The Krein-Milman theorem and elementary results in convex analysis

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Directed Reading Program, Spring 2025

Introduction

We will prove today this statement (Minkowski's theorem, or the finite case of the Krein-Milman theorem):

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is equal to the convex hull of its extreme points.

- Based on *Fundamentals of Convex Analysis* by Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal
- Thank you to Rémi Barritault for his mentorship this semester!

Convexity

For our purposes, we are working within the real vector space $\mathbb{R}^n$ equipped with the standard dot product as our scalar product $\langle \cdot, \cdot \rangle$ and the usual norm $\|x\| = \sqrt{\langle x, x \rangle}$.

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Definition

C is **convex** if the line segment $[x, x']$ is entirely contained in C whenever its endpoints x, x' are in C.

Dimension of a convex set

Definition

The **dimension** $\dim C$ of a convex set C is taken to be $\dim \operatorname{span}(C - x_0)$ for $x_0 \in C$.

This is an adaptation of how the dimension of an affine space (a translation of a linear space) is defined.
For the purposes of this talk, we will assume that if $C \subset \mathbb{R}^n$, then $\dim C = n$.

Convex combinations and hulls

Definition

A **convex combination** of elements $x_1, \ldots, x_k$ in $\mathbb{R}^n$ is a linear combination with coefficients $c_1, \ldots, c_k$ in $\mathbb{R}$ such that

$$\sum_{i=1}^{k} c_i = 1 \qquad c_i \geq 0 \ \forall i$$

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Definition

The **convex hull** of a nonempty set S, denoted $\operatorname{co} S$, is the set of all convex combinations of elements of S – i.e., the smallest convex set containing S.

Extreme points

Definition

We say that $x \in C$ is an **extreme point** of C if there are no two distinct points $x_1, x_2 \in C$ such that x lies in the line segment between x_1 and x_2.

We notate the set of extreme points of C as $\operatorname{ext} C$.

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Examples

- Every x in the unit ball $B(0,1)$ such that $\|x\| = 1$ is an extreme point of $B(0,1)$
- The vertices of a triangle (or any convex polygon in $\mathbb{R}^2$) are its extreme points

Interior, closure, boundary

Definition

- The **interior** of a set X, notated $\operatorname{int} X$, is the largest open set contained in X.
- The **closure** of X, notated $\operatorname{cl} X$, is the smallest closed set containing X.
- The **boundary** of X, notated $\operatorname{bd} X$, is $\operatorname{cl} X \setminus \operatorname{int} X$.

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Proposition

If C is convex, so are $\operatorname{int} C$ and $\operatorname{cl} C$.

Boundary of convex sets

Lemma

Let $x \in \operatorname{cl} C$ and $x' \in \operatorname{int} C$. Then the half-open segment

$$(x, x'] = \left\{\alpha x + (1-\alpha)x' : 0 \leq \alpha < 1\right\}$$

is contained in $\operatorname{int} C$.

An important consequence of this result is that the half-line issued from $x' \in \operatorname{int} C$ cannot intersect $\operatorname{bd} C$ more than once; consequently, any affine line meeting $\operatorname{int} C$ cannot intersect $\operatorname{bd} C$ more than twice.

Faces

Definition

A nonempty convex subset F of a convex set C is a **face** of C if for $x, y \in C$, $t \in (0,1)$,

$$tx + (1-t)y \in F \implies x, y \in F$$

Faces are a way to generalize extreme points; indeed, $x \in C$ is an extreme point of C iff $\{x\}$ is a face of C.

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Proposition

Let F be a face of C. Then any extreme point of F is an extreme point of C.

This property is known as the "transmission of extremality."

Hyperplanes and half-spaces

Definition

An **affine hyperplane** (or often just **hyperplane**) in $\mathbb{R}^n$ is a set $H_{s,r}$ associated with an ordered pair $(s, r) \in \mathbb{R}^n \times \mathbb{R}$:

$$H_{s,r} := \{x \in \mathbb{R}^n : \langle s, x \rangle = r\}$$

Observe that $\dim H_{s,r} = n - 1$.

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Definition

The **closed half-space** delineated by a hyperplane $H_{s,r}$ in $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n : \langle s, x \rangle \leq r\}$. When the inequality is instead strict inequality, this is instead the **open half-space**.

Supporting hyperplanes

Definition

A hyperplane $H_{s,r}$ is said to **support** the convex set C when C is entirely contained in one of the two closed half-spaces delineated by $H_{s,r}$:

$$\langle s, y \rangle \leq r \quad \forall y \in C$$

$H_{s,r}$ **supports** C **at** $x \in C$ when the above holds and $\langle s, x \rangle = r$ (that is, $x \in H_{s,r}$)

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Lemma

Let $x \in \text{bd}\, C$, where $C \neq \varnothing$ is convex in $\mathbb{R}^n$. There exists a hyperplane supporting C at x.

Exposed faces

Definition

The set $F \subset C$ is an **exposed face** of C if there is a supporting hyperplane $H_{s,r}$ of C such that $F = C \cap H_{s,r}$

Proposition

An exposed face is a face.

Krein-Milman theorem

The Krein-Milman theorem applies to any locally convex Hausdorff topological vector space, but we will prove only the finite-dimensional case today.

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is the convex hull of its extreme points – that is, $C = \text{co}(\text{ext}\, C)$.

Recall that a set is compact in $\mathbb{R}^n$ iff it is closed and bounded.

Proof

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is the convex hull of its extreme points – that is, $C = \mathrm{co}(\mathrm{ext}\, C)$.

We will prove this by induction on $\dim C$.
This is trivial when $\dim C = 0$ – i.e., C is a singleton.

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We will prove this by induction on $\dim C$.
This is trivial when $\dim C = 0$ – i.e., C is a singleton. Now suppose that $\dim C = k$ and for all $\dim C < k$, the result is true.
Take $x \in C$.

- Case 1: $x \in \text{bd } C$
- Case 2: $x \in \text{int } C \ (= C \setminus \text{bd } C)$

Case 1: $x \in \operatorname{bd} C$

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is the convex hull of its extreme points – that is, $C = \operatorname{co}(\operatorname{ext} C)$.

- Since $x \in \operatorname{bd} C$, we have a hyperplane H supporting C at x. Then the set $C \cap H$ is nonempty (as it contains x), compact, and an exposed face of C.

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Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is the convex hull of its extreme points – that is, $C = \operatorname{co}(\operatorname{ext} C)$.

- Since $x \in \operatorname{bd} C$, we have a hyperplane H supporting C at x. Then the set $C \cap H$ is nonempty (as it contains x), compact, and an exposed face of C.
- $C \cap H$ has at most dimension $k - 1$, and since it is compact and convex, our induction hypothesis tells us that $C \cap H = \operatorname{co} \operatorname{ext}(C \cap H)$.

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- Since $x \in \text{bd}\, C$, we have a hyperplane H supporting C at x. Then the set $C \cap H$ is nonempty (as it contains x), compact, and an exposed face of C.
- $C \cap H$ has at most dimension $k - 1$, and since it is compact and convex, our induction hypothesis tells us that $C \cap H = \text{co}\,\text{ext}(C \cap H)$.
- Then x is a convex combination of extreme points of $C \cap H$. By transmission of extremality, those points are also extreme in C.

Case 2: $x \in \operatorname{int} C$

Theorem

Let C be compact and convex in $\mathbb{R}^n$. Then C is the convex hull of its extreme points – that is, $C = \operatorname{co}(\operatorname{ext} C)$.

- Since $\dim C > 0$, we can take some $x' \neq x$ in C.

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Theorem

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- Since $\dim C > 0$, we can take some $x' \neq x$ in C.
- The affine line generated by x' and x cuts $\operatorname{bd} C$ in two points (since C is compact) y and z.

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- Then from Case 1, y and z are convex combinations of extreme points of C.

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- The affine line generated by x' and x cuts $\operatorname{bd} C$ in two points (since C is compact) y and z.
- Then from Case 1, y and z are convex combinations of extreme points of C.
- Since x is a convex combination of y and z, it is also a convex combination of extreme points of C.

$\square$

Closing remarks

- K-M implies existence of extreme points – what kinds of convex sets have no extreme points?

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- K-M implies existence of extreme points – what kinds of convex sets have no extreme points?
- What can happen when C is not bounded?

Questions?