt[4]{2})(3 + \omega^2\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2})$.

We note that by Theorem 17.4, the coefficient of x^3 reduces to -12.

Expanding and regrouping the coefficient of x^2 yields our coefficient as $9 \cdot 6 + 3 \cdot (3\sqrt[4]{2} + 3\omega\sqrt[4]{2} + 3\omega^2\sqrt[4]{2} + 3\omega^3\sqrt[4]{2}) + \sqrt{2}(1 + 2\omega + \omega^2 + 2\omega^3)$. Applying Theorem 17.4 again yields our coefficient of x^2 as $54 + \sqrt{2}(\omega + \omega^3)$, which polar analysis simplifies to 54.

We expand our x coefficient as $-27 \cdot 4 + 9 \cdot (3\sqrt[4]{2} + 3\omega\sqrt[4]{2} + 3\omega^2\sqrt[4]{2} + 3\omega^3\sqrt[4]{2}) - 3(2\sqrt{2} + 4\omega\sqrt{2} + 2\omega^2\sqrt{2} + 4\omega^3\sqrt{2})$. Applying Theorem 17.4 yields $-108 - 3(2\omega\sqrt{2} + 2\omega^3\sqrt{2}) = -108$.

$(3 + \sqrt[4]{2})(3 + \omega\sqrt[4]{2})(3 + \omega^2\sqrt[4]{2})(3 + \omega^3\sqrt[4]{2})$.
Our constant is expanded as
$81 + 27(\sqrt[4]{2}[1 + \omega + \omega^2 + \omega^3]) + 9(\sqrt{2}[1 + 2\omega + \omega^2 + 2\omega^3]) + 3[\sqrt[4]{2}(1 + \omega + \omega^2 + \omega^3)] + 2\omega^2$.
Using Theorem 17.4 on this for the final time, we see that our constant is $81 + 2\omega^2 = 81 - 2 = 79$.

Combining these results, our polynomial is $x^4 - 12x^3 + 54x^2 - 108x + 79$.

(This problem is completely trivialized by $(x - 3)^2 - 2$. However, this will be informative for the next problem, which cannot be "cheesed" this way.)

4. Create a polynomial with integer coefficients with a root of $\sqrt{3} + \sqrt[4]{2}$.

Solution: Multiplying the $\sqrt[4]{2}$ by the fourth roots of unity won't get rid of the $\sqrt{3}$. However, we can multiply the second roots of unity by $\sqrt{3}$ as well to fix this problem. This gives us a long list of roots; our eight roots are in the form $(\pm\sqrt{3} + \omega_n\sqrt[4]{2})$, which means our roots are $\pm(\sqrt{3} + \sqrt[4]{2}), \pm(\sqrt{3} + \omega\sqrt[4]{2}), \pm(\sqrt{3} + \omega^2\sqrt[4]{2}), \pm(\sqrt{3} + \omega^3\sqrt[4]{2})$. Multiplying them together (which we won't show here) yields an answer of $x^8 - 12x^6 + 50x^4 - 180x^2 + 49$.

5. Prove that $31 | 5^{31} + 5^{17} + 1$.

Solution: We can use Fermat's Little Theorem and note that $5^{31} \equiv 5 \ (mod\ 31)$, $5^{17} \equiv 25 \ (mod\ 31)$, and $5^{31} + 5^{17} + 1 \equiv 0 \ (mod\ 31)$. The solution to the problem below, however, will show a more ingenious proof.

6. Prove that $x^2+x+1 | x^{31}+x^{17}+1$.

Solution: Note that by Theorem 17.4, ω, ω^2 are both roots of $x^{31}+x^{17}+1$. Thus, $(x-\omega)(x-\omega^2) = x^2+x+1 | x^{31}+x^{17}+1$.

7. Prove that for positive integers a, b, c, $x^2+x+1 | x^{3a}+x^{3b+1}+x^{3c+2}$.

Solution: Again, ω, ω^2 are roots of $x^{3a}+x^{3b+1}+x^{3c+2}$. Thus, $(x-\omega)(x-\omega^2) = x^2+x+1 | x^{3a}+x^{3b+1}+x^{3c+2}$.

8. Find $\binom{12}{0}+\binom{12}{3}+\binom{12}{6}+\binom{12}{9}+\binom{12}{12}$.

Solution: Notice that binomial coefficients often go with powers of binomials (hence the name). We want to evaluate $\frac{(1+1)^{12}+(1+\omega)^{12}+(1+\omega^2)^{12}}{3}$, where $\frac{\omega^3-1}{\omega-1}=0$. The motivation for this is because the expansion looks like $\sum_{i=0}^{12}\binom{12}{i}1^i1^{12-i}+\sum_{i=0}^{12}\binom{12}{i}\omega^i1^{12-i}+\sum_{i=0}^{12}\binom{12}{i}\omega^{2i}1^{12-i}$, and except for $3|i$, all the terms cancel out by Theorem 17.4. Then notice $(1+1)^{12}=2^{12}$, $(1+\omega)^{12}=(\frac{1}{2}+\frac{\sqrt{3}}{2}i)^{12}=(1,30^\circ)^{12}=1$, and $(1+\omega^2)^{12}=(\frac{1}{2}-\frac{\sqrt{3}}{2}i)^{12}=(1,150^\circ)^{12}=1$, so $\frac{(1+1)^{12}+(1+\omega)^{12}+(1+\omega^2)^{12}}{3}=\frac{2^{12}+2}{3}=1366$.

9. Find $\binom{12}{1}+\binom{12}{4}+\binom{12}{7}+\binom{12}{10}$.

Solution: We look at our $(1+1)^{12}+(1+\omega)^{12}+(1+\omega^2)^{12}$ and look at how we can modify it. If $i \equiv 1 \ (mod\ 3)$, then the coefficients of $\binom{12}{i}$ are 1, ω, and ω^2. We want to turn the ω and ω^2 into 1. Then we notice we can find $\frac{(1+1)^{12}+\omega(1+\omega)^{12}+\omega^2(1+\omega^2)^{12}}{3}=\frac{4096+\omega+\omega^2}{3}=\frac{4095}{3}=1365$.

Now, we will start defining imaginary exponents and taking the sine of imaginary values. If you aren't familiar with the definition of e, or the trigonometric formulas, you will have trouble understanding this section. We first begin with Euler's Identity, perhaps most infamous for the $e^{i\pi} + 1 = 0$ equation. Though we will provide a proof for completeness, there is no need to fully understand it for the rest of this book, though understanding it will provide a deeper understanding of complex numbers. This proof does involve calculus, so those taking it may understand what is going on.

Euler's Formula (17.6)

For any complex θ, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Theorem 17.6's Proof

We look at the Taylor Series expansions for $\sin(x), \cos(x), e^z$. We have

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

And letting $z = i\theta$ yields

$e^{i\theta} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots) = \cos(\theta) + i\sin(\theta)$, as desired.

This result is very powerful. For instance, we can find $\cos(i\theta) + i\sin(i\theta)$. We would do well to note that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$. This means that we can express $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $i\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2}$. This becomes the basis for the hyperbolic representation of cosine and sine; we denote them as $\cosh$ and $\sinh$ respectively. Then, we let $\cosh(\theta) = \frac{e^{\theta}+e^{-\theta}}{2}$ and $\sinh(\theta) = \frac{e^{\theta}-e^{-\theta}}{2}$. Substituting $i\theta$ easily gives us $\cos(i\theta) = \cosh(\theta)$ and $\sin(i\theta) = i\sinh(\theta)$, providing us with a concrete definition of taking the cosine and sine of imaginary values. We can also use the Sum/Difference Formulas to find the cosine and sine of complex numbers as well.

We aren't done yet. We need to generalize for $x^{i\theta}$. Fortunately, we can express this as $e^{\ln(x)i\theta}$, where $\ln x = \log_e x$. This is because $x^{\log_x y} = y$ for all $x, y > 0$. Then we can substitute $\ln(x)\theta$ as the degree value, and we get $x^{i\theta} = \cos(\theta\ln(x)) + i\sin(\theta\ln(x))$. Since $x^a \cdot x^{bi} = x^{a+bi}$, we now have defined complex exponentiation as well.

A reason to use the $e^{i\theta}$ representation is for summations such as $\sum_{n=1}^{100} \cos(n\theta)$. If we let this be $\sum_{n=1}^{100} e^{in\theta}$ (and remember that we only want the real value in the end), we can let $e^{i\theta} = x$. Then by the geometric series formula, $\sum_{n=1}^{100} x^n = \frac{x^{100}-x}{x-1}$. This is especially nice if x is one of the $99th$ roots of unity (say, $x = e^{i\frac{2\pi}{33}}$ as an example), where the expression would become 0.

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1. Find the form all values of $\ln(\frac{e^3\sqrt{3}}{2} + i\frac{e^3}{2})$ can take.

2. Find $\Re(\log_2(2 + 2\sqrt{3}i))$.

3. Prove that $\cosh^2(\theta) - \sinh^2(\theta) = 1$.

4. Find $\sin(\frac{\pi}{2} + i\frac{2\pi}{3})$.

5. Find all possible values of i^i.

6. Find all possible values of $(-1)^i$.

7. Prove that $|z| + |w| \geq |z + w|$ with equality when z, w have equivalent arguments. (This means that their angle when expressed in polar form is identical, or that the line through z, w passes the origin.)

8. Prove that $|x||y| = |xy|$.

9. Consider some angle θ such that $\cos(\theta)$ is not 1, and let n be some integer such that $\cos(n\theta) = 1$. Prove that $\sum_{m=1}^{n} \cos(m\theta) = 0$.

10. Find $\sum_{n=1}^{8} n\cos(n\theta)$, where $\theta = \frac{\pi}{9}$.

11. Suppose for some angle $0 < \theta < \pi$, $\cos(\theta) = \frac{1}{7}$. Find $\sum_{n=1}^{\infty} \frac{n\cos(n\theta)}{2^n}$.

1. Find $\ln(\frac{e^3\sqrt{3}}{2} + i\frac{e^3}{2})$.

Solution: Note that this is equivalent to $\ln(e^3) + \ln(\frac{\sqrt{3}}{2} + i)$, by the logarithmic product rule. Then this becomes $3 + \ln(\frac{\sqrt{3}}{2} + i)$. We note that $e^{i(\frac{\pi}{6}+2\pi n)} = \frac{\sqrt{3}}{2} + i$ for all integer n. Thus, $\ln(\frac{e^3\sqrt{3}}{2} + i\frac{e^3}{2}) = 3 + i(\frac{\pi}{6} + 2\pi n)$.

2. Find $\Re(\log_2(2 + 2\sqrt{3}i))$.

Solution: Using the logarithmic product rule gives us $\log_2(2 + 2\sqrt{3}i)$ gives us $\log_2(2) + \log_2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 1 + \log_2(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$. Note that for $2^x = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, we need $e^{\ln(2)\cdot x} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Since $e^{i(\frac{\pi}{3}+2\pi n)} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\ln(2)\cdot x = i(\frac{\pi}{3} + 2\pi n)$, and since $\ln(2)$ is strictly real, this implies x is strictly imaginary, meaning $\Re(1 + \log_2(\frac{1}{2} + \frac{\sqrt{3}}{2}i)) = 1$.

This can be generalized to $\Re(\log_c(z)) = \log_c(|z|)$.

3. Prove that $\cosh^2(\theta) - \sinh^2(\theta) = 1$.

Solution: Putting it in terms of their definitions,
$\cosh^2(\theta) - \sinh^2(\theta) = (\frac{e^\theta+e^{-\theta}}{2})^2 - (\frac{e^\theta-e^{-\theta}}{2})^2 = 1$.

4. Find $\sin(\frac{\pi}{2} + i\frac{2\pi}{3})$.

Solution: By the Sum/Difference Identities (10.1), this is equivalent to $\sin(\frac{\pi}{2})\cos(i\frac{2\pi}{3}) + \cos(\frac{\pi}{2})\sin(i\frac{2\pi}{3})$. This is making a huge assumption, however; that this identity is still valid. We run through the calculations, noting that $\sin(a+bi) = \frac{e^{i(a+bi)}-e^{-i(a+bi)}}{2i} = \frac{e^{-b+ai}-e^{b-ai}}{2i} = \frac{e^{-b}(\cos(a)+i\sin(a))-e^{b}(\cos(a)-i\sin(a))}{2i} = \frac{(e^{-b}-e^{b})\cos(a)-i(e^{-b}+e^{b})\sin(a)}{2i}$.
We note that this looks suspiciously like $\sin(a)\cosh(b) + i\cos(a)\sinh(b)$, so our formula still works. (We can do the same thing for the other three identities.)

Plugging this in, we get $\sin(\frac{\pi}{2})\cosh(\frac{2\pi}{3}) + i\cos(\frac{\pi}{2})\sinh(\frac{2\pi}{3}) = \frac{\sqrt{2}}{2}(\cosh(\frac{2\pi}{3}) + i\sinh(\frac{2\pi}{3}))$. This becomes $\frac{\sqrt{2}}{2}(\frac{e^{\frac{2\pi}{3}}+e^{-\frac{2\pi}{3}}}{2} - \frac{e^{\frac{2\pi}{3}}-e^{-\frac{2\pi}{3}}}{2}) = \frac{e^{-\frac{2\pi}{3}}\sqrt{2}}{2}$.

5. Find all possible values of i^i.

Solution: Note $i = e^{i(\frac{\pi}{2}+2\pi n)}$ for all integer n. Thus $i^i = e^{i(\frac{\pi}{2}+2\pi n)^i} = e^{-(\frac{\pi}{2}+2\pi n)}$.

6. Find all possible values of $(-1)^i$.

Solution. Note $(-1) = i^2$, so we find $(i^i)^2 = e^{-(\pi+4\pi n)}$.

7. Prove that $|z| + |w| \geq |z + w|$ with equality when z, w have equivalent arguments. (This means that their angle when expressed in polar form is identical, or that the line through z, w passes the origin.)

Solution: Let $z = a + bi$ and $w = x + yi$. Substitution yields $\sqrt{a^2 + b^2} + \sqrt{x^2 + y^2} \geq \sqrt{(a+x)^2 + (b+y)^2}$. Squaring both sides yields $a^2 + b^2 + x^2 + y^2 + 2\sqrt{a^2 + b^2}\sqrt{x^2 + y^2} \geq a^2 + 2ax + x^2 + b^2 + 2by + y^2$, which simplifies into $2\sqrt{a^2 + b^2}\sqrt{x^2 + y^2} \geq 2ax + 2by$. Dividing through by 2 and squaring again yields $a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 \geq a^2x^2 + 2axby + b^2y^2$, which then simplifies into $a^2y^2 + b^2x^2 \geq 2axby$, which follows by AM-GM. Additionally, equality is only achieved when $ay = bx$, or when $\frac{a}{b} = \frac{x}{y}$. We note that the ratio of real to imaginary alone determines the argument of a complex number, so the equality condition is proved.

8. Prove that $|x||y| = |xy|$.

Solution: Expressed in polar form, this becomes $|x||y| = |(r_1, \theta_1)||(r_2, \theta_2)| = r_1r_2$. Remembering magnitudes multiply and angles add, $|xy| = |(r_1r_2, \theta_1\theta_2)| = r_1r_2$, and by the transitive property, $|x||y| = |xy|$, as desired.

9. Consider some angle θ such that $\cos(\theta)$ is not 1, and let n be some integer such that $\cos(n\theta) = 1$. Prove that $\sum_{m=1}^{n} \cos(m\theta) = 0$.

Solution: We are finding the real component of $\sum_{m=1}^{n} e^{im\theta}$. Let $e^{i\theta} = x$. Then we want to find $\sum_{m=1}^{n} x^m$. Since this is a finite geometric series with initial term x and final term x^n, it can be written as $\frac{x^{m+1}-x}{x-1}$. But note that $x^m = 1$ by definition, so $\sum_{m=1}^{n} x^m = \frac{x^{m+1}-x}{x-1} = \frac{x-x}{x-1} = 0$, as desired. (This also proves the same thing for sine.)

10. Find $\sum_{n=1}^{8} n\cos(n\theta)$, where $\theta = \frac{4\pi}{9}$.

Solution: Notice we are finding the real component of $\sum_{n=1}^{8} n \cdot e^{in\theta}$. Let $e^{i\theta} = x$. Then we want to find $\sum_{n=1}^{8}(\sum_{m=n}^{8} x^m)$. Since this is a finite geometric series with initial term x^n and final term x^8, the geometric series can be written as $\frac{x^9-x^n}{x-1}$, so $\sum_{n=1}^{8}(\sum_{m=n}^{8} x^m) = \sum_{n=1}^{8} \frac{x^9-x^n}{x-1}$.
Since $\sum_{n=1}^{8} x^n = \frac{x^9-x}{x-1}$, $\sum_{n=1}^{8} \frac{x^9-x^n}{x-1} = \frac{8x^9-\frac{x^9-x}{x-1}}{x-1} = \frac{8x^{10}-8x^9-x^9+x}{x-1} = \frac{8x^{10}-9x^9+x}{x-1}$. But since $x^9 = 1$,
$\frac{8x^{10}-9x^9+x}{x-1} = \frac{-9x^9+9x}{x-1} = \frac{9x-9}{x-1} = 9$.

11. Suppose for some angle $0 < \theta < \pi$, $\cos(\theta) = \frac{1}{7}$. Find $\sum_{n=1}^{\infty} \frac{n\cos(n\theta)}{2^n}$.

Solution: We first must prove this series converges. Fortunately, this is very easy. Notice that $\sum_{n=1}^{\infty} \frac{n\cos(n\theta)}{2^n} < \sum_{n=1}^{\infty} \frac{n}{2^n}$, which converges. We can do the same thing for $\sum_{n=1}^{\infty} \frac{n\sin(n\theta)}{2^n}$, which will be important as we want to ensure the imaginary component is bounded as well.

Then notice that $\sum_{n=1}^{\infty} \frac{n\cos(n\theta)}{2^n}$ is just the real component of $\sum_{n=1}^{\infty} \frac{n \cdot e^{in\theta}}{2^n} = \sum_{n=1}^{\infty}(\sum_{m=n}^{\infty} \frac{e^{im\theta}}{2^m})$. Notice that this is an infinite sum of infinite geometric series. The first term is $\frac{e^{in\theta}}{2^n}$, and the common ratio is $\frac{e^{i\theta}}{2}$. Letting $x = \frac{e^{i\theta}}{2}$, notice the sum of the infinite geometric series is $\frac{x^m}{1-x}$, so $\sum_{n=1}^{\infty}(\sum_{m=n}^{\infty} \frac{e^{im\theta}}{2^m}) = \sum_{n=1}^{\infty}(\sum_{m=n}^{\infty} x^m) = \sum_{n=1}^{\infty} \frac{x^n}{1-x}$. But $\sum_{n=1}^{\infty} x^n$ is an infinite geometric series with value $\frac{x}{1-x}$, so $\sum_{n=1}^{\infty} \frac{x^n}{1-x} = \frac{x}{(1-x)^2}$. Then we substitute in $x = \frac{e^{i\theta}}{2} = \frac{\frac{1}{7}+\frac{4\sqrt{3}}{7}i}{2} = \frac{1}{14} + \frac{4\sqrt{3}}{14}i$. So $\frac{x}{(1-x)^2} = (\frac{1}{14} + \frac{4\sqrt{3}}{14}i) \cdot (\frac{1}{\frac{13}{14}-\frac{4\sqrt{3}}{14}i})^2$. (We take out the square because multiplying the complex conjugate as soon as possible reduces calculation.) Multiplying by the complex conjugate yields

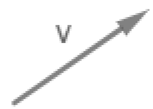
$(\frac{1}{14} + \frac{4\sqrt{3}}{14}i) \cdot (\frac{1}{\frac{13}{14} - \frac{4\sqrt{3}}{14}i})^2 = (\frac{1}{14} + \frac{2\sqrt{3}}{14}i) \cdot (\frac{\frac{13}{14} + \frac{4\sqrt{3}}{14}i}{(\frac{13}{14} - \frac{4\sqrt{3}}{14}i)(\frac{13}{14} + \frac{4\sqrt{3}}{14}i)})^2 = (\frac{1}{14} + \frac{4\sqrt{3}}{14}i) \cdot (\frac{\frac{13}{14} + \frac{4\sqrt{3}}{14}i}{\frac{169}{196} + \frac{48}{196}})^2$. Cleaning up the equation gives us

$(\frac{1}{14} + \frac{4\sqrt{3}}{14}i) \cdot (\frac{\frac{13+4\sqrt{3}i}{14}}{\frac{31}{28}})^2 = (\frac{1}{14} + \frac{4\sqrt{3}}{14}i) \cdot (\frac{13+4\sqrt{3}i}{\frac{31}{2}})^2 = (1 + 4\sqrt{3}i) \cdot \frac{2 \cdot 2}{31^2 \cdot 14} \cdot (13 + 4\sqrt{3}i)^2$. Evaluating this expression (which we will not show) yields $\frac{2}{31^2 \cdot 7}(-1127 + 588\sqrt{3}i)$. Since we only care about the real component, we need to evaluate $\frac{-1127 \cdot 2}{31^2 \cdot 7}$. In the end, this simplifies to $\frac{-322}{961}$.

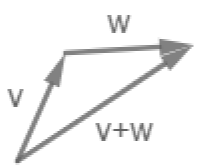
Vectors and Matrices

With the complex plane, we can now give points a "direction," so to speak. (We can consider their angle with the x axis, like we do with polar coordinates.) This gives us two important tools; vectors and matrices.

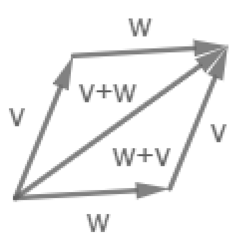
Vectors have direction and magnitude. We will denote the vector as $\vec{v}$, and its length as $||\vec{v}||$. This we already should understand, as we have studied polar coordinates in the last section. We also let the head of the vector as the end of a vector, and the tail of a vector be the start of it.



The important thing about vectors is that their starting point does not matter, and we can drag a vector wherever we want. This means that we can add $\vec{v}+\vec{w}$ together by dragging the tail of $\vec{w}$ to the head of $\vec{v}$. We then let their sum be the vector from the base of $\vec{v}$ to the head of the repositioned $\vec{w}$.

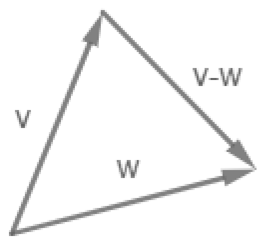


We'd like to see if vector addition holds some properties of numerical addition. We note that we create a parallelogram if we add it the other way, so we have $\vec{v}+\vec{w}=\vec{w}+\vec{v}$.



We quickly look for more things we can relate to arithmetic. We let $k\vec{v}$ be the vector with the direction of $\vec{v}$ and length $k||\vec{v}||$. This should make sense as multiplying should preserve magnitude, like it does for complex numbers. (We will show how vectors can be expressed in polar form soon.)

We note that with this definition, we get $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$. We see that by our definition of multiplying that $-\vec{w}$ has a length of $||\vec{w}||$ and the direction opposite $\vec{w}$; that is, $\vec{w} + (-\vec{w}) = 0$. We then analyse $\vec{v} - \vec{w}$. Let the angle between $\vec{v}$ and $\vec{w}$ be θ. We see that by the Law of Cosines (8.3), $||\vec{v} - \vec{w}|| = ||\vec{v}||||\vec{w}|| \cos(\theta)$. This means that the dot product, which will be expressed as $\vec{v} \cdot \vec{w}$, can be expressly stated as $\frac{||\vec{v}||^2 + ||\vec{w}||^2 - ||\vec{w} - \vec{v}||^2}{2}$.



Our definition of the dot product is commutative and distributive, and we will prove this and some other properties of the dot product. This will pop up a lot in analytic geometry; for one, it makes transformations easier to express. We will also use the dot product for matrices; more on that later.

Dot Product Commutative Property (18.1)
Given vectors $\vec{v}, \vec{w}$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

Theorem 18.1's Proof
We express $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos(\theta)$ and $\vec{w} \cdot \vec{v} = ||\vec{w}||||\vec{v}|| \cos(-\theta)$, and by the Odd/Even Functions, we know that $\cos(\theta) = \cos(-\theta)$, completing our proof.

Dot Product Multiplication Property (18.2)
For any real c, $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$.

Theorem 18.2's Proof
We note that stretching $\vec{v}$ won't change the angle, even for negative c. Then we let the angle be θ and note $(c\vec{v}) \cdot \vec{w} = c||\vec{v}||||\vec{w}|| \cos(\theta) = c(\vec{v} \cdot \vec{w})$ by the definition of vectors.

Perpendicular Vector Theorem (18.3)
Two vectors $\vec{v}, \vec{w}$ are perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$.

Theorem 18.3's Proof

We express $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos(\theta)$ and note that for the right hand side to equal zero, $\theta = 90^{\circ}$. There is no other value of θ in $0^{\circ} < \theta < 180^{\circ}$ so this condition is necessary and sufficient.

Before we prove the dot product is distributive, we'll first introduce the idea of coordinate representations of vectors. If we let the tail of the vector be the origin, then we can represent the head as a set of coordinates. We can use row vectors, which are denoted as $(x \quad y)$ in two dimensions, or column vectors, which can be represented as $\binom{x}{y}$ in two dimensions. Now this means we can represent dot products in terms of these coordinates. (If they look similar to matrices, that's because they are matrices!)

Dot Product In Terms of Coordinates (18.4)

Given $\vec{v_1} = (x_1 \quad y_1), \vec{v_2} = (x_2 \quad y_2),\ \vec{v_1} \cdot \vec{v_2} = x_1x_2 + y_1y_2.$

Theorem 18.4's Proof

Let the angle $\vec{v_1}$ forms with the positive x axis be θ_1 and the angle $\vec{v_2}$ forms be θ_2. Then $\vec{v_1} \cdot \vec{v_2} = ||\vec{v_1}||||\vec{v_2}|| \cos(\theta_1 - \theta_2)$, and by the Sum/Difference Identities (10.1), this is equivalent to $||\vec{v_1}||||\vec{v_2}||(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2))$. Rearranging yields $\vec{v_1} \cdot \vec{v_2} = (||\vec{v_1}|| \cos(\theta_1))(||\vec{v_2}|| \cos(\theta_2)) + (||\vec{v_1}|| \sin(\theta_1))(||\vec{v_2}|| \sin(\theta_2))$. Using polar coordinates and the definitions of θ_1, θ_2 gives us $\vec{v_1} \cdot \vec{v_2} = x_1x_2 + y_1y_2$, as desired.

Dot Product Distributive Property (18.5)

For vectors $\vec{u}, \vec{v}, \vec{w},\ \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}.$

Theorem 18.5's Proof

Let the head of $\vec{u}$ respective to its tail be (x_u, y_u), the head of $\vec{v}$ respective to its tail be (x_v, y_v), and the head of $\vec{w}$ respective to its tail be (x_w, y_w). Then by Theorem 18.4, $\vec{u} \cdot (\vec{v} + \vec{w}) = x_u(x_v + x_w) + y_u(y_v + y_w) = (x_ux_v + y_uy_v) + (x_ux_w + y_uy_w) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, as desired.

We can easily generalize these proofs to higher dimensions. Speaking of higher dimensions, let's talk matrices. To consider matrices, we have to consider transformations of vectors. We can easily rotate (x, y) by α degrees. Let (x, y) be (r, θ) in polar form. Then our translated coordinates are (x', y'), and we know that $x' = r\cos(\theta + \alpha)$ and $y' = r\sin(\theta + \alpha)$. Applying the Sum/Difference Formulas (10.1), we get $x' = r(\cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)) = x\cos(\alpha) - y\sin(\alpha)$ and

$y' = r(\sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha)) = x\cos(\alpha) + y\sin(\alpha)$. Since these functions are linear in x, y, they are determined by only four coefficients. We can denote this using matrices.

If the point prior to transformation is $\begin{pmatrix} x \\ y \end{pmatrix}$ and the point after transformation is $\begin{pmatrix} x' \\ y' \end{pmatrix}$, then this transformation can be expressed as $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Thus, matrices represent transformations of vectors. Similarly, we can generalize this process to larger matrices, such as 3×3 matrices.

Let's generalize and define multiplying a matrix with a vector. The process to find the first element of the product is simply multiplying the first element in the first row of the first matrix with the first element in the first row of the next matrix, and so on. For example, using this definition, we have $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xa_{11} + ya_{12} \\ xa_{21} + ya_{22} \end{pmatrix}$. Then we see that we might be able to multiply matrices with the product of a matrix and a vector, such as $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}]$. We want to find the matrix $\underline{P}$ such that $\underline{P}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}]$. Thus we define the product of matrices $\underline{A} \times \underline{B} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$. In fact, matrix multiplication is the same as matrix-vector multiplication; we restart the process for each new column of the second matrix. This means that matrix multiplication is associative; however, it is not commutative. (You can easily find a counterexample for this.) Let's take a look at a few matrix properties.

Matrices are Linear (19.1)

Given matrix $\underline{A}$ and vectors $\vec{v}, \vec{w}$, $\underline{A}(\vec{v} + \vec{w}) = \underline{A}\vec{v} + \underline{A}\vec{w}$ and $\underline{A}(c\vec{v}) = c\underline{A}\vec{v}$.

Theorem 19.1's Proof

Since the proof can easily be generalized, we prove it for vectors with two dimensions.

Let $\vec{v} = \begin{pmatrix} x_v \\ y_v \end{pmatrix}, \vec{w} = \begin{pmatrix} x_w \\ y_w \end{pmatrix}$. Then we have $A(\vec{v} + \vec{w}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_v + x_w \\ y_v + y_w \end{pmatrix} = \begin{pmatrix} a_{11}x_v + a_{12}y_v + a_{11}x_w + a_{12}y_w \\ a_{21}x_v + a_{22}y_v + a_{21}x_w + a_{22}y_w \end{pmatrix}$ and $\underline{A}\vec{v} + \underline{A}\vec{w} = \begin{pmatrix} a_{11}x_v + a_{12}y_v \\ a_{21}x_v + a_{22}y_v \end{pmatrix} + \begin{pmatrix} a_{11}x_w + a_{12}y_w \\ a_{21}x_w + a_{22}y_w \end{pmatrix} = \begin{pmatrix} a_{11}x_v + a_{12}y_v + a_{11}x_w + a_{12}y_w \\ a_{21}x_v + a_{22}y_v + a_{21}x_w + a_{22}y_w \end{pmatrix}$. We also have $\underline{A}(c\vec{v}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}c\begin{pmatrix} x \\ y \end{pmatrix} = c\underline{A}\vec{v}$, as desired.

We can factor things out of matrices, which will be useful as well. This is called scalar multiplication, when we multiply a number by a matrix. We just multiply all of the matrix's terms by our number when we conduct this multiplication.

Consider that given multiplication, exponentiation naturally is derived. The value $\underline{A}^n$ is just $\underline{A}$ multiplied by itself n times. (Though the commutative property doesn't hold, $\underline{A}$ is literally the same as itself so this works.) For example, $\underline{A}^3 = \underline{A} \times \underline{A} \times \underline{A}$.

When Matrix Multiplication Works (19.2)

For an $n \times m$ matrix $\underline{A}$ and a $j \times k$ matrix $\underline{B}$, the product $\underline{A} \times \underline{B}$ is only defined when $m = j$. Additionally, our new matrix will be an $n \times k$ matrix.

Theorem 19.2's Proof

This fact arises due to the fact that matrix multiplication is undefined when the dimension of the matrices are not the same; for example, multiplying a two-dimensional matrix by a three-dimensional one. (This isn't really a theorem, but it is an important fact, so make sure it makes sense to you!)

Let's generalize matrices even further. So far we only have defined $n \times 1$ and $n \times n$ matrices, so we will generalize and define $n \times m$ matrices. Furthermore, we will define addition and subtraction. Since matrices are generalizations of vectors, it only makes sense that for two matrices to be able to added together, they must have the same dimensions. If this condition is sufficed, we can just match the entries and add together, akin to vector addition.

As an example, $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1+2 & 1+1 \\ 1+1 & -1+2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$. This is similar for subtraction as well; we have $\underline{A} - \underline{B} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$ for the two-dimensional case. This leads us to another "theorem" which will define the possibilities of matrix addition and subtraction.

When Matrix Addition Works (19.3)

For an $n \times m$ matrix $\underline{A}$ and a $j \times k$ matrix $\underline{B}$, the sum $\underline{A} + \underline{B}$ is only defined when $n = j$ and $m = k$. Additionally, our new matrix will be an $n \times m$ matrix.

Theorem 19.3's Proof

Again, we cannot "add" vectors of a different dimension, so the same cannot be done with matrices either.

Now that we've become more acquainted with matrix multiplication and addition, this is a good time to introduce more efficient and rigorous notation, to help us explore deeper ideas. Since by Theorem 19.2 we must have the columns of the first matrix be the same as the rows of the second one, we just have $\underline{A}$ as $l \times m$ and $\underline{B}$ as $m \times n$. Then we take our messy idea and say that for any term p_{ij} as the term in the ith row and jth column of $l \times n$ matrix $\underline{P} = \underline{A} \times \underline{B}$, it is equal to $\sum_{x=1}^{m} a_{ik} b_{kj}$.

Matrix Multiplication, Formalized (19.4)

Given matrices $\underline{A} \times \underline{B} = \underline{P}$ with $\underline{P}$ being defined, we let p_{ij} be the term in $\underline{P}$ in the ith row and jth column. Then we have $p_{ij} = \sum_{x=1}^{m} a_{ik} b_{kj}$ where a_{ik} denotes the term in row i and column k of $\underline{A}$, and b_{kj} denotes the term in row k and column j of $\underline{B}$.

Theorem 19.4's Proof

We will shoddily describe the process of finding the p_{ij} term using our old definition. We just take the $1st$ term in the ith row of the first matrix and multiply it by the $1st$ term in the jth column of the second matrix, do this for the $2nd, 3rd...mth$ term of this series, and sum it all up. (Again, this just stems from our definition; this isn't really a "theorem" persay.)

We then find the last two questions to ask ourselves; what is the "size" of a matrix, and what is the matrix $\underline{I}$ such that $\underline{A} \times \underline{I} = \underline{A}$ for all $\underline{A}$ where the product is defined? (We also can have $\underline{I} \times \underline{A} = \underline{A}$, but we will have different dimensions for this $\underline{I}$ unless $\underline{A}$ is also an $n \times n$ matrix.) We'll prove that the identity matrices are just a diagonal of $1's$ with the rest of the items as 0.

The Identity Matrix (19.5)

The identity matrix can be written as $a_{ij} = \{^{1,\ i=j}_{0,\ i \neq j}$.

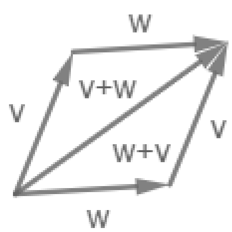
For example, the 2×2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 19.5's Proof

Note that $\underline{A} \times \underline{I} = \underline{A}$ implies that for $m \times n$ $\underline{A}$, we must have the same dimensions on the product $\underline{A} \times \underline{I}$ by the Lemma of Common Sense. This means that $\underline{I}$ must be $n \times n$ by Theorem 19.2. Then, by Theorem 19.4, we have $a_{ij} = \sum_{x=1}^{m} a_{ik} b_{kj}$. So for this to be equal to a_{ij} *across all* a_{ij} *and across all matrices*, we must have $b_{kj} = 1$ for $k = j$, and for all other k, $b_{kj} = 0$. This means that if a term have the same row and column value, then it is 1, else it is 0.

We then naturally ask ourselves; what is the value of $\underline{A}^{-1}$? Well, quite naturally, $\underline{A} \times \underline{A}^{-1} = \underline{A}^0$, and it makes sense for multiplying something by $\underline{A}^0$ to do nothing to it (if defined), meaning that $\underline{A}^0 = \underline{I}$. Thus we define the inverse of a matrix $\underline{A}$; that is, the matrix $\underline{A}^{-1}$ such that $\underline{A} \times \underline{A}^{-1} = \underline{I}$. The actual evaluation of this inverse is generally long, boring, and slow, but fortunately we will usually only use it for specific cases where it isn't too bad. To do this though, we define the determinant first. And before we define the determinant, we will need to know about another type of vector multiplication; the cross product.

As the dot product is analogous to lengths, the cross product is analogous to area. The cross product is only defined in three dimensional space with three dimensional vectors, so be careful! We define the cross product $\vec{v} \times \vec{w} = \vec{c}$ such that $\vec{c} \cdot \vec{v} = \vec{c} \cdot \vec{w} = 0$, i.e. $\vec{c}$ is perpendicular to $\vec{v}, \vec{w}$, and $||\vec{c}|| = ||\vec{v}||||\vec{w}|| \sin(\theta)$, where θ denotes the angle between $\vec{v}$ and $\vec{w}$ as it does for the dot product, i.e. $||\vec{c}||$ is the area of the parallelogram created by $\vec{v}, \vec{w}$.



However, a problem arises. You see, there are actually two possible vectors that satisfy this condition. We need a way to define the cross product such that there is only one value, and we need to make our rule "consistent," so we bring in the "right-hand rule." If you put your index finger roughly on $\vec{v}$ and your middle finger roughly on $\vec{w}$, your thumb should point roughly in the direction of $\vec{c}$. Of course, you have to do this with your right hand; after all, it's called the right hand rule for a reason. This then implies that the cross product is anticommutative, that is, $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$. Let's take a look at how to find the cross product of any two vectors.

The Cross Product Theorem (19.6)

For two vectors $\vec{v} = (x_v \quad y_v \quad z_v)$ and $\vec{w} = (x_w \quad y_w \quad z_w)$, we have

$$\vec{v} \times \vec{w} = (y_v z_w - y_w z_v \quad z_v x_w - z_w x_v \quad x_v y_w - x_w y_v).$$

Theorem 19.6's Proof

Note that there is only one vector that satisfies the conditions for the cross product, by the right hand rule. Thus we just verify that $\vec{v} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{v} \times \vec{w}) = 0.$ By Theorem 18.4, $\vec{v} \cdot (\vec{v} \times \vec{w}) = x_v y_v z_w - x_v y_w z_v + y_v z_v x_w - y_v z_w x_v + z_v x_v y_w - z_v x_w y_v = 0,$ and $\vec{w} \cdot (\vec{v} \times \vec{w}) = x_w y_v z_w - x_w y_w z_v + y_w z_v x_w - y_w z_w x_v + z_w x_v y_w - z_w x_w y_v = 0,$ as desired. (The sign of our cross product will make it point "upwards" in comparison to the other two vectors.)

Let's take a look now at how we can break vectors down. We have $\vec{v} = (x \quad y) = x(1 \quad 0) + y(0 \quad 1).$ In other words, we can express $\vec{v}$ as a sum of unit vectors. (We can generalize for higher dimensions.) For convenience, we shall express $(1 \quad 0) = \vec{i}$ and $(0 \quad 1) = \vec{j}.$ We see that $\vec{i}$ and $\vec{j}$ make a parallelogram of area $1,$ and after transforming generic matrix $\underline{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ by $\vec{i}$ and $\vec{j}$ and summing this result up, we get the area $\underline{A}\vec{i}$ and $\underline{A}\vec{j}$ spans as $|(\underline{A}\vec{i}) \times (\underline{A}\vec{j})| = |\,||\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \times \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}||\,| = |a_{11}a_{22} - a_{12}a_{21}|.$ This is the motivation for the value we call the determinant, or $a_{11}a_{22} - a_{12}a_{21}$ for matrix $\underline{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$ Thus we denote this as $\det \underline{A}$ or $|\underline{A}|,$ and we see that negative determinants reverse the direction of the cross product by comparing $\det \underline{A}$ to $(\underline{A}\vec{i}) \times (\underline{A}\vec{j}).$

Let's take a look at the transformed cross product. Given $\vec{v} = \begin{pmatrix} x_v \\ y_v \end{pmatrix}, \vec{w} = \begin{pmatrix} x_w \\ y_w \end{pmatrix},$ we can transform it by arbitrary $\underline{A}.$ Using Theorem 19.1 which states matrix multiplication is linear, we get $(x_v\underline{A}\vec{i} + y_v\underline{A}\vec{j}) \times (x_w\underline{A}\vec{i} + y_w\underline{A}\vec{j}) = (x_v y_w - x_w y_v)(\underline{A}\vec{i} \times \underline{A}\vec{j}) + x_v y_v(\underline{A}\vec{i} \times \underline{A}\vec{i}) + x_w y_w(\underline{A}\vec{j} \times \underline{A}\vec{j}).$ We see that in general, $\vec{v} \times \vec{v} = 0,$ since the area of the parallelogram created by these two vectors is obviously $0.$ (It's clearly degenerate, for crying out loud!) As thus, this simplifies to $(x_v y_w - x_w y_v)(\underline{A}\vec{i} \times \underline{A}\vec{j}) = \det A(x_v y_w - x_w y_v)\vec{r}$ for some arbitrary $r.$ Thus the area is multiplied by $|\det \underline{A}|,$ and if $\det \underline{A}$ is negative, the parallelogram is flipped. This next idea does involve some calculus; it is possible to generalize and think of any area as a bunch of parallelograms (circles can be split into infinitely many small parallelograms), meaning any transformation by $\underline{A}$ multiplies the area by $\det \underline{A}.$

Before we generalize, note that this only works for $n \times n$ matrices!

We say that the determinant of a 3×3 matrix is the factor of volume stretching, and so on for higher dimensions. This means that given some brute force, we get $\det \underline{A} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$ This can be rewritten as $\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{22}a_{31})$. These look suspiciously like determinants, which should be no surprise; this can be rewritten as $\det \underline{A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$. We can similarly factor out other combinations of coefficients. This is known as expansion by minors. We've now got all the tools we need to generalize the determinants of any $n \times n$ matrix.

Consider $n \times n$ matrix $\underline{A}$. Let M_{ij} be the i,j minor of $\underline{A}$, or the determinant of the $n - 1 \times n - 1$ matrix formed by deleting the ith row and jth column of $\underline{A}$. Then we define the i,j cofactor of scalar C_{ij} (remember scalar means a number we factor into a matrix) as $(-1)^{i+j}M_{ij}$. If we fix i,j, then we have $\det \underline{A} = \sum_{x=1}^{n} A_{ix}C_{ix}$, where A_{ix} is the item in $\underline{A}$ in the ith row and xth column. Similarly, $\det \underline{A} = \sum_{y=1}^{n} A_{yj}C_{yj}$. This theorem is known as Laplace expansion, and it actually is one of the few things in this book which will be left unproven. Let's prove a few interesting facts about matrices and some techniques that can be used to find determinants instead.

Matrix Determinant Multiplication Theorem (20.1)

Multiplying a row/column of a matrix by a number multiplies the determinant by the same number.

Theorem 20.1's Proof

This comes as a direct result of Laplace Expansion. Without loss of generality we prove this for row. Let the row we go down be row i of $\underline{A}$. By the definition of a cofactor C_{ix} does not change, and each term changes by the constant z it is multiplied by. Thus, $\det \underline{A_z} = \sum_{x=1}^{n} zA_{ix}C_{ix} = z\sum_{x=1}^{n} A_{ix}C_{ix} = z\det \underline{A}$, as desired.

Row Sum Theorem (20.2)

Consider $n \times n$ matrices $\underline{A}, \underline{B}, \underline{C}$. Given $\underline{A}$ that is identical to $\underline{B}$ and $\underline{C}$ except for row i, if $A_{ix} = B_{ix} + C_{ix}$ for all $1 \leq x \leq n$, then $\det \underline{A} = \det \underline{B} + \det \underline{C}$. (The same is true for columns.)

Theorem 20.2's Proof

Without loss of generality, we prove this for rows. Note that by Laplace expansion, $\det \underline{A} = \sum_{x=1}^{n} A_{ix}Z_{ix}$ (in this case we set Z_{ix} as the cofactor to avoid confusion) and since

$A_{ix} = B_{ix} + C_{ix},\ \det \underline{A} = \sum_{x=1}^{n} B_{ix}Z_{ix} + C_{ix}Z_{ix},$ and $\det \underline{B} = \sum_{x=1}^{n} B_{ix}Z_{ix}$ and $\det \underline{C} = \sum_{x=1}^{n} B_{ix}Z_{ix},$ summing gives $\det \underline{A} = \det \underline{B} + \det \underline{C},$ as desired.

Duplicate Rows Has Determinant of Zero (20.3)

If a matrix has two identical rows/columns, its determinant is zero.

Theorem 20.3's Proof

We proceed by induction. By the definition of a determinant, this holds true for 2×2 matrices. Then, we prove that if this holds for $n \times n,$ this holds by $n+1 \times n+1$ matrices. Let the two identical rows be i, j where $i < j.$ If we remove row i and column $1,$ then we have submatrix M_{ij} with two identical rows so this holds true, and induction finishes the proof.

Adding a Row to Another Row Keeps The Determinant (20.4)

Adding a row/column onto another row/column and multiplying it by any scalar k does not change the determinant of the matrix.

Theorem 20.4's Proof

Without loss of generality, we prove this for rows again, and for visual purposes, we display row i ahead of row $j.$ Let us add row j multiplied by k to row $i.$ Then our initial matrix becomes

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} + ka_{j1} & a_{i2} + ka_{j2} & \cdots & a_{in} + ka_{j2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Let us call this $\underline{A_x},$ and let us define $\underline{A}$ and $\underline{A_y}$ as

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{j2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Theorem 20.2 implies $\det \underline{A_x} = \det \underline{A} + \det \underline{A_y}$. We note that $\det \underline{A_y}$ is $k \det \underline{A_z}$, where $\underline{A_z}$ is

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{j2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

By Theorem 20.3, $\det \underline{A_z} = 0$, so $\det \underline{A_x} = \det \underline{A}$, as desired.

Now we take a look at some other tools we could use.

The Swapping Theorem (20.5)

Swapping two rows/columns of a matrix multiplies the determinant by -1.

Theorem 20.5's Proof

Let the two rows to be swapped be i, j. We add the jth row to the ith, subtract the jth row by the ith row, and add the jth row to the ith to get the jth row and the ith row switched, with the jth row containing the negatives of the terms that were previously in the ith row. By Theorem 20.4, none of these operations changed the determinant. We multiply the jth row by -1, and according to Theorem 20.1, this multiplies the determinant by -1, as desired.

Determinant of Diagonal Matrix (20.6)

Given square matrix $\underline{A}$ such that $a_{ij} = \{^{x\neq 0,\ i=j}_{0,\ i\neq j}$ (i.e. such that only the terms on the diagonal are nonzero), $\det \underline{A} = \prod_{x=1}^{n} a_{xx}$.

Theorem 20.6's Proof

We proceed by induction. This is clearly true for a 2×2 matrix. Now we prove it is true for an $n+1 \times n+1$. Expanding by minors on the first row gives us $\det \underline{A} = a_{11}M_{11}$, and since M_{11} is the determinant of an $n \times n$ diagonal matrix, we are done.

We generalize to show this is true if all terms are on one side of the matrix.

Determinant of Triangular Matrix (20.7)

Given square matrix $\underline{A}$ such that $a_{ij} = \{^{x\neq 0,\ i\leq j}_{0,\ i>j}$ (i.e. such that only the terms on or above the diagonal are nonzero), $\det \underline{A} = \prod_{x=1}^{n} a_{xx}$.

Theorem 20.7's Proof

We proceed by induction. This is clearly true for a 2×2 matrix. Now we prove it is true for an $n+1 \times n+1$. Expanding by minors on the first row gives us $\det \underline{A} = a_{11}M_{11}$, and since M_{11} is the determinant of an $n \times n$ triangular matrix, we are done.

Notice that the proofs were so similar, I could copy and paste the proof with minimal changes.

Now let's talk inverse matrices. We'll introduce a theorem which, once again, we will not prove. (We've proved most our theorems so far because geometry proofs are easier; however, the theorems we are not proving are fundamental to our study but require much higher maths.)

For any $n \times n$ matrices $\underline{M_1}, \underline{M_2}, \underline{M_3} \ldots \underline{M_x}$, $\det(\prod_{i=1}^{x} \underline{M_i}) = \prod_{j=1}^{x} \det \underline{M_i}$. This means that $\det \underline{AB} = \det \underline{A} \det \underline{B}$, and the like. This means that $\det \underline{A^{-1}} = (\det \underline{A})^{-1}$. So, for any matrix with a determinant of 0, it cannot have an inverse. Otherwise, it can.

As an example, we find the inverse of a 2×2 matrix. Let $\underline{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and let $\underline{A}^{-1} = \begin{pmatrix} i_{11} & i_{12} \\ i_{21} & i_{22} \end{pmatrix}$. Then, by the definition of matrix multiplication and the definition of an inverse, we get $\begin{pmatrix} a_{11}i_{11}+a_{12}i_{21} & a_{11}i_{12}+a_{12}i_{22} \\ a_{21}i_{11}+a_{22}i_{21} & a_{21}i_{12}+a_{22}i_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Term matching gives us a system of equations which we can solve for the inverse matrix with. We get $a_{11}i_{11} + a_{12}i_{21} = 1$, $a_{11}i_{12} + a_{12}i_{22} = 0$, $a_{21}i_{11} + a_{22}i_{21} = 0$, $a_{21}i_{12} + a_{22}i_{22} = 1$. This implies $a_{11}a_{22}i_{11} + a_{12}a_{22}i_{21} = a_{22}$ and $a_{12}a_{21}i_{11} + a_{12}a_{22}i_{21} = 0$, and subtracting the second equation from the first yields $a_{11}a_{22}i_{11} - a_{12}a_{21}i_{11} = a_{22}$, and rearranging yields $i_{11} = \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}}$. The expression in the denominator looks familiar; and this is because it is the determinant. Substituting, we get $i_{11} = \frac{a_{22}}{\det A}$. We solve for the rest of the matrix similarly and get $\underline{A}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

We won't prove the process for getting the inverse either, as it requires higher level maths. To find the inverse, we first define the transpose of $\underline{A}$ as $\underline{A_T}$, where every term a_{ij} in $\underline{A}$ is mapped to a_{ji} in $\underline{A_T}$. For example, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Then, we replace each entry by its cofactor, multiply by scalar $\frac{1}{\det A}$, and transpose to get $\underline{A}^{-1}$.

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1. Find $\underline{A}^6$ if $\underline{A} = \begin{pmatrix} \sqrt{6}+\sqrt{2} & \sqrt{2}-\sqrt{6} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{pmatrix}$.

2. If $\vec{v} \times \vec{w} = 0$ and $\vec{v} = (1 \quad 2 \quad 3)$, find the form all possible $\vec{w}$ can take.

3. Find

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}.$$

4. Given 3×3 matrix $\underline{A}$, we know that $\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$. What is also true though is that we can express $\det A = a_{13} \det I - a_{23} \det J + a_{33} \det K$ or some other form. Complete this factorization and evaluate $\underline{I}, \underline{J}, \underline{K}$ in terms of a_{ij}.

5. Find

$$\begin{vmatrix} 1 & 4 & 6 & 2 & 3 \\ 7 & 3 & 9 & 6 & 5 \\ 0 & 2 & 1 & 7 & 6 \\ 4 & 2 & 6 & 9 & 0 \\ 3 & 0 & 1 & 6 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 6 & 1 & 0 & 3 \\ 0 & 9 & 6 & 2 & 4 \\ 6 & 7 & 1 & 2 & 0 \\ 5 & 6 & 9 & 3 & 7 \\ 3 & 2 & 6 & 4 & 1 \end{vmatrix}.$$

6. For what value of θ is $\sqrt{3}\sin(\theta) + \cos(\theta)$ maximized?

7. Use the information discussed in this section to prove $\max_{a\sin(x)+b\cos(x)} = \sqrt{a^2 + b^2}$.

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1. Find $\underline{A}^6$ if $\underline{A} = \begin{pmatrix} \sqrt{6}+\sqrt{2} & \sqrt{2}-\sqrt{6} \\ \sqrt{6}-\sqrt{2} & \sqrt{6}+\sqrt{2} \end{pmatrix}$.

Solution: We factor out 4 from $\underline{A}$ and note that A is the rotation matrix $\frac{\pi}{12}$ scaled up by 4. Thus, $\underline{A}^6$ is the rotation matrix $6(\frac{\pi}{12}) = \frac{\pi}{2}$ scaled up by $4^6 = 4096$. Since the rotation matrix $\frac{\pi}{2}$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\underline{A}^6 = \begin{pmatrix} 0 & -4096 \\ 4096 & 0 \end{pmatrix}$.

2. If $\vec{v} \times \vec{w} = 0$ and $\vec{v} = (1 \quad 2 \quad 3)$, find the form all possible $\vec{w}$ can take.

Solution: Let $\vec{w} = (x \quad y \quad z)$. Then, by Theorem 19.6, $\vec{v} \times \vec{w} = (2z - 3y \quad 3x - z \quad y - 2x)$. Using "vector matching" we see that this implies $2z - 3y = 0, 3x - z = 0, y - 2x = 0$. This implies $y = 2x$ and $3y = 2z$. This means our vectors are of the form $(4c \quad 2c \quad 3c)$.

3. Find
$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}.$$

Solution: We can directly calculate it as $1 \cdot 2 \cdot 6 + 1 \cdot 3 \cdot 1 + 1 \cdot 1 \cdot 3 - 1 \cdot 1 \cdot 6 - 1 \cdot 2 \cdot 1 - 1 \cdot 3 \cdot 3 = 1$. Alternatively, we expand by minors and find $1\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} + 6\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$. However, not using the triangular matrix method saddens me greatly, so we'll demonstrate it.

By Theorem 20.4,
$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix}$$
By Theorem 20.7,
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix} = 1.$$

4. Given 3×3 matrix $\underline{A}$, we know that $\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$. What is also true though is that we can express $\det A = a_{13} \det I - a_{23} \det J + a_{33} \det K$ or some other form. Complete this factorization and evaluate $\underline{I}, \underline{J}, \underline{K}$ in terms of a_{ij}.

Solution: We factor out $\det A = a_{13}(a_{21}a_{32} - a_{22}a_{31}) - a_{23}(a_{11}a_{32} - a_{12}a_{31}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})$. Then we express this in the form of determinants; clearly, $\underline{I}, \underline{J}, \underline{K}$ correspond to $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$, respectively.

5. Find

$$\begin{vmatrix} 1 & 4 & 6 & 2 & 3 \\ 7 & 3 & 9 & 6 & 5 \\ 0 & 2 & 1 & 7 & 6 \\ 4 & 2 & 6 & 9 & 0 \\ 3 & 0 & 1 & 6 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 6 & 1 & 0 & 3 \\ 0 & 9 & 6 & 2 & 4 \\ 6 & 7 & 1 & 2 & 0 \\ 5 & 6 & 9 & 3 & 7 \\ 3 & 2 & 6 & 4 & 1 \end{vmatrix}.$$

Solution: To change the second matrix into the first, we can swap row 1 with row 5, row 2 with row 4, column 1 with column 5, and column 2 with column 4. By Theorem 20.5, the determinant is multiplied by $(-1)^4 = 1$, so the difference of the determinants is 0.

6. For what value of θ is $\sqrt{3}\sin(\theta) + \cos(\theta)$ maximized?

Solution: We note this is $(\sqrt{3} \quad 1) \cdot (\sin(\theta) \quad \cos(\theta))$. Let the angle between these two vectors be α. By the definition of the dot product, $(\sqrt{3} \quad 1) \cdot (\sin(\theta) \quad \cos(\theta)) = 2\cos(\alpha)$. To maximize this, we wish for $\alpha = 0$, which occurs when $\theta = \frac{\pi}{3}$.

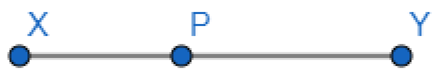
7. Use the information discussed in this section to prove $\max\limits_{a\sin(x)+b\cos(x)} = \sqrt{a^2 + b^2}$.

Solution: This can be thought of as the dot product $(a \quad b) \cdot (\sin(x) \quad \cos(x))$. These vectors have lengths $\sqrt{a^2 + b^2}$ and 1. We then note that by the geometric definition of the dot product, the dot product of two vectors with fixed lengths is maximized when the two vectors are going in the same direction (that is, their argument is the same if the tail is the origin). This is possible, so our maximum is just $\sqrt{a^2 + b^2}$.

Mass Points

In competitions, you may have heard of a method called mass points, or barycentric coordinates. While those two are not the same thing, barycentric coordinates is just an extension of mass points, so we'll be explaining that first. While mass points can be helpful, it is important to note that mass points are just a technique, and as for all techniques, there are times when using them are more detrimental than helpful.

With that out of the way, let's jump straight into it. Consider line segment XY with point P on XY.

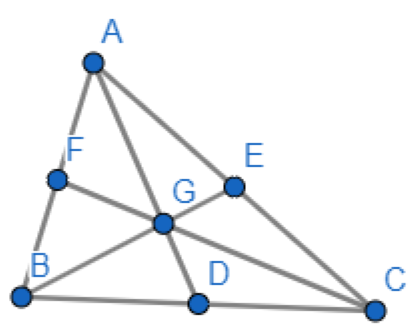


In mass points, we assign the masses with respect to P. In this example, we will assign a mass for X and a mass for Y such that the ratio of the mass of X to $\overline{PY}$ is equivalent to the ratio of the mass of Y to $\overline{PX}$.

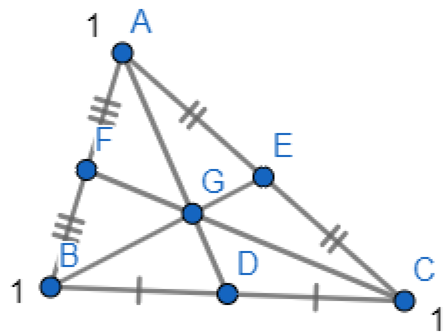
That explanation already looks long, and the text above is discussing the definition of mass points. Let's introduce some notation; let $\diamond X$ denote the mass of point X. Then, our definition is rewritten to $\frac{\frac{\diamond X}{}}{YP} = \frac{\frac{\diamond Y}{}}{XP}$. This saves us from using too many words, which is always nice. Also, we define $\diamond P = \diamond X + \diamond Y$.

Keep in mind that we can apply this to much more than a line, since our properties will be consistent no matter which point on which line we use. Also do note that these masses are ratios; we can pick arbitrary numbers at any time to get rid of fractions if we please! (We won't be so lucky with irrational numbers, but the case where we do need to use mass points with irrational numbers is rare.)

Let's take a look at a classic example; mass points with the medians.



We know that $\overline{AF} = \overline{BF}, \overline{BD} = \overline{CD}, \overline{CE} = \overline{AE},$ by definition. Thus we can label $\diamond A = \diamond B,$ $\diamond B = \diamond C,$ $\diamond C = \diamond A.$ By the transitive property, $\diamond A = \diamond B = \diamond C.$ It makes sense to set them all as $1,$ so have $\diamond A = \diamond B = \diamond C = 1.$ (We can set this number to be whatever we want, but this is probably the most convenient.)



Then we note that by the definition of mass points, $\diamond D = \diamond E = \diamond F = 2.$ (We also notice $\diamond G = 3,$ but that isn't as important.) We then get some vital information; by the definition of mass points, $\overline{AG} = 2\overline{GD},$ as $\diamond D = 2 \diamond A,$ and likewise for the other medians. We also then have $\diamond G = 3,$ implying that $\overline{AG} = 2\overline{DG}.$ This is the standard example because this is the easiest one; the scarce resources out there on mass points will use this as well.

This technique gives us information of the ratio of lengths with parts of a cevian. Below are some problems with mass points; they will make more sense as you work more of them.

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1. Consider $\triangle ABC$ where X, Y are on BC, CA such that $\frac{\overline{BX}}{\overline{CX}} = \frac{1}{4}, \frac{\overline{CY}}{\overline{YA}} = \frac{2}{3}.$ If AX, BY intersect at $Z,$ find $\frac{\overline{AZ}}{\overline{ZX}}.$

2. Consider $\triangle ABC$ with $\overline{AB} = 7, \overline{BC} = 8, \overline{AC} = 6.$ Let AD be the angle bisector of $\angle BAC$ and let E be the midpoint of $AC.$ If BE and AD intersect at $G,$ find $\overline{AG}.$

3. Consider $\triangle ABC$ with $\overline{AB} = 10, \overline{BC} = 21, \overline{CA} = 17.$ If median BE intersects altitude AD at $G,$ find $\overline{AG}.$

4. Consider $\triangle ABC$ with angle bisector AD and median $BE.$ Let them intersect at $G.$ Draw a line parallel to BC that passes $G,$ and let it intersect AB, AC at $X, Y.$ If $[ABC] = 100$ and $\frac{\overline{BD}}{\overline{BC}} = \frac{3}{7},$ find $[AXY].$

5. Consider $\triangle ABC$ such that $\overline{AB} = 17, \overline{BC} = 25, \overline{AC} = 28.$ Let the B altitude and the C angle bisector intersect at $G.$ Find $\overline{AG}.$

6. Consider rectangle $ABCD$ with $\overline{AB} = 6, \overline{BC} = 8.$ Let M be the midpoint of AD and let N be the midpoint of $CD.$ Let BM, CN intersect AC at $X, Y.$ Find $\overline{XY}.$

7. Let there be $\triangle ABC$ with points D, E, F on sides BC, CA, AB respectively such that cevians AD, BE, CF concur at point G. If $\frac{AE}{EC} = \frac{2}{3}$ and $\frac{AG}{GD} = \frac{16}{15}$, find $\frac{[CGD]}{[BCE]}$.

8. (e-dchen Mock Mathcounts) Consider $\triangle ABC$ with $\overline{AB} = 13,$ $\overline{BC} = 14,$ and $\overline{CA} = 15.$ Let AD bisect $\angle BAC$ and let E be the foot of the perpendicular of B to $AC.$ Let AD and BE intersect at $P.$ If CP intersects AB at $F,$ what is the area of quadrilateral $AFPE$?

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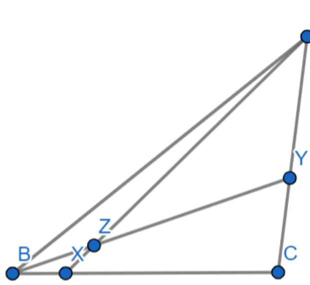
1. Consider $\triangle ABC$ where X, Y are on BC, CA such that $\frac{\overline{BX}}{CX} = \frac{1}{4}, \frac{\overline{CY}}{YA} = \frac{2}{3}$. If AX, BY intersect at Z, find $\frac{\overline{AZ}}{ZX}$.

Solution: Notice that $\diamond B = 12, \diamond C = 3, \diamond A = 2$ by the ratios given, which implies $\diamond X = 15$. Thus, $\frac{\overline{AZ}}{ZX} = \frac{\diamond X}{\diamond A} = \frac{15}{2}$.



2. Consider $\triangle ABC$ with $\overline{AB} = 7, \overline{BC} = 8, \overline{AC} = 6$. Let AD be the angle bisector of $\angle BAC$ and let E be the midpoint of AC. If BE and AD intersect at G, find $\overline{AG}$.

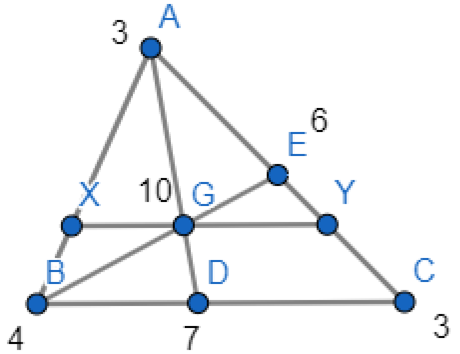
Solution: By the Angle Bisector Length Theorem (7.1.2), $\overline{AD} = \frac{21\sqrt{10}}{13}$. We then assign masses $7, 6, 7$ to A, B, C, respectively. (This comes from the Angle Bisector Proportionality Theorem (7.1.1).) The rest is routine calculation; we note that $\diamond D = 13$, $\diamond A = 7$, so $\frac{\overline{AG}}{AD} = \frac{\diamond D}{\diamond G} = \frac{13}{20}$, meaning $\overline{AG} = \frac{21\sqrt{10}}{20}$.

3. Consider $\triangle ABC$ with $\overline{AB} = 10, \overline{BC} = 21, \overline{CA} = 17$. If median BE intersects altitude AD at G, find $\overline{AG}$.

Solution: Routine calculation gives us $\overline{AD} = 8, \overline{BD} = 6, \overline{DC} = 15$. Then we have $\diamond B = 15, \diamond C = 6$, implying $\diamond D = 21$. Then we note that $\diamond A = 6$, by the definition of a midpoint. This means that $\diamond G = 27$, or $\overline{AG} = \frac{21}{27} \cdot \overline{AD} = 8$.

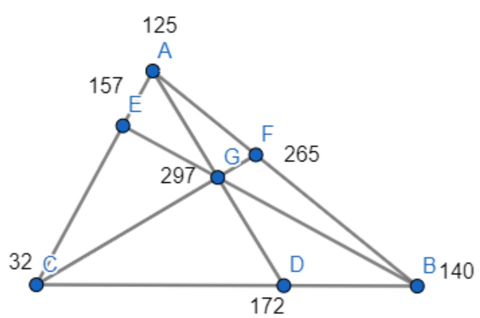
4. Consider $\triangle ABC$ with angle bisector AD and median BE. Let them intersect at G. Draw a line parallel to BC that passes G, and let it intersect AB, AC at X, Y. If $[ABC] = 100$ and $\frac{\overline{BD}}{BC} = \frac{3}{7}$, find $[AXY]$.

Solution: By similar triangles, we can just find $(\frac{\overline{AG}}{AD})^2 \cdot [ABC] = (\frac{\overline{AG}}{AD})^2 \cdot 100$. For mass points, it is important that we ignore X and Y. Then we just note that $\frac{\overline{AG}}{AD} = \frac{7}{10}$, implying $[AXY] = (\frac{7}{10})^2 \cdot 100 = 49$.



5. Consider $\triangle ABC$ such that $\overline{AB} = 17, \overline{BC} = 25, \overline{AC} = 28$. Let the B altitude and the C angle bisector intersect at G. Find $\overline{AG}$.

Solution: We set up the diagram as such. (This means BE is an altitude, and so on.)

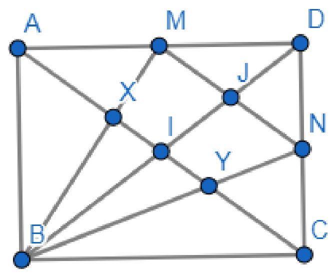


Routine calculation gives us $\overline{BE} = 15, \overline{AE} = 8, \overline{CE} = 20$. Then we assign mass points as such. We then note that this implies $\overline{CD} = 25 \cdot \frac{35}{43}, \overline{BD} = 25 \cdot \frac{8}{43}$. Applying Stewart's Theorem gives us $25 \cdot \frac{35}{43} \cdot 25 \cdot 25 \cdot \frac{8}{43} + d^2 25 = 28 \cdot 25 \cdot \frac{8}{43} \cdot 28 + 17 \cdot 25 \cdot \frac{35}{43} \cdot 17$. This simplifies to $d = \frac{21\sqrt{1201}}{43}$.

6. Consider rectangle $ABCD$ with $\overline{AB} = 6, \overline{BC} = 8$. Let M be the midpoint of AD and let N be the midpoint of CD. Let BM, CN intersect AC at X, Y. Find $\overline{XY}$.

Solution: We don't actually have to use mass points, which makes this problem all the more enlightening.

Note that $2\overline{DJ} = 2\overline{JI} = \overline{BI}$, so $\triangle BXY \sim \triangle BMN$ with a ratio of $2:3$. As thus, $\overline{XY} = \frac{2}{3}\overline{MN} = \frac{2}{3} \cdot 5 = \frac{10}{3}$.

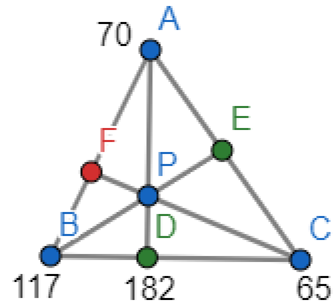


7. Let there be $\triangle ABC$ with points D, E, F on sides BC, CA, AB respectively such that cevians AD, BE, CF concur at point G. If $\frac{AE}{EC} = \frac{2}{3}$ and $\frac{AG}{GD} = \frac{16}{15}$, find $\frac{[CGD]}{[BCE]}$.

Solution: To do away with fractions, we assign $\diamond G = 31$. Then $\diamond A = 15, \diamond D = 16$, by the second condition. Then we use the first condition and notice $\diamond C = 10$, and $\diamond E = 25$. We then notice $\diamond B = 6$. By similar triangles, the ratio of the altitudes of $\triangle CGD$ and $\triangle BEC$ from G, E to BC is $\frac{25}{31}$. Then we notice $\frac{\overline{CD}}{CB} = \frac{6}{16} = \frac{3}{8}$. Thus, $\frac{[CGD]}{[BCE]} = \frac{25}{31} \cdot \frac{3}{8} = \frac{75}{248}$.

8. (e-dchen Mock Mathcounts) Consider $\triangle ABC$ with $\overline{AB} = 13$, $\overline{BC} = 14$, and $\overline{CA} = 15$. Let AD bisect $\angle BAC$ and let E be the foot of the perpendicular of B to AC. Let AD and BE intersect at P. If CP intersects AB at F, what is the area of quadrilateral $AFPE$?

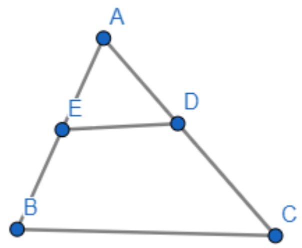
Solution: By Angle Bisector Proportionality Theorem (7.1.1) and by well-known properties of the altitude of a $13-14-15$ triangle, we may assign $\diamond A = 70, \diamond B = 117, \diamond C = 65$. Also, $\diamond D = \diamond B + \diamond C = 117 + 65 = 182$. This implies that $\frac{\overline{AF}}{AB} = \frac{117}{187}$ and $\frac{\overline{AP}}{AD} = \frac{182}{252}$. By $\frac{1}{2}ab \cdot \sin(C)$ (5.3), $[AFP] = [ABD] \cdot \frac{\overline{AF}}{AB} \cdot \frac{\overline{AP}}{AD} = 30 \cdot \frac{117}{187} \cdot \frac{182}{252} = \frac{2535}{187}$.



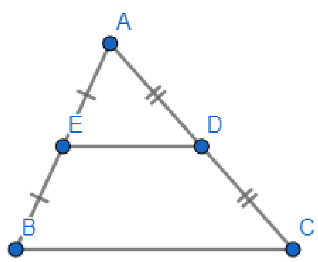
There really isn't much room for creativity with mass points problems. However, mass points is especially useful for MATHCOUNTS, as you will no longer have to coordinate bash cevian problems with multiple intersections.

Just make sure you don't use mass points at the wrong time.

Now let's take a look at transversals.

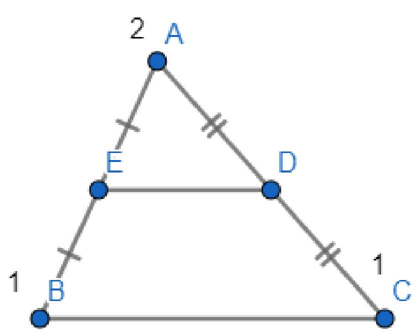


The classic analogy is having two points A_1 and A_2 at the same place. Let's use medians as an example.

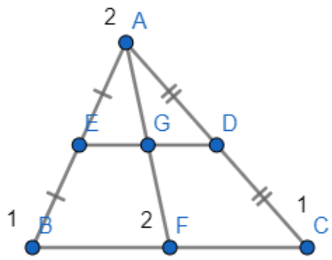


We see clearly that $\Diamond B = \Diamond C = 1.$

If we ignore $D,$ we see that A_1 has mass $1,$ and if we ignore $E,$ we see that A_2 has mass $1.$ The main idea is adding the masses together; as thus, the total mass of A is $1 + 1 = 2.$



Now let's take a look at this configuration.



This definition of transversals means we won't get a faulty result for $\frac{\overline{AG}}{\overline{AF}}$.

Again, this technique gives us some information about ratios of lengths. Once you get used to it, hard problems will seem much easier. (This is especially true for MATHCOUNTS!)

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1. Given $\triangle ABC$ with cevian AD and medians BY, CX, let AD intersect XY at M. Prove $\frac{\overline{AD}}{\overline{AM}} = 2$.

2. Given $\triangle ABC$ with E, F on line segments AC, AB such that $\overline{AE} : \overline{EC} = \overline{BF} : \overline{FA} = 1 : 3$, and with median AD that intersects EF at G, prove that $AG : GD = 2 : 5$.

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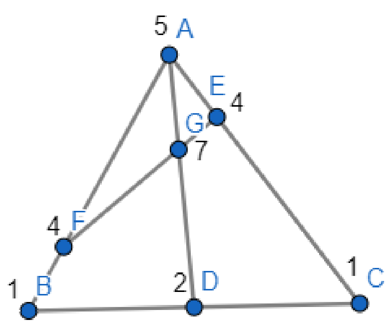
1. Given $\triangle ABC$ with cevian AD and medians $BY, CX,$ let AD intersect XY at $M.$ Prove $\frac{\overline{AD}}{AM} = 2.$

Solution: Either D is on the line segment BC or on the extension of $BC.$ We can just prove that this is an if and only if.

Let D be on $BC.$ Then, we note that $\diamond A = \diamond B + \diamond C = \diamond D,$ so $\frac{\overline{AM}}{DM} = 1,$ or $\frac{\overline{AD}}{AM} = 2.$ If D is on the extension, note that there is a unique line determined by the midpoints of AB and $AC,$ so only one position of AD works; its midpoint.

2. Given $\triangle ABC$ with E, F on line segments AC, AB such that $\overline{AE} : \overline{EC} = \overline{BF} : \overline{FA} = 1 : 3,$ and with median AD that intersects EF at $G,$ prove that $AG : GD = 2 : 5.$

Solution: Assigning $\diamond B = \diamond C = 1,$ the diagram finishes it off for us nicely.



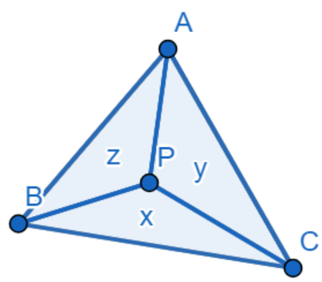
Barycentric Coordinates

We've all heard of the infamous barycentric coordinates, but my goal here is not to present a "general overview" of barycentric coordinates like Wikipedia, Math Open Reference, and the like do. I'm here to answer the infamous question, "How could I have thought of that?" And I'm going to bridge a difficult gap in one of the most useful analytic tools of olympiad geometry; this is much more useful than matrices, and even mass points.

We thus consider the most intuitive definition of barycentric coordinates, the area definition. With barycentric coordinates, we define them in relation to a triangle.

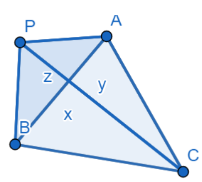
Let us define our points P with respect to $\triangle ABC$. Then we define the barycentric coordinates of P as $(\frac{[CPB]}{[ABC]}, \frac{[APC]}{[ABC]}, \frac{[BPA]}{[ABC]})$.

However, to leave no room for a lack of intuitiveness, we provide a diagram.



(We let x, y, z denote $[CPB], [APC], [BPA]$ respectively.) Then this makes our coordinates look like $(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z})$. It is very important to notice this is **ordered**; the coordinates of the exact same point P is different with respect to $[ABC]$ and $[ACB]$. (In fact, any permutation of the letters corresponds to a permutation of the coordinates.) We can also express this as an ordered ratio for convenience; $(\frac{x}{x+y+z} : \frac{y}{x+y+z} : \frac{z}{x+y+z})$ can be easier expressed as $(x : y : z)$.

Another important thing to note is that barycentric coordinates always add to one. Let's see what happens if we put P outside the triangle.



This seems like a problem until we realize that $\triangle ABC = x + y - |z|$. This means that as soon as we go to the other "side" of a line (the side of the line not containing the triangle), our area counts as negative for its opposite vertice. We should develop some notation for this.

Assuming that A, B, C are in counterclockwise order (which is true on the diagram), we see that if B, P, A is in clockwise order, then it is on the other side of the line (meaning it is negative). This is true for all the other lines. Then we let $[BPA]$ denote a **signed** area, where if $[BPA]$ is counterclockwise, its area is positive, and if $[BPA]$ is clockwise, its area is negative. This clears a lot of ambiguity.

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1. Find the barycentric coordinates of the incenter of $\triangle ABC$ with sides a, b, c.

2. Find the barycentric coordinates of A, B, C.

3. Why did we define the barycentric coordinates of P as $(\frac{[CPB]}{[ABC]}, \frac{[APC]}{[ABC]}, \frac{[BPA]}{[ABC]})$? Why can't we use $[BPC]$ in place of $[CPB]$?

4. Does our definition work if $[ABC]$ is clockwise?

5. Find the midpoint of BC in barycentric coordinates.

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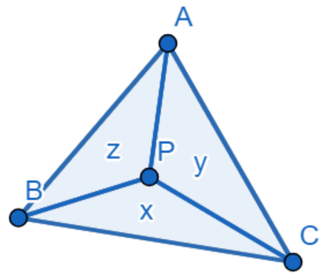
1. Find the barycentric coordinates of the incenter of $\triangle ABC$ with side lengths $a, b, c.$

Solution: By the definition of an incenter, the perpendiculars from P to AB, AC, BC are all the same. Thus we note that $[ABP] = \frac{cr}{2},$ and similar expressions arise for the other triangles. Then we note the total area is $\frac{(a+b+c)r}{2}$ and our barycentric coordinates become $(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}).$



2. Obviously if it is $A,$ then $[BPA] = [ABC]$ and the other two triangles have area $0.$ Generalizing for $B, C,$ this means that $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1).$ This is a very important tool for barycentric coordinates.

3. Why did we define the barycentric coordinates of P as $(\frac{[CPB]}{[ABC]}, \frac{[APC]}{[ABC]}, \frac{[BPA]}{[ABC]})?$ Why can't we use $[BPC]$ in place of $[CPB]?$

Solution: Because the areas are signed, so order does matter. Of course, we could've done $[BCP]$ or $[PBC]$ for the first triangle and so on.

4. Does our definition work if $[ABC]$ is clockwise?

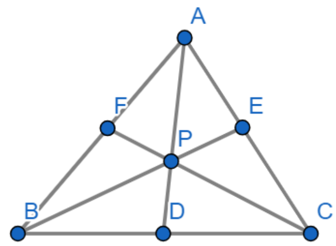
Solution: Yes! If P is in the "interior side" of AB then A, B, P is clockwise. This means the negatives cancel out, which is nice.

5. Find the midpoint of BC in barycentric coordinates.

Solution: The areas of $[BPA]$ and $[APC]$ are equivalent, and $[CPB] = 0,$ so the midpoint is $(0, \frac{1}{2}, \frac{1}{2}).$

Barycentric coordinates are known as an extension of mass points, which may not be apparent at first glance. But the analogy is certainly valid.

Let's consider reference triangle $\triangle ABC$ and a point P.



Draw an altitude from A to BC, and draw one from P to BC. Clearly, since the two triangles have the same base, the ratio of $[ABC]:[PBC]$ is the same as the ratio of their altitudes. And $PD:AD = a : a+b+c$, where $\diamond P = a+b+c, \diamond A = a$. We can do this with the other two vertices, which should illustrate the point.

We can generalize to zero and negative values if we just use directed lengths.

Thus, if $\triangle ABC$ has $\diamond A = a, \diamond B = b, \diamond C = c$, then P has barycentrics $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$. (This is due to the reciprocal nature of mass points!)

Our definitions of barycentric coordinates is certainly valid, but it also can be expanded upon to provide more uses. We introduced the area/mass points definition mostly to familiarize the reader with barycentric coordinates. We then shall introduce the vector definition.

We let $\vec{A}, \vec{B}, \vec{C}, \vec{P}$ be vectors with arbitrary tail O and heads A, B, C, P respectively. Then we assign each point P in the plane an ordered triple (x, y, z) such that $\vec{P} = x\vec{A} + y\vec{B} + z\vec{C}$ and $x + y + z = 1$. (We omit O because our choice of it is irrelevant.)

This may be hard to visualize, so we'll present an example.

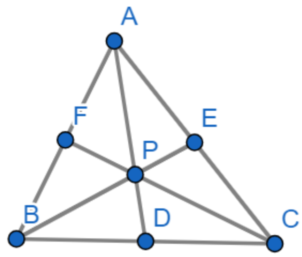
Coordinates of the Centroid (21.1)

The centroid has barycentric coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, or $(1:1:1)$.

Theorem 21.1's Proof

Without loss of generality, we can let P be the origin, or O. (This stems from the definition of vectors.) Then we wish to prove $\vec{A} + \vec{B} + \vec{C} = \vec{0}$. Note that $\vec{D} = \frac{1}{2}(\vec{B} + \vec{C})$. Then note $\vec{P} = \vec{A} + \frac{2}{3}(\vec{AD})$, and $\vec{AD} = \vec{D} - \vec{A}$. Substituting our equation for $\vec{D}$ gives us

$\vec{AD} = \frac{1}{2}\vec{B} + \frac{1}{2}\vec{C} - \vec{A}$, and substituting our equation for $\vec{AD}$ yields $\vec{P} = \frac{1}{3}\vec{A} + \frac{1}{3}\vec{B} + \frac{1}{3}\vec{C}$, as desired.



We'll prove the coordinates of the incenter, excenter, orthocenter, and circumcenter, in terms of our reference triangle. Sometimes, we'll see that our vector definition is better, sometimes the area definition will be better, sometimes the mass points definition is better, and sometimes the proof is trivial, like with the centroid.

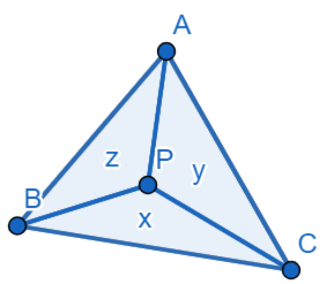
Coordinates of the Incenter (21.2)

For $\triangle ABC$ with side lengths a, b, c corresponding to the sides opposite A, B, C, respectively, the incenter I has barycentric coordinates $(a : b : c)$.

This should look familiar; we presented this as an exercise in an earlier section.

Theorem 21.2's Proof

By the definition of an incenter, the perpendiculars from P to AB, AC, BC all have the same length. Thus we note that $[BCP] = \frac{ar}{2}, [CAP] = \frac{br}{2}, [ABP] = \frac{cr}{2}$. The total area of the triangle is $\frac{(a+b+c)r}{2}$ and our barycentric coordinates become $(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}) = (a : b : c)$, as desired.

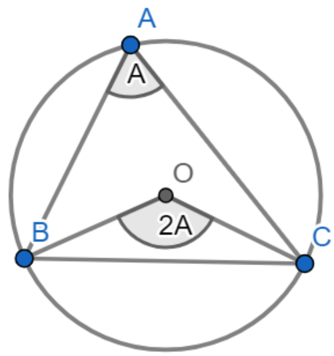


Coordinates of the Circumcenter (21.3)

For $\triangle ABC$ with angles $\angle A, \angle B, \angle C$ corresponding to its respective vertices, the circumcenter O has barycentric coordinates $\sin(2A) : \sin(2B) : \sin(2C)$.

Theorem 21.3's Proof

Let the circumcenter be O. Note that $OA = OB = OC$. Then we use the Inscribed Angle Theorem (1.1) and angle chase; this means $\angle OBC = 2\angle A, \angle OCA = 2\angle B, \angle OAB = 2\angle C$. Then we use $[ABC] = \frac{1}{2}ab \cdot \sin(C)$ (4.3) to get $[BCO] : [CAO] : [ABO] = r^2\sin(2A) : r^2\sin(2B) : r^2\sin(2C) = \sin(2A) : \sin(2B) : \sin(2C)$, as desired.



Coordinates of the Orthocenter (21.4)

For $\triangle ABC$ with angles $\angle A, \angle B, \angle C$ corresponding to its respective vertices, the orthocenter H has barycentric coordinates $\tan(A) : \tan(B) : \tan(C)$.

Theorem 21.4's Proof

Let H be the orthocenter. We use the area definition. Without loss of generality, let the circumcenter have diameter 2. Then $a = \sin(A)$, $b = \sin(B)$, $c = \sin(C)$. Then we can let D, E, F be the foots of the altitudes of A, B, C, respectively. Then we use right $\triangle ABD$ and note that $BC = \cos(B)$, $BD = \sin(C)\cos(B)$, $HD = \cos(B)\cos(C)$, $[BCH] = \frac{\sin(A)\cos(B)\cos(C)}{2}$. We use symmetry and note $[ACH] = \frac{\sin(B)\cos(A)\cos(C)}{2}, [ABH] = \frac{\sin(C)\cos(A)\cos(B)}{2}$. This implies $[BCH] : [ACH] : [ABH] = \sin(A)\cos(B)\cos(C) : \sin(B)\cos(A)\cos(C) : \sin(C)\cos(A)\cos(B)$, and dividing by $\cos(A)\cos(B)\cos(C)$ yields $[BCH] : [ACH] : [ABH] = \tan(A)\tan(B)\tan(C)$, as desired.

Coordinates of the Excenter (21.5)

The A excenter has barycentric coordinates $(-a : b : c)$ and symmetric expressions exist for the B and C excenters.

Theorem 21.5's Proof

Trivially, the perpendiculars from the A excenter to sides a, b, c all have the same absolute distance. Using $\frac{bh}{2}$ (4.2), we get $[BCE_A] : [CAE_A] : [ABE_A] = \frac{-ar}{2} : \frac{br}{2} : \frac{cr}{2} = -a : b : c,$ as desired.

Our symmetric expressions have symmetric proofs.

We'll discuss a few more advanced techniques, such as the area of a triangle, the equation of a line, and the aptly named "Evan's Favorite Forgotten Trick."

Area Formula (22.1)

Given three points P, Q, R with normalized coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3),$

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & x_3 \end{vmatrix}.$$

Theorem 22.1's Proof

We use Cartesian Coordinates; all points written in Cartesian Coordinates will be expressed as $[x, y, z]$ for clarity.

Choose O not in the plane determined by $A, B, C,$ such that $O = [0, 0, 0], A = [1, 0, 0], B = [0, 1, 0], C = [0, 0, 1].$ (We are using a three-dimensional coordinate system!) Then we note the form of the plane containing $\triangle ABC$ has the equation $x + y + z = 1.$ (This corresponds to the normalized coordinates of any point in the plane!) Then let the parallelepiped that $\vec{A}, \vec{B}, \vec{C}$ spans (remember their tails are $O = [0, 0, 0],$ which this time cannot be ignored due to no preservation of generality) be denoted as $P_{ABC},$ and similarly, let P_{PQR} denote the parallelepiped spanned by $\vec{P}, \vec{Q}, \vec{R}.$ Then we use the determinant definition of volume and note that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Then we note that by the definition of a parallelepiped, $\frac{[P_{PQR}]}{[P_{ABC}]} = \frac{2[PQR]h}{2[ABC]h} = \frac{[PQR]}{[ABC]},$ as desired.

Note that even though the h doesn't really matter so long as it is consistent, it is interesting to note that h in this case denotes the distance from O to the plane that contains $[ABC]$.

We present a collinearity corollary.

Collinearity by Barycentric Coordinates (22.2)
Three points P, Q, R are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & x_3 \end{vmatrix} = 0.$$

Theorem 22.2's Proof
This follows trivially from the area theorem; note that P, Q, R are collinear if and only if $[PQR] = 0$. A simple application of the transitive property finishes this.

Collinearity Again (22.3)
Three points P, Q, R are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Theorem 22.3's Proof
By Theorem 15.4,

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & x_1 + y_1 + z_1 \\ x_2 & y_2 & x_2 + y_2 + z_2 \\ x_3 & y_3 & x_3 + y_3 + z_3 \end{vmatrix}.$$

The observation that $x_i + y_i + z_i = 1$ finishes the proof.

This then leads us to the equation of a line.

Equation of a Line (22.4)
The general form of a line is $dx + ey + fz = 0$.

Theorem 22.4's Proof

It is well-known two points determine a line. Let the two points be $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$. By Theorem 22.2, we desire for any point (x, y, z) on the line to satisfy

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

The determinant is $xy_1z_2 + yz_1x_2 + zx_1y_2 - zy_1x_2 - z_1y_2x - x_1yz_2$. Since $x_1, y_1, z_1, x_2, y_2, z_2$ are constant, the equation becomes $x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(y_1x_2 - x_1y_2) = 0$, which is enough to finish our proof.

Remark: The equation of a line passing through $(x_1, y_1, z_1), (x_2, y_2, z_2)$ is $x(y_1z_2 - z_1y_2) + y(z_1x_2 - x_1z_2) + z(y_1x_2 - x_1y_2) = 0$.

Line Through a Vertex (22.5)

A line that passes through A has general equation $\frac{y}{z} = k$ for constant k. Symmetric expressions exist for lines passing through B, C.

Theorem 22.5's Proof

Note that $(0, d, 1-d)$ represents the point our line intersects BC. We use $(1, 0, 0), (0, d, 1-d)$, substitute, and get $y(1-d) + z(-d) = 0$, or $y(1-d) = zd$, or $\frac{y}{z} = \frac{d}{1-d}$. Since d is constant for any given line passing through A, we are done.

Concurrency by Barycentric Coordinates (22.6)

Lines $u_ix + v_iy + w_iz = 0$ for $i = 1, 2, 3$ concur if and only if

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0.$$

Theorem 22.6's Proof

This is a system of equations for given constants and variables x, y, z. It's a well-known fact that this determinant needs to be 0 for a solution to exist.

Concurrency Again (22.7)

Lines $u_ix + v_iy + w_iz = 0$ for $i = 1, 2, 3$ concur if and only if

$$\begin{vmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{vmatrix} = 0.$$

Theorem 22.7's Proof

Determinants satisfy the property that

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & u_1+v_1+w_1 \\ u_2 & v_2 & u_2+v_2+w_2 \\ u_3 & v_3 & u_3+v_3+w_3 \end{vmatrix}.$$

The observation that $u_i + v_i + w_i = 1$ finishes the proof.

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1. Prove Ceva's Theorem (6.5) using barycentric coordinates.

2. Prove Menelaus' Theorem (6.6) using barycentric coordinates.

3. Find the equation of line BC.

4. Find the equation of the A median of $\triangle ABC$.

5. Consider $\triangle ABC$ with $\angle A = 45^\circ, \angle B = 60^\circ$, and with circumcenter O. If BO intersects CA at E and CO intersects AB at F, find $\frac{[AFE]}{[ABC]}$.

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1. Prove Ceva's Theorem (6.5) using barycentric coordinates.

Solution: The mass points definition of barycentric coordinates states that $\overline{BD}/\overline{DC} = z/y, \overline{CE}/\overline{EA} = x/z, \overline{AF}/\overline{FB} = y/x,$ where D, E, F are the intersections of AP with BC, CA, AB respectively. Since AD, BE, CF are concurrent cevians by definition, then $z/y \cdot x/z \cdot y/x = 1,$ as desired.

Alternatively, note that D, E, F must have barycentric coordinates of the form $(0, d, 1-d), (1-e, 0, e), (f, 1-f, 0).$ Lines AD, BE, CF have equations $\frac{z}{y} = \frac{1-d}{d}$, $\frac{x}{z} = \frac{1-e}{e}$, $\frac{y}{x} = \frac{1-f}{f}$, implying that $1 = \frac{(1-d)(1-e)(1-f)}{def}$, as desired.

2. Prove Menelaus' Theorem (6.6) using barycentric coordinates.

Solution: This is actually quite disgusting.

Let D, E, F be on BC, CA, AB and let D, E, F have coordinates $(0, d, 1-d), (1-e, 0, e), (f, 1-f, 0).$ This means that our statement is equivalent to $|\frac{def}{(1-d)(1-e)(1-f)}| = 1,$ when using directed lengths. We then note the equation of FD is

$$\begin{vmatrix} x & 0 & f \\ y & d & 1-f \\ z & 1-d & 0 \end{vmatrix} = 0,$$

for some arbitrary $x, y, z.$ This simplifies to $yf(1-d) - x(1-f)(1-d) - zdf = 0,$ which implies $zfd = yf(1-d) - x(1-f)(1-d).$

Since x, y, z are arbitrary, we just plug in the coordinates of E to get $def = -(1-d)(1-e)(1-f)$ (the negative comes through the way we direct our lengths). Dividing both sides by $(1-d)(1-e)(1-f)$ and taking absolute values completes the proof.

3. Find the equation of line $BC.$

Solution: Substituting the points $(0, 1, 0), (0, 0, 1)$ into our remark gives us $x = 0.$ (Can you generalize for AB, BC by providing equations for them?)

4. Find the equation of the A median of $\triangle ABC.$

Solution: Since a median is a cevian, the A median passes through A. We use Corollary 1 and note that the A median intersects BC at $(0, \frac{1}{2}, \frac{1}{2}) = (0, d, 1 - d)$ where $d = \frac{1}{2}$. Thus, $\frac{y}{z} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}$, implying our equation is $y = z$.

5. Consider $\triangle ABC$ with $\angle A = 45^{\circ}, \angle B = 60^{\circ}$, and with circumcenter O. If BO intersects CA at E and CO intersects AB at F, find $\frac{[AFE]}{[ABC]}$.

Solution: Note that by $\frac{1}{2}ab \cdot \sin(C) = [ABC]$ (5.3), $[AFE] = \frac{1}{2}\sin(45^{\circ}) \cdot \overline{AE} \cdot \overline{AF}$ and $[ABC] = \frac{1}{2}\sin(45^{\circ}) \cdot \overline{AC} \cdot \overline{AB}$. Therefore, $\frac{[AFE]}{[ABC]} = \frac{\overline{AE} \cdot \overline{AF}}{\overline{AC} \cdot \overline{AB}}$. Note that O has barycentrics $(\sin 90^{\circ} : \sin 120^{\circ} : \sin 150^{\circ})$, and by the mass points definition, $\frac{\overline{EA}}{\overline{CE}} = \frac{\sin 90^{\circ}}{\sin 150^{\circ}}$ and $\frac{\overline{AF}}{\overline{FB}} = \frac{\sin 120^{\circ}}{\sin 90^{\circ}}$, so $\frac{\overline{EA}}{\overline{EA}+\overline{CE}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\sin 90^{\circ}}{\sin 90^{\circ}+\sin 150^{\circ}} = \frac{1}{1+0.5} = \frac{2}{3}$ and $\frac{\overline{AF}}{\overline{AF}+\overline{FB}} = \frac{\overline{AF}}{\overline{AB}} = \frac{\sin 120^{\circ}}{\sin 120^{\circ}+\sin 90^{\circ}} = \frac{\sqrt{3}/2}{\sqrt{3}/2+1} = \frac{\sqrt{3}/2(1-\sqrt{3}/2)}{(1+\sqrt{3}/2)(1-\sqrt{3/2})} = \frac{\sqrt{3}/2-3/4}{1/4} = 2\sqrt{3} - 3$. Then we see that $\frac{\overline{AE} \cdot \overline{AF}}{\overline{AC} \cdot \overline{AB}} = \frac{\overline{AE}}{\overline{AC}} \cdot \frac{\overline{AF}}{\overline{AB}} = \frac{2}{3} \cdot (2\sqrt{3} - 3) = \frac{4\sqrt{3}-6}{3}$, so $\frac{[AFE]}{[ABC]} = \frac{4\sqrt{3}-6}{3}$.

Before we talk about Evan's Favorite Forgotten Trick (Hi Evan!), we need to talk about displacement vectors. We define *displacement vector* $\vec{MN}$ where $M = (x_1, y_1, z_1), N = (x_2, y_2, z_2)$ as $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$. Keep in mind that the coordinates of a displacement vector sum to 0.

Lemma 1

When $\vec{O} = \vec{0}$, $\vec{A} \cdot \vec{A} = R^2$, where R denotes the circumradius.

Lemma 1's Proof

Let O be the circumcenter, then use the fact that $\vec{u} \cdot \vec{u} = |u|^2$, which comes trivially by the fact that $\theta = 0$.

Lemma 2

When $\vec{O} = \vec{0}$, $\vec{A} \cdot \vec{B} = R^2 - \frac{c^2}{2}$, where R denotes the radius. (Cyclic variations hold.)

Lemma 2's Proof

Again, let O be the circumcenter. Then we get

$$\vec{A} \cdot \vec{B} = R^2 \cos(\angle AOB) = R^2 \cos(2\angle ACB) = R^2(1 - 2\sin^2 C) = R^2 - \frac{1}{2}(2R \cdot \sin(C))^2 = R^2 - \frac{c^2}{2}.$$

These results come from the Inscribed Angle Theorem (1.1), Sum/Difference Identities (9.1) and from Extended Law of Sines (9.2).

Evan's Favorite Forgotten Trick (22.6)

Consider displacement vectors $\vec{MN}, \vec{PQ} = (x_1, y_1, z_1), (x_2, y_2, z_2)$. Then $MN \perp PQ$ if and only if $a^2(y_1z_2 + z_1y_2) + b^2(z_1x_2 + x_1z_2) + c^2(x_1y_2 + y_1x_2) = 0.$

Theorem 22.6's Proof

Let $\vec{O} = \vec{0}$. By Theorem 13.3, it is necessary and sufficient that $(x_1\vec{A}, y_1\vec{B}, z_1\vec{C}) \cdot (x_2\vec{A}, y_2\vec{B}, z_2\vec{C}) = 0$. Expanding yields

$\sum_{cyc}(x_1x_2\vec{A} \cdot \vec{A}) + \sum_{cyc}((x_1y_2 + y_1x_2)\vec{A} \cdot \vec{B}) = 0$. We use Lemma 1 and Lemma 2 to get

$\sum_{cyc}(x_1x_2R^2) + \sum_{cyc}(x_1y_2 + y_1x_2)(R^2 - \frac{c^2}{2}) = 0$, which implies

$R^2(\sum_{cyc}(x_1x_2) + \sum_{cyc}(x_1y_2 + y_1x_2)) = R^2(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = 0 = \frac{1}{2}\sum_{cyc}((x_1y_2 + x_2y_1)c^2)$. (Recall that displacement vectors have coordinates that sum up to zero!) Multiplying both sides of the final equation by 2 yields

$$0 = \sum_{cyc}(x_1y_2 + x_2y_1)c^2 = a^2(y_1z_2 + z_1y_2) + b^2(z_1x_2 + x_1z_2) + c^2(x_1y_2 + y_1x_2).$$

We present two corollaries from Evan's as exercises; the solutions will be presented afterwards.

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1. Given displacement vector $\vec{PQ} = (x_1, y_1, z_1)$, prove that $BC \perp PQ$ if and only if $a^2(z_1 - y_1) + z_1(c^2 - b^2) = 0.$

2. Prove that the perpendicular bisector of BC can be expressed as $a^2(x - y) + z(c^2 - b^2).$

---

1. Given displacement vector $\vec{PQ} = (x_1, y_1, z_1),$ prove that $BC \perp PQ$ if and only if $a^2(z_1 - y_1) + z_1(c^2 - b^2) = 0.$

Solution: We note that displacement vector $\vec{BC}$ has coordinates $(0, 1, -1).$ By Evan's, $a^2(z_1 - y_1) + z_1(c^2 - b^2) = 0,$ which comes directly by substitution.

2. Prove that the perpendicular bisector of BC can be expressed as $a^2(x - y) + z(c^2 - b^2).$

Solution: Note that $\vec{BC}$ has coordinates $(0, 1, -1)$ and that any point on PQ must at have the form $(0 - x, \frac{1}{2} - y, \frac{1}{2} - z).$ This comes by plugging the midpoint Q in and any arbitrary point on the perpendicular bisector. We can let arbitrary P have coordinates $(x, y, z).$

Plugging this into Evan's yields $a^2(\frac{1}{2} - y - \frac{1}{2} + z) + b^2(-x) + c^2(x) = 0.$ Simplifying yields $a^2(x - y) + z(c^2 - b^2) = 0,$ as desired.

Strong Evan (22.7)

Given points $M, N, P, Q,$ let $\vec{MN} = x_1\vec{AO} + y_1\vec{BO} + z_1\vec{CO}$ and let $\vec{PQ} = x_2\vec{AO} + y_2\vec{BO} + z_2\vec{CO}.$ If $x_i + y_i + z_i = 0$ for either $i = 1, 2,$ then $MN \perp PQ$ if and only if $a^2(y_1z_2 + z_1y_2) + b^2(z_1x_2 + x_1z_2) + c^2(x_1y_2 + y_1x_2) = 0.$

Theorem 22.7's Proof

We already have $\vec{O} = 0$ as our reference $\vec{A}, \vec{B}, \vec{C}$ have tail $O.$ The proof is identical until

$R^2(x_1 + y_1 + z_1)(x_2 + y_2 + z_2) = \frac{1}{2}\sum_{cyc}((x_1y_2 + x_2y_1)c^2).$ Then, we have $R^2 \cdot 0 \cdot (x_i + y_i + z_i)$ instead of $R^2 \cdot 0 \cdot 0$ for the left side. However, this is still $0,$ so the proof proceeds identically.

We'll discuss the distance formula and circles now, and provide a few corollaries as exercises.

Distance Formula (22.8)

Given displacement vector $\vec{PQ} = (x, y, z),$ $|PQ|^2 = -a^2yz - b^2zx - c^2xy.$

Theorem 22.8's Proof

We utilize the fact that $PQ^2 = (x\vec{A} + y\vec{B} + z\vec{C}) \cdot (x\vec{A} + y\vec{B} + z\vec{C})$. This yields

$$PQ^2 = (x\vec{A} + y\vec{B} + z\vec{C}) \cdot (x\vec{A} + y\vec{B} + z\vec{C}) =$$

$$(x + y + z)(|A|^2 x + |B|^2 y + |C|^2 z) - yz|B - C|^2 - xz|A - C|^2 - xy|A - B|^2 =$$

$-a^2yz - b^2zx - c^2xy,$ as desired. (The reason we can get rid of $x + y + z$ is because $x + y + z = 0$ by definition.)

Equation of a Circle (22.9)

The general equation of a circle is $-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0.$

Theorem 22.9's Proof

Let the circle have center (i, j, k) and radius r. Then we use the Distance formula and note that this is $-a^2(y - j)(z - k) - b^2(z - k)(x - i) - c^2(x - i)(y - j) = r^2$. Expanding yields $-a^2yz - b^2zx - c^2xy + Lx + My + Nz = C$ for constants L, M, N, C. Since $x + y + z = 1,$ we rewrite the right side as $C(x + y + z),$ and subtracting yields $-a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$ where $u = L - C, v - M - C, w = N - C$.

---

1. Prove the circumcircle has equation $a^2yz + b^2zx + c^2xy = 0.$

---

1. Prove the circumcircle has equation $a^2yz + b^2zx + c^2xy = 0$.

Solution: Three points signify a unique circle; plugging in A, B, C completes the proof.

Finally, as a warning against mindlessly using barycentric coordinates, I present the following exercise.

---

1. Consider $\triangle ABC$ with $\overline{AB} = 13, \overline{BC} = 15, \overline{CA} = 14$. If M is the midpoint of BC and P is a point on AC such that $MP \perp AC$, find $\overline{MP}$.

---

1. Consider $\triangle ABC$ with $\overline{AB} = 13, \overline{BC} = 15, \overline{CA} = 14.$ If M is the midpoint of BC and P is a point on AC such that $MP \perp AC,$ find $\overline{MP}.$

Solution: This is the well-known $13-14-15$ triangle, so the B altitude has length $12.$ Using similar triangles, we see there's a ratio of $\frac{1}{2},$ so $MP = 12 \cdot \frac{1}{2} = 6.$

Don't you think this problem would've been annoying if we used barycentric coordinates?

We'll smuggle in a couple more points for this chapter which we have not introduced yet. (The most prominent is probably the isogonal conjugate.)

The nine-point center has barycentrics $(a\cos(B-C) : b\cos(C-A) : c\cos(A-B)).$

Isogonal Conjugates in Barycentrics (22.10)

Given a point P with barycentrics $(x, y, z),$ its isogonal conjugate P^* has barycentrics $(\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}).$

Theorem 22.10's Proof

Let AP, AP^* intersect BC at $D, E.$ It is a property of isogonal conjugates that $\frac{\overline{BD}\cdot\overline{BE}}{\overline{CE}\cdot\overline{CD}} = \frac{c^2}{b^2}.$ This implies $\frac{\overline{BD}}{\overline{CD}} = \frac{c^2\overline{CE}}{b^2\overline{BE}},$ and by the mass points definition, this means $\frac{\overline{BD}}{\overline{CD}} = \frac{y}{b^2} \cdot \frac{c^2}{z}.$ Thus, the barycentrics of P^* satisfy $\frac{z^*}{y^*} = \frac{c^2/z}{b^2/y}$ by the mass points definition, and symmetrically, $\frac{y^*}{x^*} = \frac{b^2/y}{a^2/x},$ implying $(x^* : y^* : z^*) = (\frac{a^2}{x}, \frac{b^2}{y}, \frac{c^2}{z}),$ as desired.

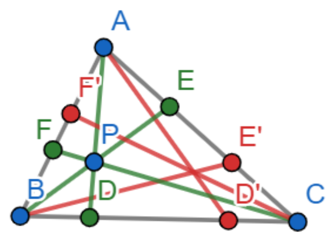
Isotomic Conjugates in Barycentrics (22.11)

The isotomic conjugate of a point with barycentrics (x, y, z) has barycentrics $(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$

Theorem 22.11's Proof

We use the mass points definition.

Let AP, BP, CP intersect BC, CA, AB at D, E, F respectively. Reflect D, E, F about their respective medians to get $D', E', F'.$ The mass points definition should then make it obvious that the barycentric coordinates reciprocate, as $\overline{BD} = \overline{CD'}$ and likewise.



Symmedian Point (22.12)

The symmedian point has barycentrics $(a^2 : b^2 : c^2)$.

Theorem 22.12's Proof

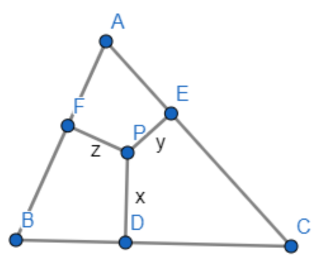
Trivial by Theorem 22.10.

The isogonal conjugate's barycentrics is an alternate proof for the fact that the circumcenter and orthocenter are isogonal conjugates, but these basic properties would need to be developed regardless.

Trilinear Coordinates

Trilinear coordinates are the less well-known cousin of Barycentric coordinates. Whereas barycentric coordinates describe ratios of areas, trilinear coordinates describe ratios of lengths.

The trilinear coordinates of a point P with respect to $\triangle ABC$ are (x, y, z), which denote the directed distances from P to BC, CA, AB respectively. The triple (x, y, z) is normalized if x, y, z denote the actual directed lengths, and it is not normalized if $(x : y : z)$ reflects the ratios of the lengths but not the actual lengths.



It's fairly easy to see if a certain coordinate is positive or negative; the process is identical to the counter-clockwise process with barycentric coordinates.

Instead of giving the trilinear coordinates for well-known points, I will instead give the general conversion formula.

Barycentric to Trilinear (23.1)

If a point P with respect to $\triangle ABC$ has barycentric coordinates $(x : y : z)$, then it has trilinear coordinates $[\frac{x}{a} : \frac{y}{b} : \frac{z}{c}]$. Conversely, if P has trilinear coordinates $[x : y : z]$, then it has barycentric coordinates $(xa : yb : zc)$.

Theorem 23.1's Proof

Use $[ABC] = \frac{bh}{2}$ (4.2) and let the distances from P to BC, CA, AB respectively be x, y, z. Then it has trilinears $[x : y : z]$ and barycentric coordinates $(xa : yb : zc)$ by definition.

A few straightforward conversion exercises are presented.

---

1. Find the trilinear coordinates of the incenter.

2. Find the trilinear coordinates of the centroid.

3. Find the trilinear coordinates of the orthocenter.

4. Find the trilinear coordinates of the circumcenter.

5. Find the trilinear coordinates of the vertices.

---

1. Find the trilinear coordinates of the incenter.

Solution: By the definition of an incenter, it is equidistant from the three sides of the triangle. Thus it has trilinears $(1 : 1 : 1)$.

2. Find the trilinear coordinates of the centroid.

Solution: Conversion from barycentrics gives us $(\frac{1}{a} : \frac{1}{b} : \frac{1}{c})$.

3. Find the trilinear coordinates of the orthocenter.

Solution: Note $\tan(A) = \frac{\sin(A)}{\cos(A)}$ and symmetric expressions for B, C. Dividing by a yields $\frac{\sin(A)}{a} \cdot \frac{1}{\cos(A)} = \frac{1}{R} \cdot \frac{1}{\cos(A)}$, by the Law of Sines (9.1). Thus the orthocenter has trilinears $(\frac{1}{R\cdot\cos(A)} : \frac{1}{R\cdot\cos(B)} : \frac{1}{R\cdot\cos(C)})$ which can be further simplified to $(\frac{1}{\cos(A)} : \frac{1}{\cos(B)} : \frac{1}{\cos(C)})$.

4. Find the trilinear coordinates of the circumcenter.

Solution: Note that by the Double Angle Identities (10.2), $\sin(2A) = 2\sin(A)\cos(A)$ and symmetric identities. Then dividing by a and symmetric for the other two coordinates, we get $\frac{2\sin(A)}{a} \cdot \cos(A) = \frac{2\cos(A)}{R}$ by the Law of Sines (9.1). Thus the circumcenter has trilinears $(\cos(A) : \cos(B) : \cos(C))$.

5. Find the trilinear coordinates of the vertices.

Solution: We don't even need to convert since the distance of a vertex from two lines is zero. Thus we have A, B, C as $(1,0,0), (0,1,0), (0,0,1)$, respectively.

We see that the orthocenter and circumcenter are relatively nicer. Any other points you could think of are equally trivial; the excenters are an example of this.

Another formula that might be useful is the normalized form of the trilinears $(x : y : z)$.

Normalizing Trilinears (23.2)

Given trilinears $(x : y : z)$, the actual distances are $(x\frac{2[ABC]}{ax+by+cz}, y\frac{2[ABC]}{ax+by+cz}, z\frac{2[ABC]}{ax+by+cz})$.

Theorem 23.2's Proof

Let the actual distances be (x', y', z'), and let $Kx = x', Ky = y', Kz = z'$. We write the barycentrics of our point as $[\triangle A, \triangle B, \triangle C]$. Then it is obvious $\triangle A + \triangle B + \triangle C = [ABC]$. Then note $\triangle A = \frac{1}{2}ax', \triangle B = \frac{1}{2}by', \triangle C = \frac{1}{2}cz'$, implying $[ABC] = \frac{1}{2}(ax' + by' + cz')$. Then our substitution comes into play, giving us $[ABC] = \frac{1}{2} \cdot K(ax + by + cz)$. Rearranging yields $K = \frac{2[ABC]}{ax+by+cz}$, meaning that $x' = xK, y' = yK, z' = zK$ becomes $x' = x\frac{2[ABC]}{ax+by+cz}, y' = \frac{2[ABC]}{ax+by+cz}, z' = z\frac{2[ABC]}{ax+by+cz}$, as desired.

This theorem will be useful to calculate actual distances when ratios are specifically given for a known triangle, or if barycentrics are given (since conversion is straightforward).

Converting all of the barycentric formulas (EFFT, Area, Distance) is an exercise in uselessness. As far as I know, people don't use trilinears for Olympiad problems because barycentric is a much better technique. However, there are a couple of techniques that are nicer in trilinears.

Trilinear Isogonal Conjugates (23.3)

The isogonal conjugate of a point P with trilinears (x, y, z) has trilinear coordinates $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$.

Theorem 23.3's Proof

Conversion from barycentrics proves this.

Isogonal Conjugate of a Function (23.4)

Given trilinear function $f(x, y, z) = 0$, its isogonal conjugate is $f(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}) = 0.$

Note: The isogonal conjugate of a shape is the mapping of every point on it to its isogonal conjugate respective to reference $\triangle ABC$.

Theorem 23.4's Proof

This is obvious due to Theorem 23.3.

---

1. Show that the isogonal conjugate of the circumcircle is the line at infinity. (The line of infinity is the line where all the points at infinity lie, and the points at infinity can be thought of as intersections of parallel lines.)

2. Find the isogonal conjugate of line BC.

1. Show that the isogonal conjugate of the circumcircle is the line at infinity. (The line of infinity is the line where all the points at infinity lie, and the points at infinity can be thought of as intersections of parallel lines.)

Solution: The circumcircle has equation $f(x,y,z) = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$ The isogonal conjugate of the circumcircle is $f(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}) = \frac{a}{\frac{1}{x}} + \frac{b}{\frac{1}{y}} + \frac{c}{\frac{1}{z}} = ax + by + cz = 0.$ This is equivalent to $x + y + z = 0$ for barycentrics, and this has no solutions (thus it is the line at infinity).

2. Find the isogonal conjugate of line $BC.$

Solution: Line BC has barycentrics $x = 0,$ so it has trilinears $\frac{x}{a} = 0,$ which is still $x = 0.$ Since $\frac{1}{x} = 0$ has no solutions, this is the line at infinity.

Looking at these theorems about isogonal conjugates, we see that lines become equations with degree 2 after multiplying by $xyz.$ In Cartesian and trilinear coordinates, things with degree 1 are lines. Cartesian equations with degree 2 are conics. It is very natural to ask if this holds true for trilinears. The answer turns out to be yes.

Second Degree Conic (23.5)

Any degree 2 equation in trilinears is a conic.

Theorem 23.5's Proof

By a modified vector definition of barycentrics, trilinears get converted to Cartesians at a constant ratio. Thus the degree 2 equation in trilinears becomes a degree 2 equation in Cartesian, which is obviously a conic.

Conic by Intersections to Line at Infinity (23.6)

If a conic intersects the line of infinity at 0 points, it is an ellipse.
If a conic intersects the line of infinity at 1 point, it is a parabola.
If a conic intersects the line of infinity at 2 points, it is a hyperbola.

Theorem 23.6's Proof

An ellipse never "shoots off" into infinity, so it never intersects the line of infinity.
A parabola is one continuous curve, so it only "shoots off" once, intersecting once.
A hyperbola is two curves, so it "shoots off" twice, intersecting twice.

Though trilinear coordinates seem like a “repeat” of barycentrics (and this is very close to how I treat it), they are very helpful for isogonal conjugates, which will come later on.

Miscellaneous Algebraic Problems

After complex numbers, vectors and matrices, and barycentric coordinates comes some lighter reading. We will discuss a couple of common types of problems, and some "geometrical problems" that can be reduced to algebra.

Type 1 - Distance Between Two Points

The shortest distance between two points is a line. The distance between a point and a line/plane is the length of the perpendicular.

---

1. What is the distance between the points $(5, 7)$ and $(8, 3)$?

2. What is the distance between the origin and the line $2x + 3y = 12$?

---

1. What is the distance between the points $(5,7)$ and $(8,3)$?

Solution: The Distance Formula (10.3) gives us a distance of
$\sqrt{(5-8)^2+(7-3)^2}=\sqrt{9+16}=\sqrt{25}=5.$

2. What is the distance between the origin and the line $2x+3y=12$?

Solution: The shortest distance between a point and a line is the length of the perpendicular from the point to the line (see Theorem 10.3).

The perpendicular line has a slope of the negative reciprocal and must pass through the origin, so our perpendicular line is $3x-2y=0$. Solving this system gives us $x=\frac{24}{13}, y=\frac{36}{13}$, and Distance Formula (10.3) gives us a distance of $\frac{12\sqrt{13}}{13}$.

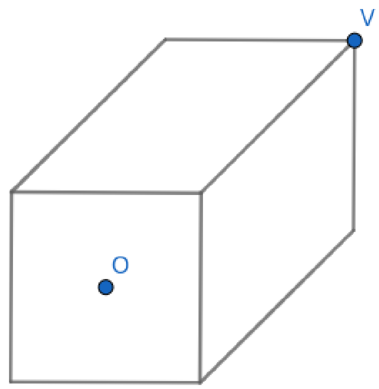
These types of problems are relatively straightforward.

Type 2 - Crawling on a Cube

This usually involves an ant crawling on the surface from one point of a rectangular prism to another point on rectangular prism. Though the ant cannot go directly from one point to the other, a transformation that preserves distance and makes the problem “distance of two points” usually solves it.

---

1. Given a $10 \times 10 \times 7$ rectangular prism, let O the center of the 10×10 square and let V a vertice of the opposite 10×10 square. If the ant crawls from O to V on the surface of the prism, what is the shortest distance he could travel?

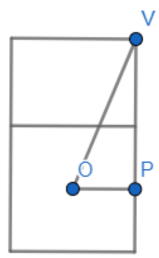


2. Given a $10 \times 10 \times 7$ tank, let O the center of the 10×10 square and let V a vertice of the opposite 10×10 square. If the ant crawls two units per second from O to V on the surface of the prism and swims one unit per second, what is the shortest time he could travel?

---

1. Given a $10 \times 10 \times 7$ rectangular prism, let O the center of the 10×10 square and let V a vertice of the opposite 10×10 square. If the ant crawls from O to V on the surface of the prism, what is the shortest distance he could travel?

Solution: Take the square with O and a rectangle with V and flatten it out. Note that $\overline{IP} = 5$, and adding $7 + \frac{10}{2} = 12$ gives us a $5 - 12 - 13$ triangle, so the shortest distance is 13.



2. Given a $10 \times 10 \times 7$ tank, let O the center of the 10×10 square and let V a vertice of the opposite 10×10 square. If the ant crawls two units per second from O to V on the surface of the prism and swims one unit per second, what is the shortest time he could travel?

Solution: By a modified version of the Triangle Inequality, either the shortest path is a direct swim or a direct crawl. We already know the direct crawl takes 13 units, or 6.5 seconds. If the ant directly swims, it takes $\sqrt{5^2 + 5^2 + 7^2} = \sqrt{99} = 3\sqrt{11}$ seconds to get there. Thus, the shortest path is the crawl, taking 6.5 seconds.

The second one is usually heuristically solved, as in a competition it would "make sense" for the shortest path to either be the crawl, or the swim. Additionally, a third travelling method (walking on the edges) may be added, which is straightforward to factor in as well.

Type 3 - X-quidistant Locus of Points

Given line segment AB of known length $x,$ the locus of points P such that $\overline{AP} = c\overline{BP}$ for some known c can be found by letting A be the origin and B be on the x axis and using the distance formula.

---

1. What is the locus of points such that $\overline{AP} = \overline{BP}$?

2. Given $\overline{AB} = 6,$ what is the size of the region bounded by the locus of points P such that $\overline{AP} = \frac{1}{2}\overline{BP}$?

---

1. What is the locus of points such that $\overline{AP} = \overline{BP}$?

Solution: Let A be the origin and let B be $(k, 0)$. Then let the locus of points be $P = (x, y)$ such that $\overline{AP} = \overline{BP}$ be expressed with the distance formula. We desire $\sqrt{x^2 + y^2} = \sqrt{(k - x)^2 + y^2}$. Clearly, we desire $x = k - x \rightarrow k = \frac{1}{2}x$ which is the perpendicular bisector of AB.

2. Given $\overline{AB} = 6$, what is the size of the region bounded by the locus of points P such that $\overline{AP} = \frac{1}{2}\overline{BP}$?

Solution: Let A be the origin and let B be $(6, 0)$. Then the locus of points $P = (x, y)$ can be expressed with the distance formula as thus:
$\sqrt{x^2 + y^2} = \frac{1}{2}\sqrt{(6 - x)^2 + y^2} = \frac{1}{2}\sqrt{x^2 - 12x + 36 + y^2}$. Squaring yields $x^2 + y^2 = \frac{1}{4}x^2 - 3x + 9 + \frac{1}{4}y^2 \rightarrow 4x^2 + 4y^2 = x^2 - 12x + 36 + y^2 \rightarrow 3x^2 + 3y^2 + 12x = 36$ and rearranging yields $x^2 + y^2 + 4x + 4 = 12 \rightarrow (x + 2)^2 + y^2 = 16$. Thus, our area is $(\sqrt{16})^2\pi = 16\pi$.

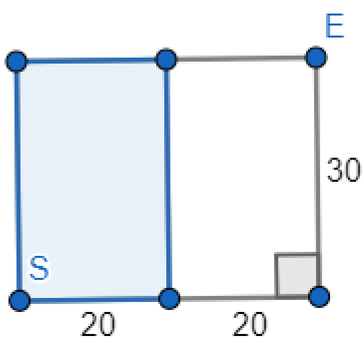
This type of process is identical for higher-dimension spaces, such as three-dimensional x-quidistant problems.

Type 4 - Driving Through Mud

This usually starts as a driver starting at mud and trying to get to a point on the highway where he drives faster or a driver starting on the highway and trying to get to a point on the mud where he drives slower. (This is identical because it's the same path regardless.) This can be solved using the Pythagorean Theorem and scaling.

---

1. If a driver is in mud 20 feet away from the highway and the driver needs to go 40 miles east and 30 miles north, what is the shortest time he could take, if he goes 1 feet per second in the mud and 5 feet per second on the highway? (Refer to the diagram. Assume mud is sky-blue in this universe.)



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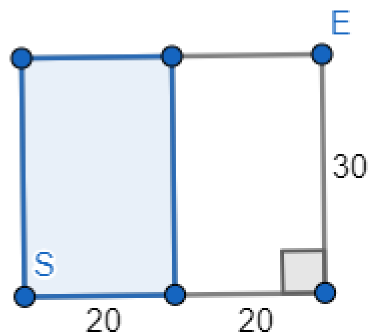
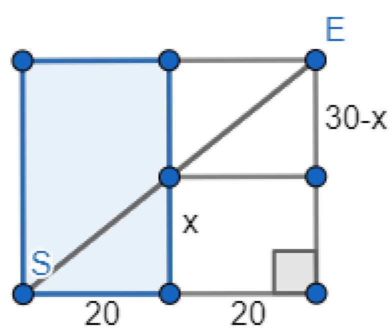
1. If a driver is in mud 20 feet away from the highway and the driver needs to go 40 miles east and 30 miles north, what is the shortest time he could take, if he goes 1 feet per second in the mud and 2 feet per second on the highway? (Refer to the diagram. Assume mud is sky-blue in this universe.)



Solution: Refer to the diagram below; this is how our driver has to drive since the shortest distance between two points is a line and it only matters how he approaches the barrier between mud and highway. Then note that our time is $\sqrt{x^2+20^2}+\frac{1}{2}\sqrt{(30-x)^2+20^2}$. By weighted QM-AM, our minimum is $\sqrt{x^2+20^2}=\sqrt{(30-x)^2+20^2}$, or $x=15$ (note x is non-negative) which gives us a straight line. Thus, we have $30+\frac{1}{2}\cdot 30=45$ as our answer.



This method is generalizable. It does not matter how many different regions there are, or what speeds you go in them; the shortest time between two points will still always be a line.

<u>Transformations</u>

Generic Transformations

We will study two different types of generic transformations, talk about how they can be used for problems in math competitions, and explore an interesting definition for congruence for weirder shapes.

A *rigid transformation* is a transformation such that for any two points A, B and their images $A', B,$ $\overline{AB} = \overline{A'B'}$.

A *homothety* preserves similarity between points A, B, C and A', B', C' with a constant ratio.

A *non-rigid transformation* is a transformation that is not necessarily rigid. We will explore examples of things that are generally non-rigid but are rigid in a few specific cases.

The three general rigid transformations are reflections, rotations, and translations.

A *reflection about a line* is defined as thus; for point A and line $l,$ its image A' is the point such that l is the perpendicular bisector of AA'.

A *reflection about a point* is defined as thus; for point A and point P to reflect about, its image A' is the point such that P is the midpoint of AA'. Note that in two dimensions, this is a rotation about 180°.

A *rotation* is defined as thus; to rotate A around P by θ degrees (this is assumed to be counterclockwise unless otherwise stated), its image A' is on the circle formed with P as the center and A as a point on the circle where $arc(AA')$ is θ degrees (this is directed the same way the rotation is, which is counterclockwise unless otherwise stated.)

A *translation* is a transformation such that for any A and its image $A',$ the vector $\vec{A'A}$ is the same.

The two generic *non-rigid transformations* are homotheties and stretches.

We use directed distances for homothety. A *homothety* with center P and ratio k sends A to A' such that $\overline{PA'} = k\overline{PA}$. This is denoted as $H(P, k)$. (We will explore homotheties more in-depth later.)

Stretches about $P = (x, y)$ for a point $A = (x_a, y_a)$ gives us its image A'. If this is a horizontal stretch with factor k, then we have $A' = (k(x_a - x), y_a)$, and if this is a vertical stretch with factor k, then we have $A' = (x_a, k(y_a - y))$. This only is applicable for a coordinate plane, as we need a reference for our perpendicular directions “horizontal” and “vertical,” which comes from the perpendicular axes. Note that though some people use shrink for $-1 < k < 1$, we will use stretch for any value of k.

Now we go over some problems that can be solved with transformations.

Type 1 - Running to a River

What is the shortest path from $A = (x_a, y_a)$ to $B = (x_b, y_b)$ that passes through the line $ax + by = c$?

Solution of Type 1

This problem has two cases.

Case 1: The direct path between A, B passes through the line.

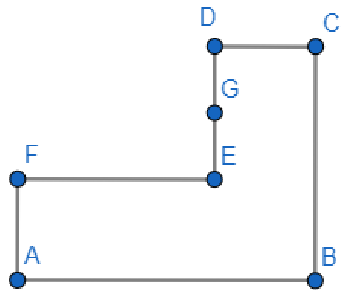
Solution: Then the shortest path is just $\overline{AB} = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$.

Case : The direct path between A, B does not pass the line.

Solution: Reflect B across the line to get B'. Then find $\overline{AB'}$ using the distance formula.

---

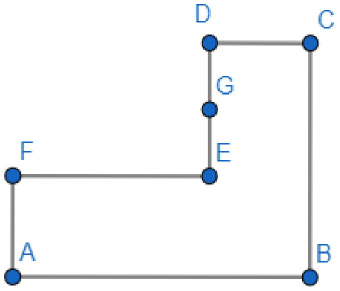
1. In the diagram, assume all angles are 90°. If we are bouncing a laser from A and we want it to hit G, what is the shortest path the laser can take, if it can bounce off of walls? Assume that $\overline{AB} = 9, \overline{BC} = 7, \overline{CD} = 3, \overline{DG}, = 2, \overline{GE} = 2, \overline{EF} = 6, \overline{FA} = 3.$



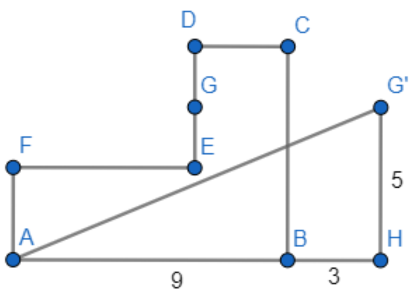
2. Consider $\triangle ABC$ with $\overline{AB} = 5$, $\overline{BC} = 7$, and $\overline{CA} = 4\sqrt{2}$. Let H be the foot of the altitude from A to BC. If P is a point on AC, find the minimum value of $\overline{BP} + \overline{HP}$.

3. Consider $\triangle ABC$ with $\overline{AB} = 13$, $\overline{BC} = 14$, and $\overline{CA} = 15$. Let H be the foot of the altitude from A to BC. If P is a point on AC, find the minimum value of $\overline{BP} + \overline{HP}$.

1. In the diagram, assume all angles are 90°. If we are bouncing a laser from A and we want it to hit G, what is the shortest path the laser can take, if it can bounce off of walls? Assume that $\overline{AB} = 9, \overline{BC} = 7, \overline{CD} = 3, \overline{DG}, = 2, \overline{GE} = 2, \overline{EF} = 6, \overline{FA} = 3.$

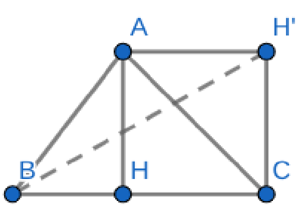


Solution: Note that the direct path from A to G is not available, so we want to "run to the river," or go to BC, to arrive at G. We use our solution and reflect G across CB to get G', and by Pythagorean's Theorem, $\overline{AG'} = 13$, which is our desired answer.



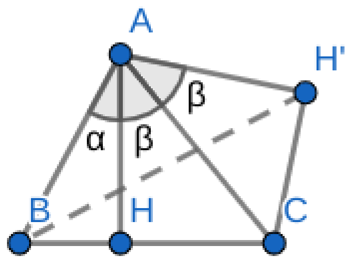
2. Consider $\triangle ABC$ with $\overline{AB} = 5$, $\overline{BC} = 7$, and $\overline{CA} = 4\sqrt{2}$. Let H be the foot of the altitude from A to BC. If P is a point on AC, find the minimum value of $\overline{BP} + \overline{HP}$.

Solution: Notice that $\overline{BH} = 3$, $\overline{AH} = 4$, and $\overline{CH} = 4$. Reflect H about AC to create square $AHCH'$. By the definition of reflection, $\overline{BP} + \overline{HP} = \overline{BP} + \overline{H'P} \geq \overline{BH'}$. By the Pythagorean Theorem, $\overline{BH'} = \sqrt{7^2 + 4^2} = \sqrt{65}$.



3. Consider $\triangle ABC$ with $\overline{AB} = 13$, $\overline{BC} = 14$, and $\overline{CA} = 15$. Let H be the foot of the altitude from A to BC. If P is a point on AC, find the minimum value of $\overline{BP} + \overline{HP}$.

Solution: The difference here is that $\overline{AH}$ is not equal to $\overline{HC}$. Nevertheless, we will still reflect. Notice that $\overline{AH} = 12$, $\overline{BH} = 5$, and $\overline{HC} = 9$. We then use the Law of Cosines (9.2) to find the length of BH'. Let $\angle BAH = \alpha$ and let $\angle CAH = \beta$. Then we want to find $\cos(\alpha + 2\beta)$. To do this, we first find $\cos(2\beta)$. By the Double Angle Identity (10.2), $\cos(2\beta) = 2\cos^2(\beta) - 1$. By the definition of cosine, $\cos(\beta) = \frac{4}{5}$, so $\cos(2\beta) = \frac{7}{25}$. By the Sum/Difference Identities (10.1), $\cos(\alpha + 2\beta) = \cos(\alpha)\cos(2\beta) - \sin(\alpha)\sin(2\beta) = \frac{12}{13} \cdot \frac{7}{25} - \frac{5}{13} \cdot \frac{24}{25} = \frac{-36}{13 \cdot 25}$. Then by the Law of Cosines (9.2), $\overline{BH'} = \frac{\sqrt{8689}}{5}$, which is our answer.



Type 2 - Paper Folding

Folding a piece of paper causes a reflection about the line of folding.

If you forget your compass on a test, you can fold a triangle's line segments in half to get the circumcenter. Folding point A to point B draws the perpendicular bisector of AB. (Remember that reflecting A across any line to A' means our line is the perpendicular bisector of AA', and that the converse is true as well.)

Similarly, you can fold the A altitude of $\triangle ABC$ by folding a line such that A maps to itself and line BC maps to itself. (Notice this is line, not line segment!)

The median can be created easily as well. (It cannot be folded, however, unless you directly fold the line with A and its corresponding midpoint.)

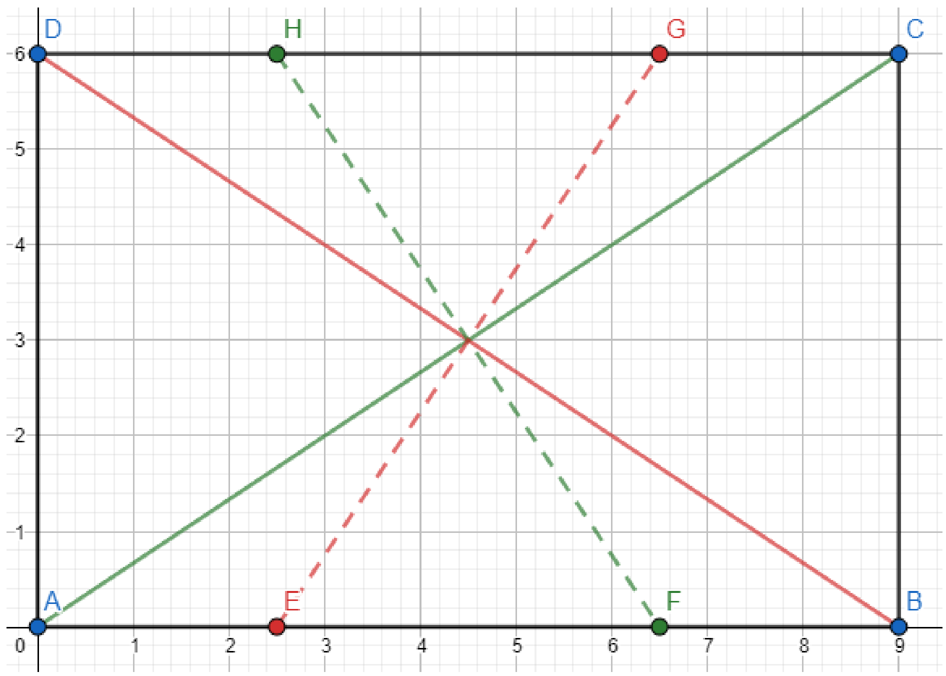
---

1. Consider a 6×9 rectangular piece of paper $ABCD$. Fold A onto C and B onto D. Our two folds and the sides of the rectangle will create two triangles. What is the sum of the areas of our two triangles?

2. (MPP Entrance Exam) Have a square $ABCD$ with side lengths of 1. Point E is the midpoint of segment $\overline{AB}$. Take the vertice D and fold the square so that D coincides with E. Let our crease intersect BC, DA at F, G, respectively. Find $\overline{AG} + \overline{BF}$.

1. Consider a 6×9 rectangular piece of paper $ABCD$. Fold A onto C and B onto D. Our two folds and the sides of the rectangle will create two triangles. What is the sum of the areas of our two triangles?

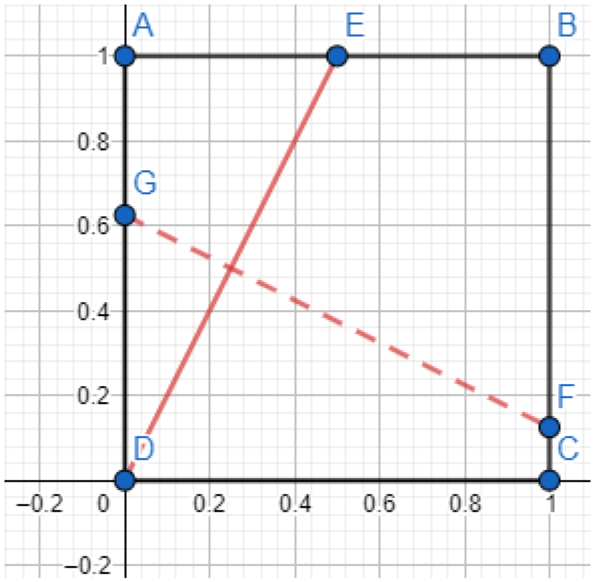
Solution: We put this on a coordinate plane.



Notice that since the slope of AC is $\frac{9}{6}$, the slope of its perpendicular bisector EF is $\frac{-3}{2}$, and similarly, the slope of GE is $\frac{3}{2}$. With this, we notice $EF = GE = 6 \cdot \frac{1}{\frac{3}{2}} = 4$, and $\frac{bh}{2}$ (4.2) gives us the sum of the areas as $\frac{4 \cdot 3}{2} + \frac{4 \cdot 3}{2} = 12$.

2. (MPP Entrance Exam) Have a square $ABCD$ with side lengths of 1. Point E is the midpoint of segment $\overline{AB}$. Take the vertice D and fold the square so that D coincides with E. Let our crease intersect BC, DA at F, G, respectively. Find $\overline{AG} + \overline{BF}$.

Solution: We can graph this. Let $A = (0, 1), B = (1, 1), C = (1, 0), D = (0, 0)$. Then $E = (\frac{1}{2}, 1)$. We graph the perpendicular bisector of DE as that is the line of reflection, and this ends up being $y = -\frac{1}{2}x + \frac{5}{8}$. We have $G = (0, \frac{5}{8})$ and $F = (1, \frac{1}{8})$, so the sum of the lengths is $\frac{3}{8} + \frac{7}{8} = \frac{3}{2}$.



Type 3 - Rotating (x, y) *around* (l, h)

An easier way to do this is let (l, h) be the origin and add back (l, h) after rotating (x, y).

---

1. Rotate $(2, 7)$ about $(\sqrt{3}, 5)$ 15° counterclockwise.

2. Rotate $(8, 0)$ about $(4\sqrt{3}, 0)$ 15° counterclockwise. Then rotate our new point about $(4\sqrt{3} - 2\sqrt{2}, 4\sqrt{6} - 6\sqrt{2})$ by 105°.

---

1. Rotate $(2,7)$ about $(\sqrt{3},5)$ 15° counterclockwise.

Solution: Let $(\sqrt{3},5)$ be the origin. Then, our translation gives us our point to be rotated as $(2-\sqrt{3},2)$. Notice that its magnitude is $\sqrt{11+4\sqrt{3}}$. Rotating $(\sqrt{11+4\sqrt{3}},0)$ around the origin by 30° yields $(\cos(30^{\circ})\sqrt{11+4\sqrt{3}},\sin(30^{\circ})\sqrt{11+4\sqrt{3}})$, and the translation yields $(2+\cos(30^{\circ})\sqrt{11+4\sqrt{3}},7+\sin(30^{\circ})\sqrt{11+4\sqrt{3}})$, which is equivalent to $(2+\frac{\sqrt{33+12\sqrt{3}}}{2},7+\frac{\sqrt{11+4\sqrt{3}}}{2})$.

2. Rotate $(8,0)$ about $(4\sqrt{3},0)$ 15° counterclockwise. Then rotate our new point about $(4\sqrt{3}-2\sqrt{2},4\sqrt{6}-6\sqrt{2})$ by 105°.

Solution: Notice that the horizontal distance is $8-4\sqrt{3}$ and the vertical distance is 0. With our knowledge of $15^{\circ}-75^{\circ}-90^{\circ}$ triangles, we notice that our new point is $(4\sqrt{3}+\sin 15^{\circ}[8-4\sqrt{3}],\cos 15^{\circ}[8-4\sqrt{3}])=(4\sqrt{3}+\sqrt{6}-\sqrt{2},3\sqrt{6}-5\sqrt{2})$.

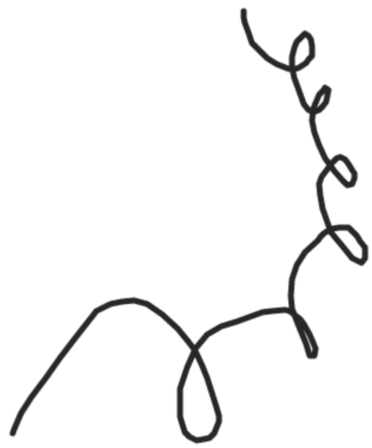
Notice then that a 105° rotation is a 75° rotation and a 30° rotation. Then notice the distance between $(4\sqrt{3}+\sqrt{6}-\sqrt{2},3\sqrt{6}-5\sqrt{2})$ and $(4\sqrt{3}-2\sqrt{2},4\sqrt{6}-6\sqrt{2})$ is $2\sqrt{3}+2$, and that the angle the center point creates with the rotated point is 75°, so our point after a 75° rotation is $(4\sqrt{3}-2\sqrt{2}-[2\sqrt{3}+2],4\sqrt{6}-6\sqrt{2})=(2\sqrt{3}-2\sqrt{2}-2,4\sqrt{6}-6\sqrt{2})$.

Set this as the origin and notice that we create a $30-60-90$ triangle, but the x and y value go down (that is, the rotated point is in the third quadrant). So our point is $(-\sqrt{3}-1,-3-\sqrt{3})$. Translating using our original point $(2\sqrt{3}-2\sqrt{2}-2,4\sqrt{6}-6\sqrt{2})$ yields $(\sqrt{3}-2\sqrt{2}-3,4\sqrt{6}-\sqrt{3}-6\sqrt{2}-3)$ as our final answer.

Reflections can be done the same way, but it is generally trivial enough to find the reflected point.

We end this section with a formal definition of congruence. While our normal definition of congruence works with polygons and ellipses, we can extend it to cover all sorts of shapes.

We define *congruence* of two objects as the ability to map one onto the other via a set of rigid transformations. (Remember that they preserve length, angle, and type!) Now we have a definition for weird shapes like this.

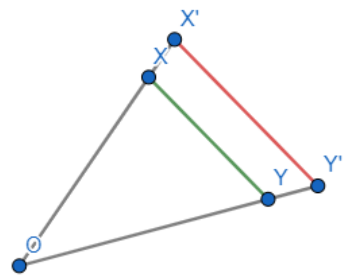


The converse is true. To make a congruent copy of an object, we must use rigid transformations as well.

A *direct isometry* preserves orientation as well as congruence. This is the ability to map one onto the other via a set of rotations and translations. (This is irrelevant at some times, but for things where orientation matters, such as barycentrics and trilinears, the difference is key.)

Homothety

A *homothety* is a dilation about point O. The homothety about O with ratio k is denoted $H(O,k)$. The dilation sends $\vec{OX} \to k\vec{OX} = \vec{OX'}$. For formatting's sake, we omit the O and say $\vec{X} \to k\vec{X} = \vec{X'}$.



There are many interesting properties that make such a simple transformation so powerful.

Property 1: Collinearity
If a homothety centered at O sends $X \to X'$, then O, X, X' are collinear.

Property 2: Parallelism
Parallelism is preserved. (This is trivial from Theorem 1.)

Property 3: Angles
Angles are preserved. (See Theorem 2 for a direct proof.)

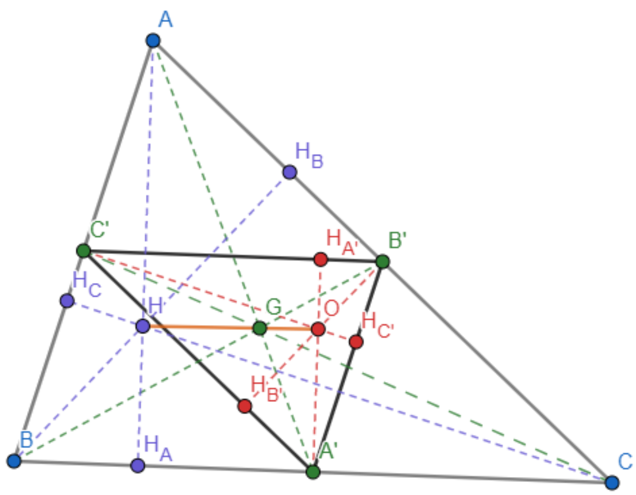
Property 4: Similarity
The image is similar to the preimage. (Either use a combination of Property 3 and the definition, or just use the similarity definition provided earlier by similitude.)

These tools (and some common sense) will be very powerful.

Euler Line (24.1)
Let the orthocenter, centroid, and circumcenter of $\triangle ABC$ be H, G, O. Then H, G, O are collinear and $\overline{HG} = 2\overline{GO}$.

Theorem 24.1's Proof

Draw the medial triangle of $\triangle ABC$, and let A' be the midpoint of BC, B' be the midpoint of CA, and C' the midpoint of AB. Then let the centroid be G. Notice the homothety $H(G, -\frac{1}{2})$ sends H to O, the orthocenter of $\triangle A'B'C'$. (Notice that $A'O, B'O, C'O$ are altitudes of $\triangle A'B'C'$ and are perpendicular bisectors of $\triangle ABC$, justifying the claim.) By Property 1, we are done. Furthermore, the ratio is also proven as $\frac{\overline{GO}}{GH} = |-\frac{1}{2}|$.



Nine-Point Circumcircle (24.2)

Consider $\triangle ABC$ with orthocenter H. Then let A_H be the reflection of H about BC and let A_M be the reflection of H about the midpoint of BC. Then A_H, A_M lie on the circumcircle of $\triangle ABC$.

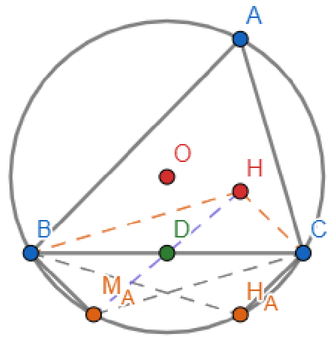
Since there is nothing special about A, the same holds true for B, C.

Theorem 24.2's Proof

First, we prove A_H lies on the circumcircle. This is equivalent to proving ABA_HC is cyclic, or that $\angle CA_HB = 180^\circ - \angle A$.

Notice that $\angle CHB = \angle CA_HB$ by the definition of a reflection. Letting D be the foot of the A altitude, notice that $\angle CHB = \angle CHD + \angle BHD = 180^\circ - \angle HCB - \angle HBC$. Notice that $\angle HBC = 90^\circ - \angle C$ and $\angle HCB = 90^\circ - \angle B$ due to right triangles made by extending BH, CH, respectively. Plugging this in yields $\angle BHC = \angle B + \angle C = 180^\circ - \angle A$, as desired.

Now we prove that A_M lies on the circumcircle. This is equivalent to proving $\angle CA_MB = 180^\circ - \angle A$. But CA_MBH is a parallelogram because both diagonals bisect one another, so the result is trivially true.



Nine-Point Circle (24.3)

Let H be the orthocenter of $\triangle ABC$, and D, E, F be the midpoints of BC, CA, AB. Then let X, Y, Z be the midpoints of AH, BH, CH. Also, let the feet of the A, B, C altitudes be P, Q, R. Then $D, E, F, X, Y, Z, P, Q, R$ are concyclic, and their circumcircle is the midpoint of OH, where O is the circumcenter of $\triangle ABC$.

Theorem 24.3's Proof

Apply the homothety $H(H, \frac{1}{2})$ to the Nine-Point Circumcircle (24.2). Our reflections will be the aforementioned points, and the circumcenter will shift to the midpoint of OH.

The secret that most geometry textbooks won't tell you is that the Euler line and Nine-Point Circle configurations, among other common homothety problems, is that these aren't based on the concept of homothety. The concept of homothety does make it much easier to understand the main idea, but the main idea isn't homothety. It's the fact that the circumcenter of a triangle is the orthocenter of its medial triangle. Notice how most, if not all, of these problems rely somehow on this fact.

---

1. A homothety about O sends $\triangle ABC \to \triangle DEF$. Prove AD, BE, CF concur at O.

2. Consider $\triangle ABC$ with orthocenter H and circumcenter O. Let X, Y, Z be the midpoints of AH, BH, CH and let D, E, F be the midpoints of BC, CA, AB. Prove that XD, YE, ZF are concurrent.

3. Let $\triangle ABC$ have circumcenter O and centroid G. Then let A', B', C' be on segments AO, BO, CO such that $\frac{\overline{OA'}}{\overline{OA}} = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{OC'}}{\overline{OC}} = \frac{1}{3}$. Prove that G is the orthocenter of $\triangle A'B'C'$.

4. Consider $\triangle ABC$ with circumcenter O and orthocenter H. If M is the midpoint of BC and $\overline{AH} = 8$, find $\overline{MO}$.

5. Consider $\triangle ABC$ with circumcenter O. Let AO, BO, CO intersect the circumcircle at D, E, F. If H is the orthocenter of $\triangle ABC$, and G' is the centroid of $\triangle DEF$, find $\frac{HO}{G'O}$.

6. Consider $\triangle ABC$ and points P, Q. For which pairs of P, Q is it possible for the image of $\triangle ABC$ after a non-identity dilation about P and Q to be identical?

7. Consider $\triangle ABC$ with $AB = 5, BC = 6$, and $CA = 7$. Let G be the centroid of $\triangle ABC$, and G_A, G_B be the reflections of A, B respectively over G. Find $[CG_AGG_B]$.

8. Consider $\triangle ABC$ with incenter I. Let the incircle of $\triangle ABC$ intersect sides BC, CA, AB at D, E, F respectively. Then let AD, BE, CF intersect the incircle again at X, Y, Z respectively. Let G' and H' be the centroid and orthocenter of $\triangle XYZ$. Prove that I, G', H' are collinear.

---

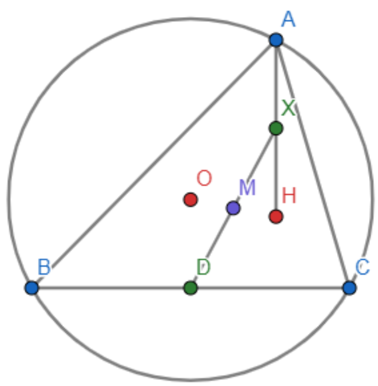
1. A homothety about O sends $\triangle ABC \to \triangle DEF$. Prove AD, BE, CF concur at O.

Solution: By Property 1, O, A, D, O, B, E, and O, C, F are collinear.

2. Consider $\triangle ABC$ with orthocenter H and circumcenter O. Let X, Y, Z be the midpoints of AH, BH, CH and let D, E, F be the midpoints of BC, CA, AB. Prove that XD, YE, ZF are concurrent.

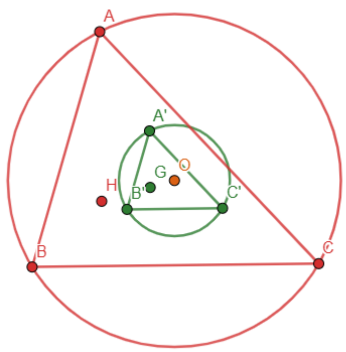
Solution: The conditions of X, Y, Z remind us of the nine-point circle, so let us set it up. Let M be the center of the nine-point circle. If we apply the homothety $H(H, 2)$, we send M to O, X to A, and D to a specific point A_M on the circumcircle, by the Nine-Point Circle (24.3).

Then the crucial observation is that A, O, A_M are collinear, which would imply X, M, D are collinear and completing the problem (due to no loss of generality). A little bit of angle chasing indeed proves that AA_M is a diameter of the circumcircle, as desired.



3. Let $\triangle ABC$ have circumcenter O and centroid G. Then let A', B', C' be on segments AO, BO, CO such that $\frac{\overline{OA'}}{\overline{OA}} = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{OC'}}{\overline{OC}} = \frac{1}{3}$. Prove that G is the orthocenter of $\triangle A'B'C'$.

Solution: Notice that A', B', C' are the results of A, B, C when applying $H(O, \frac{1}{3})$. However, by the Euler Line (24.1), notice that G is the result of $H(O, \frac{1}{3})$ upon H, so we are done.



4. Consider $\triangle ABC$ with circumcenter O and orthocenter H. If M is the midpoint of BC and $\overline{AH} = 8,$ find $\overline{MO}.$

Solution: Notice that O is the result of H and M is the result of A after the homothety $H(G,-\frac{1}{2}).$ Since lengths are altered based on the ratio of homothety, $\overline{MO} = |-\frac{1}{2}| \cdot \overline{AH} = 4.$

5. Consider $\triangle ABC$ with circumcenter $O.$ Let AO, BO, CO intersect the circumcircle at D, E, F. If H is the orthocenter of $\triangle ABC,$ and G' is the centroid of $\triangle DEF$, find $\frac{\overline{HO}}{\overline{G'O}}.$

Solution: Notice that $\triangle DEF$ is the result of $\triangle ABC$ after the homothety $H(O,-1).$ Thus, if G is the centroid of $\triangle ABC,$ it is well-known that $\frac{\overline{HO}}{\overline{GO}} = 3,$ so $\frac{\overline{HO}}{\overline{G'O}} = 3$ as $\overline{GO} = \overline{G'O}.$

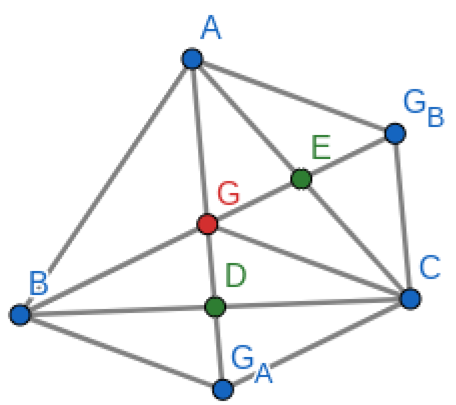
6. Consider $\triangle ABC$ and points $P, Q.$ For which pairs of P, Q is it possible for the image of $\triangle ABC$ after a non-identity dilation about P and Q to be identical?

Solution: We claim that only $P = Q$ works.

Assume otherwise. Notice that we can treat P, Q as triangle centers. Thus, if we can align the images of $\triangle ABC,$ we need to get the images of P and Q to align. However, a dilation of P about P has image P and a dilation of P about Q cannot have image P. Of course, this is untrue if $P = Q,$ which is our only solution.

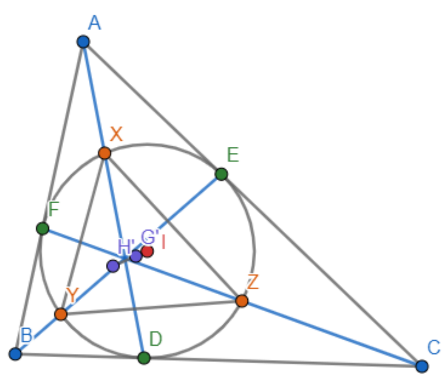
7. Consider $\triangle ABC$ with $AB = 5, BC = 6$, and $CA = 7.$ Let G be the centroid of $\triangle ABC,$ and G_A, G_B be the reflections of A, B respectively over $G.$ Find $[CG_AGG_B].$

Solution: For notation, refer to the diagram below. By the properties of the centroid, $\frac{\overline{GD}}{\overline{GA}} = \frac{1}{2}$, so $\overline{GD} = \overline{G_AD}$. This implies that $[CDG_A] = \frac{[BCG_A]}{2} = \frac{[BCG]}{2} = \frac{[ABC]}{6}$. Similarly, $[CEG_B] = \frac{[ABC]}{6}$. It is also a property of the centroid that $[CGD] = [CGE] = \frac{1}{6}$. Note that $[CG_AGG_B] = [CDG_A] + [CGD] + [CGE] + [CEG_B] = 4 \cdot \frac{[ABC]}{6} = \frac{2[ABC]}{3}$. Now we apply Heron's Formula (5.6) on $\triangle ABC$, which yields $[ABC] = 6\sqrt{6}$. Thus, $[CG_AGG_B] = \frac{2 \cdot 6\sqrt{6}}{3} = 4\sqrt{6}$.



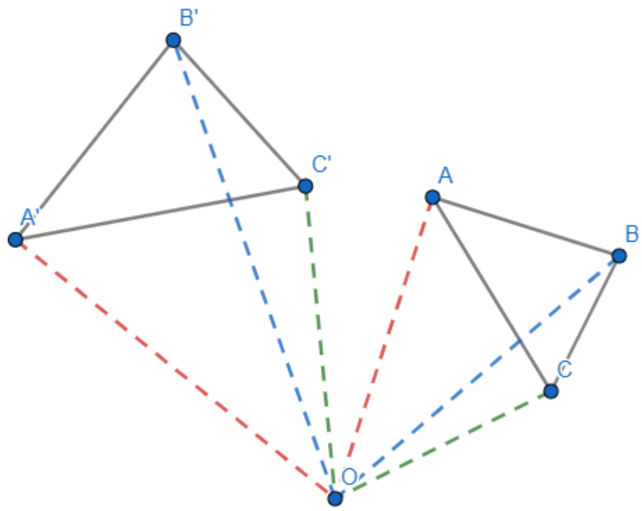
8. Consider $\triangle ABC$ with incenter I. Let the incircle of $\triangle ABC$ intersect sides BC, CA, AB at D, E, F respectively. Then let AD, BE, CF intersect the incircle again at X, Y, Z respectively. Let G' and H' be the centroid and orthocenter of $\triangle XYZ$. Prove that I, G', H' are collinear.

Solution: Note that I is the circumcenter of $\triangle XYZ$ by definition. By the Euler Line (24.1), H', G', I concur.



Similitude

A *similitude* is the combination of a rotation and dilation about a shared center O. We call O the center of similitude. (It is also known as a *spiral similarity*, because spiral similarities cause similar triangles with the same orientation.)



1. Consider a similitude centered around O. If $\measuredangle AOA' = 30^\circ$ (the angle AOA' is 30° counterclockwise) and $2\overline{BB'} = \overline{BO}$, find the scale of similitude.

1. Consider a similitude centered around O. If $\measuredangle AOA' = 30^\circ$ (the angle AOA' is 30° counterclockwise) and $2\overline{BB'} = \overline{BO}$, find the scale of similitude.

Solution: Without loss of generality, let $\overline{BO} = 2, \overline{BB'} = 1$. Then let $\overline{B'O} = r$. By the Law of Cosines, $2^2 + r^2 - 4r\cos(30^\circ) = 1$. Simplifying gives us $r^2 - 2r\sqrt{3} + 3 = 0$. The only value of r is $\sqrt{3}$, so $\frac{\sqrt{3}}{2}$ is the scale of our similitude.

Taking a look at the complex plane, a similitude centered around $O = (0,0)$ takes the form $f(z) = \alpha z$, where $|\alpha|$ is the scale and $\arg(\alpha)$ is the angle of rotation. Similarly, if z_o is the origin, then the similitude about z_o takes the shape $f(z) = z_o + \alpha(z - z_o)$.

Now we'll prove some facts about similitudes.

Unique Similitude Theorem (25.1)

Given four points A, B, C, D such that $ABCD$ is not a parallelogram, there is a unique similitude that sends $A(|\alpha|, \arg\alpha) \mapsto B$ and $C(|\alpha|, \arg\alpha) \mapsto D$.

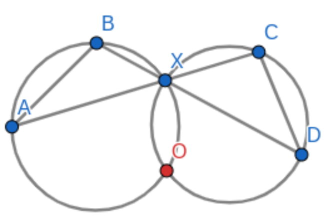
Theorem 25.1's Proof

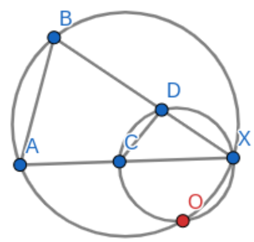
Let $f(z) = z_o + \alpha(z - z_o)$ and let a, b, c, d be the complex representations of A, B, C, D. Then notice this amounts to solving $z_o + \alpha(a - z_o) = c$ and $z_o + \alpha(b - z_o) = d$ for known a, b, c, d. Solving yields $\alpha = \frac{c-d}{a-b}$ and $z_o = \frac{ad-bc}{a+d-b-c}$.

The uniqueness condition follows trivially from the complex interpretation.

Locating the Center of Similitude (25.2)

Consider A, B, C, D such that AC is not parallel to BD. Let AC intersect BD at X. Then let the circumcircles of $\triangle ABX$ and $\triangle CDX$ meet again at O. Then O is the center of the similitude such that $A \to C, B \to D$.





Theorem 25.2's Proof

Because similitude cares about the orientation of a triangle, we use directed angles mod 180°. (This means lines, not rays!) This means four points $A, B, C, D,$ are concyclic if and only if $\angle(AB, BC) = \angle(AD, DC)$.

Notice $\angle(OA, AC) = \angle(OA, AX) = \angle(OB, BX) = \angle(OB, BD)$ and $\angle(OC, CA) = \angle(OC, CX) = \angle(OD, DX) = \angle(OD, DB)$ by the Inscribed Angle Theorem (1.1). It then follows that $\triangle AOC \sim \triangle BOD$ and the two triangles have the same orientation. Thus, there is a similitude centered at O such that $A \to C, B \to D$.

Similitude Pairs (25.3)

If O is the center of the similitude from $A \to C, B \to D,$ then O also is the center of the similitude from $A \to B, C \to D$.

Theorem 25.3's Proof

We use directed angles once more due to orientation.

Notice $\angle(AO, OB) = \angle(CO, OD)$ as similitudes preserve angles. Also, $r = \frac{OC}{OA} = \frac{OD}{OB}$. Rearranging yields $\frac{OB}{OA} = \frac{OD}{OC}$. Then the similitude of angle $\angle(AO, OB) = \angle(CO, OD)$ and dilation of $\frac{OB}{OA} = \frac{OD}{OC}$ centered at O sends $A \to B, C \to D$.

---

1. Prove that any $\triangle ABC \sim \triangle DEF$ with the same orientation has a center O such that some similitude about O sends $\triangle ABC \to \triangle DEF$ given that none of the 15 lines formed by the 6 points A, B, C, D, E, F are parallel.

2. Prove that two triangles related by a similitude are similar.

3. Consider $\triangle ABC$ and its medial triangle $\triangle A'B'C'$. Prove that both triangles share the same centroid G and that a similitude on $\triangle ABC$ yields $\triangle A'B'C'$.

4. Consider normal (i.e. non self-intersecting convex) quadrilateral $ABCD$. Let O be the unique point such that $\triangle ABO \sim \triangle CDO$ with the two triangles having the same orientation, and let X be the intersection of AC and BD. Prove $\angle AXB = \angle COD$.

5. Given $A_1A_2...A_n \sim B_1B_2...B_n$ with the same orientation, prove that for all $M_1, M_2...M_n$ such that $\frac{\overline{A_iM_i}}{\overline{M_iB_i}} = r$ where r is constant for all $1 \le i \le n$, $M_1M_2...M_n \sim A_1A_2...A_n \sim B_1B_2...B_n$.

6. Consider directly similar $\triangle ABC \sim \triangle DEF$. Let X, Y, Z be the intersection of AD and BE, BE and CF, and CF and AD, respectively. Prove that $\triangle XYZ$ is isosceles.

7. A *complete quadrilateral* is defined by four lines and their six points of intersection. Prove that the four circumcircles of the four triangles determined by each triplet of lines are concurrent.

8. Consider $\triangle ABC$ and points P, Q. For which pairs of P, Q is it possible for the image of $\triangle ABC$ after a non-identity dilation about P and Q to be identical?

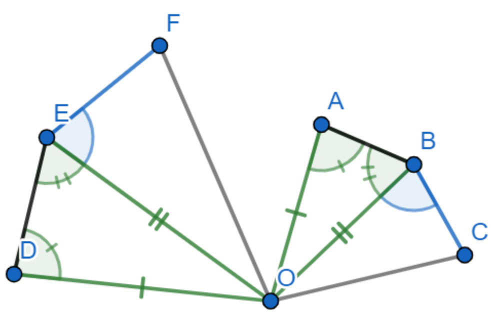
---

1. Prove that any $\triangle ABC \sim \triangle DEF$ with the same orientation has a center O such that some similitude about O sends $\triangle ABC \rightarrow \triangle DEF$ given that none of the 15 lines formed by the 6 points A, B, C, D, E, F are parallel.

Solution: We use directed angles because we care about orientation.

Let the center of similitude from $A, B \rightarrow D, E$ be O. We know this exists by Theorem 25.1.

Then we are given that $\triangle ABO \sim \triangle DEO$, meaning that the sides share a common ratio and the angles are congruent. This gives us $\angle(OA, AB) = \angle(OD, DE)$ and $\angle(OB, BA) = \angle(OE, ED)$ as well as $\frac{\overline{AB}}{DE} = \frac{\overline{OA}}{OD} = \frac{\overline{OB}}{OE} = r$. Then we notice that the second condition gives us $\angle(OB, BC) = \angle(OE, EF)$ by simple addition. Using SAS gives us $\triangle BCO \sim \triangle EFO$, and $\frac{\overline{BO}}{EO} = \frac{\overline{CO}}{FO}$. This gives us our constant ratio of similarity r and common rotational angle of $\angle(AO, OD) = \angle(BO, OE) = \angle(CO, OF)$, as desired.



2. Prove that two triangles related by a similitude are similar.

Solution: Without loss of generality, let our two triangles be $\triangle ABC \sim \triangle DEF$ such that $A \rightarrow B \rightarrow C, D \rightarrow E \rightarrow F$. Then by SAS, we have $\triangle ABO \sim \triangle BCO \sim \triangle DEO \sim \triangle EFO$. Since $\angle DEO = \angle ABO$ and $\angle FEO = \angle CBO$, we have $\angle DEF = \angle ABC$. By SAS, $\triangle ABC \sim \triangle DEF$.

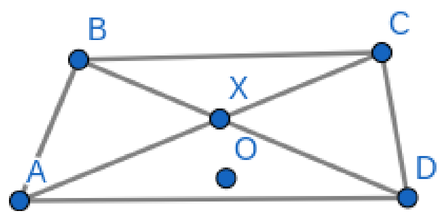
3. Consider $\triangle ABC$ and its medial triangle $\triangle A'B'C'$. Prove that both triangles share the same centroid G and that a similitude on $\triangle ABC$ yields $\triangle A'B'C'$.

Solution: The two triangles are clearly similar by a ratio of $\frac{1}{2}$, so there exists some center of similitude.

Then notice that by similar triangles, AA' passes through the midpoint of $B'C'$, so the medians of $\triangle ABC$ are also the medians of $\triangle A'B'C'$.

4. Consider normal (i.e. non self-intersecting convex) quadrilateral $ABCD$. Let O be the unique point such that $\triangle ABO \sim \triangle CDO$ with the two triangles having the same orientation, and let X be the intersection of AC and BD. Prove $\angle AXB = \angle COD$.

Solution: Notice that our condition on O implies $\angle(AO, OB) = \angle(CO, OD)$ and $r = \frac{AO}{BO} = \frac{CO}{DO}$, meaning O is the center of spiral similarity from $A \to B, C \to D$. By Theorem 25.3, O also is the center of spiral similarity from $A \to C, B \to D$. By Theorem 25.2, A, B, X, O and C, D, X, O are concyclic. It is obvious that $\angle AXB = \angle CXD$ and by the Inscribed Angle Theorem (1.1), $\angle CXD = \angle COD$, and by the transitive property, $\angle AXB = \angle COD$, as desired.



5. Given $A_1A_2...A_n \sim B_1B_2...B_n$ with the same orientation, prove that for all $M_1, M_2...M_n$ such that $\frac{\overline{A_iM_i}}{\overline{M_iB_i}} = r$ where r is constant for all $1 \le i \le n$, $M_1M_2...M_n \sim A_1A_2...A_n \sim B_1B_2...B_n$.

Solution: Let O be the center of similitude of $A_1A_2 \to B_1B_2$. Then we know that O is also the center of similitude such that $A_1B_1 \to A_2B_2$, or $\triangle A_1B_1O \sim \triangle A_2B_2O$, by Theorem 25.3. But $\triangle OA_1M_1 \sim \triangle OA_2M_2$ which gives us $\angle M_1OM_2 = \angle M_2OA_1 - \angle M_1OA_1 = \angle M_2OA_1 - \angle M_2OA_2 = \angle A_1OA_2$. Combining this with our ratio condition proves $\triangle OA_1A_2 \sim \triangle OM_1M_2$. We can repeat this for every triangle and we are done.

6. Consider directly similar $\triangle ABC \sim \triangle DEF$. Let X, Y, Z be the intersection of AD and BE, BE and CF, and CF and AD, respectively. Prove that $\triangle XYZ$ is isosceles.

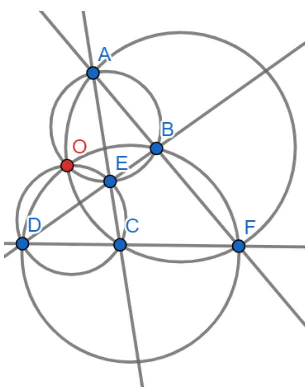
Solution: Let O be the center of similitude that sends $\triangle ABC \to \triangle DEF$. Without loss of generality, let there exist a similitude such that $A \to B \to C, D \to E \to F$.

But by Theorem 25.2, X lies on the circumcircles of $\triangle ABO$ and $\triangle DEO$, and Y lies on the circumcircles of $\triangle DEO$ and $\triangle CFO$. By the Inscribed Angle Theorem (1.1), $\angle AOB = \angle AXB$ and $\angle DOE = \angle DYF$. Then notice that $\angle AXB = \angle DXE = \angle ZXY$ and $\angle EYF = \angle CYB = \angle ZYX$, and $\triangle XYZ$ is isosceles, as desired.

7. A *complete quadrilateral* is defined by four lines and their six points of intersection. Prove that the four circumcircles of the four triangles determined by each triplet of lines are concurrent.

Solution: Refer to the diagram below for how the points are named.

Notice that the circumcircles of $\triangle ABE$ and $\triangle ECD$ intersect again at the center of similitude O that sends $A \to C, B \to D$. But the circumcircles of $\triangle AFC$ and $\triangle BFD$ also intersect at O by Theorem 25.3, as O is the center of similitude that sends $A \to B, C \to D$.



8. Consider $\triangle ABC$ and points P, Q. For which pairs of P, Q is it possible for the image of $\triangle ABC$ after a non-identity dilation about P and Q to be identical?

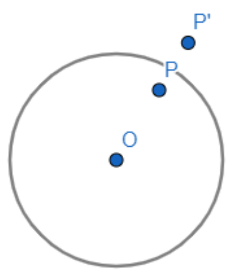
Solution: We claim that only $P = Q$ works.

Assume otherwise. Notice that we can treat P, Q as triangle centers. Thus, if we can align the images of $\triangle ABC$, we need to get the images of P and Q to align. However, a dilation of P about P has image P and a dilation of P about Q cannot have image P. Of course, this is untrue if $P = Q$, which is our only solution.

Inversion

Inversion about a circle is a useful *involution* that reveals a wealth of information about certain geometric configurations. Generally, an involution refers to a function f such that $f(f(x)) = x$ for all x. In this case, it refers to the fact that two inversions about the same circle yields the original diagram.

To invert a point P about circle ω with center O and radius r, we take the unique point P' such that that $\vec{OP'} = \vec{OP} \cdot (\frac{r}{|\vec{OP}|})^2$. (Note: This notation is vector notation, not rays.)



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1. Consider circle ω with center O and radius r. Prove that any inversion about ω such that P is sent to P', $\overline{OP} \cdot \overline{OP'} = r^2$.

2. Prove that for an inversion about circle ω centered at O that sends P to P', O, P, P' are collinear.

3. If you invert point P on circle ω about ω, what point do you get?

---

1. Consider circle ω with center O and radius r. Prove that any inversion about ω such that P is sent to P', $\overline{OP} \cdot \overline{OP'} = r^2$.

Solution: Use the vector definition and take magnitudes. Notice that $\overline{OP'} = \overline{OP} \cdot \frac{r^2}{\overline{OP}^2} = \frac{r^2}{\overline{OP}}$, which implies that $\overline{OP} \cdot \overline{OP'} = r^2$, as desired.

2. Prove that for an inversion about circle ω centered at O that sends P to P', O, P, P' are collinear.

Solution: This is trivially true by the vector definition. If you multiply $\vec{OP}$ by a constant to get $\vec{OP'}$, then O, P, P' are collinear, which is just a property of vectors.

3. If you invert point P on circle ω about ω, what point do you get?

Solution: Let the center be O. Notice that $\overline{OP} = r$, so we want $\overline{OP'} = r$, implying $P = P'$ due to the collinearity condition. Thus you get P itself when inverting.

Consider circle ω with center O. It is relatively well known that inverting a circle not passing through O yields another circle, inverting a circle passing through O yields a line, and inverting a line yields a circle passing through O. However, we should prove this.

We glossed over what happens when we invert the center of a circle. First, we need to discuss the concept of the *point at infinity*. This discussion arises when we ask the question, “What happens if you invert the center of a circle about said circle?” The seemingly obvious answer is, “You can’t.” However, we instead let the point be “the point at infinity.”

Here’s an intuitive explanation why. For standard transformations, if two pre-images intersect, their images intersect. (This is because transformations are a function. The point of intersection cannot go to two different places, after all.) But two circles who pass through the center do intersect, but after an inversion, they become parallel lines. Instead of saying parallel lines do not intersect, we say they intersect at the *point at infinity*.

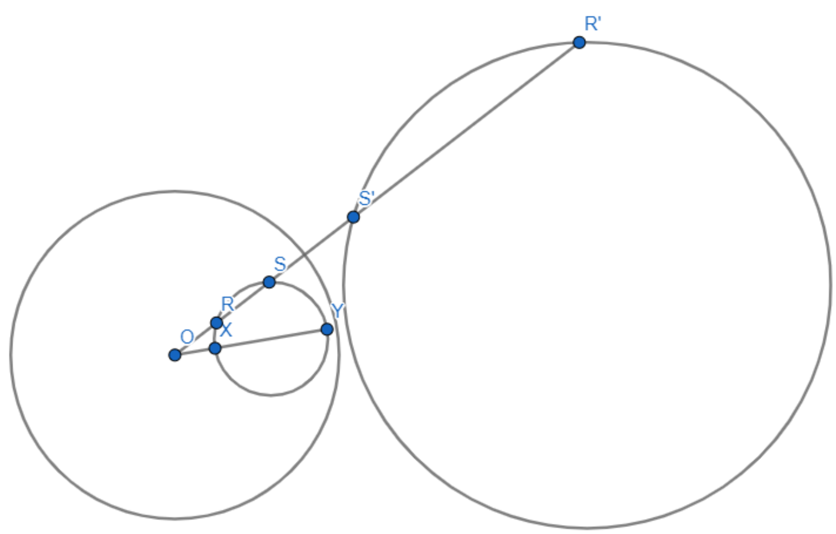
This answers two questions. Where does the center of the circle go? And where does the intersection point of two circles who intersect at the center go?

Circle to Circle (26.1)

Consider circle ω with center O. Then an inversion of a circle that does not pass through O about ω sends said circle to another circle.

Theorem 26.1's Proof

Let the initial circle be Γ. Then draw line OXY such that XY is a diameter of Γ. Then we draw arbitrary ray OR such that OR intercepts Γ at R, S. By the definition of inversion, $\overline{OR} \cdot \overline{OR'} = \overline{OS} \cdot \overline{OS'} = r^2$. Rearranging gets us $\overline{OR'} = \frac{r^2}{\overline{OR} \cdot \overline{OS}} \cdot \overline{OS}$. By Power of a Point (3.2), $\overline{OR} \cdot \overline{OS} = \overline{OX} \cdot \overline{OY}$. So $\overline{OR'} = \frac{r^2}{\overline{OX} \cdot \overline{OY}} \cdot \overline{OS}$. At this point we notice that R' is the result of a dilation of S about O with scale factor $\frac{r^2}{\overline{OX} \cdot \overline{OY}}$, which has been established as a constant. As S traces out the circle, so does R', completing the proof.

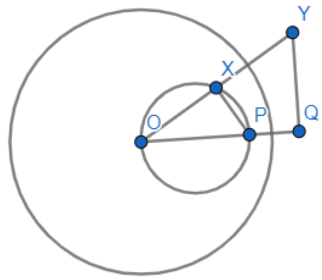


Circle to Line (26.2)

Consider circle ω with center O. Then consider circle Γ passing through O. An inversion of Γ about ω yields a line.

Theorem 26.2's Proof

Let OP be a diameter of Γ. Then let the inversion of P about ω be Q. Then pick any other point X on Γ and let its inversion about ω be Y. By the definition of inversion, $\overline{OP} \cdot \overline{OQ} = \overline{OX} \cdot \overline{OY} = r^2$. Rearranging yields $\frac{\overline{OP}}{\overline{OX}} = \frac{\overline{OY}}{\overline{OQ}}$. This implies $\triangle OPX \sim \triangle OYQ$. By the Inscribed Angle Theorem (1.1), $\angle OXP = 90^\circ$, implying $\angle OQY = 90^\circ$. So the locus of Y is the locus of points such that $OQ \perp QY$, which is a line.

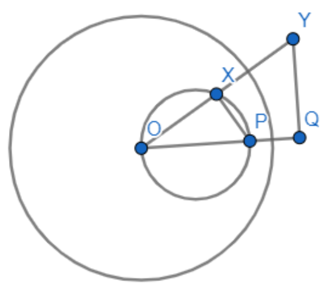


Line to Circle (26.3)

Consider circle ω with center O. Then the inversion of a line about ω yields a circle passing through O.

Theorem 26.3's Proof

Let the line be l. Drop a perpendicular from O to l, and let the foot be Q. Then pick some other point Y on l. Let the inverse of Q be P, and the inverse of Y be X. Then notice $\triangle OXP \sim \triangle OQY$. (This was proved in the proof of Theorem 26.2.) Thus, X is the locus of points such that $\angle OXP = 90^\circ$, which is also known as a circle.



Notice that these proofs are so basically identical, I reused the diagram.

Because of the nature of these transformations and the concept of the point at infinity, sometimes it is desirable to think of lines as circles with infinite radius. We can say three points determine a circle - a line is determined by two points and the point at infinity. Then we call the combination of lines and circles as *generalized circles*.

Now we have a tool to turn collinearity problems into concyclic problems and vice versa. Before we dive a little deeper with poles and polars, let's investigate some of the generic inversions.

1. Consider circle ω with diameter AB. What do you get when you invert line AB about ω?

2. What about inverting segment AB?

3. Construct the inversion of a line.

4. Construct the inversion of a circle passing through the center of the circle of inversion.

5. Construct the inversion of a circle not passing through the center of the circle of inversion.

6. Consider $\triangle ABC$, and let its incircle touch BC, CA, AB at D, E, F, respectively. Prove that an inversion of the circumcircle of $\triangle ABC$ about the incircle of $\triangle ABC$ yields the nine-point circle of $\triangle DEF$.

---

1. Consider circle ω with diameter AB. What do you get when you invert line AB about ω?

Solution: You get line AB. Let the center of ω be O. Then you can pair up all of the points on ray OA into X, X' such that $\overline{OX} \cdot \overline{OX'} = \overline{OA}^2$ with no leftover or overlap. You can do the same thing for ray OB.

2. What about inverting segment AB?

Solution: You get all of line AB except for segment AB. This is because the aforementioned pairs have a group of points inside and a group of points outside the circle. You take the points inside the circle and make them the points outside the circle. (Notice that A and B remain.)

3. Construct the inversion of a line.

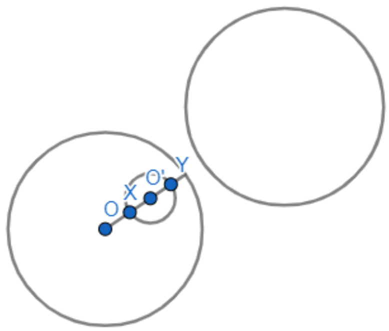
Solution: Let the circle of inversion ω have center O. Then choose the point Q on the line such that OQ is perpendicular to the line. Let P be the result of an inversion of Q about ω. Then draw the circle with diameter OP. (Notice how similar this looks to our diagram for Theorem 26.3?)

4. Construct the inversion of a circle passing through the center of the circle of inversion.

Solution: Let our circle of inversion be ω and let the circle we want to invert be Γ. Let P be on Γ such that OP is a diameter of Γ. Then invert P about ω to get Q. Draw the line passing Q perpendicular to OQ to get your inversion. (Again, notice this is the diagram for Theorem 26.2.)

5. Construct the inversion of a circle not passing through the center of the circle of inversion.

Solution: Let our circle of inversion be ω and let the circle we want to invert be Γ. Let the center of ω be O and let the center of Γ be O'. Then let OO' intersect Γ at X, Y. Then dilate Γ by a factor of $\frac{r^2}{\overline{OX} \cdot \overline{OY}}$, where r is the radius of ω.

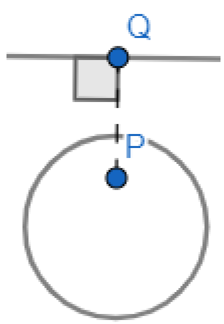


Notice that all these construction problems resulted from the **proofs** of the theorems!

6. Consider $\triangle ABC$, and let its incircle touch BC, CA, AB at D, E, F, respectively. Prove that an inversion of the circumcircle of $\triangle ABC$ about the incircle of $\triangle ABC$ yields the nine-point circle of $\triangle DEF$.

Solution: Since circles go to circles (26.1), we only need to prove three points of the circumcircle of $\triangle ABC$ belong on the nine-point circle of $\triangle DEF$. The easiest points to do this with are A, B, C. Notice that the inverse of A is the midpoint of EF as AI perpendicularly bisects EF. (This is because AE, AF are tangents to the incircle.) Analogously, the inverse of B and C are the midpoints of CA and AB, respectively. Since the midpoints of $\triangle DEF$ are on the nine-point circle of $\triangle DEF$, we are done.

Now we will discuss poles and polars. The *pole* of a point P with respect to ω is the point Q that results from an inversion about ω. (This is merely the inversion point.) The *polar* of point P with respect to ω is the line l through its pole Q such that $PQ \perp l$.



Here's a crucial theorem about polars that most books neglect to mention, let alone prove.

La Hire's (26.4)

If P lies on the polar of Q, then Q lies on the polar of P.

Theorem 26.4's Proof

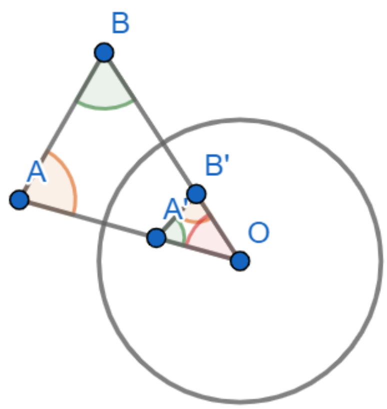
By Power of a Point (3.2), P, P', Q, Q' are concyclic. Since P is on the polar of Q, $\angle PQ'Q = 90^\circ$. By the Inscribed Angle Theorem (1.1), $\angle PP'Q = \angle PQ'Q = 90^\circ$. Thus Q is on the polar of P.

Inversion Distance Formula (26.5)

Consider circle ω with center O and radius r and points A, B. Let A', B' be the results of inverting A, B around ω, respectively. Then $\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA \cdot OB}}$.

Theorem 26.5's Proof

Note that $\triangle OAB \sim \triangle OB'A'$ as $\overline{OA'} = \frac{r^2}{\overline{OA}}$ and $\overline{OB'} = \frac{r^2}{\overline{OB}}$. Then notice that $\overline{OA} : \overline{OA'} = \overline{OA} : \frac{r^2}{\overline{OB}} = 1 : \frac{r^2}{\overline{OA \cdot OB}}$. Since $\overline{AB} : \overline{A'B'} = 1 : \frac{r^2}{\overline{OA \cdot OB}}$, $\overline{A'B'} = \overline{AB} \cdot \frac{r^2}{\overline{OA \cdot OB}}$, as desired.



As a final note, straight Cartesian coordinate bashing is possible using inversion. Without loss of generality, you should have ω be the circle $x^2 + y^2 = 1$, where you intend to invert around ω. The inversion will transform $P = (x, y)$ to $Q = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. I personally have not found any use for this, but perhaps someone out there can use coordinate inversion somehow. I find synthetic solutions better, and inversion in itself is already obscure enough already.

---

1. Verify the coordinate transformation for inversion.

2. Let the poles of A, B with respect to ω be A', B'. Prove that $ABB'A'$ is cyclic.

3. Let P inside circle ω have pole Q. Then let there be points X, Y on ω such that QX, QY are tangent to ω. Prove that Q, X, Y are collinear.

4. Consider circle ω and points A, B. Let the tangents from A to ω intersect ω at A_1, A_2 and let the tangents from B to ω intersect ω at B_1, B_2. Let the midpoint of A_1A_2 be M_A, and let the midpoint of B_1B_2 be M_B. Prove that ABM_BM_A is cyclic.

5. Two circles ω and Γ with centers O and O' that intersect at X, Y are considered *orthogonal* if and only if $OX \perp O'X$ and $OY \perp O'Y$. Prove that if ω is orthogonal with Γ, then an inversion about ω preserves Γ.

6. Consider $\triangle PAB$ with circumcenter X. Then consider an inversion about some circle ω with center P. If A', B', X' are the poles of A, B, X with respect to ω, prove that X' is the result of reflecting P about $A'B'$.

7. Consider $\triangle ABC$ with point D on BC. Let M, N be the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

8. Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z. Then choose M, N on the A excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.

---

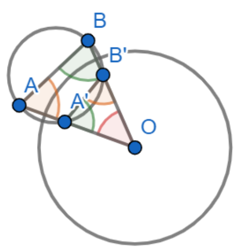
1. Verify the coordinate transformation for inversion.

Solution: Let $O=(0,0)$. Notice that Q is P dilated by $\frac{1}{x^2+y^2}$, so O,P,Q are collinear. Also, note that $\overline{OQ}=\frac{\overline{OP}}{x^2+y^2}$, and $\overline{OP}=\sqrt{x^2+y^2}$, so $\overline{OQ}=\frac{\sqrt{x^2+y^2}}{x^2+y^2}=\frac{1}{\sqrt{x^2+y^2}}$. Thus, $\overline{OP}\cdot\overline{OQ}=1^2$, verifying that Q is indeed the result of inverting P about ω.

For those of you who want vocabulary practice, we can say Q is the pole of P with respect to ω.

2. Let the poles of A,B with respect to ω be A',B'. Prove that $ABB'A'$ is cyclic.

Solution: Since $\triangle OAB \sim \triangle OB'A'$, $\angle OAB=\angle OB'A'$. Thus, $ABB'A'$ is cyclic as desired.

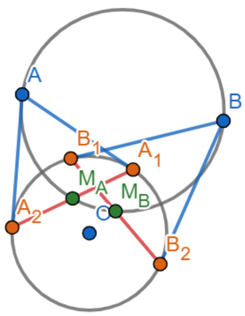


3. Let P inside circle ω have pole Q. Then let there be points X,Y on ω such that QX,QY are tangent to ω. Prove that Q,X,Y are collinear.

Solution: By similarity, $XP\perp OQ$ and $YP\perp OQ$. So XP and YP are either parallel or are the same line; since they intersect at P, they are the same line.

4. Consider circle ω and points A,B. Let the tangents from A to ω intersect ω at A_1,A_2 and let the tangents from B to ω intersect ω at B_1,B_2. Let the midpoint of A_1A_2 be M_A, and let the midpoint of B_1B_2 be M_B. Prove that ABM_BM_A is cyclic.

Solution: This is a direct result of Problem 2 and Problem 3. Notice that M_A and M_B are the poles of A,B with respect to ω, so the quadrilateral is cyclic.



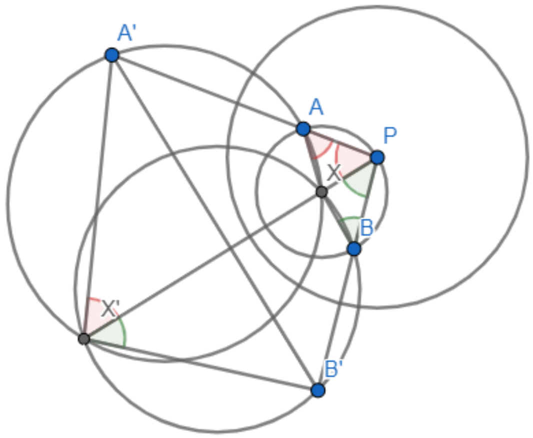
5. Two circles ω and Γ with centers O and O' that intersect at X, Y are considered *orthogonal* if and only if $OX \perp O'X$ and $OY \perp O'Y$. Prove that if ω is orthogonal with Γ, then an inversion about ω preserves Γ.

Solution: Let a line passing through O intersect Γ at P, Q. But by Power of a Point (3.2), $\overline{OX}^2 = \overline{OP} \cdot \overline{OQ}$, implying that Q is the polar of P with respect to ω. As P traces out Γ, so will Q.

6. Consider $\triangle PAB$ with circumcenter X. Then consider an inversion about some circle ω with center P. If A', B', X' are the poles of A, B, X with respect to ω, prove that X' is the result of reflecting P about $A'B'$.

Solution: By the definition of inversion, $\overline{PA} \cdot \overline{PA'} = \overline{PX} \cdot \overline{PX'}$. Applying Power of a Point (3.2) yields that $AA'X'X$ and $BB'X'X$ are cyclic quadrilaterals. Then notice that $\triangle PAX \sim \triangle PX'A'$.

Since X is the center of a circle, $\overline{AX} = \overline{PX}$. By similarity, $\overline{X'A'} = \overline{PA'}$, so $\angle PXA = \angle APX = \angle PX'A$. Similarly, $\angle BPX = \angle PX'B$. This implies that $\angle A'X'B' = \angle PA'B'$. Since P, X, X' are collinear, X' is the reflection of P about AB, as desired.

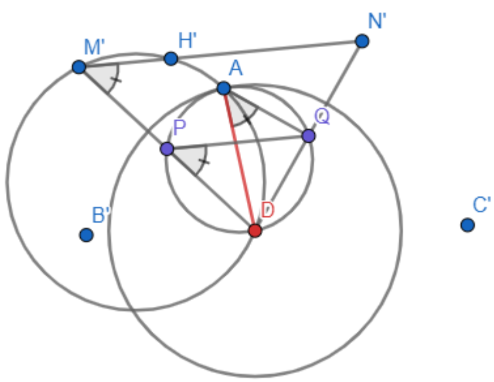


7. Consider $\triangle ABC$ with point D on BC. Let M, N be the circumcenters of $\triangle ABD$ and $\triangle ACD$, respectively. Let the circumcircles of $\triangle ACD$ and $\triangle MND$ intersect at $H \neq D$. Prove A, H, M are collinear.

Solution: Invert about the circle with center D and radius DA. This sends B, C, M, N, H to B', C', M', N', H', respectively.

By Problem 4, M' is the reflection of D about AB' and N' is the reflection of D about AB'. Then notice that H' is the intersection of $M'N'$ and AC'.

Let P be the midpoint of DM' and let Q be the midpoint of DN'. Notice that by Problem 4, $\angle APD = \angle AQD = 90^\circ$. By Theorem 2.2, $APDQ$ is cyclic because $\angle APD + \angle AQD = 90^\circ + 90^\circ = 180^\circ$. So $\angle QAD = \angle QPD = \angle N'M'D = \angle H'M'D$, by Inscribed Angle (1.1) and because $\triangle DPQ \sim \triangle DM'N'$. Since $\angle H'AD = 180^\circ - \angle QAD = 180^\circ - \angle H'M'D$, we notice that $H'M'DA$ is cyclic, as desired.



8. Consider scalene $\triangle ABC$ with incenter I. Let the A excircle of $\triangle ABC$ intersect the circumcircle of $\triangle ABC$ at X, Y. Let XY intersect BC at Z. Then choose M, N on the A

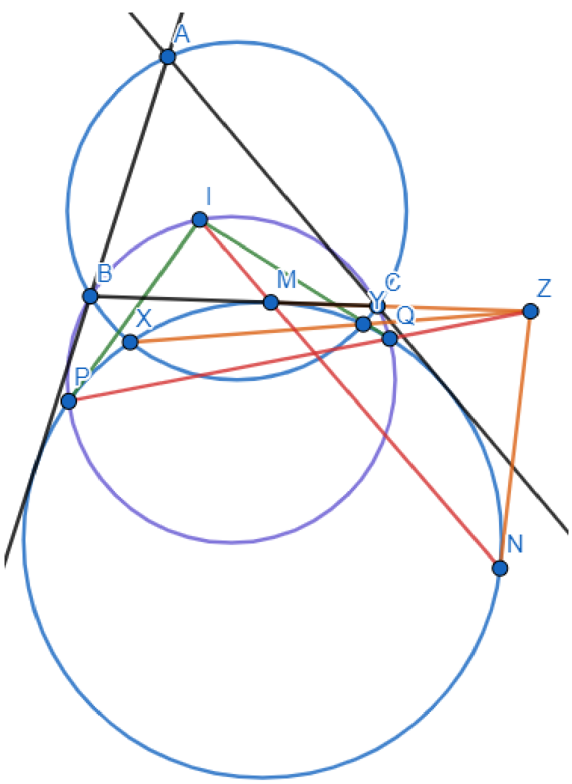
excircle of $\triangle ABC$ such that ZM, ZN are tangent to the A excircle of $\triangle ABC$. Prove I, M, N are collinear.

Solution: Notice that we wish to prove that I lies on the polar of Z with respect to the A excircle, so we instead use La Hire's (26.4) by proving Z lies on the polar of I. (The motivation is problem 2. Notice that MN is the polar of Z with respect to the A excircle.)

Let the A excenter be I_A and let the tangents from I to the A excircle be P, Q. Then notice $\angle IPI_A = \angle IQI_A = 90^\circ = \angle IBI_A = \angle ICI_A$, implying that B, C, P, Q are concyclic.

Letting the circumcircle be ω_1, the excircle be ω_2 and the circumcircle of $BCPQ$ be ω_3, notice that $\pi(Z, \omega_1) = \pi(Z, \omega_2)$. But by the Radical Axis Theorem (4.2), Z is on the radical axis of ω_2, ω_3, also known as PQ.

Then notice that PQ is the polar of I with respect to the A excircle, so Z lies on the polar of I, as desired. La Hire's (26.4) finishes the problem.

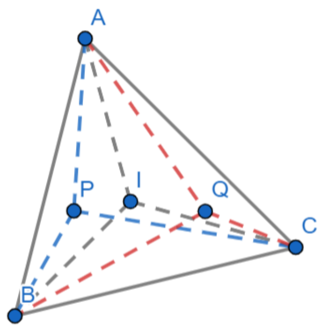


Fun fact: The diagram of the **problem** (not the solution, so points P, Q are left out) is the figure on the cover of this book.

Isogonal and Isotomic Conjugates

Isogonal conjugation is a useful involution that can be used to generalize certain well-known configurations and to relate well-known triangle centers, such as the circumcenter and orthocenter.

Consider $\triangle ABC$ with incenter I. Then consider point P. The *isogonal conjugate* of P with respect to $\triangle ABC$ is the point of concurrence of the reflection of AP over AI, BP over BI, and CP over CI.



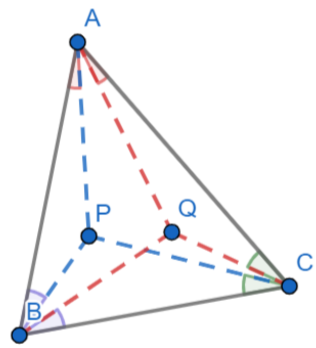
By the definition of reflecting about a line, $\angle PAI = \angle QAI$. Since $\angle BAI = \angle CAI$, $\angle BAI = \angle BAP + \angle PAI$ and $\angle CAI = \angle CAQ + \angle QAI$, we establish $\angle BAP = \angle CAQ$. This is under one huge assumption, of course: Q exists for all P. Fortunately, this is pretty much trivial to prove.

The Isogonal Conjugate Exists (27.1)

Consider $\triangle ABC$ with incenter I, and some point P. Then the reflections of AP, BP, CP about AI, BI, CI concur.

Theorem 27.1's Proof

Notice that $\angle BAQ = \angle QAC$, $\angle ACQ = \angle QCB$, and $\angle CBQ = \angle QBA$. By Sine Ceva, $\frac{\sin(\angle BAP)}{\sin(\angle PAC)} \cdot \frac{\sin(\angle ACP)}{\sin(\angle PCB)} \cdot \frac{\sin(\angle CBP)}{\sin(\angle PBA)} = 1$. Substituting, we get $\frac{\sin(\angle BAQ)}{\sin(\angle QAC)} \cdot \frac{\sin(\angle ACQ)}{\sin(\angle QCB)} \cdot \frac{\sin(\angle CBQ)}{\sin(\angle QBA)} = 1$, so the isogonal conjugate exists.



Now let's investigate a couple of general properties of isogonal conjugates.

Isogonal Proportionality Theorem (27.2)

Let Q be the isogonal conjugate of P with respect to $\triangle ABC$. Let AP, AQ intersect BC at X, Y, respectively. Then $\frac{\overline{BX}\cdot\overline{BY}}{\overline{CX}\cdot\overline{CY}} = \frac{\overline{AB}^2}{\overline{CA}^2}$.

You should try this proof yourself. Hint: Lots of angles are equal or supplementary. Try to take advantage of this using the Sine Law.

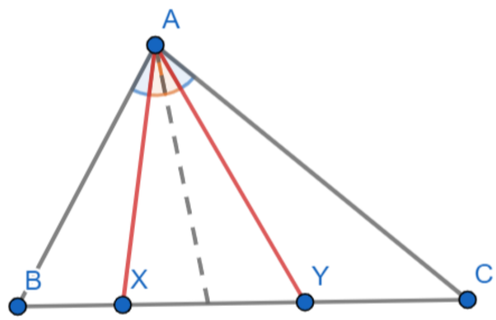
This gives us a nice way to bash the isogonal lines, and by extension, the isogonal conjugate; notice that the barycentric coordinates of the isogonal conjugate result directly from this.

Theorem 27.2's Proof

Let the bisector of $\angle A$ intersect BC at P. By the Law of Sines (4.1), $\frac{\sin(\angle BAX)}{\sin(\angle AXB)} = \frac{\overline{BX}}{AB}$, $\frac{\sin(\angle BAY)}{\sin(\angle AYB)} = \frac{\overline{BY}}{AB}$, $\frac{\sin(\angle CXA)}{\sin(\angle CAX)} = \frac{\overline{CA}}{CX}$, and $\frac{\sin(\angle CYA)}{\sin(\angle CAY)} = \frac{\overline{CA}}{CY}$. But notice $\sin(\angle BAX) = \sin(\angle CAX)$, $\sin(\angle AXB) = \sin(\angle CXA)$, $\sin(\angle BAY) = \sin(\angle CAY)$, and $\sin(\angle AYB) = \sin(\angle CYA)$.

Multiplying all of the ratios we got from the Law of Sines (4.1) yields $\frac{\sin(\angle BAX)}{\sin(\angle AXB)} \cdot \frac{\sin(\angle BAY)}{\sin(\angle AYB)} \cdot \frac{\sin(\angle CXA)}{\sin(\angle CAX)} \cdot \frac{\sin(\angle CYA)}{\sin(\angle CAY)} = \frac{\overline{BX}}{AB} \cdot \frac{\overline{BY}}{AB} \cdot \frac{\overline{CA}}{CX} \cdot \frac{\overline{CA}}{CY}$, implying $1 = \frac{\overline{BX}}{AB} \cdot \frac{\overline{BY}}{AB} \cdot \frac{\overline{CA}}{CX} \cdot \frac{\overline{CA}}{CY}$.

Rearranging yields $\frac{\overline{BX}\cdot\overline{BY}}{\overline{CX}\cdot\overline{CY}} = \frac{\overline{AB}^2}{\overline{CA}^2}$, as desired.



This should remind you of the Angle Bisector Proportionality Theorem (7.1.1).

On that note, it would be helpful to define the pedal triangle of P with respect to $\triangle ABC$. Let the feet of the perpendiculars from P to BC, CA, AB be D, E, F, respectively. Then the *pedal triangle* of P with respect to $\triangle ABC$ is $\triangle DEF$.

Isogonal Angle Chase (27.3)

Consider $\triangle ABC$ and point P with isogonal conjugate Q with respect to $\triangle ABC$. Then $\angle BPC + \angle BQC = 180^\circ + \angle A$.

Theorem 27.3's Proof

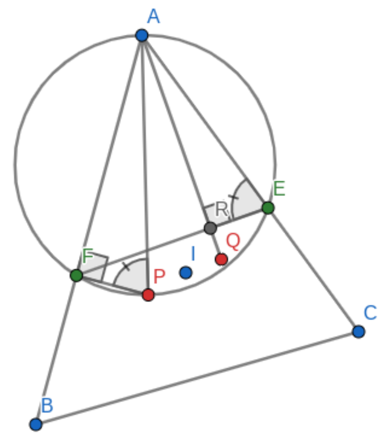
Let I be the incenter of $\triangle ABC$. Notice that $\angle PBC = \angle B - \angle ABP = \angle B - \angle QBC$, so $\angle PBC + \angle QBC = \angle B$. Analogously, $\angle PCB + \angle QCB = \angle C$. Notice that $\angle PBC + \angle PCB + \angle BPC + \angle QBC + \angle QCB + \angle BQC = 360^\circ$. Substituting yields $\angle BPC + \angle BQC + \angle B + \angle C = 360^\circ$, implying $\angle BPC + \angle BQC = 180^\circ + \angle A$, as desired.

Isogonal Perpendicularity Theorem (27.4)

Consider $\triangle ABC$ and point P with isogonal conjugate Q, and let P have pedal triangle $\triangle DEF$. Then $AQ \perp EF$.

Theorem 27.4's Proof

Let R be the foot of the perpendicular from A to EF. Then notice $\angle PAF = 90^\circ - \angle FPA = 90^\circ - \angle FEA = \angle EAR$. Thus, R lies on the reflection of AP about AI, proving the assertion.

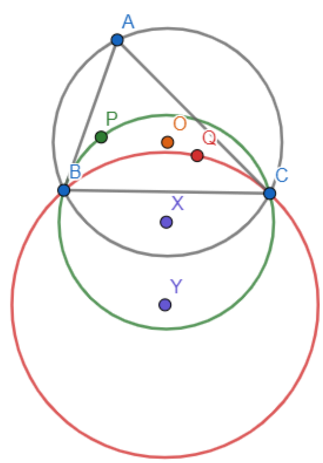


Generalized Incenter-Excenter (27.5)

Consider $\triangle ABC$ and isogonal conjugates P, Q with respect to $\triangle ABC$. Then the circumcircles of $\triangle BCP$ and $\triangle BCQ$ are inverses with respect to the circumcircle of $\triangle ABC$.

Theorem 27.5's Proof

Let the circumcenter of $\triangle BCP$ be X and the circumcenter of $\triangle BCQ$ be Y. For obvious reasons, X, Y lie on the perpendicular bisector of BC. Then by Theorem 27.4, $\angle BPC + \angle BQC = 180^{\circ} + \angle A$. But notice $\angle BXO = 180^{\circ} - \angle BPC$ and $\angle BYO = 180^{\circ} - \angle BQC$. So $\angle BXO = \angle OBY$, implying $\triangle BXO \sim \triangle OBY$. This means $\overline{OB}^2 = \overline{OX} \cdot \overline{OY}$, or that Y is the pole of X with respect to $\triangle ABC$. By Circle to Circle (21.1), the inversion is valid because B, C are on both circles.

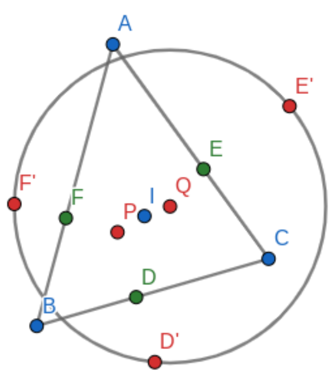


Circumcircle of Dilated Pedal Triangle (27.6)

Consider $\triangle ABC$ and point P. Let Q be the isogonal conjugate of P and let $\triangle DEF$ be the pedal triangle of P with respect to $\triangle ABC$. Then dilate D, E, F about P to get D', E', F'. Then the circumcenter of $\triangle D'E'F'$ is Q.

Theorem 27.6's Proof

By homothety, $EF \parallel E'F'$. By Theorem 27.4, $AQ \perp E'F'$. But since $PE \perp AB$ by definition, and $PE = EE'$, we have $\overline{PE}^2 + \overline{EA}^2 = \overline{EE'}^2 + \overline{EA}^2$. Thus, by Pythagorean's, $\overline{PA} = \overline{PE'}$. Analogously, $\overline{PA} = \overline{PF'}$. Since $AQ \perp E'F'$ and $\overline{AE'} = \overline{AF'}$, it stands that AQ is the perpendicular bisector of $E'F'$. Similarly, BQ is the perpendicular bisector of $F'D'$, and CQ is the perpendicular bisector of $D'E'$. Since AQ, BQ, CQ obviously concur at Q, the circumcenter of $\triangle D'E'F'$ is Q, as desired.



Ellipse Pedal Triangle Theorem (27.7)

Consider $\triangle ABC$ and ellipse ω tangent to AB, BC, CA with foci P, Q. Then P, Q are isogonal conjugates.

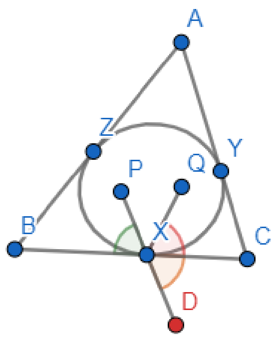
Remarkably, this is basically a corollary of Theorem 27.5. The converse is also true, which we will leave as a projective exercise for the next chapter.

Theorem 27.7's Proof

Let the ellipse be tangent to BC, CA, AB at X, Y, Z, respectively. Then by the definition of an ellipse, $\overline{PX} + \overline{QX} = \overline{PY} + \overline{QY} = \overline{PZ} + \overline{QZ}$.

Let D, E, F be the reflection of Q about BC, CA, AB, respectively. But due to the properties of tangents to ellipses, $\angle BXP = \angle CXQ = \angle CXD$. Thus, P, X, D are collinear.

So $\overline{PD} = \overline{PX} + \overline{XD} = \overline{PX} + \overline{QX}$. Similarly, $\overline{PE} = \overline{PY} + \overline{QY}$, and $\overline{PF} = \overline{PZ} + \overline{QZ}$. Substituting, $\overline{PD} = \overline{PE} = \overline{PF}$, implying P, Q are isogonal conjugates by Theorem 27.5.



Of course, the incenter is the special case $P = Q = I$.

Also, if you dilate the ellipse into a circle, you see that AX, BY, CZ concur at the Gergonne Point. Thus AX, BY, CZ will concur for the ellipse.

This theorem plus the concurrency is pretty much the direct solution to ISL G3 2002, which asks to prove if the ellipse with foci O, H is tangent to $\triangle ABC$ at D, E, F, then AD, BE, CF concur.

Pascal's Theorem (27.8)

Consider hexagon $ABCDEF$ inscribed in conic ω. Let AC intersect BD at P, let BE intersect CD at X, and let AE intersect DF at Q. Then P, Q, X are collinear.

Theorem 27.8's Proof

First, we notice we can transform the conic into a circle by projecting. Thus we only need to prove this for a circle.

Then notice that $\triangle XBC \sim \triangle XDE$. By Inscribed Angle (1.1), $\angle CBP = \angle XFQ$, so P, Q correspond to isogonal conjugates. Thus $\angle BXP = \angle QXE$, so P, Q, X are collinear.

Many of these theorems should remind you of relations of common triangle centers. (In fact, some problems will ask you to apply a general theorem more specifically.) It seems curious that Theorem 27.5 can be made into the nine-point circle, and $AH \perp XY$ where X, Y are the midpoints of AB, AC, which looks suspiciously like Theorem 27.4. Since these are not coincidences, let's investigate the isogonal conjugates of some common triangle centers.

Fixed Isogonal Conjugates (28.1)

Consider $\triangle ABC$ with incenter I and excenters I_A, I_B, I_C. Then the only fixed points when applying isogonal conjugation are I, I_A, I_B, I_C.

Theorem 28.1's Proof

Clearly, AI reflected about AI results in AI, and so on, so I is its own isogonal conjugate. Then, notice that AI_A reflected about AI results in AI_A because the two lines are perpendicular. The only points such that AP, BP, CP are the same when reflected about AI, BI, CI are I, I_A, I_B, I_C, because the lines either have to be identical or perpendicular.

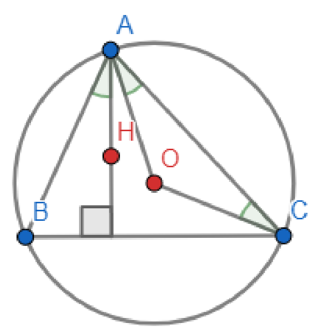
Circumcenter and Orthocenter are Isogonal (28.2)

The circumcenter and orthocenter of a triangle are isogonal conjugates with respect to said triangle.

Theorem 28.2's Proof

By the definition of isogonal conjugates, this is analogous to proving $\angle BAH = \angle CAO$.

Let the reference triangle be $\triangle ABC$ and let it have orthocenter and circumcenter O and H, respectively. By the Inscribed Angle Theorem (1.1), $\angle AOC = 2\angle B$. Due to isosceles triangles, $\angle OAC = \frac{1}{2}(180^\circ - 2\angle B) = 90^\circ - \angle B$. But notice by right triangles, $\angle BAH = 90^\circ - \angle B$, so $\angle OAC = \angle BAH$.



Of the significant triangle centers, we notice that the isogonal conjugate of I is I, and the isogonal conjugate of O is H (and vice versa). But what is the isogonal conjugate of the centroid?

First, let us define the reflection of a median across the respective angle bisector as a *symmedian*. Then, the symmedians of a triangle intersect at the *symmedian point*. By Theorem 27.1, they must intersect. We'll investigate some properties of the symmedian.

Symmedian Proportionality Theorem (28.3)

Let the A symmedian intersect BC at S_a. Then $\frac{\overline{BS_A}}{\overline{CS_A}} = (\frac{c}{b})^2$.

This can be generalized for other symmedians.

Theorem 28.3's Proof

The barycentric coordinates of the centroid are $(1 : 1 : 1)$. Notice the barycentric coordinates of the symmedian point are $(a^2 : b^2 : c^2)$, by Theorem 17.10. Then by the mass points definition, $\frac{\overline{BS_A}}{\overline{CS_A}} = (\frac{c}{b})^2$, as desired.

With this follows a trivial corollary from the trilinear definition.

Symmedian Point Proportionality Theorem (28.4)

The distance of the symmedian point to each of the sides is proportional to the lengths of the sides.

Theorem 28.4's Proof

By Theorem 18.1, the symmedian point has trilinears $(a : b : c)$. The definition of trilinears proves this.

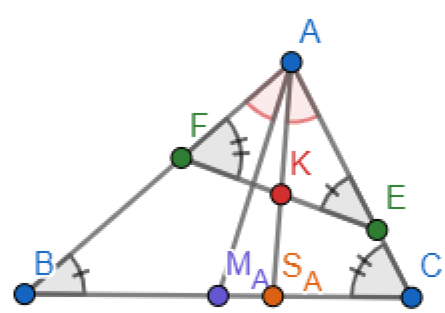
Symmedian Bisects Antiparallels (28.5)

Let any point on the A symmedian of $\triangle ABC$ be denoted as K.

Let the A antiparallel through K be the line through K that intersects CA at E and AB at F such that $\angle AEF = \angle B$ and $\angle AFE = \angle C$. Then $\overline{EK} = \overline{FK}$.

Theorem 28.5's Proof

By the definition of an antiparallel line, $\triangle ABC \sim \triangle AEF$.



Then notice that because $\angle M_A AB = \angle KAE$ by the definition of isogonal conjugates, AM_A is mapped to AK, so AK is a median of EF, as desired.

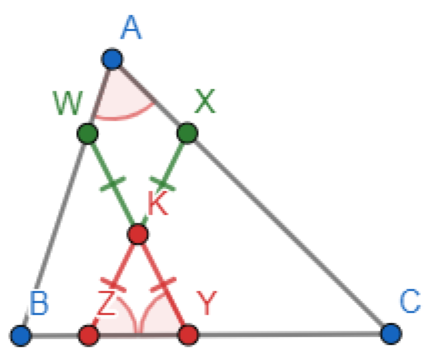
Equal Antiparallels from Symmedian Point (28.6)

Let the symmedian point of $\triangle ABC$ be denoted as K. Then the A, B, C antiparallels through K have equal length.

Theorem 28.6's Proof

Without loss of generality, we can just prove the B, C antiparallels of K have the same length. Let the B antiparallel intersect AB at W and BC at Y, and let the C antiparallel intersect CA at X and BC at Z.

By the definition of an antiparallel, notice $\angle KZY = \angle A = \angle KYZ$, so $\overline{KZ} = \overline{KY}$. But by Theorem 28.5, $\overline{KW} = \overline{KY} = \overline{KZ} = \overline{KX}$, so $\overline{WY} = \overline{XZ}$, as desired.



Point of Minimal Squared Distance (28.7)

Consider $\triangle ABC$ with symmedian point K. Then the point which minimizes the sum of the squares of the distances from said point to the sides of the triangle is K.

Theorem 28.7's Proof

We use signed distances.

It is fairly easy to confirm that

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2 + (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2.$$

Letting a, b, c be the side lengths of $\triangle ABC$ and x, y, z being the distance of some point from the three respective sides, we notice $ax + by + cz = 2[ABC]$, so it is fixed. It is also obvious that $a^2+b^2+c^2$ is fixed, so all that remains is to minimize $(bz-cy)^2+(cx-az)^2+(ay-bx)^2.$

But this non-negative value achieves a value of 0 when $x:y:z=a:b:c,$ so the trilinear coordinates of our desired point are $(a:b:c).$ This confirms it indeed is the symmedian point, so we are done.

Though this proof technically does not rely heavily on barycentric or trilinear coordinates, it still “feels” very trilinear.

It would be very annoying to construct the symmedians by reflecting medians about the angle bisector every time. Thus, here are five alternate methods to construct a symmedian.

Symmedian by Tangents (28.8)

Let $\triangle ABC$ have circumcircle ω and let the tangents of ω at B and C intersect at $S.$ Then AS is a symmedian of $\triangle ABC.$

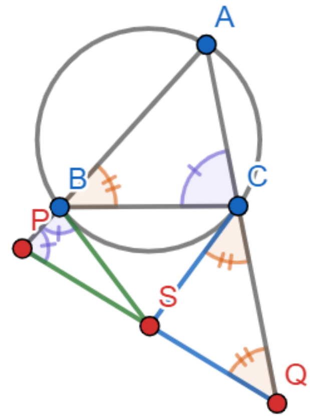
This construction is useful when you want to draw all three symmedians, because you can “reuse” tangents in a sense.

Theorem 28.8’s Proof

This proof mostly utilizes Theorem 28.5. Notice the A symmedian is the locus of midpoints of antiparallels of $\angle BAC.$

Let S be the intersection of the B and C tangents to the circumcircle. Then draw an antiparallel of $\angle BAC$ through $S,$ and let it intersect AB, AC at $P, Q.$ Notice $\angle SBP=\angle ACB=\angle APQ.$ But this implies $\triangle BSP$ and $\triangle CSQ$ are isosceles, so $\overline{BP}=\overline{BS}$ and $\overline{CS}=\overline{CQ}.$ But by the Two Tangent Theorem (3.5), $\overline{BS}=\overline{CS},$ implying $\overline{PS}=\overline{QS}.$

This implies S is on the A symmedian of $\triangle ABC.$ For obvious reasons, A is also on the A symmedian of $\triangle ABC,$ so AS is the A symmedian of $\triangle ABC,$ as desired.



Symmedian by Squares (28.9)

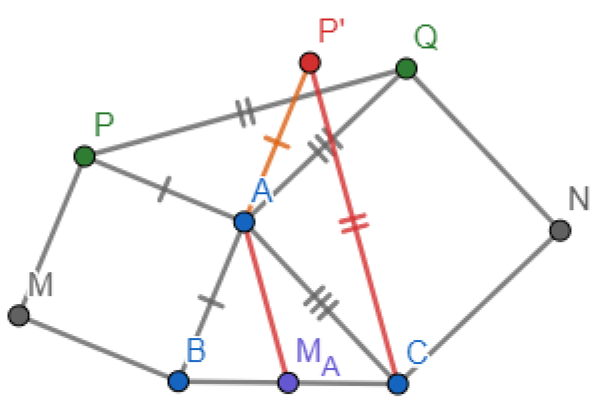
Construct squares $ABMP$ and $ACNQ$. Let O be the circumcenter of $\triangle APQ$. Then AO is a symmedian of $\triangle ABC$.

This one is not quite as useful for constructing, but may appear as a configuration. Besides, it is nice to know.

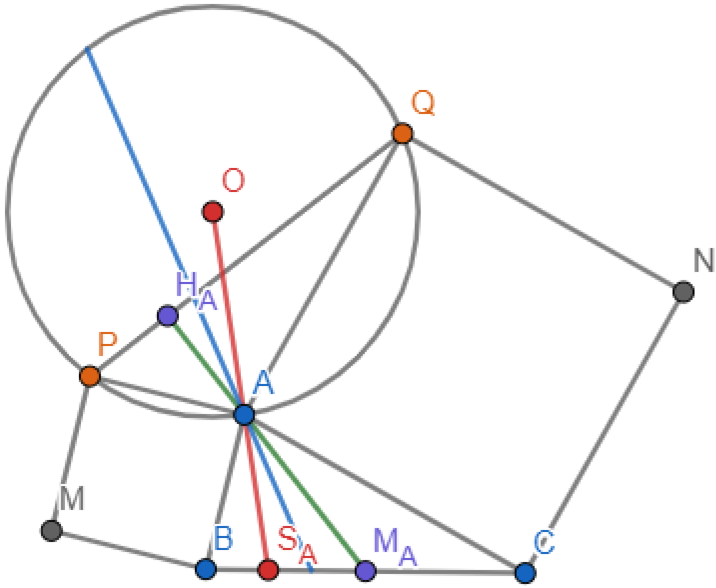
Theorem 28.9's Proof

First, we prove that the A median of $\triangle ABC$ is the A altitude of $\triangle APQ$. To do this, rotate P about A such that the same rotation would make Q coincide about C. (This is a 90° rotation, either counterclockwise or clockwise.)

Then notice that since PQ is rotated to $P'C$, $PQ \perp P'C$. But since AM_A is a midsegment of $\triangle BP'C$, $AM_A \parallel P'C$, so $AM_A \perp PQ$, as desired.



But since $\angle PAQ$ and $\angle BAC$ share an angle bisector, O is on the isogonal line of H_AM_A, implying that S_A (the foot of the A symmedian) is also on the isogonal line of H_AM_A, as desired.

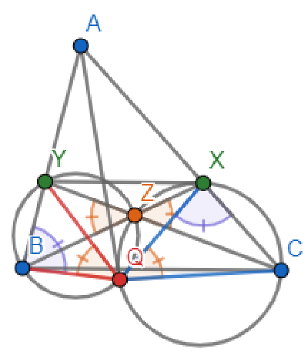


Symmedian by a Transversal (28.10)

Consider $\triangle ABC$ and transversal XY such that X is on CA and Y is on AB. Then let BX, CY intersect at Z. If the circumcircles of $\triangle BYZ$ and $\triangle CXZ$ intersect at another point Q, then AQ is a symmedian.

Theorem 28.10's Proof

Notice that by the Inscribed Angle Theorem (1.1), $\angle BQY = \angle BZY = \angle XZC = \angle XQC$ and $\angle YBQ = 180^\circ - \angle YZQ = \angle CZQ = \angle CXQ$, implying $\triangle BQY \sim \triangle XQC$. So the distances from Q to BY and Q to CX are proportional to BY and CX. But notice that by the definition of a transversal, $\frac{\overline{BY}}{CX} = \frac{\overline{AB}}{CA}$, which implies Q lies on the A symmedian as desired.



Symmedian by Intersection of Tangent and Side (28.11)

Consider $\triangle ABC$. Let the tangent of A to the circumcircle of $\triangle ABC$ intersect BC at K, and let the tangents from K to the circumcircle be A and S. Then AS is a symmedian.

Theorem 28.11's Proof

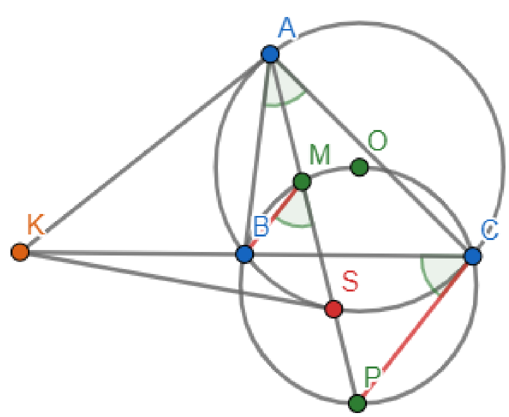
Let the tangents from B and C to the circumcircle intersect at P, and let the midpoint of AS be M. Without loss of generality, let $\angle B > \angle C$.

First, we prove that M and P lie on the circumcircle of $\triangle BCO$.

For M, notice that the inversions of M, B, C must be collinear for B, M, O, C to be concyclic, by Circle to Line (21.2). But the inversion of M is K, and K, B, C are collinear by definition.

For P, notice that by Theorem 1.4, $\angle BCP = \angle CBP = \angle A$. Since there are 180° in a triangle, $\angle BPC = 180^\circ - 2\angle A$. Since $\angle BOC = 2\angle A$, P lies on the circumcircle of $\triangle BCO$ as desired.

By the Inscribed Angle Theorem (1.1), $\angle BMP = \angle BCP$ and $\angle BMS = 2\angle A$. We want to prove that $\angle BMS = \angle CMS$. Since MS and KO are perpendicular, this is equivalent to proving $\angle KMB = \angle OMC$. Notice that $\angle KMB = 180^\circ - \angle OMB = \angle OCB = \angle OCK$. By the Inscribed Angle Theorem (1.1), $\angle OMC = \angle OBC$. But O is the arc midpoint of BC, implying $\angle OBC = \angle OCB = \angle OCK$. So $\angle KMB = \angle OMC$, implying $\angle BMS = \angle A = \angle BMP$, showing that M, S, P are collinear, or that AS is the symmedian, as desired.



Now what happens if we take the isogonal conjugate of a locus of points with respect to a triangle? Turns out, the result is quite fascinating to watch. We will have to turn to our old friend barycentric (or trilinear) coordinates for this one.

Note: The isogonal conjugate of a point on the circumcircle of $\triangle ABC$ with respect to P is the point at infinity.

Line to Ellipse (29.1)

Consider $\triangle ABC$ with line l that does not intersect its circumcircle. Then l is sent to an ellipse.

Theorem 29.1's Proof

The line is $dx + ey + fz = 0$ in trilinears, where d, e, f are constant. By Theorem 18.4, this becomes $d\frac{1}{x} + e\frac{1}{y} + f\frac{1}{z} = 0.$ Multiplying by $xyz,$ we get $dyz + ezx + fxy = 0.$ It is a property of trilinears that if the intersection of a conic and the line at infinity has no real solutions, then the conic is an ellipse. Since the line does not intersect the circle, it is an ellipse.

Line to Parabola (29.2)

Consider $\triangle ABC$ with line l that is tangent to its circumcircle. Then l is sent to a parabola.

Theorem 29.2's Proof

By Theorem 18.4, the circumcircle is sent to the line at infinity and the line is sent to a conic. Since the circumcircle and conic intersect once, so do the line of infinity and the conic. Thus, the conic is a parabola.

Line to Hyperbola (29.3)

Consider $\triangle ABC$ with line l that intersects its circumcircle twice. Then l is sent to a hyperbola.

Theorem 29.3's Proof

We proceed as the first and second proofs do. Since there are two intersections, the conic is a hyperbola.

The converse holds true as well, because isogonal conjugation is an involution. (However, a circle may be sent to multiple parabolas or hyperbolas.)

With all the theory and the interesting tidbit about conjugation of a locus out of the way, here are some problems. Note: Symmedians pretty much only exist due to isogonal conjugation. For this reason, symmedian configurations will be used even if it does not directly pertain to isogonal conjugation.

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1. Consider $\triangle ABC$ with orthocenter H. Let AB have midpoint X and AC have midpoint Y. Prove that $AH \perp XY$.

2. Consider $\triangle ABC$ with incenter I. Then let the incircle touch AB, AC at X, Y, respectively. Prove that $AI \perp XY$.

3. Consider $\triangle ABC$. Let point P have pedal triangle $\triangle DEF$ and let the isogonal conjugate of P with respect to $\triangle ABC$ be Q. Prove that the circumcenter of $\triangle DEF$ is the midpoint of PQ.

4. Consider $\triangle ABC$ with incenter I and a point P in the interior of $\triangle ABC$. Then let the pedal triangle of point P be $\triangle DEF$ and let the isogonal conjugate of P be Q, with respect to $\triangle ABC$. Prove that I is not the midpoint of PQ unless P and Q are the same point.

5. Let P have pedal triangle $\triangle DEF$ and isogonal conjugate Q with pedal triangle $\triangle XYZ$ with respect to $\triangle ABC$. Prove that D, E, F, X, Y, Z are concyclic.

6. Consider $\triangle ABC$ with circumcenter O, and let $\overline{AB} = 13, \overline{BC} = 14, \overline{CA} = 15$. Let AO intersect BC at X. Find $\overline{AX}$.

7. Consider $\triangle ABC$ with circumcircle ω and consider circle Γ with center P. Let ω be tangent to Γ at X. If the isogonal conjugate Γ^* of Γ intersects ω at B, C but is not inside ω, prove that P lies on the bisector of $\angle A$.

8. Consider $\triangle ABC$ with incenter I. Let the pedal triangle of I with respect to $\triangle ABC$ be $\triangle DEF$. Prove that the Gergonne Point (the point where AD, BE, CF meet) of $\triangle ABC$ is the symmedian point of $\triangle DEF$.

9. Prove that the symmedian point is the centroid of its own pedal triangle.

10. Consider $\triangle ABC$ with symmedian point K. Let the A antiparallel intersect AB, AC at A_1, A_2, let the B antiparallel intersect BC, BA at B_1, B_2, and let the C antiparallel

intersect CA, CB at C_1, C_2. Prove that K is the center of a circle passing through $A_1, A_2, B_1, B_2, C_1, C_2$.

11. Consider $\triangle ABC$ and some point P. Let the pedal triangle of P with respect to $\triangle ABC$ be $\triangle DEF$. Prove that $\overline{DE}^2 + \overline{EF}^2 + \overline{FD}^2$ attains its minimum when P is the symmedian point of $\triangle ABC$.

12. Let $\triangle ABC$ have squares $ABC_BC_A, BCA_CA_B, CAB_AB_C$ constructed on the exterior of $\triangle ABC$. Let A_BC_B meet A_CB_C at A', let B_CA_C meet B_AC_A at B', and let C_AB_A meet C_BA_B at C'. Prove that $\triangle A'B'C'$ is the result of a homothety about the symmedian point of $\triangle ABC$.

13. Consider $\triangle ABC$ and a circle ω that intersects all of its sides twice. Consider points X_1, Y_1, Z_1 on ω with isogonal conjugates X_2, Y_2, Z_2 also on ω. Prove that for the right choice of X_1 and Y_1, the angle bisector of $\angle A$, X_1X_2, and Y_1Y_2 concur.

---

1. Consider $\triangle ABC$ with orthocenter H. Let AB have midpoint X and AC have midpoint Y. Prove that $AH \perp XY$.

Solution: Let $\triangle ABC$ have circumcenter O. Since X, Y are vertices of the pedal triangle of O, and H is the isogonal conjugate of O, Theorem 27.4 finishes the problem.

2. Consider $\triangle ABC$ with incenter I. Then let the incircle touch AB, AC at X, Y, respectively. Prove that $AI \perp XY$.

Solution: Notice that X, Y are vertices of the pedal triangle of I. Since the isogonal conjugate of I is itself, Theorem 27.4 finishes this problem.

3. Consider $\triangle ABC$. Let point P have pedal triangle $\triangle DEF$ and let the isogonal conjugate of P with respect to $\triangle ABC$ be Q. Prove that the circumcenter of $\triangle DEF$ is the midpoint of PQ.

Solution: Dilate by a factor of 2 with center P. Then, you get Theorem 27.5, which finishes the problem, since all our steps are reversible.

4. Consider $\triangle ABC$ with incenter I and a point P in the interior of $\triangle ABC$. Then let the pedal triangle of point P be $\triangle DEF$ and let the isogonal conjugate of P be Q, with respect to $\triangle ABC$. Prove that I is not the midpoint of PQ unless P and Q are the same point.

Solution: The pedal triangle suggests Theorem 27.5. Notice that the midpoint of PQ, which we shall denote as M, is the circumcenter of $\triangle DEF$ because of Theorem 27.5 and dilation. (See the previous problem.) But for I to be M, the circumcircle of $\triangle DEF$ must be tangent to $\triangle ABC$. However, this is only possible for one specific point of P, which is the incenter. If $P = Q$, it is obvious that they are both the incenter.

(The interior condition yields that P, Q cannot be an excenter.)

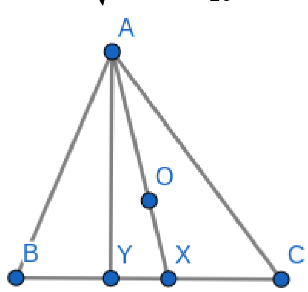
5. Let P have pedal triangle $\triangle DEF$ and isogonal conjugate Q with pedal triangle $\triangle XYZ$ with respect to $\triangle ABC$. Prove that D, E, F, X, Y, Z are concyclic.

Solution: By Theorem 27.5, $\triangle DEF$ dilated by a factor of 2 about P has circumcenter Q. Letting M be the midpoint of PQ and taking a homothety, we see that $\triangle DEF$ has circumcenter M. Since there is no loss of generality, Theorem 27.5 shows that the circumcenter of $\triangle XYZ$ has center M as well. Due to similar triangles, we see that the perpendicular from M to BC bisects DX, and so on, so Pythagorean Theorem verifies that the two circumcircles indeed have the same radius.

This should remind you of the nine-point circle!

6. Consider $\triangle ABC$ with circumcenter O, and let $\overline{AB} = 13, \overline{BC} = 14, \overline{CA} = 15$. Let AO intersect BC at X. Find $\overline{AX}$.

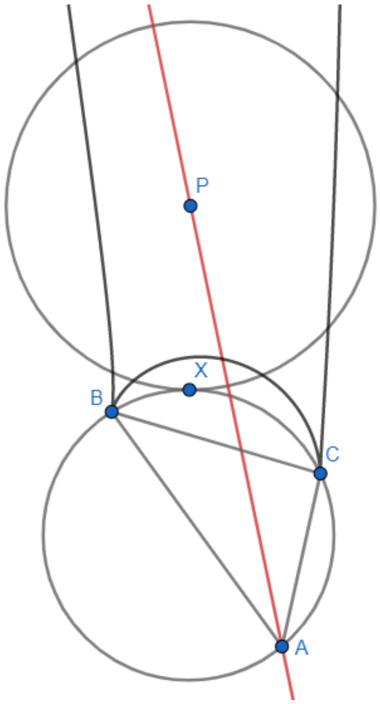
Solution: Let H be the orthocenter of $\triangle ABC$. Because O, H are isogonal conjugates, and the A altitude has a length of 12, it is very tempting to let AH intersect BC at Y. Then we notice that $\overline{AX}^2 = \overline{AY}^2 + \overline{XY}^2$. So we want to find BY to find XY. By Theorem 27.2, we notice that $\frac{\overline{BX}\cdot\overline{BY}}{\overline{CX}\cdot\overline{CY}} = \frac{\overline{AB}^2}{\overline{CA}^2} = \frac{169}{225}$. Then notice that $\overline{BY} = 5, \overline{CY} = 9$, because this is a well-known property of the 13-14-15 triangle. So $\frac{5\overline{BX}}{9\overline{CX}} = \frac{169}{225}$, implying that $\frac{\overline{BX}}{\overline{CX}} = \frac{169}{125}$. Then $\frac{\overline{BX}+\overline{CX}}{\overline{CX}} = \frac{\overline{BC}}{\overline{CX}} = \frac{14}{\overline{CX}} = \frac{294}{125}$. Thus, $\overline{CX} = \frac{125}{21}$. Since X is closer to C than Y is, we notice $\overline{CX} + \overline{XY} = \overline{CY}$, implying $\frac{125}{21} + \overline{XY} = 9$, or $\overline{XY} = \frac{64}{21}$. Then
$\overline{AX} = \sqrt{12^2 + (\frac{64}{21})^2} = 4\sqrt{3^2 + (\frac{16}{21})^2} = 4\sqrt{\frac{4225}{441}} = \frac{260}{21}$.



7. Consider $\triangle ABC$ with circumcircle ω and consider circle Γ with center P. Let ω be tangent to Γ at X. If the isogonal conjugate Γ^* of Γ intersects ω at B, C but is not inside ω, prove that P lies on the bisector of $\angle A$.

Solution: Let the A conjugate of any point N on Γ be the reflection of N about AI, and define the B and C conjugate analogously.

Realize that by symmetry, the A conjugate of P must be P to "balance" how "big" each of the curves are. (This is because the curves are centered around the angle bisector.) Then notice that this implies P lies on the $\angle A$ bisector.



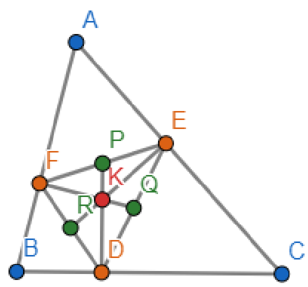
8. Consider $\triangle ABC$ with incenter I. Let the pedal triangle of I with respect to $\triangle ABC$ be $\triangle DEF$. Prove that the Gergonne Point (the point where AD, BE, CF meet) of $\triangle ABC$ is the symmedian point of $\triangle DEF$.

Solution: This is a direct application of Theorem 28.8.

9. Prove that the symmedian point is the centroid of its own pedal triangle.

Solution: Let K be the symmedian point of $\triangle ABC$, and let $\triangle DEF$ be the pedal triangle of K with respect to $\triangle ABC$. Then let DK, EK, FK intersect EF, FD, DE at P, Q, R, respectively. By the Law of Sines (9.1), $\frac{PE}{\sin(\angle PKE)} = \frac{KE}{\sin(\angle KPE)}$ and $\frac{PF}{\sin(\angle PKF)} = \frac{KF}{\sin(\angle KPF)}$. As $\sin(\angle KPE) = \sin(\angle KPF)$, note that $\frac{PE}{PF} = \frac{KE}{KF} \cdot \frac{\sin(\angle PKE)}{\sin(\angle PKF)}$. But by Theorem 28.4, $\frac{KE}{EF} = \frac{CA}{AB}$, so $\frac{PE}{PF} = \frac{CA}{AB} \cdot \frac{\sin(\angle PKE)}{\sin(\angle PKF)}$.

Notice that as $KDCE$ is cyclic, $\angle PKE = 180^\circ - \angle EKD = 180^\circ - (180^\circ - \angle C) = \angle C$. Similarly, $\angle PKF = \angle B$. Then $\frac{PE}{PF} = \frac{CA}{AB} \cdot \frac{\sin(\angle C)}{\sin(\angle B)} = \frac{CA}{\sin(\angle B)} \cdot \frac{\sin(\angle C)}{AB} = 1$, by the Law of Sines (9.1). Thus, $\overline{PE} = \overline{PF}$, as desired.



10. Consider $\triangle ABC$ with symmedian point K. Let the A antiparallel intersect AB, AC at A_1, A_2, let the B antiparallel intersect BC, BA at B_1, B_2, and let the C antiparallel intersect CA, CB at C_1, C_2. Prove that K is the center of a circle passing through $A_1, A_2, B_1, B_2, C_1, C_2$.

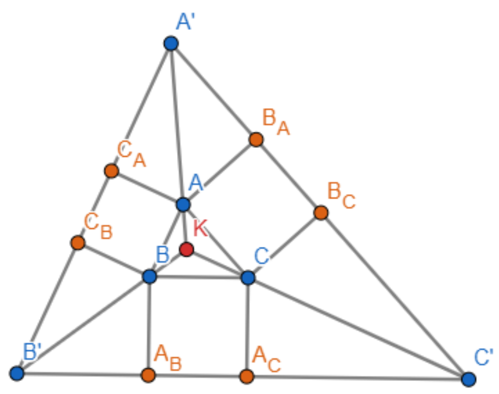
Solution: By Theorem 28.5 and 28.6, $\overline{KA_1} = \overline{KA_2} = \overline{KB_1} = \overline{KB_2} = \overline{KC_1} = \overline{KC_2}$. By the definition of a circle, $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle centered at K, as desired.

This is known as the First Lemoine Circle, and its existence is very obvious.

11. Consider $\triangle ABC$ and some point P. Let the pedal triangle of P with respect to $\triangle ABC$ be $\triangle DEF$. Prove that $\overline{DE}^2 + \overline{EF}^2 + \overline{FD}^2$ attains its minimum when P is the symmedian point of $\triangle ABC$.

12. Let $\triangle ABC$ have squares $ABC_BC_A, BCA_CA_B, CAB_AB_C$ constructed on the exterior of $\triangle ABC$. Let A_BC_B meet A_CB_C at A', let B_CA_C meet B_AC_A at B', and let C_AB_A meet C_BA_B at C'. Prove that $\triangle A'B'C'$ is the result of a homothety about the symmedian point of $\triangle ABC$.

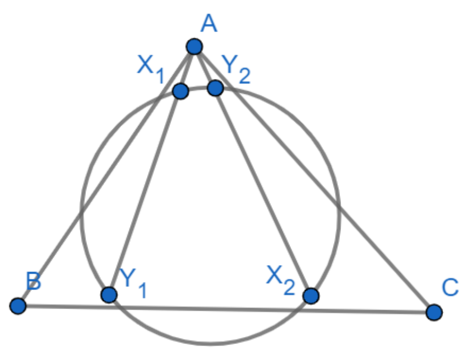
Solution: This is equivalent to proving that $\triangle ABC \sim \triangle A'B'C'$ and the two triangles share a symmedian point. The first is fortunately very easy to prove, as all the sides are parallel to each other. To prove the second, let the distances from K to BC, CA, AB be α, β, γ, respectively. But notice the distances from K to $B'C', C'A', A'B'$ are a, b, c, by the definition of a square. By the definition of the symmedian point, $\alpha : \beta : \gamma = a : b : c$, so $\alpha : \beta : \gamma = \alpha + a : \beta + b : \gamma + c$, and K is the symmedian point of $\triangle A'B'C'$, as desired.



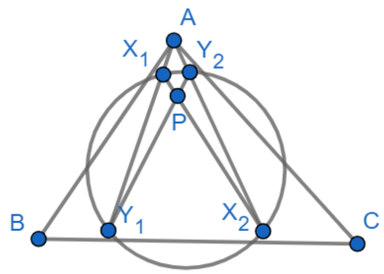
13. Consider $\triangle ABC$ and a circle ω that intersects all of its sides twice. Consider points X_1, Y_1, Z_1 on ω with isogonal conjugates X_2, Y_2, Z_2 also on ω. Prove that for the right choice of X_1 and Y_1, the angle bisector of $\angle A$, X_1X_2, and Y_1Y_2 concur.

Solution: First, we claim that A, X_1, Y_1 and A, X_2, Y_2 are collinear, where the points have been denoted as such in the diagram below.

Notice that by definition, $\angle BAX_1 = \angle CAX_2$ and $\angle BAY_1 = \angle CAY_2$. By Theorem 2.2, $\angle Y_1X_1X_2 = \angle Y_1Y_2X_2$ and $\angle X_1Y_1Y_2 = \angle X_1X_2Y_2$. What we want to prove is that $\angle AY_1Y_2 = \angle X_1Y_1Y_2$. Since $\angle X_1Y_1Y_2 = \angle X_1X_2Y_2$, this is the same as proving $\angle AY_1Y_2 = \angle X_1X_2Y_2$. Some angle chasing shows this to be true.



Now that we've proven collinearity, the rest is very simple. Let P be the intersection of X_1X_2 and Y_1Y_2. As $\angle BAX_1 = \angle CAX_2$ and $\angle BAY_1 = \angle CAY_2$, the angle bisector of $\angle X_1AX_2$ and $\angle Y_1AY_2$ is the same as the angle bisector of $\angle BAC$. Since by definition, $\angle X_1AP = \angle X_2AP$, then X_1Y_1 and Y_2X_2 are symmetric about the angle bisector of $\angle X_1AY_2$, implying the concurrency as desired.

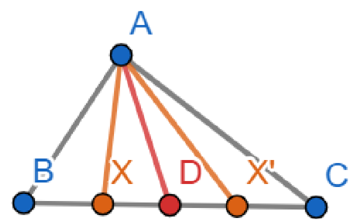


Now for the less popular cousin of isogonal conjugation - isotomic conjugation.

Consider $\triangle ABC$ with centroid G and point P. The *isotomic conjugate* of P with respect to $\triangle ABC$ is the point of concurrence of the reflection of AP over AG, BP over BG, and CP over CG. The proof of isotomic conjugation is also fairly obvious.

The Isotomic Conjugate Exists (30.1)

Let AP meet BC at X, and let AD be a median. Then by the definition of reflection, $\overline{XD} = \overline{X'D}$. Since $\overline{BD} = \overline{CD}$, $\overline{BX} = \overline{CX'}$. Letting BP intersect CA at Y and letting CP intersect AB at Z, analogous results follow. By Ceva's Theorem (6.5), $\frac{\overline{AZ}}{\overline{ZB}} \cdot \frac{\overline{BX}}{\overline{XC}} \cdot \frac{\overline{CY}}{\overline{YC}} = 1$.



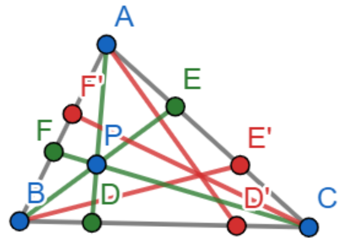
Products of Areas with Isotomic Conjugates (30.2)

If P, Q are isotomic conjugates with respect to $\triangle ABC$, then

$$[ABP][ABQ] = [BCP][BCQ] = [CAP][CAQ].$$

Theorem 30.2's Proof

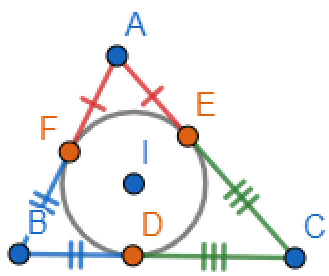
We use Theorem 17.11. By the area definition, if $[BPC] = k_1x$, then $[BQC] = \frac{k_2}{x}$, and similar results follow for the other triangles. Thus $[BPC][BQC] = k_1k_2$, which follows analogously for the other two area products, as desired.



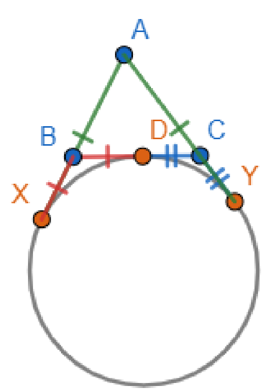
Nagel and Gergonne Points are Isotomic Conjugates (30.3)
The Nagel and Gergonne points of a triangle are isogonal conjugates.

Theorem 30.3's Proof
We use the mass points definition of barycentric coordinates.

Let's find the coordinates of the Gergonne Point. By the Two Tangent Theorem (3.4) and some algebra, $\overline{AE} = \overline{AF} = s - a$, $\overline{BF} = \overline{BD} = s - b$, and $\overline{CD} = \overline{CE} = s - c$. We can assign $\diamond A = \frac{1}{s-a}$, $\diamond B = \frac{1}{s-b}$, and $\diamond C = \frac{1}{s-c}$, so the coordinates of the Gergonne Point are $(\frac{1}{\diamond A} : \frac{1}{\diamond B} : \frac{1}{\diamond C}) = (s - a : s - b : s - c.)$



Now let's find the coordinates of the Nagel Point. By the Two Tangent Theorem (3.4), $\overline{AX} = \overline{AY}$, $\overline{BD} = \overline{BX}$ and $\overline{CE} = \overline{CY}$. Thus $\overline{AB} + \overline{BD} = \overline{AC} + \overline{CD} = s$. Then $\overline{BD} = s - c$ and $\overline{CD} = s - b$. Cyclic variants hold. Thus, we can assign $\diamond A = s - a$, $\diamond B = s - b$, and $\diamond C = s - c$, implying the coordinates of the Nagel Point are $(\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c})$.



By Theorem 17.11, the Gergonne Point and Nagel Point are isotomic conjugates.

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1. Let P, Q inside of $\triangle ABC$ be isotomic conjugates with respect to $\triangle ABC$. If $[ABC] = 1,$ find the maximum value of $[ABP][ABQ]$.

2. Let N, Ge be the Nagel and Gergonne points of $\triangle ABC,$ respectively. Prove that $[ABN] \cdot [ABGe] = \frac{[ABC]^4}{ab+bc+ca-s^2}$.

---

1. Let P, Q inside of $\triangle ABC$ be isotomic conjugates with respect to $\triangle ABC.$ If $[ABC] = 1,$ find the maximum value of $[ABP][ABQ].$

Solution: We claim the answer is $\frac{1}{9}.$

Let $[ABP] = a,$ $[BCP] = b,$ and $[CAP] = c,$ and let $[ABQ] = x,$ $[BCQ] = y,$ and $[CAQ] = z.$ Then we have the restrictions $a + b + c = x + y + z = 1$ and $ax = by = cz.$ To maximize $ax,$ we maximize $ax \cdot by \cdot cz = abc \cdot xyz.$ As by AM-GM, $\frac{a+b+c}{3} = \frac{1}{3} \geq \sqrt[3]{abc}$ and $\frac{x+y+z}{3} = \frac{1}{3} \geq \sqrt[3]{xyz},$ so $\frac{1}{3^6} \geq abc \cdot xyz.$ As $ax = by = cz,$ $\frac{1}{3^6} \geq a^3x^3,$ implying $\frac{1}{9} \geq ax.$ Equality occurs when P and Q are the centroid of $\triangle ABC.$

2. Let N, Ge be the Nagel and Gergonne points of $\triangle ABC,$ respectively. Prove that $[ABN] \cdot [ABGe] = \frac{[ABC]^4}{ab+bc+ca-s^2}.$

Solution: More barycentric coordinates.

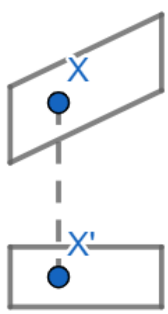
Notice that by the area definition, $[ABN] = [ABC] \cdot \frac{1}{s-a} \cdot \frac{1}{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}}$ and $[ABGe] = [ABC] \cdot (s-a) \cdot (s-a+s-b+s-c) = (s-a) \cdot s.$ Thus $[ABN] \cdot [ABGe] = [ABC]^2 \cdot s \cdot \frac{1}{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}} = [ABC]^2 \cdot s \cdot \frac{(s-a)(s-b)(s-c)}{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)}.$ Notice that $(s-a)(s-b) = s^2 - s(a+b) + ab,$ so cyclically summing yields $[ABC]^2 \cdot s \cdot \frac{(s-a)(s-b)(s-c)}{3s^2-s(2a+2b+2c)+ab+bc+ca} = [ABC]^2 \cdot \frac{s(s-a)(s-b)(s-c)}{3s^2-4s^2+ab+bc+ca} = [ABC]^2 \cdot \frac{s(s-a)(s-b)(s-c)}{ab+bc+ca-s^2}.$ By Heron's Formula (5.6), $s(s-a)(s-b)(s-c) = [ABC]^2,$ so $[ABN] \cdot [ABGe] = \frac{[ABC]^4}{ab+bc+ca-s^2},$ as desired.

Projective Geometry

Projective geometry has two important parts. First is the intuitive and fundamental definition of projections - this is useful in particular because lines are sent to lines, but conics can be manipulated into circles. Then is the idea of harmonic quadrilaterals and bundles, cross ratios, and "pencils." Of course, we will first be introducing the intuitive definition.

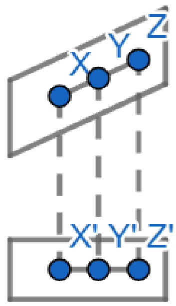
There are three types of projections. There are orthogonal projections, central projections, and parallel projections.

We'll explore the orthogonal projection first. Consider planes P and Q. A point X on plane P will *project* to a point X' on plane Q such that XX' is perpendicular to plane Q. (Important distinction: Projecting X from P to Q yields X', but projecting X' from Q to P does not necessarily yield X. The only time this is true is when P and Q are parallel!)



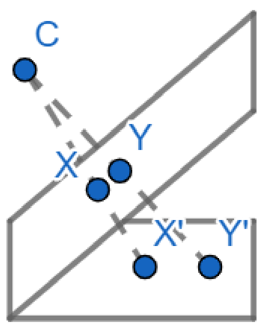
Notice that an orthogonal projection is the same as a *stretch*. This is because it preserves ratios of lengths in the same direction (which means it also preserves ratios of areas.)

Consider collinear points X, Y, Z on plane P. If their projections to some point Q are X', Y', Z', then $\frac{\overline{XY}}{\overline{XZ}} = \frac{\overline{X'Y'}}{\overline{X'Z'}}$, since quadrilaterals $XYY'X'$ and $XYY'X'$ are similar. (Since all the lines are perpendicular to plane Q and make a certain angle with plane P, angles are preserved.)



The second type of projections, *parallel projections*, are similar to orthogonal projections. Rather than XX' being perpendicular to Q, it must make a specific angle. This means that $XX' \parallel YY'$. Thus, parallel projections do the same thing as orthogonal projections; they are just distortions.

The final type of projections are *central projections*. Consider some point C. Then to project a point X on a plane P about C to plane Q, let CX intersect plane Q at X'. Then the projection of X is X'.



Unlike parallel projections, central projections do not preserve ratios!

That's it for basic projections. Now, we will introduce the idea of the projective plane, and the analogy of a sphere with antipodes (diametrically opposite points) being the same.

The projective plane should not be a foreign concept; we explained it in the Inversion chapter. In the Euclidean plane, non-parallel lines meet. In the projective plane, we extend the Euclidean plane such that **parallel lines meet at a point at infinity**. Each pair of parallel lines meet at a *point at infinity*. And the line containing all the points at infinity is the *line at infinity*.

Now let's talk about the sphere analogy. Let the projective plane be a sphere instead. (This is okay because the surface of a sphere is 2D, just like a plane.) Think of lines as great circles on the projective sphere and pairs of antipodal points as a point. **Here, we define antipodal points A and A' as identical**. We see that two great circles (two lines) intersect at a pair of antipodal points (a point), and two distinct pairs of antipodal points (two distinct points) determine a great circle (a line). Now, we can tangibly visualize the projective plane.

This explains why you can project conics to conics. If you let the sphere be inscribed within a double cone and consider the plane containing a great circle, you see that it cuts off the cross section of the double cone (creating a conic). This means that if you have collinearity/concurrency problems involving conics, you can just treat the conic as a circle because you can project the conic to a circle. (See Pascal's Theorem.) **In fact, in general, a projection is any transformation that takes lines to lines and conics to conics.**

Given our basic definition of projections, **areas scale linearly**. So if you want to find the maximum area of a triangle inscribed within an ellipse, you merely have to find the maximum area of a triangle inscribed within a circle (very easy) and multiply by a factor (also very easy).

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1. A square is inscribed within an ellipse with a major axis of length 2 and a minor axis of length $1.$ What is its side length?

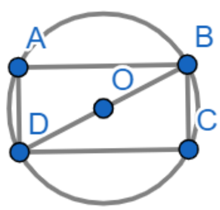
2. Consider two planes P and Q that form a 30° angle. Let X be a point on both planes and let Y be a point such that $\overline{XY} = 1$ and XY is perpendicular to $Q.$ Let A be a point on P such that $\triangle XYA$ is equilateral, and let A' be the central projection of A about Y onto plane $Q.$ Find the value of $\overline{XA'}.$

---

1. A square is inscribed within an ellipse with a major axis of length 2 and a minor axis of length $1.$ What is its side length?

Solution: Project to a circle with radius $1.$ Then you have a rectangle with sides of ratio $1:2.$

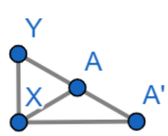
Let $\overline{AB} = 2\overline{AD}.$ By the Pythagorean Theorem, $\overline{AD}^2 + \overline{AB}^2 = 5\overline{AD}^2 = 4.$ Thus, $\overline{AD} = \frac{2\sqrt{5}}{5}$ and $\overline{AB} = \frac{4\sqrt{5}}{5}.$ Projecting back, we see that the length of the square is $\frac{4\sqrt{5}}{5}.$



2. Consider two planes P and Q that form a 30° angle. Let X be a point on both planes and let Y be a point such that $\overline{XY} = 1$ and XY is perpendicular to $Q.$ Let A be a point on P such that $\triangle XYA$ is equilateral, and let A' be the central projection of A about Y onto plane $Q.$ Find the value of $\overline{XA'}.$

Solution: Let Y' be the foot of the altitude from Y to P. Since $\angle YXA = 60^\circ$ by definition, A lies on $XY'.$ Thus, X, Y, A, A' are coplanar and we may take a cross section.

Notice that $\angle XA'Y = 30^\circ$ and $\overline{XY} = 1,$ so $\overline{XA'} = \sqrt{3}.$



Now let's get into the meat.

Consider collinear points A, B, C, D. We define the *cross ratio* $(A, B; C, D)$ as $\frac{\vec{CA}}{\vec{CB}} : \frac{\vec{DA}}{\vec{DB}}$. **Notice that we are using directed lengths; future uses of cross ratio will not include the vector symbol.** When $(A, B; C, D) = -1$, we call A, B, C, D a *harmonic bundle*.

Let P be a point not collinear with A, B, C, D. Then we call PA, PB, PC, PD a *pencil* and denote it as $P(A, B, C, D)$.

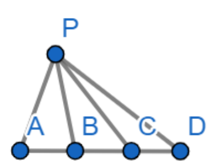
The Pencil's Cross Ratio (31.1)

Let P be a point. Then $(A, B; C, D) = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)}$.

Theorem 31.1's Proof

Notice that $(A, B; C, D) = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$. By the Law of Sines (9.1), $\frac{\sin(\angle CPA)}{\overline{CA}} = \frac{\sin(\angle PCA)}{\overline{AP}}$ and $\frac{\sin(\angle CPB)}{\overline{CB}} = \frac{\sin(\angle PCA)}{\overline{BP}}$. Also, $\frac{\sin(\angle DPA)}{\overline{DA}} = \frac{\sin(\angle PDA)}{\overline{AP}}$ and $\frac{\sin(\angle DPB)}{\overline{DB}} = \frac{\sin(\angle PDA)}{\overline{BP}}$. (Note how we just switched a C with a D for our second set of equations.)

This implies that $\sin(\angle PCA) = \frac{\overline{AP}}{\overline{CA}} \cdot \sin(\angle CPA) = \frac{\overline{BP}}{\overline{CB}} \cdot \sin(\angle CPB)$ and $\sin(\angle PDA) = \frac{\overline{AP}}{\overline{DA}} \cdot \sin(\angle DPA) = \frac{\overline{BP}}{\overline{DB}} \cdot \sin(\angle DPB)$. Dividing the first set of equations by the second yields $\frac{\overline{DA}}{\overline{CA}} \cdot \frac{\sin(\angle CPA)}{\sin(\angle DPA)} = \frac{\overline{DB}}{\overline{CB}} \cdot \frac{\sin(\angle CPB)}{\sin(\angle DPB)}$. Rearranging yields the desired conclusion.



Now given that we have an invariant no matter what P is, we can define the cross ratio of $(PA, PB; PC, PD)$ as $(A, B; C, D)$. If (A, B, C, D) is a harmonic bundle, so is $P(A, B, C, D)$. This also allows us to prove another lemma.

Line Invariant (31.2)

Consider pencil $P(A, B, C, D)$ and another line l. If PA, PB, PC, PD intersect l at A', B', C', D' respectively, then $(A, B; C, D) = (A', B'; C', D')$.

Theorem 31.2's Proof

Obvious consequence of Theorem 31.1.

We can also define cross-ratios for four concyclic points. There are two reasons for this; one, cross-ratios stay invariant under inversion (more on that later), and two, the cross ratio of the pencil $P(A, B, C, D)$ is also invariant if P is on the circumcircle of $ABCD$ (which we will prove right now).

Circle Invariant (31.3)

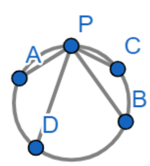
Consider concyclic points P, A, B, C, D. Then

$$(PA, PB; PC, PD) = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)} = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}} = (A, B; C, D).$$

Theorem 31.3's Proof

Assume all angles are directed to avoid configuration issues.

By the Inscribed Angle Theorem (1.1), $\angle PAC = \angle PBC$ and $\angle PAD = \angle PBD$. By the Law of Sines (9.1), $\frac{\sin(\angle CPA)}{\overline{CA}} = \frac{\sin(\angle PAC)}{\overline{PC}}$ and $\frac{\sin(\angle CPB)}{\overline{CB}} = \frac{\sin(\angle PBC)}{\overline{PC}}$. Since $\angle PAC = \angle PBC$, $\frac{\sin(\angle CPA)}{\overline{CA}} = \frac{\sin(\angle CPB)}{\overline{CB}}$. Analogously, $\frac{\sin(\angle DPA)}{\overline{DA}} = \frac{\sin(\angle DPB)}{\overline{DB}}$. Dividing the first set of equations by the second, we get $\frac{\sin(\angle CPA)}{\overline{CA}} : \frac{\sin(\angle DPA)}{\overline{DA}} = \frac{\sin(\angle CPB)}{\overline{CB}} : \frac{\sin(\angle DPB)}{\overline{DB}}$, which rearranges to $\frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}} = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)}$, as desired.



If $(A, B; C, D) = -1$, we call $ACBD$ a *harmonic quadrilateral*. (The order matters!) So if A, C, B, D are on a circle **in that order**, and $\frac{CA}{CB} : \frac{DA}{DB} = 1$ (lengths aren't directed this time), then $ACBD$ is harmonic.

Now we have cross ratios for circles, and we have them for lines. Circles and lines. That brings up memories of inversion! Inversion is symmetric to an extent (you get antiparallel rather than parallel lines), so let's give it a go. What happens to the cross ratio?

Inversion Invariant (31.4)

Consider collinear points (A, B, C, D). Invert them about a circle ω and let the inversions be A', B', C', D'. Then $(A, B; C, D) = (A', B'; C', D')$.

Try using the Inversion Distance Formula (26.5) to prove that the cross-ratio doesn't change.

Theorem 31.4's Proof

Let ω have radius r and center O. By the Inversion Distance Formula (26.5), $\overline{C'A'} = \overline{CA} \cdot \frac{r^2}{\overline{OC} \cdot \overline{OA}}$, $\overline{C'B'} = \overline{CB} \cdot \frac{r^2}{\overline{OC} \cdot \overline{OB}}$, $\overline{D'A'} = \overline{DA} \cdot \frac{r^2}{\overline{OD} \cdot \overline{OA}}$, and $\overline{D'B'} = \overline{DB} \cdot \frac{r^2}{\overline{OD} \cdot \overline{OB}}$. Thus, $\frac{\overline{C'A'}}{\overline{C'B'}} : \frac{\overline{D'A'}}{\overline{D'B'}} = \frac{\overline{OB}}{\overline{OA}} \cdot \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{OB}}{\overline{OA}} \cdot \frac{\overline{DA}}{\overline{DB}} = \frac{\overline{CA}}{\overline{CB}} : \frac{\overline{DA}}{\overline{DB}}$, as desired.

This also means that in the two-dimensional sense, we can project from a line to a line, a line to a circle, or a circle to a circle, and it will keep the same cross-ratio (by Theorem 31.2 and 31.4). **We may only project with circles if the point of projection P is on the circumcircle of cyclic quadrilateral $ABCD$.**

Diagram of line to line.

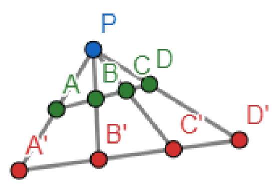


Diagram of circle to line.

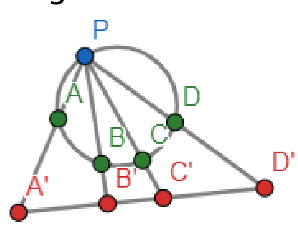
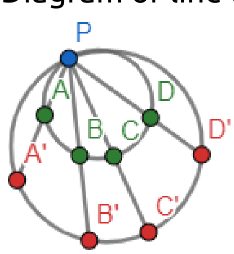
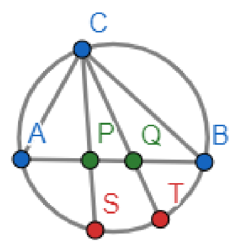


Diagram of line to circle.



This completely trivializes 2016 AIME II #10.

Given $\triangle ABC$ with circumcircle ω with points P, Q on AB such that $\overline{AP} = 4$, $\overline{PQ} = 3$, $\overline{QB} = 6$, and $\overline{AP} < \overline{AQ}$, let CP, CQ intersect ω again at S, T. If $\overline{AS} = 7$ and $\overline{BT} = 5$, find $\overline{ST}$.



Project A, P, Q, B about C onto the circumcircle of $\triangle ABC$ to get A, S, T, Q. Notice that $(A, Q; P, B) = (A, T; S, B)$. By the definition of cross ratios, $(A, Q; P, B) = \frac{\overline{PA}}{\overline{PQ}} \div \frac{\overline{BA}}{\overline{BQ}} = \frac{4}{3} \div \frac{13}{6} = \frac{8}{13}$, and $(A, T; S, B) = \frac{\overline{SA}}{\overline{ST}} \div \frac{\overline{BA}}{\overline{BT}} = \frac{7}{\overline{ST}} \div \frac{13}{5} = \frac{35}{13\overline{ST}}$. Thus, $\frac{8}{13} = \frac{35}{13\overline{ST}}$, and $\overline{ST} = \frac{35}{8}$.

Now we'll introduce some common configurations in projective geometry and some lemmas.

Midpoint and Point at Infinity Bundle (31.5)

Consider points A, B, and let M be the midpoint of AB. If P_∞ is the point at infinity on line AB, then $(A, B; M, P_\infty) = -1$.

This is useful because it can be inverted, creating harmonic quadrilaterals with one of the vertices being the center of inversion. The proof is obvious, and is only included for completeness.

Theorem 31.5's Proof

By the definition of cross ratios, $\frac{\overline{MA}}{\overline{MB}} : \frac{\overline{P_\infty A}}{\overline{P_\infty B}} = -1 : 1 = -1$, as desired.

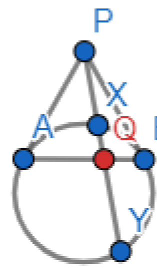
Harmonic Bundle With Polar (31.6)

Consider circle ω and P outside of ω. Let the tangents from P to ω intersect ω at A, B. Let line l through P intersect ω at distinct points X, Y such that $PX < PY$. Also, let l intersect AB at Q. Then $XAYB$ is harmonic, and $(P, Q; X, Y) = -1$.

Theorem 31.6's Proof

Notice that we want to prove $(X,Y;A,B)=\frac{\overline{AX}}{AY}:\frac{\overline{BX}}{BY}=-1.$ We don't need to worry about signed directions anymore. Since $\triangle PXA\sim\triangle PAY$, $\frac{\overline{XA}}{AY}=\frac{\overline{PX}}{PA}$. Since $\triangle PXB\sim\triangle PBY$, $\frac{\overline{XB}}{BY}=\frac{\overline{PX}}{PB}$. Since $\overline{PA}=\overline{PB}$, then $\frac{\overline{XA}}{AY}=\frac{\overline{XB}}{BY}$, as desired.

To prove that $(P,Q;X,Y)=-1,$ we project from the circumcircle of $XAYB$ to line l about point A. Then $A\rightarrow P,B\rightarrow Q,X\rightarrow X,Y\rightarrow Y$. (Note that as A' approaches A, AA' approaches the tangent line.) Since cross-ratios are preserved upon projection, we are done.



This theorem can prove that P is the pole of Q with respect to the circumcircle of $AXBY$, which is very useful for collinearity/concurrency problems involving circles. This is probably the strongest theorem in this chapter, as problems involving most other theorems will fall pretty easily. (The exception is Brokard's Theorem, which relies on this.)

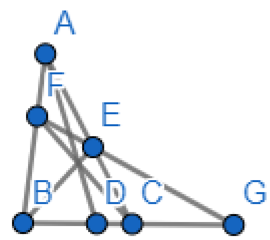
Ceva-Menelaus Harmonic Bundle (31.7)

Consider $\triangle ABC$ with cevians AD,BE,CF. Let EF intersect BC at G. Then $(B,C;D,G)$ is a harmonic bundle if and only if AD,BE,CF concur.

Theorem 31.7's Proof

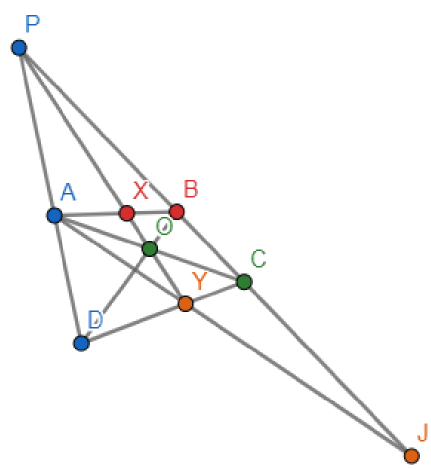
Using directed lengths, for any two points $A,B,$ there is only one point X such that $\frac{\overline{AX}}{BX}=x.$ Thus, proving the if proves the only if, and vice versa. We prove the only if condition.

If $(B,C;D,G)=-1,$ then $\frac{\overline{DB}}{DC}\cdot\frac{\overline{GC}}{GB}=-1.$ Notice that this implies $\frac{\overline{GB}}{GC}=-\frac{\overline{DB}}{DC}$. By Menelaus (6.6), $\frac{\overline{AE}}{BF}\cdot\frac{\overline{BG}}{CG}\cdot\frac{\overline{CE}}{AE}=-1.$ Substituting yields $\frac{\overline{AE}}{BF}\cdot\frac{\overline{DB}}{DC}\cdot\frac{\overline{CE}}{AE}=1.$ By Ceva's (6.5), AD,BE,CF concur, as desired.



Complete Quadrilateral Harmonic Bundle (31.8)
Consider quadrilateral $ABCD$. Let AC, BD meet at O and let AD, BC meet at P. Let OP meet AB, CD at X, Y, respectively. Prove that $(P, O; X, Y) = -1$.

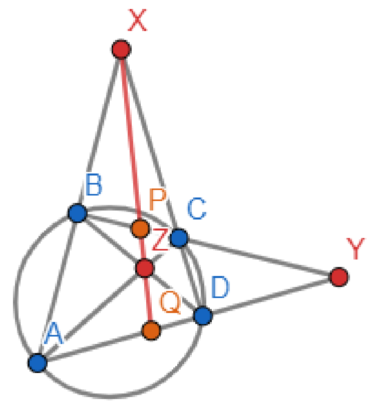
Theorem 31.8's Proof
Let AY intersect PC at J. If we consider $\triangle PCD$ and cevians PY, DB, CA, then by Theorem 31.7, $(P, C; B, J) = -1$. Project from BC to PO through P to get $(P, O; X, Y) = -1$, as desired.



Brokard's Lemma (31.9)
Consider cyclic quadrilateral $ABCD$ with center O. Let AB intersect CD at X, let BC intersect DA at Y, and let AC intersect BD at Z. Then X is the polar of YZ, Y is the polar of ZX, and Z is the polar of XY. Also, O is the orthocenter of $\triangle XYZ$.

Theorem 31.9's Proof
The configuration looks a lot like Theorem 31.8. Let ZX intersect BC at P and DA at Q. Then we want to prove that PQ is the polar of Y. By Theorem 31.8, $(Y, P; C, B) = (Y, Q; D, A) = -1$. By Theorem 31.6, P, Q lie on the polar of Y, as desired.

By the definition of a polar, $OY \perp ZX$, so O is the orthocenter. Symmetry applies because projective geometry doesn't care about orientation.



Let's look at Apollonian circles. Apollonian circles are the locus of points X such that $\frac{\overline{AX}}{\overline{BX}} = k$, where AB is a line segment and k is some constant. The most simple proof of the fact that the Apollonian circle is a circle is through Cartesian coordinates, but projective geometry also holds a way of proving this. To do this, we first state a lemma.

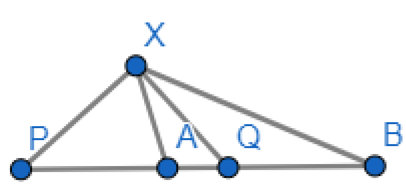
Harmonic Bundles in Right Triangles (31.10)

Consider collinear points P, A, Q, B in that order, and consider some other point X. Then any two of the following three implies the third.

1. $(A, B; P, Q) = -1$.
2. $\angle PXQ = 90^\circ$.
3. $\angle APQ = \angle BPQ$.

Theorem 31.10's Proof

Notice that $(A, B; P, Q) = \frac{\overline{PA}}{\overline{PB}} : \frac{\overline{QA}}{\overline{QB}} = \frac{\overline{PA}}{\overline{PB}} \cdot \frac{\overline{QB}}{\overline{QA}}$. By the Angle Bisector Proportionality Theorem (7.1.1), $\frac{\overline{QB}}{\overline{QA}} = \frac{\overline{XB}}{\overline{XA}}$. Thus we want to prove that $\frac{\overline{PA}}{\overline{PB}} \cdot \frac{\overline{XB}}{\overline{XA}} = 1$, or that $\frac{\overline{XB}}{\overline{PB}} = \frac{\overline{XA}}{\overline{PA}}$. By the Law of Sines (9.1), $\frac{\overline{XB}}{\overline{PB}} = \frac{\sin(\angle XPB)}{\sin(\angle PXB)}$ and $\frac{\overline{XA}}{\overline{PA}} = \frac{\sin(\angle XPB)}{\sin(\angle PXA)}$. Since $\angle PXB + \angle PXA = 180^\circ$, $\sin(\angle PXA) = \sin(\angle PXB)$, as desired.



Try proving this synthetically. (Hint: Draw the line through Q parallel to PX to create an isosceles triangle.)

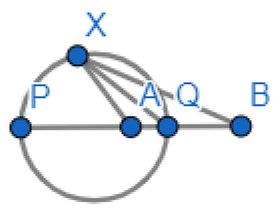
Now let's look at Apollonian circles.

Apollonian Circles (31.11)

The locus of points X such that $\frac{\overline{AX}}{\overline{BX}} = k$, where AB is a line segment and k is some constant, is a circle.

Theorem 31.11's Proof

Let P, Q be the points on AB that satisfy this condition. Then by definition, $(A, B; P, Q)$ is a harmonic bundle. If $\angle PXQ = 90^\circ$, then by Theorem 31.10, $\angle AXQ = \angle BXQ$. By the Angle Bisector Proportionality Theorem (7.1.1), $\frac{\overline{AX}}{\overline{BX}} = \frac{\overline{AQ}}{\overline{BQ}} = k$, as desired.



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1. Let P be a point outside of circle ω with center O, and let the tangents from P to ω intersect ω at A, B. Prove that $APBO$ is a harmonic quadrilateral.

2. Consider $\triangle ABC$ with cevians AD, BE, CF. Let EF intersect BC at G. If B, C, D are fixed and A, E, F vary such that AD, BE, CF concur, prove that G is a fixed point.

3. Consider segment AB and point P on segment AB. Let X be a point on the circle with diameter AB. Let the reflection of XP about BP intersect AB at P'. Prove that as X varies, P' stays constant.

4. Consider $\triangle ABC$ with circumcircle ω. Let the tangents to ω at B, C intersect at S. Prove that AS is the symmedian.

5. Consider $\triangle ABC$ with X on AC and Y on AB. Let BX and CY intersect at P, and let the incircle of $\triangle PBC$ be ω. Let M be on ω such that XM is tangent to ω and M is not on BX, and let N be on ω such that YN is tangent to ω and M is not on CY. Let XM and YN intersect at Z. Prove that XN, YM, PZ concur.

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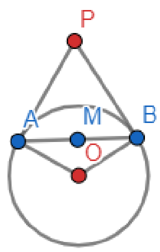
1. Let P be a point outside of circle ω with center $O,$ and let the tangents from P to ω intersect ω at $A, B.$ Prove that $APBO$ is a harmonic quadrilateral.

Solution: Invert about ω to get $A, B, M, P_\infty,$ where M is the midpoint of AB and P_∞ is the point at infinity of $AB.$ Then use Theorem 31.5 and Theorem 31.4. Since $(A, B; M, P_\infty) = (A, B; P, O) = -1,$ $APBO$ is harmonic as desired.

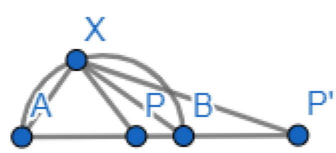


2. Consider $\triangle ABC$ with cevians $AD, BE, CF.$ Let EF intersect BC at $G.$ If B, C, D are fixed and A, E, F vary such that AD, BE, CF concur, prove that G is a fixed point.

Solution: By Theorem 31.7, $(B, C; D, G) = -1.$ There is only one point G where this is true, so G is fixed.

3. Consider segment AB and point P on segment $AB.$ Let X be a point on the circle with diameter $AB.$ Let the reflection of XP about BP intersect AB at $P'.$ Prove that as X varies, P' stays constant.

Solution: By Theorem 31.10, $(A, B; P, P') = -1,$ so P' is fixed.



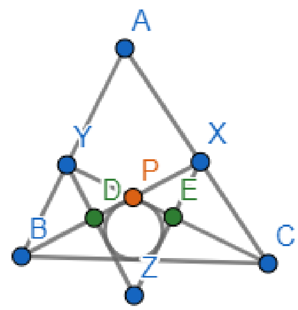
4. Consider $\triangle ABC$ with circumcircle $\omega.$ Let the tangents to ω at B, C intersect at $S.$ Prove that AS is the symmedian.

Solution: Let AS intersect BC at P and ω at $Q.$ By Theorem 31.6, $(A, Q; P, S) = -1.$ Project from B to ω to get that $(A, Q; C, B) = -1.$ By the definition of cross ratios, $\frac{\overline{CA}}{CQ} : \frac{\overline{BA}}{BQ} = -1,$ which is a property of the symmedian.

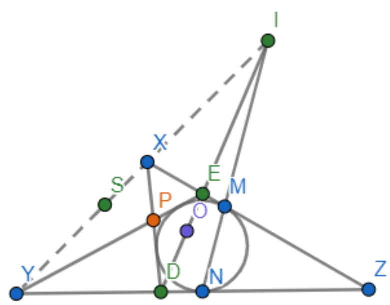
5. Consider $\triangle ABC$ with X on AC and Y on $AB.$ Let BX and CY intersect at $P,$ and let the incircle of $\triangle PBC$ be $\omega.$ Let M be on ω such that XM is tangent to ω and M

is not on BX, and let N be on ω such that YN is tangent to ω and M is not on CY. Let XM and YN intersect at Z. Prove that XN, YM, PZ concur.

Solution: Let BX intersect YZ at D and let CY intersect XZ at E. Now we can just consider tangential quadrilateral $PDZE$ and erase A, B, C from our minds.



Let PZ and DE intersect at O. Then let OP intersect XY at S, and let RQ intersect XY at I. By Theorem 31.7, $(Y, X; S, I) = -1$. It is also obvious that XD, YE, ZP concur. Thus, for XN, YM, ZS to concur, MN must concur with XY, ED.



To do this, we use La Hire's (26.4) and prove that Z lies on the polar of I instead. For obvious reasons, S, O, Z are collinear. We want to prove that S lies on the polar of I and that O lies on the polar of I. Use La Hire's (26.4) again and prove that I lies on the polar of O and the polar of P. We claim that XY is the polar of O and ED is the polar of S, or in other words, O lies on the polar of S and vice versa. This is because $(S, O; P, Z)$ is a harmonic bundle. Obviously, I lies on XY and ED, and SO is the polar of I, so Z lies on the polar of I, as desired.

<u>The Third Dimension</u>

Volume and Surface Area

Volume, like area, is built on perpendicularity. Before we can explore the third dimension, we must first talk about *perpendicularity to a plane*.

Consider plane N with point P. A line l passing through P is said to be perpendicular to N if and only if any line k within plane N passing through P is perpendicular to l.

Now we can define the volume of a figure with a constant base, such as a cylinder or rectangular prism. (A figure with a constant base is one such that the cross-section of said figure with any plane parallel to the base is constant. This definition may sound intimidating at first, but it will make more sense.) If the altitude is h and the area of the base is B, the area of the figure is Bh.

Volume of a Cube (32.1)

Given a cube with side length x, it has volume x^3.

Theorem 32.1's Proof

Let one of its sides be the base. Then $B = x^2$ and $h = x$, implying the volume is

$$Bh = x^2 \cdot x = x^3.$$

Now we introduce Cavalieri's Principle, a powerful method of giving us the volume of a cone, parallelepiped, sphere, and so on. (A parallelepiped is a solid that has 6 parallelogram faces.)

Cavalieri's Principle states that if two solids have identical cross-sectional areas when a plane is moved parallel to an arbitrary base, then the two solids have the same volume. This also provides an integral calculus approach to finding the volume.

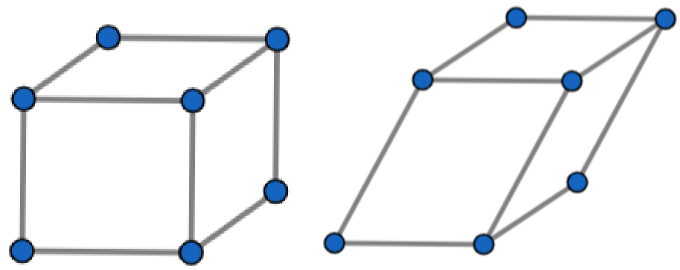
Volume of a Parallelepiped (32.2)

Consider a parallelepiped that has a base of area B and a height of h. Then its area is

$$Bh.$$

Theorem 32.2's Proof

By Cavalieri's Principle, the parallelepiped has the same volume as a parallel prism with area B and height h, thus it has volume Bh.



Volume of a Pyramid/Cone (32.3)
Given a pyramid or a cone with base of area B and a height of $h,$ its volume is $\frac{Bh}{3}$.

If you are not familiar with calculus, feel free to ignore the proof. (You can probably ignore all of these proofs, though some of them may be interesting.)

Theorem 32.3's Proof
Let the plane we "scan" with be parallel to the base. If k is the distance from our plane and the apex (the top) of the pyramid/cone, then the area of the cross-section is $B\frac{k^2}{h^2}$.

So we integrate $V = \int_0^h B\frac{k^2}{h^2}dk = \frac{B}{h^2}\int_0^h k^2 dk = \frac{B}{h^2} \cdot \frac{h^3}{3} = \frac{Bh}{3}$.

Of course, to find the volume of any figure, you can split it into other figures as deemed necessary.

The surface area of a figure is simply the area exposed on the exterior of said figure. Usually, a typical figure's surface area is found by adding the different surface pieces together.

Surface Area of a Cube (32.4)
The surface area of a cube with side length x is $6x^2$.

Theorem 32.4's Proof
It has 6 faces of equal area. Each face has area x^2. Multiplication yields the result.

Surface Area of a Rectangular Prism (32.5)
The surface area of a rectangular prism with side lengths l, w, h is $2(lw + wh + hl)$.

Theorem 23.5's Proof

There are two faces with dimensions $l \times w$, $w \times h$, and $h \times l$. Multiplication yields the desired result.

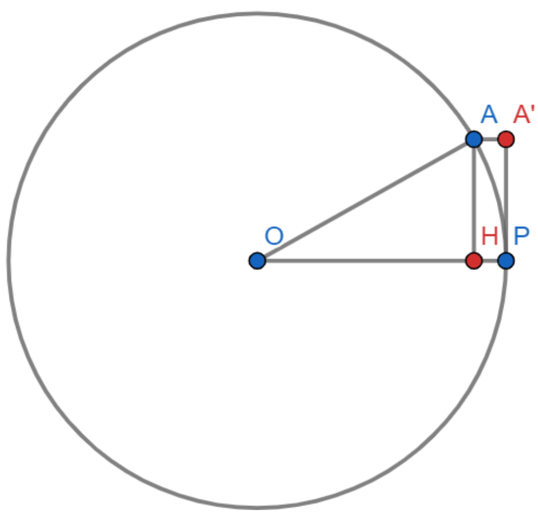
Volume of a Sphere (32.6)
The volume of a sphere with radius r is $\frac{4\pi}{3}r^3$.

Theorem 32.6's Proof
Instead we prove that the volume of a hemisphere is $\frac{2\pi}{3}r^3$. We let the altitude be perpendicular to the great circle. Then at an altitude of h, the radius of the circle created by the cross section of the plane parallel to the great circle is $\sqrt{r^2 - h^2}$, by the Pythagorean Theorem. Then the area of the circle is $\pi(r^2 - h^2)$. Integrating, the volume of the hemisphere is $V = \int_0^r \pi(r^2 - h^2 dh) = \pi(r^3 - \int_0^r h^2 dh) = \pi(r^3 - \frac{1}{3}r^3)$, as desired.

Surface Area of a Sphere (32.7)
The surface area of a sphere with radius r is $4\pi r^2$.

Theorem 32.7's Proof
We encapsulate the sphere with a cylinder of height $2r$ and radius r. The interior surface area of the cylinder is $2r \cdot 2\pi r = 4\pi r^2$. Now we form a bijection between the cylinder and sphere.

Take cross-sections perpendicular to the base of the cylinder through the center of the sphere. Then notice that since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, this analogy holds, as desired.



With the basics out of the way, here are some problems.

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1. Consider two right cylinders P and Q with the same volume. Cylinder P has a radius 30% longer than Cylinder $Q.$ What percent larger is the height of Cylinder Q than that of Cylinder P?

2. A parallelepiped with a volume of 32000 and a base area of $80.$ When the parallelepiped is cut by a line parallel and equidistant to both bases, what is the combined surface area of the two remaining figures?

3. Consider square pyramid with base $ABCD$ and apex P. If $AB = \sqrt{3}$ and $\overline{AP} + \overline{BP} = 2,$ find the maximum area of the pyramid.

---

1. Consider two right cylinders P and Q with the same volume. Cylinder P has a radius 30% longer than Cylinder Q. What percent larger is the height of Cylinder Q than that of Cylinder P?

Solution: Without loss of generality, let P have radius 1.3 and height 1. (These numbers were chosen deliberately!) Then notice the area of P is $Bh = (1.3^2 \cdot \pi) \cdot 1 = 1.69\pi$. By the given, Q has radius 1. Let Q have height x. Then the area of Q is $Bh = (1^2 \cdot \pi) \cdot x = x\pi = 1.69\pi$, as the area of Q is equal to the area of P. Thus, $x = 1.69$, so the height of Q is 69% greater than that of P.

2. A parallelepiped with a volume of 32000 and a base width of 8 and a base length of 10. When the parallelepiped is cut by a line parallel and equidistant to both bases, what is the combined surface area of the two remaining figures?

Solution: The volume of a parallelepiped is Bh. Thus, this yields $80h = 32000 \rightarrow h = 400$. To find the combined surface area, we only need to find the surface of one of the figures, then double our answer. The surface area would be $2(80 \cdot 2 + 3200 \cdot 2 + 4000 \cdot 2) = 29120$.

3. Consider square pyramid with base $ABCD$ and apex P. If $AB = \sqrt{3}$ and $\overline{AP} + \overline{BP} = 2$, find the maximum area of the pyramid.

Solution: The foot of the altitude from P to $ABCD$ is less than or equal to the slant height from AB, so the foot should land on AB. Let $\overline{AP} = x$. By Heron's Formula (5.6), $[ABP] = \frac{\sqrt{(2+\sqrt{3})(2-\sqrt{3})(2-2x+\sqrt{3})(2-2(2-x)+\sqrt{3})}}{4} = \frac{\sqrt{(2+\sqrt{3})^2 - 2(2) + 4x(2-x)}}{4}$. By AM-GM, $x(2-x)$ is maximized when $x = 1$. Then the area is $\frac{\sqrt{3}}{4}$, so the height is $\frac{1}{2}$ by $\frac{bh}{2}$ (5.2).

The intersection of a solid and a plane is known as a cross section of the solid. We can use cross sections to give us a planar view of a three dimensional problem, which is useful for length bashing, similarity, angles, and other things.

There are no theorems to be taught, so a couple of tips before we get into our problems.

If we have tangencies, take cross sections that contain the point of tangency. This is good for spheres because both cross sections will be circles.

Angles are tricky. Taking cross sections can help. Trigonometry will be your friend if you don't spot some sort of obvious symmetry (isosceles or equilateral triangles). Consider lengths individually.

The Pythagorean Theorem (and the three dimensional variant) is your friend here. After all, lengths are very important.

---

1. Inside a cone of radius 5 and height 12 there is a sphere inscribed. What is its radius?

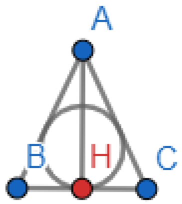
2. Consider cube $ABCDEFGH$ with dimensions $1 \times 1 \times \sqrt{3}.$ Let AE, BF, CG, DH be perpendicular to planes $ABCD$ and $EFGH,$ and let $\overline{AE} = \overline{BF} = \overline{CG} = \overline{DH} = 1.$ Furthermore, let $\overline{AB} = 1$ and $\overline{BC} = \sqrt{3}.$ Find $\angle ACG.$

3. Inside a cylinder of radius 16 and height 25 are packed two spheres of radius 12 and $r.$ Find $r.$

---

1. Inside a cone of radius 5 and height 12 there is a sphere inscribed. What is its radius?

Solution: Take a cross-section perpendicular to the base through the center of the base. Let the apex be A and the diameter of the circle which the cross section cuts off be BC. Then notice that $\overline{AH} = 12$ and $\overline{BH} = 5$, so $\overline{AB} = 13$. By $\frac{bh}{2}$ (5.2), $[ABC] = \frac{12 \cdot 10}{2} = 60$. By $[ABC] = rs$ (5.4), $60 = r \cdot \frac{1}{2}(13 + 13 + 10) = 13r$, so $r = \frac{60}{13}$.

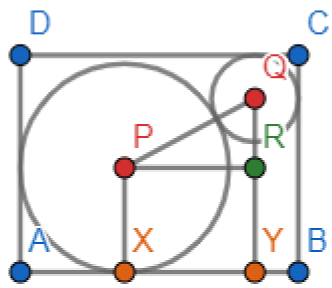


2. Consider cube $ABCDEFGH$ with dimensions $1 \times 1 \times \sqrt{3}$. Let AE, BF, CG, DH be perpendicular to planes $ABCD$ and $EFGH$, and let $\overline{AE} = \overline{BF} = \overline{CG} = \overline{DH} = 1$. Furthermore, let $\overline{AB} = 1$ and $\overline{BC} = \sqrt{3}$. Find $\angle ACG$.

Solution: By the Pythagorean Theorem, $\overline{AC} = 2$ and $\overline{AG} = \sqrt{5}$. Since $\overline{CG} = 1$, $\overline{AC}^2 + \overline{GC}^2 = \overline{AG}^2$, implying that $\angle ACG = 90^\circ$.

3. Inside a cylinder of radius 16 and height 25 are packed two spheres of radius 12 and r. Find r.

Solution: Take the cross section that includes the center of both spheres and the center of the base. Then we have a 32×25 rectangle and two circles of radius 12 and r. Let P be the center of the circle with radius 12 and Q be the center of the circle with radius r. Let X, Y be the feet of the altitudes from P, Q respectively to AB and let R be the foot of the altitude from P to QY. Notice that $\overline{XY} = \overline{AB} - 12 - r = 32 - 12 - r = 20 - r$, and $\overline{QR} = 25 - \overline{PX} - r = 13 - r$, and $\overline{PQ} = 12 + r$. By the Pythagorean Theorem, $(20 - r)^2 + (13 - r)^2 = (12 + r)^2$, implying that $r = 5$.


D
C
Q
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Tetrahedron Centers

You've probably heard about the centroid of a triangle before. Some of you might've heard about the concept of a centroid in general for polygons. But what is the centroid of a tetrahedron? (And what's the incenter, circumcenter, or orthocenter of a tetrahedron? It turns out there isn't an orthocenter in general, but there is a Monge Point.)

The centroid of an object in general is the center of mass. Okay, so that meant nothing. Something that might be more useful is that the centroid is the average of the coordinates.

So a notationally complex way to denote this in a j dimension plane would be letting the points be $P_i = (d_{i1}, d_{i2} \cdots d_{ij})$ for $1 \le i \le n$ for some n that denotes the total amount of points. Then the centroid G is $(\frac{1}{n}\sum_{x=1}^{n} d_{x1}, \frac{1}{n}\sum_{x=1}^{n} d_{x2} \cdots \frac{1}{n}\sum_{x=1}^{n} d_{xj})$. (If this is confusing, try this in two dimensions to get something you can recognize.)

Clearly we see that this is more than a little bit ugly, and we can't do anything about it with complex numbers as it is three dimensional. Why not make it look better with vectors?

Vector Formula for Centroids (33.1)

The centroid G of $P_1P_2 \cdots P_n$ satisfies $\frac{1}{n}\sum_{x=1}^{n} \vec{P}_i = \vec{G}$.

There being no tail implies that this is true regardless of the tail.

Theorem 33.1's Proof

Let the tail O have coordinates $(a_1, a_2 \cdots a_j)$ with respect to some arbitrary origin. Then $\vec{P}_i = (d_{i1} - a_1, d_{i2} - a_2 \cdots d_{ij} - a_j)$, and $\frac{1}{n}\sum_{x=1}^{n} \vec{P}_i = (\frac{1}{n}\sum_{x=1}^{n} d_{x1} - a_1, \frac{1}{n}\sum_{x=1}^{n} d_{x2} - a_2 \cdots \frac{1}{n}\sum_{x=1}^{n} d_{xi} - a_i)$. But this is the definition of $\vec{G}$, so we are done.

Tetrahedron Centroid Collinearity (33.2)

Consider tetrahedron $ABCD$ with centroid G. Let the centroid of $\triangle BCD$ be P. Then A, G, P are collinear.

Theorem 33.2's Proof

By Theorem 33.1, $\frac{1}{4}(\vec{A}+\vec{B}+\vec{C}+\vec{D})=\vec{G}.$ Let P be the common tail of all of these vectors. Then by Theorem 33.1, $\vec{B}+\vec{C}+\vec{D}=0,$ so $\vec{G}=\frac{1}{4}\vec{A},$ as desired.

This shows that the centroid cuts the medians in a $3:1$ ratio. In a triangle, the ratio is $2:1.$ Notice that a tetrahedron has three dimensions and a triangle has two! Try proving that this holds in general.

Now let's talk about the *bimedian*. The bimedian is a line segment connecting the midpoints of two opposite sides. The centroid is the midpoint of all three bimedians.

Bimedian through the Centroid (33.3)

The bimedians are bisected by the centroid.

Theorem 33.3's Proof

Let the vertices P_i have coordinates (x_i, y_i) for $1 \le i \le 4.$ Notice that the midpoint of each bimedian is $(\frac{1}{4}(x_1+x_2)+\frac{1}{4}(x_3+x_4), \frac{1}{4}(y_1+y_2)+\frac{1}{4}(y_3+y_4))$ and the centroid is $(\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4)),$ as desired.

Now let's talk about the incenter/circumcenter of a tetrahedron. In general, it takes $n+1$ points to uniquely determine an n sphere, provided that no $k+1$ points are in the same k dimensional space. In this case, it takes four points to determine a sphere provided they aren't coplanar and no three points are collinear.

$[ABCD]=\frac{1}{3}r(A+B+C+D)$ *(33.4)*

Let r be the inradius of $ABCD,$ $A=[BCD],$ $B=[CDA],$ $C=[DAB],$ and $D=[ABC].$ Then $[ABCD]=\frac{1}{3}r(A+B+C+D).$

Theorem 33.4's Proof

Let I be the incenter of $ABCD.$ By Volume of a Pyramid (32.3), $[IBCD]=\frac{1}{3}rA.$ Similar expressions follow for the other three faces. Thus, $[ABCD]=\frac{1}{3}r(A+B+C+D),$ as desired.

Of course, this result can be generalized, but the proof is basically identical.

The coordinates of the incenter are a bit harder to prove. We'll prove that $\frac{[BCD]\vec{A}+[CDA]\vec{B}+[DAB]\vec{C}+[ABC]\vec{D}}{[BCD]+[CDA]+[DAB]+[ABC]} = \vec{I}$ is equidistant from all of the planes rather than deriving it.

Incenter of a Tetrahedron (33.5)

The incenter I of tetrahedron $ABCD$ satisfies $\frac{[BCD]\vec{A}+[CDA]\vec{B}+[DAB]\vec{C}+[ABC]\vec{D}}{[BCD]+[CDA]+[DAB]+[ABC]} = \vec{I}.$

Theorem 33.5's Proof

Let's use Cartesian Coordinates this time. Let $A = (x_1, y_1, z_1),\ B = (x_2, y_2, z_2),$ $C = (x_3, y_3, z_3),$ and $D = (x_4, y_4, z_4).$ Also, let $a = [BCD],\ b = [CDA],\ c = [DAB],$ and $d = [ABC].$ Then $I = \frac{1}{a+b+c+d}(ax_1 + bx_2 + cx_3 + dx_4, ay_1 + by_2 + cy_3 + dy_4, az_1 + bz_2 + cz_3 + dz_4).$ By the Point to Plane Distance Formula (which involves finding the equation of the plane and converting it to Hessian normal form), I is equidistant from all the planes. (The distance also turns out to be $\frac{3[ABCD]}{A+B+C+D},$ which is exactly what Theorem 33.3 says.)

Now, we will define the tetrahedron's analogy to the orthocenter, the Monge point.

We define the *midplanes* of a tetrahedron as the plane through the midpoint of one edge perpendicular to the opposite edge. (There are six midplanes.) They concur at the *Monge point*.

The Monge Point Exists (33.6)

The midplanes of $ABCD$ concur.

Theorem 33.6's Proof

Let P, Q, R be the midpoints of AB, AC, AD and let the feet of the altitudes from P, Q, R intersect $\triangle BCD$ at $P', Q', R'.$ Then the plane perpendicular to CD through the midpoint of P passes $P'.$ We can do similar things with $Q, R.$ Then notice that the altitude from P' to $Q'R'$ is the same as the altitude from P' to $CD.$ Thus we can let H be the orthocenter of $PQR.$ Notice that the intersection of our three midplanes is the line through H perpendicular to plane $BCD.$ We can do this to all the planes - let's call this line the A Monge line, and define the other Monge lines similarly.

Let the B and D Monge lines intersect at $M.$ Then M must lie on every midplane except for the one perpendicular to $BD.$ But M lies on the intersection of the midplanes perpendicular to AB and $AD,$ which is the C Monge line, which lies on the midplane perpendicular to $BD,$ as desired.

With the centroid, circumcenter, and the analogous point to the orthocenter (the Monge point) defined, we will discuss the *twelve point sphere*, the analogy to the nine point circle.

Euler Line of a Tetrahedron (33.7)

Consider tetrahedron $ABCD$ with circumcenter $O,$ centroid $G,$ and Monge point $M.$ Then O, G, M are collinear.

Theorem 33.7's Proof

We prove that the feet altitudes of the circumcenter, centroid, and Monge point to BCD are collinear. (This is because we can do this without loss of generality.) Let the circumcenter be $O = (0,0,0)$ and let BCD be parallel to the xz plane. Then let $A = (x,y,z)$ and let $B = (x_1, y_b, z_1),$ $C = (x_2, y_b, z_2),$ $D = (x_3, y_b, z_3).$ Then the centroid is $(\frac{1}{4}(x + x_1 + x_2 + x_3), \frac{1}{4}(y + 3y_b), \frac{1}{4}(z + z_1 + z_2 + z_3)).$ The foot of the centroid is $(\frac{1}{4}(x + x_1 + x_2 + x_3), y_b, \frac{1}{4}(z + z_1 + z_2 + z_3)).$ Also, the foot of the circumcenter is $(0, y_b, 0).$ Then the foot of the Monge Point is $(\frac{1}{2}(x + x_1 + x_2 + x_3), y_b, \frac{1}{2}(z + z_1 + z_2 + z_3)).$ Thus the centroid is the midpoint of $OM.$

The Twelve Point Sphere (33.8)

Consider tetrahedron $ABCD$ with Monge point $M.$ The sphere through the four centroids of $ABCD$ also pass through the points A', B', C', D' such that $\overline{A'M} = \frac{1}{3}\overline{AM},$ $\overline{B'M} = \frac{1}{3}\overline{BM},$ $\overline{C'M} = \frac{1}{3}\overline{CM},$ and $\overline{D'M} = \frac{1}{3}\overline{DM},$ and it also passes through the feet of the altitudes of the Monge point.

Theorem 33.8's Proof

We take a homothety about the Monge point with a factor of $3.$ This sends A', B', C', D' to $A, B, C, D.$ Now we just have to prove that the dilation of the foot of the altitude from the Monge point to BCD and the dilation of the centroid of BCD lies on the circumsphere of $ABCD.$

Let the circumcenter be $O = (0,0,0)$ and let $A = (x_1, y_1, z_1),$ $B = (x_2, y_2, z_2),$ $C = (x_3, y_3, z_3),$ $D = (x_4, y_4, z_4).$ Then $G = (\frac{1}{4}\sum_{i=1}^{4} x_i, \frac{1}{4}\sum_{i=1}^{4} y_i, \frac{1}{4}\sum_{i=1}^{4} z_i)$ and $M = (\frac{1}{2}\sum_{i=1}^{4} x_i, \frac{1}{2}\sum_{i=1}^{4} y_i, \frac{1}{2}\sum_{i=1}^{4} z_i).$ Then notice the centroid of BCD is $(\frac{1}{3}\sum_{i=2}^{4} x_i, \frac{1}{3}\sum_{i=2}^{4} y_i, \frac{1}{3}\sum_{i=2}^{4} z_i).$

Thus the dilation of the centroid of BCD about M is $(-x_1, -y_1, -z_1)$, which is obviously on the circumsphere.

For the Monge point's foot, we can assume that BCD is parallel to the xz plane. Then the foot is $(\frac{1}{2}\sum_{i=1}^{4} x_i, y_2, \frac{1}{2}\sum_{i=1}^{4} z_i)$. Then the dilation is $(\frac{1}{2}\sum_{i=1}^{4} x_i, -\frac{9}{4}y_i, \frac{1}{2}\sum_{i=1}^{4} z_i)$. It turns out that this is equal to $x_i^3 + y_i^3 + z_i^3$ for all i, as desired.

---

1. Consider n points $P_1P_2 \cdots P_n$ in $n-1$ dimensional space. Let G be the centroid of $P_1P_2 \cdots P_n$ and let Q be the centroid of $P_2P_3 \cdots P_n$. Prove that $\vec{PG} = \frac{1}{n}\vec{PA}$.

2. A regular tetrahedron has an inradius of 1. What is its side length?

3. Generalize the formula for the incenter to higher dimensions.

---

1. Consider n points $P_1P_2\cdots P_n$ in $n-1$ dimensional space. Let G be the centroid of $P_1P_2\cdots P_n$ and let Q be the centroid of $P_2P_3\cdots P_n$. Prove that $\vec{PG} = \frac{1}{n}\vec{PA}$.

Solution: Notice that by Theorem 33.1, $\frac{1}{n}(\sum_{i=1}^{n}\vec{P}_i) = \vec{G}$. Since Q is the tail, $\sum_{i=2}^{n}\vec{P}_i = 0$, so $\frac{1}{n}\vec{A} = \vec{G}$, as desired.

2. A regular tetrahedron has an inradius of 1. What is its side length?

Solution: Let its sidelength be x. Then by the Pythagorean Theorem and Volume of a Pyramid/Cone (32.3), $V = \frac{x^3\sqrt{2}}{12}$. Then by $[ABCD] = \frac{1}{3}r(A+B+C+D)$ (33.3), $V = \frac{1}{3}(4\frac{x^2\sqrt{3}}{4})$, implying that $\frac{x^3\sqrt{2}}{12} = \frac{x^2\sqrt{3}}{4}$, or that $x = \frac{3\sqrt{6}}{2}$.

3. Generalize the formula for the incenter to higher dimensions.

Solution: The area of a hyper-pyramid in n dimensions with a base of b and a height of h is $\frac{bh}{n}$. (Proof - integral calculus.) Since $h = r$ for all the hyper-pyramids, $V = (\sum_{i=1}^{n+1} b_i)\frac{r}{n}$.

Extra

Pythagorean Theorem

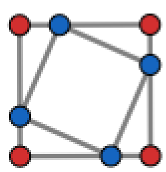
We discuss several ways to prove the Pythagorean Theorem. We will give diagrams as hints as to where we want the reader to start, and leave the solutions below.

The Pythagorean Theorem states that for right $\triangle ACB$ with $\angle C = 90^\circ$, that $\overline{BC}^2 + \overline{AC}^2 = \overline{AB}^2$. This is commonly denoted as $a^2 + b^2 = c^2$.

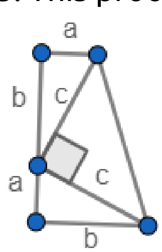
---

1. Let the foot of the altitude from C to AB be H. Use similar triangles.

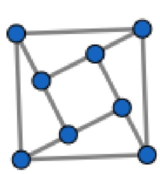
2. Construct a square with side length c. Then, construct four right triangles with side lengths a, b, c such that the hypotenuse is a side of the square and the four right triangles make a square.



3. This proof is due to the 20th President of the United States, James Garfield.



4. This proof is due to Bhaskara. Let the larger square have side length c and the smaller one have side length $|a - b|$.



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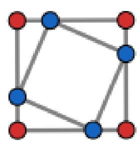
1. Let the foot of the altitude from C to AB be $H.$ Use similar triangles.

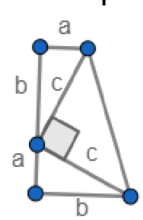
Solution: Notice that $\triangle ABC \sim \triangle ACH \sim \triangle CBH.$ Thus, $\frac{\overline{AH}}{AC} = \frac{\overline{AC}}{AB}$ and $\frac{\overline{BH}}{CB} = \frac{\overline{CB}}{AB}.$ This implies that $\overline{AH} \cdot \overline{AB} = \overline{AC}^2$ and $\overline{BH} \cdot \overline{AB} = \overline{CB}^2.$ Adding these up yields $\overline{AB}(\overline{AH} + \overline{BH}) = \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2,$ as desired.

2. Construct a square with side length $c.$ Then, construct four right triangles with side lengths a, b, c such that the hypotenuse is a side of the square and the four right triangles make a square.



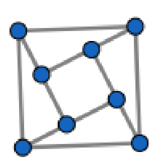
Solution: The area of the entire square can be expressed as $(a+b)^2$ and as $c^2 + 4\frac{ab}{2} = c^2 + 2ab.$ If $(a+b)^2 = c^2 + 2ab,$ then $a^2 + b^2 = c^2,$ as desired.

3. This proof is due to the 20th President of the United States, James Garfield.



Solution: The area of this entire figure can either be expressed as $\frac{(a+b)^2}{2}$ or as $\frac{1}{2}c^2 + 2\frac{ab}{2}.$ This implies that $a^2 + b^2 = c^2$ as desired.

4. This proof is due to Bhaskara. Let the larger square have side length c and the smaller one have side length $|a - b|.$



Solution: This implies that $c^2 = (a-b)^2 + 4\frac{ab}{2} = a^2 + b^2$ as desired.

For more proofs of the Pythagorean Theorem, see Cut the Knot's page of 115 proofs.

Constructions

Constructions are one of the most important skills in Olympiad Geometry. Making an accurate diagram (or diagrams) can help you see certain “coincidences” and lead you on the right path, or correct you when you’re wrong on something.

Informally constructions are anything you can do with a straightedge and compass. Formally construction consists of five operations.

1. Given two points, you may draw the line connecting them.
2. Given two points, you may draw a circle with the center at one point and the other point on the circle.
3. You can create the intersection of two non-parallel lines.
4. You can create the intersection of a line and a circle.
5. You can create the intersection of two circles.

Surprisingly, you can construct pretty much everything you care about (altitudes, medians, perpendicular bisectors, midpoints, circumcircles, and more).

There are also two “types” of compasses. One is the non-collapsing compass (you can keep it opened up to a certain length) and the other is the collapsing compass (once it leaves the page, the length is “lost”). There is no difference between the things you can construct, so we will just use the non-collapsing compass as it provides more practicality.

Duplicating the Line Segment (34.1)

Given a line segment of a certain length, you can make another line segment of the same length.

Theorem 34.1’s Proof

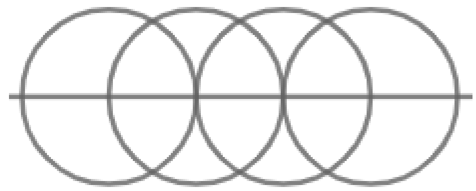
Open up the compass to be as wide as the line segment. Draw a circle and pick any point on the circle and its center.

Multiplying the Line Segment (34.2)

Given a line segment of length $1,$ you may make a line segment of length n for any integer $n.$

Theorem 34.2’s Proof

Make the line segment a line (while keeping track of the endpoints). Then draw a circle with radius 1 and any center O on the line and let it intersect the line at A_1. Then draw a circle with radius 1 with center A_1 and let it intersect at A_2 such that $\overline{OA_2} > \overline{OA_1}$. Repeat this process such that $\overline{OA_{i+1}} > \overline{OA_i}$. Then $\overline{OA_n} = n$, as desired.



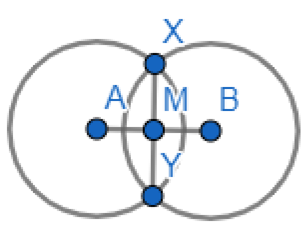
Constructing a Triangle (34.3)
Given three line segments, you can construct a triangle with them as side lengths, provided they satisfy the Triangle Inequality.

Theorem 34.3's Proof
Let the line segments have lengths a, b, c, and let the line segment of length a have endpoints X, Y. Then draw a circle with radius b centered at X and draw a circle with radius c centered at Y. Their intersections create the desired triangle.

Now for what we came for - cevians.

Constructing the Midpoint (34.4)
You can construct the midpoint of a line segment.

Theorem 34.4's Proof
Construct two circles centered at A and B with equal radius such that the circles intersect. (This means the radius must be longer than half the line segment by the Triangle Inequality.) Then notice that $\overline{AX} = \overline{BX}$ and $\overline{AY} = \overline{BY}$, so XY is the perpendicular bisector of AB. They intersect at M, which is the midpoint by definition.



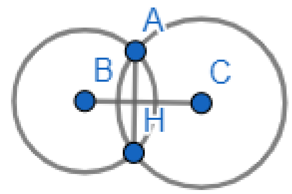
This also doubles as the construction for the perpendicular bisector.

Constructing the Altitude (34.5)

You can construct the altitude of a triangle.

Theorem 34.5's Proof

Let the triangle be $\triangle ABC$. Let the circle with center B through A and the circle with center C through A intersect at another point H. Then H is the reflection of A about BC, implying that AH is perpendicular to BC.

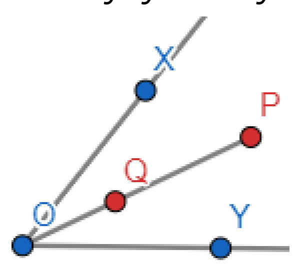


Constructing the Angle Bisector (34.6)

Given an angle, you can construct its angle bisector.

Theorem 34.6's Proof

Let the angle have vertex O. Then let a circle with center O intersect the angle at X, Y. Then draw two congruent circles with centers X, Y such that they intersect at P, Q. Then PQ bisects the angle as the distance from P, Q to OX and OY are the same by symmetry.



Perpendicular through a Point on the Line (34.7)

Given a line and point P on it, you can construct a perpendicular through P.

Theorem 34.7's Proof

Draw a circle with center P and let it intersect the line at X, Y. Then construct the perpendicular bisector of XY. As $\overline{PX} = \overline{PY}$, this will work.

Parallel Lines (34.8)

Given a line and a point, you can construct a parallel line through that point.

Theorem 34.8's Proof

This is different from the copy an angle method (which personally annoys me).

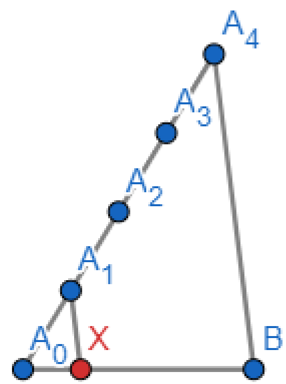
Construct a perpendicular from the point to the line. Then, through the point, construct a perpendicular through the perpendicular line. If two lines are perpendicular to the same line, they are parallel.

The p/q Theorem (34.9)

Provided a line segment of length $1,$ you can construct a line segment of length $\frac{p}{q}$ for positive integers $p, q.$

Theorem 34.9's Proof

To prove this we merely need to construct a line segment of length $\frac{1}{q}.$ Let the line segment of length 1 be $A_0B.$ Through A_1 draw a line $l.$ Then using a compass, draw points $A_1, A_2, A_3 \cdots A_q$ such that $\overline{A_iA_{i+1}}$ is constant. Then draw line A_qB and through A_1 draw a line parallel to $A_qB.$ Then let it intersect A_0B at $X.$ Then $\overline{A_0X} = \frac{1}{q}$ by similar triangles.



The $\sqrt{p/q}$ Theorem (34.10)

Provided a line segment of length $1,$ you can construct a line segment of length $\sqrt{\frac{p}{q}}$ for positive integers $p, q.$

Theorem 34.10's Proof

Let $\sqrt{\frac{p}{q}} = \frac{\sqrt{m}}{n}.$ Then if we make a segment of length $\frac{1}{n}$ by the p/q Theorem (34.9), we can let it be our base segment and make a segment of length $\sqrt{m}$ (with respect to the base segment).

We proceed by induction. Notice a segment of length 1 is possible. Then notice that a segment of length $\sqrt{m+1}$ is possible if a segment of $\sqrt{m}$ is possible, since you can make a right triangle with legs of lengths $\sqrt{m}$ and $1.$

As for the problems, we will be discussing a *locus of points*, which is the set of all points that satisfy a given condition. When given an angle condition, use the Inscribed Angle Theorem (1.1). When told that a line is a certain distance away from a point, use Theorem 3.4.

Similar triangles are your best friend!

---

1. Consider circle ω with diameter AB and radius $r.$ Let C be a point on the circle. What is the area of the locus of the centroid of $\triangle ABC$?

2. Construct the center of a circle.

3. Consider line AB and line l parallel to $AB.$ Let X be a point on l and let H be the orthocenter of $\triangle ABX.$ What is the locus of points H?

4. Consider the parabola $y = x^2.$ Let $O = (0,0)$ and let $G = (0,c).$ Find points A, B on the parabola such that G is the centroid of $\triangle OAB.$

5. Consider circle ω with points A, B on it such that the measure of minor arc AB is $60^\circ.$ Let C be a point on major arc $AB.$ Prove that the angle bisector of $\angle ACB$ passes through a fixed point.

---

1. Consider circle ω with diameter AB and radius r. Let C be a point on the circle. What is the area of the locus of the centroid of $\triangle ABC$?

Solution: Let M be the midpoint of AB. The centroid of $\triangle ABC$ is the point G such that $\overline{MG} = \frac{1}{3}\overline{MC}$, so ω is dilated into a circle with radius $\frac{r}{3}$. Thus the area is $\frac{r^2\pi}{9}$.

2. Construct the center of a circle.

Solution: Pick any two distinct chords. Their perpendicular bisectors intersect at the center.

3. Consider line AB and line l parallel to AB. Let X be a point on l and let H be the orthocenter of $\triangle ABX$. What is the locus of points H?

Solution: We claim that the locus is a parabola. We coordinate bash.

Let $A = (0,0)$ and let $B = (1,0)$. Let l have equation $y = q$. Then let $X = (p,q)$. Then the altitude from X to AB is $x = p$, and the altitude from A to BX is $y = \frac{-(p+1)}{q}x$. Solving, we get $y = \frac{-p(p+1)}{q}$, which is a quadratic about p, as desired.

4. Consider the parabola $y = x^2$. Let $O = (0,0)$ and let $G = (0,c)$. Find points A, B on the parabola such that G is the centroid of $\triangle OAB$.

Solution: Let M be the midpoint of AB. Notice that $M = (0, \frac{3c}{2})$. Then draw line $y = \frac{3c}{2}$, and where it intersects the parabola, you have your points A, B.

5. Consider circle ω with points A, B on it such that the measure of minor arc AB is 60°. Let C be a point on major arc AB. Prove that the angle bisector of $\angle ACB$ passes through a fixed point.

Solution: This fixed point is the arc midpoint of AB.

Directed Angles

One of the most annoying things in olympiad geometry are configuration issues with circles. How does this occur? Let's take a look at the Opposite Angles of Cyclic Quadrilaterals Theorem (2.1) and the Diagonal Angles of Cyclic Quadrilaterals Theorem (2.2), the fundamental theorems of angles in a circle. Basically, two angles are congruent only when they're on the same side, otherwise they're supplementary. That's annoying. Why don't we fix that?

We introduce the idea of *directed angles*. Consider two lines $m, n.$ Then $\measuredangle(m, n)$ is the amount of degrees you have to rotate m **counterclockwise** about their point of intersection for it to overlap with $n.$ Of course, if you rotate the line by $180^{\circ},$ it stays the same, so we can direct angles *modulo* $180.$ This means that negative angles also make sense. They're just clockwise.

Of course, this means that $\measuredangle(m, n) = -\measuredangle(n, m).$ This also fixes another annoyance - the Angle Addition Property.

Usually, $\angle AOP + \angle POB = \angle AOB$ only when P is "inside" $\angle AOB.$ Now, this is no longer the case.

For concurrent lines $l, m, n,$ $\measuredangle(l, m) + \measuredangle(m, n) = \measuredangle(l, n).$ You can do this with more lines if desired. In general, for concurrent lines $l_1, l_2 \cdots l_n,$ $\sum_{i=1}^{n-1} \measuredangle(l_i, l_{i+1}) = \measuredangle(l_1, l_n).$

We can use three points to denote an angle as well.

For points $A, B, C, D,$ $\measuredangle ACB = \measuredangle ADB$ if and only if A, B, C, D are concyclic.

Also, for $\triangle ABC,$ $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0.$ It should be obvious why.

Points A, B, C are collinear when $\measuredangle XAB = \measuredangle XAC$ for all points $X.$

Directed angles are useful with problems involving circles (which usually also involve configuration issues) and collinearity/concurrency problems. They should be avoided with angle bisector problems, since you can't take "half an angle" modulo $180.$ They're also unnecessary with problems that don't have any configuration issues (mostly problems that don't involve circles or angles in any way.) You'll see that there are some problems (such as the last problem in Circles and Angles) that are much easier to do with one configuration than the other, and that can save a lot of headache.

No problems for this section - use directed angles on pure angle chasing problems that involve circles.

Geometric Inequalities

This chapter assumes the knowledge of inequalities in olympiad math.

The fundamental inequality for geometric inequalities is the Triangle Inequality.

The Triangle Inequality (35.1)

Given non-collinear points $A, B, C,$ $\overline{AB} < \frac{1}{2}(\overline{AB} + \overline{BC} + \overline{CA}).$

Theorem 35.1's Proof

We prove that $\overline{AB} < \overline{BC} + \overline{CA}.$ The shortest distance between two points is a line, so we are done.

When equality is achieved, A, C, B are collinear, **in that order**.

Now let's discuss two harder theorems. One concerning the circumradius and inradius of a triangle, and the second concerning the lengths of a quadrilateral.

Euler's Inequality (35.2)

Consider $\triangle ABC$ with circumradius R and inradius $r.$ Then $R \geq 2r$ with equality if and only if $\triangle ABC$ is equilateral.

This is really a disguised version of Euler's Equality, which states that $\overline{OI} = \sqrt{R(R-2r)}.$ (We won't prove this; we'll use a different method to prove our theorem.)

Theorem 35.2's Proof

Let $\triangle ABC$ have side lengths $a, b, c.$ Then $R = \frac{abc}{4[ABC]}$ and $r = \frac{2[ABC]}{a+b+c}.$ We want to prove that $\frac{abc}{4[ABC]} \geq \frac{4[ABC]}{a+b+c}$, or that $abc(a+b+c) \geq 16[ABC]^2.$ By Heron's Formula (5.6), $16[ABC]^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$ Thus we want to prove $abc \geq (-a+b+c)(a-b+c)(a+b-c).$ Notice that $a+b+c = (-a+b+c)+(a-b+c)+(a+b-c).$ Assume $a \leq b \leq c.$ Since $(-a+b+c) \geq c$ and $(a+b-c) \geq a,$ $abc \geq (-a+b+c)(a-b+c)(a+b-c)$ as desired.

Ptolemy's Inequality (35.3)

Consider convex quadrilateral $ABCD.$ Then $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{DA} \geq \overline{AC} \cdot \overline{BD}$ with equality when $ABCD$ is cyclic.

Theorem 35.3's Proof

Let P be the point such that $\triangle ABC \sim \triangle ADP$. Then $\triangle ABD \sim \triangle ACP$. Then by the Triangle Inequality, $\overline{PD} + \overline{DC} \geq \overline{PC}$. Notice that by similar triangles, $\overline{PD} = \overline{BC} \cdot \frac{\overline{AD}}{\overline{AB}}$ and $\overline{PC} = \overline{DB} \cdot \frac{\overline{AC}}{\overline{AB}}$. Substituting and cross multiplying yields $\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{DA} \geq \overline{AC} \cdot \overline{BD}$, as desired. Equality occurs when P, D, C are collinear, or when $ABCD$ is cyclic.

Also, remember the Pythagorean Inequality. If $\triangle ABC$ is acute, $a^2 + b^2 > c^2$. And if $\triangle ABC$ is obtuse, then $a^2 + b^2 < c^2$. (The case of a right angle is obvious.)

Remember the formulas for a triangle. The theorems concerning the inradius, circumradius, nine point circle, Euler line, and the like may be used. Also remember that the area of a triangle with a fixed perimeter is maximized when it is equilateral, and the perimeter of a triangle with a fixed area is minimized when it is equilateral.

---

1. Prove that in $\triangle ABC$, $\sin(A) + \sin(B) + \sin(C) \leq \frac{3\sqrt{3}}{2}$.

2. Prove that $3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}}{2}R$.

3. Prove that $a^2 + b^2 + c^2 \geq 4\sqrt{3}[ABC]$.

4. If $x^4 + x^2(2\sin x + 1) + 2x\cos x + 1 = 0$, find $\sin(x)$. (All trigonometric functions are in radians.)

5. Prove that $9r \leq a\sin A + b\sin B + c\sin C \leq \frac{9R}{2}$.

---

1. Prove that in $\triangle ABC,\ \sin(A)+\sin(B)+\sin(C) \le \frac{3\sqrt{3}}{2}.$

Solution: We use a method called perturbation. We show that if we change anything from the equality state, the result gets smaller. Notice that $(\sin(A+B)+\sin(A-B)) = 2\sin(A)\cos(B) < 2\sin(A).$ Similarly we can prove that $\cos(A)+\cos(B)+\cos(C) \le \frac{3}{2}.$

2. Prove that $3\sqrt{3}r \le s \le \frac{3\sqrt{3}}{2}R.$

Solution: Notice that multiplying both sides by s yields $3\sqrt{3}[ABC] \le s^2$ by $[ABC] = rs$ (5.4). Then by Heron's Formula (5.6) we want to prove $27s(s-a)(s-b)(s-c) \le s^4,$ or $27(s-a)(s-b)(s-c) \le s^3.$ By AM-GM, $\frac{(s-a)+(s-b)+(s-c)}{3} \ge \sqrt[3]{(s-a)(s-b)(s-c)}.$ This implies that $\frac{s}{3} \ge (s-a)(s-b)(s-c).$ Multiply both sides by 3 and cubing yields the desired conclusion.

Notice that $s = \frac{a}{2}+\frac{b}{2}+\frac{c}{2}.$ Thus we want to prove $\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R} \le \frac{3\sqrt{3}}{2}.$ By the Extended Law of Sines (9.2), $\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R} = \sin(A)+\sin(B)+\sin(C).$ We have already proved $\sin(A)+\sin(B)+\sin(C) \le \frac{3\sqrt{3}}{2},$ so we are done.

(This is also another possible proof for Euler's Inequality! Unfortunately, thinking of this intermediate step would be kind of contrived.)

3. Prove that $a^2+b^2+c^2 \ge 4\sqrt{3}[ABC].$

Solution: Notice that by QM-AM, $\sqrt{\frac{a^2+b^2+c^2}{3}} \ge \frac{a+b+c}{3},$ or $a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3}.$ Notice that $\frac{(a+b+c)^2}{[ABC]} \ge 12\sqrt{3}.$ This is because if we fix $a+b+c,$ the area of $[ABC]$ is maximized when $a=b=c.$ Then $\frac{(a+b+c)^2}{[ABC]} \ge \frac{(3a)^2}{\frac{\sqrt{3}}{4}a^2} = 12\sqrt{3},$ as desired.

4. If $x^4+x^2(2\sin x+1)+2x\cos x+1=0,$ find $\sin(x).$ (All trigonometric functions are in radians.)

Solution: Notice this is $(x^2 + \sin x)^2 + (x + \cos x)^2$. Thus $\sin x = -x^2$ and $\cos x = -x$, implying that $-\cos^2 x = -x^2 = \sin x$, or that $\sin^2 x - 1 = \sin x$. By the Quadratic Formula, $\sin x = \frac{\sqrt{5}-1}{2}$.

5. Prove that $9r \leq a \sin A + b \sin B + c \sin C \leq \frac{9R}{2}$.

Solution: By the Extended Law of Sines (9.2), $a \sin A = 2R\sin^2 A$. Thus the quantity is $2R(\sin^2 A + \sin^2 B + \sin^2 C)$. Notice that $\sin^2 A + \sin^2 B + \sin^2 C \geq \frac{9}{4}$. Thus, $a \sin A + b \sin B + c \sin C \leq \frac{9R}{2}$.

For the left side, notice that $[ABC] = rs$ (5.4) and $\frac{1}{2}ab \cdot \sin(C) = [ABC]$ (5.3). Thus, we want to prove that $\frac{18[ABC]}{a+b+c} \leq \frac{2(a^2+b^2+c^2)[ABC]}{abc}$, or that $9abc \leq (a+b+c)(a^2+b^2+c^2)$, which is true by applying AM-GM and then multiplying.

The Problem Cauldron

Here are additional problems that I wrote that may be of interest. They don't necessarily have to be geometry, and ordering doesn't matter. Solutions will **not** be included.

---

1. For which integers x from 1 to 100 is $x^2+2x+15$ divisible by 10?

2. What is the smallest positive integer k such that there exists no positive integer n such that $\lfloor \frac{n^2}{36} \rfloor = k$?

3. How many 10 digit numbers divisible by 5 also have the sum of their digits divisible by 5?

4. If $a^2+8a+b^2-6b+c^2-10c+d^2+14d=70,$ find the sum of the minimum and maximum values of $a^2+b^2+c^2+d^2$.

5. Consider a number line with integers $-65, -64 \cdots 62, 63.$ Every second, a particle at the origin randomly moves to an adjacent integer. Find the expected amount of seconds for the particle to reach either -65 or 63.

6. Consider a number line with a drunkard at $0,$ and two cops at -2019 and $1000.$ Each second, the drunkard will randomly move to an adjacent integer with equal probability. The cops must move to an adjacent integer of their choice every second as well, and the movements of the cops and drunkard happen simultaneously. If the goal of the cops is to occupy the same number as the drunkard, what is the expected amount of seconds it will take the cops to occupy the same space as the drunkard? Assume optimal movement from the cops.

7. Consider a monic cubic polynomial $P(x)$ with roots $a, b, c \leq 1.$ If the constant term of the polynomial is 8 and there is one root at least 4 greater than another, find the maximum possible value of the sum of the coefficients of $P(x)$. (The constant term is included.)

8. Find the remainder of $(1^3)(1^3+2^3)(1^3+2^3+3^3)\cdots(1^3+2^3+\cdots+99^3)$ when divided by 101.

9. The expansion of $\frac{1}{7}$ is $0.\overline{142857}$, which is a repeating decimal with a period of 6. What is the period of the expansion of $\frac{1}{13}$?

10. Dan and Tom are playing a coin-flipping game, and Dan flips first. The first person to flip a heads is the winner. Dan's probability of flipping heads is $\frac{1}{3}$, and Tom's chance is n. If Dan and Tom have an equal probability of winning, what is n?

11. A secret spy organization needs to spread some secret knowledge to all of its members. In the beginning, only 1 member is informed. Every informed spy will call an uninformed spy such that every informed spy is calling a different uninformed spy. After being called, an uninformed spy becomes informed. The call takes 1 minute, but since the spies are running low on time, they call the next spy directly afterward. However, to avoid being caught, after the third call an informed spy makes, the spy stops calling. How many minutes will it take for every spy to be informed, provided that the organization has 600 spies?

12. Let $\lfloor x \rfloor$ be the largest integer such that $\lfloor x \rfloor \le x$, and let $\{x\} = x - \lfloor x \rfloor$. How many values of x satisfy $x + \lfloor x \rfloor \cdot \{x\} = 23$?

13. Prove that $-\sqrt{2}\sin x \ge \cos^2 x - \frac{3}{2}$.

14. Find $\frac{1}{\sqrt{1-1}+\sqrt{1+1}} + \frac{1}{\sqrt{2-1}+\sqrt{2+1}} + \frac{1}{\sqrt{3-1}+\sqrt{3+1}} + \cdots + \frac{1}{\sqrt{2018-1}+\sqrt{2018+1}}$.

15. Consider cubic $p(x)$ such that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, and $p(4) = 0$. Find $p(5)$.

16. Let $f(x) = x^2 - 12x + 36$. For $k \ge 2$, find the sum of all real n such that $f^{k-1}(n) = f^k(n)$.

17. Consider the set $\{1, 2, 3 \cdots 12, 13\}$. It is possible to create S distinct sums by adding together N distinct numbers. Find the sum of all values of N that maximize the value of S.

18. A tweenie is a natural number that is the mean of two distinct powers of two. Find the tenth smallest tweenie.

19. There are 3 six-sided dice, one red, white, and blue. They are considered distinct. How many ways can the sum of the 15 faces showing on the three die equal 56 if each

die orientation is only considered unique if the sum of its faces that are showing are unique?

20. Have $p(n)$ be the probability that after rolling a regular 6 sided die n times, you get at least one $6.$ Find $p(1) + p(2) + \cdots + p(10),$ to the nearest integer.

21. What is the largest integer value of n such that $1.01^2 - \frac{n^2}{10000} \geq 1?$

22. Have $\frac{1}{a} + \frac{1}{b} = \frac{2}{5}$ for integers $a, b.$ Find all values of a that have a corresponding value of b that satisfies this equality.

23. Prove that $\gcd(2n + 8, 3n - 2)$ is never equal to $8.$

24. Chennis and Den are playing a game with a cursed coin. They take turns flipping the coin, and the winner of the game is the first person to get heads. At first, its probability of coming up heads is $\frac{1}{2}.$ However, after every flip, its probability of coming up heads is halved. For example, if Chennis flips the coin, his probability of getting heads is $\frac{1}{2},$ and if Den then flips the coin afterwards, his probability of getting heads is $\frac{1}{4}.$ If Chennis flips first, what is his probability of winning?

25. Find the amount of ordered pairs of positive integers (a, b) such that $\gcd(a, b) = 20$ and $lcm(a, b) = 19!$

26. Consider 6 people where exactly 6 pairs of people are friends. Call two people acquainted if they are friends or have a mutual acquaintance. How many possible arrangements of friendship are there such that everyone is acquainted? (Note: Two arrangements are considered distinct only if one pair of people who were friends in the first arrangement are not in the second arrangement.)

27. What is the sum of all odd n such that $\frac{1}{n}$ expressed in base 8 is a repeating decimal with period $4?$

28. Consider polynomial $f(x) = (x - 1)(x - 2) \cdots (x - 8).$ Let a, b be integers such that $a \neq b,$ a, b are not roots of $f(x),$ and the remainder of $f(x)$ when divided by $x - a$ and $x - b$ are equal. What is $a + b?$

29. Mark takes the first $3n$ non-negative integers and adds them up. Kathy then takes the first n perfect cubes and adds them up. If Mark and Kathy get the same sum, what is n?