UNIT – IV

APPLICATIONS OF LAPLACE TRANSFORM

1.1 INTRODUCTION

The Laplace Transform is a powerful integral transform,introduced by Laplace a French mathematician, astronomer, and physicist who applied the Newtonian theory of gravitation to the solar system (an important problem of his day). He played a leading role in the development of the metric system.

The Laplace Transform is widely used in solving linear Differential equations with initial conditions such as those arising in the analysis of electronic circuits. It can be greatly used to find the solution of problems of both ordinary and partial differential equations, system of simultaneous differential equations, and it is applied to evaluate some definite integrals.

Ordinary and partial differential equations describe the way certain quantities vary with time such as thecurrent in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through aninsulated conductor these equations are generally coupled with initial conditions that describe the state ofthe system at time t = 0. A very powerful technique for solving these problems is that of Laplace transformwhich transform the differential equation into an algebraic equation from which we get the solution.

Solutions of Differential Equations using Laplace Transform

The following results will be used in solving differential and integral equations using Laplace transforms.

Theorem

If $f(t)$ is continuous in $t \geq 0$, $f'(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$and$f(t)$ and $f'(t)$ are of exponential order, then

$$L(f'(t)) = sL(f(t)) - f(0)$$

Proof

The given conditions ensure the existence of the Laplace transforms of $f(t)$ and $f'(t)$

By definition $L[f'(t)] = \int_0^\infty e^{-st} f(t)dt$

$= \int_0^\infty e^{-st} d(f(t))$

$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt$,on integration by parts

$= \lim_{t\to\infty}[e^{-st} f(t)] - f(0) + s.L(f(t))$

= 0 – (f(0) + sL(f(t))   [$\because$ $f(t)$ is of exponential order]

= sL(f(t)) – f(0)

Corollary 1

In the above theorem if we replace $f(t)$ by $f'(t)$ we get,

$L(f''(t)) = sL(f'(t)) - f'(0)$

$= s[sL(f(t)) - f(0)] - f'(0)$

$= s^2L(\text{f(t)}) - sf(0) - f'(0)$

Repeated application of the above theorem gives the following result:

$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f^1(0) - \ldots - f^{n-1}(0)$

Solved Problems

1. Using Laplace transform, solve $y' - y = t$, $y(0) = 0$.

Solution:

Given $y' - y = t$, $y(0) = 0$

Taking Laplace transform on both sides,

$L(y') - L(y) = L(t)$

$sL(y) - y(0) - L(Y) = \frac{1}{s^2}$

$$L(y) = \frac{1}{s^2(s-1)}$$

$$\therefore y = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$y = \int_0^t \int_0^t L^{-1}\left(\frac{1}{s-1}\right)dt \;\; dt$$

$$y = \int_0^t \int_0^t e^t dt \;\; dt$$

$$= \int_0^t [e^t]_0^t \;\; dt$$

$$= \int_0^t [e^t - 1]dt$$

$$= (e^t - 1)_0^t$$

$$= e^t - t - 1$$

2. Solve $y'' - 4y' + 8y = e^{2t}, y(0) = 2$ and $y'(0) = -2$

Solution:

Taking Laplace transform on both sides of the equation, we get

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$[s^2L(y) - sy(0) - y^1(0)] - 4[sL(y) - y(0)] + 8L(y) = \frac{1}{s-2}$$

$$\text{i.e.,} [s^2 - 4s + 8]L(y) = \frac{1}{s-2} + 2s - 10$$

$$L(y) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

Solving we get $A=\frac{1}{4}, \quad B=\frac{-1}{4}, \quad C=\frac{1}{2}$

$$= \frac{\frac{1}{4}}{s-2}+\frac{\frac{-1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$

$$= \frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$

$$= \frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$y = \frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}\left(\frac{\frac{7}{4}s-6}{s^2+4}\right)$$

$$= \frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$

$$= \frac{1}{4}e^{2t}(1+7\cos 2t-12\sin 2t)$$

3. Use Laplace transform to solve $y'-y=e^t$ given $y(0) = 1$

Solution:

$$y'-y=e^t$$

Taking Laplace transform on both sides of the equation,

we get $L(y')-L(y)=L(e^t)$, $y(0) = 1$

$$sL(y)-y(0)-L(y)=\frac{1}{s-1}$$

$$L(y)[s-1]=\frac{1}{s-1}+1$$

$$L(y) = \frac{s}{(s-1)^2}$$

$$y = L^{-1}\left[\frac{s}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{(s-1)+1}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\frac{1}{(s-1)^2}$$

$$= e^t + te^t$$

$$= e^t(1+t)$$

4. Solve $\frac{d^2y}{dt^2} + 9y = 18t$ given that $y(0) = 0 = y\left(\frac{\pi}{2}\right)$

Solution:

$y'' + 9y = 18t$ where $y'' = \frac{d^2y}{dt^2}$

Taking Laplace transform on both sides of the equation, we get

$$L(y'') + 9L(y) = 18L(t)$$

$$\left[s^2L(y) - sy(0) - y'(0)\right] + 9L(y) = \frac{18}{s^2}$$

$L(y)[s^2+9] = \frac{18}{s^2} + y'(0)$ [$\because y'(0)$ is not given we can take it to be a constant a]

$$= \frac{18}{s^2} + a$$

$$= \frac{as^2+18}{s^2}$$

$$L(y) = \frac{as^2+18}{s^2(s^2+9)}$$

$$= \frac{a}{s^2+9} + \frac{18}{s^2(s^2+9)}$$

$$y = L^{-1}\left(\frac{a}{s^2+9}\right) + L^{-1}\left(\frac{18}{s^2(s^2+9)}\right)$$

$$= L^{-1}\left(\frac{a}{s^2+9}\right) + L^{-1}\left(\frac{2}{s^2} - \frac{2}{(s^2+9)}\right) \qquad \text{(using partial fractions)}$$

$$= \frac{a\sin 3t}{3} + 2t - \frac{2\sin 3t}{3}$$

Now, using the conditions $t = 0$ and $t = \frac{\pi}{2}$ we have

$$0 = \frac{a}{3}\sin\left(\frac{3\pi}{2}\right) + \pi - \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$= -\frac{a}{2} + \pi + \frac{2}{3}$$

$$\frac{a}{3} = \frac{3\pi+2}{3}$$

Hence $a = 3\pi + 2$

$$\therefore y = \frac{(3\pi+2)\sin 3t}{3} + 2t - \frac{2\sin 3t}{3}$$

$$= \pi\sin 3t + 2t$$

5. Using Laplace transform, $y'' + 4y' + 3y = \sin t, y(0) = y'(0) = 0$

Solution:

Given $y'' + 4y' + 3y = \sin t$

Taking Laplace transform on both sides

$$L(y'') + 4L(y') + 3L(y) = L(\sin t)$$

$$[s^2 L(y) - sy(0) - y'(0)] + 4(sL(y) - y(0)] + 3L(y) = \frac{1}{s^2+1}$$

$$L(y)[s^2 + 4s + 3] = \frac{1}{s^2+1}$$

$$L(y) = \frac{1}{(s^2+4s+3)(s^2+1)}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+3)(s^2+1)}\right) \qquad \ldots (1)$$

Now, $$\frac{1}{(s+1)(s+3)(s^2+1)} = \frac{A}{(s+1)} + \frac{B}{(s+3)} + \frac{C_2 + D}{(s^2+1)}$$

$$1 = A(s+3)(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s+1)(s+3) \qquad \ldots(2)$$

Put S = -3 in (2)

$$1 = B(-2)(10) \Rightarrow B = \frac{-1}{20}$$

Put S = -1 in (2)

$$1 = A(2)(2) \Rightarrow A = \frac{1}{4}$$

Comparing the coefficient of s^3,

$$0 = A + B + C$$

$$\therefore C = -A - B = -\frac{1}{4} + \frac{1}{20} = -\frac{4}{20} = -\frac{1}{5}$$

$$\therefore C = -\frac{1}{5}$$

Put $S = 0$ in (2)

$$1 = 3A + B + 3D$$

$\therefore 3D = 1 - 3A - B$

$$= 1 - \frac{3}{4} + \frac{1}{20} = \frac{3}{10}$$

$$\therefore \frac{1}{(s+1)(s+3)(s^2+1)} = \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{20}}{s+3} + \frac{\frac{-1}{5}s + \frac{3}{10}}{(s^2+1)}$$

$$\therefore L^{-1}\left(\frac{1}{(s+1)(s+3)(s^2+1)}\right) = \frac{1}{4}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{20}L^{-1}\left(\frac{1}{s+3}\right)$$

$$\frac{-1}{5}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{3}{10}L^{-1}\left(\frac{1}{(s^2+1)}\right)$$

$$= \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} - \frac{1}{5}\cos t + \frac{3}{10}\sin t$$

6. Using Laplace transform solve $y'' - 3y' + 2y = 4$ given that $y(0) = 2, y'(0) = -3$.

Solution:

$$y'' - 3y' + 2y = 4$$

Taking Laplace transform on both sides

$$L(y'') - 3L(y') + 2L(y) = L(4)$$

$$[s^2L(y) - sy(0) - y^1(0)] - 3[sL(y) - y(0)] + 2L(y) = \frac{4}{s}$$

$$s^2L(y) - 2s + 3 - 3sL(y) + 6 + 2L(y) = \frac{4}{s}$$

$$L(y)[s^2 - 3s + 2] = \frac{4}{s} + 2s - 3 = \frac{4 + 2s^2 - 3s}{s}$$

$$L(y) = \frac{4 + 2s^2 - 3s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \qquad \ldots(1)$$

$$2s^2 - 3s + 4 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

Put $s = 1$ in (1), $3 = -B \Rightarrow B = -3$

Put $s = 2$ in (1), $6 = 2C \Rightarrow C = 3$

Put $s = 0$ in (1), $4 = -2A \Rightarrow A = 2$

$$Y = L^{-1}\left(\frac{2s^2 - 3s + 4}{s(s-1)(s-2)}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{s-2}\right) = 2 - 3e^{t} + 3e^{2t}$$

7. Solve using Laplace transform the differented equation $\frac{d^2y}{dt^2} + \frac{2dy}{dt} + 5y = 0$ where $y = 2, \frac{dy}{dt} = -4$ at $t = 0$.

Solution:

$y'' + 2y' + 5y = 0$ where $y(0) = 2, y'(0) = -4$

Taking Laplace transform on both sides

$$L(y'') + 2L(y') + 5L(y) = 0$$

$$[s^2L(y) - sy(0) - y^1(0)] + 2[sL(y) - y(0)] + 5L(y) = 0$$

$$L(y)[s^2 + 2s + 5] - 2s + 4 - 4 = 0$$

$$L(y) = \frac{2s}{s^2 + 2s + 5}$$

$$y = 2L^{-1}\left(\frac{(s+1) - 1}{(s+1)^2 + 4}\right)$$

$$= 2L^{-1}\left(\frac{(s+1)}{(s+1)^2 + 4}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2 + 4}\right)$$

$$= 2e^{-1}L^{-1}\left(\frac{s}{s^2 + 4}\right) - e^{-t}L^{-1}\left(\frac{2}{s^2 + 4}\right)$$

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

$$= e^{-t}(2\cos 2t - \sin 2t)$$

$$L(y'') + 2L(y') + 5L(y) = 0$$

$$[s^2 L(y) - sy(0) - y^1(0)] + 2[sL(y) - y(0)] + 5L(y) = 0$$

$$L(y)[s^2 + 2s + 5] - 2s + 4 - 4 = 0$$

$$L(y) = \frac{2s}{s^2 + 2s + 5}$$

$$y = 2L^{-1}\left(\frac{(s+1) - 1}{(s+1)^2 + 4}\right)$$

$$= 2L^{-1}\left(\frac{(s+1)}{(s+1)^2 + 4}\right) - 2L^{-1}\left(\frac{1}{(s+1)^2 + 4}\right)$$

$$= 2e^{-1}L^{-1}\left(\frac{s}{s^2 + 4}\right) - e^{-t}L^{-1}\left(\frac{2}{s^2 + 4}\right)$$

$$= 2e^{-t}\cos 2t - e^{-t}\sin 2t$$

$$= e^{-t}(2\cos 2t - \sin 2t)$$

8. Using Laplace transform, solve $\frac{d^2 y}{dt^2} + \frac{2dy}{dt} + y = te^{-t}$ given $y = (0) = 1, y'(0) = -2.$

Solution:

$$y'' + 2y' + y = te^{-t}$$

Taking Laplace transform on both sides

$$L(y'') + 2L(y') + L(y) = L(te^{-t})$$

$$s^2 L(y) - sy(0) - y^1(0) + 2[sL(y) - y(0)] + L(y) = \frac{1}{(s+1)^2}$$

$$L(y)[s^2 + 2s + 1] = \frac{1}{(s+1)^2} + s$$

$$L(y) = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)^2}\right) + L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$= e^{-t}L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{s+1}{(s+1)^2}\right) - L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= \frac{e^{-t}}{3!}L^{-1}\left(\frac{3!}{s^4}\right) + e^{-t} - e^{-t}t$$

$$= \frac{e^{-t}t^3}{6} + e^{-t} - e^{-t}t$$

$$= e^{-1}\left[\frac{t^3}{6} + t + 1\right]$$

9. Solve using L.T $y'' - 2y' + y = (t+1)^2$ given $y(0) = 4$ and $y'(0) = -2$.

Solution:

Given $y'' - 2y' + y = (t+1)^2$

$$L(y'') - 2L(y') + L(y) = L(t+1)^2$$

$$[s^2L(y) - sy(0) - y^1(0)] - 2[sL(y) - y(0)] + L(y) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

$$s^2L(y) - 4s + 2 - 2sL(y) + 8 + L(y) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

$$L(y)(s-1)^2 = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} + 4s - 10$$

$$L(y) = \frac{2}{s^3(s-1)^2} + \frac{2}{s^2(s-1)^2} + \frac{1}{s(s-1)^2} + \frac{4s}{(s-1)^2} - \frac{10}{(s-1)^2}$$

$$y = 2L^{-1}\left(\frac{1}{s^3(s-1)^2}\right) + 2L^{-1}\left(\frac{1}{s^2(s-1)^2}\right) + L^{-1}\left(\frac{1}{s(s-1)^2}\right) +$$

$$4L^{-1}\left(\frac{s}{(s-1)^2}\right)-10L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= 2\int_0^t\int_0^t\int_0^t L^{-1}\left(\frac{1}{(s-1)^2}\right)dtdtdt+2\int_0^t\int_0^t L^{-1}\left(\frac{1}{(s-1)^2}\right)dtdt+\int_0^t L^{-1}\left(\frac{1}{(s-1)^2}\right)dt$$

$$+4L^{-1}\left(\frac{s-1+1}{(s-1)^2}\right)-10e^tL^{-1}\left(\frac{1}{s^2}\right)$$

$$y \quad = 2\int_0^t\int_0^t\int_0^t e^tL^{-1}\left(\frac{1}{s^2}\right)dtdtdt+2\int_0^t\int_0^t e^tL^{-1}\left(\frac{1}{s^2}\right)dtdt+\int_0^t e^tL^{-1}\left(\frac{1}{s^2}\right)dt+$$

$$4L^{-1}\left(\frac{s-1}{(s-1)^2}\right)+4L^{-1}\left(\frac{1}{(s-1)^2}\right)-10e^t.t$$

$$= 2\int_0^t\int_0^t\int_0^t e^t.tdtdtdt+2\int_0^t\int_0^t e^tt\,dtdt+\int_0^t e^tt\,dt+4L^{-1}\left(\frac{1}{(s-1)^2}\right)+4e^tL^{-1}\left(\frac{1}{s^2}\right)-10e^tt$$

$$= 2\int_0^t\int_0^t (te^t-e^t)_0^t\,dtdt+2\int_0^t (e^tt-e^t)_0^t\,dt+(te^t-e^t)_0^t+4e^t+4e^tt-10e^tt$$

$$= 2\int_0^t\int_0^t (te^t-e^t+1)dtdt+2\int_0^t te^t-e^t+1)dt+(te^t-e^t+1)+4e^t-6e^t.t$$

$$= 2\int_0^t (te^t-e^t-e^t+t)_0^t\,dt+2(te^t-e^t-e^t+t)_0^t+(te^t-e^t+1)+4e^t-6e^tt$$

$$y \quad = 2\int_0^t (te^t-2e^t+t+2)dt+2(te^t-2e^t+t+2)+(te^t-e^t+1)+4e^t-6e^tt$$

$$y \quad = 2\left[te^t-e^t-2e^t+\frac{t^2}{2}+2t\right]_0^t-3e^tt-e^t+2t+5$$

$$y \quad = 2\left[te^t-3e^t+\frac{t^2}{2}+2t+3\right]-3e^tt-e^t+2t+5$$

$$y \quad = -te^t-7e^t+t^2+6t+11$$

10. Using Laplace Transform, solve $\frac{d^2y}{dt^2}-\frac{4dy}{dt}+8y=e^{2t}$ $\quad y(0)=2, y'(0)=-2$

Solution:

Given $y'' - 4y' + 8y = e^{2t}$

Taking Laplace Transform on both sides,

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$[s^2 L(y) - sy(0) - y'(0)] - 4[sL(y) - y(0)] + 8L(y) = \frac{1}{s-2}$$

$$[s^2 - 4s + 8]L(y) - 2s + 10 = \frac{1}{s-2}$$

$$L(y)[s^2 - 4s + 8] = \frac{1}{s-2} + 2s - 10 = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$y = L^{-1}\left[\frac{1}{(s-2)(s^2-4s+8)}\right] + 2L^{-1}\left[\frac{s-5}{(s-2)^2+4}\right] \qquad \ldots(1)$$

$$\frac{1}{(s-2)(s^2-4s+8)} + \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8}$$

$$1 = A(s^2 - 4s + 8) + (s-2)(Bs + C) \qquad \ldots(2)$$

Put $S = 2$ in (2)

$1 = 4A \therefore A = 1/4$

Compare the coefficient of s^2,

$0 = A + B \therefore B = -1/4$

Compare the constant terms, we have

$1 = 8A - 2C$

$$\therefore 2C = 8A - 1 = 8\left(\frac{1}{4}\right) - 1 = 1$$

$C = \frac{1}{2}$

$$y = \frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left(\frac{\frac{-s}{4}+\frac{1}{2}}{s^2-4s+8}\right) + 2L^{-1}\left(\frac{s-2-3}{(s-2)^2+4}\right)$$

$$= \frac{1}{4}e^{2t} - \frac{1}{4}L^{-1}\left(\frac{s-2}{(s-2)^2+4}\right) + 2L^{-1}\left(\frac{s-2}{(s-2)^2+4}\right) - 6L^{-1}\left(\frac{1}{(s-2)^2+4}\right)$$

$$= \frac{1}{4}e^{2t} - \frac{1}{4}e^{2t}L^{-1}\left(\frac{s}{s^2+4}\right) + 2e^{2t}L^{-1}\left(\frac{s}{s^2+4}\right) - 6e^{2t}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{e^{2t}}{4} - \frac{1}{4}e^{2t}\cos 2t + 2e^{2t}\cos 2t - 3e^{2t}L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$y = \frac{e^{2t}}{4} + \frac{7}{4}e^{2t}\cos 2t - 3e^{2t}\sin 2t$$

$$y = \frac{e^{2t}}{4}(1 + 7\cos 2t - 12\sin 2t)$$

11. Solve $\frac{d^2y}{dt^2} = f(t)$ with $y(0) = 0$ $y'(0) = 1$ and $f(t) = \begin{cases} 1 \; for \; 0 < t < 1 \\ 0 \; for \; t > 1 \end{cases}$

Solution:

Given $y''(t) = f(t)$

Taking Laplace transform on both sides we get

$$L(y'') = L(f(t))$$

$$s^2L(y) - sy(0) - y'(0) = L(f(t))$$

$$s^2L(y) - 1 = L(f(t))$$

Now

$$L(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st}dt + \int_1^\infty e^{-st} f(t)dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1 = \frac{1-e^{-s}}{s}$$

$$\therefore s^2 L(y) - 1 = \frac{1-e^{-s}}{s}$$

$$L(y) \quad = \frac{1-e^{-s}+s}{s^3} = \frac{1}{s^3} + \frac{1}{s^2} - \frac{e^{-s}}{s^3}$$

$$y \quad = L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^2}\right) - L^{-1}\left(\frac{e^{-s}}{s^3}\right)$$

$$= \frac{t^2}{2!} + t - L^{-1}\left(\frac{e^{-s}}{s^3}\right)$$

By second shifting theorem $L^{-1}(e^{-as}F(s) = f(t-a)U_a(t)$ where $U_a(t) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$

$$\therefore L^{-1}\left(\frac{e^{-s}}{s^3}\right) \quad = f(t-1)U_1(t)$$

$$= \frac{(t-2)^2}{2!} U_1(t) \qquad \left(\because f(t) = \frac{t^2}{2!}\right)$$

$$\therefore y = \frac{t^2}{2!} + t - \frac{(t-1)^2}{2!} U_1(t)$$

12. Using Laplace transform, solve the following equation $L\frac{di}{dt} + Ri = Ee^{-at}; i(0) = 0,$ where L, R, E and a are constants.

Solution:

Taking Laplace transform on both sides of the equation

$$L(L(i'(t) = RL(i(t)) = EL(e^{-at})$$

$$L(sL(i(t)) - i(0)) + RL(i(t)) = \frac{E}{s+a}$$

$$(Ls+R)L(i(t)) = \frac{E}{s+a}$$

$$L(i(t)) = \frac{E}{(s+a)(Ls+R)}$$

$$= \frac{A}{s+a} + \frac{B}{Ls+R} \qquad \ldots (1)$$

$$E = A(Ls+R) + B(s+a)$$

Put $\quad S = -a$

$$E = A(-aL + R)$$

$$\Rightarrow A = \frac{E}{R-aL}$$

Comparing the coefficient of S on both sides

$$0 = AL + B$$

$$B = \frac{-EL}{R-aL}$$

Substituting the values of A and B in (1)

$$L(i(t)) = \frac{\frac{E}{R-aL}}{s+a} - \frac{\frac{EL}{R-aL}}{Ls+R}$$

$$L(i(t)) = \frac{E}{R-aL}\left[\frac{1}{s+a}\frac{L}{L(s+R/L)}\right]$$

$$i(t) = \frac{E}{R-aL}\left[L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+R/L}\right)\right]$$

$$= \frac{E}{R-aL}\left[e^{-at} - e^{\frac{-R}{L}t}\right]$$

Exercise

1. Solve $y'' - 4y' + 8y = e^{2t}$, $y(0) = 2$ and $y'(0) = -2$

2. Solve $y'' + 4y = \sin wt, y(0) = 0$ and $y'(0) = 0$

3. Solve $y'' + y' - 2y = 3\cos 3t - 11\sin 3t, y(0) = 0$ and $y'(0) = 6$

4. Solve $(D^2 + 4D + 13)y = e^{-t}\sin t, y = 0$ and $Dy = 0$ at $t = 0$ where $D = \dfrac{d}{dt}$

5. Solve $(D^2 + 6D + 9)x = 6t^2 e^{-3t}, x = 0$ and $Dx = 0$ at $t = 0$

6. Solve $x'' + 3x' + 2x = 2(t^2 + t + 1), x(0) = 2, x'(0) = 0$

7. Solve $y'' - 3y' - 4y = 2e^t, y(0) = y'(0) = 1$

8. Solve $x'' + 9x = 18t,\ x(0) = 0,\ x\left(\dfrac{\pi}{2}\right) = 0$

9. $y'' + 4y' = \cos 2t, y(\pi) = 0, \quad y'(\pi) = 0$

10. $x'' - 2x + x = t^2 e^{-3t},\ x(0) = 2,\ x'(0) = 3$

Answers

1. $y = \dfrac{1}{4}e^{2t}(1 + 7\cos 2t - 12\sin 2t)$

2. $y = \dfrac{1}{8}(\sin 2t - 2t\cos 2t)$

3. $y = \sin 3t - e^{-2t} + e^t$

4. $y = \dfrac{1}{85}[e^{-t}\{-2\cos t + 9\sin t\}] + e^{-st}\left\{2\cos 3t = -\dfrac{7}{3}\sin 3t\right\}$

5. $x = \dfrac{1}{2}t^4 e^{-3t}$

6. $x = t^2 - 2t + 3 - e^{-2t}$

7. $y = \dfrac{1}{25}(13e^{-t} - 10te^{-t} + 12e^{4t})$

8. $x = 2t + \pi\sin 3t$

9. $y = \frac{1}{4}(t-\pi)\sin 2t$

10. $x = \left(\frac{t^4}{12} + t + 2\right)e^t$

Solution of Integral equations using Laplace transform

Theorem

If $f(t)$ is a piecewise continuous in everyfinite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Proof

Let $g(t) = \int_0^t f(t)dt$

$\therefore g'(t) = f(t)$

$\therefore L(g'(t)) = sL(g(t)) - g(0)$

i.e $L(f)(t)) = sL\left(\int_0^t f(t)dt\right) - \int_0^0 f(t)dt$

$$\therefore L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Corollary:

$$L\left[\int_0^t \int_0^t f(t)dtdt\right] = \frac{1}{s^2}L(f(t))$$

In general

$$L\left[\int_0^t \int_0^t \ldots\ldots\int_0^t f(t)(dt)^n\right] = \frac{1}{s^n}L(f(t))$$

Problems

1. Solve $y + \int_0^t y\,dt = t^2 + 2t$

Solution:

Given $y + \int_0^t y\,dt = t^2 + 2t$

Taking Laplace Transform on both sides

$$L(y) + L\left(\int_0^t y\,dt\right) = L(t^2) + L(2t)$$

$$L(y) + \frac{1}{s}L(y) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L(y)\left[1 + \frac{1}{s}\right] = 2\left[\frac{1+s}{s^3}\right]$$

$$L(y)\left[\frac{s+1}{s}\right] = 2\left[\frac{s+1}{s^3}\right]$$

$$L(y) = 2\left[\frac{s+1}{s^3}\right]\left[\frac{s}{s+1}\right]$$

$$= \frac{2}{s^2}$$

$$y = L^{-1}\left(\frac{2}{s^2}\right) = 2t$$

2. Solve $\dfrac{dy}{dt} + 2y + \int_0^t y\,dt = 2\cos t, \quad y(0) = 1$

Solution:

Given $y' + 2y + \int_0^t y\,dt = 2\cos t$

Taking Laplace Transform on both sides

$$L(y') + 2Ly + L\left(\int_0^t ydt\right) = 2L(\cos t)$$

$$sL(y) - y(0) + 2L(y) + \frac{1}{s}L(y) = \frac{2s}{s^2+1}$$

$$L(y)\left[s + 2 + \frac{1}{s}\right] - 1 = \frac{2s}{s^2+1}$$

$$L(y)\left[\frac{s^2+2s+1}{s}\right] = \frac{2s}{s^2+1} + 1$$

$$L(y) = \left[\frac{s^2+2s+1}{s^2+1}\right]\left[\frac{s}{s^2+2s+1}\right]$$

$$= \frac{s}{s^2+1}$$

$$(y) = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

3. Using Laplace Transform solve $y + \int_0^t y(t)dt = e^{-t}$

Solution:

Given $y + \int_0^t y(t)dt = e^{-t}$

Taking Laplace transform on both sides,

$$L(y) + L\left(\int_0^t y(t)dt\right) = L(e^{-t})$$

$$L(y) + \frac{1}{s}L(y) = \frac{1}{s+1}$$

$$L(y)\left[1 + \frac{1}{s}\right] = \frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$y=L^{-1}\left(\frac{s}{(s+1)^2}\right)=L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$=L^{-1}\left(\frac{1}{(s+1)}\right)-e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$y=e^{-t}-e^{-t}t$$

$$y=e^{-t}(1-t)$$

4. Using Laplace transform, solve $x+\int_0^t x(t)dt=\cos t+\sin t$

Solution:

$$x+\int_0^t x(t)dt=\cos t+\sin t$$

Taking Laplace transform on both sides,

$$L(x)+L\left(\int_0^t x(t)dt\right)=L(\cos t+\sin t)$$

$$L(x)\left[1+\frac{1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)\left[\frac{s+1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)=\left(\frac{s+1}{s^2+1}\right)\left(\frac{s}{s+1}\right)$$

$$L(x)=\frac{s}{s^2+1}$$

$$\therefore x = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

5. Solve using Laplace transform $y' + 3y + 2\int_0^t y\,dt = t, \quad y(0) = 0$

Solution:

$$y' + 3y + 2\int_0^t y\,dt = t$$

Taking Laplace Transform on both sides,

$$L(y') + L(3y) + 2L\left(\int_0^t y\,dt\right) = L(t)$$

$$sL(y) - y(0) + 3L(y) + 2\frac{1}{s}L(y) = \frac{1}{s^2}$$

$$L(y)\left[s + 3 + \frac{2}{s}\right] = \frac{1}{s^2}$$

$$L(y)\left[\frac{s^2+3s+2}{s}\right] = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2}.\frac{s}{s^2+3s+2}$$

$$= \frac{1}{s(s+1)(s+2)}$$

$$y = L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) \qquad \ldots(1)$$

Now, $\dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1) \qquad \ldots(2)$$

Put $s = -1$ in (2)

$B = -1$

Put $s = -2$ in (2)

$C = \frac{1}{2}$

Put $s = 0$ in (2)

$A = \frac{1}{2}$

$\therefore$ (1) becomes

$$y = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$y = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t}$$

Solving Integral Equations using convolution

Theorem

By the definition of convolution, we have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$ and by convolution theorem $L(f(t) * g(t)) = L(f)(t)L(g)(t))$

Problems

1. Solve $y = 1 + 2\int_0^t e^{-2t} y(t-u)du$ … (1)

Solution:

$\int_0^t e^{-2t} y(t-u)du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t) = e^{-2t}, g(t) = y(t)$

Taking Laplace Transform on both sides of (1),

$$L(y) = L(1) + 2L\left[\int_0^t e^{-2u} y(t-u)du\right]$$

$$= \frac{1}{s} + 2L\left[e^{-2t} * y(t)\right] \quad \text{(Definition of convolution)}$$

$$= \frac{1}{s} + 2L\left(e^{-2t}\right)L(y) \quad \text{(Convolution theorem)}$$

$$= \frac{1}{s} + 2\left(\frac{1}{s+2}\right)L(y)$$

$$L(y) = \frac{1}{s} + \frac{2}{s+2}L(y)$$

$$L(y) = \left[1 - \frac{2}{s+2}\right] = \frac{1}{s}$$

$$L(y) = \left[\frac{s}{s+2}\right] = \frac{1}{s}$$

$$L(y) = \frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2}$$

$$y = L^{-1}\left(\frac{1}{s} + \frac{2}{s^2}\right)$$

$$y = 1 + 2t$$

2. Using Laplace transform solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

Solution:

Given $y = 1 + \int_0^t y(u)\sin(t-u)du$

Taking Laplace transform on both sides,

$$L(y) = L(1) + L\left[\int_0^t y(u)\sin(t-u)du\right] \qquad \ldots(1)$$

Now the integral $\int_0^t y(u)\sin(t-u)du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t) = y(t), g(t) = \sin t$

$\therefore$ (1) becomes

$$L(y) = \frac{1}{s} + L(y(t) * \sin t)$$

$$L(y) = \frac{1}{s} + L(y).\frac{1}{s^2+1}$$

$$L(y)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

$$L(y)\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s}$$

$$L(y) = \frac{s^2+1}{s^3}$$

$$= \frac{1}{s} + \frac{1}{s^3}$$

$$y = L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^3}\right)$$

$$y = 1 + \frac{1}{2}t^2$$

3. Using Laplace transform solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Solution:

Given $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$ … (1)

Taking Laplace transform on both sides of (1),

$$L(f)(t)) = L(\cos t) + \left[L\int_0^t e^{-u} f(t-u)du\right]$$

$$= \frac{s}{s^2+1} + L(e^{-t} * f(t))$$

$$= \frac{s}{s^2+1} + L(e^{-t})\ Lf(t))$$

$$= \frac{s}{s^2+1} + \frac{1}{s+1} L(f(t))$$

$$L(f(t))\left[1 - \frac{1}{s+1}\right] = \frac{s}{s^2+1}$$

$$L(f(t))\left[\frac{1}{s+1}\right] = \frac{s}{s^2+1}$$

$$L(f(t)) = \frac{s+1}{s^2+1}$$

$$f(t) = L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$f(t) = \cos t + \sin t$$

4. Solve the integral equation $y(t) = t^2 + \int_0^t y(t)\sin(t-u)du$

Solution:

$$y(t) = t^2 + \int_0^t y(t)\sin(t-u)du$$

Taking Laplace transform obn both sides,

$$L(y(t)) = L(t^2) + L\left[\int_0^t y(t)\sin(t-u)du\right]$$

$$L(y) = \frac{2}{s^3} + L(y(t) * \sin t)$$

$$= \frac{2}{s^3} + L(y)L(\sin t)$$

$$= \frac{2}{s^3} + L(y)\left(\frac{1}{s^2+1}\right)$$

$$L(y)\left(1-\frac{1}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y)\left(\frac{s^2}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y) \quad = \frac{2(s^2+1)}{s^5}=\frac{2}{s^3}+\frac{2}{s^5}$$

$$(y) \quad =L^{-1}\left(\frac{2}{s^3}\right)+\frac{2}{4!}L^{-1}\left(\frac{4!}{s^5}\right)$$

$$(y) \quad = t^2+\frac{1}{12}t^4$$

5. Using Laplace transform solve the integral equation $y+\int_0^t y(u)du=e^{-t}$

Solution:

$$y+\int_0^t y(u)du=e^{-t}$$

$$\therefore y+y(t)*1=e^{-t}$$

Applying Laplace transform on both sides we get

$$L(y)+L[y(t)*1]=L(e^{-t})$$

$$\therefore L(y)+L(y)L(1)=L(e^{-t})$$

$$\therefore L(y)\left[1+\frac{1}{s}\right]=\frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$
\begin{aligned}
y &= L^{-1}\left(\frac{s}{(s+1)^2}\right) \\
&= L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right) \\
&= L^{-1}\left(\frac{1}{s+1}-\frac{1}{(s+1)^2}\right) \\
&= e^{-t}-e^{-t}L^{-1}\left(\frac{1}{s^2}\right) \\
&= e^{-t}-e^{-t}.t \\
&= e^{-t}(1-t)
\end{aligned}
$$

Exercise

1. Solve $x' + 3x + 2\int_0^t x\,dt = t, \quad x(0) = 0$

2. Solve $y' + 4y + 5\int_0^t y\,dt = e^{-t}, \quad y(0) = 0$

3. Solve $x' + 2x + \int_0^t x\,dt = \cos t, \quad x(0) = 1$

4. Solve $y' + 4y + 13\int_0^t y\,dt = 3e^{-t}, \sin 3t \quad y(0) = 3$

5. Solve $x(t) = 4t - 3\int_0^t x(u)\sin(t-u)du$

6. Solve $y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)du$

7. Solve $\int_0^t y(u)y(t-u)du = 2y(t) + t - 2$

8. Solve $y(t) = t + \int_0^t \sin u \; y(t-u)du$

9. Solve $y = 1 + \int_0^t y(u)\sin(t-u)du$

10. Solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Answers

1. $x = \frac{1}{2}(1+e^{-2t}) - e^{-t}$

2. $y = \frac{-1}{2}e^{-t} + \frac{1}{2}e^{-t}(\cos t + 3\sin t)$

3. $x = \frac{1}{2}[(1-t)e^{-t} + \cos t]$

4. $y = e^{2t}\left[3\cos 3t - \frac{7}{3}\sin 3t + \frac{3}{2}t\sin 3t + t\cos 3t\right]$

5. $x = t + \frac{3}{2}\sin 2t$

6. $y(t) = e^{-t}(1-t)^2$

7. $y(t) = 1$

8. $y = t + \frac{t^3}{6}$

9. $y = 1 + \frac{t^2}{2}$

10. $f(t) = \cos t + \sin t$

Simultaneous differential equations

1. Using Laplace transform solve

$$\frac{dx}{dt} + y = \sin t$$

$$\frac{dy}{dt} + x = \cos t$$

given$x(0) = 2$ and $y(0) = 0$

Solution:

Applying Laplace transform to the given equations

We get, $L(x') + L(y) = L(\sin t)$

$$L(y') + L(x) = L(\cos t)$$

$$\therefore sL(x) - x(0) + L(y) = \frac{1}{s^2 + 1}$$

$$sL(y) - y(0) + L(x) = \frac{s}{s^2 + 1}$$

$$\therefore sL(x) + L(y) = \frac{1}{s^2 + 1} + 2$$

$$= \frac{2s^2 + 3}{s^2 + 1} \qquad \ldots(1)$$

Also $sL(y) + L(y) = \dfrac{s}{s^2 + 1}$ $\qquad \ldots(2)$

$$(1) \times s \Rightarrow s^2 L(x) + sL(y) = \frac{(2s^2 + 3)s}{s^2 + 1} \qquad \ldots(3)$$

$$(2) \Rightarrow \quad L(x) + sL(y) = \frac{s}{s^2 + 1} \qquad \ldots(4)$$

$$(3) - (4)(s^2 - 1)L(x) = \frac{(2s^3 + 3s)}{s^2 + 1} - \frac{s}{s^2 + 1}$$

$$= \frac{2s^3 + 2s}{s^2 + 1}$$

$$L(x) = \frac{2s}{s^2 - 1} \qquad \ldots(5)$$

Substituting (5) in (2), we get

$$sL(y) = \frac{s}{s^2+1} - \frac{2s}{s^2-1} = \frac{s(s^2-1) - 2s(s^2+1)}{(s^2+1)(s^2-1)}$$

$$= \frac{-s^3 - 3s}{(s^2+1)(s^2-1)}$$

$$= \frac{-s(s^2+3)}{-(s^2+1)(1-s^2)}$$

$$L(y) = \frac{(s^2+3)}{(s^2+1)(1-s^2)} \qquad \ldots(6)$$

From (5), $\quad x = L^{-1}\left(\frac{2s}{s^2-1}\right)$

$$= 2\cosh t$$

$$y = L^{-1}\left(\frac{(s^2+3)}{(1-s^2)(s^2+1)}\right)$$

Consider $\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{A}{1-s} - \frac{B}{1+s} + \frac{Cs+D}{s^2+1} \qquad \ldots(7)$

$$s^2 + 3 = A(1+s)(s^2+1) + B(1-s)(s^2+1) + (Cs+D)(1-s)(1+s)$$

Put $\quad s = 1, \quad 4 = A(2)(2)$

$\Rightarrow 4 = 4A \Rightarrow A = 1$

Put $\quad s = -1, \ 4 = B(2)(2)$

$\Rightarrow B = 1$

Put $\quad s = 0, \quad 3 = A + B + D$

$3 = 1 + 1 + D$

$$\Rightarrow D = 1$$

Comparing the coeffcient of S,

$0 = A - B + C$

$\Rightarrow C = 0$

Substituting the values of A, B, C, D in (7) we get

$$\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{1}{1-s} - \frac{1}{1+s} + \frac{1}{s^2+1}$$

$$\therefore y = L^{-1}\left(\frac{1}{1-s}\right) + L^{-1}\left(\frac{1}{1+s}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$y = e^t + e^{-t} + \sin t$$

Hence the solution is $x = 2\cos ht$ and $y = e^t + e^{-t} + \sin t$

2. Solve $\dfrac{dx}{dt} + ax = y$

$$\frac{dy}{dt} + ay = x$$

given that $x = 0$ and $y = 1$ when $t = 0$

Solution:

Applying Laplace transform we get

$$L(x') + aL(x) = L(y)$$

$$L(y') + aL(y) = L(x)$$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$

$$sL(y) - y(0) + aL(y) = L(x)$$

Given that $x(0) = 0, \quad y(0) = 1$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$

$$sL(y) - y(0) + aL(y) = L(x)$$

$$\therefore sL(x) + aL(x) = L(y)$$

$$sL(y) - 1 + aL(y) = L(x)$$

$$\therefore (s+a)L(x) = L(y)$$

$$(s+a)L(x) - L(y) = 0 \qquad \ldots(1)$$

$$-L(x) + (s+a)L(y) = 1 \qquad \ldots(2)$$

$$(1) + (s+a) \times (2) \Rightarrow L(y)[(s+a)^2 - 1] = s + a$$

$$\therefore L(y) = \frac{s+a}{(s+a)^2 - 1}$$

Also by (1) $L(x) = \dfrac{1}{(s+a)^2 - 1}$

$$\therefore x = L^{-1}\left(\frac{1}{(s+a)^2 - 1}\right)$$

$$= e^{-at} L^{-1}\left(\frac{1}{s^2 - 1}\right)$$

$$= e^{-at} \sin\ ht$$

$$y = L^{-1}\left(\frac{s+a}{(s+a)^2 - 1}\right)$$

$$= e^{-at} L^{-1}\left(\frac{s}{s^2 - 1}\right)$$

$$= e^{-at} \cos ht$$

3. Solve $\dfrac{dy}{dt} + 2x = \sin 2t$ and

$$\frac{dx}{dt} - 2y = \cos 2t, \qquad x(0) = 1 \qquad y(0) = 0$$

Solution:

Taking Laplace Transform on both sides

$$L(y') + 2L(x) = L(\sin\ 2t)$$

$$L(x') - 2L(y) = L(\cos 2t)$$

$$sL(y) - y(0) + 2L(x) = \frac{2}{s^2 + 4}$$

$$sL(x) - x(0) + 2L(y) = \frac{s}{s^2 + 4}$$

$$sL(x) - 2L(y) = \frac{s}{s^2 + 4} + 1 \qquad \dots (1)$$

$$2L(x) + sL(y) = \frac{2}{s^2 + 4} \qquad \dots (2)$$

$$(1) \times (2) \Rightarrow 2sL(x) - 4L(y) = \frac{2s}{s^2 + 4} + 2 \qquad \dots (3)$$

$$(2) \times s \Rightarrow 2sL(x) + s^2 L(y) = \frac{2s}{s^2 + 4} \qquad \dots (4)$$

$$(3) - (4) \Rightarrow -(s^2 + 4)L(y) = 2$$

$$L(y) = \frac{-2}{s^2 + 4}$$

$$\Rightarrow y = -\sin\ 2t$$

$$\therefore y' = -2\ \cos\ 2t$$

Substituting y' in $y' + 2x = \sin\ 2t$

$$2x = \sin\ 2t + 2\cos 2t$$

$$x = \frac{1}{2}[\sin 2t + 2\cos 2t]$$

4. Solve $\frac{dx}{dt} + 3x - 2y = 1, \qquad \frac{dy}{dt} - 2x + 3y = e^t, \qquad x(0) = 0, \qquad y(0) = 0$

Solution:

Taking Laplace transform on both sides

$$L(x') + 3L(x) - 2L(y) = L(1)$$

$$L(y') - 2L(x) + 3L(y) = L(e^t)$$

$$sL(x) - x(0) + 3L(x) - 2L(y) = \frac{1}{s}$$

$$sL(y) - y(0) - 2L(x) + 3L(y) = \frac{1}{s-1}$$

$$(s+3)L(x) - 2L(y) = \frac{1}{s} \qquad \ldots (1)$$

$$-2L(x) + (s+3)L(y) = \frac{1}{s-1} \qquad \ldots (2)$$

$$(1) \times 2 \Rightarrow 2(s+3)L(x) - 4L(y) = \frac{2}{s}$$

$$(s+3) \times (2) \Rightarrow -2(s+3)L(x) + (s+3)^2 L(y) = \frac{s+3}{s-1}$$

Adding $\left[(s+3)^2 - 4\right]L(y) = \frac{s+3}{s-1} + \frac{2}{s}$

$$(s^2 + 6s - 5)L(y) = \frac{s^2 + 5s - 2}{s(s-1)}$$

$$L(y) = \frac{s^2 + 5s - 2}{(s^2 + 6s + 5)(s(s-1))} \qquad \ldots (3)$$

Now, $$\frac{s^2+5s-2}{s(s-1)(s+1)(s+5)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+1}+\frac{D}{s+5}$$

$$s^2+5s-2=A(s-1)(s+1)(s+5)+Bs(s+1)(s+5)$$

$$+C(s(s-1)(s+5))+D(s(s-1)(s+1))$$

when, $s=1,\quad 4=12B\Rightarrow B=\frac{1}{3}$

when, $s=-1,\quad -6=8C\Rightarrow C=\frac{-3}{4}$

when, $s=-5,\quad -2=-120D\Rightarrow D=\frac{1}{60}$

when, $s=0,\quad -2=-5A\Rightarrow A=\frac{2}{5}$

$\therefore$ From (3) $$y=\frac{2}{5}L^{-1}\left(\frac{1}{s}\right)+\frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right)-\frac{3}{4}L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{60}L^{-1}\left(\frac{1}{s+5}\right)$$

$$\therefore y(t)=\frac{2}{5}+\frac{1}{3}e^{t}-\frac{3}{4}e^{-t}+\frac{1}{60}e^{-5t} \qquad \ldots (4)$$

$$\therefore y'(t)=\frac{1}{3}e^{t}+\frac{3}{4}e^{-t}-\frac{1}{12}e^{-5t}$$

Substituting (4) and (5) in

$y'-2x+3y=e^{t}$ we get,

$2x=y'+3y-e^{t}$

$$2x = \frac{1}{3}e^{t}+\frac{3}{4}e^{-t}-\frac{1}{12}e^{-5t}+3\left(\frac{2}{5}+\frac{e^{t}}{3}-\frac{3}{4}e^{-t}+\frac{e^{-5t}}{60}\right)-e^{t}$$

$$=\frac{6}{5}+e^{t}\left(\frac{1}{3}+1-1\right)+e^{-t}\left(\frac{3}{4}-\frac{9}{4}\right)+e^{-5t}\left(\frac{-1}{12}+\frac{1}{20}\right)$$

$$= \frac{6}{5} + \frac{1}{3}e^{t} - \frac{6}{4}e^{-t} - \frac{2}{60}e^{-5t}$$

$$x(t) = \frac{3}{5} + \frac{1}{6}e^{t} - \frac{3}{4}e^{-t} - \frac{1}{60}e^{-5t}$$

5. Using Laplace transform solve

$Dx + Dy = t$ and $D^2x - y = e^{-t}$, $x = 3$, $Dx = -2$ and $y = 0$ at $t = 0$

Solution:

Taking Leplace transform on both sides, we get

$$L(x') + L(y') = L(t)$$

$$L(x'') - L(y) = L(e^{-t})$$

$$sL(x) - x(0) + sL(y) - y(0) = \frac{1}{s^2}$$

$$s^2L(x) - sx(0) - x'(0) - L(y) = \frac{1}{s+1}$$

i.e., $$sL(x) + sL(y) = \frac{1}{s^2} + 30$$

$$\Rightarrow L(x) + L(y) = \frac{1}{s^3} + \frac{3}{s} \qquad \ldots (1)$$

and $$s^2L(x) - L(y) = \frac{1}{s+1} + 3s - 2 \qquad \ldots (2)$$

$$(1) + (2) \Rightarrow (s^2+1)L(x) = \frac{1}{s^3} + \frac{3}{s} + \frac{1}{s+1} + 3s - 2$$

$$L(x) = \frac{1}{s^3(s^2+1)} + \frac{3}{s(s^2+1)} + \frac{1}{(s^2+1)(s+1)} + \frac{3s-2}{s^2+1} \qquad \ldots (3)$$

Consider $$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2 = 1) + (Bs + C)(s + 1)$$

Put, $s = -1, \quad 1 = 2A \Rightarrow A = 1/2$

Put, $s = 0, \quad 1 = A + C \Rightarrow C = 1/2$

Comparing coefficients of S;

$$0 = B + C \Rightarrow B = -1/2$$

(3) becomes

$$x(\mathrm{t}) = \int_0^t \int_0^t \int_0^t \sin t \, dt \, dt \, dt + 3\int_0^t \sin t \, dt + \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1} - \frac{s}{s^2+1}\right)$$

$$+ 3L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$x(\mathrm{t}) = \int_0^t \int_0^t (-\cos t)_0^t \, dt \, dt + 3(-\cos t)_0^t + \frac{1}{2} e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t + 3\cos t - 2\sin t$$

$$= \int_0^t \int_0^t (-\cos t + 1) \, dt \, dt + 3(-\cos + 1) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t - \frac{5}{2}\cos t$$

$$= \int_0^t (t - \sin t)_0^t \, dt + 3(-\cos + 1) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t + \frac{5}{2}\cos t$$

$$= \int_0^t (t - \sin t) dt + 3(1 - \cos t) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t - \frac{5}{2}\cos t$$

$$= \left(\frac{t^2}{2} + \cos t\right)_0^t + 3(1 - \cos t) + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t + \frac{5}{2}\cos t$$

$$= \frac{t^2}{2} + \frac{1}{2}\cos t + 2 + \frac{1}{2} e^{-t} - \frac{3}{2}\sin t$$

$$\therefore x'' = \frac{1}{2} e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t + 1$$

Substituting x'' in $y = x''(\mathrm{t}) - e^{-t}$

$$y = 1 - \frac{1}{2}e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t$$

6. Solve $x' - 2x + 3y = 0$

$y' - y + 2x = 0$

given that $x(0) = 8$ and $y(0) = 3$

Solution:

Applying Laplace transform to the given equations we get,

$$L(x') - 2L(x) + 3L(y) = 0$$

$$L(y') - L(y) + 2L(x) = 0$$

i.e., $sL(x) - x(0) - 2L(x) + 3L(y) = 0$

$$sL(y) - y(0) - L(y) + 2L(x) = 0$$

The above equations reduce to

$$(s-2)L(x) + 3L(y) = 8 \quad \dots (1)$$

$$2L(x) + (s-1)L(y) = 3 \quad \dots (2)$$

$$(1)\times 2 \Rightarrow 2(s-2)L(x) + 6L(y) = 16 \quad \dots (3)$$

$$(2)\times(s-2) \Rightarrow 2(s-2)L(x) + (s-1)(s-2)L(y) = 3(s-2) \quad \dots (4)$$

$$(3)-(4) \Rightarrow [6-(s-1)(s-2)]L(y) = 16 - 3(s-2)$$

$$\Rightarrow -[s^2 - 3s - 4]L(y) = -[3s - 22]$$

$$L(y) = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)}$$

$$= \frac{A}{s+1} + \frac{B}{s-4}$$

$3s - 22 = A(s - 4) + B(s + 1)$ … (5)

Put $s = 4$ in (5),

$-10 = 5B \Rightarrow B = -2$

Put $s = -1$ in (5),

$-25 = -5A \Rightarrow A = 5$

$$\therefore L(y) = \frac{5}{s+1} + \frac{2}{s-4}$$

$$y = L^{-1}\left(\frac{5}{s+1}\right) - L^{-1}\left(\frac{2}{s-4}\right)$$

$$y = 5e^{-t} - 2e^{4t}$$

$$\Rightarrow y' = -5e^{-t} - 8e^{4t}$$

Substituting y and y' in $y' - y + 2x = 0$

we get $2x = y - y'$

$= (5e^{-t} - 2e^{4t}) - (-5e^{-t} - 8e^{4t})$

$= 10e^{-t} + 6e^{4t}$

$$x = \frac{1}{2}[10e^{-t} + 6e^{4t}] = 5e^{-t} + 3e^{4t}$$

7. Solve $x'' + y = -5 \cos 2t$

$y'' + x = 5 \cos 2t$

given that $x(0) = 0$, $x'(0) = 0$, $y'(0) = 0$, $y(0) = 0$

Solution:

Applying Laplace transform to the given equations

$$L(x'') + L(y) = -5L(\cos 2t)$$

$$L(y'') + L(x) = -5L(\cos 2t)$$

$$\therefore s^2 L(x) - sx(0) - x'(0) + L(y) = \frac{-5s}{s^2+4}$$

$$s^2 L(y) - sy(0) - y'(0) + L(x) = \frac{5s}{s^2+4}$$

Given that $x(0) = x'(0) = y'(0) = y(0) = 0$

$$\Rightarrow s^2 L(x) + L(y) = \frac{-5s}{s^2+4} \qquad \ldots (1)$$

$$L(x) + s^2 L(y) = \frac{5s}{s^2+4} \qquad \ldots (2)$$

$$(1)\times 1 \Rightarrow s^2 L(x) + L(y) = \frac{-5s}{s^2+4} \qquad \ldots (3)$$

$$(2)\times s^2 \Rightarrow s^2 L(x) + s^4 L(y) = \frac{5s^3}{s^2+4} \qquad \ldots (4)$$

$$(3)-(4) \Rightarrow (1-s^4)L(y) = \frac{-5s}{s^2+4} - \frac{5s^3}{s^2+4}$$

$$= \frac{-5s-5s^3}{s^2+4}$$

$$L(y) = \frac{5s(s^2+1)}{(s^4-1)(s^2+4)}$$

$$= \frac{5s(s^2+1)}{(s+1)(s-1)(s^2+1)(s^2+4)}$$

$$y = L^{-1}\left(\frac{5s}{(s+1)(s-1)(s^2+4)}\right)$$

Now $\dfrac{5s}{(s+1)(s-1)(s^2+4)} = \dfrac{A}{s+1} + \dfrac{B}{s-1} + \dfrac{Cs+D}{s^2+4}$

$$5s = A(s-1)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-1)$$

Put $s = 1$, $5 = B(2)(5)$

$$\Rightarrow B = \frac{1}{2}$$

Put $s = -1$, $-5 = A(-2)(5)$

$$\Rightarrow A = \frac{1}{2}$$

Put $s = 0$, $0 = -4A + 4B - D$

$$0 = -4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) - D$$

$$\Rightarrow D = 0$$

Comparing the coefficient of s^3,

$$A + B + C = 0$$

$$\frac{1}{2} + \frac{1}{2} + C = 0$$

$$\Rightarrow C = -1$$

$$\therefore y = \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - \cos 2t$$

$$= \cos ht - \cos 2t$$

$$y' = \sin ht + 2\sin 2t$$

$$y'' = \cos ht + 4\cos 2t$$

From the given equation

$x = 5\cos 2t - y''$

$= 5 \cos 2t - (\cos ht + 4 \cos 2t)$

$x = \cos 2t - \cos ht$

Exercise

1. Solve the simultaneous equations

   $2x' - y' + 3x = 2t$ and $x' + 2y' - 2x - y = t^2 - t$, $x(0) = 1$, $y(0) = 1$

2. Solve the simultaneous equations

   $D^2x - Dy = \cos t$ and $Dx + D^2y = -\sin t$; $x = 1$, $Dx = 0$, $y = 0$, $Dy = 1$ at $t = 0$

3. Solve $x' - y = e^t$ and $y' + x = \sin t$; $x(0) = 1$, $y(0) = 0$.

4. Solve $x' - y = \sin t$, $y' - x = -\cos t$; $x = 2$ and $y = 0$ at $t = 0$.

5. Solve $D^2x + y = -5 \cos 2t$, $D^2y + x = 5 \cos 2t$, $x = Dx = Dy = 1$ and $y = -1$ and $t = 0$.

Answer

1. $x = -1 + \frac{9}{8}e^{-t} + \frac{7}{8}e^{\frac{3t}{5}}$

   $y = -\frac{9}{8}e^{-t} + \frac{49}{8}e^{\frac{3t}{5}} - t^2 - 3t - 4$

2. $x = 1 + t \sin t, \quad y = t \cos t$

3. $x = \frac{1}{2}(e^t + 2\sin t + \cos t - t\cos t)$

   $x = \frac{1}{2}(-e^t - \sin t + \cos t - t\sin t)$

4. $x = 2\cos ht, \quad y = 2\sin\ ht - \sin t$

5. $x = \sin t + \cos 2t, \quad y = \sin\ t - \cos 2t$