UNIT – V

FOURIER TRANSFORM

Introduction

Jean-Baptiste Joseph Fourier (21st March 1768 – 16th May1830)was a Frenchmathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law of conduction are also named in his honour. **Joseph Fourier** introduced the **transform**in his study of heat transfer, where Gaussian functions appear as solutions of the heat equation.

In the study of **Fourier** series, complicated but periodic functions are written as the sum of **simple** waves mathematically represented by sine and cosine functions. The **Fourier transform** is an extension of the **Fourier** series that results when the period of the represented function is lengthened and allowed to approach infinity. Fourier Transform maps a time series (eg. audio samples) into the series of frequencies (their amplitudes and phases) that composed the time series. Inverse Fourier Transform maps the series of frequencies (their amplitudes and phases) back into the corresponding time series. The two functions are inverses of each other. Shortly, The Fourier Transform is a mathematical technique that transforms a function of time, $f(t)$, to a function of frequency, $F(s)$.

Applications

- The Fourier transform has many applications, in fact any field of physical science that uses sinusoidal signals, such as engineering, physics, applied mathematics, and chemistry, will make use of Fourier series and Fourier transforms. Here are some examples from physics, engineering, and signal processing.

  - Communication
  - Astronomy
  - Geology
  - Optics

- Fourier Transforms helps to analyze spectrum of the signals, helps in find the response of the LTI systems. (Continuous Time Fourier Transforms is for Analog signals and Discrete time Fourier Transforms is for discrete signals)
- Discrete Fourier Transforms are helpful in Digital signal processing for making convolution and many other signal manipulation.

Integral Transform

The integral of a function f(x) is defined by

$$I[f(x)] = \int_a^b f(x)k(s,x)dx$$

Where *k(s,x)* is the kernel of the integral transform and s is the parameter. If $k(s,x) = e^{-sx}$, the integral transform leads to Laplace transform of f(x).

$$L[f(x)] = \int_a^\infty f(x)e^{-sx}dx$$

When $k(s,x) = e^{isx}$, then the Integral transform become complex form of Fourier transform.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

If we replace k(s,x) by sine and cosine functions, we get Fourier Sine and Cosine Transform.

Fourier Integral Theorem

A function f(x) which is piece-wise continuous in every finite interval in (-∞,∞) and is absolutely integrable in (-∞,∞) can be expressed as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda \qquad \ldots(1)$$

This integral is known as Fourier integral of the function f(x).

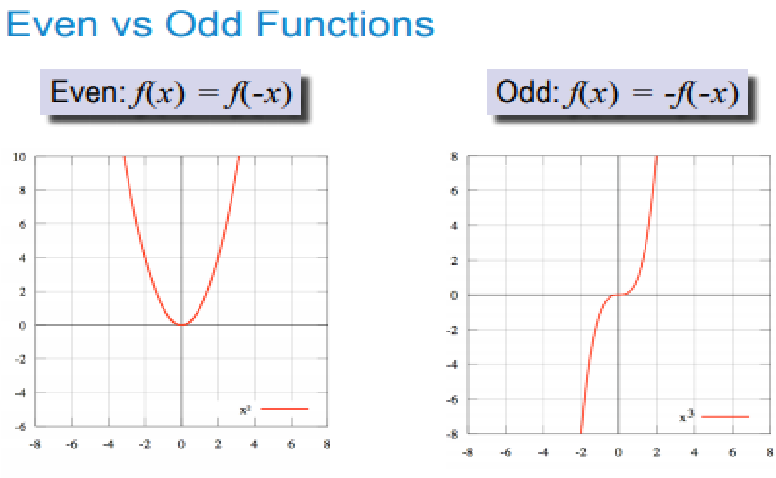
Definition of odd and even function

Odd Function

A function $f(x)$ is said to odd if $f(-x)= -f(x)$. Ex: $f(x)=x$, $\sin x$, $\tan x$, x^3.

Even Function

A function $f(x)$ is said to even if $f(-x)= f(x)$. Ex: $f(x)=x^2, x^4, \cos x, \sec x$.



Fourier Sine and cosine Integrals

The Fourier integral of f(x) is

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda \qquad \ldots(2)$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda + \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\lambda t\sin\lambda x\,dt\,d\lambda$$

Case (i)

If $f(t)$ is an odd function, then $f(t)\cos\lambda t$ is also an odd function and hence the first integral in equation (2) becomes zero.

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\sin\lambda t\sin\lambda x\,dt\,d\lambda$$

This is known as Fourier sine integral.

Case (ii)

If $f(t)$ is an even function, then $f(t)\sin\lambda t$ is odd function and hence the second integral in equation (2) becomes zero.

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda$$

This is known as Fourier cosine integral.

Let us look at the definition of Fourier transform and some basic properties of it without getting into mathematical rigor.

Fourier Transforms

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty,\infty)$ $f: R \to C$ and be piece-wise continuous in each finite partial interval then the complex Fourier transform of $f(x)$ is defined by

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Inverse Fourier Transform

Inverse complex Fourier transform of F(s) is given by

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

Properties of Fourier Transforms

1. Linearity property

If $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, then

$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$

$= aF(s) + bG(s)$, where a and b are constants.

Proof: Given $F(s) = F[f(x)]$, $G(s) = F[g(x)]$

$$F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x)+bg(x)]e^{isx}dx$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{isx}dx$$

$= aF[f(x)] + bF[g(x)] = aF(s) + bG(s)$

2. Shifting property

If $F[f(x)] = F(s)$ then $F(f(x - a) = e^{ias} F[f(x)] = e^{ias} F(s)$

Proof: Given $F[f(x)] = F(s)$

$$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$$

Put $u = x - a \therefore du = dx$. When x = -∞, u = -∞ and when x = ∞, u = ∞

$$\therefore F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{is(u+a)}du$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{isu}.e^{isa}du$$

$$= e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{isu}du = e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Changing the dummy variable from u to x

$\Rightarrow F[f(x\text{-}a)$ $= e^{isa}\, F[f(x)] = e^{isa}F(s)$

3. Changeof scale property

If $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{|a|}F\left(\frac{s}{a}\right)$ where $a \neq 0$

Proof: Given $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax).e^{isx}dx$

$$Consider \quad u = ax$$

$$\therefore du = a\,dx \Rightarrow dx = \frac{du}{a}$$

Case (i) If a>0, then when x = -∞, u = -∞ and x = ∞, u = ∞

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{\frac{isu}{a}}\frac{du}{a}$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{i\left(\frac{s}{a}\right)u}du = \frac{1}{a}F\left(\frac{s}{a}\right) \qquad \text{...(i)}$$

Case (ii) If a<0, then when x = -∞, u = ∞ and x = ∞, u = -∞

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(u).e^{\frac{isu}{a}}\frac{du}{a}$$

$$= -\frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u).e^{i\left(\frac{s}{a}\right)u}du = -\frac{1}{a}F\left(\frac{s}{a}\right) \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$F[f(ax)] = \frac{1}{|a|}F\left(\frac{s}{a}\right).\ if\ \ a \neq 0$$

Note*:* Put $a = -1$, then $F[f(-x) = F(-s)$

It can be seen that, if *f(x)* is even, then *F(s)* is even and if *f(x)* is odd, then *F(s)* is odd.

4. Shifting in s

If $F[f(x)] = F(s)$ then $F(e^{iax}f(x)) = F(s+a)$

Proof: Given $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}e^{iax}\ f(x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx = F(s+a)$$

5. Modulation Property

If $F(f(x)) = F(s)$ then $F[\cos ax f(x)] = \frac{1}{2}[F(s+a)+F(s-a)]$

Proof Given $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$\therefore F[\cos ax f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos ax f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{e^{iax}+e^{-iax}}{2} f(x)e^{isx}\right]dx$$

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}e^{iax}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}e^{-iax}dx\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}dx\right\}$$

$F[\cos axf(x)] = \frac{1}{2}\{F(s+a)+F(s-a)\}$

6. Fourier transform of Derivative

If $F[f(x)] = F(s)$ and derivative $f'(x)$ is continuous, absolutely integrable on $(-\infty, \infty)$, then $F[f'(x)] = -(is)\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

Proof Given $F(f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(e^{isx}f(x)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(is)e^{isx}f(x)dx\right\}$$

Applying integration by parts, taking $u = e^{isx}$, $dv = f'(x)dx$

$$\therefore \quad du = sx\, e^{isx}\, dx,\ v = f(x)$$

We have $\left|e^{isx}\right| = \left|\cos sx + i\sin sx\right| = 1$

Since $f(x) \to 0$ as $x \to \pm\infty$, we have $e^{isx}f(x) \to 0$ as $x \to \pm\infty$

$$\therefore \quad F[f'(x)] = \frac{1}{\sqrt{2\pi}}\left[0 - is\int_{-\infty}^{\infty} f(x)e^{isx}dx\right]$$

$$= (-is)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx = -isF[f(x)] = -isF(s)$$

Note: Similarly, we can prove that

$F(f''(x)] = -isF[f'(x)]$

$= -is(-is)\, F[f(x)] = (-is)^2 F[f(x)]$

Generally, for any positive integer n, $F[f^{(n)}(x)] = (-is)^n F(f(x))$

if $f(x), f'(x)\ldots.f^{n-1}(x)$ approaches 0 as $x \to \pm\infty$.

7. Derivative of transform

If $F[f(x)] = F(s)$, then $F(x^n f(x)) = (-i)^n \dfrac{d^n F(s)}{ds^n}$

Proof Given $F[f(x)] = F(s) \Rightarrow F(s) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

Differentiating w.r to s we get,

$$\frac{dF(s)}{ds} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{\partial}{\partial s}(e^{isx}) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} ix dx$$

$$\frac{dF(s)}{ds} = i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{isx} dx \qquad \ldots(1)$$

We again differentiating (1) w.r. to s, we get

$$\frac{d^2 F(s)}{ds^2} = i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{isx} ix dx$$

$$= i^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 f(x) e^{isx} dx = i^2 F[x^2 f(x)]$$

$$F[x^2 f(x)] = (-i)^2 \frac{d^2 F(s)}{ds^2}$$

Continuing this way, $F[x^n f(x)] = (-i)^n \dfrac{d^n F(s)}{ds^n}$

8. Fourier transform of an integral function

If $f(x)$ is an integral function with $F(f(x)) = F(s)$, then $F\left[\int_a^x f(x) dx\right] = \dfrac{i}{s} F(s)$

Proof Given $F[f(x)] = F(s)$ and $f(x)$ is integrable.

Let $\int_a^x f(x) dx = g(x)$, then $f(x) = g'(x)$ by fundamental theorem of integral calculus.

$$F[f(x)] = F[g'(x)]$$

$$= -is\, F[g(x)] = -is\, F\left[\int_a^x f(x) dx\right] \qquad \text{[by property 6]}$$

$$F\left[\int_a^x f(x) dx\right] = \frac{1}{-is} F[f(x)] = \frac{i}{s} F[f(x)]$$

9. If $F[f(x)] = F(s)$, then $F[\overline{f(x)}] = \overline{F(-s)}$ where bar denotes complex conjugate.

Proof Given $F[f(x)] = F(s) \Rightarrow F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$\therefore F(-s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}dx \qquad [\because \bar{\bar{z}} = z]$$

$$\therefore \overline{F(-s)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)e^{-isx}}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}\, e^{isx}dx = F[\overline{f(x)}]$$

$$F[\overline{f(x)}] = \overline{F(-s)}$$

Note: $F[\overline{f(-x)}] = \overline{F(s)}$

Definition: Convolution of two functions.

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$$

PROBLEMS

Problem 1. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$. Hence evaluate $\int_{0}^{\infty} \frac{\sin s}{s} ds$.

Solution: Fourier transform of $f(x)$ is $F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \cos sx\, dx \qquad (\therefore \sin sx \text{ is an odd } fn.)$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{a} \cos sx\, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin sx}{s}\right]_0^a$$

$$F(s) = \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin as}{s}\right]$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\,\frac{\sin as}{s}(\cos sx - i\sin sx)\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as}{s}\cos sx\,ds \qquad \left[\because \frac{\sin as}{s}\sin\ sx\ is\ odd\right]$$

$$\text{f(x)} = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds$$

Put $a = 1$, $x = 0$

$$f(0) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin s}{s}\,ds$$

$$\frac{\pi}{2}\times 1 = \int_0^{\infty}\frac{\sin s}{s}\,ds \ (\therefore\ f(x) = 1, -a \le x \le a)$$

$$\therefore \int_0^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}$$

Problem 2: Find the Fourier transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b; \\ 0, & x < a\ and\ x > b \end{cases}$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b e^{ikx}e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(k+s)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(k+s)x}}{i(k+s)} \right]_a^b$$

$$F[f(x)] = \frac{-i}{(k+s)\sqrt{2\pi}} \left[e^{i(k+s)b} - e^{i(k+s)a} \right]$$

Definition: If the fourier transform of f(x) is equal to f(s) then the function f(x) is called **self-reciprocal**. i.e. F(f(x)) = f(s)

Problem3: Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2)+isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)} dx \qquad \ldots (1)$$

Consider $a^2x^2 - isx = (ax)^2 - 2(ax)\frac{(is)}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2} \qquad \ldots (2)$$

Substitute (2) in (1), we get

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} \qquad \text{Let } t = ax - \frac{is}{2a}, dt = adx$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \sqrt{\pi} \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{\frac{s^2}{4a^2}} \qquad \ldots (3)$$

Put $a = \frac{1}{\sqrt{2}} in$ (3)

$$F[e^{-x^2/2}] = e^{-s^2/2}$$

$\therefore e^{-s^2/2}$ is self-reciprocal with respect to Fourier Transform.

Problem 4: State and Prove convolution theorem on Fourier transform.

Solution:

Statement: If $F(s)$ and $G(s)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, Then the Fourier transform of the convolutions of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i.e. $F[f(x) * g(x)] = F[f(x)]F[g(x)] = F(s)G(s)$

Proof:

$$F(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx} dx dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) F(g(x-t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, F(g(t)) dt \qquad [\therefore\ f(g(x-t) = e^{ist} F(g(t))]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt G(s) \qquad [\therefore F(g(t)) = G(s)]$$

$$F(f * g) = F(s).G(s). \qquad [\therefore F(f(t)) = F(s)].$$

Problem5: State and Prove Parseval's Identity in Fourier Transform.

Solution:

Statement: If $F(s)$ is the Fourier transform of $f(x)$, then $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{\infty}^{\infty} |F(s)|^2$

Proof by convolution theorem $F\,[f(x) * g(x)] = F(s)G(s)$

$$f(x) * g(x) = F^{-1}\,[F(s)G(s)]$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s) e^{isx} ds$$

Put $x = 0$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(-t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s) e^{0} ds --- (1)$$

Let $g(-t) = \overline{f(t)}$, then it follows that $G(s) = \overline{F(s)}$ (by property 9)

$\therefore$ (1) becomes

$$\int_{-\infty}^{\infty} \left[f(t)\overline{f(t)}\right] dt = \int_{-\infty}^{\infty} \left[F(s)\overline{F(s)}\right] ds \quad \because z\bar{z} = |z|^2$$

i.e. $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

Problem 6: Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$ and hence evaluate

(i) $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)dt$ **(ii)** $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt$

Solutions:

Fourier transform of $f(x)$ is

$$F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 + \int_{-a}^{a}(a^2 - x^2)e^{isx}dx + 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a^2 - x^2)(\cos sx + i\sin x)\,dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^a (a^2 - x^2)\cos sx\,dx \left[\because (a^2 - x^2)\sin sx \text{ is an odd fn.}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (2)\left(\frac{\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-2as\cos as + 2\sin as}{s^3}\right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right] \quad \ldots (1)$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin as - as\cos as}{s^3}\cos sx\,dx$$ (the second term is on odd function)

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin as - as\cos as}{s^3}\cos sx\,dx$$

Put $a = 1$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos sx\,dx \qquad \left[f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases} \right]$$

Put $x = 0$

$$f(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\,dx \qquad \left[\begin{aligned} f(0) &= 1-0 \\ &= 1 \end{aligned} \right]$$

$$1 = \frac{4}{\pi}\int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds$$

$$\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt = \frac{\pi}{4} \quad \text{[by changing s} \rightarrow \text{t]}$$

Using Parseval's identify

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3} \right) \right]^2 ds = \int_{-\infty}^{\infty} |a^2 - x^2|^2\,dx$$

$$\int_{-\infty}^{\infty} \frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \int_{-1}^{1} (1-x^2)^2\,dx \text{ (put a = 1)}$$

$$2 \times \frac{8}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = 2\int_0^1 (1-x^2)^2\,dx$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3} \right]_0^1$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \frac{\pi}{16} \times 2\left(\frac{8}{15} \right) = \frac{\pi}{15}$$

Put $s = t$, $\displaystyle\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 dt = \frac{\pi}{15}$.

Problem 7: Find the Fourier transform of $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0, & |x|\geq 1\end{cases}$ and hence find the value of **(i)** $\int_0^\infty \frac{\sin^2 t}{t^2}dt.$ **(ii)** $\int_0^\infty \frac{\sin^4 t}{t^4}dt.$

Solution:

The Fourier transform of $F(f(x))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\,(\cos sx+i\sin sx)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos sx\,dx\ [\because (1-|x|)\sin\ sx\ is\ an\ odd\ fn.]$$

$$=\frac{2}{\sqrt{2\pi}}\left\{(1-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{\cos sx}{s^2}\right)\right\}_0^1$$

$$=\frac{2}{\sqrt{2\pi}}\left\{-\frac{\cos s}{s^2}+\frac{1}{s^2}\right\}$$

$$F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]\quad (1)$$

(i) By inverse Fourier transform

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right](\cos sx-i\sin\ sx)ds\ (by\ (1))$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\,ds\ \text{ (Second term is odd)}$$

$$f(x)=\frac{2}{\pi}\int_0^\infty\left(\frac{1-\cos s}{s^2}\right)\cos sx\,ds$$

Put $x=0$

$$f(0)=1-|0|=\frac{2}{\pi}\int_0^\infty\left(\frac{1-\cos s}{s^2}\right)ds$$

$$\int_0^\infty \left(\frac{1-\cos s}{s^2}\right) ds = \frac{\pi}{2}$$

$$\int_0^\infty \frac{2\sin^2(s/2)}{s^2}\, ds = \frac{\pi}{2}$$

Put $t = s/2$ $ds = 2dt$

$$\int_0^\infty \frac{2\sin^2 t}{(2t)^2} 2dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}.$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right]^2 ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\frac{4}{\pi}\int_0^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_0^1 (1-x)^2\, dx$$

$$\frac{4}{\pi}\int_0^{\infty} \left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^2 ds = \left[2\left(\frac{1-x}{-3}\right)^3\right]_0^1$$

$$\frac{16}{\pi}\int_0^{\infty} \left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^4 ds = \frac{2}{3}; Let\ t = s/2, dt = \frac{ds}{2}$$

$$\frac{16}{\pi}\int_0^{\infty} \left(\frac{\sin t}{2t}\right)^4 2dt = \frac{2}{3}$$

$$\frac{16}{16\pi}\int_0^\infty \left(\frac{\sin t}{2t}\right)^4 dt = \frac{1}{3}$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

i.e. $\int_{-\infty}^{\infty} |f(t)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

Problem 8: Find the Fourier transform of $f(x) = \begin{cases} a-|x|, & |x| < a \\ 0, & |x| \geq a \end{cases}$ and hence prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\, dx + \int_{-a}^{-a} (a-|x|)e^{isx}dx + \int_a^\infty 0dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{-a} (a-|x|)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (a-|x|)\cos sx\, dx + 0 \qquad \left[\therefore \int [a-|x|]\sin sx\, dx = 0\; odd\;\; function\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a-x)\cos sx\, dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{\cos as}{s^2} + \frac{1}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]$$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right] \qquad \ldots (1)$$

By inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}ds.$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2\frac{as}{2}}{s^2}\right]e^{-isx}ds \text{ , Put } x = 0$$

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2\frac{as}{2}}{s^2}ds$$

$$\frac{\pi a}{4} = \int_{0}^{\infty} \frac{\sin^2\left(\frac{as}{2}\right)}{s^2}ds \quad [f(0) = a\text{-}0 = a]$$

Put $a = 2$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin^2 s}{s^2}ds \qquad [\because s \text{ is a dummy variable, we can replace it by `}t\text{'}]$$

$$\text{i.e} \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Problem 9: Find the Fourier transform of $e^{-a|x|}, a > 0$ and hence deduce that

(a) $\int_{0}^{\infty} \frac{\cos xt}{a^2 + t^2}dt = \frac{\pi}{2a}e^{-a|x|}$ **(b)** $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}}\frac{2as}{(s^2+a^2)^2}.$

Using Parseval's Theorem find the value of $\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)^2}dx, a > 0.$

Solution: $F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}(\cos sx + i \sin sx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-a|x|}(\cos sx\, dx) \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\, dx = 0,\ odd\ \ function\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \cos sx\, dx$$

$$F(e^{-a|x|}) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + s^2}\right).$$

(a) Using Fourier inverse transform,

$$\text{f(x)} = e^{-a|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2}\right](\cos sx - i \sin sx) ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx}{a^2 + s^2} ds + 0 \qquad \left[\therefore \frac{\sin sx}{s^2 + a^2} is\ an\ odd\ \ fn.\right]$$

$$= \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos xt}{a^2 + t^2} dt \qquad \text{(Replace ‘}s\text{’ by ‘}t\text{’)}$$

$$= \int_{0}^{\infty} \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

(b) To prove $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$

Property:

$$F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$$

$$F[xf(x)] = (-i) \frac{dF(s)}{ds}$$

$$F[e^{-a|x|}] = (-i) \frac{dF(e^{-a|x|})}{ds}$$

$$= -i\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right)$$

$$= ia\sqrt{\frac{2}{\pi}}\left(\frac{2s}{(a^2+s^2)^2}\right) = i\sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}$$

Parseval's identity is $\int\limits_{-\infty}^{\infty} |f(x)|^2 dx = \int\limits_{-\infty}^{\infty} |F(s)|^2 ds$

Result: $F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$

$$\int\limits_{-\infty}^{\infty}\left[e^{-|ax|}\right]^2 dx = \int\limits_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right)^2 ds$$

$$2\int\limits_{0}^{\infty}(e^{-ax})^2\ dx = 2\int\limits_{0}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left(\frac{e^{-2ax}}{-2a}\right)_0^{\infty} = \frac{2}{\pi}\int\limits_{0}^{\infty}\frac{s^2}{(s^2+a^2)^2}ds$$

i.e., $\int\limits_{0}^{\infty}\frac{s^2}{(s^2+a^2)^2}ds = \frac{\pi}{2}\left(\frac{0+1}{2a}\right) = \frac{\pi}{4a}.$

$$\therefore \int\limits_{0}^{\infty}\frac{x^2}{(x^2+a^2)^2}dx = \frac{\pi}{4a}.$$

Problem 10: Derive the relation between Fourier transform and Laplace transform.

Solution:

Consider $f(t) = \begin{cases} e^{-xt}g(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$ … (1)

The Fourier transformer of $f(x)$ is given by

$$F[f(t)] = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{0}^{\infty} e^{-xt}g(t)e^{ist}dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(is-x)t} g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-pt} g(t)\, dt \text{ where } p = x - is$$

$$= \frac{1}{\sqrt{2\pi}} L(g(t)) \quad \left[\because L\left[f(t)\right] = \int_0^\infty e^{-st} f(t) dt \right]$$

$\therefore$ Fourier transform of $f(t) = \frac{1}{\sqrt{2\pi}} \times$ Laplace transform of $g(t)$ where $g(t)$ is defined by (1).

Fourier Sine and Cosine Transform

Fourier sine and cosine transform are related to Fourier sine and cosine integrals. The Fourier transform applies to the problems concerning the real axis or the interval ($-\infty,\infty$) whereas sine and cosine transform apply to the problem concerning the interval $(0,\infty)$.

The Fourier Sine Integral of f(x) in $(0,\infty)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \sin st \sin sx dt ds \text{ [Replace } \lambda \text{ by s]}$$

$$= \frac{2}{\pi} \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin st dt \right] \sin sx ds$$

We denote $F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$

This is known as **Infinite Fourier Sine Transform** of f(x).

Inverse Fourier Sine Transform is

$$f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, ds$$

The Fourier cosine Integral of f(x) in $(0,\infty)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos st \cos sx dt ds \text{ [Replace } \lambda \text{ by s]}$$

$$= \frac{2}{\pi}\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt\right]\cos sx\,ds$$

We denote $F_C[f(x)] = F_C(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

This is known as **Infinite Fourier Cosine Transform** of f(x).

The **Inverse Fourier Cosine Transform** is

$$f(x) = F^{-1}[F_C(s)] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_C(s)\cos sx\,ds$$

Properties of Fourier Sine and Cosine Transforms

1. Linearity Property

(i) $F_c[af(x) + bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

(ii) $F_s[af(x) + bg(x)] = aF_s[f(x)] + bF_s[g(x)]$ where a and b are constants.

Proof: (i) By definition, $F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = F_c(s)$

$$F_c[af(x) + bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(af(x) + bg(x)\right)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(a\,f(x)\right)\cos sx\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty \left(bg(x)\right)\cos sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty \left(f(x)\right)\cos sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty \left(g(x)\right)\cos sx\,dx$$

$$= aF_c[f(x)] + bF_c[g(x)]$$

(ii) By definition, $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = F_s(s)$

$$F_s[af(x) + bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(af(x) + bg(x)\right)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left(a\,f(x)\right)\sin sx\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty \left(bg(x)\right)\sin sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty (f(x))\sin sx dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty (g(x))\sin sx dx$$

$$= aF_s[f(x)] + bF_s[g(x)]$$

2. Modulation property

If $F_c[f(x)] = F_c[s]$ and $F_s[f(x)] = F_s[s]$, then

(i) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(ii) $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iii) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(iv) $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

(i) To Prove: $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

Proof: We have, $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int_0^\infty f(x)\{\cos(s+a)x + \cos(s-a)x\}dx$$

$$= \frac{1}{2}.\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\cos(s+a)x + f(x)\cos(s-a)x]dx$$

$$= \frac{1}{2}.\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\cos(s+a)x dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x dx\right\}$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(ii) **To prove:** $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

Proof: We have, $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$

$$\therefore F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}\int_0^\infty f(x)\{\sin(s+a)x+\sin(s-a)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\sin(s+a)x+f(x)\sin(s-a)x]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)xdx+\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)xdx\right\}$$

$$= \frac{1}{2}[F_s(s+a)+F_s(s-a)]$$

(iii) **To prove:** $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(s+a)-F_s(s-a)]$

Proof We have, $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$

$$\therefore F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\cos sx dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty f(x)\{\sin(s+a)x-\sin(s-a)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\sin(s+a)x-f(x)\sin(s-a)x]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s+a)xdx-\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(s-a)xdx\right\}$$

$$= \frac{1}{2}[F_S(s+a)-F_S(s-a)]$$

(iv) **To prove:** $F_s[f(x)\sin ax] = \frac{1}{2}[F_C(s-a)-F_C(s+a)]$

Proof We have, $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$

$\therefore F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax \sin sx dx$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty f(x)\{\cos(s-a)x - \cos(s+a)x\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty [f(x)\cos(s-a)x - f(x)\cos(s+a)x]dx$$

$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)xdx - \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)xdx\right\}$$

$$= \frac{1}{2}[F_C(s-a) - F_C(s+a)]$$

3. Change of Scale Property

(i) $F_C[f(ax)] = \frac{1}{a}F_C\left(\frac{s}{a}\right)$ $if\ a > 0$ (ii) $F_S[f(ax)] = \frac{1}{a}F_S\left(\frac{s}{a}\right)$ $if\ a > 0$

(i) To prove: $F_C[f(ax)] = \frac{1}{a}F_C\left(\frac{s}{a}\right)$ **if** $a > 0$

Proof We have $F_C[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sx dx$

Put $t = ax$. $\therefore dt = adx \Rightarrow dx = \frac{dt}{a}$

when $x = 0$, $t = 0$ and when $x = \infty$, $t = \infty$

$$\therefore F_C[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{st}{a}\right)\frac{dt}{a} = \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\left(\frac{s}{a}\right)tdt = \frac{1}{a}F_c\left(\frac{s}{a}\right) \quad [\because a > 0]$$

(i) To prove: $F_S[f(ax)] = \frac{1}{a}F_s\left(\frac{s}{a}\right)$ $if\ a > 0$ **if** $a > 0$

Proof We have $F_S(f(ax)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\sin sx dx$

Put $t = ax$ $\quad \therefore dt = adx \Rightarrow dx = \dfrac{dt}{a}$

when $x = 0$, $t = 0$ and when $x = \infty$, $t = \infty$

$$\therefore F_S[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin\left(\frac{st}{a}\right)\frac{dt}{a} = \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin\left(\frac{s}{a}\right)tdt = \frac{1}{a}F_S\left(\frac{s}{a}\right)$$

4. Differentiation of sine and cosine transform

(i) $F_C[xf(x)] = \dfrac{d}{ds}[F_S(s)] = \dfrac{d}{ds}[F_S(f(x))]$

(ii) $F_S[xf(x)] = -\dfrac{d}{ds}[F_C(s)] = -\dfrac{d}{ds}[F_C(f(x))]$

(i) To prove: $F_C[xf(x)] = \dfrac{d}{ds}[F_S(s)]$

Proof: We know $F_S(s) = F_S[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x)\sin sxdx$

Differentiating w.r to s, we get $\quad \dfrac{d}{ds}[F_S(s)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x)\frac{\partial}{\partial s}(\sin sx)dx$

$$\frac{d}{ds}[F_S(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x).(x\cos sx)dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty xf(x)\cos sxdx = F_C[xf(x)]$$

$$F_C[xf(x)] = \frac{d}{ds}[F_S(s)]$$

(ii) To prove: $F_S[xf(x)] = -\dfrac{d}{ds}[F_C(s)]$

Proof: We know $F_C(s) = F_c[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x)\cos sxdx$

Differentiating w.r to s, we get $\quad \dfrac{d}{ds}[F_C(s)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x)\frac{\partial}{\partial s}(\cos sx)dx$

$$\frac{d}{ds}[F_C(s)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x).(-\sin sx)x\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty x f(x)\sin sx dx = -F_s[xf(x)]$$

$$F_s[xf(x)] = -\frac{d}{ds}[F_C(s)]$$

5. Cosine and sine transforms of derivative

If *f(x)* is continuous and absolutely integrable in (-∞, ∞) and if *f(x)* → 0 as *x* → ∞,

then (i) $F_C[f'(x)] = sF_S[f(x)] - \sqrt{\frac{2}{\pi}}f(0)$

(ii) $F_S[f'(x)] = -sF_C[f(x)]$

Proof (i) by definition of Fourier cosine transform

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx \quad \therefore F_C[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos sx dx$$

Applying integration by parts, taking $u = \cos sx$, $dv = f'(x)dx$

$\therefore$ $du = -\sin sx.\ s\ dx.$ $v = f(x)$

We get,

$$\therefore F_C[f'(x)] = \sqrt{\frac{2}{\pi}}\left\{[\cos sx. f(x)]_0^\infty - \int_0^\infty f(x)(-s\sin sx)dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\{[0-\cos 0. f(0)]\} + s\int_0^\infty f(x)\sin sx dx \quad [\because f(x)\to 0, as\ x\to\infty]$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + s\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$

$$F_C[f'(x)] = sF_S[f(x)] - \sqrt{\frac{2}{\pi}}f(0)$$

(ii) $$F_S[f'(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin sx dx$$

Applying integration by parts, taking $u = \sin sx$, $dv = f'(x)dx$, we get

$$F_S[f'(x)] = \sqrt{\frac{2}{\pi}}\left\{[\sin sx.f(x)]_0^\infty - \int_0^\infty f(x)s\cos sx dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{0 - s\int_0^\infty f(x)\cos sx dx\right\} \qquad [\text{as } x \to \infty, \text{f(x)} \to 0]$$

$$= -s\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$F_S[f'(x)] = -sF_C[f(x)]$$

Note:

1. $$F_C[f''(x)] = sF_s[f'(x)] - \sqrt{\frac{2}{\pi}}f'(0) = s(-sF_C[(x)]) - \sqrt{\frac{2}{\pi}}f'(0)$$

2. $$F_C[f''(x)] = -s^2F_C[f(x)] - \sqrt{\frac{2}{\pi}}f'(0)$$

3. $$F_S[f''(x)] = -sF_C[f'(x)]$$

$$= -s\left\{sF_S[f(x)] - \sqrt{\frac{2}{\pi}}f(0)\right\} = -s^2F_S[f(x)] + s\sqrt{\frac{2}{\pi}}f(0)$$

These formulae are useful in solving differential equations.

6. Identities

If $F_c(s)$ and $G_c(s)$ are the Fourier cosine transforms and $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively then

i) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(s)G_c(s)ds$$

ii) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s(s)G_s(s)ds$$

iii) $$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |F_s(s)|^2\, ds$$

Problem 1: Find the Fourier Sine Transform of e^{-3x}**.**

Solution:

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$$

$$F_s(e^{-3x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3x} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-3x}}{s^2+9}(-3\sin sx - s\cos sx) \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+9} \right) \quad \left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] \right].$$

Problem 2: Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Solution:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \left[\frac{\cos(s+1)x + \cos(s-1)x}{2} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1} \right], \text{ provided } s \neq 1, s \neq -1.$$

Problem 3: Find the Fourier cosine transform of $e^{-2x} + 3e^{-x}$.

Solution:

Let $f(x) = e^{-2x} + 3e^{-x}$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c[e^{-2x} + 3e^{-x}] = \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-2x} \cos sx\, dx + \int_0^\infty 3e^{-x} \cos sx\, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4}+\frac{3}{s^2+1}\right].$$

Problem 4: Find the Fourier cosine transform of f(*x*) defined as

$$f(x)=\begin{cases} x & for \quad 0<x<1 \\ 2-x & for \quad 1<x<2 \\ 0 & for \quad x>2 \end{cases}$$

Solution: By definition of Fourier Cosine Transform

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\, dx + \int_1^2 (2-x)\cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(x\frac{\sin sx}{s}-\frac{\cos sx}{s^2}\right)_0^1+\left((2-x)\frac{\sin sx}{s}+\frac{\cos sx}{s^2}\right)_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{\sin s}{s}-\frac{\cos s-\cos 0}{s^2}\right)+\left(0-(1)\frac{\sin s}{s}+\frac{\cos 2s-\cos s}{s^2}\right)\right]$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos 2s-2\cos s+1}{s^2}\right]$$

Problem 5: Find the Fourier sine transform of $\frac{1}{x}$.

Solution:

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\, sx\, dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{1}{x}\sin\, sx\, dx$$

Let $sx = \theta$, $sdx = d\theta$;

X	0	∞
θ=sx	0	∞

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{s}{\theta}\sin\,\theta\frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} d\theta \left[\therefore \int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}.$$

Problem 6: Find the Fourier cosine and sine transformation of $f(x) = e^{-ax}$, $a > 0$. Hence deduce that $\int_0^\infty \frac{x \sin \alpha\, x}{1+x^2} dx = \frac{\pi}{2} e^{-\alpha}$. and $\int_0^\infty \frac{\cos xt}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$

Solution:

The Fourier cosine transform is $F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx dx$

$$\Rightarrow F_C[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2}(-a+0)\right] = \sqrt{\frac{2}{\pi}}.\frac{a}{a^2+s^2}$$

The Fourier sine transform is $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin sx\, dx$

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2}(0-s)\right] = \sqrt{\frac{2}{\pi}}.\frac{s}{a^2+s^2}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)$$

By inverse Sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2 + a^2} \right) \sin sx\, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

$$\frac{\pi}{2} e^{-ax} = \int_0^\infty \frac{s \sin sx}{s^2 + a^2} ds$$

Put $a = 1$, $x = \alpha$

$$\frac{\pi}{2} e^{-a} = \int_0^\infty \frac{s \sin sx}{s^2 + 1} ds$$

Replace 's' by 'x' and 'x' by 's'

$$\int_0^\infty \frac{x \sin sx}{1 + x^2} dx = \frac{\pi}{2} e^{-\alpha}.$$

Using Fourier inverse cosine transform,

$$\text{f(x)} = e^{-a|x|} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \cos sx\, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2 + s^2} ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos xt}{a^2 + t^2} dt \qquad \text{(Replace '}s\text{' by '}t\text{')}$$

$$\int_0^\infty \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

Problem 7: Find Fourier cosine transform of (i) $e^{-ax}\sin ax$ (ii) $e^{-ax}\cos ax$

Solution: (i) $F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_c[e^{-ax}\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin ax\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\infty e^{-ax}[\sin(s+a)x - \sin(s-a)x]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{s+a}{a^2+(s+a)^2} - \frac{s-a}{a^2+(s-a)^2}\right\} \qquad \left[\therefore \int_0^\infty e^{-ax}\sin bx dx = \frac{b}{a^2+b^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{(a^2+(s-a)^2)(s+a)-(s-a)(a^2+(s+a)^2}{(a^2+(s+a)^2)(a^2+(s-a)^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{2a^2s+s^3-2as^2+2a^3+as^2-2a^2s-2s^2-s^3-2as^2+2a^3+s^2a+2sa^2}{4a^4+2a^2s^3-4a^3s+2a^2s^2+s^4-2as^3+4a^3s+2as^2-4a^2s^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{2a^3-as^2}{4a^4+s^4}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{a(2a^2-s^2}{4a^4+s^4}\right)$$

(ii) $F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

By Modulation Theorem.

$$F_c[f(x)\cos ax] = \frac{1}{2}[F_c(a+s)+F_c(a-s)]$$

$$F_c[e^{-ax}\cos ax] = \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left\{\frac{a}{a^2+(a+s)^2}+\frac{a}{a^2+(a-s)^2}\right\}\right]$$

$$= \frac{1}{2}\times\sqrt{\frac{2}{\pi}}\times a\left\{\frac{a^2+(a-s)^2+a^2+(a+s)^2}{[a^2+(a+s)^2](a^2+(a-s)^2]}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{4a^2+2s^2}{s^4+4a^4}\right]$$

$$F_c[e^{-ax}\cos ax] = \frac{2a}{\sqrt{2\pi}}\left[\frac{2a^2+s^2}{s^4+4a^4}\right]$$

Problem 8: Find $F_c(xe^{-ax})$ and $F_s(xe^{-ax})$

Solution:

$$F_c(xe^{-ax}) = \frac{d}{ds}F_s[f(x)]$$

$$F_c(xe^{-ax}) = \frac{d}{ds}F_s[e^{-ax}]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx\right]$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$F_c(xe^{-ax}) = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$F_s[xe^{-ax}] = -\frac{d}{ds}[F_c e^{-ax}] \quad \left(\because F_s(xf(x)) = -\frac{d}{ds}\left(F_c(f(x))\right)\right)$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx\right]$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]$$

$$F_s[xe^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]$$

Problem9. Find (i) $Fs\left(\frac{e^{-ax}}{x}\right)$ and (ii) $F_c\left(\frac{e^{-ax}}{x}\right)$

(i) To find $F_s\left[\frac{e^{-ax}}{x}\right]$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx \qquad \ldots (1)$$

Diff. on both sides w. r. to 's' we get

$$\frac{d}{ds}(F_s(s)) = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx \quad \left[\therefore \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{b^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left\{\frac{e^{-ax}}{x}\sin sx\right\}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{xe^{-ax}\cos sx}{x}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\frac{d}{ds}F_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating w. r. to 's' we get

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int \frac{a}{s^2+a^2}\,ds + c$$

$$= \sqrt{\frac{2}{\pi}}a.\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$$

But $F_s(s) = 0$ When $s = 0$ $\therefore c = 0$ from (1)

$$\therefore F_s\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

(ii) To find $F_c\left[\frac{e^{-ax}}{x}\right]$

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx \qquad \ldots (1)$$

Diff. on both sides w. r. to 's' we get

$$\frac{d}{ds}(F_c(s)) = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\, dx \quad \left[\therefore \int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{b^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left\{\frac{e^{-ax}}{x}\cos sx\right\}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{-xe^{-ax}\sin sx}{x}dx$$

$$= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\, dx$$

$$\frac{d}{ds}F_c(s) = -\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

Integrating w. r. to 's' we get

$$F_c(s) = -\sqrt{\frac{2}{\pi}}\int \frac{s}{s^2+a^2}ds$$

$$= -\sqrt{\frac{2}{\pi}}.\frac{1}{2}\log\left(s^2+a^2\right)$$

$$\therefore F_c\left(\frac{e^{-ax}}{x}\right) = -\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)$$

Problem 10: Find (i) $Fs\left(\frac{e^{-ax}-e^{-bx}}{x}\right)$ and (ii) $F_c\left(\frac{e^{-ax}-e^{-bx}}{x}\right)$

Solution: (i) $Fs\left(\frac{e^{-ax}-e^{-bx}}{x}\right) = Fs\left(\frac{e^{-ax}}{x}\right) - Fs\left(\frac{e^{-bx}}{x}\right)$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) - \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)\right] \qquad \ldots\text{(ii)}$$

$$Fc\left(\frac{e^{-ax}-e^{-bx}}{x}\right)=Fc\left(\frac{e^{-ax}}{x}\right)-Fc\left(\frac{e^{-bx}}{x}\right)$$

$$=-\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)+\frac{1}{\sqrt{2\pi}}\log\left(s^2+b^2\right)$$

$$=\frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Problem11: Using Parseval's Identity calculate

(a) $\int\limits_0^\infty \frac{1}{(a^2+x^2)^2}dx$ **(b)** $\int\limits_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$

Solution: (a) By Parseval's identity.

$$\int\limits_0^\infty |f(x)|^2\,dx=\int\limits_0^\infty |F_c(s)|^2\,ds$$

$$\int\limits_0^\infty e^{-2ax}dx=\int\limits_0^\infty\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]^2ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty=\frac{2}{\pi}a^2\int\limits_0^\infty\frac{ds}{(a^2+s^2)^2}$$

$$\frac{1}{2a}=\frac{2a^2}{\pi}\int\limits_0^\infty\frac{ds}{a^2+s^2}$$

i.e. $\int\limits_0^\infty\frac{dx}{(a^2+x^2)^2}=\frac{\pi}{4a^3}$ [Replace, s by x]

(b) By Parseval's identity.

$$\int\limits_0^\infty |f(x)|^2\,dx=\int\limits_0^\infty |F_s(f(x))|^2\,ds$$

$$\int\limits_0^\infty (e^{-ax})^2dx=\frac{2}{\pi}\int\limits_0^\infty\left(\frac{s}{a^2+s^2}\right)^2ds$$

i.e $\int\limits_0^\infty\frac{s^2}{(a^2+s^2)^2}ds=\frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty=\frac{\pi}{2}\times\frac{1}{2a}$

$\int\limits_0^\infty\frac{x^2}{(a^2+x^2)^2}dx=\frac{\pi}{4a}$ [Replace, s by x]

Problem 12. Evaluate **(a)** $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$, using Fourier cosine and sine transform.

Solution: **(a)** Let $f(x) = e^{-x}$ and $g(x) = e^{-2x}$

$$F_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{s^2+1}(-\cos x + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right] \qquad \dots (1)$$

$$F_c(e^{-2x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{2}{s^2+4}\right) \qquad \dots (2)$$

$$\therefore \int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(f(x))F_c(g(x))ds$$

$$\int_0^\infty e^{-x}e^{-2x}dx = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{s^2+1}\cdot\frac{2}{s^2+4}\right)ds \text{ (} from \text{ (1) \& (2))}$$

$$\int_0^\infty e^{-3x}dx = \frac{4}{\pi}\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)}ds$$

$$\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \frac{\pi}{4}\left[\frac{e^{-3x}}{-3}\right]_0^\infty = \frac{\pi}{4}\left(\frac{1}{3}\right)$$

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12} \text{ [Replace s to x]}$$

(b) To find $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx.$

Let

$$f(x) = e^{-ax}, g(x) = e^{-bx}$$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \qquad \ldots (1)$$

$$F_s(g(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right) \qquad \ldots (2)$$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s[f(x)].F_s[g(x)]ds \text{ From (1) and (2)}$$

$$\int_0^\infty e^{-ax}e^{-bx}dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

i.e $$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2(a+b)} \text{ [Replace s to x]}$$

Problem 13: Find the Fourier sine and cosine transform of x^{n-1}. Hence deduce that $\frac{1}{\sqrt{x}}$ is self-reciprocal under sine and cosine transform. Also find $F\left(\frac{1}{\sqrt{|x|}}\right)$.

Solution: We know that gamma function is given by $\Gamma(n) = \int_0^\infty e^{-y}y^{n-1}dy$ ….(1)

Put $y = ax$, we get $\int_0^\infty e^{-ax}(ax)^{n-1}a\,dx = \Gamma(n)$

$$\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}$$

Put a = is

$$\therefore \int_0^\infty e^{-isx}x^{n-1}dx = \frac{\Gamma(n)}{(is)^n}$$

$$\int_0^\infty (\cos sx - i\sin sx)x^{n-1}dx = \frac{\Gamma(n)}{s^n}i^{-n}$$

$$= \frac{\Gamma(n)}{s^n}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-n}$$

$$\int_0^\infty (\cos sx - i \sin sx) x^{n-1} dx = \frac{\Gamma(n)}{s^n}\left[\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right]$$

Equating real and imaginary parts, we get

$$\int_0^\infty x^{n-1} \cos sx dx = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \qquad \ldots (2)$$

$$\int_0^\infty x^{n-1} \sin sx dx = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \qquad \ldots (3)$$

X[ly] $\sqrt{\frac{2}{\pi}}$ on both sides of equation (2) and (3)

$$F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\, dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \qquad \ldots (3)$$

$$F_s(x^{n-1}) = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\, dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \qquad \ldots (4)$$

Put $n = \frac{1}{2}$ in (3) and (4)

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{s}}\frac{1}{\sqrt{2}} \qquad \left[\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$$

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\sin\frac{\pi}{4}$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{s}}\frac{1}{\sqrt{2}} \qquad \left[\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self-reciprocal under Fourier sine and cosine transform.

To find $F\left\{\frac{1}{\sqrt{1x1}}\right\}$

$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e\frac{isx}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{x}}(\cos sx + i\sin sx)\,dx$$

$$F\left\{\frac{1}{\sqrt{|x|}}\right\} = \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} \frac{1}{\sqrt{x}}\cos sx\,dx \qquad \text{….(5)}[\because \text{The second term odd}]$$

Put n = 1/2 in (2), we get

$$\int_0^\infty \frac{\cos sx}{\sqrt{x}}\,dx = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}}\cos\frac{\pi}{4}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2s}} \qquad \text{...(6)}$$

Substitute (6) in (5)

$$\therefore F\left\{\frac{1}{\sqrt{|x|}}\right\} = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{\sqrt{2s}} = \frac{1}{\sqrt{s}}$$

Problem 14: Find $f(x)$ if its sine transform is e^{-as}, $a > 0$.

Solution:

$$F_s(f(x)) = F(s)$$

Given that $F_s(f(x)) = e^{-as}$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin\, x\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\sin\, sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+s^2}(-a\sin sx - x\cos sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{x}{a^2+x^2}\right).$$

Problem 15: Find $f(x)$ if its Fourier sine Transform is $\frac{e^{-as}}{s}$.

Solution:

Let $F_s(f(x)) = \frac{e^{-as}}{s}$

$$\text{Then } f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{e^{-as}}{s}\sin sx\, ds \qquad \ldots (1)$$

$$\therefore \frac{df}{dx} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\partial}{\partial x}\left(\frac{e^{-as}}{s}\sin sx\right) ds =$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-as}}{s} s\cos sx\, ds = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-as}\cos sx\, ds = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+x^2}$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}a\int\frac{dx}{a^2+x^2}$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right) + c \qquad \ldots (2)$$

At $x = 0$, $f(0) = 0$ using (1)

$$(2) \Rightarrow f(0) = \sqrt{\frac{2}{\pi}}\tan^{-1}(0) + c \quad \therefore c = 0$$

$$\text{Hence } f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$$

Problem 16. Find the Fourier Cosine Transform of e^{-x^2} and hence Show that $xe^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier sine transform.

Solution:

The Fourier Cosine Transform of f(x) is

$$F_c\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$=\sqrt{\frac{2}{\pi}}.\frac{1}{2}.2\int_0^\infty e^{-x^2}\cos sx dx$$

$$=\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\text{ R.P of }\ e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty}\frac{e^{-x^2+isx}}{e^{\frac{-s^2}{4}}}dx$$

$$=\frac{1}{\sqrt{2\pi}}\text{ R.P of }\ e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty}e^{-x^2+isx+\frac{s^2}{4}}dx$$

$$=\frac{1}{\sqrt{2\pi}}\text{ R.P of }\ e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty}e^{-\left(x-\frac{is}{2}\right)^2}dx$$

Put $x-\dfrac{is}{2}=y;\quad dx=dy$

When $x=-\infty,\quad y=-\infty$

$x=\infty,\quad y=\infty$

$$=\frac{1}{\sqrt{2\pi}}\text{ R.P of }\ e^{-\frac{s^2}{4}}\int_{-\infty}^{\infty}e^{-y^2}dy$$

$$=\frac{1}{\sqrt{2\pi}}\text{ R.P of }\ e^{-\frac{s^2}{4}}2\int_{0}^{\infty}e^{-y^2}dy$$

$$=\frac{1}{\sqrt{2\pi}}\text{ R.P of }\ e^{-\frac{s^2}{4}}2\frac{\sqrt{\pi}}{2}\qquad\qquad\left(\because\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\right)$$

$$F_c\left(e^{-x^2}\right)=\frac{1}{\sqrt{2}}e^{\frac{-s^2}{4}}$$

Result : $$F_s\left[xe^{\frac{-x^2}{2}}\right] = -\frac{d}{ds}F_c\left[e^{\frac{-x^2}{2}}\right]$$

But $$F_c\left[e^{\frac{-x^2}{2}}\right] = e^{\frac{-s^2}{2}}$$

$$F_s\left[xe^{\frac{-x^2}{2}}\right] = -\frac{d}{ds}\left(e^{\frac{-s^2}{2}}\right)$$

$$= -e^{\frac{-s^2}{2}}.\left(\frac{-2s}{2}\right)$$

$$= se^{\frac{-s^2}{2}}$$

$\therefore$ $xe^{\frac{-s^2}{2}}$ is self reciprocal with respect to sine transform

Exercise 1.

1. Find the Fourier sine and cosine transforms for $f(x) = \begin{cases} 1, & if\ \ 0 < x < 1 \\ 0, & if\ \ x \geq 1 \end{cases}$

2. Find the Fourier cosine transform of $2e^{-5x} + 5e^{-2x}$.

3. Find the Fourier sine and cosine transform of e^{-x} and hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}$ and $\int_0^\infty \frac{\cos mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}$

4. Find the Fourier cosine transform of $f(x) = \begin{cases} \cos sx & if\ \ 0 < x < a \\ 0 & if\ \ x \geq a \end{cases}$

5. Find the Fourier cosine transform of e^{-x^2}.

6. Find the Fourier sine transform of $\frac{x}{1+x^2}$.

7. Find the Fourier cosine transform of $\frac{x}{1+x^2}$.

8. If $F_S[f(x)] = \frac{e^{-as}}{s}$, find f(x) and $F_S^{-1}\left[\frac{1}{s}\right]$.

9. If $F_S[f(x)] = \begin{cases} 1, & 0 \le s \le 1 \\ 2, & 1 \le s \le 2, \\ 0, & if \ s \ge 2 \end{cases}$ find $f(x)$.

10. Show that $xe^{-x^2/2}$ is self reciprocal with respect to Fourier sine transform.

11. Find the Fourier sine transform of $e^{-|x|}$, $x \ge 0$, and hence evaluate $\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx$.

12. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

13. Find the Fourier sine and cosine transform of $e^{-ax} \cos ax$, $a > 0$.

14. Find the Fourier sine transform of the function $f(x) = \begin{cases} \sin x, & 0 \le x < a \\ 0, & x > a \end{cases}$

15. Find the Fourier cosine transform of $\frac{1}{x^2 + a^2}$.

Answer

1. $F_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos s}{s} \right), \quad F_C(s) \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}.$

2. $\frac{\sqrt{2}}{\pi} \left[\frac{10}{s^2 + 25} + \frac{10}{s^2 + 4} \right]$

4. $\frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1} \right], s \ne \pm 1.$

5. $\frac{1}{\sqrt{2}} e^{\frac{s^2}{4}}$

6. $\sqrt{\frac{\pi}{2}} e^{-s}$

7. $\sqrt{\frac{\pi}{2}} e^{-s}$

8. $\sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a}; \sqrt{\frac{\pi}{2}}$

9. $\frac{2}{\pi x}[1+\cos x-2\cos 2x]$

11. $\frac{\pi}{2}e^{-m}$

12. $\frac{\sqrt{2}}{\pi}\tan^{-1}\frac{s}{a}$

13. $\frac{1}{\sqrt{2\pi}}\left\{\frac{s+a}{a^2+(s+a)^2}+\frac{s-a}{a^2+(s-a)^2}\right\},\frac{1}{\sqrt{2\pi}}\left[\frac{a}{a^2+(s+a)^2}+\frac{a}{a^2+(s-a)^2}\right]$

14. $\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)a}{s-a}-\frac{\sin(s+1)a}{s+1}\right], \quad s\neq\pm 1.$

15. $\sqrt{\frac{\pi}{2}}\frac{e^{-as}}{a}, a>0$

Exercise 2.

1. Using the Fourier transform $e^{-|x|}$, prove that $\int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}$

2. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ using transform methods.

   [Hint: Consider $f(x) = e^{-x}, g(x) = e^{-2x}$]

3. If $f(x) = \begin{cases}\cos x, & |x|<\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}$ using parseval's Identity evaluate $\int_0^\infty \frac{-\cos^2\frac{\pi x}{2}}{(1-x)^2}dx$

4. Using Parseval's identity, evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx.$

   $\left(\text{Hint: } f(x) = \begin{cases}1, & if\ 0\le x<1\\ 0 & if\ x\ge 1\end{cases}\text{. Find } F_C[f(x)] \text{ and use Parseval's identity}\right)$

5. If $F_s[f(x)] = \frac{e^{-as}}{s}, a>0$ find $f(x)$ and $F_S^{-1}\left[\frac{1}{s}\right]$

6. Find the Fourier transform of *f(x)* given by *f(x)* = $\begin{cases} 1 & for\,|x|<2 \\ 0 & for\,|x|>2 \end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$ and $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx.$

7. Using Parseval's identity evaluate $\int_0^\infty \frac{x^2}{(a^2+x^2)}dx.$

8. Using Parseval's identities, prove that $\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}dx = \frac{\pi}{2}.\frac{(1-e^{-a^2})}{a^2}$

9. Prove that $\int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}, a>0, b>0$

10. Solve the integral equation $\int_0^\infty f(\theta)\cos\alpha\theta d\theta = \begin{cases} 1-\alpha, & if\; 0\le\alpha\le 1 \\ 0, & if\; \alpha>1 \end{cases}$

Answer

1. $\frac{\pi}{2}$ 3. $\frac{\pi^2}{8}$ 4. $\frac{\pi}{2}$ 5. $\sqrt{\frac{2}{\pi}}\tan^{-1}\frac{x}{a}, \sqrt{\frac{\pi}{2}}$

6. $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ 7. $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx = \frac{\pi}{2}$ 10. $\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$

Finite Fourier Transforms

If *f(x)* is a function defined in the interval (0,*l*)then **the finite Fourier sine transform** of *f(x)* in 0 <*x*<*l* is defined as

$$F_S[f(x)] = \int_0^l f(x).\sin\frac{n\pi x}{l}dx \text{ where `}n\text{' is an integer.}$$

The **inverse finite Fourier sine transform** of $F_s[f(x)]$ *f(x)* and given by

$$f(x) = \frac{2}{l}\sum_{n=1}^\infty F_s[f(x)]\sin\frac{n\pi x}{l}$$

The **finite Fourier cosine transform** of *f(x)* in 0 <*x* < *l* is defined as

$$F_C[f(x)] = \int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

where *'n'* is an integer.

The **inverse finite Fourier cosine transform** of $F_c[f(x)]$ is $f(x)$ and is given by

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} F_c[f(x)] \cos \frac{n\pi x}{l}$$

Example 1. Find the finite Fourier sine and cosine transform of $f(x) = x^2$ in $0 < x < 1$.

Solution:

The finite Fourier sine transform is

$$F_S[f(x)] = \int_0^l f(x) . \sin \frac{n\pi x}{l} dx$$

Here $f(x) = x^2$

$$\therefore \quad F_s[x^2] = \int_0^l x^2 \sin \frac{n\pi x}{l} dx$$

$$= \left[x^2 \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right] - 2x \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + \left(2 \frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right)_0^l$$

$$= \frac{-l^3}{n\pi} \cos n\pi + \frac{2l^3}{n^3\pi^3} \cos n\pi - \frac{2l^3}{n^3\pi^3} \quad , \cos n\pi = (-1)^n, \ \sin n\pi = 0$$

$$= \frac{l^3}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}[(-1)^n - 1]$$

The finite Fourier cosine transform is

$$F_c[f(x)] = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

Here $f(x) = x^2$

$$\therefore \quad F_c[x^2] = \int_0^l x^2 \cos \frac{n\pi x}{l} dx$$

$$= \left[x^2 \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - 2x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + 2 \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l$$

$= \frac{2l^3}{n^2\pi^2}\cos n\pi$, $\cos n\pi = (-1)^n$, $\sin n\pi = 0$

$= \frac{2l^3}{n^2\pi^2}(-1)^n$

Example 2: Find the finite Fourier sine and cosine transform of $f(x) = x$ in $(0, \pi)$

Solution:

The finite Fourier sine transform of $f(x)$ in $(0, \pi)$ is

$$F_S[f(x)] = \int_0^{\pi} f(x).\sin nx\,dx$$

Here $f(x) = x$ in $(0, \pi)$

$$\therefore \quad F_s[x] = \int_0^{\pi} x\sin nx\,dx$$

$$= \left[x\left(\frac{-\cos nx}{n}\right) - 1\frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{\pi}{n}\cos n\pi = (-)^{n+1}.\frac{\pi}{n}$$

The finite Fourier cosine transform of $f(x)$ in $(0, \pi)$ is

$$F_C[f(x)] = \int_0^{\pi} f(x).\cos nx\,dx$$

$$\therefore F_C[x] = \int_0^{\pi} (x)\cos nx\,dx$$

$$= \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{n^2}[\cos n\pi - 1]$$

$$= \frac{1}{n^2}[(-1)^n - 1]$$

Example 3: Find the finite Fourier sine and cosine transforms of $f(x) = e^{ax}$ in $(0,l)$

Solution:

We know that $F_S[f(x)] = \int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

Here $f(x) = e^{ax}$

$$\therefore F_s[e^{ax}] = \int_0^l e^{ax} \sin \frac{n\pi x}{l} dx$$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left(a \sin \frac{n\pi x}{l} - \frac{n\pi}{l} . \cos \frac{n\pi x}{l} \right) \right\}_0^1$$

$$= \frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}} \left(\frac{-n\pi}{l} \cos n\pi \right) + \frac{\frac{n\pi}{l}}{a^2 + \frac{n^2\pi^2}{l^2}}$$

$$= \frac{n\pi l}{a^2 l^2 + n^2\pi^2} [(-1)^{n+1} e^{al} + 1]$$

$$F_C\,[e^{ax}] = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \int_0^l e^{ax} \cos \frac{n\pi x}{l} dx$$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left[a \cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l} \right] \right\}_0^l$$

$$= \frac{e^{al} . l^2}{a^2 l^2 + n^2\pi^2} (a \cos n\pi) - \frac{al^2}{a^2 l^2 + n^2\pi^2}$$

$$= \frac{al^2}{al^2 + n^2\pi^2} [e^{al} . (-1)^n - 1]$$

Example 4: Find the finite Fourier cosine transform of $f(x) = \sin ax$ in $(0, \pi)$.

Solution:

$$F_C[\sin ax] = \int_0^\pi \sin ax . \cos nx \, dx$$

$$= \frac{1}{2} \int_0^\pi [\sin(a+n)x + \sin(a-n)x] dx$$

$$= \frac{1}{2}\left[\frac{-\cos(a+n)x}{a+n} - \frac{\cos(a-n)x}{a-n}\right]_0^{\pi}$$

$$= \frac{-1}{2}\left[\frac{\cos(a+n)\pi}{a+n} + \frac{\cos(a-n)\pi}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

$$= \frac{-1}{2}\left[\frac{(-1)^{a+n}}{a+n} + \frac{(-1)^{a-n}}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

if both n and a are even

$$F_c(\sin ax) = \begin{cases} 0, & \textit{if both n and a are even} \\ \frac{1}{2}\left[\frac{2}{a+n} + \frac{2}{a-n}\right], & \textit{if n or a is odd} \end{cases}$$

$$F_c(\sin ax) = \begin{cases} 0, & \textit{if both n and a is odd} \\ \frac{2n}{a^2-n^2}, & \textit{if n or a is odd} \end{cases}$$

Example 5: Find *f(x)* if its finite sine transform is given by $\frac{2\pi(-)^{p-1}}{p^3}$, where p is positive integer 2 <x< π.

Solution:

We know that the inverse Fourier since transform is given by

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{\infty} F_S[f(x)]\sin nx \qquad \dots (1)$$

Here $$F_S[f(x)] = \frac{2\pi(-1)^{p-1}}{p^3} \qquad \dots (2)$$

Substituting (2) in (1) we get

$$f(x) = \frac{2}{\pi}\sum_{p=1}^{\infty} \frac{2\pi(-1)^{p-1}}{p^3}\sin px$$

$$= 4\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^3}\sin px$$

Example 6: If $f(p) = \dfrac{\cos\left(\dfrac{2p\pi}{3}\right)}{(2p+1)^2}$ find F_c^{-1} $[f(p)]$ if $0 < x < 1$.

Solution:

$$F_C^{-1}[f(p)] = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_C[f(x)]\cos\frac{n\pi x}{l}$$

Here $f(p) = \dfrac{\cos\left(\dfrac{2p\pi}{3}\right)}{(2p+1)^2}$

Let $F_C[f(x)] = f(p)$

$$\therefore Fc^{-1}[f(p) = \frac{1}{1} f_c(0)\frac{2}{1}\sum_{n=1}^{\infty} f(p).\cos\frac{n\pi x}{l} \qquad [\because l = 1]$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{\cos\left(\dfrac{2p\pi}{3}\right)}{(2p+1)^2}.\cos n\pi x$$

Exercise

1. Find finite Fourier sine and cosine transform of

1. $f(x) = x in (0, l)$ [**Ans.** $\dfrac{1-\cos n\pi}{n}, 0$]

2. $f(x) = x^3 in\ (0, l)$ [**Ans.** $\dfrac{l^4}{n\pi}(-l)^{n+1} + \dfrac{6l^4}{n^3\pi^3}(-l)^n; \dfrac{3l^4}{n^2\pi^2}(-1)^n - \dfrac{6l^4}{n^4\pi^4}[(-1)^n - 1]$]

3. $f(x) = \begin{cases} 1\ in\ 0 < x < \dfrac{\pi}{2} \\ -1\ in \dfrac{\pi}{2} < x < \pi \end{cases}$ [**Ans.** $\dfrac{1}{n}[\cos n\pi - 2\cos\dfrac{n\pi}{2} + 1]; \dfrac{2}{n}\sin\dfrac{n\pi}{2}$]

4. $f(x) = x^3, 0 < x < 4$ [**Ans.** $\dfrac{-64}{n\pi}\cos n\pi + \dfrac{128}{n^2\pi^2}(\cos n\pi - 1); \dfrac{128}{n^2\pi^2}\cos n\pi$]

2. Find the finite cosine transform of $\left(1 - \dfrac{x}{\pi}\right)^2$ [**Ans.** $\begin{Bmatrix} \dfrac{2}{\pi^{2,}} s > 0 \\ \dfrac{\pi}{3}, s = 0 \end{Bmatrix}$]