The Krein-Milman theorem and elementary results in convex analysis

lynne homann cure '27

University of Maryland, College Park

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Introduction

We will prove today this statement (Minkowski's theorem, or the finite case of the Krein-Milman theorem):

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is equal to the convex hull of its extreme points.

- Based on *Fundamentals of Convex Analysis* by Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal
- Thank you to Rémi Barritault for his mentorship this semester!

Contents

Convexity

For our purposes, we are working within the real vector (affine) space $\mathbb{R}^n$ equipped with the standard dot product as our scalar product $\langle \cdot, \cdot \rangle$ and the usual norm $\|x\| = \sqrt{\langle x, x \rangle}$.

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The set $C \subset \mathbb{R}^n$ is **convex** if $\alpha x + (1-\alpha)x'$ is in C for all $x, x' \in C$ and $\alpha \in (0, 1)$.

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Intuitive definition

C is convex if the line segment $[x, x']$ is entirely contained in C whenever its endpoints x, x' are in C.

Affine combinations

Definition

An **affine combination** of elements $x_1, \dots, x_k$ of $\mathbb{R}^n$ is an element $\sum_{i=1}^{k} \alpha_i x_i$ where the coefficients α_i satisfy $\sum_{i=1}^{k} \alpha_i = 1$.

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Definition

The $k+1$ points $x_0, x_1, \dots, x_k$ are said to be **affinely independent** if $\dim \operatorname{aff} \{x_0, \dots, x_k\} = k$. Up to $n+1$ points can be affinely independent in $\mathbb{R}^n$.

Simplices

Definition

We define the **unit simplex** Δ_k in $\mathbb{R}^k$ as

$$\Delta_k := \left\{ \alpha \in \mathbb{R}^k : \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0 \right\}$$

Simplices are closely related to affine combinations – in fact, they are used to construct the most important kind of affine combination for our purposes.

Complex combinations and hulls

Definition

A **convex combination** of elements $x_1, \ldots, x_k$ in $\mathbb{R}^n$ is an affine combination where all the coefficients are nonnegative – that is, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \Delta_k$.

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Proposition

$\operatorname{co} S$ is equal to the set of all convex combinations of elements of S.

Extreme points

Definition

We say that $x \in C$ is an **extreme point** of C if there are no two distinct points $x_1, x_2 \in C$ such that x lies in the line segment between x_1 and x_2.

We notate the set of extreme points of C as $\operatorname{ext} C$.

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Examples

- Every x in the unit ball $B(0,1)$ such that $\|x\| = 1$ is an extreme point of $B(0,1)$
- The vertices of a triangle (or any convex polygon in $\mathbb{R}^2$) are its extreme points

Interior, closure, boundary

Definition

- The **interior** of a set X, notated $\operatorname{int} X$, is the union of all open subsets of X.
- The **closure** of X, notated $\operatorname{cl} X$, is the intersection of all closed sets containing X. Observe that $\operatorname{int} X \subset X \subset \operatorname{cl} X$.
- The **boundary** of X, notated $\operatorname{bd} X$, is $\operatorname{cl} X \setminus \operatorname{int} X$.

Note that $\operatorname{int} X$ is open, $\operatorname{cl} X$ is closed, and $\operatorname{cl} X = (\operatorname{int} X)^C$.

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Proposition

If C is convex, so are $\operatorname{int} C$ and $\operatorname{cl} C$.

Relative interior

Definition

The **relative interior** of a set X, notated $\operatorname{ri} X$, is the interior of X with respect to its affine hull:

$$\operatorname{ri} X := \{x \in X : \exists \delta \text{ s.t.} B(\delta, x) \cap \operatorname{aff} X \subset X\}$$

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Examples

- The interior of a sheet of paper w.r.t. $\mathbb{R}^3$ is empty, but its relative interior is the whole sheet without the edge
- The interior of a point is empty, but its relative interior is the point itself

Relative boundary

Definition

The **relative boundary** of a set X, notated $\operatorname{rbd} X$, is equal to $\operatorname{cl} X \setminus \operatorname{ri} X$.

Observe that the closure of a lower-dimensional set doesn't encounter the same usefulness issue as the interior does; that is, the "relative closure" of a set is equal to its ordinary closure.

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Lemma

Let $x \in \operatorname{cl} C$ and $x' \in \operatorname{ri} C$. Then the half-open segment

$$(x, x'] = \{\alpha x + (1 - \alpha)x' : 0 \leq \alpha < 1\}$$

is contained in $\operatorname{ri} C$.

An important consequence of this result is that the half-line issued from $x' \in \operatorname{ri} C$ cannot intersect $\operatorname{rbd} C$ more than once; consequently, any affine line meeting $\operatorname{ri} C$ cannot intersect $\operatorname{rbd} C$ more than twice.

Faces

Definition

A nonempty convex subset F of a convex set C is a **face** of C if every segment in C which contains an element of F is entirely contained within F.

That is, if a segment between two points in C is inside F anywhere, the entire segment is contained within F.

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Proposition

Let F be a face of C. Then any extreme point of F is an extreme point of C.

This property is known as the "transmission of extremality."

Hyperplanes and half-spaces

Definition

An **affine hyperplane** (or just **hyperplane**) in $\mathbb{R}^n$ is a set $H_{s,r}$ associated with an ordered pair $(s, r) \in \mathbb{R}^n \times \mathbb{R}$:

$$H_{s,r} := \{x \in \mathbb{R}^n : \langle s, x \rangle = r\}$$

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Definition

The **closed half-space** delineated by a hyperplane $H_{s,r}$ in $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n : \langle s, x \rangle \leq r\}$. When the inequality is instead strict inequality, this is instead the **open half-space**.

Supporting hyperplanes

Definition

A hyperplane $H_{s,r}$ is said to **support** the convex set C when C is entirely contained in one of the two closed half-spaces delineated by $H_{s,r}$:

$$\langle s, y \rangle \leq r \quad \forall y \in C$$

$H_{s,r}$ **supports** C **at** $x \in C$ when the above holds and $\langle s, x \rangle = r$ (that is, $x \in H_{s,r}$)

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Lemma

Let $x \in \operatorname{bd} C$, where $C \neq \varnothing$ is convex in $\mathbb{R}^n$. There exists a hyperplane supporting C at x.

Exposed faces

Definition

The set $F \subset C$ is an **exposed face** of C if there is a supporting hyperplane $H_{s,r}$ of C such that $F = C \cap H_{s,r}$

Proposition

An exposed face is a face.

Krein-Milman theorem

The Krein-Milman theorem applies to any locally convex Hausdorff topological space, but we will prove only the finite-dimensional case today.

Theorem

Let C be compact and convex (in $\mathbb{R}^n$). Then C is the convex hull of its extreme points – that is, $C = \text{co}(\text{ext } C)$.

Recall that a set is compact in $\mathbb{R}^n$ iff it is closed and bounded.

Proof

Theorem

Let C be compact and convex (in $\mathbb{R}^n$). Then C is the convex hull of its extreme points – that is, $C = \text{co}(\text{ext } C)$.

We will prove this by induction on $\dim C$ $(= \dim \text{aff } C)$.
This is trivial when $\dim C = 0$ – i.e., C is a singleton.

Proof

Theorem

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We will prove this by induction on $\dim C$ ($= \dim \operatorname{aff} C$).
This is trivial when $\dim C = 0$ – i.e., C is a singleton. Now suppose that $\dim C = k$ and for all $\dim C < k$, the result is true.
Take $x \in C$.

- Case 1: $x \in \operatorname{rbd} C$
- Case 2: $x \in \operatorname{ri} C$ ($= C \setminus \operatorname{rbd} C$)

Case 1: $x \in \operatorname{rbd} C$

Theorem

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- Since $x \in \operatorname{rbd} C \subset \operatorname{bd} C$, we have a hyperplane H supporting C at x. Then the set $C \cap H$ is nonempty (as it contains x), compact, and an exposed face of C.

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- Since $x \in \operatorname{rbd} C \subset \operatorname{bd} C$, we have a hyperplane H supporting C at x. Then the set $C \cap H$ is nonempty (as it contains x), compact, and an exposed face of C.
- $C \cap H$ has at most dimension $k - 1$, and since it is compact and convex, our induction hypothesis tells us that $C \cap H = \operatorname{co}\operatorname{ext}(C \cap H)$.

Case 1: $x \in \operatorname{rbd} C$

Theorem

Let C be compact and convex (in $\mathbb{R}^n$). Then C is the convex hull of its extreme points – that is, $C = \operatorname{co}(\operatorname{ext} C)$.

- Since $x \in \operatorname{rbd} C \subset \operatorname{bd} C$, we have a hyperplane H supporting C at x. Then the set $C \cap H$ is nonempty (as it contains x), compact, and an exposed face of C.
- $C \cap H$ has at most dimension $k - 1$, and since it is compact and convex, our induction hypothesis tells us that $C \cap H = \operatorname{co} \operatorname{ext}(C \cap H)$.
- Then x is a convex combination of extreme points of $C \cap H$. By transmission of extremality, those points are also extreme in C.

Case 2: $x \in \operatorname{ri} C$

Theorem

Let C be compact and convex (in $\mathbb{R}^n$). Then C is the convex hull of its extreme points – that is, $C = \operatorname{co}(\operatorname{ext} C)$.

- Since $\dim C > 0$, we can take some $x' \neq x$ in C.

Case 2: $x \in \operatorname{ri} C$

Theorem

Let C be compact and convex (in $\mathbb{R}^n$). Then C is the convex hull of its extreme points – that is, $C = \operatorname{co}(\operatorname{ext} C)$.

- Since $\dim C > 0$, we can take some $x' \neq x$ in C.
- The affine line generated by x' and x cuts $\operatorname{rbd} C$ in two points (since C is compact) y and z.

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- Then from Case 1, y and z are convex combinations of extreme points of C.

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Theorem

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- Since $\dim C > 0$, we can take some $x' \neq x$ in C.
- The affine line generated by x' and x cuts rbd C in two points (since C is compact) y and z.
- Then from Case 1, y and z are convex combinations of extreme points of C.
- Since x is a convex combination of y and z, it is also a convex combination of extreme points of C.

$\square$

Questions?