Convex Analysis and the Krein-Milman Theorem

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Contents

Introduction

We will prove today this statement (Krein-Milman):

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is equal to the convex hull of its extreme points.

- Based on *Fundamentals of Convex Analysis* by Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal
- Thank you to Rémi Barritault for his mentorship this semester!

Convexity

For our purposes, we are working within the real vector (affine) space $\mathbb{R}^n$ equipped with the standard dot product as our scalar product $\langle \cdot, \cdot \rangle$ and the usual norm $\|x\| = \sqrt{\langle x, x \rangle}$.

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Formal definition

The set $C \subset \mathbb{R}^n$ is **convex** if $\alpha x + (1-\alpha)x'$ is in C if $x, x' \in C$ and $\alpha \in (0,1)$.

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Intuitive definition

C is convex if the line segment $[x, x']$ is entirely contained in C when its endpoints x, x' are in C.

Intersection

Proposition

Let $\{C_j\}_{j \in J}$ be an arbitrary family of convex sets. Then their intersection $C = \bigcap_{j \in J} C_j$ is convex.

Many important convex sets are constructed by intersecting many convex sets, some of which we will encounter later.

Affine transformation

Definition

An **affine mapping** $A : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping that can be characterised by a linear mapping $A_0 : \mathbb{R}^n \to \mathbb{R}^m$ and a point $y_0 := A(0) \in \mathbb{R}^m$ such that

$$A(x) = A_0(x) + y_0.$$

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Proposition

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine mapping and C a convex set of $\mathbb{R}^n$. The image $A(C)$ of C under A is convex in $\mathbb{R}^m$.

This means that the sum of two convex sets is also convex – a nice property that isn't true for, e.g., closedness.

Interior and closure

Definition

The **interior** of a set X, notated $\operatorname{int} X$, is the union of all open subsets of X.
The **closure** of X, notated $\operatorname{cl} X$, is the intersection of all closed sets containing X.
Note that $\operatorname{int} X$ is open, $\operatorname{cl} X$ is closed, and $\operatorname{cl} X = (\operatorname{int} X)^C$.

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Proposition

If C is convex, so are $\operatorname{int} C$ and $\operatorname{cl} C$.

Affine combinations

Definition

An **affine combination** of elements $x_1, \ldots, x_k$ of $\mathbb{R}^n$ is an element $\sum_{i=1}^k \alpha_i x_i$ where the coefficients α_i satisfy $\sum_{i=1}^k \alpha_i = 1$.

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The $k+1$ points $x_0, x_1, \ldots, x_k$ are said to be **affinely independent** if $\dim \operatorname{aff} \{x_0, \ldots, x_k\} = k$. Up to $n+1$ points can be affinely independent in $\mathbb{R}^n$.

Simplices

Definition

We define the **unit simplex** Δ_k in $\mathbb{R}^k$ as

$$\Delta_k := \left\{ \alpha \in \mathbb{R}^k : \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0 \right\}$$

Simplices are closely related to affine combinations – in fact, they are used to construct the most important kind of affine combination for our purposes.

Complex combinations and hulls

Definition

A **convex combination** of elements $x_1, \ldots, x_k$ in $\mathbb{R}^n$ is an affine combination where all the coefficients are nonnegative – that is, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \Delta_k$.

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Proposition

A set S is convex if and only if $S = \operatorname{co} S$.

Carathéodory's theorem

Theorem

Any $x \in \operatorname{co} S \subset \mathbb{R}^n$ can be represented as a convex combination of $n+1$ elements of S.

Proving this theorem essentially consists of expressing an arbitrary convex combination in terms of an *affine* combination, then using the upper bound of $n+1$ on $\dim \operatorname{aff} S$.

Extreme points

Definition

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Examples

- Every x in the unit ball $B(0,1)$ such that $\|x\| = 1$ is an extreme point of $B(0,1)$
- The vertices of a triangle (or any convex polygon) are its extreme points

We notate the set of extreme points of C as $\operatorname{ext} C$.

The Krein-Milman Theorem

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is the convex hull of its extreme points – that is, $C = \text{co}(\text{ext}\, C)$.

Recall that a set is compact iff it is closed and bounded.