UNIT – III

LAPLACE TRANSFORMS

1. Introduction

A transformation is mathematical operations, which transforms a mathematical expressions into another equivalent simple form. For example, the transformation logarithms converts multiplication division, powers into simple addition, subtraction and multiplication respectively.

The Laplace transform is one which enables us to solve differential equation by use of algebraic methods. Laplace transform is a mathematical tool which can be used to solve many problems in Science and Engineering. This transform was first introduced by Laplace, a French mathematician, in the year 1790, in his work on probability theory. This technique became very popular when Heavisidefunctions was applied to the solution of ordinary differential equation in electrical Engineering problems.

Many kinds of transformation exist, but Laplace transform and fourier transform are the most well known. The Laplace transform is related to fourier transform, but whereas the fourier transform expresses a function or signal as a series of mode of vibrations, the Laplace transform resolves a function into its moments.

Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In Physics and Engineering it is used for analysis of linear time invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation form the time domain in which inputs and outputs are functions of time, to the frequency domain, where the same inputs and outputs are functions of complex angular frequency in radius per unit time. Given a simple mathematical or functional discription of an input or output to a system, the Laplace transform provides an alternative functional discription that often simplifies the process of analyzing the behaviour of the system or in synthesizing a new system based on a set of specification. The Laplace transform belongs to the family of integral transforms. The solutions of mechanical or electrical problems involving discontinuous force function are obtained easily by Laplace transforms.

1.1 Definition of Laplace Transforms

Let $f(t)$ be a functions of the variable t which is defined for all positive values of t. Lets be the real constant. If the integral $\int_0^\infty e^{-st} f(t)dt$ exist and is equal to F(s), then F(s) is called the Laplace transform of $f(t)$ and is denoted by the symbol L[f(t)].

i.e. $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F[s]$

The Laplace Transform of $f(t)$ is said to exist if the integral converges for some values of s, otherwise it does not exist.

Here the operator L is called the Laplace transform operator which transforms the functions $f(t)$ into F(s).

Remark: $\underset{S\to\infty}{Lim} F(s) = 0$.

1.2 Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous in any interval $[a,b]$ if it is defined on that interval, and the interval can be divided into a finite number of sub intervals in each of which $f(t)$ is continuous.

In otherwords piecewise continuous means *f(t)* can have only finite number of finite discontinuities.

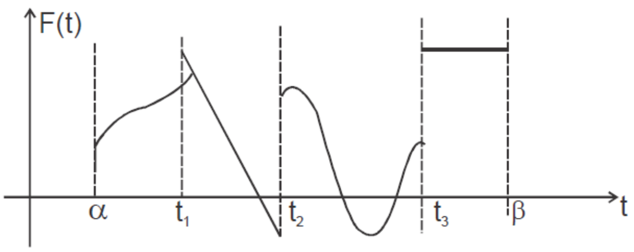


Figure 1.1

An example of a function which is periodically or sectional continuous is shown graphically in Fig 1.1 above. This function has discontinuities at t_1, $t_2$and t_3.

1.3 Definition of Exponential order

A function $f(t)$ is said to be of exponential order if $\underset{t\to\infty}{Lim}\, e^{-st} f(t) = 0$.

1.4 Sufficient conditions for the existence of the Laplace Transforms

Let $f(t)$ be defined and continuous for all positive values of t. The Laplace Transform of $f(t)$ exists if the following conditions are satisfied.

(i) $f(t)$ is piecewise continuous (or) sectionally continuous.

(ii) $f(t)$ should be of exponential order.

1.5 Seven Indeterminates

1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0 \times \infty$
4. $\infty \times \infty$
5. 1^{∞}
6. ∞^{o}.
7. 0^{o}.

Example

Check whether the following functions are exponential or not $(a)\ f(t) = t^2$ $(b)\ f(t) = e^{t^2}$

Solution:

(a) $f(t) = t^2$

By the definition of exponential order

$$\underset{S\to\infty}{Lim}\, e^{-st} f(t) = 0$$

$$\Rightarrow \underset{t\to\infty}{Lim}\, e^{-st}.\, t^2$$

$$\Rightarrow \underset{t\to\infty}{Lim} \frac{t^2}{e^{st}} \Rightarrow \left(\frac{\infty}{\infty}\right) \text{ which is indeterminate form}$$

Apply L – Hospital Rule

$$\underset{t\to\infty}{Lime} \frac{2t}{e^{st} \times S} \Rightarrow \left(\frac{\infty}{\infty}\right) \text{ which is indeterminate form}$$

Again apply L – Hospital Rule.

$$\Rightarrow \underset{t\to\infty}{Lim} \frac{2}{S^2 e^{st}} \Rightarrow \underset{t\to\infty}{Lim} \frac{2}{S^2}.e^{-st} = 0 \text{ (finite)}$$

$$\therefore \underset{t\to\infty}{Lim}\, e^{-st}.\, t^2 = 0 \text{ (finite numbers)}$$

Hence $f(t) = t^2$ is exponential order.

$(b)\ f(t) = e^{t^2}$

Solution:

By the definition of exponential order.

$$\Rightarrow \lim_{t\to\infty} e^{-st} f(t) = 0$$

$$\Rightarrow \lim_{t\to\infty} e^{-st}.e^{t^2} \Rightarrow \lim_{t\to\infty} e^{-st+t^2} = e^{\infty} = \infty$$

$\therefore f(t) = e^{t^2}$ is not of exponential order.

2. Laplace Transform of Standard functions

1. Prove that $L[e^{-at}] = \frac{1}{s+a}$ where $s + a > 0$ or $s > -a$

Proof:

By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[e^{-at}] = \int_0^\infty e^{-st}.e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)}\, dt$$

$$= \left[\frac{-e^{-(s+a)}}{s+a}\right]_0^\infty = \frac{-1}{s+a}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-1}{s+a}$$

Hence $L[e^{-at}] = \frac{1}{s+a}$

2. Prove that $L[e^{at}] = \frac{1}{s-a}$ where $s > a$

Proof:

By the defn of $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[e^{+at}] = \int_0^\infty e^{-st}.e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{s+a}$$

Hence $L[e^{at}] = \dfrac{1}{s-a}$

3. $L(\cos at) = \int_0^\infty e^{-st}\ \cos at\,dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin\ at)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-S)$$

$$= \frac{s}{s^2+a^2}$$

$$\because \int\ e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\int\ e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx - b\sin bx]$$

Hence $L(\cos at) = \dfrac{s}{s^2+a^2}$

4. $L(\sin at) = \int_0^\infty e^{-st}\ \sin at\,dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin\ at + a\cos at)\right]_0^\infty$$

$$= \quad 0 - \frac{1}{s^2 + a^2}(0 - a)$$

L(sin at) $= \quad \dfrac{a}{s^2 + a^2}$

5. *L(cos hat)* $= \quad \dfrac{1}{2} L\left(e^{at} + e^{-at}\right)$

$$= \quad \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) + \frac{1}{2}\left(\frac{s+a+s-a}{(s+a)(s-a)}\right)$$

$$= \quad \frac{s}{s^2 - a^2}$$

L(cos hat) $= \quad \dfrac{s}{s^2 - a^2}$

6. *L(sin hat)* $= \quad \dfrac{1}{2} L\left(e^{at} - e^{-at}\right)$

$$= \quad \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \quad \frac{1}{2}\left(\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right)$$

$$= \quad \frac{a}{s^2 - a^2}$$

L(sin hat) $= \quad \dfrac{a}{s^2 - a^2}$

7. *L(1)* $= \quad \int_0^\infty e^{-st} .1 . dt$

$$= \quad \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \quad \left(0 - \frac{1}{-s}\right) = \frac{1}{s}$$

L(1) $= \quad \dfrac{1}{s}$

8. *L(t^n)* $= \quad \int_0^\infty e^{-st} t^n \; dt$

$$= \left[\left(t^n\right)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty nt^{n-1}\left(\frac{e^{-st}}{-s}\right)dt$$

$$= (0-0) + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

8. $L(t^t) = \int_0^\infty e^{-st}t^n dt$

$$= \left[\left(t^n\right)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty nt^{n-1}\left(\frac{e^{-st}}{-s}\right)dt$$

$$= (0-0) + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}L\left(t^{n-1}\right)$$

$$L(t^n) = \frac{n}{s}L\left(t^{n-1}\right)$$

$$L(t^{n-1}) = \frac{n-1}{s}L\left(t^{n-2}\right)$$

$$L(t^3) = \frac{3}{s}L\left(t^2\right)$$

$$L(t^2) = \frac{2}{s}L(t)$$

$$L(t^n) = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.....\frac{3}{s}.\frac{2}{s}.\frac{1}{s}.L(1).$$

$$= \frac{n!}{s^n}L[1] = \frac{n!}{s^n}.\frac{1}{s}$$

$$L(t^n) = \frac{n!}{s^{n+1}} \; or \; \frac{\sqrt{(n+1)}}{s^{n+1}}$$

In particular $n = 1,2,3….$

we get $L(t) = \frac{1}{s^2}$

$$L(t^2) = \frac{2!}{s^3}$$

$$L(t^3) = \frac{3!}{s^4}$$

2.1 Linear property of Laplace Transform

1. $L(f(t) \pm g(t)) = L(f(t) \pm L(g(t))$

2. $L(Kf(t) \pm g(t)) = KL(f(t)$

Proof (1): By the defn of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L(f(t) \pm g(t)] = \int_0^\infty e^{-st}\left[f(t) \pm g(t)\right]dt$$

$$= \int_0^\infty e^{-st} f(t)dt \pm \int_0^\infty e^{-st} g(t)dt$$

$$= L\left[f(t)\right] \pm L\left[g(t)\right]$$

Hence $L[f(t) \pm g(f)] = L[f(t)] \pm L[g(t)]$

(2) $L[Kf(t)] = KL[f(t)]$

By the defn of L.T

$$L[Kf(t)] = \int_0^\infty e^{-st} Kf(t)dt$$

$$= K\int_0^\infty e^{-st} f(t)dt$$

$$= KL[f(t)]$$

Hence $L[K(t)] = KL[f(t)]$

2.2 Recall

1. 2 sin A cos B= sin$(A+B)$ + sin$(A-B)$

2. 2 cos A sin B= sin$(A+B)$ - sin$(A-B)$

3. $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

4. $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

5. $\sin^2 A = \dfrac{1-\cos 2A}{2}$

6. $\cos^2 A = \dfrac{1+\cos 2A}{2}$

7. $\sin 3A = 3\sin A - 4\sin^3 A$

8. $\cos 3A = 4\cos^3 A - 3\cos A$

9. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

10. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

11. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

12. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

3.1 Problems

1. Find Laplace Transform of $\sin^2 t$

Solution:

$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2} L(1-\cos 2t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

2. Find $L(\cos^3 t)$

Solution:

We know that $\cos^3 A = 4\cos^3 A - 3\cos A$

hence $\cos^2 A = \dfrac{3}{4}\cos A + \dfrac{1}{4}\cos 3A$

L($\cos^2 t$) $= \frac{1}{4} L(3\cos t + \cos 3t)$

$= \frac{1}{4}\left(\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right)$

3. Find $L(\sin 3t \cos t)$

Solution:

We know that $\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

hence $\sin 3t \cos t = \frac{1}{2}(\sin 4t + \sin 2t)$

$L(\sin 3t \cos t) = \frac{1}{2} L(\sin 4t + \sin 2t)$

$= \frac{1}{2}\left(\frac{4}{s^2+16} + \frac{2}{s^2+4}\right)$

$= \frac{2}{s^2+16} + \frac{1}{s^2+4}$

4. Find $L(\sin t \sin 2t \sin 3t)$

Solution:

We know that $\sin t \sin 2t \sin 3t = \sin t \frac{1}{2}(\cos t - \cos 5t)$

$= \frac{1}{2}\sin t \cos t - \frac{1}{2}(\sin t \cos 5t)$

$= \frac{1}{4}\sin 2t \cos 2t - \frac{1}{4}(\sin 6t - \sin 4t)$

$L(\sin t \sin 2t \sin 3t) = \frac{1}{4} L(\sin 2t + \sin 4t - \sin 6t)$

$= \frac{1}{4}\left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2 36}\right]$

5. Find $L(1 + e^{-3t} - 5e^{4t})$

Solution:

$$L[1+e^{-3t} - 5e^{4t}] = L[1]\ L[e^{-3t}] + 5L\ (e^{4t}]$$

$$= \frac{1}{s} + \frac{1}{s+3} - \frac{5}{s-4}$$

6. Find $L(3 + e^{6t} + \sin 2t - 5\cos 3t)$

Solution:

$$L(3 + e^{6t} + \sin 2t - 5\cos 3t) = 3L(1) + L(e^{6t}) + L(\sin 2t) - 5L(\cos 3t)$$

$$= 3.\frac{1}{s} + \frac{1}{s-6} + \frac{2}{s^2+4} - \frac{5s}{s^2+9}$$

7. Find $L(\sin(2t + 3))$

Solution:

$$L(\sin(2t + 3)) = L(\sin 2t \cos 3 + \sin 3 \cos 2t)$$

$$= \cos 3L\ (\sin 2t) + \sin 3L(\cos 2t)$$

$$= \cos 3\frac{2}{s^2+4} + \sin\ 3\frac{s}{s^2+4}$$

8. Find $L(\sin 4t + 3\sin h2t - 4\cos h5t + e^{-5t})$

Solution:

$$L(\sin 4t + 3\sin h2t - 4\cos h5t + e^{-5t})$$

$$= L(\sin 4t) + 3L\ (\sin h2t) - 4L\ (\cos h5t) + \mathrm{L}(e^{-5t})$$

$$= \frac{4}{s^2+16} + 3.\frac{2}{s^2-4} - 4.\frac{s}{s^2-25} + \frac{1}{s+5}$$

$$= \frac{4}{s^2+16} + \frac{6}{s^2-4} - \frac{4s}{s^2-25} + \frac{1}{s+5}$$

9. Find $L((1 + t)^2)$

Solution:

$$L((1 + t)^2) = L(1 + 2t + t^2)$$

$$= \quad L(1) + 2L(t) + L(t^2)$$

$$= \quad \frac{1}{s} + 2.\frac{1}{s^2} + \frac{2!}{s^3}$$

10. Find the Laplace Transform $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$

Solution:

By definition,

$$L(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^\pi e^{-st} f(t) dt + \int_\pi^\infty e^{-st} f(t) dt$$

$$= \int_0^\pi e^{-st} \sin t \ dt + \int_\pi^\infty e^{-st} (0) dt$$

$$= \int_0^\pi e^{-st} \sin t \ dt$$

$$= \left[\frac{e^{-st}}{(-s)^2 + 1^2} (-s \sin - \cos t) \right]_0^\pi \quad \because \int e^{ax} \sin bx \ dx = \frac{e^{ax}}{a^2 + b^2} (\sin bx - b \cos bx)$$

$$= \quad \frac{e^{-s\pi}}{s^2 + 1} (-s \sin \ \pi - \cos \pi) - \frac{e^0}{s^2 + 1} (0 - 1)$$

$$= \quad \frac{e^{-s\pi}}{s^2 + 1} (1) + \frac{1}{s^2 + 1}$$

$$= \quad \frac{1}{s^2 + 1} \left(e^{-s\pi} + 1 \right)$$

11. Find the Laplace Transform $f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution:

By definition, $\quad L(f(t)) = \int_0^\infty e^{-st} f(t) dt$

$$= \int_0^1 e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt$$

$$= \int_0^1 e^{-st} e^t dt + \int_1^\infty e^{-st} 0\ dt$$

$$= \int_0^1 e^{(-st+1)t} dt$$

$$= \left[\frac{e^{(1-s)t}}{1-s}\right]_0^1$$

$$= \frac{1}{1-s}\left(e^{1-s} - 1\right)$$

3.2 Note

1. $\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$ (By definition)

   $\Gamma(n+1) = n!$, $n = 1,2,3,\ldots$

   $\Gamma(n+1) = n\Gamma(n)$, $n > 0$

12. Find $L\left(\frac{1}{\sqrt{t}} + t^{3/2}\right)$

Solution:

$$L\left(\frac{1}{\sqrt{t}} + t^{3/2}\right) = L(t^{-1/2}) + L(t^{3/2})$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} + \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} + \frac{3}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{5/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} + \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}}$$

4. First Shifting Theorem (First translation)

1. If $L(f(t)) = F(s)$. then $L(e^{-at} f(t)) = F(s+a)$

Proof

By definition, $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}.e^{-at} f(t)dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)dt$$

$$= F(s+a)$$

Hence $L[e^{-at} f(t)] = F(s+a)$

4.1 Corollary: $L(e^{at} f(t)) = F(s-a)$

4.2 Note

1. $L(e^{-at}f(t)) = L[f(t)]_{s\to s+a}$

$$= [F(s)]_{s\to s+a}$$

$$= F(s+a)$$

2. $L(e^{at}f(t)) = L[f(t)]_{s\to s-a}$

$$= [F(s)]_{s\to s-a}$$

$$= F(s-a)$$

4.3 Problems

1. Find $L(te^{2t})$

Solution:

$$L(te^{2t}) = [L(t)]_{s\to s-2}$$

$$= \left(\frac{1}{s^2}\right)_{s\to s-2} = \frac{1}{(s-2)^2}$$

2. Find $L(t^5e^{-t})$

Solution:

$$L(t^5e^{-t}) = [L(t^5)]_{s\to s+1}$$

$$= \left(\frac{5!}{s^6}\right)_{s\to s+1}$$

$$= \frac{5!}{(s+1)^6}$$

3. Find $L(e^{-2t}\sin 3t)$

Solution:

$$L(e^{-2t}\sin 3t) = L(\sin 3t)]_{s\to s+2}$$

$$= \left(\frac{3}{s^2+9}\right)_{s\to s+2}$$

$$= \frac{3}{(s+2)^2+9}$$

4. Find $L(e^{-t}\cos h4t)$

Solution:

$$L(e^{-t}\cos h4t) = L(\cos h4t)]_{s\to s+1}$$

$$= \left(\frac{3}{s^2+16}\right)_{s\to s+1}$$

$$= \frac{s+1}{(s+1)^2-16}$$

5. Find $L(e^{3t}\sin^2 4t)$

Solution:

$$L(e^{3t}\sin^2 4t) = L(\sin^2 4t)_{s\to s-3}$$

$$= L\left(\frac{1-\cos 8t}{2}\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(L(1) - L(\cos 8t)\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+64}\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(\frac{1}{s-3} - \frac{s-3}{(s-3)^2+64}\right)$$

6. Find $L(e^{-2t} \sin 4t \cos 6t)$

Solution:

$$L(e^{-2t} \sin 4t \cos 6t) = L(\sin 4t \cos 6t)_{s\to s+2}$$

$$= \frac{1}{2}\left(L(2\sin 4t\cos 6t)\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(L(\sin(4t+6t) + (\sin 4t - 6t)\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(L(\sin 10t - \sin 2t)\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(\frac{10}{s^2+100} - \frac{2}{s^2+4}\right)_{s\to s+2}$$

$$= \frac{1}{2}\left(\frac{10}{(s+2)^2+100} - \frac{2}{(s+2)^2+4}\right)$$

7. Find $L(e^{4t} (\sin^3 3t + \cosh^3 3t)$

Solution:

$$L(e^{4t} (\sin^3 3t + \cosh^3 3t) = L(\sin^3 3t + \cosh^3 3t)_{s\to s-4}$$

$$= L\left(\frac{3\sin 3t - \sin 9t}{4} + \frac{3\cosh 3t + \cosh 9t}{4}\right)_{s\to s-4}$$

$$\because \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}, \cosh^3\theta = \frac{3\cosh\theta + \cosh 3\theta}{4}$$

$$= \left[\frac{3}{4}L(\sin 3t) - \frac{1}{4}L(\sin 9t) + \frac{3}{4}L(\cosh 3t) + \frac{1}{4}(\cosh 9t)\right]_{s\to s-4}$$

$$= \left(\frac{3}{4}.\frac{3}{s^2+9} - \frac{1}{4}\frac{9}{s^2+81} + \frac{3}{4}.\frac{s}{s^2-9} + \frac{1}{4}\frac{s}{s^2-81}\right)_{s \to s-4}$$

$$= \frac{3}{4}.\frac{3}{(s-4)^2+9} - \frac{1}{4}\frac{9}{(s-4)^2+81} + \frac{3}{4}.\frac{s-4}{(s-4)^2-9} + \frac{1}{4}\frac{s-4}{(s-4)^2-81}$$

8. Fine $L(\cos ht \cos 2t)$

Solution:

$$L(\cos ht \cos 2t) = \left(\left(\frac{e^t + e^{-t}}{2}\right)\cos 2t\right)$$

$$= \frac{1}{2}L(e^t \cos 2t + e^{-t}\cos 2t)$$

$$= \frac{1}{2}L(\cos 2t)_{s \to s-1} + L(\cos 2t)_{s \to s+1}$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+4}\right)_{s \to s-1} + \left(\frac{s}{s^2+4}\right)_{s \to s+1}\right]$$

$$= \frac{1}{2}\left(\frac{s-1}{(s-1)^2+4} + \frac{s+1}{(s+1)^2+4}\right)$$

5. Theorem

If *L(f(t)) = F(s)*, then *L(tf(t))* = $\frac{-d}{ds}(F(s))$

Proof:

Given *F(s) = L(f)(t))*

differentiate both sides, w.r. to 's'

$$\frac{d}{ds}(F(s)) = \frac{d}{ds}(L(f)(t)))$$

$$= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)dt\right)$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)dt$$

$$= -\int_0^\infty tf(t)^{-st} dt$$

$$\frac{d}{ds}(F(s)) = -L(tf(t))$$

$$\therefore L(tf(t)) = \frac{-d}{ds} F(s)$$

or $L(tf(t)) = F'(s)$ where $F(s) = L(f(t))$

similarly we can show that,

$$L\left(t^2 f(t)\right) = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$L\left(t3 f(t)\right) = (-1)^3 \frac{d^3}{ds^3} F(s)$$

In general, $L\left(t^n f(t)\right) = (-1)^n \dfrac{d^n}{ds^n} F(s)$

5.1 Problems

1. Fine $L\left(te^{3t}\right)$

Solution:

We know that $L\left(tff(t)\right) = \dfrac{-d}{ds} L(f(t))$

Here $f(t) = e^{3t}$

$$\text{L}\left(te^{3t}\right) = \frac{-d}{ds} L\left(e^{3t}\right)$$

$$= \frac{-d}{ds}\left(\frac{1}{s-3}\right)$$

$$= \left(-\frac{(s-3)(0)-(1)}{(s-3)^1}\right)$$

$$= \frac{1}{(s-3)^2}$$

2. Find $L(t\sin 3t)$

Solution:

$$L(tf(t)) = \frac{-d}{ds}L(f(t))$$

$$L(tf(t)) = \frac{-d}{ds}L(\sin 3t)$$

$$= \frac{-d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \left(\frac{-(s^2+9)(0)+3(2s)}{(s^2+9)^2}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

3. Find $L(t\cos^2 3t)$

Solution:

$$L(t\cos^2 3t) = \frac{-d}{ds}L(\cos^2 3t)$$

$$= \frac{-d}{ds}L\left(\frac{1+\cos 6t}{2}\right)$$

$$= \frac{-1}{2}\frac{d}{ds}(L(1)+L(\cos 6t))$$

$$= \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+16}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{(s^2+16).1-s(2s)}{(s^2+16)^2}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{16-s^2}{(s^2+16)^2}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right)$$

4. Find $L\left(te^{-2t}\sin 3t\right)$

Solution:

$$L\left(e^{-2t}(t\sin 3t\right) = L(t\sin 3t)_{s\to s+2}$$

$$= \left\{\frac{-d}{ds}L(\sin 3t)\right\}_{s\to s+2}$$

$$= \left\{\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right\}_{s\to s+2}$$

$$= \left\{\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right\}_{s\to s+2}$$

$$= \frac{6(s+2)}{\left((s+2)^2+9\right)^2}$$

5. Find $L\left(te^{-2t}\sin 2t\sin 3t\right)$

Solution:

$$L\left(te^{-2t}\sin 2t\sin 3t\right) = L\left(t\sin 2t\sin 3t\right)_{s\to s+2}$$

$$= \left[\frac{1}{2}\times L(t.2\sin 2t\sin 3t)\right]_{s\to s+2}$$

$$= \left[\frac{1}{2}\times L(t(\cos(2t-3t)-\cos(2t+3t)))\right]_{s\to s+2}$$

$$= \frac{1}{2}\times L(t\cos t - t\cos 5t)_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L(\cos t)+\frac{d}{ds}L(\cos 5t)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+1}\right)+\frac{d}{ds}\left(\frac{s}{s^2+25}\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[-\left(\frac{\left(s^2+1\right).1-s(2s)}{\left(s^2+1\right)^2}\right)+\frac{d}{ds}\left(\frac{\left(s^2+25\right).1-s(2s)}{\left(s^2+25\right)^2}\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[-\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)+\left(\frac{25-s^2}{\left(s^2+25\right)^2}\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{s^2-1}{\left(s^2+1\right)^2}+\frac{25-s^2}{\left(s^2+25\right)^2}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{(s+2)^2-1}{\left((s+2)^2+1\right)^2}+\frac{25-(s+2)^2}{\left(\left(s+2^2\right)+25\right)^2}\right]$$

6. Find $L\left(t^2e^{-t}\cos h2t\right)$

Solution:

$$L\left(e^{-t}\left(t^2\cos h2t\right)\right) = L\left(t^2\cos h2t\right)_{s\to s+1}$$

$$= \left((-1)^2\frac{d^2}{ds^2}L(\cos h2t)\right)_{s\to s+1}$$

$$= \left(\frac{d^2}{ds^2}\left(\frac{s}{s^2-4}\right)\right)_{s\to s+1}$$

$$= \left(\frac{d}{ds}\left(\frac{\left(s^2-4\right)\,.\,1-s(2s)}{\left(s^2-4\right)^2}\right)\right)_{s\to s+1}$$

$$= \frac{d}{ds}\left(\frac{-4-s^2}{\left(s^2-4\right)^2}\right)_{s\to s+1}$$

$$= \frac{-d}{ds}\left(\frac{4+s^2}{\left(s^2-4\right)^2}\right)_{s\to s+1}$$

$$= \left(\frac{\left(s^2-4\right)^2(2s)-\left(4+s^2\right)2\left(s^2-4\right).(2s)}{\left(s^2-4\right)^2}\right)_{s\to s+1}$$

$$= \left(-2s\left(s^2-4\right)\left(\frac{s^2-4-2\left(4+s^2\right)}{\left(s^2-4\right)^4}\right)\right)_{s\to s+1}$$

$$= \left(\frac{-2s\left(s^2 - 4 - 8 - 2s^2\right)}{\left(s^2 - 4\right)^3} \right)_{s \to s+1}$$

$$= \left(\frac{2s\left(s^2 + 12\right)}{\left(s^2 - 4\right)^3} \right)_{s \to s+1}$$

$$= \frac{2(s+1)\left((s+1)^2 + 12\right)}{\left((s+1)^2 - 4\right)^3}$$

6. Theorem

If $L(f(t)) = F(s)$ and if $\underset{t \to 0}{Lt} \frac{f(t)}{t}$ exist then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty e^{-st} f(t) ds$

Proof:

By definition, $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt$

Integrate both sides w.r.t. 'S' from S → ∞

$$\int_s^\infty F(s) dt = \int_s^\infty \int_0^\infty e^{-st} f(t) dt ds$$

$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t) ds \right] dt$ (Changing the order of integration since 's' and 't' are independent variable)

$$= \int_0^\infty f(t) \left(\int_s^\infty e^{-st} ds \right) dt$$

$$= \int_0^\infty f(t) dt \left(\frac{e^{-st}}{-t} \right)_s^\infty$$

$$= \int_0^\infty f(t) dt \left(\frac{-1}{t} \left(0 - e^{-st}\right) \right)$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left(\frac{f(t)}{t} \right)$$

$$\therefore L\left(\frac{f(t)}{t} \right) = \int_s^\infty L(f(t)) ds$$

Similarly we can provide that $L\left(\frac{f(t)}{t^2}\right) = \int_s^\infty \int_s^\infty L(f(t))ds\ ds$

In general $L\left(\frac{f(t)}{t_n}\right) = \underbrace{\int_s^\infty \int_s^\infty \cdots\cdots \int_s^\infty}_{n \quad times} L(f(t))\ \underbrace{ds\ ds \ldots ds}_{n\, times}$

Recall

1. $\log(AB) = \log A + \log B$

2. $\log\left(A/_B\right) = \log A - \log B$

3. $\log A^B = B \log A$

4. $\log 1 = 0$

5. $\log 0 = -\infty$

6. $\log \infty = \infty$

7. $\int \frac{1}{x} dx = \log x$

8. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

9. $\tan^{-1}(\infty) = \frac{\pi}{2}$

10. $\cot^{-1}\left(s/_a\right) = \frac{\pi}{2} - \tan^{-1}\left(s/_a\right)$

Problems

1. Find $L\left(\frac{1-e^{2t}}{t}\right)$

Solutions:

$$\underset{t\to 0}{Lim} \frac{1-e^{2t}}{t} = \frac{0}{0} \text{ (Interminate form)}$$

Apply L – Hospital Rule

$$\lim_{t\to 0}\frac{-2e^{2t}}{1} = -2$$

$\therefore$ the given function exists in the limit t $\to$ 0

$$L\left(\frac{1-e^{2t}}{t}\right) = \int_s^\infty L\left(1-e^{2t}\right)ds$$

$$= \int_s^\infty \left(L(1) - L\left(e^{2t}\right)\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-2}\right)ds$$

$$= \left(\log s - \log(s-2)\right)_s^\infty$$

$$= \left[\log\left(\frac{s}{s-2}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s\left(1-\cancel{2}/\cancel{s}\right)}\right)\right]_s^\infty = \log\left(\frac{1}{1-\cancel{2}/\cancel{s}}\right)_s^\infty$$

$$= 0 - \log\frac{s}{s-2}$$

$$= \log\left(\frac{s}{s-2}\right)^{-1}$$

$$= \log\left(\frac{s-2}{s}\right)$$

2. Find $L\left(\frac{1-\cos at}{t}\right)$

Solution:

$$\lim_{t\to 0}\frac{1-\cos at}{t} = \frac{0}{0} \text{ (Indeterminte form)}$$

Apply L – Hospital Rule.

$$\underset{t\to 0}{Lim}\frac{a\ \ \sin at}{1} = 0 \ \ \text{(finite)}$$

$\therefore$ the given function exists in the limit t $\to$ 0

$$L\left(\frac{1-\cos at}{t}\right) = \int_s^\infty L(1-\cos at)ds$$

$$= \int_s^\infty \left(L(1) - L(\cos at)\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds$$

$$= \left(\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right)_s^\infty$$

$$= \left(\log s - \log\left(s^2+a^2\right)^{1/2}\right)_s^\infty$$

$$= \left(\log\frac{s}{\sqrt{s^2+a^2}}\right)_s^\infty$$

$$= \left(\log\frac{s}{s\sqrt{1+a^2/s^2}}\right)_s^\infty$$

$$= \left(\log\frac{1}{\sqrt{1+a^2/s^2}}\right)_s^\infty$$

$$= \left(\log 1 - \log\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{s}{\sqrt{s^2+a^2}}\right) = \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)^{-1} = \log\left(\frac{\sqrt{a^2+s^2}}{s}\right)$$

3. Find $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)$

Solution:

$$\underset{t\to 0}{Lim}\frac{e^{-at}-e^{-bt}}{t}=\frac{0}{0} \text{ (Indeterminte form)}$$

Apply L – Hospital Rule.

$$\underset{t\to 0}{Lim}\frac{-ae^{-at}+be^{-bt}}{1}=b-a$$

$\therefore$ the given function exists in the limit t → 0

$$L\left(\frac{e^{-at}-e^{-bt}}{t}\right) = \int_s^\infty L\left(e^{-at}-e^{-bt}\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$= \left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{(s+a)}{(s+b)}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1+a/s)}{1+b/s}\right)\right]_s^\infty$$

$$= \log 1-\log\left(\frac{1+a/s}{1+b/s}\right)$$

$$= \log 1-\log\left(\frac{s+a}{s+b}\right)$$

$$= -\log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+a}{s+b}\right)^{-1}$$

$$= \log\left(\frac{s+a}{s+b}\right)$$

4. Find $L\left(\frac{\cos at - \cos bt}{t}\right)$

Solution:

$$\underset{t\to 0}{Lim}\frac{\cos at - \cos bt}{t} = \frac{0}{0} \text{ (Interminate form)}$$

Apply L – Hospital Rule

$$\underset{t\to 0}{Lim}\frac{-a\sin at + b\sin bt}{1} = 0 \text{ (finite)}$$

$\therefore$ the given function exists in the limit $t \to 0$

$$L\left(\frac{\cos at - \cos bt}{1}\right) = \int_s^\infty L(\cos at - \cos bt)ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log\left(s^2+a^2\right) - \frac{1}{2}\log\left(s^2+b^2\right)\right]_s^\infty$$

$$= \left[\frac{1}{2}\log\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2\left(1+a^2/s^2\right)}{s^2\left(1+b^2/s^2\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{\left(1+a^2/s^2\right)}{\left(1+b^2/s^2\right)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{\left(s^2+a^2\right)}{\left(s^2+b^2\right)}\right)$$

5. Find $L\left(\frac{e^{at}-\cos bt}{t}\right)$

Solution:

Since $\lim_{t\to 0}\frac{e^{at}-\cos bt}{t}$ exists

$$L\left(\frac{e^{at}-\cos bt}{t}\right) = \int_s^\infty L\left(e^{at}-\cos bt\right)ds$$

$$= \int_s^\infty \left(\frac{1}{s-a}-\frac{1}{s^2+b^2}\right)ds$$

$$= \left[\log(s-a)-\frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \left[\log(s-a)-\log\sqrt{s^2+b^2}\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s\left(1-a/s\right)}{s\sqrt{1+b^2/s^2}}\right)\right]_s^\infty$$

$$= \log 1-\log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= -\log\frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right)$$

6. Find $L\left(\frac{\sin^2 t}{t}\right)$

Solution:

Since $\lim_{t\to 0}\frac{\sin^2 t}{t}$ exists

$$L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty L(\sin^2 t)\,ds$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(L(1) - L(\cos 2t)\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{s\sqrt{1+4/s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{s\sqrt{1+4/s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{s}{\sqrt{1+4/s^2}}\right]$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

7. Find $L\left(\frac{\sin 3t\cos 2t}{t}\right)$

Solution:

$$\lim_{t\to 0}\left(\frac{\sin 3t\cos 2t}{1}\right) \text{ exists}$$

$$L\left(\frac{\sin 3t\cos 2t}{t}\right) = \int_s^\infty L(\sin 3t\cos 2t)ds$$

$$= \frac{1}{2}\int_s^\infty L(2\sin 3t\cos 2t)ds$$

$$= \frac{1}{2}\int_s^\infty L(\sin 5t+\sin t)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)ds$$

$$= \frac{1}{2}\left[5.\frac{1}{5}\tan^{-1}{s/5}+\tan^{-1}{s/1}\right]_s^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}{s/5}+\tan^{-1}{s/1}\right]_s^\infty$$

$$= \frac{1}{2}\left(\tan^{-1}(\infty)+\tan^{-1}(\infty)-\tan^{-1}\left(s/5\right)-\tan^{-1}\left(s/1\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\tan^{-1}\left(s/5\right)-\tan^{-1}\left(s/1\right)\right)$$

$$= \frac{1}{2}\left(\pi\tan^{-1}\left(s/5\right)-\tan^{-1}s\right)$$

8. Find $L\left(\frac{\sin at}{t}\right)$. Hence find the value of $\int_0^\infty \frac{\sin t}{t}dt$

Solution:

Since $\lim_{t\to 0}\frac{\sin at}{t}$ exists

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty L(\sin at)\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \left(a.\frac{1}{a}\tan^{-1}\frac{s}{a}\right)_s^\infty$$

$$= \left(\tan^{-1}\frac{s}{a}\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= L\left(\frac{\sin at}{t}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Deduction:

By definition

$$\int_0^\infty e^{-st}\frac{\sin at}{t} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Put s = 0, a =1

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2} - \tan^{-1}(0)$$

$$= \frac{\pi}{2}$$

9. Find $L\left(\frac{\cos at}{t}\right)$

Solution:

$$\underset{t\to 0}{Lt}\frac{\cos at}{t} = \frac{1}{0} = \infty$$

$\therefore \underset{t\to 0}{Lt}\frac{\cos at}{t}$ does not exist.

Hence $L\left(\frac{\cos at}{t}\right)$ does not exist.

10. Find $L\left(\frac{e^{at}}{t}\right)$

Solution:

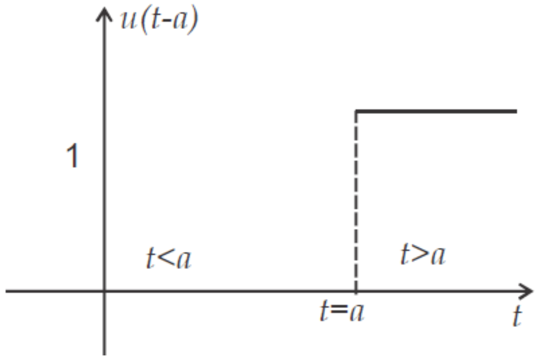
$$\underset{t\to 0}{Lt}\frac{e^{at}}{t}=\frac{1}{0}=\infty$$

$\therefore L\left(\frac{e^{at}}{t}\right)$ does not exist.

7. Unit Step function (or) heavisides unit step function

The unit step function about the point $t = a$ is defined as $U(t\text{-}a) = \begin{cases} 0 & for\ t<a \\ 1 & for\ t>1 \end{cases}$

It can also be denoted by *H(t-a)*



7.1 Find the Laplace transform of unit step function.

Solution:

The Laplace transform of unit step function is

$$L(U(t-a)) = \int_0^\infty e^{-st}U(t-a)dt$$

$$= \int_0^a e^{-st}0.dt + \int_a^\infty e^{-st}(1)dt$$

$$= \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^{\infty}$$

$$= \frac{-1}{s}\left(e^{-\infty} - e^{-as}\right)$$

$$L(U(t-a)) = \frac{-1}{s}\left(0 - e^{-at}\right) = \frac{e^{-as}}{s}$$

$$\therefore L(U(t-a)) = \frac{e^{-as}}{s}$$

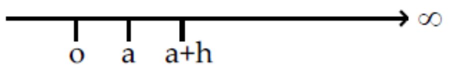
8. Dirac delta function (or) Unit Impulse function

8.1 Dirac delta function or unit impulse function about the point t = a is defined as

$$\delta(t-a) = \begin{cases} \underset{h \to 0}{Lt} \dfrac{1}{h} & a < t < a+h \\ 0 & otherwise \end{cases}$$

Find the Laplace transform of Dirac delta function.

Solution:



$$L[\delta(t-a)] = \int_0^{\infty} e^{-st}\delta(t-a)dt$$

$$= \int_0^{\infty} e^{-st}0dt + \underset{h \to 0}{Lt}\frac{1}{h}\int_a^{a+h} e^{-st}dt + \int_{a+h}^{\infty} e^{-st}0dt$$

$$= \underset{h \to 0}{Lt}\frac{1}{h}\int_a^{a+h} e^{-st}dt$$

$$= \underset{h \to 0}{Lt}\frac{1}{h}\left[\frac{-1}{s}\left(e^{-(a+h)s} - e^{-as}\right)\right]$$

$$= \underset{h \to 0}{Lt}\frac{1}{h}\left[\frac{e^{-as}}{s} \frac{e^{-(a+h)s}}{s}\right]$$

$$= \underset{h \to 0}{Lt}\frac{1}{h}\frac{e^{-as}\left(1 - e^{-hr}\right)}{sh} = \frac{0}{0} \text{ (Indeterminate form)}$$

Applying L′ Hospital Rule.

$$= \quad \underset{h\to 0}{Lt}\, \frac{e^{-as}\left(e^{-hs}s\right)}{s} = e^{-as}$$

$$L(\delta(t-a)) \quad = \quad e^{-as} \text{ when } a = 0, L(\delta(t)) = 1$$

8.2 Note

The dirac delta function is the derivative of unit step function.

9. Second shifting Theorem (Second Translation)

If $L(f(t)) = F(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0 & t > a \end{cases},$

Then $L(G)(t)) = e^{-as}F(s)$

Proof:

$$L(G)(t)) \quad = \quad \int_0^\infty e^{-st}G(t)dt$$

$$= \quad \int_0^a e^{-st}G(t)dt + \int_a^\infty e^{-st}G(t)dt$$

$$= \quad \int_0^a e^{-st}0\,.\,dt + \int_a^\infty e^{-st}f(t-a)dt$$

$$= \quad \int_0^\infty e^{-st}f(t-a)dt$$

Put $t - a = u$ When $t = a, \quad u = 0$

$dt = du$ $\quad t = \infty, \quad u = \infty$

$$\therefore L(G(t)) \quad = \quad \int_0^\infty e^{-st(u+a)}f(u)du$$

$$= \quad e^{-sa}\int_0^\infty e^{-su}f(u)du$$

In $\int_0^a e^{-su}f(u)du, u$ is a dummy variable. Hence we can replace it by the variable t.

$$\therefore L(G)(t)) \quad = \quad e^{-sa}\int_0^\infty e^{-st}f(t)dt$$

$$= \quad e^{-sa}L(f(t))$$

$$= \quad e^{-as}F(S)$$

Another form of second shifting theorem

If *L(f)(t)) = F(s)* and *a*> 0 then

L(f(t-a)U(t-a) = e^{-as}*F(s)* where *U(t-a)* is the unit step function.

Proof:

We know that by the definition of unit step function.

$$U(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

$$\therefore\ f(t-a)U(t-a) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases} \qquad \ldots(1)$$

Let *f(t-a)U(t-a) = G(t)*

$$\therefore (1) \text{ becomes, } G(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

which is precisely the same as the first form of second shifting theorem, as discussed above

$\therefore$ *L(G)(t))* = e^{-as}*F(s)*

9.1 Problems

1. Find the Laplace transform of $G(t)$, where

$$G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & if \quad t > \frac{2\pi}{3} \\ 0 & if \quad t < \frac{2\pi}{3} \end{cases}$$

Solution:

We know that by second shifting if *L(f(t)) = F(s)* and *G(t)* $\begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

then *L(G)(t))* = e^{-as-} *F(s)* …(1)

Here *f(t-a) = cos* $\left(t = \frac{2\pi}{3}\right)$

(ie) $f(t) = \cos t \;\&\; a\dfrac{2\pi}{3}$ … (2)

$\therefore L(f(t)) = L(\cos t) = \dfrac{s}{s^2+1}$ … (3)

Submitting (2) & (3) in (1), we get

$\therefore L(G(t)) = e^{\frac{2\pi}{3}s} \cdot \dfrac{s}{s^2+1}$

2. Find the Laplace transform using second shifting theorem for

$$G(t) = \begin{cases} (t-a)^3; & t > a \\ 0 & t < a \end{cases}$$

Solution:

Here $a = 2$, $f(t-a) = t-2)^3$

$f(t) = t^3$

$L(f(t)) = L(t^3) =) \dfrac{3!}{s^4} = F(s)$

$\therefore L(G)(t)) = e^{-as} F(s)$

$= e^{-as} \dfrac{3!}{s^4}$

3. Using second shifting theorem, find the Laplace transform of

$$G(t) = \begin{cases} \sin t - \pi/3; & t > \pi/3 \\ 0 & t < \pi/3 \end{cases}$$

Solution:

Here $a = \dfrac{\pi}{3}$, $f(t-a) = \sin\left(t - \pi/3\right)$

$\therefore f(t) = \sin t$

$\therefore L(f(t)) = L(\sin t)$

$= \dfrac{1}{s^2+1} = F(s)$

$$\therefore L(G(t)) = e^{-as}F(s)$$

$$= e^{-\pi/3 s} \cdot \frac{1}{s^2+1}$$

$$= e^{-\pi/3 s} \frac{1}{s^2+1}$$

10. Change of Scale Property

If $L(f(t)) = F(s)$, Then $L(f)at)) = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof:

By definition, $L(f(t) = \int_0^\infty e^{-st} f(t)dt$

$$\therefore L(f)(at) = \int_0^\infty e^{-st} f(a\,t)dt$$

Put $at = y$ when $t = 0, \quad y = 0$

$adt = dy$ $t = \infty, \quad y = \infty$

$$L(f(at)) = \int_0^\infty e^{-s(y/a)} f(y)\frac{dy}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)y} f(y)dy$$

$= \frac{1}{a}\int_0^\infty e^{-(s/a)y} f(t)dt$ (Replacing the dummy variable y by t)

$$L(f(at)) = \frac{1}{a}F\left(s/a\right)$$

10.1 Corollary

$$L\left[f\left(t/a\right)\right] = aF(as)$$

10.2 Problems

1. Assuming $L(\sin t)$. Find $L(\sin 2t)$ and $L\left(\sin t/2\right)$

Solution:

We know that $L(\sin t) = \dfrac{1}{s^2+1}$ … (1)

$\therefore L(\sin 2t) = \dfrac{1}{2}.\dfrac{1}{\left(s/2\right)^2+1}$ Using (1) (Replace S by s/2)

$L(\sin 2t) = \dfrac{1}{2}\left(\dfrac{4}{s^2+4}\right)$

$= \dfrac{2}{s^2+4}$ … (2)

$\therefore L\left(\sin t/2\right) = 2\dfrac{1}{(2s)^2+1} = \dfrac{2}{4s^2+1}$ Using (2) (Replace s by 2s)

2. Give that $L(t\cos t) = \dfrac{s^2-1}{\left(s^2+1\right)^2}$

Find (i) $L(t\cos at)$ and (ii) $L\left(t\cos\dfrac{t}{a}\right)$

Solution:

(i) Given $L(t\cos t) = \dfrac{s^2-1}{\left(s^2+1\right)^2}$

Replacing t by at

$\therefore L(at\cos at) = \dfrac{1}{a}\dfrac{\left(s/a\right)^2-1}{\left(\left(s/a\right)^2+1\right)}$ ($\because$ Replacing s by s/a)

$L(at\cos at) = \dfrac{a^4(s^2-a^2)}{a^3(s^2+a^2)^2}$

$\therefore L(t\cos at) = \dfrac{a^4(s^2-a^2)}{a^4(s^2+a^2)^2} = \dfrac{s^2-a^2}{(s^2+a^2)^2}$

(ii) Given $L\left(t\cos\dfrac{t}{a}\right) = \dfrac{s^2-1}{\left(s^2+1\right)^2}$

Replacing by t/a, $L\left(t/a \cos t/a\right) = a\left(\frac{(as)^2-1}{\left((as)^2+1\right)^2}\right)$

$$L\left(t\cos\frac{t}{a}\right) = a^2\left(\frac{a^2s^2-1}{\left(a^2s^2+1\right)^2}\right)$$ Replace s by as.

11. Laplace Transform of Derivations

Here, we explore how the Laplace transform interacts with the basic operators of calculus differentation and integration. The greatest interest will be in the first identity that we will derive. This relates the transform of a derivative of a function to the transform of the original function, and will allow to convert many initial - value problems to easily solved algebraic Equations. But there are useful relations involving the Laplace transform and either differentiation (or) integration. So we’ll look at them too.

11.1. Theorem

If $L(f(t)) = F(s)$ Then

(i) $L(f'(t)) = sL(f(t)) - f$

(ii) $L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0)$

and in genereal

$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots\ldots f^{n-1}(0)$

Proof:

(i) By definition,

$$L(f'(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty e^{-st}d(f(t)$$

$$= \left(e^{-st} f(t)\right)_0^\infty - \int_0^\infty f(t)d\left(e^{-st}\right)$$

$$= \left(0 - f(0)\right) - \int_0^\infty f(t)e^{-st}(-s)dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + sL\left(f(t)\right)$$

$\therefore L(f'(t)) = SL(f(t)) - f(0)$ …. (1) which proves (i)

(ii) Again by definition,

$$L(f''(t)) = \int_0^\infty e^{-st} f''(t)dt$$

$$= \int_0^\infty e^{-st} d(f'(t))$$

$$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty f'(t)e^{-st}(-s)dt$$

$$= \left[0 - f'(t)\right] + S\int_0^\infty e^{-st} f'(t)dt$$

$$= -f'(0) + sL(f'(t))$$

$$= sL(f'(t)) - f'(0)$$

$$= s(sL(f(t)) - f(0)) - f'(0) \qquad \text{Using (1)}$$

$$L(f''(t)) = s^2Lf(t) - sf(0) - f'(0) \qquad \ldots(2)$$

Similarly proceeding like this, we can show that

$$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots f^{n-1}(0) \qquad \ldots(3)$$

The above results (1), (2) and (3) are very useful in solving linear differential equations with constant coefficients.

11.2 Note

We have, $L(f'(t)) = sL(f(t)) - f(0)$ …. (1)

and

$$L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0) \qquad \ldots(2)$$

When $f(0) = 0$ and $f'(0) = 0$

(1) & (2) becomes

$Lf'(t) = sLf(t)$ and $Lf''(t) = s^2Lf(t)$

This shows that under certain conditions, the process of Laplace transform replaces differentiation bymultiplication by the factor s and *s* 2 respectively.

12. Laplace Transform of integrals

Analogous to the differentiation identities $L[f'(t)] = sF(s) - f(0)$ and $L[tf(t) = -F'(s)$are a pair of identities concerning transforms of integrals and integrals of transforms. These identities will not be nearly as important to us as the differentiation identities, but they do have their uses and are considered to be part of the standard set of identities for the Laplace Transform.

Before we start, however, take another look at the above differentiation identities. They show that, under the Laplace transform, the differentiation of one of the functions, f(t) or F(S) corresponds to the multiplication of the other by the approprate variable.

This may lead to suspect that the analogous integrations identities. They show that, under Laplace transform integration of one of the functions f(t) or F(S), corresponds to the division of the other by the appropriate variables.

12.1 Theorem: If $L[f(t)] = F(s)$ *then* $L\left[\int_0^r f''(t)dt\right] = \frac{1}{s}L[f(t)]$

Proof:

Let $\int_0^t f(t)dt = \phi(t)$ …(1)

Differentiate both sides with respect to '*t*'

$\therefore f(t) = \phi(t)$ …(2)

and $\phi(0) = \int_0^t f(t)dt = 0$

We know that $L[\phi(t)] = sL[\phi(t)] - \phi(0)$

$L[\phi(t)] = sL[\phi(t)]$ $\therefore \phi(0) = 0$

$$\therefore L[f(t)] = sL\left[\int_0^t f(t)dt\right] \qquad \text{by (1) \& (2)}$$

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

Similarly we can prove that

$$L\left[\int^{t}\int^{t} f(t)dt\right]=\frac{1}{s^2}L[f(t)]$$

$$\therefore \text{ In general } \quad L\left[\underbrace{\int^{t}\int^{t}\cdots\cdots\int^{t} f(t)dt}_{n\,items}\right]=\frac{1}{sn}L[f(t)]$$

12.2 Note

The above result expresses that the integral between the limits from '0' to '*t*' is transformed into simple division by the factor '*S*' using Laplace transform.

12.3 Problems

1. Find $L\left(e^{-t}\int_{0}^{t} t\cos t dt\right)$

Solution:

$$L\left(e^{-t}\int_{0}^{t} t\cos dt\right)=\left[L\left(\int_{0}^{t} t\cos t dt\right)\right]_{s\to s+1}$$

$$=\left(\frac{1}{s}L(t\cos t)\right)_{s\to s+1}$$

$$=\left(\frac{1}{s}\left(\frac{-d}{ds}(L(\cos t))\right)\right)_{s\to s+1}$$

$$=\left[\frac{-1}{s}\left(\frac{s^2+1-s(2s)}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$=\left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$=\left(\frac{s^2-1}{s(s^2+1)^2}\right)_{s\to s+1}$$

$$= \left(\frac{(s+1)^2 - 1}{(s+1)((s+1)^2+1)^2} \right)_{s \to s+1}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

2. Find $L\left(e^{-t} \int_0^t \frac{\sin t}{t} dt \right)$

Solution:

$$L\left(e^{-t} \int_0^t \frac{\sin t}{t} dt \right) = \left[L\left(\int_0^t \frac{\sin t}{t} dt \right) \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} L\left(\frac{\sin t}{t} \right) \right]_{s \to s+1}$$

Since $\underset{t \to 0}{Lim} \frac{\sin t}{t}$ exist

$$= \left[\frac{1}{s} \int_s^\infty L(\sin t) ds \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \int_s^\infty \frac{1}{s^2+1} ds \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \left(\tan^{-1} s \right)_s^\infty \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \left(\tan^{-1} \infty - \tan^{-2}(s) \right) \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \left(\frac{\pi}{2} - \tan^{-1}(s) \right) \right]_{s \to s+1}$$

$$= \left[\frac{1}{s} \cot^{-1} s \right]_{s \to s+1} = \frac{\cot^{-1}(s+1)}{s+1}$$

3. Find the Laplace Transform of $\int\limits_0^t te^{-t}\sin t dt$

Solution:

$L(te^{-t} \sin tdt) = (L(t \sin t))_{s\to s+1}$

$$= \left(\frac{-d}{ds} L(\sin t)\right)_{s\to s+1}$$

$$= \left(\frac{-d}{ds}\left(\frac{1}{s^2+1}\right)\right)_{s\to s+1}$$

$$= \left(\frac{(s^2+1)0-2S}{(s^2+1)^2}\right)_{s\to s+1}$$

$$= \left(\frac{2s}{(s^2+1)^2}\right)_{s\to s+1}$$

$$= \frac{2(s+1)}{((s+1)^2+1)^2}$$

$$= \frac{2(s+1)}{s^2+2\,s+2}$$

4. Find $L\left(\int\limits_0^t \frac{e^{-t}\sin t}{t} dt\right)$

Solution:

$$L\left(\int\limits_0^t \frac{e^{-t}\sin t}{t} dt\right) = \frac{1}{s} L\left(\frac{e^{-t}\sin t}{t}\right)$$

Since $\lim\limits_{t\to 0} \frac{e^{-t}\sin t}{t}$ exist.

$$= \frac{1}{s}\left[\int\limits_s^\infty L(e^{-t}\sin t)\right] ds$$

$$= \frac{1}{s}\left[\int\limits_s^\infty L(\sin t)\right]_{s\to s+1} ds$$

$$= \frac{1}{s}\left[\int_s^{\infty}\left(\frac{1}{s^2+1}\right)\right]_{s\to s+1} ds$$

$$= \frac{1}{s}\left[\int_s^{\infty}\left(\frac{1}{(s+1)^2+1}\right)\right] ds$$

$$= \frac{1}{s}\left[\int_s^{\infty}\left(\frac{ds}{(s+1)^2+1}\right)\right]$$

$$= \frac{1}{s}\left(\tan^{-1}(s+1)\right)_s^{\infty}$$

$$= \frac{\cot^{-1}(s+1)}{s}$$

Problems

1. Find $L\left(\int_0^t e^{2t}dt\right)$

Solution:

$$L\left(\int_0^t e^{2t}dt\right) = \frac{1}{s}L\left(e^{2t}\right)$$

$$= \frac{1}{s}\cdot\frac{1}{s-2}$$

$$= \frac{1}{s(s-2)}$$

2. Find $L\left(\int_0^t \sin 3t dt\right)$

Solution:

$$L\left(\int_0^t \sin 3t dt\right) = \frac{1}{s}L(\sin 3t)$$

$$= \frac{1}{s}\cdot\frac{3}{s^2+9}$$

$$= \frac{3}{s(s^2+9)}$$

3. Find $L\left(\int_0^t e^{-2t}\cos 3t\,dt\right)$

Solution:

$$L\left(\int_0^t e^{-2t}\cos 3t\,dt\right) = \frac{1}{s}L(e^{-2t}\cos 3t)$$

$$= \frac{1}{s}L(\cos 3t)_{s\to s+2} \quad \text{(Using first shifting theorem)}$$

$$= \frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s\to s+2}$$

$$= \frac{1}{s}\left(\frac{s+2}{(s+2)^2+9}\right)$$

4. Find $L\left(\int_0^t e^{-t}\sin h2t\,dt\right)$

Solution:

$$L\left(\int_0^t e^{-t}\sin h2t\,dt\right) = \frac{1}{s}L\left(e^{-t}\sin h2t\right)$$

$$= \frac{1}{s}L\left(\sin h2t\right)_{s\to s+1}$$

$$= \frac{1}{s}\left(\frac{2}{s^2-4}\right)_{s\to s+1}$$

$$= \frac{1}{s}\left(\frac{2}{(s+1)^2-4}\right)$$

5. Find $L\left(\int_0^t \sin 3t\cos 2t\,dt\right)$

Solution:

$$L\left(\int_0^t \sin 3t\cos 2t\,dt\right) = \frac{1}{s}L\left(\sin 3t\cos 2t\right)$$

$$= \frac{1}{2s}L\left(2\sin 3t\cos 2t\right)$$

$$= \frac{1}{2s} L(\sin 5t + \sin t)$$

$$= \frac{1}{2s}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)$$

6. Find $L\left(e^{-3t}\int_0^t t\sin^2 t\right)dt$

Solution:

$$L\left(e^{-3t}\int_0^t t\sin^2 t\right)dt = L\left(\int_0^t t\sin^2 t dt\right)_{s\to s+3}$$

$$= \left[\frac{1}{s} L(t\sin^2 t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds} L(\sin^2 t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds} L\left(\frac{1-\cos 2t}{2}\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\frac{d}{ds} L(1-\cos 2t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\left(\frac{-1}{s^2} - \frac{(s^2+4).1 - s(2s)}{(s^2+4)^2}\right)\right]_{s\to s+3}$$

$$= \left[\frac{+1}{2s}\left(\frac{+1}{s^2} - \frac{4-s^2}{(s^2+4)^2}\right)\right]_{s\to s+3}$$

$$= \frac{1}{2(s+3)}\left(\frac{+1}{(s+3)^2} + \frac{4-(s+3)^2}{((s+3)^2+4)^2}\right)$$

$$= \frac{1}{2(s+3)^3}\left(\frac{4-(s+3)^2}{2(s+3)(s^2+6s+13)^2}\right)$$

7. Find $L\left(e^{4t}\left(\int_0^t \frac{\sin 3t\cos 2t}{t}dt\right)\right)$

Solution:

$$L\left(e^{4t}\left(\int_0^t \frac{\sin 3t\cos 2t}{t}dt\right)\right) = \frac{1}{s}L\left(\frac{\sin 3t\cos 2t}{t}\right)_{s\to s-4}$$

$$= \left[\frac{1}{s}L\left(\frac{\sin 3t\cos 2t}{t}\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{s}\int_s^\infty L(\sin 3t\cos 2t)dt\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty L(\sin 3t\cos 2t)ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty L(\sin 5t+\sin t)ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty \left(\frac{5}{s^2+25}+\frac{1}{s^2+1}\right)ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(5.\frac{1}{5}\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)_s^\infty\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left((\tan^{-1}\infty+\tan^{-1}\infty)-\left(\tan^{-1}\frac{s}{5}+\tan^{-1}s\right)\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-\tan^{-1}\frac{s}{5}-\tan^{-1}s\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\pi-\tan^{-1}\frac{s}{5}-\tan^{-1}s\right)\right]_{s\to s-4}$$

$$= \frac{1}{2(s-4)}\left(\pi-\tan^{-1}\frac{s-4}{5}-\tan^{-1}(s-4)\right)$$

13. Periodic Functions

Laplace transform of periodic functions have a particular structure. In many applications the nonhomogeneous term in a linear differential equation is a periodic function. In this section, we desire a formulafor the Laplace transform of such periodic functions.

13.1 Definition of Periodic functions

A function f(t) is said to have a period T or to be periodic with period T if for all t, f(t+T)=f(t) where T is a positive constant. The least value of T>0 is called the period of f(t).

Example 1

Consider $f(t) = \sin t$

$$f(t + 2\pi) = \sin(t + 2\pi)$$

$$= \sin t$$

(ie) $$f(t) = f(t + 2\pi)$$

$$= \sin t$$

sin *t* t is a periodic function with period 2π.

Example 2

tan *t* is a periodic function with period π.

13.2 Laplace Transform of Periodic functions

Let $f(t)$ be a periodic function with period a

$f(t) = f(t + a) = f(t + 2a) = f(t + 3a) \ldots\ldots$

Now $$L(f(t)) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty e^{-st} f(t)dt + \int_a^{2a} e^{-st}dt + \int_{2a}^{3a} e^{-st} f(t)dt$$

$$+ \int_{3a}^{4a} e^{-st} f(t)dt + \ldots\ldots..$$

Put in the second integral $t = T + a;\ dt = dT$

in the Third integral $t = T + 2a$; $dt = dT$

in the Fourth integral $t = T + 3a$; $dt = dT$

When $t = a$, $T = 0$

$t = 2a$, $T = a$

When $t = 2a$, $T = 0$

$t = 3a$, $T = a$

When $t = 3a$, $T = 0$

$t = 4a$, $T = a$

$$\therefore L(f(t)) = \int_0^a e^{-st} f(t)dt + e^{-as}\int_0^a e^{-sT} f(T+a)dT$$

$$+ e^{-2as}\int_0^a e^{-st} f(T+2a)dt + ...$$

$$= \int_0^a e^{-st} f(t)dt + e^{-sa}\int_0^a e^{-st} f(t+a)dt + e^{-2as}\int_0^a e^{-st} f(t+2a)dt$$

$$= (1 + e^{-as} + (e^{-as}) + ...)\int_0^a e^{-st} f(t)dt$$

$$= (1 - e^{-as})^{-1} + \int_0^a e^{-st} f(t)dt \qquad \left(\because (1-x)^{-1} = 1 + x + x^2 + ...\right)$$

$$L(f(t)) = \frac{1}{1-e^{-as}} + \int_0^a e^{-st} f(t)dt$$

13.3 Problems

1. Find the Laplace Transform of the square wave given by

$$f(t) = \begin{matrix} E & for & 0 < t < a/2 \\ -E & for & a/2 < t < a \end{matrix}$$

and $f(t+a) = f(t)$

Solution:

Given that $f(t+a) = f(t)$

Hence $f(t)$ is a periodic function with period $p = a$

$$L(f(t)) = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} E dt + \int_{a/2}^{a} e^{-st}(-E)dt \right]$$

$$= \frac{1}{1-e^{-as}} \left[E \int_0^{a/2} e^{-st} dt - E \int_{a/2}^{a} e^{-st} dt \right]$$

$$= \frac{E}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{a/2} - \left(\frac{e^{-st}}{-s} \right)_{a/2}^{a} \right]$$

$$= \frac{E}{s(1-e^{-as})} \left[\left(-e^{-sa/2} + 1 \right) + \left(e^{-sa} - e^{-sa/2} \right) \right]$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - e^{-sa/2} - e^{sa/2} + e^{-sa} \right)$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - e^{-2sa/2} - e^{sa} \right)$$

$$= \frac{E}{s(1-e^{-\frac{as}{2}})(1+e^{-sa/})} \left(1 - e^{\frac{-sa}{2}} \right)^2$$

$$= \frac{E\left(1 - e^{\frac{-sa}{2}}\right)}{s(1-e^{-sa/2})}$$

$$= \frac{E}{s} \tan h\left(\frac{sa}{4} \right)$$

2. Find the Laplace transform of the function $f(t) = \begin{matrix} t & 0 < t < b \\ 2b - t & b < t < 2b \end{matrix}$

Solution:

The given function is a periodic function with period 2b

$$\therefore L(f(t)) = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\int_0^{b} e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} t dt + \int_b^{2b} e^{-st}(2b-t)dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left\{\begin{array}{c}\left[t\left(\frac{e^{-st}}{-s}\right)-1\left(\frac{e^{-st}}{s^2}\right)\right]_0^b + \\ \left[(2b-1)\left(\frac{e^{-st}}{-s}\right)-(-1)\frac{e^{-st}}{s^2}\right]_b^{2b}\end{array}\right\}$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{-be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^{-2bs}}{s^2} + \frac{b}{s}e^{-bs}\,\frac{-e^{-bs}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left(\frac{1-2e^{bs}+e^{-2bs}}{s^2}\right)$$

$$= \frac{(1-e^{-bs})^2}{s^2(1+e^{-bs})(1-e^{-bs})}$$

$$= \frac{1-e^{-bs}}{s^2(1+e^{-bs})}$$

$$= \frac{1}{s^2}\cdot\left(\frac{\left(1-e\frac{-bs}{2}\right).e^{\frac{-bs}{2}}}{\left(1+e^{\frac{-bs}{2}}\right).e^{\frac{-bs}{2}}}\right)$$

$$= \frac{1}{s^2}\cdot\frac{e^{\frac{bs}{2}}-e^{\frac{-bs}{2}}}{e^{\frac{bs}{2}}+e^{\frac{-bs}{2}}}$$

$$= \frac{1}{s^2}\tan h\left(\frac{bs}{2}\right)$$

3. Find the Laplace transform of $f(s)\ \begin{matrix} \sin t & in & 0 < t < \pi \\ 0 & in & \pi < t < 2\pi \end{matrix}$ and $f(t+2\pi) = f(t)$.

Solution:

Given that $f(t + 2\pi) = f(t)$

Hence $f(t)$ is a periodic function with period $P = 2\pi$.

$$L(f(t)) = \frac{1}{1-e^{-sP}} \int_0^P e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\int_0^{\pi} e^{-st} \sin t dt + \int_{\pi}^{2\pi} e^{-st}.0dt\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\frac{1}{s^2+1}\left(e^{-st}(s\sin t - 1.\cos t)\right)_0^{\pi}\right]$$

$$= \frac{1}{s^2+1}.\frac{1}{1-e^{-2s\pi}}\left(e^{-s\pi}(0+1) - 1(0-1)\right)$$

$$= \frac{1}{s^2+1}\frac{1}{(1-e^{-2s\pi})}(e^{-s\pi}+1)$$

$$= \frac{1}{s^2+1}.\frac{1}{(1-e^{-s\pi)}}\frac{(1+s^{-s\pi)}}{(1+e^{-s\pi})}$$

$$= \frac{1}{s^2+1}.\frac{1}{1-e^{-s\pi}}$$

4. Find the Laplace transform of the Half-wave rectifier function

$$f(t) = \begin{cases} \sin wt, & 0 < t < \dfrac{\pi}{w} \\ 0, & \dfrac{\pi}{w} < t < \dfrac{2\pi}{w} \end{cases}$$

Solution:

Given $f(t) = \begin{cases} \sin wt, & 0 < t < \dfrac{\pi}{w} \\ 0, & \dfrac{\pi}{w} < t < \dfrac{2\pi}{w} \end{cases}$

This ia a periodic function with period $\frac{2\pi}{w}$ in the interval $\left(0, \frac{2\pi}{w}\right)$.

$$\therefore L(f(t)) = \frac{1}{1-e^{\frac{-2\pi s}{w}}} \int_0^{\frac{2\pi}{w}} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}} \left[\int_0^{\frac{\pi}{w}} e^{-st} f(t) dt + \int_{\frac{\pi}{w}}^{\frac{2\pi}{w}} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}} \left[\int_0^{\frac{\pi}{w}} e^{-st} \sin wt dt + \int_{\frac{\pi}{w}}^{\frac{2\pi}{w}} e^{-st} .0 dt \right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}} \left[\frac{e^{-st}(-s \sin wt - w \cos wt}{s^2 + w^2} \right]_0^{\frac{\pi}{w}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}} \left[\frac{e^{\frac{-s\pi}{w}}(w) + w}{s^2 + w^2} \right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{w}}} \frac{w\left(1+e^{-s\pi/w}\right)}{s^2 + w^2}$$

$$= \frac{w}{(1+e^{-s\pi/2})(1-e^{-s\pi/w})} \cdot \frac{(1+e^{-s\pi/w})}{s^2 + w^2}$$

$$= \frac{w}{\left(1-e^{-s\pi/w}\right)s^2 + w^2}$$

5. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t, & for \quad 0 < t < 1 \\ 2-t & for \quad 1 < t < 2 \end{cases} \text{ and } f(t+2) = f(t)$$

Solution:

The given function is a periodic function with period 2.

$$\therefore L(f(t)) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}tdt + \int_1^2 (2-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left(\frac{te^{-st}}{-s} - 1.\frac{e^{-st}}{s^2}\right)_0^1 + \left(\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right)_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2s}}\left(\frac{1-2e^{-s}+e^{-2s}}{s^2}\right)$$

$$= \frac{(1-e^{-s})^2}{(1-e^{-s})(1+e^{-s})s^2} = \frac{1}{s^2}\left(\frac{(1-e^{-s})}{(1+e^{-s})}\right)$$

$$= \frac{1}{s^2}\frac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/}} = \frac{1}{s^2}\tan h\left(s/2\right)$$

6. Find the Laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < \frac{\pi}{2} \\ \pi - t, & \frac{\pi}{2} < t < \pi \end{cases} \quad f(\pi + r) = f(t)$$

Solution:

$$\therefore L(f(t)) = \frac{1}{1-e^{-s\pi}}\left[\int_0^{\pi/2} te^{-st}tdt + \int_{\pi/2}^{\pi} (\pi-t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-s\pi}}\left[\left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_0^{\pi/2} + \left((\pi-1)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{1-e^{-s\pi}}\left[\frac{\pi/2\, e^{-s\pi/2}}{-s} - \frac{e^{-s\pi/s}}{s^2} + \frac{1}{s^2} + \frac{e^{-s\pi}}{s^2} - \frac{\pi/2\, e^{-s\pi/2}}{-s} + \frac{s^{-s\pi/s}}{s^2}\right]$$

$$= \frac{1}{1-e^{-s\pi}}\left[\frac{1-2e^{-s\pi/s}+e^{-s\pi}}{s^2}\right]$$

$$= \frac{(1-e^{-s\pi/2})^2}{s^2(1-e^{-s\pi/2})(1+e^{-s\pi/2})}$$

$$= \frac{1-e^{-s\pi/2}}{s^2(1+e^{-s\pi/2})}$$

7. Find the Laplace transform of the rectangular wave given by

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$$

Solution:

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic the interval (0, 2b) with period 2b.

$$\therefore L(f(t)) = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} (1)dt + \int_b^{2b} e^{-st} (-1)dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} dt + \int_b^{2b} e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\left(\frac{e^{-st}}{-S} \right)_0^b - \left(\frac{e^{-st}}{-S} \right)_b^{2b} \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\frac{e^{-sb}}{-S} + \frac{1}{S} + \frac{e^{-2sb}}{S} - \frac{e^{-sb}}{S} \right]$$

$$= \frac{1}{S} \left[\frac{1-2e^{-sb} + e^{-2sb}}{1-e^{-2bs}} \right]$$

$$= \frac{1}{S} \left[\frac{(1-e^{-sb})^2}{(1+e^{-sb})(1-e^{-sb})} \right]$$

$$= \quad \frac{1}{S}\frac{1-e^{-sb}}{1+e^{-sb}}$$

$$= \quad \frac{1}{S}\frac{(1-e^{-sb})e^{-sb/2}}{(1+e^{-sb})(e^{-sb2})}$$

$$= \quad \frac{1}{S}\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}$$

$$= \quad \frac{1}{S}\tan h\left(\frac{sb}{2}\right)$$

14. Initial value theorem

If $L(f(t)) \quad = \quad F(s)$, then $\underset{t\to 0}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ sF(s)$

Proof:

We know that $L[f(t)] = sL[f(t)] - f(0))$

Take the limit as S → ∞ on both sides, we have

$$\underset{s\to\infty}{Lt}\ L(f'(t)) = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$

$$\underset{s\to\infty}{Lt}\int_0^\infty e^{-st}dt = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$ ($\because$ By definition of Laplace Transform)

$$\int_0^\infty \underset{s\to\infty}{Lt}\ e^{-st}f'(t)dt = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$ ($\because$ s is independent of t, we can take the limit in the L.H.S. before integration)

$$0 = \underset{s\to\infty}{Lt}(sF(s) - f(0))$$

$$\therefore \underset{s\to\infty}{Lt}\ sF(s) = f(0)$$

$$= \underset{t\to 0}{Lt}\ f(t)$$

$$\therefore \underset{s\to\infty}{Lt}\ sF(s) = \underset{t\to 0}{Lt}\ f(t)$$

15. Final value Theorem

If $L(f(t)) = F(s)$, then $\underset{s\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

Proof:

We know that $L(f'(t)) = sL[f(t)] - f(0)$

$L(f'(t)) = sF(s) - f(0)$

$$\int_0^\infty e^{-st} f' dt = sF(s) - f(0)$$

Take the limit as $s \to 0$ on both sides,

$$\underset{s\to 0}{Lt} \int_0^\infty e^{-st} f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$\int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$ ($\because$ s is independent of t, we can take the limit in the L.H.S. before integration)

$$\int_0^\infty f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$(f(t))_0^\infty = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$\underset{t\to\infty}{Lt}\, f(t) - f(0) = \underset{s\to 0}{Lt}\, sF(s) - f(0)$$

Since $f(0)$ is n ot a function of 's' (or) 't' it can be cancelled both sides,

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

15.1 Problems

1. If $L(f)(t)) = \dfrac{1}{s(s+a)}$ find $\lim\limits_{t\to\infty} f(t)$ and $\lim\limits_{t\to 0} f(t)$

Solution:

$$\begin{aligned}
\lim_{t\to 0} f(t) &= \lim_{s\to\infty} sF(s) \\
&= \lim_{s\to\infty} s \times \frac{1}{s(s+a)} \\
&= \lim_{s\to\infty} \frac{1}{(s+a)} \\
&= 0
\end{aligned}$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s\times\frac{1}{s(s+a)}$$

$$= \lim_{s\to 0} \frac{1}{(s+a)}$$

$$= \frac{1}{a}$$

2. If $L(e^{-t}\cos^2 t) = F(s)$. Find $\lim_{s\to 0}(sF(s))$ and $\lim_{s\to\infty}(sF(s))$

Solution:

$L(e^{-t}\cos^2 t) = F(s)$

(ie)., $f(t) = e^{-t}\cos^2 t$

By final value theorem,

$$\lim_{s\to 0}(sF(s)) = \lim_{t\to\infty}(e^{-t}\cos^2 t) = 0$$

By initial value theorem,

$$s\lim_{s\to 0} sF(s) = \lim_{t\to 0}(e^{-t}\cos^2 t) = 1$$

3. Verify the initial and final value theorem for the function $f(t) = 1 - e^{-at}$

Solution:

Given that $f(t) = 1 - e^{-at}$ …(1)

$L(f(t)) = L(1-e^{-at})$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$SF(s) = s\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

$$= 1 - \frac{1}{s+a} \qquad \text{… (2)}$$

From (1), $\underset{t\to 0}{Lt}\ f(t) \quad = \quad \underset{t\to 0}{Lt}\, 1-e^{-at}$

$$= \quad 1 - 1$$

$$= \quad 0 \qquad \ldots(3)$$

$$\underset{t\to\infty}{Lt}\ f(t) \quad = \quad \underset{t\to\infty}{Lt}\, 1-e^{-at}$$

$$= \quad 1 - 0$$

$$= \quad 1 \qquad \ldots(4)$$

From (2), $\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\, 1-\dfrac{s}{s+a} = 1 \qquad \ldots (5)$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ 1-\frac{s}{s+a}$$

$$= \underset{s\to\infty}{Lt}\ 1-\frac{s}{s\left(1+{a}/{s}\right)} = 0 \qquad \ldots (6)$$

From (3) & (6), we have

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$

and from (4) & (5)

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$$

4. Verify initial and final value theorem for the function $f(t) = e^{-2t}\cos 3t$

Solution:

Given $f(t) = e^{-2t}\cos 3t$

$$L(f(t)) = L(e^{-2t}\cos 3t)$$

$$= L(\cos 3t)_{s\to s+2}$$

$$F(s) = \left(\frac{s}{s^2+9}\right)_{s\to s+2} = \frac{s+2}{(s+2)^2+9}$$

$$SF(s) = \frac{s(s+2)}{s^2+4s+13} = \frac{s^2+2s}{s^2+4s+13}$$

$$\underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\ e^{-2t}\cos 3t = 1 \qquad \ldots(1)$$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ e^{-2t}\cos 3t = 0 \qquad \ldots(2)$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ \frac{s^2+2s}{s^2+4s+13} = 0 \qquad \ldots(3)$$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ \frac{s^2(1+2/s)}{s^2(1+4/s+13/s^2)} = 1 \qquad \ldots(4)$$

From (1) and (4), $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$

From (2) and (3), $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

5. Verify initial and final value theorem for $f(t) = t^2e^{-3t}$

Solution:

$f(t) = t^2e^{-3t}$

$L(f(t)) = [L(t^2)]_{s\to s+3}$

$$= \left(\frac{2!}{s^3}\right)_{s\to s+3} = \frac{2}{(s+3)^3}$$

$$sF(s) = \frac{2}{(s+3)^3}$$

$$\underset{s\to 0}{Lt}\ f(s) = \underset{t\to 0}{Lt}\ t^2e^{-3t} = 0 \qquad \ldots(1)$$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ t^2e^{-3t} = 0 \qquad \ldots(2)$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ \frac{2s}{(s+3)^3} = 0 \qquad \ldots(3)$$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ \frac{2s}{(s+3)^3} = \underset{s\to 0}{Lt}\ \frac{2s}{\left(1+\frac{3}{s}\right)^3}$$

$$= \underset{s\to\infty}{Lt}\ \frac{2}{s^2\left(1+\frac{3}{s}\right)^3} = 0 \qquad \ldots(4)$$

From (1) & (4)

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$$

From (2) & (3)

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s).$$

Exercise - 1 (a)

Find the Laplace transform of the following

1. $5 - 3t - 2\, e^{-t}$ Ans: $\dfrac{3s^2+2s-3}{s^2(s+1)}$

2. $6 \sin 2t - 5 \cos 2t$ Ans: $\dfrac{12-5s}{s^2+4}$

3. e^{3t-5} Ans: $\dfrac{e^5}{s-3}$

4 $\cos(wt+\infty)$ Ans: $\dfrac{s\cos\infty - w\sin\infty}{s^2+w^2}$

5. $7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2$

Ans: $\dfrac{7}{s-2}+\dfrac{9}{s+2}+\dfrac{5s}{s^2+1}+\dfrac{42}{s^4}+\dfrac{15}{s^2+9}\,\dfrac{2}{s}$

6. $\sin 2t \cos 3t$ Ans: $\dfrac{2(s^2-5)}{(s^2+1)(s^2+25)}$

7. $\cos h2t - \cos h3t$ Ans: $\dfrac{-5s}{(s-4)(s-9)}$

8. $\sin^2 at$ Ans: $\dfrac{2a^2}{s(s^2+4a^2)}$

9. $(t^2+1)^2$ Ans: $\dfrac{24}{s^5}+\dfrac{4}{s^3}+\dfrac{1}{s}$

10. $a + bt + \dfrac{c}{vt}$ Ans: $\dfrac{a}{s}+\dfrac{b}{s^2}+c\sqrt{\pi/s}$

11. $\sin^3 2t$ Ans: $\dfrac{48}{(s^2+4)(s^2+36)}$

12. $(\sin t - \cos t)^2$ Ans: $\dfrac{s^2-2s+4}{s(s^2+4)}$

13. $\cos \pi t + 4e^{2t/3}$ Ans: $\dfrac{s}{s^2+\pi^2}+\dfrac{12}{3s^2}$

Exercise - 1 (b)

Find the Laplace transform of the following functions.

1. t^3e^{-3t} Ans: $\dfrac{6}{(s+3)^4}$

2. $e^{-2t}(\cos 4t + 3\sin 4t)$ Ans: $\dfrac{s+10}{s^2-4s+20}$

3. $e^t(t+2)$ Ans: $\dfrac{2}{(s-1)^3}+\dfrac{4}{(s-1)^2}+\dfrac{4}{s-1}$

4. $e^{-at}t^2$ Ans: $\dfrac{2}{(s+a)^3}$

5. $e^{-t}\cos^2 t$ Ans: $\dfrac{1}{2s+2}+\dfrac{s+1}{2s^2+4s+10}$

6. $e^{-2t}(1-2t)$ Ans: $\dfrac{6}{(s+2)^2}$

7. $e^{-2t}\cos t$ Ans: $\dfrac{s+2}{s^2+4s+5}$

8. $e^t \sin t \cos t$ Ans: $\dfrac{1}{(s-1)^2+4}$

9. $e^{-t}\cos ht$ Ans: $\dfrac{s+1}{s^2+2s}$

10. $e^{at}t^n$ Ans: $\dfrac{n!}{(s-a)^{n+1}}$

11. $t^2 \sin h2t$ Ans: $\frac{1}{(s-2)^3}+\frac{1}{(s+2)^3}$

12. $\sin h2t \sin 3t$ Ans: $\frac{1}{2}\left[\frac{3}{s^2-4s+13}-\frac{3}{s^2+4s+13}\right]$

13. $\cosh t \cos 3t \cos 4t$

Ans: $\frac{1}{4}\left[\frac{s-2}{s^2-4s+53}-\frac{s+2}{s^2+4s+53}+\frac{s-2}{s^2-4s+5}-\frac{s+2}{s^2+4s+53}\right]$

14. $\sin h2t \sin^2 t$ Ans: $\frac{1}{4}\left[\frac{1}{s-2}-\frac{1}{s+2}-\frac{S-2}{s^2-4s+8}+\frac{S+2}{s^2+4s+18}\right]$

15. $\sin h3t \sin 3t \sin 4t$

Ans: $\frac{1}{4}\left[\frac{s+3}{s^2+6s+10}-\frac{s+3}{s^2+6s+58}+\frac{s-3}{s^2-6s+58}-\frac{s-3}{s^2-6s+10}\right]$

Exercise - 1 (c)

Find the Laplace transform of the following functions.

1. $t \cos at$ Ans: $\frac{s^2-a^2}{(s^2+a^2)^2}$

2. $t^3 \sin at$ Ans: $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

3. $t^3 \sin t$ Ans: $\frac{24s(1-s)^2}{(1-s^2)^4}$

4. $t^3 e^{-3t}$ Ans: $\frac{3!}{(s+3)^4}$

5. $t^3 \cos hat$ Ans: $\frac{2s(s^2+3a^2)}{(s^2-a^2)^3}$

6. $(1+te^{-t})^3$ Ans: $\frac{1}{s}+\frac{3}{(s+1)^3}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$

7. $te^{at} \sin at$ Ans: $\frac{2a(s-a)}{(s^2-2as+2a^2)^2}$

8. $te^{-t}\sin^2 t$ Ans: $\frac{1}{2}\frac{1}{(s+2)^2}+\frac{(s+1)^2+4+-2(s+1)^2}{((s+1)^2+4)^2}$

9. $t\cos t\cos 2t$ Ans: $\frac{1}{2}\left[\frac{s^2-9}{(s^2+9)^2}+\frac{s^2-1}{(s^2+1)^2}\right]$

10. $t\cos^2 2t$ Ans: $\frac{1}{2}\left[\frac{1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right]$

11. $r\cos h2t\sin 2t$ Ans: $\frac{1}{2}\left[\frac{4(s-2)}{(s^2-4s+8)^2}+\frac{4(s+2)}{(s^2+4s+8)^2}\right]$

12. $r\cos ht\sin 3t$ Ans: $\frac{1}{2}\left[\frac{s^2-2s-8}{(s^2-2s+10)^2}+\frac{s^2+2s-8}{(s^2+2s+10)^2}\right]$

13. $r^2e^{-t}\cos t$ Ans: $\frac{2(s+1)(s^2+2s-2)}{(s^2+2s+2)^3}$

14. $te^{-t}\cos ht$ Ans: $\frac{s^2+2s+2}{(s^2+2s)^3}$

15. $\frac{t\sin 2t}{e^{-2t}}$ Ans: $\frac{4s-8}{(s^2-4s-8)^2}$

Exercise 1 - (d)

Find the Laplace transform of the following functions

1. $\frac{\sin t}{t}$ Ans: $\cos^{-1}s$

2. $\frac{e^{2t}-e^{bt}}{t}$ Ans: $\log\frac{s-b}{s-a}$

3. $\frac{e^{2t}-e^{-3t}}{t}$ Ans: $\log\frac{s+3}{s-2}$

4. $\frac{1-\cos at}{t}$ Ans: $\frac{1}{2}\log\left(\frac{s^2+a^a}{s^2}\right)$

5. $\frac{\sin^2 t}{t}$ Ans: $\frac{1}{4}\log\frac{s^2+4}{s^2}$

6. $\dfrac{\sin t \sin 2t}{t}$ Ans: $\dfrac{1}{4}\log\dfrac{s^2+9}{s^2+1}$

7. $\dfrac{e^t - \cos 2t}{t}$ Ans: $\log\dfrac{\sqrt{s^2+4}}{s-1}$

8. $\dfrac{\sin 3t \cos t}{t}$ Ans: $\dfrac{1}{2}\left[\pi - \tan^{-1}\left(\dfrac{s}{4}\right) - \tan^{-1}\left(\dfrac{s}{2}\right)\right]$

9. $\dfrac{e^{-t} - e^{-2t}}{t}$ Ans: $\log\dfrac{s+2}{s+1}$

10. $\dfrac{e^{-at} - e^{-bt}}{t}$ Ans: $\log\dfrac{s+b}{s+a}$

11. $\dfrac{\cos 4t \sin 2t}{t}$ Ans: $\dfrac{1}{2}\left[\tan^{-1}\left(\dfrac{s}{2}\right) - \tan^{-1}\left(\dfrac{s}{6}\right)\right]$

12. $\dfrac{\cos 2t - \cos 3t}{t}$ Ans: $\log\dfrac{\sqrt{s^2+9}}{\sqrt{s^2+4}}$

13. $\dfrac{\sin\ ht}{t}$ Ans: $\dfrac{\log\sqrt{s+1}}{\log\sqrt{s-1}}$

14. $\dfrac{1-e^{-2t}}{t}$ Ans: $\log\dfrac{s+2}{s}$

15. $\dfrac{e^{at} - \cos bt}{t}$ Ans: $\dfrac{1}{2}\log\left(\dfrac{s^2+s^2}{(s-a)^2}\right)$

Exercise 1 (e)

Find the Laplace transform of the following functions.

1. $\int_0^t e^t \cos^2 t dt$ Ans: $\dfrac{s^2-2s+3}{s(s-1)(s^2-2s+5)}$

2. $\int_0^t t \sin t \sin 2t dt$ Ans: $\dfrac{1}{2s}\left[\dfrac{s^2-1}{(s^2+1)^2} + \dfrac{s^2-9}{(s^2+9)^2}\right]$

3. $\int_0^t \frac{\sin ht}{t} dt$ Ans: $\frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$

4. $\int_0^t e^{2t}\sin 3t dt$ Ans: $\frac{1}{s}\left(\frac{3}{s^2-4s+13}\right)$

5. $\int_0^t e^{-2t}\sin^3 t dt$ Ans: $\frac{3}{2s}\left[\frac{s+2}{(s^2+4s+5)^2}+\frac{3(s+2)}{(s^2+4s+13)^2}\right]$

6. $\int_0^t \frac{\sin^2 t}{t} dt$ Ans: $\frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$

7. $\int_0^t \frac{e^{-t}\sin t dt}{t}$ Ans: $\cot^{-1}(s+1)$

8. $e^t\int_0^t \frac{\sin t}{t} dt$ Ans: $\frac{\cot^{-1}(s-1)}{s-1}$

9. $\int_0^t te^t \sin t dt$ Ans: $\frac{1}{s}.\frac{2(s+1)}{s^2+2s+2}$

10. $e^{-t}\int_0^t t\cos t dt$ Ans: $\frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$

Exercise - 1 (f)

Find the Laplace transform of the following

1. $f(t) = t$ for $0 < t < 4$, $f(t+4) = f(t)$ Ans: $\frac{1-4Se^{-4s}-e^{-4s}}{(1-e^{-4s})s^2}$

2. $f(t)=\begin{cases} t & 0<t<1 \\ 2-t & 1<1<2 \end{cases}$ *and* $f(t+2)=f(t)$ Ans: $\frac{1}{s^2}\tan h\left(s/2\right)$

3. $f(t)=\begin{cases} 1 & 0<t<a/2 \\ -t & a/2<t<a \end{cases}$ *and* $f(a+t)=f(a)$ Ans: $\frac{1}{1-e^{-as}}\left(\frac{1+e^{-as}-2e^{-as/2}}{s}\right)$

4. $f(t)=\begin{cases} \sin t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$ *and* $f(t+2\pi)=f(t)$ Ans: $\frac{1}{1-e^{-\pi/s}}\frac{1}{s^2+1}$

5. $f(t)=\begin{cases} t & 0<t<1 \\ 0 & 1<1<2 \end{cases} and\ f(t+2)=f(t)$ Ans: $\dfrac{1-e^{-s}(s+1)}{s^2(1-e^{-2s})}$

6. $f(t)=\begin{cases} 0 & 0<t<{}^{w}\!/_{2} \\ -\sin wt & \dfrac{\pi}{w}<t<\dfrac{2\pi}{w}, \end{cases} f\left(t+\dfrac{2\pi}{w}\right)=f(t)$ Ans: $\dfrac{w}{(w^2+S^2)(e^{\pi S/w}-1)}$

7. $f(t)=e^{-t}, 0\le t<2, f(t+2)=f(t)$ Ans: $\dfrac{1-e(s+1)}{(s+1)(1-e^{-2s})}$

8. $f(t)=\begin{cases} 1 & 0<t<1 \\ -1 & a<t<2a \end{cases} given\ f(t+2a)=f(t)$

9. $f(t)=\begin{cases} \sin wt & 0<t<{}^{\pi}\!/_{w} \\ 0 & {}^{\pi}\!/_{w}<t<{}^{2\pi}\!/_{w} \end{cases}$ given that $f\left(t+{}^{2\pi}\!/_{w}\right)=f(t)$

10. $f(t)=\sin wt 0<1<{}^{\pi}\!/_{w}\ f\left(t+{}^{\pi}\!/_{w}\right)=f(t)$

16.1. Definition

If the Laplace transform of a function $f(t)$ is $F(S)$ (ie) $L(f(t)) = F(S)$ then $f(t)$ is called an inverse laplace transform of $F(s)$ and is denoted by

$f(t) = L^{-1}(F(s))$

Here $L^{-1}1$ is called the inverse Laplace transform operator.

17. Standard results in inverse Laplace transforms

Laplace Transform	**Inverse Laplace Transform**
$L(1)=\dfrac{1}{s}$	$L^{-1}\left(\dfrac{1}{s}\right)=1$
$L(e^{at})=\dfrac{1}{s-a}$	$L^{-1}\left(\dfrac{1}{s-a}\right)=e^{at}$
$L(e^{-at})=\dfrac{1}{s+a}$	$L^{-1}\left(\dfrac{1}{s+a}\right)=e^{-at}$
$L(t)=\dfrac{1}{s^2}$	$L^{-1}\left(\dfrac{1}{s^2}\right)=t$

$$L(t^2) = \frac{2!}{s^3} \qquad L^{-1}\left(\frac{2!}{s^3}\right) = t^2$$

$$L(t^3) = \frac{3!}{s^4} \qquad L^{-1}\left(\frac{3!}{s^4}\right) = t^3$$

$$L(t^n) = \frac{n!}{s^{n+1}} \qquad L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$$

where n is a +ve integer

$$L(\sin\ at) = \frac{a}{s^2 + a^2} \qquad L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin\ at$$

$$L(\cos at) = \frac{s}{s^2 + a^2} \qquad L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$$

$$L(\sin\ hat) = \frac{a}{s^2 - a^2} \qquad L^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sin\ hat$$

$$L(\cos hat) = \frac{s}{s^2 - a^2} \qquad L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cos hat$$

$$L(t \sin\ at) = \frac{2as}{(s^2 + a^2)^2} \qquad L^{-1}\left(\frac{2as}{(s^2 + a^2)^2}\right) = t \sin\ at$$

$$L(t \cos at) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \qquad L^{-1}\left(\frac{s^2 - a^2}{(s^2 + a^2)^2}\right) = t \cos at$$

$$L(t \sin\ hat) = \frac{2as}{(s^2 - a^2)^2} \qquad L^{-1}\left(\frac{2as}{(s^2 + a^2)^2}\right) = t \sin\ hat$$

$$L(t \cos hat) = \frac{s^2 + a^2}{(s^2 - a^2)^2} \qquad L^{-1}\left(\frac{s^2 + a^2}{(s^2 - a^2)^2}\right) = t \cos hat$$

$$L(e^{at} \sin\ bt) = \frac{b}{(s-a)^2 + b^2} \qquad L^{-1}\left(\frac{b}{(s+a)^2 + b^2}\right) = e^{at} \sin bt$$

$$L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2} \qquad L^{-1}\left(\frac{s-a}{(s-a)^2 + b^2}\right) = e^{at} \cos bt$$

$$L(e^{at}\sin hbt) = \frac{b}{(s-a)^2+b^2} \qquad L^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin hbt$$

$$L(e^{at}\cos hbt) = \frac{s-a}{(s-a)^2+b^2} \qquad L^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right) = e^{at}\cos hbt$$

$$L(te^{-at}) = \frac{1}{(s+a)^2} \qquad L^{-1}\left(\frac{1}{(s+a)^2}\right) = te^{-at}$$

$$L(t^2e^{-at}) = \frac{2!}{(s+a)^3} \qquad L^{-1}\left(\frac{2!}{(s+a)^3}\right) = t^2e^{-at}$$

18. Properties of Inverse Laplace Transforms

18.1 Linear Property

If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$L^{-1}(c_1F_1(s) + c_2F_2(s)) = c_1L^{-1}(F_1(s)) + c_2L^{-1}(F_2(s))$ where c_1 &$c_2$are constants.

Proof:

We know that

$$
\begin{aligned}
L(c_1f_1(t) + c_2f_2(t)) &= c_1L(f_1(t)) + c_2L(f_2(t)) \\
&= c_1F_1(s) + c_2F_2(s) \\
&= [\because L(f_1(t)) + F_1(s) \text{ and } L(f_2(t)) = F_2(s)] \\
c_1f_1(t) + c_2f_2(t) &= L^{-1}(c_1F_1(s)) + c_2F(s)) \\
&= L^{-1}(c_1F_1(s)) + L^{-1}(c_2F(s)) \\
&= c_1L^{-1}(F_1(s)) + c_2L^{-1}(F_2(s))
\end{aligned}
$$

Problems

1. Find $L^{-1}\left(\frac{1}{s-3}+s+\frac{s}{s^2-4}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s-3}+s+\frac{s}{s^2-4}\right) = L^{-1}\left(\frac{1}{s-3}\right)+L^{-1}(s)+L^{-1}\left(\frac{s}{s^2-4}\right)$$

$= \quad e^{3t} + 1 + \cos h2t$

$= \quad e^{3t} + \cos h2t + 1$

2. Find $L^{-1}\left(\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right)$$

$$= \quad L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s+4}\right) + L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left(\frac{s}{s^2-9}\right)$$

$$= \quad t + e^{-4t} + \frac{\sin 2t}{2} + \cos h3t$$

3. Find $L^{-1}\left(\frac{1}{s} + \frac{2}{s^2} - \frac{3s}{s^2+4} + \frac{4}{s^2+16}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s} + \frac{2}{s^2} - \frac{3s}{s^2+4} + \frac{4}{s^2+16}\right)$$

$$= \quad L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{2}{s^2}\right) - L^{-1}\left(\frac{3s}{s^2+4}\right) + L^{-1}\left(\frac{4}{s^2+16}\right)$$

$$= \quad 1 + 2t - 3\cos 2t + \sin 4t$$

4. Find $L^{-1}\left(\frac{4}{s^6} - \frac{2}{s^{10}} + \frac{2}{s^2-9} + \frac{3s}{s^2+25}\right)$

Solution:

$$L^{-1}\left(\frac{4}{s^6} - \frac{2}{s^{10}} + \frac{2}{s^2-9} + \frac{3s}{s^2+25}\right)$$

$$= \quad \frac{4}{5!}L^{-1}\left(\frac{5!}{s^6}\right) - \frac{2}{9!}L^{-1}\left(\frac{9!}{s^{10}}\right) + \frac{2}{3}L^{-1}\left(\frac{3}{s^2-9}\right) + 3L^{-1}\left(\frac{s}{s^2+25}\right)$$

$$= \quad \frac{1}{36}t^5 - \frac{1}{181440}t^9 + \frac{2}{3}\sin\ h3t + 3\cos 5t$$

5. Find $L^{-1}\left(\frac{2}{s^5}-\frac{3}{s^4}+\frac{2}{s^2-3}+\frac{5}{s^2+100}+\frac{s}{s^2+10}\right)$

Solution:

$$L^{-1}\left(\frac{2}{s^5}-\frac{3}{s^4}+\frac{2}{s^2-3}+\frac{5}{s^2+100}+\frac{s}{s^2+10}\right)$$

$$=\frac{2}{4!}L^{-1}\left(\frac{4!}{s^s}\right)-\frac{3}{3!}L^{-1}\left(\frac{3!}{s^4}\right)+\frac{3}{\sqrt{3}}L^{-1}\left(\frac{\sqrt{3}}{s^2-\sqrt{3^2}}\right)+\frac{5}{10}L^{-1}\left(\frac{10}{s^2-100}\right)+L^{-1}\left(\frac{s}{s^2+10}\right)$$

$$=\frac{1}{12}t^4\ \frac{1}{2}t^3\sqrt{3}\sin\ \sqrt{3t}+\frac{1}{2}\sin\ h10t+\cos\sqrt{10t}$$

6. Find $L^{-1}\left(\frac{5}{s^5-25}+\frac{4s}{s^2-16}+\frac{s}{s^2+9}+\frac{s}{s^2-25}\right)$

Solution:

$$L^{-1}\left(\frac{5}{s^5-25}+\frac{4s}{s^2-16}+\frac{s}{s^2+9}+\frac{s}{s^2-25}\right)$$

$$= \quad L^{-1}\left(\frac{5}{s^2-25}\right)+4L^{-1}\left(\frac{s}{s^2-16}\right)+L^{-1}\left(\frac{s}{s^2-9}\right)+L^{-1}\left(\frac{s}{s^2-25}\right)$$

= sin $h5t$ + 4 cos $h4t$ + cos $3t$ – cos $h5t$

7. Find $L^{-1}\left(\frac{1}{2s+3}\right)$

Solution:

$$L^{-1}\left(\frac{1}{2s+3}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+3/2}\right)$$

$$= \frac{1}{2}e^{-3/2t}$$

19. First Shifting Property

(i) If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(F(s-a)) = e^{at}L^{-1}(F(s))$

Proof:

We know that $L(f(s)) = f(t)$ then $L^{-1}(F(s\text{-}a)) = e^{at}L^{-1}(F(s))$

Hence $e^{at}f(t) = L^{-1}(F(s\text{-}a))$

$e^{at}L^{-1}(F(s)) = L^{-1}(F(s\text{-}a))$

(ii) If $L^{-1}(F(s)) = f(t)$ Then $L^{-1}(F(s+a)) = e^{-at}L^{-1}(F(s))$

Proof:

We know that $L(f(f)) = F(s)$ Then $L(e^{-at}f(t)) = F(s+a)$

Hence $e^{-at}f(t) = L^{-1}(F(s+a))$

$e^{-at}L^{-1}(F(s)) = L^{-1}(F(s+a))$

19.1 Problems

1. Find $L^{-1}\left(\dfrac{1}{(s+1)^2}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t}t$$

2. Find $L^{-1}\left(\dfrac{1}{(s+1)^2+1}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= e^{-t}\sin t$$

3. Find $L^{-1}\left(\dfrac{s-3}{(s-3)^2+4}\right)$

Solution:

$$L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) = e^{3t}L^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= \quad e^{3t}\cos 2t$$

4. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) \quad = \quad L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$

$$= \quad L^{-1}\left(\frac{s+2}{(s+2)^2}-\frac{2}{(s+2)^2}\right)$$

$$= \quad L^{-1}\left(\frac{1}{(s+2)}\right)-2L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= \quad e^{-2t} - 2e^{-2t}\,.\,t$$

$$= \quad e^{-2t}\,(1-2t)$$

5. Find $L^{-1}\left(\frac{s}{(s-1)^2+3}+\frac{3s}{(s+2)^2-5}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s-1)^2+3}+\frac{3s}{(s+2)^2-5}\right) = L^{-1}\left(\frac{s}{(s-1)^2+3}\right)+3L^{-1}\left(\frac{s}{(s+2)^2-5}\right)$$

$$= \quad L^{-1}\left(\frac{s-1+1}{(s-1)^2+3}\right)+3L^{-1}\left(\frac{s+2-2}{(s+2)^2-5}\right)$$

$$= \quad L^{-1}\left(\frac{s-1}{(s-1)^2+3}\right)+L^{-1}\left(\frac{1}{(s-2)^2+3}\right)$$

$$+3L^{-1}\left(\frac{s+2}{(s+2)^2-5}\right)-6L^{-1}\left(\frac{1}{(s+2)^2-5}\right)$$

$$= e^t L^{-1}\left(\frac{s}{s^2+3}\right) + e^t L^{-1}\left(\frac{1}{s^2+3}\right) + 3e^{-2t} L^{-1}\left(\frac{s}{s^2-5}\right)$$

$$-6e^t L^{-1}\left(\frac{1}{s^2-5}\right)$$

$$= e^t L^{-1}\left(\frac{s}{s^2+\sqrt{3}^2}\right) + \frac{e^t}{\sqrt{3}} L^{-1}\left(\frac{\sqrt{3}}{s^2+\sqrt{3}^2}\right)$$

$$= 3e^{-2t} L^{-1}\left(\frac{s}{s^2+\sqrt{5}^2}\right) + \frac{6}{\sqrt{5}} e^{-2t} L^{-1}\left(\frac{\sqrt{5}}{s^2+\sqrt{5}^2}\right)$$

$$= e^t \cos\sqrt{3}t + \frac{e^t}{\sqrt{3}} \sin\sqrt{3}t + 3e^{-2t} \cos h\sqrt{5}t$$

$$= \frac{6}{\sqrt{5}} e^{-2t} \sin\ h\sqrt{5}t$$

6. Find $L^{-1}\left(\frac{3s-4}{s^2-8s+65}\right)$

Solution:

$$L^{-1}\left(\frac{3s-4}{s^2-8s+65}\right) = L^{-1}\left(\frac{3s-4}{(s-4)^2+49}\right)$$

$$= L^{-1}\left(\frac{3\left(s-4/3\right)}{(s-4)^2+49}\right) = 3L^{-1}\left(\frac{3-4+4-4/3}{(s-4)^2+49}\right)$$

$$= 3L^{-1}\left(\frac{s-4+8/3}{(s-4)^2+49}\right)$$

$$= 3L^{-1}\left(\frac{s-4}{(s-4)^2+49}\right) + 3.\,8/3\, L^{-1}\left(\frac{1}{(s-4)^2+49}\right)$$

$$= 3e^{4t} L^{-1}\left(\frac{s}{s^2+49}\right) + 8e^{4t} L^{-1}\left(\frac{1}{s^2+49}\right)$$

$$= \quad 3e^{4t}\cos 7t + \frac{8}{7}e^{4t}L^{-1}\left(\frac{7}{s^2+49}\right)$$

$$= \quad 3e^{4t}\cos 7t + \frac{8}{7}e^{4t}\sin 7t$$

20. Change of Scale Property

If $L(f(t)) = F(s)$, then $L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0$

Proof:

$F(s) \quad = L(f(t))$

$$= \int_0^\infty e^{-st} f(t)dt$$

$$F(as) \quad = \quad \int_0^\infty e^{-ast} f(t)dt$$

Let $at = t_1$ When $t = 0, t_1 = 0$

$dt = \frac{dt_1}{a}$ $t = \infty, t_1 = \infty$

$$F(as) \quad = \quad \int_0^\infty e^{-st} f\left(\frac{t_1}{a}\right)\frac{dt_1}{a}$$

$$= \quad \frac{1}{a}\int_0^\infty e^{-st_1} f\left(\frac{t_1}{a}\right)dt_1$$

$$= \quad \frac{1}{a}\int_0^\infty e^{-st} f\left(\frac{t}{a}\right)dt_1 \qquad \left(\because \int_a^b f(t)dt = \int_a^b f(t_1)dt_1\right)$$

$$= \quad \frac{1}{a}L\left(f\left(t/a\right)\right)$$

$$\therefore L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right)$$

20.1 Problems

1. If $L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right)=t\cos t$, then find $L^{-1}\left(\frac{9s^2-1}{(9s^2+1)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right)=t\cos t$$

writing as for S,

$$L^{-1}\left(\frac{a^2s^2-1}{(a^2s^2+1)^2}\right)=\frac{1}{a}.\frac{t}{a}\cos\left(\frac{t}{a}\right)$$

Put a = 3, $L^{-1}\left(\frac{9s^2-1}{(9s^2+1)^2}\right)=\frac{1}{3}.\frac{t}{3}\cos\left(\frac{t}{3}\right)$

$$=\frac{t}{9}\cos\left(\frac{t}{3}\right)$$

2. Find $L^{-1}\left(\frac{s}{(2s^2-8)}\right)$

Solution:

We know that $L^{-1}\left(\frac{s}{(s^2-4^2)}\right)=\cos h4t$

Putting as for S,

$$L^{-1}\left(\frac{2s}{(2s)^2-4^2)}\right)=\frac{1}{2}\cos h\left(\frac{4t}{2}\right)$$

$$L^{-1}\left(\frac{2s}{4s^2-16}\right)=\frac{1}{2}\cos h\ 2t$$

(ie)

$$L^{-1}\left(\frac{s}{2s^2-18}\right)=\frac{1}{2}\cos h\ 2t$$

3. Find $L^{-1}\left(\dfrac{s}{s^2a^2+b^2}\right)$

Solution:

$$\frac{s}{s^2a^2+b^2} = \frac{1}{a}\frac{as}{s^2a^2+b^2}$$

$$= \frac{1}{a}F(as) \text{ where } F(as) = \frac{1}{s^2+b^2}$$

$$\therefore L^{-1}\left(\frac{s}{s^2a^2+b^2}\right) = \frac{1}{a}L^{-1}\frac{sa}{s^2a^2+b^2}$$

$$= \frac{1}{a}L^{-1}(F(as))$$

$$= \frac{1}{a}\cdot\frac{1}{a}f\left(\frac{t}{a}\right)$$

where $f(t) = L^{-1}(F(s)) = L^{-1}\left(\dfrac{s}{s^2+b^2}\right) = \cos bt$

$$\therefore f\left(\frac{t}{a}\right) = \cos\left(\frac{bt}{a}\right)$$

$$\therefore L^{-1}\left(\frac{s}{s^2+b^2}\right) = \frac{1}{a}\cdot\frac{1}{a}\cos\left(\frac{bt}{a}\right)$$

$$= \frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$$

21. Result

We know that if $L(f(t)) = F(S)$, then $L(tf(t)) = \dfrac{-d}{ds}F(s)$

$$L(tf(t)) = -F'(s)$$

Hence $L^{-1}(F'(s)) = tf(t)$

$$= tL^{-1}(F(s))$$

$$\therefore L^{-1}(F'(s)) = -tL^{-1}(F(s))$$

21.1 Problems

1. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Solution:

$$\text{Let} \quad F'(s) = \frac{s}{(s^2+a^2)^2}$$

$$\frac{d}{ds}F(s) = \frac{s}{(s^2+a^2)^2}$$

$$\therefore F(s) = \int \frac{s}{(s^2+a^2)^2}\,ds$$

$$\text{put } s^2 + a^2 = u$$

$$2sds = du$$

$$\therefore \int \frac{s}{(s^2+a^2)^2}\,ds = \int \frac{\frac{du}{2}}{u^2}$$

$$= \frac{-1}{2u} = \frac{-1}{2(s^2+a^2)}$$

$$\therefore F(s) = \frac{-1}{2(s^2+a^2)}$$

We know that $L(F'(s)) = -tL^{-1}(F(s))$

$$\therefore L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = -tL^{-1}\left(\frac{1}{2(s^2+a^2)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+a^2)}\right)$$

$$= \frac{t}{2}\frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{t}{2a}\sin at$$

2. Find $L^{-1}\left(\dfrac{s+3}{(s^2+6s+13)^2}\right)$

Solution:

Let $\left(\dfrac{s+3}{(s^2+6s+13)^2}\right) = F'(s)$

$$\frac{dF(s)}{ds} = \frac{s+3}{(s^2+6s+13)^2}$$

$$\therefore F(s) = \frac{(s+3)ds}{(s^2+6s+13)^2}$$

Put $s^2 + 6s + 13 = u$

$(2s + 6)ds = du$

$2(s + 3)ds = du$

(ie) $F(s) = \int \dfrac{\frac{du}{2}}{u^2} = \dfrac{-1}{2u}$

$$= \frac{-1}{2(s^2+6s+13)}$$

We know that $L^{-1}(F'(s)) = -tL^{-1}(F(s))$

$$\therefore L^{-1}\frac{s+3}{(s^2+6s+13)^2} = -tL^{-1}\left(\frac{-1}{2(s^2+6s+13)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{-1}{(s^2+6s+13)}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s+3)^2+2^2}\right)$$

$$= \frac{t}{2}e^{-3t}L^{-1}\left(\frac{1}{(s^2+2^2)}\right)$$

$$= \frac{t}{2}e^{-3t}\frac{1}{2}L^{-1}\left(\frac{2}{(s^2+2^2)}\right)$$

$$= \frac{t}{4}e^{-3t}\sin 2t$$

3. Find $L^{-1}\left(\dfrac{2(s+1)}{(s^2+2s+2)^2}\right)$

Solution:

$$F'(s) = \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$\frac{dF(s)}{ds} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$F(s) = \int \frac{2(s+1)}{(s^2+2s+2)^2}\, ds$$

Put $s^2 + 2s + 2 = u$

$(2s + 2)ds = du$

$2(s + 2)ds = du$

$$\therefore F(s) = \int \frac{du}{u^2}$$

$$= \frac{-1}{u}$$

$$= \frac{-1}{s^2+2s+2}$$

$$\therefore L^{-1}\left(\frac{2(s+1)}{(s^2+2s+2)^2}\right) = -tL^{-1}\left(\frac{1}{s^2+2s+2}\right)$$

$$= -tL^{-1}\left(\frac{1}{s^2+2s+2}\right) = tL^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= te^{-t}L^{-1}\left(\frac{1}{s^2 1}\right)$$

$$= te^{-t}\sin^{t}$$

4. Find $L^{-1}\left(\dfrac{s+2}{(s^2+4s+5)^2}\right)$

Solution:

Let $F'(s) = \left(\dfrac{s+2}{(s^2+4s+5)^2}\right)$

Integrate both sides w.r.t. 'S'

$$F'(s) = \frac{s+2}{(s^2+4s+5)^2}$$

$$\int F'(s) = \int \frac{(s+2)ds}{(s^2+4s+5)^2}$$

$$F(s) = \int \frac{(s+2)ds}{(s^2+4s+5)^2} \quad \text{Let } y = s^2 + 4s + 5$$

$$F(s) = \int \frac{dy/2}{y^2} \quad dy = (2s+4)\,ds$$

$$= \frac{1}{2}\int \frac{dy}{y^2} \quad = \frac{dy}{2} = (s+2)ds$$

$$= \frac{1}{2}\int y^{-2}dy$$

$$F(s) = \frac{1}{2}\left(\frac{y^{-2+1}}{-2+1}\right)$$

$$= \frac{-1}{2y}$$

$$= \frac{-1}{2(s^2+4s+5)}$$

We know that

$$L^{-1}(F'(s)) = -tL^{-1}(F(s))$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = tL^{-1}\left(\frac{-1}{2(s^2+4s+5)}\right)$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = \frac{t}{2}L^{-1}\left(\frac{1}{(s^2+4s+5)}\right)$$

$$= \frac{t}{2} L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$= \frac{t}{2} e^{-2t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{t}{2} e^{-2t} \sin t$$

5. Find $L^{-1}\left(\tan^{-1}\left(1/s\right)\right)$

Solution:

Let $F(s) = \tan^{-1}\left(1/s\right)s$

$$F'(s) = \frac{1}{1+\left(1/s\right)^2}\left(\frac{-1}{s^2}\right) \quad \left[\because \frac{d(\tan^{-1} x)}{dzx} = \frac{1}{1+x^2}\right]$$

$$F'(s) = \frac{s^2}{s^2+1}\left(\frac{-1}{s^2}\right)$$

$$= \frac{-1}{s^2+1}$$

We know that $L^{-1}(F'(s)) = -tL^{-1}(F(s))$

or

$$L^{-1}(F(s)) = \frac{-1}{t} L^{-1}(F'(s)) \qquad \ldots(1)$$

$\therefore$ (1) becomes, $L^{-1}\left(\tan^{-1}\left(1/s\right)\right) = \frac{-1}{t} L^{-1}F(s)$

$$= \frac{1}{t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$L^{-1}\left(\tan^{-1}\left(1/s\right)\right) = \frac{1}{t}\sin t$$

6. Find $L^{-1}\left(\tan^{-1}\left(a/s\right) + \cot^{-1}\left(s/b\right)\right)$

Solution:

Let $F(s) = \tan^{-1}\left(a/s\right) s + \cot^{-1}\left(s/b\right)$

$$F(s) = \frac{1}{1+\left(a/s\right)^2}\left(\frac{-a}{s^2}\right) + \frac{-1}{1+\left(s/b\right)^2}\left(\frac{1}{b}\right)$$

$$F'(s) = \frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right) - \frac{b^2}{b^2+s^2}\left(\frac{1}{b}\right)$$

$$F'(s) = \frac{-a}{s^2+a^2} - \frac{b}{b^2+s^2}$$

We know that $L^{-1}(F(s)) = \frac{-1}{t} L^{-1}(F'(s))$

$$L^{-1}\left(\tan^{-1}\left(a/s\right) + \cot^{-1}\left(s/b\right)\right) = \left(\frac{-a}{s^2+a^2} - \frac{b}{b^2+s^2}\right)$$

$$= \frac{1}{t} L^{-1}\left(\frac{a}{s^2+a^2} - \frac{b}{b^2+s^2}\right)$$

$$= \frac{1}{t} L^{-1}\left(L^{-1}\left(\frac{a}{s^2+a^2}\right) - L^{-1}\left(\frac{b}{b^2+s^2}\right)\right)$$

$$= \frac{1}{t}(\sin at + \sin bt)$$

7. Find $L^{-1}\left(\log\left(1 + a^2/s^2\right)\right)$

Solution:

Let $F(s) = \log\left(1 + a^2/s^2\right)$

$$\therefore F(s) = \log\left(\frac{s^2+a^2}{s^2}\right)$$

$$F(s) = \log\left(s^2+a^2\right) - \log s^2$$

$$F(s) = \log\left(s^2+a^2\right) - 2\log s^2$$

$$\therefore F(s) = \frac{2s}{s^2+a^2} - \frac{2}{s}$$

We know that $L^{-1-}(F(s))$ $= \dfrac{-1}{t} L^{-1}(F'(s))$

$$L^{-1}\left(\log(1+\left(a^2/s^2\right)\right) = \frac{-1}{t} L^{-1}\left(\frac{2s}{s^2+a^2} - \frac{2}{s}\right)$$

$$= \frac{-2}{t}\left(L^{-1}\left(\frac{s}{s^2+a^2}\right) - L^{-1}\left(\frac{1}{s}\right)\right)$$

$$= \frac{-2}{t}(\cos at - 1)$$

$$= \frac{2}{t}(1-\cos at)$$

8. Find $L^{-1}\left(\log\dfrac{(s+a)}{(s+b)}\right)$

Solution:

Let $F(s) = \log\dfrac{(s+a)}{(s+b)}$

$= \log(s+a) - \log(s+b)$

$F'(s) = \dfrac{1}{s+a} - \dfrac{1}{s+b}$ $\qquad \because L^{-1}(F(s)) = \dfrac{-1}{t} L^{-1}(s))$

$$L^{-1}\left(\log\frac{(s+a)}{(s+b)}\right) = \frac{-1}{t} L^{-1}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$= \frac{-1}{t}(e^{-ar} - e^{-bt})$$

9. Find $L^{-1}\left(\log\dfrac{s(s^2+a^2)}{(s^2+b^2)}\right)$

Solution:

Let $F(s) = \log\dfrac{s(s^2+a^2)}{(s^2+b^2)}$

$F(s) = \log(s(s^2+a^2) - \log(s^2+b^2))$

$$F(s) = \log s + \log(s(s^2+a^2) - \log(s^2+b^2))$$

$$F'(s) = \frac{1}{s} + \frac{2s}{(s^2+a^2)} - \frac{2s}{(s^2+^2)}$$

We know that $L^{-1}(F(s)) = \frac{-1}{t} L^{-1}(F'(s))$

$$L^{-1}\log\frac{s(s^2+a^2)}{s(s^2+b^2)} = \frac{-1}{t}L^{-1}\left(\frac{1}{s} + \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right)$$

$$= \frac{-1}{t}\left(L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{2s}{s^2+a^2}\right) - L^{-1}\left(\frac{2s}{s^2+b^2}\right)\right)$$

$$= \frac{-1}{t}[1 + 2\cos at - 2\cos bt]$$

10. Find $L^{-1}\left(\log\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)$

Solution:

Let $F(s) = \log\left(\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)$

$$= \log(s(s^2+1)(s-4)^2 - \log((s^2-9)(s^2+4))$$

$$F(s) = \log s + \log(s^2+1) + \log(s-4)^2 - \log(s^2-9) - \log(s^2+4)$$

$$F'(s) = \frac{1}{s} + \frac{2s}{s^2+1} + \frac{2(s-4)}{(s-4)^2} - \frac{2s}{s^2-9} - \frac{2s}{s^2+4}$$

we know that, $L^{-1}(F(s) = \frac{-1}{1}L^{-1}(F'(s))$

$$L^{-1}\log\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)} = \frac{-1}{1}L^{-1}\left(\frac{1}{s} + \frac{2s}{s^2+1} + \frac{2}{s-4} - \frac{2s}{s^2-9} - \frac{2s}{s^2+4}\right)$$

$$= \frac{-1}{1}(1 + 2\cos t + 2e^{4t} - 2\cos h3t - 2\cos 2t)$$

11. Find $L^{-1}\left(\log\frac{s-a}{(s^2+a^2)}\right)$

Solution:

$$\text{Let } F(s) = \log\frac{s-a}{s^2+a^2}$$

$$= \log(s-a) - \log(s^2+a^2)$$

$$F'(s) = \frac{1}{s-a} - \frac{2s}{s^2+a^2}$$

We know that $L^{-1}(F(s)) = \frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\log\frac{s-a}{(s^2+a^2)}\right) = \frac{-1}{t}L^{-1}\left(\frac{1}{s-a} - \frac{2s}{s^2+a^2}\right)$$

$$= \frac{-1}{t}L^{-1}\left(\frac{2s}{s^2-a^2} - \frac{1}{s-a}\right)$$

$$= \frac{-1}{t}\left(L^{-1}\left(\frac{2s}{s^2+a^2}\right) - L^{-1}\left(\frac{1}{s-a}\right)\right)$$

$$= \frac{1}{t}(2\cos at - e^{at})$$

22. Theorem

If $L(f(t)) = F(s)$ and $\varphi(t)$ is a function such that $L(\varphi(t)) = F(s)$ and $\varphi(0) = 0$, then $f(t) = \varphi'(t)$, (ie) $L^{-t}(sf(s)) = \frac{d}{dt}L^{-1}(F(s))$.

Proof:

We know that

$$L(\varphi'(t)) = sL(\varphi(t)) - \varphi(0)$$

$$= sF(s) \qquad (\because \varphi(0) = 0$$

$$\text{(ie) } L(\varphi'(t)) = L(f(t))$$

$$\therefore \varphi'(t) = f(t)$$

From this result, we get

$$L^{-1}(s(s)) = f(t)$$

$$= \varphi'(t)$$

$$= \frac{d}{dt}\varphi(t)$$

$$= \frac{d}{dt}L^{-1}(F(s)) \qquad (\because L\varphi(t)) = F(s))$$

Provided $L^{-1}(F(s)) = 0$ as $t \to 0$

Problems

1. Find $L^{-1}\left(\frac{s}{(s+2)^2+4}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2+4}\right) = L^{-1}\left(s.\frac{1}{(s+2)^2+4}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{(s+2)^2+4}\right) \quad \text{(using the above result)}$$

$$= \frac{d}{dt}e^{-2t}L^{-t}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{d}{dt}e^{-2t}L^{-t}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\frac{1}{2}\sin 2t\right)$$

$$= \frac{1}{2}\left(2e^{-2t}\cos 2t + \sin 2te^{-2t}(-2)\right)$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

Aliter:

$$L^{-1}\left(\frac{s}{(s+2)^2+4}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2+4}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2+4} - \frac{s}{(s+2)^2+4}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2+4}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2+4}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{s}{s^2+2^2}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= e^{-2t}\cos 2t - 2e^{-2t}\frac{1}{2}\sin 2t$$

$$= e^{-2t}(\cos 2t - \sin 2t)$$

2. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s}{(s+2)^2}\right)$$

$$= L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-2t} + t(e^{-2t}(-2))$$

$$= e^{-2t}(1-2t)$$

Aliter:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2}\right) - L^{-1}\left(\frac{2}{(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+2)}\right) - 2e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-2t} - 2e^{-2t}t$$

$$= e^{-2t}(1-2t)$$

3. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right) = L^{-1}\left(s.\frac{s}{(s^2+a^2)}\right)$$

$$= \frac{d}{dt}L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$= \frac{d}{dt}\left(\frac{t}{2a}\sin at\right)$$

(By the Previous Section 21.1 Problem No.1)

$$= \frac{1}{2a}(at\cos at + \sin at)$$

4. Find $L^{-1}\left(\frac{s^2}{(s-1)^4}\right)$

Solution:

$$L^{-1}\left(\frac{s^2}{(s-1)^4}\right) = L^{-1}\left(s.\frac{s}{(s-1)^4}\right)$$

$$= \frac{d}{dt} L^{-1}\left(\frac{s}{(s-1)^4}\right)$$

$$= \frac{d}{dt} L^{-1}\left(\frac{s-1+1}{(s-1)^4}\right)$$

$$= \frac{d}{dt}\left(L^{-1}\left(\frac{s-1}{(s-1)^4}\right)+L^{-1}\left(\frac{1}{(s-1)^4}\right)\right)$$

$$= \frac{d}{dt}\left(L^{-1}\left(\frac{1}{(s-1)^3}\right)+L^{-1}\left(\frac{1}{(s-1)^4}\right)\right)$$

$$= \frac{d}{dt}\left(e^t L^{-1}\left(\frac{1}{s^3}\right)+e^t L^{-1}\left(\frac{1}{s^4}\right)\right)$$

$$= \frac{d}{dt}\left(e^t\,\frac{t^2}{2}+e^t\,\frac{t^3}{6}\right)$$

$$= \frac{1}{2}(e^t 2t + t^2 e^t) + \frac{1}{6}(e^t 3t^2 + t^3 e^t)$$

$$= te^t + e^t t^2 + \frac{t^3 e^t}{6}$$

5. Find $L^{-1}\left(\dfrac{s-3}{s^2+4s+13}\right)$

Solution:

$$L^{-1}\left(\frac{s-3}{s^2+4s+13}\right) = L^{-1}\left(\frac{s-3}{s^2+4s+13}\right) - L^{-1}\left(\frac{3}{s^2+4s+13}\right)$$

$$= \frac{d}{dt} L^{-1}\left(\frac{1}{s^2+4s+13}\right) - 3L^{-1}\left(\frac{1}{s^2+4s+13}\right)$$

$$= \frac{d}{dt} L^{-1}\left(\frac{1}{(s+2)^2+9}\right) - 3L^{-1}\left(\frac{1}{(s+2)^2+3^2}\right)$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left(\frac{1}{s^2+3^2}\right) - 3e^{-2t} L^{-1}\left(\frac{1}{s^2+3^2}\right)$$

$$= \frac{d}{dt}\left(e^{-2t}\frac{\sin 3t}{3}\right) - 3^{-2t}\left(\frac{\sin 3t}{3}\right)$$

$$= \frac{1}{3}(3e^{-2t}\cos 3t - 2\sin 3te^{-2t}) - 3^{-2t}\sin 3t$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$$

23. Theorem

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t L^{-1}(F(s))dt$$

Proof:

We know that,

$$L\left(\int_0^t f(x)dx\right) = \frac{1}{s}L(f(t))$$

$$\therefore \int_0^t f(x)dx = L^{-1}\left(\frac{1}{s}L(f(t))\right)$$

(ie) $$L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(t)dt \qquad s[\therefore F(s) = L(f)(t))]$$

$$= \int_0^t L^{-1}(F(s))dt$$

$$\therefore L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t L^{-1}(F(s))dt$$

Note:

Similarly $$L^{-1}\left(\frac{1}{s^2}F(s)\right) = \int_0^t\int_0^t L^{-1}(F(s))dtdt$$

$$L^{-1}\left(\frac{1}{s^3}F(s)\right) = \int_0^t\int_0^t\int_0^t L^{-1}(F(s))dtdtdt$$

$$L^{-1}\left(\frac{1}{s^n}F(s)\right) = \underbrace{\int_0^t\int_0^t\cdots\int_0^t}_{n\,times} L^{-1}(F(s))\underbrace{dtdt\cdots dt}_{n\,times}$$

23.1 Problems

1. Find $L^{-1}\left(\frac{1}{s(s+1)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s+1)}\right) = \int_0^t L^{-1}\left(\frac{1}{(s+1)}\right)dt \quad \text{(by the above theorem)}$$

$$= \int_0^t e^{-t}dt$$

$$= \left(-e^{-t}\right)_0^t$$

$$= -\left(e^{-t}-1\right)$$

$$= 1 - e^{-t}$$

2. Find $L^{-1}\left(\frac{1}{s(s+2)^3}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s+2)^3}\right) = \int_0^t \left(\frac{1}{(s+2)^3}\right)dt$$

$$= \int_0^t e^{-2t}L^{-1}\left(\frac{1}{s^3}\right)dt$$

$$= \int_0^t \frac{e^{-2t}}{2}L^{-1}\left(\frac{2}{s^3}\right)dt$$

$$= \frac{1}{2}\int_0^t e^{-2t}t^2dt$$

$$= \frac{1}{2}\left[(t^2)\left(\frac{e^{-2t}}{2}\right)-(2t)\left(\frac{e^{-2t}}{4}\right)+2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

$$\left[\because \int udv = uv - u'v_1 + u''v_2 \ldots\right]$$

$$= \frac{1}{2}\left[\frac{-t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{-e^{-2t}}{2}\left(t^2 + t + \frac{1}{2}\right) + \frac{1}{4}\right]$$

$$= \frac{1}{8}\left(1 - \left(2t^2 + 2t + 1\right)e^{-2t}\right)$$

3. Find $L^{-1}\left(\frac{54}{s^3(s-3)}\right)$

Solution:

$$L^{-1}\left(\frac{54}{s^3(s-3)}\right) = 54\int_0^t\int_0^t\int_0^t L^{-1}\left(\frac{1}{(s-3)}\right)dtdtdt$$

$$= 54\int_0^t\int_0^t\int_0^t e^{3t}\,dtdtdt$$

$$= 54\int_0^t\int_0^t \left(\frac{3^{3t}}{(3)}\right)_0^t dtdt$$

$$= 18\int_0^t\int_0^t \left(e^{3t} - 1\right)dtdt$$

$$= 18\int_0^t \left(\frac{e^{3t}}{3} - t\right)_0^t dt$$

$$= 18\int_0^t \left(\frac{e^{3t}}{3} - t\right) - \left(\frac{1}{3} - 0\right)dt$$

$$= 18\int_0^t \left(\frac{e^{3t}}{3} - t - \frac{1}{3}\right)dt$$

$$= 18\left(\frac{e^{3t}}{9} - \frac{t^2}{2} - \frac{t}{3} - \frac{1}{9}\right)$$

$$= 2e^{3t} - 9t^2 - 6t - 2$$

4. Find $L^{-1}\left(\dfrac{1}{s(s^2+a^2)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right) dt$$

$$= \int_0^t \frac{1}{a} L^{-1}\left(\frac{a}{s^2+a^2}\right) dt$$

$$= \frac{1}{a}\int_0^t \sin at\, dt$$

$$= \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t$$

$$= \frac{-1}{a^2}(\cos at - 1)$$

$$= \frac{+1}{a^2}(\cos at)$$

5. Find $L^{-1}\left(\dfrac{1}{s(s^2+a^2)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{s}{s(s^2+a^2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right)$$

$$= \int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) dt$$

$$= \int_0^t \frac{t\sin at}{2a} dt$$

$$= \frac{1}{2a}\left(t\left(\frac{-\cos at}{a}\right) - 1\left(\frac{-\sin at}{a^2}\right)\right)_0^t$$

$$= \frac{1}{2a}\left(\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right)$$

(By the previous section 21.1 Problem no.1)

6. Find $L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right) = L^{-1}\left(\frac{1}{s}.\frac{1}{s^2-2s+5}\right)$$

$$= \int_0^t L^{-1}\left(\frac{1}{s^2-2s+5}\right)dt$$

$$= \int_0^t L^{-1}\left(\frac{1}{(s-1)^2+2^2}\right)dt$$

$$= \int_0^t e^t L^{-1}\left(\frac{1}{s^2+2^2}\right)dt$$

$$= \int_0^t e^t \frac{\sin 2t}{2} t$$

$$= \frac{1}{2}\int_0^t e^t \sin 2t\, dt$$

$$= \frac{1}{2}\left[\frac{e^t}{1^2+2^2}(\sin 2t - 2\cos 2t)\right]_0^t$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t\right]_0^t$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t - 0 + 2\right]$$

$$= \frac{1}{10}\left[e^t \sin 2t - 2e^t \cos 2t + 2\right]$$

7. Find $L^{-1}\left(\dfrac{1}{s(s^2-6s+13)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2-6s+13)}\right) = L^{-1}\left(\frac{1}{s}.\frac{1}{s^2+6s+13}\right)$$

$$= \int_0^t L^{-1}\left(\frac{1}{(s+3)^2+4}\right)dt$$

$$= \int_0^t e^{-3t}L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

$$= \frac{1}{2}\int_0^t e^{-3t}L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

$$= \frac{1}{2}\int_0^t e^{-3t}\sin 2t\ dt$$

$$= \frac{1}{2}\left\{\frac{e^{-3t}}{(-3)^2+2^2}(-3\sin 2t-2\cos 2t)\right\}_0^t$$

$$= \frac{-1}{26}\left\{e^{-3t}(3\sin 2t+2\cos 2t)-2\right\}$$

8. Find $L^{-1}\left(\dfrac{1}{s(s^2+a^2)^2}\right)$

Solution:

$$L^{-1}\left(\frac{1}{s(s^2+a^2)^2}\right) = L^{-1}\left(\frac{1}{s^2(s^2+a^2)^2}\right)$$

$$= \int_0^t\int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)dtdt$$

$$= \int_0^t\int_0^t \frac{t}{2a}\sin atdtdt \quad \text{(refer the above problem)}$$

$$= \frac{1}{2a}\int_0^t\int_0^t t\sin atdtdt$$

$$= \frac{1}{2a}\int_0^t \left(\left(t\frac{-\cos at}{a}\right)-(1)\left(\frac{-\sin at}{a^2}\right)\right)dt$$

$$= \frac{1}{2a}\int_0^t \left(\frac{\sin at}{a^2}-\frac{t\cos at}{a}\right)_0^t dt$$

$$= \frac{1}{2a^3}\int_0^t (\sin at - at\cos at)dt$$

$$= \frac{1}{2a^3}\left[\left(\frac{-\cos at}{a}\right)_0^t - a\left(t\left(\frac{\sin at}{a}\right)-(1)\left(\frac{-\cos at}{a^2}\right)\right)_0^t\right]$$

$$= \frac{1}{2a^3}\left[\frac{-\cos at}{a} - t\sin at - \frac{-\cos at}{a}\right]_0^t$$

$$= \frac{-1}{2a^3}\left[\frac{2\cos at}{a} + t\sin at\right]_0^t$$

$$= \frac{-1}{2a^3}\left[\frac{2\cos at}{a} + t\sin at - \frac{2}{a}\right]$$

$$= \frac{-1}{2a^4}(2-2\cos at - at\sin at)$$

Inverse Laplace Transform using Second Shifting Theorem

If $L(f(t)) = F(s)$, then $L(f(t-a)) = U(tc - a)) = e^{-as} F(s)$ where 'a' is a positive constant and $U(t-a)$ isthe unit step function.

The above property can be written in terms of inverse Laplace operator as,

If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(e^{-as}F(s)) = f(t-a)U(t-a)$

$\therefore L^{-1}(e^{-as}F(s)) = L^{-1}(F(s))_{t\to t-a}.\ U(t-a)$ where U is the unit step function.

Thus we want to find the Laplace inverse transform of the product of two factors one of which is e^{-as}, ignore e^{-as}, find the inverse transform of the other function and then replace t by $t - a$ in it and multiply by $U(t-a)$

Problems

1. Find $L^{-1}\left(\frac{e^{-s}}{s+2}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-s}}{s+2}\right) = L^{-1}\left(\frac{1}{s+2}\right)_{t\to t-1} \quad U(t-1).$$

$$= \left(e^{-2t}\right)_{t\to t-1} \quad U(t-1) \text{ where U is the unit step function.}$$

$$= e^{-2(t-1)}U(t-1).$$

2. Find $L^{-1}\left(\frac{e^{-2s}}{s-1}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{s-1}\right) = \left\{L^{-1}\left(\frac{1}{s-1}\right)\right\}_{t\to t-2} .U(t-2)$$

$$= (e^{t})_{t\to t-2}U(t-2) \text{ where U is the unit step function}$$

$$= e^{t-2}U(t-2)$$

3. Find $L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right) = \left\{L^{-1}\left(\frac{1}{(s+1)^{5/2}}\right)\right\}_{t\to t=1} U(t-1) \qquad \ldots (1)$$

Now, $$L^{-1}\left(\frac{1}{(s+1)^{5/2}}\right) = e^{-t}L^{-1}\left(\frac{1}{s^{5/2}}\right) \text{ Using first shifting property.}$$

$$= e^{-t}\frac{1}{\Gamma\left(5/2\right)}t^{3/2} \qquad \left(\because L^{-1}\left(\frac{1}{s^{n}}\right) = \frac{1}{\Gamma(n)}t^{n-1}\right)$$

$$= \quad e^{-t} \frac{1}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} t^{3/2}$$

$$= \quad \frac{4}{3.\sqrt{\pi}} 3^{-t} t^{3/2} \qquad \ldots (2)$$

Substituting (2) in (1)

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right) = \left(\frac{4}{3\sqrt{\pi}} e^{-t} t^{3/2}\right)_{t \to t-1} U(t-1)$$

$$L^{-1}\left(\frac{e^{-s}}{(s+1)^{5/2}}\right) = \left(\frac{4}{3\sqrt{\pi}}\right) . e^{-(t-1)} (t-1)^{3/2} . U(t-1)$$

4. Find $L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right), a > 0$

Solution:

$$L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right) = \left\{L^{-1}\left(\frac{s}{s^2 - w^2}\right)\right\}_{t \to t-a} U(t-a)$$

$$= (\cosh wt)_{t \to t-a} . U(t-a)$$

$$= \cosh wt(t-a) . U(t-a)$$

5. Find $L^{-1}\left(\frac{e^{-2s}}{(s+1)^3}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{(s+1)^3}\right) = \left\{L^{-1}\left(\frac{1}{(s+1)^3}\right)\right\}_{t \to t-2} . U(t-2) \qquad \ldots (1)$$

Now, $L^{-1}\left(\frac{1}{(s+1)^3}\right) = e^{-r} L^{-1}\left(\frac{1}{s^3}\right)$

$$= \frac{e^{-t}}{2!} L^{-1}\left(\frac{2!}{s^2}\right)$$

$$= \frac{e^{-t}}{2} t^2 \qquad \ldots (2)$$

Substituting (2) in (1)

$$L^{-1}\left(\frac{e^{-2s}}{(s+1)^3}\right) = \left(\frac{e^{-t}}{2}t^2\right)_{t\to t-2} U(t-2)$$

$$= \frac{e^{-(t-2)}.(t-2)^2 U(t-2)}{2}$$

6. Find $L^{-1}\left(\left(\frac{3a-4s}{s^2+a^2}\right)e^{-5s}\right)$

Solution:

$$L^{-1}\left(e^{-5s}\left(\frac{3a-4s}{s^2+a^2}\right)\right) = L^{-1}\left(\left(\frac{3a-4s}{s^2+a^2}\right)\right)_{t\to t-s} .U(t-5)$$

$$= \left[3L^{-1}\left(\frac{a}{a^2+s^2}\right) - 4L^{-1}\left(\frac{s}{a^2+s^2}\right)\right]_{t\to t-5} .U(t-5)$$

$$= (3\sin at - 4\cos at)_{t\to t-5} U(t-5)$$

$$= 3\sin a(t-5) - 4\cos a(t-5).U(t-5)$$

7. Find $L^{-1}\left(\frac{e^{-\pi S}}{(s-2)(S+5)}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-\pi S}}{(s-2)(S+5)}\right) = L^{-1}\left(\frac{1}{(s-2)(S+5)}\right)_{t\to t-\pi}$$

Now, $\frac{1}{(s-2)(s+5)} = \frac{A}{s-2} + \frac{B}{s+5}$

$1 = A(\text{s}+5) + B(s-2)$

Put s = -5 Put s = 2

$\therefore B = \frac{-1}{7}$ $\therefore A = \frac{1}{7}$

$$\therefore L^{-1}\left(\frac{1}{(s-2)(s+5)}\right) = \frac{1}{7}L^{-1}\left(\frac{1}{s-2}\right)-\frac{1}{7}L^{-1}\left(\frac{1}{s+5}\right)$$

$$= \frac{1}{7}e^{2t}-\frac{1}{7}e^{-5t}$$

$$\therefore L^{-1}\left(\frac{e^{-\pi s}}{(s-2)(s+5)}\right) = \left(\frac{e^{2t}}{7}-\frac{e^{5t}}{7}\right)_{t\to t-\pi} U(t-\pi)$$

$$= \left(\frac{e^{2(t-\pi)}}{7}-\frac{e^{-5(t-\pi)}}{7}\right)U(t-\pi)$$

Exercise - 1(g)

Find the inverse Laplace transform of the following functions.

1. $\frac{e^{-as}}{s^2}, a>0$ Ans: $\begin{cases} 0 & if \quad t<a \\ \frac{t-a}{1!} & if \quad t>a \end{cases}$

2. $\frac{e^{-2s}-e^{-3s}}{s}$ Ans: $\begin{Bmatrix} 0 & if & t<2 \\ 1 & if & t>2 \end{Bmatrix}+\begin{Bmatrix} 0 & if & t<3 \\ 1 & if & t>3 \end{Bmatrix}$

3. $\frac{e^{-3s}}{s-2}$ Ans: $\begin{cases} 0 & if \quad t<3 \\ e^{2(t-3)} & if \quad t>2 \end{cases}$

4. $\frac{se^{-s}}{s^2-9}$ Ans: $\begin{cases} 0 & if \quad t<1 \\ \cos 3(t-1) & if \quad t>1 \end{cases}$

5. $\frac{1+e^{-\pi s}}{s^2-1}$ Ans: $\sin t+\begin{cases} 0 & if \quad t<\pi \\ \sin(t-\pi) & if \quad t>\pi \end{cases}$

6. $\frac{1}{(s+1)^3}$ Ans: $e^{-t}\frac{t^2}{2!}$

7. $\frac{s^2+2s+3}{s^3}$ Ans: $1+2t+\frac{t^2}{2!}$

8. $\frac{s}{(s-2)^3}$ Ans: $e^{2t}\frac{t^3}{3!}$

9. $\dfrac{2s+3}{s^2+5}$ Ans: $2\cos 2t + 6\sin 2t$

10. $\dfrac{s+6}{s^2-16}$ Ans: $\cos h4t + 24\sin\ h4t$

Exercise - 1 (h)

Find the inverse Laplace transform of the following functions.

1. $\dfrac{1}{s^2-6s+10}$ Ans: $e^{3t}\sin t$

2. $\dfrac{1}{s^2-8s+16}$ Ans: te^{-4t}

3. $\dfrac{3s-2}{s^2-4s+20}$ Ans: $3e^{2t}\cos 4t + e^{2t}\sin 4t$

4. $\dfrac{3s+7}{s^2-4s+20}$ Ans: $4e^{3t} = e^{-t}$

5. $\dfrac{s+a}{(s+a)^2+a^2}$ Ans: $e^{-at}(b\cos bt - (d-ca)\sin bt)$

6. $\dfrac{s}{(s-a)^2+a^2}$ Ans: $e^{bt}\cos at$

7. $\dfrac{s+1}{s^2+6s+25}$ Ans: $e^{-3t}\left(\cos 4t - \dfrac{1}{2}\sin 4t\right)$

8. $\dfrac{1}{s^2+8s+16}$ Ans: te^{-4t}

9. $\dfrac{s}{(s+3)^2}$ Ans: $e^{-3t}(1-2t)$

10. $\dfrac{s}{(s^2+1)^2}$ Ans: $\dfrac{t}{2}\sin t$

Exercise - 1(i)

Find the inverse Laplace transform of the following functions.

1. $\dfrac{s}{(s-4)^5}$ Ans: $\dfrac{e^{4t}t^3(4-3t)}{24}$

2. $\dfrac{1}{(s^2+9)^2}$ Ans: $\dfrac{\sin 3t-3t\cos 3t}{54}$

3. $\dfrac{s+2}{(s^2+4s+5)^2}$ Ans: $\dfrac{t}{2}e^{-2t}\sin t$

4. $\dfrac{s^2+2s}{(s^2+2s+2)^2}$ Ans: $te^{-t}\cos t$

5. $\dfrac{1}{s(s+2)^3}$ Ans: $\dfrac{1}{S}(1-(1+2t+2t^2)e^{-2t}$

6. $\dfrac{s^2-s+2}{s(s-3)(s+2)}$ Ans: $\dfrac{1}{3}+\dfrac{8}{15}e^{st}+\dfrac{4}{5}e^{-2t}$

7. $\dfrac{2s-1}{s^2(s-1)^2}$ Ans: $t(e^t-1)$

8. $\dfrac{1}{s^2(s^2+a^2)^2}$ Ans: $\dfrac{at-\sin at}{a^3}$

9. $\dfrac{s+1}{s(s+2)}$ Ans: $\dfrac{1+e^{-t}}{2}$

10. $\dfrac{1}{(s*s^2+2s+2)}$ Ans: $\dfrac{1}{2}(1-\sin t+\cos t)e^t$

Exercise - 1(j)

Find the inverse Laplace Transform of the following functions.

1. $\log\dfrac{s-1}{s}$ Ans: $\dfrac{1-e^t}{t}$

2. $\log\dfrac{1+s}{s^2}$ Ans: $\dfrac{2-e^t}{t}$

3. $\log\left(1-\frac{a}{s}\right)$ Ans: $\frac{1-e^{at}}{t}$

4. $\log\frac{s^2+a^2}{s^2+b^2}$ Ans: $\frac{2}{t}(\cos bt-\cos at)$

5. $\log\frac{s^2+1}{s(s+1)}$ Ans: $\frac{1}{t}(1+e^{-r}-2\cos t)$

6. $\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$ Ans: $\frac{1}{t}(e^{at}-\cos bt)$

7. $\frac{1}{2}\log\frac{s^2+1}{(s+1)^2}$ Ans: $\frac{1}{t}(e^{-t}-\cos t)$

8. $\log\frac{s+3}{s(s-2)}$ Ans: $\frac{1}{t}(1+et^{2t}-e^{-3t})$

9. $\cot^{-1}(as)$ Ans: $\frac{1}{t}\sin\left(t/a\right)$

10. $\cot^{-1}\left(\frac{2}{s+1}\right)$ Ans: $\frac{-1}{t}(e^{-t}\sin 2t)$

11. $\cot^{-1}(1+s)$ Ans: $\frac{1}{t}e^{-t}\sin t$

12. $\tan^{-1}\left(\frac{s+a}{b}\right)$ Ans: $\frac{-1}{t}e^{at}\sin bt$

24. Partial Fraction

The rational fraction P(x)/Q(x) is said to be resolved into partial fraction if it can be expressed as the sum ofdifference of simple proper fractions.

Rules for resolving a Proper Fraction P(x) / Q(x) into partial fractions

Rule 1

Corresponding to every non repeated, linear factor (ax+b) of the denomiator Q(x), there exists a partial fraction of the form $\frac{A}{ax+b}$ where A is a constant, to be determined.

For Example

(i) $$\frac{2x-7}{(x-2)(3x-5)} = \frac{A}{x-2} + \frac{B}{3x-5}$$

(ii) $$\frac{5x^2+18x+22}{(x-1)(x+2)(2x+3)} = \frac{A}{x-1} + \frac{A}{x+2} + \frac{C}{2x+3}$$

Rule 2

Corresponding to every repeated linear factor $(ax\ b)^k$of the denominator Q(x), there exist k partial fractions of the forms,

$$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \frac{A_3}{(ax+b)^3}, \cdots \frac{A_k}{(ax+b)^k}$$

where $A_1, A_2,$ ..... A_k are constants to be determined.

For example

(i) $$\frac{4x-3}{(x+2)(2x-3)^2} = \frac{A}{x+2} + \frac{B}{2x-3} + \frac{C}{(2x-3)^2}$$

(ii) $$\frac{x+2}{(x-1)(2x-1)^3} = \frac{A}{x-1} + \frac{B}{(2x+1)} + \frac{C}{(2x-1)^2} + \frac{D}{(2x+1)^3}$$

Rule 3

Corresponding to every non-repeated irreducible quadratic factor $ax^{2\,+}\,bx + c$ of the denominator Q(x)there exists a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.

$(ax^2 + bx + c)$ is said to be an irreducible quadratic factor, if it cannot be factorized into two linear factors with real coefficients.

Example

(i) $$\frac{x^2+1}{(x^2+4)(x^2+9)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+9}$$

(ii) $$\frac{8x^3-5x^2+2x+4}{(2x-1)^2(3x^2+4)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{Cx+D}{3x^2+4}$$

In the case of an improper fraction, by division, it can be expressed as the sum of integral function and a proper fraction and then proper fraction is resolved into partial fractions.

Inverse Laplace Transform using Partial Fractions

1. Find $L^{-1}\left(\dfrac{1}{(s+1)(s+3)}\right)$

Solution:

Let *F(s)* $= \left(\dfrac{1}{(s+1)(s+3)}\right)$

Let us split $F(S)$ into partial fractions,

$$\frac{1}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$1 = A(S+3) + B(S+1)$$

Putting $S = -1$ Putting $S = -3$

$A = \frac{1}{2}$ $B = -\frac{1}{2}$

$$\therefore \frac{1}{(s+1)(s+3)} = \frac{\frac{1}{2}}{(s+1)} + \frac{-\frac{1}{2}}{(s+3)}$$

$$\therefore \left(\frac{1}{(s+1)(s+3)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

$$= \frac{1}{2}\left(e^{-t} - e^{-3t}\right)$$

2. Find $L^{-1}\left(\dfrac{s^2+s-2}{s(s+3)(s-2)}\right)$

Solution:

Consider, $\dfrac{s^2+s-2}{s(s+3)(s-2)} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{s-2}$

$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A(s+3)(s-2)+Bs(s-2)+Cs(s+3)}{s(s+3)(s-2)}$$

$$s^2+s-2=A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$$

put s = -3 put s = 2 put s = 0

9 – 3 – 2 -= B(-3)(5) 4 + 2 – 2 -= C(2)(5) -2 = A(3)(-2)

4=15B 4 = 10C $A = \frac{1}{3}$

$B = \frac{4}{15}$ $\therefore C=\frac{4}{10}$ $C = \frac{2}{5}$

$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{1}{3}.\frac{1}{s}+\frac{4}{15}.\frac{1}{s+3}+\frac{2}{5}.\frac{1}{s-2}$$

$$\therefore L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)+\frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$=\frac{1}{3}(1)+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

3. Find $L^{-1}\left(\frac{s}{s^2+5s+6}\right)$

Solution:

Consider, $\frac{s}{s^2+5s+6}=\frac{s}{(s+2)(s+3)}=\frac{A}{(s+2)}+\frac{B}{(s+3)}$

S =A (s + 3) + B(s + 2)

Put s = -3 Put s = -2

-3 = A(0) + b(-1) -2 = A(1) + B(0)

-3 = -B A = -2

B = 3

$$\frac{s}{(s+2)(s+3)}=\frac{-2}{(s+2)}+\frac{3}{(s+3)}$$

$$\therefore L^{-1}\left(\frac{1}{(s+2)(s+3)}\right) = 2L^{-1}\left(\frac{1}{(s+2)}\right) + 3L^{-1}\left(\frac{B}{(s+3)}\right)$$

$$= -2e^{-2t} + 3e^{-3t}$$

4. Find $L^{-1}\left(\frac{s}{(s+1)^2}\right)$

Solution:

Consider, $\frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$

$$\frac{s}{(s+1)^2} = \frac{A(s+1)+B}{(s+1)^2}$$

$s = A(s + 1) + B$

Put $s = -1$

$B = -1$

Put $s = 0$

$0 = A + B$

$0 = A - 1$

$A = 1$

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$L^{-1}\left(\frac{s}{(s+1)^2}\right) = L^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$= L^{-1}\left(\frac{1}{(s+1)}\right) - L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= e^{-t} - e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t} - e^{-t}(t) = e^{-t}(1-t)$$

5. Find $L^{-1}\left(\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right)$

Solution:

$$\frac{5s^2-15S-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

put s = -1	Put s = 2	Equating the	Equating the
-27A = 9	3D = -21	coefficient of s^3	constant coefficient
$A = \frac{-9}{27}$	D = -7	$A + B = 0$	$-8A + 4B - 2C + D = -11$
$A = \frac{-1}{3}$		$B = \frac{1}{3}$	$\frac{8}{3}+\frac{4}{3}-2C-7=-11$
			-2C = -8
			C = 4

$$\therefore \frac{5s^2-15S-11}{(s+1)(s-2)^2}=\frac{\frac{-1}{3}}{s+1}+\frac{\frac{1}{3}}{s-2}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left(\frac{5s^2-15S-11}{(s+1)(s-2)^2}\right)=\frac{-1}{3}L^{-1}\left(\frac{1}{s+1}\right)+\frac{-1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$+4L^{-1}\left(\frac{1}{(s-2)^2}\right)-7L^{-1}\left(\frac{1}{(s-2)^3}\right)$$

$$=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left(\frac{1}{s^2}\right)-7e^{2t}L^{-1}\left(\frac{1}{s^3}\right)$$

$$=\frac{-1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}.t-\frac{7}{2}e^{2t}t^2$$

6. Find $L^{-1}\left(\frac{2s^2+5s+2}{(s-3)^4}\right)$

Solution:

To resolve $\frac{2s^2+5s+2}{(s-3)^4}$ into partial fraction we substitute $s - 3 = y$ (or) $s = y + 3$

$$\therefore \frac{2s^2+5s+2}{(s-3)^4} = \frac{2(y+3)^2+5(y+3)+2}{y^4}$$

$$= \frac{2(y^2+6y+9)+5y+15+2}{y^4}$$

$$= \frac{2y^2+17y+35}{y^4}$$

$$= \frac{2}{y^2}+\frac{17}{y^3}+\frac{35}{y^4}$$

$$\frac{2s^2+5s+2}{(s-3)^4} = \frac{2}{(s-3)^2}+\frac{17}{(s-3)^3}+\frac{35}{(s-3)^4}$$

$$\therefore L^{-1}\left(\frac{2s^2+5s+2}{(s-3)^4}\right) = 2L^{-1}\left(\frac{1}{(s-3)^2}\right)+17L^{-1}\left(\frac{1}{(s-3)^3}\right)+35L^{-1}\left(\frac{1}{(s-3)^4}\right)$$

$$= 2e^{3t}L^{-1}\left(\frac{1!}{s^2}\right)+\frac{17}{2}e^{3t}L^{-1}\left(\frac{2!}{s^3}\right)+\frac{35}{6}e^{3t}L^{-1}\left(\frac{3}{s^4}\right)$$

$$= 2e^{3t}.t+\frac{17}{2}e^{3t}t^2+\frac{35}{6}t^3e^{3t}$$

7. Find $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s+b^2)}\right)$

Solution:

$$\frac{s^2}{(s^2+a^2)(s+b^2)} = \frac{A}{(s^2+a^2)}+\frac{B}{(s^2+b^2)}$$

$$s^2 = A(s^2+b^2)+B(s^2+a^2)$$

Put $s^2 = -a^2$, $\quad -a^2 = A(-a^2+b^2)$

$$A = \frac{-a^2}{b^2-a^2} = \frac{a^2}{a^2-b^2}$$

Put $s^2 = -b^2$, $\quad -b^2 = B(-b^2 + a^2)$

$$B = \frac{-b^2}{a^2 - b^2}$$

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{\frac{a^2}{a^2-b^2}}{(s^2+a^2)} + \frac{\frac{-b^2}{a^2-b^2}}{(s^2+b^2)}$$

$$= \frac{1}{a^2-b^2}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right)$$

$$L^{-1}\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}L^{-1}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right)$$

$$= \frac{1}{a^2-b^2}\left(L^{-1}\left(\frac{a^2}{s^2+a^2}\right) - L^{-1}\frac{b^2}{s^2+b^2}\right)$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt)$$

8. Find $L^{-1}\left(\frac{1-s}{(s+1)^2(s^2+4s+13)}\right)$

Solution:

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{S+1} + \frac{Bs+C}{s^2+4s+13}$$

$$1-s = A(s^2+4s+13) + (Bs+C)(s+1)$$

Putting s = -1 | Equating coefficient of s^2 | Equating constant coefficient

$2=10A$ | $A + B = 0$ | $13A + C = 1$

$A = \frac{1}{5}$ | $A = \frac{-1}{5}$ | $C = 1 - \frac{13}{5}$

$C = \frac{-8}{5}$

(ie), $\dfrac{1-s}{(s+1)(s^2+4s+13)} = \dfrac{\frac{1}{5}}{s+1}+\dfrac{\frac{-1}{5}s-\frac{8}{5}}{s^2+4s+13}$

$$L^{-1}\left(\frac{1-s}{(s+1)(s^2+4s+13)}\right) = \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)-\frac{1}{5}L^{-1}\left(\frac{s+8}{s^2+4s+13}\right)$$

$$= \frac{1}{5}e^{-t}-\frac{1}{5}L^{-1}\left(\frac{s+2+6}{(s+2)^2+9}\right)$$

$$= \frac{1}{5}e^{-t}-\frac{1}{5}L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right)-\frac{1}{5}L^{-1}\left(\frac{6}{(s+2)^2+3^2}\right)$$

$$= \frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 3t=\frac{6}{5}e^{-2t}\frac{\sin 3t}{3}$$

$$= \frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 3t=\frac{2}{5}e^{-2t}\sin 3t$$

9. Find $L^{-1}\left(\dfrac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right)$

Solution:

$$\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{A}{s+1}+\frac{Bs+C}{s^2-3s+2}$$

$$4s^2-3s+5 = A(s^2-3s+2)+(Bs+C)(s+1)$$

Putting	$s = -1$	Equating coefficient s^2	Equating constant coefficients
	$6A = 12$	$4 = A + B$	$5 = 2A + C$
	$A = 2$	$B = 2$	$C = 5 - 2A$
			$C = 1$

$$\therefore \frac{4s^2-3s+5}{(s+1)(s^2-3s+2)} = \frac{2}{s+1}+\frac{2s+1}{s^2 3s+2}$$

$$L^{-1}\left(\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right) = L^{-1}\left(\frac{2}{s+1}\right)L^{-1}\left(\frac{2s+1}{s^2-3s+2}\right)$$

$$= \quad 2L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{2s+1}{(s-3/2)^2 - 1/4}\right)$$

$$= \quad 2e^{-t} + 2L^{-1}\frac{s+1/2}{\left(s-3/2\right)^2 - 1/4}$$

$$= \quad 2e^{-t} + 2L^{-1}\left(\frac{s+1/2-2+2}{\left(s-3/2\right)^2 - 1/4}\right)$$

$$= \quad 2e^{-t} + 2L^{-1}\left(\frac{s+3/2}{\left(s-3/2\right)^2 - 1/4}\right) + 4L^{-1}\left(\frac{1}{\left(s-3/2\right)^2 - 1/4}\right)$$

$$= \quad 2e^{-t} + 2e^{\left(\frac{3}{2}\right)}L^{-1}\left(\frac{s}{s^2 - \left(1/2\right)^2}\right) + 4e^{\left(\frac{3}{2}\right)t}\sin\, h\left(\frac{t}{2}\right).2$$

$$= \quad 2e^{-t} + 2e^{\left(\frac{3}{2}\right)}\cosh\left(\frac{t}{2}\right) + 8e^{\left(\frac{3}{2}\right)t}\sin\, h\left(\frac{t}{2}\right)$$

Exercise - 1 (c)

Find the inverse Laplace transform of the following by Partial fraction method.

1. $\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$ Ans: $-3e^{-3t} + 2e^{4t} + e^{\left(1/5\right)t}$

2. $\dfrac{2-5s}{(s-6)(s^2+11)}$ Ans: $\dfrac{1}{45}\left(-28e^{-6t} + 28\cos\cos\sqrt{11}t - \dfrac{67}{\sqrt{11}}\sin\sqrt{11}r\right)$

3. $\dfrac{25}{s^3(s^2+4s+5)}$ Ans: $\dfrac{1}{5}\left(11 - 20t\dfrac{25}{2}t^2 - 11e^{-2t}\cos t - 2e^{-2t}\sin t\right)$

4. $\dfrac{1}{(s+1)(s^2+2s+2)}$ Ans: $e^{-1}(1-\cos t)$

5. $\dfrac{1}{(s-1)(s+3)}$ Ans: $\dfrac{1}{4}(e^t - e^{-3t})$

6. $\dfrac{1}{(s+1)(s^2+1)}$ Ans: $\dfrac{1}{2}(\sin t - \cos t + e^{-t})$

7. $\dfrac{1}{(P+2)^2(P-2)}$ Ans: $\dfrac{1}{16}(e^{2t} - (4t+1)e^{-2t})$

8. $\dfrac{1}{s(s+1)^3}$ Ans: $1 - e^{-t} - \left(\dfrac{t^2}{2} + t + 1\right)$

9. $\dfrac{3s+1}{(s-2)(s^2+1)}$ Ans: $\dfrac{1}{5}(7e^{2t} - 7\cos t + \sin t)$

10. $\dfrac{1}{(s+1)^2(s^2+4)}$ Ans: $\dfrac{e^{-t}}{50}(te^{-t} - 3\sin 2t - 4\cos 2t)$

11. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ Ans: $\dfrac{1}{2}e^{t} - e^{2t} + \dfrac{5}{2}e^{3t}$

12. $\dfrac{19s+37}{(s+1)(s-2)(s+3)}$ Ans: $5e^{2t} - 3e^{-t} - 2e^{-3t}$

13. $\dfrac{1}{s^2(s^2+1)}$ Ans: $t - \sin t$

14. $\dfrac{1}{s^2(s^2+1)(s^2+9)}$ Ans: $\dfrac{t}{9} - \dfrac{\sin t}{8} + \dfrac{1}{72}\left(\dfrac{\sin 3t}{3}\right)$

15. $\dfrac{2s^2+5s+4}{s^3+s^2-2s}$ Ans: $2 + e^{t} - e^{2t}$

25. Convolution of two functions

If $f(t)$ and $g(t)$ are given functions, then the convolution of $f(t)$ and $g(t)$ is defined as $\int_0^t f(u)g(t-u)du.$ It is denoted by $f(t) * g(t)$.

25.1 Convolution Theorem

If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$, then $L(f(t) * g(t)) = L(f(t))L(g(t))$

(ie) $L(f(t) * g(t)) = F(s).\ G(s)$

where $F(s) = L(f(t)),\ G(s) = L(g(t))$

Proof:

By definition of Laplace Transform,

We have $L(f(t))*g(t) = \int_0^\infty \{e^{-st} f(t)*g(t)\}dt$

$$= \int_0^\infty e^{-st}\left\{\int_0^t f(u)(t-u)du\right\}dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)dudt$$

on changing the order of integration,

$$= \int_0^\infty f(u)\left\{\int_u^\infty e^{-st}g(t-u)du\right\}dt$$

Put $t-u = v$ When $t = u, v = 0$

$dt = dv$ When $t = \infty, v = \infty$

$$L(f(t))*g(t) = \int_0^\infty f(u)\left\{\int_0^\infty e^{-s(u+v)}g(v)dv\right\}du$$

$$= \int_0^\infty f(u)e^{-su}\left\{\int_0^\infty e^{sv}g(v)dv\right\}du$$

$$= \int_0^\infty e^{su} f(u)du \int_0^\infty e^{-sv}g(v)dv$$

$$= \int_0^\infty e^{-st} f(t)dt \int_0^\infty e^{-st}g(t)dv$$

$$= L(f)(t))L(g(t))$$

$$\therefore L(f(t))*g(t) = F(s).G(s)$$

Corollary

Using the above theorem

We get,

$$L^{-1}(F(s).G(s)) = f(t)*g(t)$$

$$= L^{-1}(F(s)*L^{-1}(G(s))$$

Note

$$f(t) * g(t) \quad = \quad g(t) * f(t)$$

1. Find the value of $1*e^{-t}$

Solution:

Let $f(t) = 1, g(t) = e^{-t}$

$f(u) = 1, g(t-u) = e^{-(t-u)}$

$= e^{-t}e^{u}$

By definition, $f(t) * g(t) = \int_0^t f(u)g)t-u)du$

$$1*e^{t} = \int_0^r 1e^{-t}e^{u}du$$

$$= e^{-t}(e^{u})_0^t$$

$$= e^{-t}(e^{t}-1)$$

$$= 1-e^{-t}$$

2. Evaluate $1*\sin t$

Solution:

Let $f(t) = \sin t \quad g(t) = 1$

$f(t) = \sin u \quad g(t-u) = 1$

By definition, $f(t) * g(t) = \int_0^t f(u)g)t-u)du$

$$t*e^{t} = \int_0^r \sin u 1.du$$

$$= (\cos u)_0^t$$

$$= (\cos t - 1)$$

$$= 1-\cos t$$

3. Evaluate $e^t * \cos t$

Solution:

Let $f(t) = cos\ t$ $\quad g(t) = e^t$

$f(t) = cos\ u$ $\quad g(t\text{-}u) = e\text{-}u^{t-u}$

$= e^t.e^{-u}$

$$f(t)*g(t) = \int_0^t f(u)g)t-u)du$$

$$e^t * cos\ t = \int_0^t \cos u e^t e^{-u} du$$

$$e^t * cos\ t = e^t \int_0^t e^{-u} \cos du$$

$$= e^t \left(\frac{e^{-u}}{(-1)^2 + 1^2} (-\cos u + \sin u) \right)_0^t$$

$$\left[\therefore \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right]$$

$$= e^t \left[\frac{e^{-t}}{2} (-\cos t + \sin t) - \frac{1}{2}(-1) \right]$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2} e^t$$

$$= \frac{1}{2}(\sin t - \cos t + e^t)$$

4. Use convolution theorem to find $L^{-1}\left(\frac{1}{(s+a)(s+b)} \right)$

Solution:

$$L^{-1}\left(\frac{1}{(s+a)(s+b)} \right) = L^{-1}\left(\frac{1}{(s+a)} \right) * L^{-1}\left(\frac{1}{(s+b)} \right)$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)} du$$

$$= \int_0^t e^{-au} e^{-bt+bu} du$$

$$= e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)} \right]_0^t$$

$$= \frac{e^{-bt}}{-(a-b)} (e^{-(a-b)t} - 1)$$

$$= \frac{e^{-bt}}{-(a-b)} + \frac{e^{-bt}}{(a-b)}$$

$$= \frac{1}{(a-b)} (e^{-bt} e^{-at})$$

5. Use convolution theorem to find $L^{-1} \frac{1}{s(s^2+1)}$

Solution:

$$L^{-1} \frac{1}{s(s^2+1)} = L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 1 * \sin t$$

$$= \int_0^t \sin(t-u) du$$

$$= \left[\frac{-\cos(t-u)}{-1} \right]_0^t$$

$$= \cos 0 - \cos t$$

$$= 1 - \cos t$$

6. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$ using convolution theorem

Solution:

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right)$$

$$= L^{-1}\left(\frac{s}{(s^2+a^2)}\right) * L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right)$$

$$= \cos at * \frac{1}{a}\sin at$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)du$$

$$= \frac{1}{a}\int_0^t \left(\frac{\sin a(t-u+u)+\sin a(t-u-u)}{2}\right)du$$

$$= \frac{1}{2a}\int_0^t (\sin at + \sin a(t-du))du$$

$$= \frac{1}{2a}\left[u\sin at + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{t\sin at}{2a}$$

7. Find $L^{-1}\left(\frac{1}{s(s^2-a^2)}\right)$ using convolution theorem.

Solution:

$$L^{-1}\left(\frac{1}{s(s^2-a^2)}\right) = L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2-a^2}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2-a^2}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) * \frac{1}{a}L^{-1}\left(\frac{1}{s^2-a^2}\right)$$

$$= 1 * \frac{1}{a}\sin hat$$

Let $f(t) = \sin hat$; $g(t) = 1$

$f(u) = \sin hau$; $g(t-u) = 1$

$$1 * \frac{1}{a}\sin hat = \frac{1}{a}\int_0^t \sin hau.1du$$

$$= \frac{1}{a}\left(\frac{\cos hau}{a}\right)_0^t$$

$$= \frac{1}{a^2}(\cosh at - 1)$$

$$\therefore L^{-1}\left(\frac{1}{s(s^2 - a^2)}\right) = \frac{1}{a^2}(\cos hat - 1)$$

8. Find $L^{-1}\left(\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right)$ using convolution theorem.

Solution:

$$L^{-1}\left(\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right) = L^{-1}\left(\frac{s}{(s^2 + a^2)} \cdot \frac{s}{s^2 + b^2}\right)$$

$$= L^{-1}\left(\frac{s}{(s^2 + a^2)}\right) * L^{-1}\left(\frac{s}{s^2 + b^2}\right)$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au.\cos b(t-u)du$$

$$= \int_0^t \left(\frac{\cos(au + bt - bu) + \cos(au - bt + bu)}{2}\right)du$$

$$= \frac{1}{2}\int_0^t \left(\cos((a-b)u + bt)) + \cos((a+b)u - bt)\right)du$$

$$= \frac{1}{2}\int_0^t \left[\frac{\sin(bt + (a-b)u}{a-b} + \frac{\sin(a+b)u - bt}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(bt+at-bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

9. Using convolution theorem find $L^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right)$

Solution:

$$L^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right) = L^{-1}\left(\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+a^2}\right) * L^{-1}\left(\frac{1}{s^2+b^2}\right)$$

$$= \frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right) * \frac{1}{b}L^{-1}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{1}{a}\sin at * \frac{1}{b}\sin bt$$

Let $f(t) = \frac{1}{a}\sin at;$ $\quad g(t) = \frac{1}{b}\sin bt$

$$f(u) = \frac{1}{a}\sin au; \quad g(t-u) = \frac{1}{b}\sin b(t-u) = \frac{1}{b}\sin(bt-bu)$$

$$\frac{1}{a}\sin at * \frac{1}{b}\sin bt = \int_0^t \frac{1}{a}\sin au\,\frac{1}{b}\sin(bt-bu)du$$

$$= \frac{1}{ab}\int_0^t \sin au\sin(bt-bu)du$$

$$= \frac{1}{2ab}\int_0^t 2\sin au\sin(bt-bu)du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu) - \cos(au+bt-bu))du$$

$$= \frac{1}{2ab}\left[\frac{\sin(au - bt + bu}{a+b} - \frac{\sin(au + bt - bt}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at - bt + bt)}{a+b} - \frac{\sin(at + bt - bt)}{a-b} - \left(\frac{\sin bt}{a+b} - \frac{\sin bt}{a-b}\right)\right]$$

$$= \frac{1}{2ab}\left[\sin at\left(\frac{1}{a+b} - \frac{1}{a-b}\right) + \sin bt\left(\frac{1}{a+b} + \frac{1}{a-b}\right)\right]$$

$$= \frac{1}{2ab}\left[\sin at\left(\frac{-2b}{a^2 - b^2}\right) + \sin bt\left(\frac{2a}{a^2 + b^2}\right)\right]$$

$$= \frac{a\sin bt - b\sin at}{ab(a^2 - b^2)}$$

$$= \frac{1}{2ab}\left[\sin at\left(\frac{-2a}{a^2 + b^2}\right) + \sin bt\left(\frac{2a}{a^2 - b^2}\right)\right]$$

$$= \frac{2[a\sin bt - b\sin at]}{2aba(a^2 - b^2)}$$

$$= \frac{a\sin bt - b\sin at}{ab(a^2 - b^2)}$$

$$\therefore L^{-1}\left(\frac{1}{(s^2 + a^2)(s^2 + b^2)}\right) = \frac{a\sin bt - b\sin at}{ab(a^2 - b^2)}$$

10. Find $L^{-1}\left(\frac{1}{s^2(s+1)}\right)$ using convolution theorem

Solution:

$$L^{-1}\left(\frac{1}{s^2(s+1)}\right) = L^{-1}\left(\frac{1}{s^2}.\frac{1}{s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s^2}\right) * L^{-1}\left(\frac{1}{s+1}\right)$$

$$= t * e^{-t}$$

$$= \int_0^t ue^{-(t-u)}du$$

$$= \int_0^t ue^{-t}e^{u}du$$

$$= e^{-t}\int_0^t ueudu$$

$$= e^{-t}\left[ue^{u} - (1)(e^{u})\right]_0^t$$

$$= e^{-t}\left[(te^{t} - et) - (0-1)\right]$$

$$= e^{-t}\left[(te^{t} - e^{t} + 1\right]$$

$$= t - 1 + e^{-t}$$

Exercise - 1 (l)

Find the inverse Laplace transforms using convolution theorem.

1. $\dfrac{1}{s(s^2+4)^2}$ Ans: $\dfrac{1}{16}(1-\cos 2t - t\sin 2t)$

2. $\dfrac{1}{s(s^2+9)}$ Ans: $\dfrac{1}{6}(1-\cos 3t)$

3. $\dfrac{s^2}{(s^2+4)^2}$ Ans: $\dfrac{1}{2}\left(t\cos 2t + \dfrac{1}{2}\sin 2t\right)$

4. $\dfrac{1}{(s^2+4)(s+2)}$ Ans: $\dfrac{1}{8}(\sin 2t - \cos 2t + e^{-2t})$

5. $\dfrac{1}{s^2(s^2+a^2)}$ Ans: $\dfrac{1}{a^3}(at - \sin at)$

6. $\dfrac{4s^2}{(s^2+a^2)^2}$ Ans: $\dfrac{t}{2}\sin t$

7. $\dfrac{1}{(s^2-a^2)^2}$ Ans: $\dfrac{1}{2a^3}(at\cos hat - \sin hat)$

8. $\dfrac{1}{(s^2+4)^2}$ Ans: $\dfrac{1}{s}\left(\dfrac{\sin 2t}{2} - t\cos 2t\right)$