UNIT – II

VECTOR CALCULUS

Scalars

The quantities which have only magnitude and are not related to any direction in space are called scalars. Examples of scalars are (i) mass of a particle (ii) pressure in the atmosphere (iii) Temperature of a heated body (iv) speed of a Train.

Vectors

The quantities which have both magnitude and direction are called Vectors.

Examples of vectors are (i) The gravitational force on a particle in space (ii) The velocity at any point in a moving fluid.

Representation and notation of a Vector

A vector is often denoted by two letters with an arrow over them ie., $\overrightarrow{AB}$, A is called the origin (initial point) and B is the terminus (end point). Its magnitude is given by the length AB and direction is from A to B as indicated by the arrow. We write vector quantities also in single letter like $\vec{a}, \vec{b}, \vec{c},$ and the corresponding letters a, b, c denote their magnitudes.

The magnitude $\left|\vec{a}\right|$ of a vector $\vec{a}$ is called its modulus or module.

Collinear or Parallel Vectors

Two or more vectors are said to be collinear or parallel when they act along the same line or along parallel lines.

Coplanar Vectors

Three or more vectors are said to be coplanar when they are parallel to the same plane or lie in the same plane whatever their magnitude may be.

Unit Vectors

A vector whose magnitude is of unit length is called a unit vector. If $\vec{a}$ is a vector whose magnitude is ‘a’ then the unit vector in the direction of $\vec{a}$ is denoted by $\hat{n}$ and is obtained by dividing the vector $\vec{a}$ by its magnitude ‘a’ i.e. $\hat{n} = \frac{\vec{a}}{|\vec{a}|}$

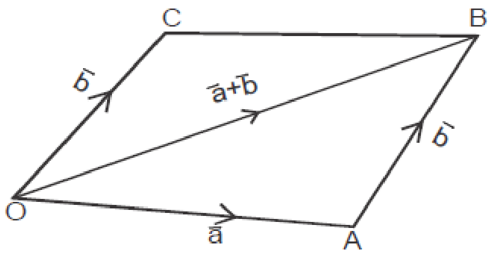
Position Vector

If O be a fixed origin and P any point, then the vector $\overrightarrow{OP}$ is called the position vector of the point P(x,y,z) with respect to the origin O(0,0,0).

Addition of Vectors

Let $\vec{a}$ and $\vec{b}$ be any two vectors. Choose any point O as origin and draw the vectors $\vec{a}$ and $\vec{b}$ so that the terminals of $\vec{a}$ coincides with the origin of $\vec{b}$ ie., $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Then the vector given by $\overrightarrow{OB}$ is defined as the sum of vectors $\vec{a}$ and $\vec{b}$.

The above law is called triangle law of addition.



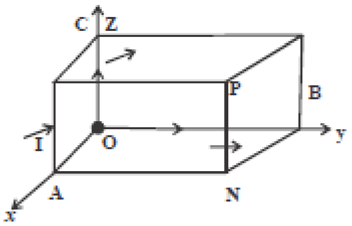
The Unit Vectors $\vec{i}, \vec{j}, \vec{k}$ (Orthonormal system of unit vectors)

Let OX, OY and OZ be three mutually perpendicular straight lines in the right handed orientation. These three mutually perpendicular lines can uniquely determine the position of a point. Hence these lines can be taken as the co-ordinate axes with O as origin. The planes XOY, YOZ and ZOX are called co-ordinate planes.

Let $\overrightarrow{OP}$ represents a vector $\vec{r}$. With OP as diagonal construct a rectangular paralleloiped whose three coterminous edges OA, OB, OC lie along OX, OY and OZ respectively. Let OA = x, OB = y, OC = z. Then $\overrightarrow{OA} = x\vec{i}$, $\overrightarrow{OB} = y\vec{j}$ and $\overrightarrow{OC} = z\vec{k}$. Now, we have

$$\vec{r} = \overrightarrow{op} = \overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OA} + \overrightarrow{AN} + \overrightarrow{NP}$$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\vec{i} + y\vec{j} + z\vec{k}$$



Thus $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Here x, y, z are called the co-ordinate of the point P referred to the axes OX, OY and OZ. Also $x\vec{i}, y\vec{j}$ and $z\vec{k}$ are called resolved parts of the vector $\vec{r}$ in the direction of $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively. The modulus of $\vec{r}$ is given by $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

Scalar Product or dot Product

Let a $\vec{a}$ and $\vec{b}$ be two vectors. The scalar product or dot product of $\vec{a}$ and $\vec{b}$ is defined to be $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}| \times \cos\theta$, where θ is the angle between the two vectors when drawn from a common origin.

Note:

(i) $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

(ii) $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$

(iii) $\vec{a}\,.\,\vec{b} = 0$ *if* $\vec{a}$ *and* $\vec{b}$ are perpendicular vectors.

(iv) $\vec{a}.\left(\vec{b} + \vec{c}\right) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$

(v) $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1.$

(vi) $\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0.$

since $\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular vectors.

(vii) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector product or Cross product

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then the vector product or cross product of $\vec{a}$ and $\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ with magnitude ab sin θ. Here $0 \le \theta \le \pi$ is the angle between $\vec{a}$ and $\vec{b}$. The direction is along a unit vector $\hat{n}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ form right handed system. Thus $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$.

Note

(i) $\left|\vec{a}\times\vec{b}\right|$ = area of the parallelogram with sides a and b

(ii) $\vec{a}\times\vec{b} = -\left(\vec{b}\times\vec{a}\right)$

(iii) $\vec{a}\times\vec{b} = 0$ if $\vec{a}$ and $\vec{b}$ are parallel.

(iv) $\vec{a}\times\left(\vec{b}+\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)$

(v) $\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$ ($\therefore$ Parallel Vector)

(vi) $\vec{i}\times\vec{j} = \vec{k}, \vec{j}\times\vec{k} = \vec{i}, \vec{k}\times\vec{i} = \vec{j}$

(vii) $\vec{j}\times\vec{i} = -\vec{k}, \vec{k}\times\vec{j} = -\vec{i}, \vec{i}\times\vec{k} = -\vec{j}$

(viii) $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $b = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$,

then $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Definition

The scalar triple product or box product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined to be the scalar $\vec{a}.\left(\vec{b}\times\vec{c}\right)$. It is usually denoted by $\left\lfloor\vec{a},\vec{b},\vec{c}\right\rfloor$.

It can be easily verified that

$$\vec{a}.\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}, \vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$ and $c = c_1\vec{i}+c_2\vec{j}+c_3\vec{k}$

Note

(i) $\vec{a}.\left(\vec{b}\times\vec{c}\right)$ represents the volume of the parallelepiped formed by the co-terminus edges $\vec{a},\vec{b}$ and $\vec{c}$.

(ii) $[\vec{a},\vec{b},\vec{c}]=[\vec{b},\vec{c},\vec{a}]=[\vec{c},\vec{a},\vec{b}]$

(iii) $[\vec{a},\vec{b},\vec{c}]=-[\vec{b},\vec{a},\vec{c}]=-[\vec{c},\vec{b},\vec{a}]=-[\vec{a},\vec{c},\vec{b}]$

(iv) The vector $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar if and only if $[\vec{a},\vec{b},\vec{c}]=0$.

Results

(i) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

(ii) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$

(iii) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$

(iv) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{b},\vec{d}]\vec{c}-(\vec{a},\vec{b}.\vec{c})\vec{d}$

In this chapter, we introduce a vector differential operator which is used to obtain gradient of a scalar valued function, divergence and curl of a vector valued function and discuss briefly the properties arising out of these concepts. We see the general rules for differentation of a vector functions.

Rules

If $\vec{a},\vec{b}$ are vector functions of a scalar 't' and 'ϕ' is a scalar function of 't', then

(i) $\dfrac{d}{dt}(\vec{a}\pm\vec{b})=\dfrac{d\vec{a}}{dt}+\dfrac{d\bar{b}}{dt}$

(ii) $\dfrac{d}{dt}(\vec{a}.\vec{b})=\vec{a}.\dfrac{d\vec{b}}{dt}+\dfrac{d\vec{a}}{dt}.\vec{b}$

(iii) $\dfrac{d}{dt}(\vec{a}\times\vec{b})=\vec{a}\times\dfrac{d\vec{b}}{dt}+\dfrac{d\vec{a}}{dt}\times\vec{b}$

(iv) $\dfrac{d}{dt}(\phi\vec{a})=\phi\dfrac{d\vec{a}}{dt}+\dfrac{d\phi}{dt}\vec{a}$

Scalar Point function

If to each point P (x,y,z) of a region R in space there corresponds a unique scalar f(P) then f is called a scalar point function.

Example

The temperature distribution in a heated body, density of a body and potential due to a gravity.

Vector point function

If to each point P (x,y,z) of a region R in space there corresponds a unique vector $\vec{f}$ (P), then $\vec{f}$ is called a vector point function.

Example

The velocity of a moving fluid, Gravitational force.

Vector differential operator (∇)

The vector differential operator Del, denoted by ∇is defined as

$$\nabla = \vec{i}\,\frac{\delta}{\delta x} + \vec{j}\,\frac{\delta}{\delta y} + \vec{k}\,\frac{\delta}{\delta z}$$

Level Surface

Let the surface ϕ(x,y,z) = c passes through a point P. If the value of the function at each point on the surface is the same as at P, then such a surface is called a level surface through P.

Example

ϕ (x,y,z) represents potential at the point P, then equipotential surface ϕ(x,y,z) =c is a level surface.

Gradient of a scalar point function

Let ϕ(x,y,z) be a scalar point function defined in a region R of space. Then the vector point function given by

$$\nabla\phi = \left(\vec{i}\,\frac{\delta}{\delta x} + \vec{j}\,\frac{\delta}{\delta y} + \vec{k}\,\frac{\delta}{\delta z}\right)\phi$$

$= \vec{i}\,\frac{\delta\phi}{\delta x} + \vec{j}\,\frac{\delta\phi}{\delta y} + \vec{k}\,\frac{\delta\phi}{\delta z}$ is defined as the gradient of $\phi$and denoted as grad ϕ.

Directional Derivative : (D.D)

The directional derivative of a scalar point function ϕ at point (x,y,z) in the direction of a vector $\vec{a}$ is given by

$$D.D = \nabla\phi.\frac{a}{|\vec{a}|}$$

Definition

The unit vector normal to the surface ϕ (x,y,z) = c is given by $\frac{\nabla\phi}{|\nabla\phi|}.\hat{n}$

Definition

The Directional derivative at a point is maximum in the direction of the normal to the level surface at P and its magnitude is $|\nabla\phi|$ (ie) maximum of $D.D = |\nabla\phi|$

Definition

Angle between the normal to surface is given by $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Example: 1

If $\phi(x,y,z) = x^2y - 2y^2z^3$ find $\nabla\phi$ at the point (1, -1, 2)

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy, \frac{\partial\phi}{\partial y} = x^2 - 4yz^3, \frac{\partial\phi}{\partial z} = -6y^2z^2$$

$$\therefore \nabla\phi = 2xy\vec{i} + (x^2 - 4yz^3)\vec{j} - 6y^2z^2\vec{k}$$

$$\nabla\phi_{(1,-1,2)} = 2(1)(-1)\vec{i} + (1 - 4(-1)(2)^3\vec{j} - 6 - (-1)^2(2)^2\vec{k}$$

$$= 2\vec{i} + 33\vec{j} - 24\vec{k}$$

Example: 2

If $\phi(x,y,z) = x^2yz^3$ find $\nabla\phi$ at the point (1, 1, 1)

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3, \frac{\partial\phi}{\partial y} = x^2z^3, \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\therefore \nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} - 3x^2yz^2\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 2\vec{i} + \vec{j} + 3\vec{k}$$

Example: 3

Find the unit vector normal to the surface $x^2y + 2xz^2 = 8$ at (1,0,2).

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy + 2z^2, \quad \frac{\partial\phi}{\partial y} = x^2, \quad \frac{\partial\phi}{\partial z} = 4xz$$

$$\nabla\phi = (2xy + 2z^2)\vec{i} + x^2\vec{j} + 4xyz\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\phi| = \sqrt{8^2 + 1^2 + 8^2} = \sqrt{129}$$

Since unit vector normal to the surface is $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\left|\sqrt{129}\right|}$$

Example: 4

Find the unit vector normal to the surface $z = x^2 + y^2$ at the point (-1,-2,-5).

Solution:

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Given $z = x^2 + y^2 \Rightarrow \phi(x,y,z) = x^2 + y^2 - z$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} - 1\vec{k}$$

$$\nabla\phi_{(-1,-2,-5)} = 2(-1)\vec{i} + 2(-2)\vec{j} - \vec{k}$$

$$= 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2 + (-4)^2 + (-1)^2} = \sqrt{21}$$

$\therefore$ Unit vector normal to the surface is $\hat{n} = \dfrac{-2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$

Example: 5

Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2,-1,2).

Solution:

Given the surface $\phi_1(x,y,z) = x^2 + y^2 + z^2 - 9$

$$\nabla\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi_1}{\partial z}$$

$$\frac{\partial\phi_1}{\partial x} = 2x, \quad \frac{\partial\phi_1}{\partial y} = 2y, \quad \frac{\partial\phi_1}{\partial z} = 2z$$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(2,-1,2)} = 2(2)\vec{i} + 2(-1)\vec{j} + 2(2)\vec{k}$$

$$= 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{4^2 + (-2)^2 + 4^2} = \sqrt{36} = 6$$

Given the surface $\phi_2(x,y,z) = x^2 + y^2 - z^2 - 9$

$$\frac{\partial\phi_2}{\partial x} = 2x, \quad \frac{\partial\phi_2}{\partial y} = 2y, \quad \frac{\partial\phi_2}{\partial z} = -1$$

$$\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_{2(2,-1,2)} = 2(2)\vec{i} + 2(-1)\vec{j} - \vec{k} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_2| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$$

Since the angle between the surfaces

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}).(4\vec{i} - 2\vec{j} - \vec{k})}{6 \times 3}$$

$$= \frac{16 + 4 - 4}{18} = \frac{16}{18}$$

$$\theta = \cos^{-1}\left(\frac{8}{9}\right)$$

Example 6

Find the angle between the normal to the surface $xy - z^2 = 0$ at the points (1,4,-2) and (-3,-3,3).

Solution:

Given the surface $\phi(x,y,z) = xy - z^2$

$$\nabla\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi_1}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x, \quad \frac{\partial\phi}{\partial z} = -2z$$

$$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$\nabla\phi_1 = \nabla\phi_{(1,4,-2)} = 4\vec{i} + \vec{j} - 2(-2)\vec{k}$$

$$\nabla\phi_1 = 4\vec{i} + \vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$$

$$\nabla\phi_2 = \nabla\phi_{(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 2(3)\vec{k}$$

$$\nabla\phi_2 - 3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$|\nabla\phi_2| = \sqrt{(-3)^2 + (-3)^2 + (-6)^2} = \sqrt{54}$$

Since the angle between the normal is

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}).(-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33}\sqrt{54}}$$

$$\cos\theta = \frac{-12 - 3 - 24}{9\sqrt{22}} \quad \Rightarrow \cos\theta = \frac{-39}{9\sqrt{22}}$$

$$\Rightarrow \cos\theta = \frac{-13}{3\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{-13}{3\sqrt{22}}\right)$$

$$\Rightarrow \theta = \cos^{-1} = \left(\frac{13}{3\sqrt{22}}\right)$$

Example 7

Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point (1,-2,-1) in the directional of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given the surface $\phi(x,y,z) = x^2yz + 4xz^2$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2 z, \quad \frac{\partial\phi}{\partial z} = x^2 y + 8xy$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2 z\vec{j} + (x^2 y + 8xy)\vec{k}$$

$$\nabla\phi_{(1,-2,-1)} = (2(1)(-2)(-1) + 4(-1)^2\vec{i} + 1^2(-1)\vec{j} + [1^2(-2) + 8(1)(-1)]\vec{k}$$

$$= 8\vec{i} - \vec{j} - 10\vec{k}$$

To find the Directional Derivative of ϕ in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$

Find the unit vector along the direction

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k} \Rightarrow |\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

Directional Derivative along the direction $\vec{a}$ at the point (1,-2,-1) = $\nabla\phi.\dfrac{\vec{a}}{|\vec{a}|}$

$$= (8\vec{i} - \vec{j} - 10\vec{k}).\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{16+1+20}{3}$$

$$= \frac{37}{3} \text{ units.}$$

Example 8

Find the directional derivative of $\phi\,(x,y,z) = xy^2+yz^3$ at (1,-1,2) towards the point (2,1,-1).

Solution:

Given the surface $\phi\,(x,y,z) = xy^2+yz^3$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = y^2, \quad \frac{\partial \phi}{\partial y} = 2xy + z^3, \quad \frac{\partial \phi}{\partial z} = 3yz^2$$

$$\nabla \phi = y^2 \vec{i} + (2xy + z^3)\vec{j} + 3yz^2 \vec{k}$$

$$\nabla \phi_{(1,-1,2)} = (-1)^2 \vec{i} + [2(1)(-1) + 2^3]\vec{j} + 3(-1)2^2 \vec{k}$$

$$\nabla \phi = \vec{i} + 6\vec{j} - 12\vec{k}$$

To find the Directional derivative along the point P (1,-1,2) towards the point Q(2,1,-1)

Find the Vector $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= (2\vec{i} + \vec{j} - \vec{k}) - (\vec{i} - \vec{j} + 2\vec{k})$$

$$\overrightarrow{PQ} = \vec{i} + 2\vec{j} - 3\vec{k}$$

$$\overrightarrow{PQ} = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

Directional Derivative = $\nabla \phi . \dfrac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}$

$$= (\vec{i} + 6\vec{j} - 12\vec{k}) . \frac{(\vec{i} + 2\vec{j} - 3\vec{k})}{\sqrt{14}}$$

$$= \frac{1 + 12 + 36}{\sqrt{14}}$$

$$= \frac{49}{\sqrt{14}} \text{ Units.}$$

Example 9

Find the directional derivative of the scalar function $\phi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3,1,3).

Solution:

Given the surface $\phi = (x,y,z) = xyz$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = xz, \quad \frac{\partial\phi}{\partial z} = xy$$

$$\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$\nabla\phi_{(3,1,3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$$

Given the surface $\phi_1\,(x,y,z) = xy - z$

$$\frac{\partial\phi_1}{\partial x} = y, \frac{\partial\phi_1}{\partial y} = x, \frac{\partial\phi_1}{\partial z} = -1$$

The normal to the surface is

$$\nabla\phi_1 = y\vec{i} + x\vec{j} - \vec{k}$$

$$\nabla\phi_{(3,1,3)} = \vec{i} + 3\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}$$

Directional derivative of $\phi(x,y,z)$ along the direction of outward to the surface $\phi_1(x, y, z)$ at (3,1,3) is

Directional Derivative = $\nabla\phi . = \dfrac{\nabla\phi_1}{|\nabla\phi_1|}$

$$= (3\vec{i} + 9\vec{j} + 3\vec{k}).\frac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}$$

$$= \frac{3 + 27 - 3}{\sqrt{11}}$$

$$= \frac{27}{\sqrt{11}} \text{ Units.}$$

Example 10

Find the maximum value of the directional derivative of $\phi = x^3yz$ at the point (1,4,1).

Solution:

Given the surface $\phi_1(x, y, z) = x^3yz$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2yz \quad \frac{\partial\phi}{\partial y} = x^3z \quad \frac{\partial\phi}{\partial z} = x^3y$$

$$\nabla\phi = 3x^2yz\vec{i} + x^3z\vec{j} + x^3y\vec{k}$$

$$\nabla\phi_{(1,4,1)} = 3(1)^2(4)(1)\vec{i} + (1)^3(1)\vec{j} + (1)^3(4)\vec{k}$$

$$\nabla\phi_{(1,4,1)} = 3(1)^2(4)(1)\vec{i} + (1)^3(1)\vec{j} + (1)^3(4)\vec{k}$$

$$\nabla\phi = 12\vec{i} + \vec{j} + 4\vec{k}$$

Since the maximum of the Directional derivative $|\nabla\phi|$

$\therefore$ Maximum Directional Derivative at (1,4,1) = $\sqrt{12^2 + 1^2 + 4^2}$

= $\sqrt{161}$

Example 11

In what direction from the point (1, 1, -2) is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ maximum? Also find the value of the maximum directional derivative.

Solution:

Given the surface $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -4y, \quad \frac{\partial\phi}{\partial z} = 8z$$

$$\nabla\phi = 2x\vec{i} - 4y\vec{j} + 8z\vec{k}$$

$$\nabla\phi_{(1,1,-2)} = 2\vec{i} - 4\vec{j} - 16\vec{k}$$

Maximum of the directional derivative at the point (1,1,-2) = $|\nabla\phi|$

$$= \sqrt{2^2 + (-4)^2 + (-16)^2}$$

$$= \sqrt{4 + 16 + 256}$$

$$= \sqrt{276}$$

Example 12

Find the Directional Derivative of $\phi = xy + yz + zx$ at the point (1,2,3) along the x-axis.

Solution:

Given the surface $\phi(x, y, z) = xy + yz + zx$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y + z, \quad \frac{\partial\phi}{\partial y} = x + z, \quad \frac{\partial\phi}{\partial z} = x + y$$

$$\nabla\phi = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$$

$$\nabla\phi_{(1,2,3)} = (2+3)\vec{i} + (1+3)\vec{j} + (1+2)\vec{k}$$

$$= 5\vec{i} + 4\vec{j} + 3\vec{k}$$

Directional Derivative of ϕ along the direction of x-axis at the point (1,2,3)

$$= \nabla\phi.\frac{\vec{i}}{|\vec{i}|}$$

$$= (5\vec{i} + 4\vec{j} + 3\vec{k}).\vec{i}$$

$$= 5$$

Example 13

If $\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k},$, find the scalar potential ϕ.

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Equating like coefficients we get

$$\frac{\partial\phi}{\partial x} = 2xyz \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = x^2z \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = x^2y \qquad \ldots (3)$$

Partially integrating (1), (2) and (3) with respect to x, y and z respectively, we get

$$\phi = x^2yz + f(y,z) \qquad \ldots (4)$$

$$\phi = x^2zy + f(x,z) \qquad \ldots (5)$$

$$\phi = x^2yz + f(x,y) \qquad \ldots (6)$$

From (4), (5) and (6) we get

$\phi = x^2yz + C$ (union of all the three results)

Example: 14

Find the equations of the tangent plane and normal line to the surface $x^2 + y^2 - z = 0$ at the point (2,-1,5).

Solution:

Given the surface $\phi(x, y, z) = x^2 + y^2 - z$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$\nabla\phi_{(2,-1,5)} = 4\vec{i} - 2\vec{j} - \vec{k}$ which is normal to the surface.

$\therefore$ Direction ratio of the normal to the surface at the point (2,-1,5) are (4,-2,-1)

$\therefore$ Equation of the tangent plane is

$4(x-2) - 2(y+1) - (z-5) = 0$

$4x - 8 - 2y - 2 - z + 5 = 0$

$4x - 2y - z - 5 = 0$

$4x - 2y - z = 5$ which is a tangent plane.

Equation of normal line passing through point (2,-1,5) and having Direction ratio (4,-2,-1) is

$$\frac{x-2}{4} = \frac{y+1}{-2} = \frac{z-5}{-1}$$

Exercise

1. If $\phi(x, y, z) = x^2y + y^2x + z^2$ find $\nabla\phi$ at the point (1,1,1).
2. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find grad ϕ at the point (1,-2,-1).
3. Find the unit vector normal to the surface $x^3+y^3+3xyz = 3$ at the point (1,2,-1).
4. Find the unit vector normal to the surface $x^2y + 2xz = 4$ at the point (2,-2,3)
5. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1,-2,1)
6. Find the angle between the normals to the surface $xy^3z^2=4$ at the point (-1,-1,2) and (4,1,-1)
7. Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at the point (6,4,3)
8. Find the directional derivative of $\phi = x^3+y^3+z^3$ at the point (1,-1,2) in the direction of the vector $\vec{i} + 2\vec{j} + \vec{k}$

9. Find the directional derivative of $\phi = (x, y, z) = \mathbf{x}^2 - 2y^2 + 4z^2$ at the point (1,1,-1) in the direction $2\vec{i} - \vec{j} - \vec{k}$

10. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P(1,2,3) in the direction of the line PQ where Q (5,0,4)

11. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point P (1,-2,-1) along the direction of PQ where Q(3,-3,-2).

12. Find the directional derivative of $\phi = xy^2 + yz^2$ at the point (2,-1,1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at the point (-1,2,1).

13. Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point (1,-1,2) in the direction of the normal to the surface $x^2 + y^2 + z^2 = 9$ at the point (1,2,2).

14. Find the maximum value of the directional derivative of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at the point (1,1,-4).

15. Find the maximum directional derivative of $\phi = x^3y^2z$ at the point (1,1,1).

16. In what direction is the directional derivative of the function $\phi = x^2 - 2y^2 + 4z^2$ from the point (1,1,-1) is maximum and what is its value?

17. Find the direction along which the directional derivative of the function $\phi = xy + 2yz + 3xz$ is greatest at the point (1,1,1). Also find the greatest directional derivative.

18. Find the function ϕ if grad $\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$

19. If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$, find $<j)(x, y, z)$ if $\phi(l, -2, 2) = 4$

20. Find the equation of the tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 25$ at the point (4,0,3)

Answer

1.	$3\vec{i} + 3\vec{j} + 2\vec{k}$	11.	$\dfrac{27}{\sqrt{6}}$
2.	$-(16\vec{i} + 9\vec{j} + 4\vec{k})$	12.	$\dfrac{15}{\sqrt{17}}$

3. $\dfrac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{14}}$

4. $\dfrac{1}{3}(\vec{i}+2\vec{j}+2\vec{k})$

5. $\cos^{-1}\left(\dfrac{3}{7\sqrt{6}}\right)$

6. $\cos^{-1}\left(\dfrac{45}{\sqrt{2299}}\right)$

7. $\cos^{-1}\left(\dfrac{-24}{\sqrt{5246}}\right)$

8. $\dfrac{7\sqrt{6}}{2}$

9. $\dfrac{8}{\sqrt{6}}$

10. $\dfrac{4\sqrt{21}}{3}$

13. -7

14. $\sqrt{1652}$

15. $\sqrt{14}$

16. $2\vec{i}-4\vec{j}-4\vec{k}, \quad 2\sqrt{21}$

17. $4\vec{i}+3\vec{j}+5\vec{k}, \quad 5\sqrt{21}$

18. $\phi(x,y,z)=xy^2-x^2yz^3+3y+\dfrac{3}{2}z^4+C$

19. $\phi(x,y,z)=x^2yz^3+20$

20. $4x+3z=25$

$$\frac{x-4}{4}=\frac{z-3}{3}, y=0$$

Divergence of a vector point function

The divergence of a differentiable vector point function $\vec{F}$ is denoted by div $\vec{F}$

$$\text{div } \vec{F}=\nabla.\vec{F}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\vec{F} \qquad \text{If } \vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$$

$$=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$$

Curl of a vector point function

The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and is defined bycurl

$$\vec{F} = \nabla \times \vec{F} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \right) \times \vec{F}$$

if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Definition

A vector point function $\vec{F}$ is said to be solenoidal if div $\vec{F}$ = 0 and it is said to be irrotational if curl $\vec{F}$ = 0.

Example

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\text{div } \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

i.e., $\nabla . \vec{r} = 3$

$$\text{curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}\,(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

Hence $\nabla \times \vec{r} = 0.$

1. Find the divergence and curl of the vector $\overline{V} = xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point (2,-1,1)

Solution:

Given $\overline{V} = xyz\vec{i} + 3xy^2\vec{j} + (xz^2 - y^2z)\vec{k}$

$$\text{div } \overline{V} = \nabla.\overline{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3xy^2) + \frac{\partial}{\partial z}(xz^2 - y^2 z)$$

$$= yz + 6xy + 2xz - y^2$$

At (2,-1,1), $\nabla.\overline{V} = (-1).1 + 6(2)(-1) + 2(2)(1) - (-1)^2$

$= -1 + 12(-1) + 4 - 1$

$= -1 + 12 + 4 - 1$

$= (-10)$

$$\text{curl } \overline{V} = \nabla \times \overline{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3xy^2 & xz^2 - y^2 z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(3xy^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xz^2 - y^2 z) - \frac{\partial}{\partial z}(xyz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(xy)\right]$$

$$= \vec{i}[-2yz] - \vec{j}[z^2 - xy] + \vec{k}[3y^2 - xy]$$

At (2,-1,1), $\nabla \times \overline{V} = \vec{i}[-2(-1)(1)] - \vec{j}[1 - 2(-1)] + \vec{k}[3(-1)3^2 - 2(1)]$

$$= 2\vec{i} - 3\vec{j} + \vec{k}$$

2. If $\vec{F} = (x^2 - y^2 + 2xy)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, find

$\nabla.\vec{F}, \nabla(\nabla.\vec{F}), \nabla \times \vec{F}, \nabla.(\nabla \times \vec{F})$ and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

$$\vec{F} = (x^2 - y^2 + 2xy)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xy) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$$

$$= (2x + 2z) + (-x + z) + 2z = x + 5z$$

$$\nabla(\nabla.\vec{F}) = \frac{\partial}{\partial x}(x + 5z)\vec{i} + \frac{\partial}{\partial y}(x + 5z)\vec{j} + \frac{\partial}{\partial z}(x + 5z)\vec{k}$$

$$= \vec{i} + 0\vec{j} + 5\vec{k}$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-y^2+2xz & xz-xy+yz & z^2+x^2\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}(z^2+x^2)-\frac{\partial}{\partial z}(xz-xy+yz)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial y}(z^2+x^2)-\frac{\partial}{\partial z}(x^2-y^2+2xz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial y}(xz-xy+yz)-\frac{\partial}{\partial z}(x^2-y^2+2xz)\right]$$

$$=\vec{i}[0-(x+y)]-\vec{j}[2x-2x]+\vec{k}[z-y+2y]$$

$$=-(x+y)\vec{i}+(y+z)\vec{k}$$

$$\nabla.(\nabla\times\vec{F})=\frac{\partial}{\partial x}(-(x+y)]+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}[y+z]$$

$$=1+0+1=0$$

$$\nabla\times(\nabla\times\vec{F})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -(x+y) & 0 & y+z\end{vmatrix}$$

$$=\vec{i}\left[\frac{\partial}{\partial y}(y+z)-0\right]-\vec{j}\left[\frac{\partial}{\partial x}(y+z)-\frac{\partial}{\partial z}(-x+y))\right]+\vec{k}\left[0+\frac{\partial}{\partial x}(x+y)\right]$$

$$=\vec{i}+\vec{k}$$

$$\therefore(\nabla.\vec{F})_{(1,1,1)}=6$$

$$[\nabla(\nabla.\vec{F})_{(1,1,1)}=\vec{i}+5\vec{k}$$

$$(\nabla\times\vec{F})_{(1,1,1)}=2\vec{i}+2\vec{k}$$

$$[\nabla.(\nabla\times\vec{F})_{(1,1,1)}=0$$

$$[\nabla\times(\nabla\times\vec{F})_{(1,1,1)}=\vec{i}+\vec{k}$$

3. Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is both solenoidal and irrotational.

Solution:

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xy + 2z)$$

$$= -2 + 2x - 2x + 2$$

= 0 for all points (x,y,z)

$\therefore \vec{F}$ is solenoidal vector.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xy + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xy - 2xy + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$$= \vec{i}[3x - 3x] - \vec{j}[3y - 2z + 2z - 3y] + \vec{k}[3z + 2y - 2y - 3z]$$

= 0 for all points (x, y, z)

$\therefore \vec{F}$ is an irrotational vectors.

4. Find the constants a, b, c so that

$\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + Cy + 2z)\vec{k}$ is irrotational.

Solution:

Given $\nabla \times \vec{F} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+Cy+2z) \end{vmatrix} = 0$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(4x+Cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(4x+Cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$(C+1)\vec{i} - (4-a)\vec{j}(b-2)\vec{k}$$

$= 0$

i.e. $C + 1 = 0$, $\therefore C = -1$

$4 - a = 0$, $\therefore a = 4$

$b - 2 = 0$, $\therefore b = 4$

$\therefore a = 4$, $b = 2$, $C = -1$.

5. Determine the constant m so that the vector

$\vec{F} = (x+y)\vec{i} + (3x+my)\vec{j} + (x-5z)\vec{k}$ is such that its divergence is zero.

Solution:

div $\vec{F} = 0$

i.e., $\nabla.\vec{F} = 0$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(3x+my) + \frac{\partial}{\partial z}(x-5z)$$

$\Rightarrow 1 + m - 5 = 0$

$\Rightarrow m - 4 = 0$

$\Rightarrow m = 4$

Laplacian operator ∇^2

The operator $\nabla^2$is called the laplacian operator. If ϕ is a scalar function of *x, y, z* then

$$\nabla^2\phi == \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

6. Prove that $\nabla.\nabla\phi = \nabla^2\phi$

Solution:

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

$$\nabla.\nabla\phi = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left(\vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \nabla^2\phi$$

7. Prove that curl (grad ϕ) = 0

Solution:

Curl (grad ϕ) = $\nabla \times (\nabla\phi)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)\right] + \vec{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]$$

8. Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2) + \vec{k}$ is irrotational and fine its scalar potential.

Solution:

$$\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2) + \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2 \cos x + z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2 \cos x + z^3)\right]$$

$$= \vec{i}[0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2z\cos x - 2y\cos x]$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0.$$

Hence $\vec{F}$ is irrotational

$$\vec{F} = \nabla\phi$$

i.e., $(y^2 \cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2)\vec{k}$

$$= \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

Equating the coefficients $= \vec{i}, \vec{j}, \vec{k}$, we get

$$\frac{\partial\phi}{\partial x} = y^2 \cos x + z^3 \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 \qquad \ldots (3)$$

Integrating (1) with respect to 'x' treating 'y' and 'z' as constants we get.

$$\phi = y^2 \sin x + z^3 x + f(y,z) \qquad \ldots (4)$$

Integrating (2) with respect to 'y' treating 'x' and 'z' as constants we get.

$$\phi = 2\left(\frac{y^2}{2}\right)\sin x - 4y + f(x,z) \qquad \ldots (5)$$

Integrating (3) with respect to 'z' treating 'x' and 'y' as constants we get.

$$\phi = \frac{3xz^3}{3} + f(x,y) \qquad \ldots (6)$$

from equations (4), (5) and (6) we get

$$\phi = y^2 \sin x + xz^3 - 4y + C$$

9. If $r = |\bar{r}|$, where $\bar{r}$ is the position vector of the point (x, y, z), prove that $\nabla^2(r^n) = (n+1).r^{n-2}$

Solution:

$$\nabla^2(r^n) = \nabla.(\nabla r^n)$$

$$\nabla r^n = \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\left[nr^{n-1}\frac{\partial r}{\partial x}\right] + \vec{j}\left[nr^{n-1}\frac{\partial r}{\partial y}\right] + \vec{k}\left[nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= \vec{i}\left[nr^{n-1}.\frac{x}{r}\right] + \vec{j}\left[nr^{n-1}.\frac{y}{r}\right] + \vec{k}\left[nr^{n-1}.\frac{z}{r}\right]$$

$$\because r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$= nr^{n-2}[x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= nr^{n-2}\bar{r}$$

$$\nabla.\nabla r^n = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left(nr^{n-2}x\vec{i} + y\vec{j} + 2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(nr^{n-2}x) + \frac{\partial}{\partial y}(nr^{n-2)}y) + \frac{\partial}{\partial z}(nr^{n-2)}z)$$

$$= n\left[r^{n-2} + x.(n-2)r^{n-3}\left(\frac{x}{r}\right)\right] + n\left[r^{n-2} + y.(n-2)r^{n-3}\left(\frac{y}{r}\right)\right] + n\left[r^{n-2} + z.(n-2)r^{n-3}\left(\frac{z}{r}\right)\right]$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}.r^2$$

$$= 3nr^{n-2} + n(n-2)r^{n-2}$$

$$= nr^{n-2}[3 + n - 2] = n(n-2)r^{n-2}$$

10. Find f® if the vector $f(r)\ \vec{r}$ is both solenoidal and irrotational.

Solution:

$f(r)\ \vec{r}$ is solenoidal. $\nabla.f(r)\vec{r} = 0$

i.e., $\nabla.f(r).\vec{r} + f(r)\nabla.\vec{r} = 0$

since $\nabla.f(r) = \frac{f'(r)}{r}\vec{r}$ we get

i.e, $\frac{f'(r)}{r}\vec{r}.\vec{r} + 3f(r) = 0$

since $\vec{r}.\vec{r} = r^2$ we get

i.e. $rf'(r) = 3f(r) = 0$

i.e. $\frac{f'(r)}{f(r)} + \frac{3}{r} = 0$ (on division by 3r)

Integrating both sides with respect to r

log $f(r)$ + 3 log r = log C

log $r^3 f(r)$ = log C

$$f(r) = \frac{C}{r^3} \quad \dots (1)$$

$f(r)\ \vec{r}$ is also irrotational

$\therefore \nabla \times (f(r)\vec{r}) = 0$

i.e, $\nabla f(r) \times \vec{r} + f(x) \nabla \times \vec{r} = 0$

i.e., $\frac{f'(r)}{r}(\vec{r} \times \vec{r}) + 0 = 0$ since $(\nabla \times \vec{r} = 0)$

since $\vec{r} \times \vec{r} = 0$

$$\frac{f'(r)}{r}(0) + 0 = 0 \quad \dots (2)$$

This is true for all values of $f(r)$

From (1) & (2) $f(r) = \frac{C}{r^3}$ we get $f(r)\vec{r}$ is both solenoidal and irrotational.

Exercise

1. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ find div $\bar{r}$ and curl $\bar{r}$.
2. If $\bar{r} = x^2 y\bar{i} + y^2 z\bar{j} + z^2 x\bar{k}$, find curl $\bar{F}$.
3. If $\phi = x^2 + y^2 + z^2$ prove that curl (grad ϕ) = 0.
4. For what value of 'a' the vector $\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is solenoidal.
5. Prove that div (curl $\bar{F}$) = 0
6. If $\bar{A}$ and $\bar{B}$ are irrotatioanl, prove that $\bar{A} \times \bar{B}$ is solenoidal.

7. Find the value of 'c' so that the vector $\overline{F} = (cxy - z^3)\overline{i} - (c-2)x^2\overline{j} + (1-c)xz^2\overline{k}$ is irrotational.

8. Find the values of the constants a, b, c so that $\overline{F} = (axy + bz^3)\overline{i} + (3x^2 - cz)\overline{j} + (3xz^2 - y)\overline{k}$ may be irrotational for these values of a, b, c. Also find the scalar potential of $\overline{F}$.

9. Find div $\overline{F}$ and curl $\overline{F}$,where $\overline{F}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$.

10. Prove that $f = (2x + yz)\overline{i} + (4y + zx)\overline{j} - (6z - xy)\overline{k}$ is solenoidal as well as irrotational. Also find the scalar potential.

11. $\overline{F}$ is solenoidal, prove that curl curl curl $\overline{F} = \nabla^4 F$.

12. If $\overline{F} = 3x^2\overline{i} + 5xy^2\overline{j} + xyz^3\overline{k}$, find $\nabla.\overline{F}, \nabla(\nabla.\overline{F}), \nabla\times\overline{F}, \nabla.(\nabla\times\overline{F})$ and $\nabla\times(\nabla\times\overline{F})$ at the point (1, 2, 3).

13. Show that $\overline{F} = (z^2 + 2x + 3y)\overline{i} + (3x + 2y + z)\overline{j} + (y + 2zx)\overline{k}$ is irrotational, but not solenoidal. Find also its scalar potential.

14. Prove that $f(r)\,\overline{r}$ is irrotational.

15. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Answers

1. 3, 0

2. $-y^2\overline{i} - z^2\overline{j} - x^2\overline{k}$

3. - 2

7. 4

8. $a = 6$, $b = 1$, $c = 1$

9. $6(x + y + z)$, 0

10. $\phi = x^2 + 2y^2 + 3z^2 + xyz + k$

12. $80, 80\overline{i} + 37\overline{j} + 36\overline{k}, 27\overline{i} - 54\overline{j} + 20\overline{k}, 0, 74\overline{i} + 27\overline{j}$

13. $x^2 + y^2 + 3xy + yz + z^2x + C$

LINE INTEGRALS

Any integral which is to be evaluated along a curve is called a line integral.

Let $\overline{F}\,(x, y, z)$ be a vector point function defined at all points in some region of space and let C be a curve in thatregion. The integral $\int_C \overline{F}.d\overline{r}$ is defined as the line integral of $\overline{F}$ along the curve C.

Note

(1) Physically $\int_C \overline{F}.d\overline{r}$ denotes the total work done by the force $\overline{F}$ in displacing a particle from A to B along the curve C.

(2) $\int_A^B \overline{F}.d\overline{r}$ depends not only on the curve C but also on the terminal points A and B.

(3) If the path of integration C is a closed curve, the line integral is denoted as $\oint_C \overline{F}.d\overline{r}$.

(4) If the value of $\int_A^B \overline{F}.d\overline{r}$ does not depend on the curve C, but only on the terminal points A and B, than $\overline{F}$ is called a conservative vector or conservative force.

(5) If $\overline{F}$ is irrotational (conservative) and C is a closed curve then $\int_C \overline{F}.d\overline{r} = 0$.

(6) If $\int_C \overline{F}.d\overline{r}$ is independent of the path C then curl $\overline{F} = \overline{0}$.

(1) If $\overline{F} = 3xy\overline{i} - y^2\overline{j}$, evaluate $\int_C \overline{F}.d\overline{r}$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution:

Given $\overline{F} = 3xy\vec{i} - y^2\vec{j}$

$$d\overline{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\overline{F}.d\overline{r} = 3xydx - y^2dy$$

Given $y = 2x^2$

$dy = 4xdx$

$\therefore \overline{F}.d\overline{r} = 3x(2x^2)dx - (2x^2)^2(4xdx)$

$= (6x^3 - 16x^5)dx$

$\int_C \overline{F}.d\overline{r} = \int_0^1 (6x^3 - 16x^5)dx$

$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_{.0}^1$

$= \frac{6}{4} - \frac{16}{6}$

$= \frac{-7}{6}$

(2) If $\overline{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$. Evaluate $\int_C \overline{F}.d\overline{r}$ from (0, 0, 0) to (1, 1, 1) along the curve $x = t$, $y = t^2$, $z = t^3$.

Solution:

Given $\overline{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

$d\overline{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\overline{F}.d\overline{r} = (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$

Given $x = t$ $y = t^2$ $z = t^3$

$dx = dt$ $dy = 2tdt$ $dz = 3t^2$dt

$\int_C \overline{F}.d\overline{r} = \int_0^1 (3t^2 + 6t^2)dt - 14(t^2)(2tdt) + 20t(t^3)^2(3t^2dt)$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)dt$

$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$

= 3 – 4 + 6

= 5 Units.

(3) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xy - y)\vec{j} - z\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, y-=t, $z = 4t^3$.

Solution:

Work done $= \int_C \bar{F}.d\bar{r}$

Given $\vec{F} = 3x^2\vec{i} + (2xy - y)\vec{j} - z\vec{k}$

$d\bar{r} = 2x\vec{i} + 2y\vec{j} + dz\vec{k}$

$\bar{F}.d\bar{r} = 3x^2 d + (2xz - y)dy - zdz$

Given $x = 2t^2$ $y = t$ $z = 4t^3$

$dx = 4tdt$ $dy = dt$ $dz = 12t^2dt$

$\int_C \bar{F}.d\bar{r} = \int_0^1 [48t^5 + (16t^5 - t) - 48t^5]dt$

$= \int_0^1 (16t^5 - t)dt$

$= \left[16.\frac{t^6}{6} - \frac{t^2}{2}\right]_0^1$

$= \frac{16}{6} - \frac{1}{2}$

$= \frac{13}{6}$ Units

(4) Find the work done by the force $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ along the curve C where C is the rectangle in the xy-plane bounded by $x = 0$, $x = a$, $y = 0$, $y = b$.

Solution:

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

$d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

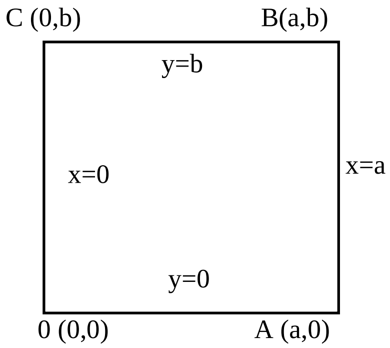
$\bar{F}.d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

The curve C is the rectangle *OABC* and *C* consists of four different paths *OA, AB, BC, CO*.

$$\therefore \int_C \bar{F}.d\bar{r} = \int_{OA} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CO} \bar{F}.d\bar{r}$$

Along *OA*,

$y = 0$ $\qquad dy = 0$



Along *AB*,

$x = a$ $\qquad dx = 0$

Along *BC*,

$y = b$ $\qquad dy = 0$

Along CO,

$x = 0$ $\qquad dx = 0$

$$\therefore \int_C \bar{F}.d\bar{r} = \int_{OA} x^2 dx + \int_{AB} -2aydy + \int_{BC} (x^2+b^2)dx + 0.$$

$$= \int_0^a x^2 dx - 2a\int_0^b ydy + \int_0^b (x^2+b^2)dx$$

$$= \left(\frac{x^3}{3}\right)_0^a - 2a\left(\frac{y^2}{2}\right)_0^b + \left(\frac{x^3}{3} + b^2 3\right)_a^0$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$= -2ab^2$

5. If $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$. Evaluate $\int_C \bar{F}.d\bar{r}$ where *C* is the straight line from *A* (0,0,0) to *B*(2,1,3).

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}.d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz$$

Equation of the straight line AB is given by

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

where $(x_1, y_1, z_1) = (0, 0, 0)$ and $(x_2, y_2, z_2) = (2, 1, 3)$

$$\therefore \frac{x-0}{0-2}=\frac{y-0}{0-1}=\frac{z-0}{0-3}$$

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{3}=t \text{ (say)}$$

$\therefore x = 2t$ $\quad y = t$ $\quad z = 3t$

$dx = 2dt$ $\quad dy = dt$ $\quad dz = 3dt$

$$\int_C \bar{F}.d\bar{r} = \int_0^1 3(2t)^2(2dt)+(12t^2-t)dt+9tdt$$

$$= \int_0^1 24t^2dt+(12t^2+8t)dt$$

$$= \int_0^1 (36t^2+8t)dt$$

$$= \left(36.\frac{t^3}{3}+8\frac{t^2}{2}\right)_0^1$$

$$= \frac{36}{3}+\frac{8}{2}$$

$= 12 + 4$

$= 16.$

6. Find the work done by the force $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, when it moves a particle along the arc of the curve $\vec{r} = \cos t 2\vec{i} + \sin t\vec{j} + t\vec{k}$ from t=0 to $t = 2\pi$.

Solution:

Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}.d\bar{r} = zdx + xdy + ydz$$

work done by $\vec{F} = \int_C \overline{F}.d\bar{r}$

$$= \int_C zdx + rdy + ydz$$

From the vector equation of the curve *C*,

$x = \cos t \qquad y = \sin t \qquad z = t$

$dx = \text{-}\sin t dt \qquad dy = \cos t dt \qquad dz = dt$

$$\therefore \int_C \vec{F}.d\bar{r} = \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]dt$$

$$= \left[t\cos t - \sin t - \frac{1}{2}\left(t + \frac{\sin 2t}{2} \right) - \cos t \right]_0^{2\pi}$$

$$= (2\pi + \pi - 1) - (-1)$$

$$= 3\pi$$

(7) Find the work done by the force $\vec{F} = y(3x^2 - y - z^2)\vec{i} + x(2x^2 y - z^2)\vec{j} - 2xyzy\vec{k}$ when it moves a particle around a closed curve *C*.

Solution:

To evaluate the work done by a force, the equation of the path C and the terminal points must be given.

Since C is a closed curve and the particle moves around this curve completely, any point (x_0, y_0, z_0) can be taken as the initial as well as the final point.

But the equation of C is not given. Hence we verify when the given force $\vec{F}$ is conservative, ie., irrotational.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2x^2 - yz^2 & 2x^3y - z^2x & -2xyz \end{vmatrix}$$

$$= (-2xz + 2xz)\vec{i} - (-2yz + 2yz)\vec{j} + (6x^2y - 6x^3y + z^3 - z^2)\vec{k}$$

$$= 0$$

Since $\nabla \times \vec{F} = \overline{0}$

$\Rightarrow \vec{F}$ is irrotatonal

$\Rightarrow \oint_C \vec{F}.d\bar{r} = \overline{0}$

(8) If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$ check whether the integral $\int_C \vec{F}.d\bar{r}$ is independent of the path C.

Solution:

$$\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$$

$\int_C \vec{F}.d\bar{r}$ is independent of the path of integration, if $\nabla \times \vec{F} = \overline{0}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^2z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-6x^2 + 6x^2z) + \vec{k}(4x - 4x)$$

$$= \overline{0}.$$

Hence the line integral is independent of the path C.

(9) If $\vec{F} = x\vec{j} - y\vec{i}$, find $\int_C \vec{F}.d\bar{r}$ along the arc of the circle $x^2 + y^2 = 1$ from (1,0) to (0,1).

Given $\vec{F} = x\vec{j} - y\vec{i}$

$$d\bar{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\overline{F}.d\bar{r} = -ydx + xdy$$

Given $x^2 + y^2 = 1$ … (1)

$$2xdx + 2ydy = 0$$

$$2xdx = -2ydy$$

$$x\,dx = -y\,dy \qquad \ldots (2)$$

$$\bar{F}.d\bar{r} = -y\,dx + x\,dy\left(\frac{-x}{y}dx\right) \qquad \text{(from 2))}$$

$$= -y\,dx - \frac{x^2}{y}dx$$

$$= \left[y + \frac{x^2}{y}\right]dx$$

$$= -\left[\frac{y^2 + x^2}{y}\right]dx$$

$$= \frac{-dx}{y} \qquad \text{(from (1))}$$

$$\bar{F}.d\bar{r} = \frac{-1}{\sqrt{1-x^2}}dx \qquad (\because x^2 + y^2 = 1)$$

$$\int_C \bar{F}.d\bar{r} = -\int_1^0 \frac{dx}{\sqrt{1-x^2}}$$

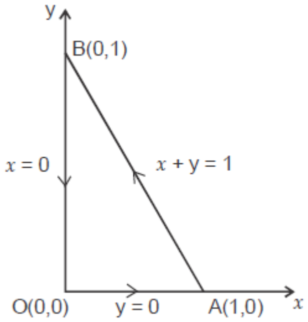
$$= \int_1^0 \frac{dx}{\sqrt{1-x^2}}$$

$$= (\sin^{-1} x)_0^1$$

$$= \frac{\pi}{2}.$$

(10) Evaluate $\int_C \bar{F}.d\bar{r}$ where C is the boundary of the region given by $x = 0$, $y = 0$, x +y = 1 and $\vec{F} = (3x^2 - 8y^2)\vec{i} + (4y - 6xy)\vec{j}$

Solution:



Given $\vec{F} = (3x^2 - 8y^2)\vec{i} + (4y - 6xy)\vec{j}$

$$d\bar{r} = dx\vec{i} + dy\vec{j} + dx\vec{k}$$

$$\bar{F}.d\bar{r} = (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Here C consists of the lines $x = 0$, $x + y = 1$.

Along AB,

$$x + y = 1$$

$$\Rightarrow y = 1 - x$$

$$dy = -dx$$

Along BO,

$$x = 0, \quad dx = 0$$

$$\int_C \bar{F}.d\bar{r} = \int_{0A} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BO} \bar{F}.d\bar{r}$$

$$= \int_0^1 3x^2 dx + \int_0^1 [3x^2 - 8(1-x)^2 dx + (4(1-x) - 6x(1-x))(-dx)] + \int_1^0 4ydy$$

$$= 3.\left(\frac{x^3}{3}\right)_0^1 + \int_0^1 (-11x^2 + 26x - 12)dx + 4.\left(\frac{y^2}{2}\right)_1^0$$

$$= 1 + \left[-11.\frac{x^3}{3} + 26.\frac{x^2}{2} - 12x\right]_1^0 + 2(0-1)$$

$$= 1 + \frac{11}{3} - 13 + 12 - 2$$

Exercise

(1) Evaluate $\int_C \bar{F}.d\bar{r}$ where $\bar{F} = x^2 y^2 \vec{i} + y\vec{j}$ and C is $y^= 4x$ in the xy-plane from (0, 0) to (4, 4).

(2) If $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$, find $\int_C \bar{F}.d\bar{r}$ along C, where C is the straight line joining the points (0, 0, 0) to (1, 1, 1).

(3) Find the work done by the force $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j}$ when it moves a particle from (1, -2, 1) to (3, 1, 4) along any path.

(4) Find the total work done in moving a particle in a force field given by $\vec{F} = (2x - y + z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$ along a circle C in the xy-plane $x^2 + y^2 = 9$, $z = 0$.

(5) Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

(6) Given the vector field $\vec{F} = xz\vec{i} + yz\vec{j} + z^2\vec{k}$, evaluate the work done in moving a particle from the point (0, 0, 0) to (1, 1, 1) along the curve C, $x = t$, $y = t^2$, $z = t^3$.

(7) If $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F}.d\bar{r}$ from (0, 0) to (1, 1) along the line $y = x$.

(8) If $\vec{F} = 3x(x+2y)\vec{i} + (3x^2 - y^3)\vec{j}$, show that $\int_C \vec{F}.d\bar{r}$ is independent of the path C.

(9) Find the work done by the force $\vec{F} = y^2\vec{i} + 2(xy+z)\vec{j} + 2y\vec{k}$, when it moves a particle around a closed curve C.

(10) Find the work done by $\vec{F} = xy\vec{i} + (y-z)\vec{j} + 2x\vec{k}$, when the particle moves along the curve $x = t$, $y = t^2$, $z = t^3$, from $t = 1$ to $t = 2$.

SURFACE INTEGRAL

Introduction

A surface integral is a definite integral taken over a surface. It can be thought of as the double integral analogue of the line integral. Given the surface, one may integrate over its scalar field (ie, functions which return scalars as value) and vector field ((ie) functions which return vectors as value). Surface integrals have applications in physics, particularly with the classical theory of electromagnetism. Various useful results for surface integrals can be derived using differential geometry and vector calculus, such as the divergence theorem and its generalization Stokes theorem.

Consider any surface (planar, curved, closed or open) and let $\vec{F} = \vec{F}(x, y, z)$ be a vector point function, defined and continuous on a region S of the surface. Then $\iint_S \vec{F}.d\vec{s}$ where ds denotes an element of the surface S is called the surface integral of $\vec{F}$ over S.

We define it as the limit of a sum as follows.

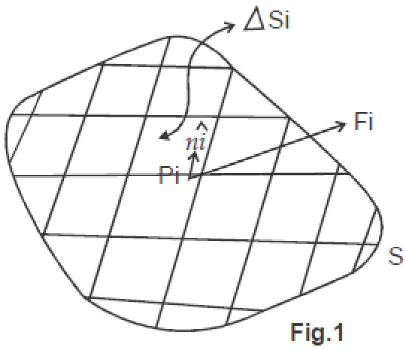


Fig.1

Subdivide S, in any manner into n elements of areas $\Delta S_i, I = 1, 2, ...n$. Let $\vec{F_i}$ be the value of $\vec{F}$ at some point, P_i inside or on the boundary of the sub-region ΔS_i.

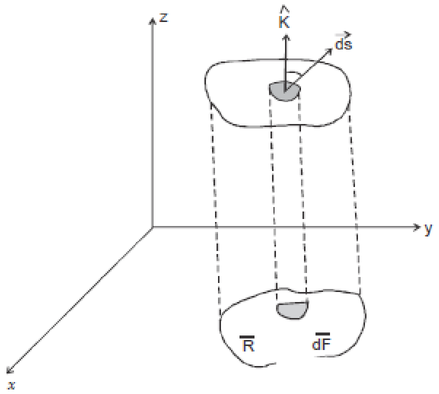
From the vector sum $In = \sum_{i=1}^{n} F_i \Delta S_i$. If the limits of the above sum exists, as $n \to \infty$ in such a way that each ΔS_i collapses ((ie) shrinks) to a point, and is independent of the mode of the sub-division of S, then this limit is called the surface integral of $\vec{F}$ over S and is denoted by $\iint_S \vec{F}.d\vec{s}$.

Normal surface Integral of $\vec{F}$ over the parts of a given surface

Consider the above Fig. 1 let P_i be any point of S and let $\hat{n}$, a unit normal vector at P_i, pointing outwardly (called the outward unit normal at P_i) to the surface ΔS_i. Then $\vec{F}.\hat{n}_i$ is the scalar complement of $\vec{F_i}$, in the direction of $\hat{n}_i$.

The limiting value of the sum $\sum_{i=1}^{n} F_i.\vec{n}_i$ as $n \to \infty$ where n is the number of subregions ΔSi, such that each ΔS_i shrinks to a point, if it exists and is independent of the manner of division of S into sub-regions, is called the normal surface integral of $\vec{F}$ over S and is denoted by $\iint_S \vec{F}.\hat{n}\, d\vec{s}$.

Evaluation of double integral



The surface S is projected onto a region R of the xy-plane, so that an element of surface area ds at point P projects onto the area element. We see that $d\vec{F} = |\cos\alpha|\, d\vec{S}$, where α is the angle between the unit vector $\hat{k}$ in the z-direction and the unit normal $\hat{n}$ to the surface at P. So at any given point of S, we have simply $d\vec{s} = \dfrac{d\vec{F}}{|\cos\alpha|} = \dfrac{d\vec{F}}{|\hat{n}.\hat{k}|}$

Now if the surface S is given by the equation $f(x,y,z) = 0$ then the unit normal at any point of the surface is simply given by $\hat{n} = \dfrac{\nabla\vec{f}}{\left|\nabla\vec{f}\right|}$ evaluated at that point. The scalar element of the surface area then becomes $d\vec{s} = \dfrac{d\vec{F}}{\hat{n}.\hat{k}} = \dfrac{|\nabla\vec{f}|\, d\vec{F}}{\left|\nabla\vec{f}\right|.\hat{k}} = \dfrac{|\nabla\vec{f}|\, d\vec{f}}{\dfrac{d\vec{f}}{dz}}$

Where $|\nabla\vec{f}|$ and $\dfrac{\partial\vec{f}}{\partial z}$ and are evaluated on the surface S. we can therefore express any surface integral over S as a double integral over the region R in the xy-plane.

Note

The projection of the elementary surface ds on the xy palne is $dxdy$. Also the projection of the vector $\hat{n}\, ds$ of magnitude ds on the XOY plane to which $\hat{k}$ is the unit normal vector.

Thus $dy = |\hat{n}.ds.\hat{k}| = |\hat{n}.\hat{k}|\, d\vec{s}$ giving $ds = \dfrac{dxdy}{|\hat{n}.\hat{k}|}$. Hence $\iint_S \vec{F}.\hat{n}d\vec{s} = \iint_{S'} \vec{F}.\hat{n}\dfrac{dxdy}{|\hat{n}.\hat{k}|}$.

where S is the projection of S on the XOY plane. Similarly $\int\limits_{S} \vec{F}.\hat{n}d\vec{s} = \iint\limits_{S'} \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{i}|}$ where $s'$is the projection of S on the YOZ plane and $\int\limits_{S} \vec{F}.\hat{n}d\vec{s} = \iint\limits_{S'},\vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{j}|}$ where $s'$is the projection S on the XOZ plane.

Flux

In physical applications the integral $\int\limits_{S} \vec{F}.\hat{n}d\vec{s}$ is called the flux of $\vec{F}$ through S.

Cylindrical and Spherical polar co-ordinates

In evaluating surface and volume integrals, in certain cases, it will be advantage to change the variable x, y, z into cylindrical or spherical polar co-ordinates. So it is better to recall the relations between these coordinates and the respective Jacobian of transformation.

Polar Co-ordinates

We know that in two dimensions, the relation between x, y and the polar co-ordinates r, θ are $x = r \cos \theta$; $y = r \sin \theta$ and $0 \le r < \infty$, $0 \le \theta \le 2\pi$. And the jacobian of the transformation is r so that $dxdy = rdrd\theta$.

Cylindrical Co-ordinates

If P is (x, y, z) (refer Fig.3) and if PL is the perpendicular from P to the XOY plane then OL angle XOL, LP are the cylindrical polar co-ordinates which are denoted by r, θ, z where $0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$.

The relations between x, y, z and r, θ, z and that between

$$dxdydz \,\&\, drd\theta dz \text{ are } \left.\begin{array}{l} x = r\cos\theta \\ y = r\cos\theta \\ x = z \end{array}\right\} \quad dxdydz = rdrd\theta dz$$

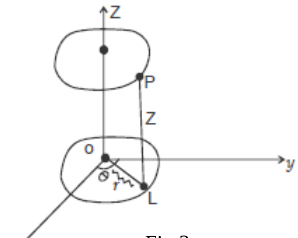


Fig.3
Cylindrical co-ordinates

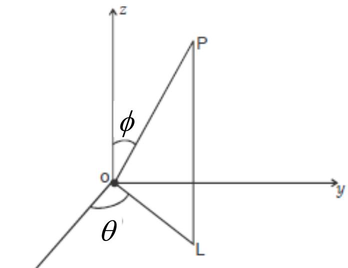


Fig.4
Spherical co-ordinates

Spherical Co-ordinates

If P is (x, y, z) (refer Fig.4) and if PL is perpendicular from P to the XOY plane, then OP, angle ZOP, angle XOL are the spherical polar co-ordinates which are denoted by r, θ, ϕ where $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. The relations between x, y, z and r, θ, ϕ.

are $x = r \sin \theta \cos \phi$

$y = r \sin \theta \cos \phi$

$z = r \cos \theta$

The relation between $dxdydz$ and $drd\theta d\phi$ is $dxdydz = r^2 \sin \theta drd\theta d\phi$. For a hemisphere the limits of θ will be from 0 to $\frac{\pi}{2}$ and the limits for ϕ will be 0 to 2π.

Note

(i) If S is a closed surface, the outer surface is usually chosen as the positive side

(ii) $\int_S \phi d\vec{s}$ and $\int_S \vec{F} \times d\vec{s}$, where ϕ is a scalar point function, are also surface integrals.

(iii) To evaluate a surface integral in the scalar form, we convert it into a double integral and then evaluate. Hence the surface integral $\int_S \overline{F}.d\vec{s}$ is also denoted as $\iint_S \overline{F}.d\vec{s}$.

(iv) The area of the region S is $\iint_S ds$.

Solved Problems

1. Obtain $\int_S \vec{F}.\hat{n}\, ds$, where $\vec{F} = (x^2 + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ over the surface of the plane $2x + y + 2z = 5$ in first octant.

Solution

Let the given surface be $\phi = 2x + y + 2z - 6$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{3}$$

Let s' be the projection is S in the XOY plane

$$\int_S \vec{F}.\hat{n}ds = \iint_S \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\vec{k}|}$$

Now $\vec{F}.\hat{n} = ((x^2+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}).\dfrac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$

$$= \frac{2(x^2+y^2) - 2x + 4yz}{3}$$

$$= \frac{2}{3}\left(x^2 + y^2 - x + 2y\left(\frac{6-2x-y}{2}\right)\right) \qquad \text{since } z = \frac{6-2x-y}{2}$$

$$= \frac{2}{3}\left(x^2 + y^2 - x + 6y - 2xy - y^2\right)$$

$$= \frac{2}{3}\left(x^2 - 2xy^2 - x + 6y\right)$$

Since the equation of the line AB is $2x+y=6$ (or) $y=6-2x$. In the region s'as x varies from 0 to 3, y varies from 0 to 6-2x.

$$\therefore \int_S \vec{F}.\hat{n}d\vec{s} = \iint_S \frac{2}{3}(x^2 - 2xy - x + 6y)\frac{dxdy}{\frac{2}{3}}$$

$$= \int_0^3 \int_0^{-2x} (x^2 - 2xy - x + 6y)dxdy$$

$$= \int_0^3 (x^2y - xy^2 - xy + 3y^2)_{y=0}^{y=6-2x} dx$$

$$= \int_0^3 (108 - 114x + 44x^2 - 6x^3)dx$$

$$= \frac{171}{2}$$

2. Given that $\vec{F} = x\hat{i} + y\hat{j} - 2z\hat{k}$ find $\int_S \vec{F}.d\vec{s}, S$ being the surface of the sphere $x^2 + y^2 + z^2 = a^2, z \geq 0$

Solution:

Let $\phi = x^2 + y^2 + z^2$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a} \qquad \text{since } x^2 + y^2 + z^2 = a^2$$

$$\int_S \vec{F}.d\vec{s} = \iint_s \vec{F}.\hat{n}ds$$

$= \iint_s \vec{F}.\hat{n}\frac{dxdy}{\hat{n}.\hat{k}}$ where s' is the projection of the spherical surface above the XY-plane on the XOY plane.

$$\vec{F}.\hat{n} = (x\hat{i} + y\hat{j} + 2z\hat{k}).\frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{1}{a}(x^2 + y^2 - 2z^2)$$

$$= \frac{1}{a}(x^2 + y^2 - 2a^2 - x^2 - y^2) \text{ since } x^2 + y^2 + z^2 = a^2 \quad \therefore z^2 = a^2 - x^2 - y^2$$

$$= \frac{1}{a}(3x^2 + 3y^2 - 2a^2)$$

$$\hat{n}.\hat{k} = (x\hat{i} + y\hat{j} + 2z\hat{k}).\hat{k}$$

Since $s'$is the circle $x^2 + y^2 = a^2$on the XOY plane. x varies from $-a$ to $+a$ and y varies from

$$\sqrt{a^2 - x^2} \; to + \sqrt{a^2 + x^2}$$

Hence $\int_S \vec{F}.d\vec{s} = \iint_s \frac{1}{a}(3x^2 + 3y^2 - 2a^2).\frac{a}{z}dxdy$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}} dxdy$$

Taking $x = r\cos\theta$, $y = r\sin\theta$, we get $x^2 + y^2 + r^2$and $dxdy = rdrd\theta$. For the circle s', r varies from 0to a and θvaries from 0 to 2π.

$$\int_S \vec{F}.d\vec{s} = \int_0^{2\pi}\int_0^{a} \frac{3r^2-2a^2}{\sqrt{a^2-r^2}} rdrd\theta = \int_0^{2\pi}\int_0^{a} \frac{-3(a^2-r^2)+a^2}{\sqrt{a^2-r^2}} rdrd\theta$$

$$= 3\int_0^{2\pi}\int_0^{a} r\sqrt{a^2-r^2}drd\theta + a^2 = \int_0^{2\pi}\int_0^{a} \frac{rdrd\theta}{\sqrt{a^2-r^2}}$$

$$= I_1 + I_2 \text{ (say)} \qquad \ldots (1)$$

Consider I_1 $\quad -3\int_0^{2\pi}\int_0^{a} \sqrt{a^2-r^2}drd\theta$

Let $\quad a^2 - r^2 = t^2$

$2rdr = -2tdt$

$rdr = -tdt$

If $\quad r = 0, t = a$

$r = a, t = 0$

$$\int_0^a r\sqrt{a^2-r^2}dr = \int_0^a \sqrt{t^2}(-t)dt$$

$$= -\int_0^a t^2 dt$$

$$= \int_0^a t^2 dt$$

$$= \left(\frac{t^3}{3}\right)_0^a$$

$$= \frac{a^3}{3}$$

$$\therefore I_1 = -3\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,drd\theta$$

$$= -3\int_0^{2\pi}\left(\frac{a^3}{3}\right)d\theta$$

$$= -a^3(\theta)_0^{2\pi}$$

$$= -a^3 2\pi$$

Consider $I_2 = a^2\int_0^{2\pi}\int_0^a \frac{rdrd\theta}{\sqrt{a^2-r^2}}$

Let $a^2 - r^2 = t^2$ If $r = 0$ $t = \text{a}$

$2rdr = -2tdt$ $r = 0$ $t = \text{a}$

$rdr = -tdt$

$$\therefore \int_0^a \frac{rdr}{\sqrt{a^2-r^2}} = \int_a^0 \frac{1}{\sqrt{t^2}}(-tdt)$$

$$= \int_a^0 -dt$$

$$= \int_a^0 dt$$

$$= (t)_0^a$$

$$= a$$

$$\therefore I_2 = a^2\int_0^{2\pi}\int_0^a \frac{rdrd\theta}{\sqrt{a^2-r^2}}$$

$$= a^2\int_0^{2\pi} ad\theta$$

$$= a^3\int_0^{2\pi} d\theta$$

$$= a^3(\theta)_0^{2\pi}$$

$$= a^3(\theta)_0^{2\pi}$$

$$= a^3 2\pi$$

$$\therefore (1) \Rightarrow \int_0 \vec{F}.d\vec{s} = -a^3 2\pi + a^3 2\pi$$

$$= 0$$

3. Obtain $\int \vec{F}.\hat{n}d\vec{s}$ over the surface of the cylinder $x^2 + y^2 = 16$ in the first octant between $z = 0$ & $z = 5$ where $\vec{F} = z\vec{i} + x\vec{j} - 3y^2 z\hat{k}$

Solution:

Let $\phi = x^2 + y^2 - 16$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{x\hat{i} + y\hat{j}}{4} \qquad \text{since } x^2 + y^2 = 16$$

Now $\vec{F}.\hat{n} = (z\vec{i} + x\vec{j} - 3y^2 zk).\frac{1}{4}(x\vec{i} + y\vec{j})$

$$= \frac{1}{4}(xz + xy)$$

$$= \frac{1}{4}x(z + y)$$

The surface S of the cylinder in the first octant can be projected onto YOZ (or ZOX) plane into a surface s'

Hence $\int_S \vec{F}.\hat{n}ds = \int_{S'} \frac{1}{4}x(y+z)\frac{dydz}{|\hat{n}.\hat{i}|}$

$$= \int_{S'} \frac{1}{4}x(y+z)\frac{dydz}{\frac{x}{4}} \qquad \text{since } \hat{n}.\hat{i} = \frac{x}{4}$$

$$= \int_{z=0}^{5}\int_{y=0}^{4} (y+z)dydz, S' \text{ is a rectangle of sides 4 \& 5 units.}$$

$$= \int_{z=0}^{5} \left(\frac{y^2}{2} + zy\right)_{y=0}^{4} dx$$

$$= \int_{z=0}^{5} (8+4z)dz$$

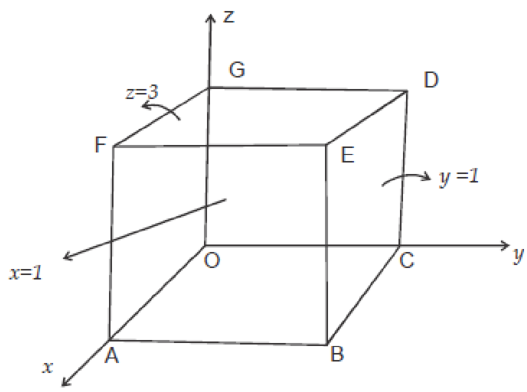
$$= \left(8z + \frac{4z^2}{2}\right)_0^5$$

$$= 40 + 50$$

$$= 90$$

4. If $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and S is the rectangle parellelopiped bounded $x = 0$, $y = 0$, $z = 0$, calculate $\iint_S \vec{F}.\hat{n}d\vec{s}$

Solution:



There are six faces of the parellelepiped and we calculate the integral over each of these faces. We denote the values of $\vec{F}$ on these faces by $\vec{F}_1, \vec{F}_2 ....\vec{F}_6$

Face	$\hat{n}$	Equation	$\overrightarrow{ds}$	$\vec{F}$
ABEF	$\hat{i}$	$x = 1$	$dydz$	$\vec{F}_1 = 2y\hat{i} + yz^2\hat{j} + z\hat{k}$
COGD	$-\hat{i}$	$x = 0$	$dydz$	$\vec{F}_2 = yz^2\hat{j}$
BCDE	$\hat{j}$	$y = 2$	$Dzdx$	$\vec{F}_3 = 4x\hat{i} + 2z^2\hat{j} + xz\hat{k}$
GOAE	$-\hat{j}$	$y = 1$	$dzdx$	$\vec{F}_4 = xz\hat{k}$
EDGE	$\hat{k}$	z = 3	$dxdy$	$\vec{F}_5 = 2xy\hat{i} + 9y\hat{j} + 3x\hat{k}$
AOCB	$-\hat{k}$	$z = 0$	$Dxdy$	$\vec{F}_6 = 2xy\hat{i}$

$$\therefore \int \vec{F}.ds = \iint_{ABEF} \vec{F}_1.\hat{n}ds + \iint_{COGD} \vec{F}_2.\hat{n}ds + \iint_{BCDE} \vec{F}_3.\hat{n}ds$$

$$+ \iint_{GOAE} \vec{F}_4.\hat{n}ds + \iint_{EDGE} \vec{F}_5.\hat{n}ds + \iint_{AOCB} \vec{F}_6.\hat{n}ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \qquad \text{(Say)} \qquad \dots (1)$$

Consider, $I_1 = \iint_{ABEF} \vec{F}_1.\hat{n}ds$

$$\vec{F}_1.\hat{n} = (2y\hat{i} + yz^2\hat{j} + z\hat{k}).\hat{i}$$

$$= 2y$$

on the surface ABEF, z varies from 0 to 3 and y varies from 0 to 2

$$\therefore I_1 = \int_{y=0}^{2} \int_{z=0}^{3} 2y dy dz$$

$$= 2\int_0^2 y dy (z)_0^3$$

$$= 2 \times 3 \int_0^2 y dy$$

$$= 6\left(\frac{y^2}{2}\right)_0^2$$

$$= 12$$

Consider $\therefore I_2 = \iint_{COGD} \vec{F}_2.\hat{n}ds$

$$\vec{F}_2.\hat{n} = yz^2\vec{j}.(-\vec{i})$$

$$= 0$$

on the surface COGD, z varies from 0 to 3 and y varies from 0 to 2.

$$\therefore I_2 = \int_{y=0}^{2} \int_{z=0}^{3} 0 \, dydz$$

$$= 0$$

Consider $\therefore I_3 = \iint\limits_{BCDE} \vec{F}_3.\hat{n}ds$

$$\vec{F}_3.\hat{n} = (4x\hat{i} + 2z^2\hat{j} + xz\hat{k}).\hat{j}$$

$$= 2z^2$$

on the surface BCDE, z varies from 0 to 3 and x varies from 0 to 1

$$\therefore I_3 = \int_{x=0}^{1}\int_{z=0}^{1} 2z^2 dxdz$$

$$= \int_{x=0}^{1}\left(\frac{2z^3}{3}\right)_0^3 dx$$

$$= \frac{2}{3}\int_{x=0}^{1}\left(z^3\right)_0^3 dx$$

$$= 18\int_0^1 dx$$

$$= 18$$

Consider $\qquad I_4 = \iint\limits_{GOAE} \vec{F}_4.\hat{n}ds$

$$\vec{F}_3.\hat{n} = xz\hat{k}.(-\hat{j})$$

$$= 0$$

on the surface GOAE, z varies from 0 to 3, x varies from 0 to 1.

$$\therefore I_4 = \int_{x=0}^{1}\int_{z=0}^{3} 0.dxdz$$

$$= 0$$

Consider $\qquad I_5 = \iint\limits_{EDGF} \vec{F}_5.\hat{n}ds$

$$\vec{F}_5.\hat{n} = (2xy\hat{i} + 9y\hat{j} + 3x\hat{k}).\hat{k}$$

$$= 3x$$

on the surface EDGE, y varies from 0 to 2, x varies from 0 to 1.

$$\therefore I_5 = \int_{x=0}^{1} \int_{y=0}^{2} 3x\,dx\,dz$$

$$= 3\int_0^1 (y)_0^2\, x\,dx$$

$$= 3 \times 2\int_0^1 x\,dx$$

$$= 6\left(\frac{x^2}{2}\right)_0^1$$

$$= 3$$

Consider $\quad I_6 = \iint_{AOCD} \vec{F}_6.\hat{n}ds$

$$\vec{F}_6.\hat{n} = (2xy\hat{i}.(-\hat{k})$$

$$= 0$$

on the surface AOCB, y varies from 0 to 2, x varies from 0 to 1.

$$\therefore I_6 = \int_{x=0}^{1} \int_{y=0}^{2} 0\,dx\,dz$$

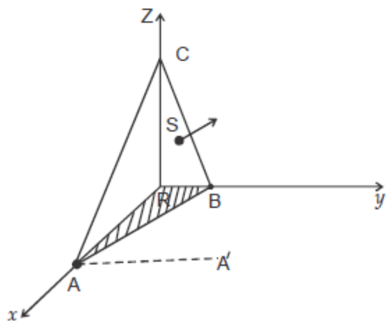
$$= 0$$

$$\therefore (1) \Rightarrow \int_S \vec{F}.d\vec{s} = 12 + 0 + 18 + 0 + 3 + 0$$

$$= 13$$

5. Evaluate the integral $\iint_S \vec{A}.\hat{n}ds$ if $A = 4y\hat{i} + 18z\hat{j} - x\hat{k}$ and S is the surface of the portion of the plane $3x + 2y + 6z = 6$ contained in the first octant.

Solution:



Let OABC be the given surface *S*. Then the projection *R* of *S* on the xoy plane is *OAB*.

$$\phi = 3x + 2y + 6z - 6$$

$$\nabla\phi = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$|\nabla\phi| = \sqrt{9+4+36}$$

$$= 7$$

$$\vec{A} = 4y\hat{i} + 18z\hat{j} - x\hat{k}$$

$$\hat{n} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{7}$$

$$\hat{n}.\hat{k} = \frac{6}{7}$$

$$\vec{A}.\hat{n} = \frac{1}{7}(12y + 36z - 6x)$$

$$\iint_S \vec{A}.\hat{n}ds = \iint_S \vec{A}.\hat{n}\frac{dxdy}{|\hat{n}.\hat{k}|}$$

$$= \iint_R \frac{1}{7}\frac{12y + 36z - 6x}{\frac{6}{7}}dxdy$$

$$= \iint_R (2y + 6z - x)dxdy$$

$$= \iint_R (2y + (-3x - 2y + 6 - x)dxdy \qquad \text{Since } 3x + 2y + 6z = 6$$

$$\therefore 6z = 6 - 3x - 2y$$

$$= \iint_R (2y - 3x - 2y + 6 - x)dxdy$$

$$= \iint_R (6 - 4xdxdy$$

Let AA'parellel to the y axis. Then R lies between the y-axis &AA', where A is (2,0,0). Thus $0 \le x \le 2$ with this restriction on the x-co-ordinate of a point of R, the y-co-ordinate varies from y =0 to $\frac{3(2-x)}{2}$. Since R is bounded by OA and AB,

$$\text{Thus } = \iint_S \vec{A}.\hat{n}ds = \int_0^2 \int_{y=0}^{\frac{3(2-x)}{2}} (6 - 4x)dydx$$

$$= \int_0^2 (6 - 4x)(y)_0^{\frac{3(2-x)}{2}} dx$$

$$= \int_0^2 (6 - 4x)\left(\frac{3(2-x)}{2}\right)dx$$

$$= \int_0^2 \frac{3}{2}(12 - 6x - 8x + 4x^2)dx$$

$$= 3\int_0^2 (2x^2 - 7x + 6)dx$$

$$= 3\left(\frac{2x^3}{3} - \frac{7x^2}{2} + 6x\right)_0^2$$

$$= 3\left(\frac{16}{3} - \frac{28}{2} + 12\right)$$

$$= 3\left(\frac{32 - 84 + 72}{6}\right)$$

$$= \frac{1}{2}(20)$$

$$= 10$$

6. Evaluate $= \iint_S \vec{F}.d\vec{s}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ contained in the first octant between the planes z=0 &z=2.

Solution:

Let $\phi = x^2 + y^2 - 9$

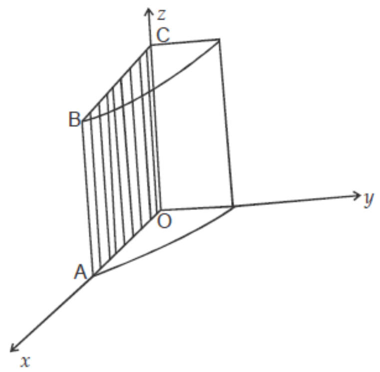
$\nabla\phi = 2x\hat{i} + 2y\hat{j}$

$|\nabla\phi| = \sqrt{4(x^2 + y^2)}$ $\qquad \therefore x^2 + y^2 = 9$

$= 6$

$\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j}}{6}$

$= \dfrac{x\hat{i} + y\hat{j}}{3}$



$\vec{F}.\hat{n} = \dfrac{xyz + 2y^3}{3}$

$\iint_S \vec{F}.d\vec{s} = \iint_S \vec{F}.\hat{n}ds$

$= \iint_S (xyz + 2y^3)ds$

$= \dfrac{1}{3}\iint_{OR} (xyz + 2y^3)\dfrac{dxdz}{\hat{n}.\hat{j}}$

where R is the rectangular region OABC in the xoz-plane, got by projecting the cylindrical surface S on the xoy plane and z varies from 0 to 2, x varies from 0 to 3.

$\therefore \iint_S \vec{F}.\hat{n}ds = \dfrac{1}{3}\iint_R (xyz + 2y^3)\dfrac{dxdz}{\frac{y}{3}}$

$= \iint_R (xy + 2y^2)dxdz$

$= \int_{y=0}^{2}\int_{x=0}^{3} (xz + 2(9 - x^2)dxdz$ $\qquad \because x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2$

$= \int_0^2 \left(\dfrac{x^2}{2}z + 18x - \dfrac{2x^3}{3}\right)_0^3 dt$

$$=\int_0^2 \left(\frac{9}{2}z+18\times3-2\times9\right)dz$$

$$=\left(\frac{9}{2}\frac{z^2}{2}+36z\right)_0^2$$

$$=9+72$$

$$=81$$

Volume Integral

In multivariable calculus, a volume integral refers to an integral over a 3-dimensional domain. Let V denote the volume enclosed by some closed surfaces and $\vec{F}$, a vector function defined throughout V. Then, $\iiint_v \vec{F}.d\vec{V}$, where $d\vec{V}$ denotes an element of the volume V, is called the volume integral $\vec{F}$ over V.

We define it as the limit of a sum as follows.

Sub-divide V into n regions of elementary volumes $\Delta V_i, i = 1, 2, \ldots n$. Let $\vec{F}_i$ be the value of $\vec{F}$ at some point $P_i$inside (or) on the boundary of the region, enclosing the volume ΔV_i.

Form the vector sum $I_n = \sum_{i=1}^{n} F_i \Delta V_i$. If the limit of $I_n$exists as $n \to \infty$, in such a way that each ΔV_i shrinks into a point, and is independent of the manner of division of V into these elementary volumes, then the limit is called the volume integral of $\vec{F}$ over V and is denoted by $\iiint_v \vec{F}.d\vec{V}$.

Remark

A volume integral is a triple integral of the constant function 1 which gives the volume of the region D (ie) the integral $Vol(D) = \iiint dxdydz.$

A triple integral within a region D in $R^3$of a function $f(x, y, z)$ is usually written as $\iiint f(x, y, z)dxdydz.$

Note

A volume integral in cylindrical co-ordinates is $\iiint\limits_D f(r,\theta,z)rdrd\theta dx$ and a volume integral in spherical co-ordinates has the form $\iiint\limits_D f(r,\theta,\phi)r^2\sin\phi drd\theta d\phi$

Remarks

Integrating the function f (x, y, z) =1 over a unit cube yields the following result $\int_0^1\int_0^1\int_0^1 1\times dxdydz = 1$ so, the volume of the unit cube is 1 as expected. That is, rather trivial however a volume integral is far more powerful. For instance if we have a scalar function $f : R^3 \to R$ describing the density of the cube at a given point (x,y,z) by $f = x+y+z$ then performing the volume integral will give the total mass of the cube

$$\int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz = \frac{3}{2}.$$

Solved Problems

1. If $\vec{F} - 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint\limits_D \vec{F}.dV$ where V is the region bounded by the surfaces $x = 0$, $y = 0$, $x = 2$, $y = 4$, $z = x^2$, $z = 2$

Solution:

$$\iiint \vec{F}.dV = \iiint\limits_V (2z\hat{i} - x\hat{j} + y\hat{k})dxdydz$$

$$= \int_{x=0}^{2}\int_{y=0}^{4}\int_{z=x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})dzdydx$$

$$= \int_0^2\int_0^4 (z^2\hat{i} - xz\hat{j} + yz\hat{k})^2 dydx$$

$$= \int_0^2\int_0^4 (4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k})dydx$$

$$= \int_0^2 \left(4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right)_0^4 dx$$

$$= \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right)dx$$

$$= \left(16x\hat{i} - 4^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right)_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$= \frac{32}{5}\hat{i} + \frac{32\hat{k}}{3}$$

$$= \frac{32}{15}(\hat{i} + 5\hat{j})$$

2. Evaluate $\iiint\limits_{v} (\nabla.\vec{F})$ if $\vec{F} = x^2\hat{i} + y^2\hat{i} + z^2\hat{i})$ and if V is the volume of the region enclosed by the cube $\theta \le x, y, z \le 1$

Solution:

$$\iiint\limits_{v} (\nabla.\vec{F})d\vec{V} = 2\iiint\limits_{v} (x + y + z)dV$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(x + y + z)dzdydx$$

$$= 2\int_{0}^{1}\int_{0}^{1}\left(xz + yz + \frac{z^2}{2}\right)_0^1 dydx$$

$$= 2\int_{0}^{1}\int_{0}^{1}\left(x + y + \frac{1}{2}\right)dydx$$

$$= 2\int_{0}^{1}\left(xy + \frac{y^2}{2} + \frac{y}{2}\right)_0^1 dx$$

$$= 2\int_{0}^{1}\left(x + \frac{1}{2} + \frac{1}{2}\right)dx = 2\left(\frac{x^2}{2} + \frac{1}{2}x + \frac{1}{2}x\right)_0^1$$

$$= 2\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)$$

$$= 3$$

3. Evaluate $\iiint\limits_{v} \nabla.\vec{A}dV$ if $\vec{A} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and S is the surface of the parellelopiped formed by the planes $x = 0$, $x = 2$, $y = 0$, $y = 1$, $z = 0$, $z = 3$.

Solution:

$$\iiint_v (\nabla.\vec{A})dV = \iiint_S (2y+z^2+x)dV$$

$$= \int_0^2 \int_0^1 \int_0^3 (2y+z^2+x)dzdydx$$

$$= \int_0^2 \int_0^1 \left(2yz+\frac{z^3}{3}+xz\right)_0^3 dydx$$

$$= \int_0^2 (3y^2 9y+3x)_0^1 dx$$

$$= \int_0^2 (6y+9+3x)dydx$$

$$= \int_0^2 (3+9+3x)dx$$

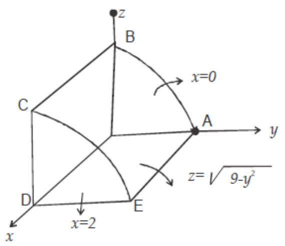
$$= \left(12x+\frac{3x^2}{2}\right)_0^2$$

$$= 24+6$$

$$= 30$$

4. Find $\int_v (\nabla.\vec{A})dV$ where $\vec{A} = 2x^2 y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}, V$ being the region in the first octant bounded by $y^2 + z^2 = 9$ & $x = 2$.

Solution:



$$\nabla.\vec{A} = 4xy - 2y + 8xz$$

To cover the volume of the region shown in the figure, we take $x = 0$ to $x = 2$, $y = 0$ to $y = 3$ and $z = 0$ to $z = \sqrt{9-y^2}$

$$\int_v \nabla.\vec{A}dV = \int_{x=0}^{2}\int_{y=0}^{3}\int_{0}^{2z\sqrt{9-y^2}}(4xy-2y+8xz)dzdydx$$

$$=\int_0^2\int_0^3 (4xyz-2yz+4xz^2)_0^{\sqrt{9-y^2}}dxdy$$

$$=\int_0^2\int_0^3 \left((4x-2)y\sqrt{9-y^2}+4x(9-y^2)\right)dxdy$$

$$=\int_0^3 \left(y\sqrt{9-y^2}\left(\frac{4x^2}{2}-2x\right)+(9-y^2)\frac{4x^2}{2}\right)_0^2 dy$$

$$=\int_0^3 \left(y\sqrt{9-y^2}(2x^2-2x)+(9-y^2)2x^2\right)_0^2 dy$$

$$=4\int_0^3 y\sqrt{9-y^2}dy+8\int_0^3 y(9-y^2)dy$$

Let $9-y^2=t^2$

$-2ydy=2tdt$

$ydy=-tdt$

If $y=0,\quad t=3$

$y=3,\quad t=0$

$$\int_v (\nabla.\vec{A})dV = 4\int_3^0 \sqrt{t^2}(-tdt)+8\left(9y-\frac{y^3}{3}\right)_0^3$$

$$=4\int_0^3 t^2dt+8(27-9)$$

$$=4\left(\frac{t^3}{3}\right)_0^3+144=4\left(\frac{27}{3}\right)+144=36+144=180$$

5. Find $\vec{F}=4x\hat{i}-2y^2\hat{j}+z^3\hat{k}$ and V is the volume enclosed by $x^2+y^2=a^2$, $z=h$ and prove that $\iiint_v (\nabla.\vec{F})dV=\pi a^2(4h+h^3)$

Solution:

Given $\vec{F}=4x\hat{i}-2y^2\hat{j}+z^3\hat{k}$

$$\nabla.\vec{F} = 4 - 4y + 3z^2$$

Also, on the circle $x^2 + y^2 = a^2$, as y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$, x varies from $-a$ to a

$$\iiint_v \nabla.\vec{F}dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h (4 - 4y + 3z^2)dzdydx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (4z - 4zy + z^3)_0^h dydx$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (4h - 4yh + h^3)dydx$$

$$= \int_{-a}^{a} (4hy - 2y^2h + h^3y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} (4y + h^3)y - 2y^2h)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} [4h + h^3)\sqrt{a^2 - x^2} - 2(a^2 - x^2)h - (4h + h^3)(-\sqrt{a^2 - x^2}) + 2(a^2 - x^2)h]dx$$

$$= \int_{-a}^{a} [4h + h^3)\left(2\sqrt{a^2 - x^2}\right) - 2h(a^2 - x^2 - a^2 + x^2)]dx$$

$$= 2\int_{-a}^{a} (4h + h^3)\sqrt{a^2 - x^2}dx$$

$$= 2(4h + h^3)\int_{-a}^{a} \sqrt{a^2 - x^2}dx$$

$$= 2(4h + h^3)\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_{-a}^{a}$$

$$= 2(4h + h^3)\left(\frac{a^2}{2}\sin^{-1}1 - \frac{a^2}{2}\sin^{-1}(-1)\right)$$

$$= 2(4h + h^3)\left(\frac{a^2}{2}.\frac{\pi}{2} - \frac{a^2}{2}\left(\frac{-\pi}{2}\right)\right)$$

$$= \pi a^2(4h + h^3)$$

6. *S.T* if $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, $\iiint_v (\nabla.\vec{A})dV = \frac{12}{5}\pi R^5$ where *V* is the volume enclosed by the sphere of radius *R* with origin as centre.

Solution:

$$\iiint_v (\nabla.\vec{A})dV = \iiint_v (3x^2 + 3y^2 + 3z^2)dV$$

$$= 3\iiint_v (x^2 + y^2 + z^2)dV$$

To evaluate the integral, we consider the transformation $x = r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z = r\cos\theta$. Then using the Jacobian, we obtain $dV = r^2 \sin\theta dr d\theta d\phi$, where r changes from 0 to R, θ from 0 to π and ϕ from 0 to 2π.

$$\iiint_v \nabla.(\vec{A}d\vec{V}) = 3\int_0^R \int_0^\pi \int_0^{2\pi} r^2 r^2 \sin\theta d\phi d\theta dr$$

$$= 3\int_0^R \int_0^\pi r^4 \sin\theta (\phi)_0^{2\pi} d\theta dr$$

$$= 3\int_0^R \int_0^\pi r^4 \sin\theta (2\pi - 0) d\theta dr$$

$$= 6\pi \int_0^R \int_0^\pi r^4 \sin\theta \, d\theta dr$$

$$= 6\pi \int_0^R r^4 (-\cos\theta)_0^\pi dr$$

$$= -6\pi \int_0^R r^4 (\cos\pi - \cos 0) \, dr$$

$$= -6\pi \times -2\int_0^R r^4 dr$$

$$= 12\pi \left(\frac{r^5}{5} \right)_0^R$$

$$= \frac{12\pi R^5}{5}$$

Exercise

1. Show that $\iint_S \vec{F}.\hat{n}d\vec{s} = \frac{3}{2}$, where $F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the x=0, x=1, y=0, y=1, z=0, and z=1.

2. Evaluate $\iint_S \vec{F}.\hat{n}d\vec{s}$ where, $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$in the first octant.

3. Evaluate $\iint_S (z\hat{i} + x\hat{j} - y^2z\hat{k})d\,\vec{s}$ where S is the surface of the cylinder $x^2 + y^2 = 1$ in the first octant between the planes z=0 &z=2.

4. Find the area of the surface of the portion of the plane 3x+2y+6z=6 contained in the first octant.

5. Evaluate $\iint_S \vec{A}.\hat{n}ds$ if $\vec{A} = (x^2 + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane 2x+y+2z=6 in the first octant.

6. Evaluate $\iiint_V (\nabla.\vec{F})d\vec{V}$ where $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ and V is the region bounded by x=0, y=0, z=0 & 2x+2y+z = 4.

7. Evaluate $\iiint_V \vec{F}.d\vec{V}$ where $\vec{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$ and V is the volume of the region enclosed by the cylinder $x^2 + y^2 = a^2$ between the planes z=0, z=c.

8. Evaluate $\iiint_V \nabla.\vec{A}d\vec{V}$ if $\vec{A} = 2x^2Y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ where V is the region bounded by the cylinder $y^2 + z^2 = 9$& the plane $x = 2$.

9. Evaluate $\iiint_V \nabla.\vec{F}d\vec{V}$ where $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped bounded by x=0, x=a, y=0, y=b, z=0, z=c.

10. Evaluate $\iiint_V 45x^2y dV$ where V is the surface bounded by the plane x=0, y=0, z=0, 4x+2y+z=8.

Answers

1. 264

2. $\frac{25}{6}$

3. 202

4. 18π

6. $\frac{86}{105}$

7. $\frac{2}{3}$

9. 0

10. $\frac{427}{20}$

12. $\frac{3}{8}$

13. 3

14. $\frac{7}{2}$

15. 81

16. $\frac{8}{3}$

17. $\frac{a^4 c\pi}{4}\hat{k}$

18. 180

19. abc (a + b + c)

20. 128

Theorem

Gauss Divergence Theorem

If $\vec{F}$ be a vector point function having continuous partial derivation in the region bounded by a closed surface S, then where $\iiint\limits_V (\nabla.\vec{F})dV = \iint\limits_s \vec{F}.\hat{n}ds$ where $\hat{n}$ is the unit outward normal at any point of the surface.

Proof

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\iiint_v (\nabla.\vec{F})dV = \iiint_v \left(i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})dV$$

$$= \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz \qquad \ldots (1)$$

Assume that a closed surface S is such that any line parallel to the coordinate axis intersects S at the most at two points. Divide the surface S into two parts S_1 the lower and S_2, the upper part.

Let $Z_1 = F_1(x, y)$ and $Z_2 = F_2(x, y)$ be the equation and $\hat{n}_1$ and $\hat{n}_2$ be the normals to the surface S_1 and S_2respectively. Let R be the projection of the surface S on xy-plane.

$$\iiint_v \frac{\partial F_3}{\partial F} dxdydz = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \left(\frac{\partial F_3}{\partial z}\right)dzdxdy\right]$$

$$= \iint_R [F_3(x, y, z)]_{f_1}^{f_2} dxdy$$

$$= \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]dxdy$$

$$= \iint_R [F_3(x, y, f_2)dxdy - \iint_R [F_3(x, y, f_1)dxdy \qquad \ldots (2)$$

$dxdy$ = projection of ds on xy-plane = $\hat{n}.\hat{k}\ ds$

For Surface S_2: $Z = F_2(x, y)\ dxdy = \hat{n}.\hat{k}\ dS_2$

For Surface S_1: $Z = F_1(x, y)\ dxdy = \hat{n}.\hat{k}\ ds$

Substituting in eqn (2)

$$= \iiint_v \frac{\partial F_3}{\partial z} dxdydz = \iint_{S_2} F_3(\hat{n}_2.\hat{k})ds_2 - \iint_{S_2} F_3.(-\hat{n}_1.\hat{k})ds_1$$

$$= \iint_{S_2} F_3(\hat{n}_2.\hat{k})ds_2 + \iint_{S_2} F_3(\hat{n}_1.\hat{k})ds_1$$

$$= \iint_{S_2} F_3(\hat{n}.\hat{k})ds \qquad \ldots (3)$$

Projecting the surface S on yz and zx plane, we get

$$\iiint_v \frac{\partial F_1}{\partial z} dxdydz = \iint_S F_1(\hat{n}.\hat{i})ds \qquad \ldots (4)$$

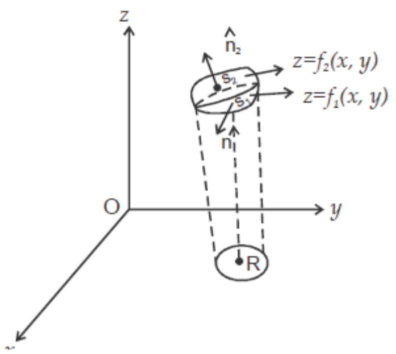
$$\iiint_v \frac{\partial F_2}{\partial z} dxdydz = \iint_S F_2(\hat{n}.\hat{i})ds \qquad \ldots (5)$$

Sub (3), (4) & (5) in (1)

$$\iiint_v (\nabla.\vec{F})dV \quad = \iint_S F_1(\hat{n}.\vec{i})ds + \iint_S F_2(\hat{n}.\vec{j})ds_2 + \iint_S F_3(\hat{n}.\vec{k})ds$$

$$= \iint_S (F_1\vec{i} + F_2\vec{j} + F_3\vec{i}).\hat{n}ds$$

$$= \iint_S \vec{F}.\hat{n}ds$$



Problems

1. Verify divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 0xy)\vec{k}$ taken over the rectangular parallopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Solution:

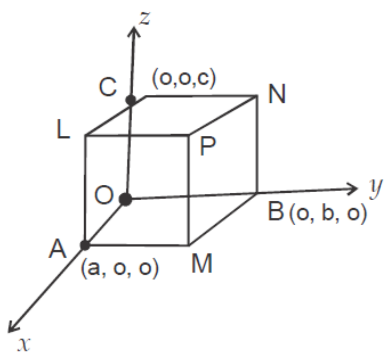
For verification of divergence theorem, we shall evaluate the volume and surfaceseparately and show that they are equal.

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 0xy)\vec{k}$

$$\nabla.\vec{F} = div\ F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right).\vec{F}$$

$= 2x + 2y + 2z$

$= 2(x + y + z)$



$dV = dxdydz$ or $dV = dzdydx$

x varies from 0 to a

y varies from 0 to b

z varies from 0 to c

$$\therefore \iiint_V (\Delta.\vec{F})dV = \int_0^a \int_0^b \int_0^c 2(x+y+z)dzdydx$$

$$= 2\int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right] dydx$$

$$= 2c\int_0^a \int_0^b \left[x + y + \frac{c}{2}\right] dydx$$

$$= 2c\int_0^a \left[xz + \frac{y^2}{2} + \frac{c}{2}y\right]_0^b dx$$

$$= 2bc\int_0^a \left[x\frac{b}{2} + \frac{c}{2}\right] dx$$

$$= 2bc\left[\frac{x^2}{2} + \frac{bx}{2} + \frac{cx}{2}\right]_0^a = abc[a+b+c]$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallopiped into 6 parts.

S_1 = face OAMB S_2 = face CLPN S_3 = face OBNC

S_4 = face AMPL S_5 = face OALC S_6 = face BNPM

$$\therefore \iint_C \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iint_{S_3} \vec{F}.\hat{n}ds \iint_{S_4} \vec{F}.\hat{n}ds + \iint_{S5} \vec{F}.\hat{n}ds \iint_{S6} \vec{F}.\hat{n}ds$$

Face $S_1 : z = 0; ds = dxdy; \hat{n} = -\vec{k}$

$$\vec{F} = x^2\bar{i} + y^2\vec{j} - xy\vec{k}$$

$$\vec{F}.\hat{n} = xy$$

$$\therefore \iint_{S_1} \vec{F}.\hat{n}ds \int_0^a \int_0^b xydxdy = \left[\frac{x^2}{2}\right]_0^a \left[\frac{x^2}{2}\right]_0^b$$

$$=\frac{1}{4}a^2b^2$$

Face $S_2: z=c; \hat{n}=\vec{k}; ds=dxdy$

$$\vec{F}=(x^2-cy)\vec{i}+(y^2-cx)\vec{j}+(c^2-xy)\vec{k}$$

$$\vec{F}.\hat{n}=(c^2-xy)$$

$$\therefore \iint_{S_2} \vec{F}.\hat{n}ds=\int_0^a\int_0^b (c^2-xy)dydx$$

$$=\int_0^a\left[c^2y-\frac{xy^2}{2}\right]_0^b dx$$

$$=\int_0^a\left[c^2b-\frac{xb^2}{2}\right]dx$$

$$=b\int_0^a\left[c^2-\frac{xb}{2}\right]dx$$

$$=b\left[c^2x-\frac{x^2}{4}b\right]_0^a$$

$$=b\left[ac^2-\frac{a^2b}{4}\right]$$

$$=abc^2-\frac{a^2b^2}{4}$$

Face $S_3: \hat{n}=-\vec{i}; ds=dydz; x=0$

$$\vec{F}=-yz\vec{i}+y^2\vec{j}+z^3\vec{k}$$

$$\vec{F}.\hat{n}=yz$$

$$\therefore \iint_{S_3} \vec{F}.\hat{n}ds=\int_0^b\int_0^c yzdydz$$

$$=\left[\frac{y^2}{2}\right]_0^b\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{1}{4} b^2 c^2$$

Face $S_4 : x = 0; \hat{n} = -\vec{i}; ds = dydz$

$$\vec{F} = -(a^2 yz)\bar{i} + (y^2 - az)\vec{j} + (z^2 - ay)\vec{k}$$

$$\vec{F}.\hat{n} = a^2 - yz$$

$$\iint_{S_4} \vec{F}.\hat{n} ds = \int_0^c \int_0^b (a^2 - yz) dydz$$

$$= \int_0^c \left[a^2 y - \frac{y^2}{2} z \right]_0^b dz$$

$$= \int_0^c \left[a^2 b - \frac{b^2}{2} z \right] dz$$

$$= \left[a^2 bz - \frac{b^2}{4} z^2 \right]_0^c$$

$$= bc\left[a^2 - \frac{1}{4} bc \right]$$

Face $S_5 : y = 0; \hat{n} = -\vec{i}; ds = dzdx$

$$\vec{F} = x^2 \bar{i} - zx\vec{j} + z^2 \vec{k}$$

$$\vec{F}.\hat{n} = zx$$

$$\iint_{S_5} \vec{F}.\hat{n} ds = \int_0^a \int_0^c dxdzdx = \int_0^a xdx . \int_0^c zdz$$

$$= \frac{1}{2} a^2 . \frac{1}{2} c^2 = \frac{1}{4} a^2 c^2$$

Face $S_6 : y = b; \hat{n} = \vec{j}; ds = dzdx$

$$\vec{F} = (x^2 - bz)\bar{i} + (b^2 - zx)\vec{j} + (z^2 = bx)\vec{k}$$

$$\vec{F}.\hat{n} = b^2 - zx$$

$$\iint_{S_6} \vec{F}.\hat{n}ds = \int_0^a \int_0^c (b^2 - zx)dzdx$$

$$= \int_0^a \left(b^2 z - \frac{z^2}{2}x\right)_0^c dx$$

$$= \int_0^a \left(b^2 c - \frac{c^2}{2}x\right) dx$$

$$= \left(b^2 cx - \frac{c^2}{2}x^2\right)_0^a$$

$$= ac\left(b^2 - \frac{1}{4}ac\right)$$

$$\iint_S \vec{F}.\hat{n}ds = abc^2 + ab^2c + a^2bc$$

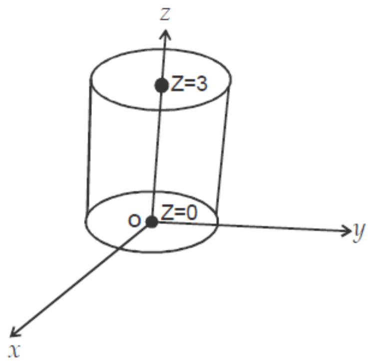
$$= abc(a+b+c)$$

$$\iint_{S_6} \vec{F}.\hat{n}ds = \iiint_V dV$$

Hence Gauss divergence theorem is verified.

2. Verify divergence theorem for $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region bounded by the cylinder $x^2 + y^2 + 4$, $z = 0$, $z = 3$

Solution:



Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

Gauss divergence theorem is

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V div\vec{F}dV$$

$$div\vec{F} = \nabla.\vec{F}$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial x}(z^2)$$

$$= 4 - 4y + 2z$$

Also given $x^2 + y^2 = 4$

$$\Rightarrow y^2 = 4 - x^2$$

$$y = \pm\sqrt{4-x^2}$$

And when $y = 0 \quad \Rightarrow 0 = \sqrt{4-x^2}$

(ie) $x = \pm 2$

$$\therefore \iiint_V \nabla.\vec{F} = \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{z=3} (4-4y+2z)\,dzdydx$$

$$= \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4-4y) + \frac{2z^2}{2}\right]_0^3 dydx$$

$$= \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4-4y)3+9]\,dydx$$

$$= \int_{-2}^{2} \left[21y - \frac{12y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[21\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6\left((4-x^2-(4-x^2)\right)\right] dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\,dx = 84\int_{0}^{2} \sqrt{4-x^2}\,dx \qquad \left[\because \sqrt{4-x^2}\; is\ even\ function\right]$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84[0 + 2\sin^{-1}1 - 0]$$

$$= 84 \times 2 \times \frac{\pi}{2}$$

$$= 84\pi.$$

We shall now compute the surface integral $\iint_S \vec{F}.\hat{n}ds$. S consists of the bottom surface S_1, top surface S_2 and the curved surface S_3 of the cylinder.

On $S_1 : z = 0;\ \hat{n} = -\vec{k}$

$\vec{F}.\hat{n} = -z^2 = 0$

$\therefore \iint_{S_1} \vec{F}.\hat{n}ds = 0$

On $S_2 : z = 3;\ \hat{n} = \vec{k}$

$\vec{F}.\hat{n} = z^2 = 9$

$$ds = \frac{dxdy}{|\vec{n}.\vec{k}|} = d\frac{dxdy}{|\vec{k}.\vec{k}|} = dxdy$$

$$\iint_{S2} \vec{F}.\hat{n}ds = \iint_{S2} 9dxdy = 9\iint_{S2} dxdy$$

$= 9 \times$ area of circle S_2

$= 9\ (\pi.2^2)$ (radius of circle = 2)

$= 36\pi$

On $S_3 : x^2 + y^2 = 4$

Let $\phi = x^2 + y^2 - 4$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial x} \qquad \left[\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2 + y^2}}\right.$$

$$= 2(x\vec{i} + y\vec{j}) \qquad \left. = \frac{1}{2}(x\vec{i} + y\vec{j})(\because x^2 + y^2 = 4, \sqrt{x^2 + y^2 = 2})\right]$$

$$\vec{F}.\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}).\frac{1}{2}(x\vec{i} - y\vec{j})$$

$$= 2x^2 - y^3$$

Since S_3 is the surface of cylinder $x^2 + y^2 + 4$, we use cylindrical polar co-ordinates to evaluate $\iint_{S_3} \vec{F}.\hat{n}ds$

$x = 2\cos\theta, y = 2\sin\theta, z = z$ $\quad$ $(\therefore x = r\cos\theta$

$ds = 2d\theta dz$ $\quad$ $y = r\sin\theta$ where $r=2$

$ds = rd\theta dz)$

θ varies from 0 to 2π

z varies from 0 to 3.

$$\therefore \iint_{S_3} \vec{F}.\hat{n}ds = \int_0^3 \int_0^{2\pi} (2\times 4\cos^2\theta - 8\sin^2\theta)d\theta dz$$

$$= 16\int_0^3 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)d\theta dz$$

$$= 16\int_0^3 \int_0^{2\pi} \left[\frac{1+\cos 2\theta}{2} - \frac{1}{4}(3\sin\theta - \sin 3\theta)\right]d\theta dz$$

$$= 16\int_0^3 \left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)\right]_0^{2\pi} dz$$

$$= 16\int_0^3 \left[\frac{1}{2}\left(2\pi + \frac{\sin 4\pi}{2} - 0\right) - \frac{1}{4}\left(-3\cos 2\pi + \frac{\cos 6\pi}{3} - 3\cos 0 + \frac{\cos 0}{3}\right)\right]dz$$

$$= 16\int_0^3 \left(\pi + \frac{3}{4} - \frac{1}{12} - \frac{3}{4} + \frac{1}{12}\right)dz$$

$$= 16\pi\int_0^3 dz = 16\pi[z]_0^3$$

$$= 16\pi\times 3$$

$$= 48\pi$$

$$\therefore \iint_S \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds + \iint_{S_3} \vec{F}.\hat{n}ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi \qquad \ldots (2)$$

Thus from (1) & (2) $\quad \therefore \iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$

Hence Gauss divergence theorem is verified.

Ex 3: Verify Gauss divergence theorem for $\vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$ over the region bounded by the upper hemisphere $x^2 + y^2 + z^2 = a^2$ and the plane $z=0$.

Solution:

Gauss divergence theorem is $\iint\limits_S \vec{F}.\hat{n}ds = \iiint\limits_V (\nabla.\vec{F})dV$

Given $\vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(a(x+y)) + \frac{\partial}{\partial y}(a(y-x)) + \frac{\partial}{\partial z}(z^2)$$

$$= a + a + 2z + 2(a+z)$$

$$\therefore \iiint\limits_V (\nabla.\vec{F})dV = \iiint\limits_V 2(a+z)dV$$

$$= 2a\iiint\limits_V dv + 2\iiint\limits_V zdv$$

$$= 2aV + 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}} zdzdydx$$

$$= 2aV + 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 2aV + \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(a^2 - x^2 - y^2)dydx$$

$$= 2aV + 2\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}(a^2 - x^2 - y^2)dydx \quad \text{(since } (a^2 - x^2 - y^2) \text{ is even)}$$

$$= 2aV + 2\int_{-a}^{a}\left((a^3 - x^2)y - \frac{y^3}{3}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= 2aV + 2\int_{-a}^{a}\left[(a^2 - x^2)\sqrt{a^2 - x^2} - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3}\right]dx$$

$$= 2aV + 2\int_{-a}^{a}\left[(a^2 - x^2) - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3}\right]dx$$

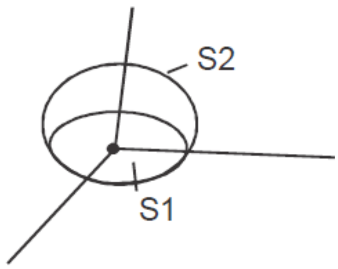
$$= 2aV + 2\int_{-a}^{a}\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}dx$$

$$= 2aV + \frac{8}{3}\int_{-a}^{a}(a^2 - x^2)^{3/2}dx \qquad (\therefore (a^2 - x^2) \text{ is even}).$$

$$= 2aV + \frac{8}{3}I$$

where $I = \int_{-a}^{a}(a^2 - x^2)^{\frac{3}{2}}dx$

Limits of x : $x = 0$ to $x = a$.



Let $x = a\sin\theta \qquad dx = a\cos\theta d\theta$

when $x = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

$x = a$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_{0}^{\frac{\pi}{2}}(a^2 - a^2\sin^2\theta)^{\frac{3}{2}}a\cos\theta d\theta$$

$$= \int_{0}^{\frac{\pi}{2}}a^3\cos^3\theta a\cos\theta = a^4\int_{0}^{\frac{\pi}{2}}\cos^4\theta d\theta$$

$$= a^4\int_{0}^{\frac{\pi}{2}}\cos^4\theta d\theta$$

$$= a^4.\frac{4-1}{4}.\frac{4-3}{4-2}.\frac{\pi}{2}$$

$$= a^4.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$I = \frac{3\pi a^4}{16}$$

$$\therefore \iiint_V \nabla.\vec{F}dV = 2aV + \frac{8}{3}\times\frac{3\pi a^4}{16}$$ (Volume of hemisphere = $\frac{2\pi}{3}a^4$)

$$= \frac{4\pi a^4}{3} + \frac{\pi a^4}{2}$$

$$= \frac{11}{6}\pi a^4 \qquad \ldots (1)$$

Now we shall compute the double integral

$$\iint_S \vec{F}.\hat{n}ds$$

S consists of S_1 and S_2 ((ie) flat bottom surface & curved surface)

$$\therefore \iint_S \vec{F}.\hat{n}ds = \iint_{S_1} \vec{F}.\hat{n}ds + \iint_{S_2} \vec{F}.\hat{n}ds$$

On $S_1 : z = 0, \hat{n} = -\vec{k}$

$$\vec{F}.\hat{n} = (a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}).(-\vec{k})$$

$$= -z^2 = 0 \quad (\because z = 0)$$

$$\therefore \iint_S \vec{F}.\hat{n}ds = 0$$

On S_2 : Surface in $x^2 + y^2 + z^2 = a^2$

Let $\phi = x^2 + y^2 + z^2 - a^2$

$$\nabla\phi = 2(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2+y^2+z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$\vec{F}.\hat{n} = \left[a(x+y)\vec{i} + a)(y-x)\vec{j} + z^2\vec{k}\right].\left[\frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}\right]$$

$$= (x+y)x + (y-x)y + \frac{z^3}{a}$$

$$= x^2 + y^2 + \frac{z^3}{a}$$

$\therefore \iint \vec{F}.\hat{n}ds = \iint_R \vec{F}.\hat{n}\frac{dxdy}{|\hat{n}.\vec{k}|}$ where R is the projection of S_2 on the xy plane

$$\therefore \iint \vec{F}.\hat{n} = \iint_R \left(x^2 + y^2 + \frac{z^3}{a}\right)\frac{dxdy}{(z/a)}$$

$$= \iint_R \left(\frac{a(x^2+y^2)}{z} + z^2\right)dxdy$$

$$= \iint_R \left(\frac{a(x^2+y^2)}{z} + (a^2 - x^2 - y^2\right)dxdy$$

Change to polar co-ordinates

$x = r\cos\theta$ $y = r\sin\theta$ $r^2 = x^2 + y^2$ $dxdy = rdrd\theta$

$$\therefore \iint_{S_2} \vec{F}.\hat{n}ds = \int_0^a \int_0^{2\pi} \left[\frac{ar^2}{\sqrt{a^2-r^2}} + (a^2 - r^2)\right]rdrd\theta$$

$$= \int_0^a \int_0^{2\pi} \left[\frac{-a(a^2-r^2)+a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right]rdrd\theta$$

$$= \int_0^a \int_0^{2\pi} \left[-a\sqrt{a^2-r^2} + \frac{a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right]rdrd\theta$$

$$= \int_0^{2\pi} d\theta.\int_0^a \left[-a\sqrt{a^2-r^2} + \frac{a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right]rdr$$

$$= [\theta]_0^{2\pi}.\int_0^a \left[-ar\sqrt{a^2-r^2} + a^3(a^2-r^2)^{-1/2}.r + (a^2-r^2).r\right]dr$$

$$= 2\pi.\int_0^a \frac{a}{2}(a^2-r^2)^{1/2}(-2r).dr - \frac{a^3}{2}\int_0^a (a^2-r^2)^{-1/2}(-2r)dr + \int_0^a (a^2-r^2)rdr$$

$$= 2\pi.\left(\left(\frac{a}{2}.\frac{(a^2-r^2)^{3/2}}{3/2}\right)_0^a - \frac{a^3}{2}.\left(\frac{(a^2-r^2)^{1/2}}{1/2}\right)_0^a + \left(\frac{a^2\times r^2}{2} - \frac{r^4}{4}\right)_0^a\right)$$

$$= 2\pi\left[\frac{-a^4}{3} + a^4 + \frac{a^4}{4}\right]$$

$$= 2\pi\left[\frac{11a^4}{12}\right] = \frac{11\pi a^4}{6}$$

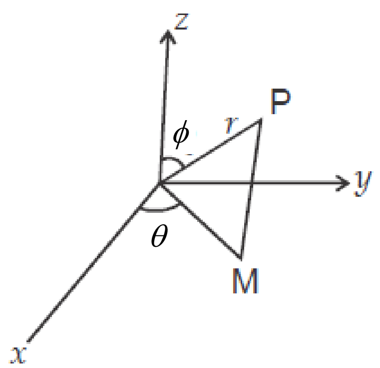
$$\therefore \iint_{S_2} \vec{F}.\hat{n}ds = \frac{11\pi a^4}{6} \qquad \dots (2)$$

From (1) & (2) $\iint_{S_2} \vec{F}.\hat{n}ds = \iiint_V \nabla.\vec{F}dV$

Hence Gauss divergence theorem is verified.

4. Using divergence theorem evaluate $\iint_S \vec{F}.\hat{n}ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the surface $x^2 + y^2 + z^2 = a^2$

Solution:



Gauss divergence theorem is

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V \nabla.\vec{F}dV$$

$$\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^3)\frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial y}(z^3)$$

$$= 3(x^2 + y^2 + z^2)$$

$$\therefore \iint_S \vec{F}.\hat{n}ds = \iiint_V 3(x^2 + y^2 + z^2)dxdydz$$

We shall evaluate this triple integral by using spherical polar co-ordinates..

$x = r \sin \theta \cos\phi$ $\qquad$ $y = r \sin \theta \sin \phi$ $\qquad$ $z = r \cos \theta$

then $dxdydz = \left|\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\right| drd\theta d\phi$

$$= r^2 \sin \theta drd\theta d\phi$$

$x^2 + y^2 + z^2 = r^2$

r varies from 0 to a

θ varies from 0 to π

ϕ varies from 0 to 2π

$$\therefore \iint_S \vec{F}.\hat{n}ds = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3r^4 \sin\theta dr d\theta d\phi$$

$$= 3\int_0^{2\pi} d\phi.\int_0^{\pi} \sin\theta d\theta.\int_0^{a} r^4 dr$$

$$= 3(\phi)_0^{2\pi}.(-\cos\theta)_0^{\pi}.\left(\frac{r^4}{5}\right)_0^a$$

$$= 3\times 2\pi(2).\frac{a^5}{5}$$

$$= \frac{12\pi a^4}{5}$$

Note

Here $div\ \vec{F}$ =3(x^2 +y^2 + z^2). Since the equation of the surface $x^2 + y^2 + z^2 = a^2$, we cannot replace $x^2 + y^2 + z^2 = a^2$ in div F, since $x^2 + y^2 + z^2 = a^2$ is true only for points on S but $\vec{F}$ is defined inside and on S.

5. If $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$ a,b,c are constant. Show that $\iint_S \vec{F}.\hat{n}ds = \frac{4\pi}{3}(a+b+c)$ where S is the surface of the unit sphere.

Solution:

By Gauss divergence theorem we have

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$$

$$= \iiint_V \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right)dV$$

$$= \iiint_V (a+b+c)dV$$

$$= (a+b+c)\times \text{ Volume of unit sphere.}$$

$$= (a+b+c)\frac{4\pi}{3}\times 1^3$$

$$= \frac{4\pi}{3}((a+b+c).$$

6. Using divergence theorem evaluate $\iint_S \nabla r^2.\hat{n}ds$ where S in a closed surface.

Let $\vec{F} = \nabla r^2$ where $r = x\vec{i} + y\vec{j} + z\vec{k}$

$\& \, r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

By Gauses divergence theorem

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V (\nabla.\vec{F})dV$$

$$= \iiint_V \nabla.(\nabla.r^2)dV$$

$$= \iiint_V \nabla^2 r^2 dV = \iiint_V \left(\Sigma\frac{\partial^2}{\partial x^2}\right)(x^2+y^2+z^2)dV$$

$$= \iiint_V (2+2+2)dV$$

$$= 6\iiint_V dV$$

= 6 × volume of closed surfaces.

Stoke's Theorem

If S be an open surface bounded by a closed curve C and $\vec{F}$ be a continuous and differentiable vector function then $\int_C \vec{F}.d\bar{r} = \iint_S Curl\vec{F}.\hat{n}ds$ where $\hat{n}$ is the unit outward normal at any point of the surfaces.

Proof

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\iint_S (\nabla\times\vec{F}).\hat{n}ds = \iint_S \nabla\times(F_1\vec{i} + F_2\vec{j} + F_3k).\hat{n}ds$$

$$= \iint_S (\nabla\times F_1\vec{i}).\hat{n}ds + \iint_S (\nabla\times F_2\vec{j})\hat{n}ds + \iint_S (\nabla\times F_3\vec{k}).\hat{n}ds \qquad \ldots (1)$$

Consider, $\iint_S (\nabla \times F_1\vec{i}).\hat{n}ds + \iint_S \left[\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times F_1\vec{i}\right].\hat{n}ds$

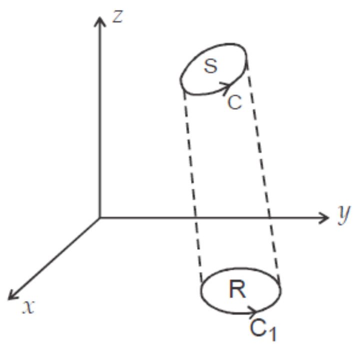
$$= \iint_S \left(-\vec{k}\frac{\partial F_1}{\partial y} + \vec{j}\frac{\partial F_1}{\partial z}\right).\hat{n}ds$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\vec{j}.\hat{n} - \frac{\partial F_1}{\partial y}\vec{k}.\hat{n}\right).ds \qquad \ldots (2)$$

Let equation of the surface S be $z=f(x,y)$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= x\vec{i} + y\vec{j} + f(x,y)\vec{k}$



differentiating partially with respect to y.

$$\frac{\partial \vec{r}}{\partial y} = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$$

Taking dot product with $\hat{n}$

$$\frac{\partial \vec{r}}{\partial y}.\hat{n} = \vec{j}.\hat{n} + \frac{\partial f}{\partial y}\vec{k}.\hat{n} \qquad \ldots (3)$$

$\frac{\partial \vec{r}}{\partial y}$ is tangential and $\hat{n}$ is normal to the surface S, $\frac{\partial \vec{r}}{\partial y}.\hat{n} = 0$ substituting in equation (3)

$$0 = \vec{j}.\hat{n} + \frac{\partial f}{\partial y}\vec{k}.\hat{n}$$

$$\vec{j}.\hat{n} = \frac{-\partial f}{\partial y}\vec{k}.\hat{n} = \frac{-\partial z}{\partial y}\vec{k}.\hat{n} \quad [\because z = f(x,y)]$$

Substituting in equation (2):

$$\iint_S (\nabla \times F_1 i).\hat{n}ds = \iint_S \left[\frac{\partial F_1}{\partial z}\left(\frac{-\partial z}{\partial y}\vec{k}.\hat{n}\right) - \frac{\partial F_1}{\partial y}\vec{k}.\hat{n}\right]ds$$

$$= -\iint_S \left(\frac{\partial F_1}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial F_1}{\partial y}\right)\vec{k}.\hat{n}ds \qquad \ldots (4)$$

Equation of the surface is $z = f(x,y)$

$F_1(x,y,z) = F_1(x,y,f(x,y)) = G(x,y)$ say

differentiating partially with respect to y,

$$\frac{\partial G}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \cdot \frac{\partial z}{\partial y}$$

Substituting in equation (4).

$$= \iint_S (\nabla \times F_1 \vec{i}).\hat{n}ds = -\iint_S \frac{\partial G}{\partial y} \vec{k}.\hat{n}ds$$

Let R is the protection of S on xy plane and dxdy is the protection of ds on xy plane then $\vec{k}.\hat{n} = dxdy$

$$= \iint_S (\nabla \times F_1 \vec{i}).\hat{n}ds = -\iint_S \frac{\partial G}{\partial y} dxdy$$

$$= \iint_{C_1} Gdx$$

Since the value of G at each point (x,y) of 1 C is some as the value of F_1at the each point (x,y,z) of C and dx is same for both the $C_1$and C we get $\iint_S (\nabla \times F_1 \vec{i}).\hat{n}ds = \int_C F_1 dx$ … (5)

Similarly, by projecting surface Son to yz and zx planes.

$$\iint_S (\nabla \times F_2 \vec{j}).\hat{n}ds = \int_C F_2 dy \qquad \dots (6)$$

$$\iint_S (\nabla \times F_3 \vec{k}).\hat{n}ds = \int_C F_3 dz \qquad \dots (7)$$

Substituting equations (5), (6) and (7) in equation (1)

$$\iint_S \nabla \times \vec{F}.\hat{n}ds = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C \vec{F}.dr$$

Problems

1) Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by the lines x=0, x=a, y=0, y=b

Solution:

By Stoke's theorem $\iint\limits_S Curl\,\vec{F}.\hat{n}ds = \int\limits_C \vec{F}.dr$

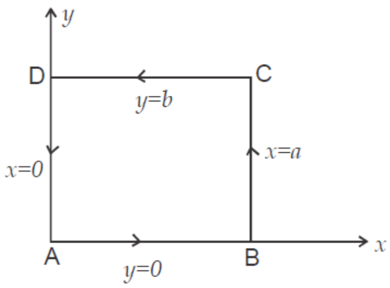
To find $\int\limits_C \vec{F}.dr$

$\int\limits_C \vec{F}.dr = \int\limits_{AB} \vec{F}.\overrightarrow{dr} + \int\limits_{BC} \vec{F}.\overrightarrow{dr} + \int\limits_{CD} \vec{F}.\overrightarrow{dr} \int\limits_{DA} \vec{F}.\overrightarrow{dr}$

Now

$\vec{F}.dr = [(x^2 - y^2)\vec{i} + 2xy\vec{j}\,].[dx\vec{i} + dy\vec{j} + dz\vec{k}\,]$

$= (x^2 - y^2)dx + 2xydz$



Along AB:

y =0, dy = 0

$\int\limits_{AB} \vec{F}.\overrightarrow{dr} = \int\limits_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$

Along BC:

$x = a$, $dx = 0$

$\int\limits_{BC} \vec{F}.\overrightarrow{dr} = \int\limits_0^b 2aydy = 2a\left(\frac{y^2}{2}\right)_0^b = ab^2$

Along CD:

$y = b$, $dy = 0$

$\int\limits_{CD} \vec{F}.\overrightarrow{dr} = \int\limits_a^a (x^2 - b^2)dx$

$= \left(\frac{x^3}{3} - xb^2\right)_a^0$

$= 0 - \left(\frac{a^3}{3} - ab^2\right)$

$= \frac{-a^3}{3} + ab^2$

Along DA:

$x = 0,\ dx = 0$

$$\int_{DA} \vec{F}.\overrightarrow{dr} = 0$$

$$\int_{C} \vec{F}.\overrightarrow{dr} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0$$

$$= 2ab^2 \qquad \dots (1)$$

To find $\iint_S Curl\ \vec{F}.\hat{n}ds$

$$\text{Now, } Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y + 2y)$$

$$= 4\vec{k}$$

Surface S is the rectangle ABCD in xy plane

$$\hat{n} = \vec{k} \text{ and } ds = \frac{dxdy}{|\hat{n}.\vec{k}|} = \frac{dxdy}{|\vec{k}.\vec{k}|} = dxdy$$

$$\iint_S curl\vec{F}.\hat{n}ds \quad = \iint_S 4y\vec{k}.\vec{k}\ dxdy$$

$$= \int_0^a \int_0^b 4y\ dxdy$$

$$= \int_0^a 4\left(\frac{y^2}{2}\right)_0^b dx$$

$$= 2b^2(x)_0^a$$

$$= 2ab^2 \qquad \dots (2)$$

From equation (1) and (2).

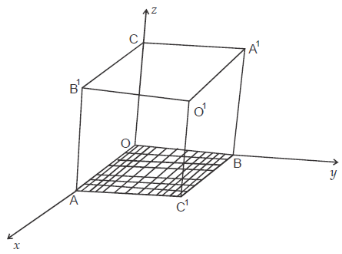
$$\iint_S curl\ \vec{F}.\hat{n}ds = \iint_S \vec{F}.\overrightarrow{dr}$$

Hence Stoke's theorem is verified.

2. Verify Stoke's theorem for $\vec{F} = (xy\vec{i} - 2yz\vec{j} - xz\vec{k})$ where S is the open surface of the rectangular parallelopiped formed by the planes x=0, x=1, y=0, y=2 and z=3 above the xoy plane.

Solution:

Stoke's Theorem is, $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$



To find $\int_C \vec{F}.\overrightarrow{dr}$

$$\vec{F}.\overrightarrow{dr} = (xy\vec{i} - 2yz\vec{j} - xz\vec{k}).(dx\vec{i} - dy\vec{j} - dz\vec{k})$$

$$= xydx - 2yzdy - xzdz$$

The boundary C lies on the plane z=0, $\vec{F}.\overrightarrow{dr} = xydx$

Along OA

y=0, dy=0

$$\int_{OA} \vec{F}.\overrightarrow{dr} = 0$$

Along AC^1

x=1, dx=0

$$\int_{AC^1} \vec{F}.\overrightarrow{dr} = 0$$

Along C^1B

y=2, dy=0

$$\int_{C^1B} \vec{F}.\overrightarrow{dr} = \int_1^0 2xdx = 2\left(\frac{x^2}{2}\right)_1^0 = -1$$

Along BO

x=0, dx=0

$$\int_{BO} \vec{F}.\overrightarrow{dr} = 0$$

$$\int_{C} \vec{F}.\overrightarrow{dr} = \int_{OA} \vec{F}.\overrightarrow{dr} + \int_{AC^1} \vec{F}.\overrightarrow{dr} + \int_{C^1B} \vec{F}.\overrightarrow{dr} + \int_{BO} \vec{F}.\overrightarrow{dr}$$

$$= 0 + 0 - 1 + 0 = 1 \qquad \dots (1)$$

To find $\iint_S curl\ \vec{F}.\hat{n}ds$

Now $\quad Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$

$$= 2y\vec{i} + z\vec{j} - x\vec{k}$$

$$\iint_S curl\ \vec{F}.\hat{n}ds = \iint_{\substack{x=0 \\ \hat{n}=-\vec{i}}} curl\ \vec{F}.\hat{n}ds + \iint_{\substack{x=1 \\ \hat{n}=\vec{i}}} curl\ \vec{F}.\hat{n}ds + \iint_{\substack{y=0 \\ \hat{n}=\vec{j}}} curl\ \vec{F}.\hat{n}ds$$

$$+ \iint_{\substack{y=2 \\ \hat{n}=\vec{j}}} curl\ \vec{F}.\hat{n}ds + \iint_{\substack{z=0 \\ \hat{n}=\vec{k}}} curl\ \vec{F}.\hat{n}ds$$

$$= \int_0^3 \int_0^2 2ydydz + \int_0^3 \int_0^2 2ydydz - \int_0^1 \int_0^3 zdzdx + \int_0^1 \int_0^3 zdzdx - \int_0^2 \int_0^1 xdxdy$$

$$= -\int_0^2 \int_0^1 xdxdy$$

$$= \int_0^2 \left(\frac{x^2}{2}\right)_0^1 dy$$

$$= -\frac{1}{2}\int_0^3 dy$$

$$= -\frac{1}{2}(y)_0^2$$

$$= -\frac{1}{2}(2) = -1 \qquad \dots (2)$$

From equations (1) and (2), $\int_C \vec{F}.\overrightarrow{dr} = \int_S curl\vec{F}.\hat{n}ds$

Hence Stoke's theorem is verified.

3. Verify Stoke's theorem for $\vec{F} = -y\vec{i} + 2yz\vec{j} + y^2\vec{k})$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ and C is the circular boundary on the xoy plane.

Solution:

Stoke's Theorem is $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$

To find $\int_C \vec{F}.\overrightarrow{dr}$

Here C is the circle in the xoy plane whose equation is $x^2 + y^2 = a^2$ and whose parametric equations are $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$, $dz = 0$.

$$\therefore \int_C \vec{F}.d\vec{r} = \int_C (-y\vec{i} + 2yz\vec{j} + y^2\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C (-ydx + 2yzdy + y^2dz)$$

$$= \int_C -ydx$$

$$= \int_{x^2+y^2=a^2} -ydx$$

$$= \int_0^{2\pi} (-a\sin\theta)(-a\sin\theta)d\theta$$

$$= a^2\int_0^{2\pi} \sin^2\theta d\theta = a^2\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$$= \frac{a^2}{2}.2\pi$$

$$= \pi a^2 \qquad \dots (1)$$

To find $\iint_S curl\,\vec{F}.\hat{n}ds$

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{vmatrix}$$

$$= \vec{i}[2y - 2y] - \vec{j}[0 - 0] + \vec{k}[0 - (-1)]$$

$$curl\ \vec{F} = \vec{k}$$

Now, $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = x^2 + y^2 + z^2 - a^2 = 0$

Here $\nabla\phi = \vec{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2 - a^2) + \vec{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2 - a^2) + \vec{k}\frac{\partial}{\partial x}(x^2 + y^2 + z^2 - a^2)$

$$= \vec{i}\,2x + \vec{j}\,2y + 2z\vec{k}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

and $|\nabla\phi| = \sqrt{(2x^2 + (2y)^2 + (2z)^2} = \sqrt{4(x^2 + y^2 + z^2} = 2\sqrt{a^2} = 2a$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2a} + \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$\iint_S cur\vec{F}.\hat{n}dt = \iint_s \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}\right).\vec{k}\ ds$$

$$= \iint_s \frac{z}{a} ds$$

$= \iint_R \frac{z}{a}\frac{dx.dy}{\hat{n}.\vec{k}}$ where R is the projection of S on the xoy plane.

$$= \iint_R \frac{z}{a}\frac{dx.dy}{z/a}$$

$= \iint_R dxdy$ where R is the region enclosed by $x^2 + y^2 = a^2$

$$= \pi a^2 \qquad \ldots (2)$$

From (1) and (2); Stoke's theorem is verified.

4. Evaluate by Stoke's theorem $\int_C (e^x dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 + 4$, $z = 2$.

Solution:

By Stoke's Theorem $\iint_C \vec{F}.\overrightarrow{dr} = \iint_S \vec{F}.\hat{n}ds$

Here $\vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$

$$curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0$$

$$\therefore \int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$$

$$\int_C (e^x dx + 2ydy - dz) = 0$$

5. Evaluate $\int_C$ (sin zdx – cosxdy + sinydz)by using Stoke's theorem where C is the boundary of the rectangle defined by $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$

Solution:

By Stoke's Theorem $\int_C \vec{F}.\overrightarrow{dr} = \iint_S curl\vec{F}.\hat{n}ds$

Here $\vec{F} = \sin z\vec{i} - \cos x\vec{j} + \sin y\vec{k}$

$$\text{Now } curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= \vec{i}[\cos y - 0] - \vec{j}[0 - \cos z] + \vec{k}[\sin x - 0]$$

$$= \cos y\vec{i} + \cos z\vec{j} + \sin x\vec{k}$$

Since S is the rectangle in the z = 3 plane, $\therefore \hat{n} = \vec{k}$

$$curl\ \vec{F}.\hat{n} = (\cos y\vec{i} + \cos z\vec{j} + \sin x\vec{k}).\vec{k}$$

$$= \sin x$$

$$\int_C (\sin zdx - \cos xdy + \sin ydz)$$

$$= \iint_S \sin x\, dxdy$$

$$= \int_0^1 \int_0^{\pi} \sin x\, dxdy$$

$$= (-\cos x)_0^{\pi} (y)_0^1$$

$$= (-\cos \pi - (-\cos 0))(1-0)$$

$$= (1+1)(1)$$

$$= 2.$$

Green's Theorem in the Plane

If C is a regular closed curve in the xy-plane and R be the region bounded by C, then

$$\int_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$

Where $F_1(x, y)$ and $F_2(x, y)$ are continuously differentiable functions inside and on C.

Proof

From Stoke's theorem, we have

$$\int_C \vec{F}.\overrightarrow{dr} = \iint_R Curl\vec{F}.\hat{n}ds \qquad \ldots (1)$$

Let $\vec{F} = F_1\vec{i} + F_2\vec{j}$, then

$$curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}$$

$$= \vec{i}\left(0 - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(0 - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right)$$

$$= \vec{i}\left(-\frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial z}\right)$$

Also as C is a closed curve in the *xy*-plane, we have.

$$\hat{n} = \vec{k}$$

$$\therefore curl\ \vec{F}.\hat{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Also on xy-plane, we have

$$ds = \frac{dxdy}{|\hat{n}.\vec{k}|} = dxdy$$

$$\therefore \iint_S curl\ \vec{F}.\hat{n}ds = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy \qquad \ldots (1)$$

Where R is the projection of S on xy-plane.

Also, $\int_c \vec{F}.\overrightarrow{dr} = \int_c (F_1\vec{i} + F_2\vec{j}).(dx\vec{i} + dy\vec{j})$

$$= \int_c F_1dx + F_2dy \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$\int_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$

Corollary: If $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, the value of the integral $\int_C (F_1dx + F_2dy)$ is independent f the path of integration.

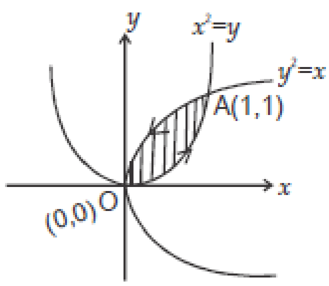
ILLUSTRATIVE EXAMPLES

1. Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of theregion defined by (a) $y = \sqrt{x}$ $y = x^2$ (b) $x=0$, $y=0$, $x+y=1$

Solution:

The Green's theorem is

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy$$



Here $F_1 = 3x^2 - 9y^2$. $F_2 = 4y - 6x$

(a) C is $y = \sqrt{x}$, $y = x^2$

(i.e) $y^2 = x$, $y = x^2$

$$\therefore \int_C F_1 dx + F_2 dy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= I_1 + I_2 \qquad \ldots (1)$$

$$I = \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along OA, $y = x^2$

$dy = 2xdx$

x varies from 0 to 1

$$\therefore I_1 = \int_0^1 (3x^2 - 8x^4)dx + (4x^2 - 6x^3)(2xdx)$$

$$= \int_0^1 (3x^2 + 8x^4 - 20x^4)dx$$

$$= (x^2 + 2x^4 - 4x^5)_0^1$$

$$= 1 + 2 - 4$$

$$\therefore I_1 = -1$$

$$I_2 = \int_{AO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along AO, $x = y^2$

$dx = 2ydy$

y varies from 1 to 0

$$\therefore I_2 = \int_1^0 (3y^4 - 8y^2)(2ydy) + (4y - 6y^3)dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dy$$

$$= 6\left(\frac{y^6}{6}\right) - 22\left(\frac{y^4}{4}\right) + 4\left(\frac{y^2}{2}\right)_1^0$$

$$= \left(y^6 - \frac{11}{2}y^4 + 2y^2\right)_1^0$$

$$= -1 + \frac{11}{2} - 2$$

$$\therefore I_2 = \frac{5}{2}$$

$\therefore$ from (1),

$$\int_C F_1 dx + F_2 dy = I_1 + I_2$$

$$= -1 + \frac{5}{2}$$

$$= \frac{3}{2} \qquad \ldots (2)$$

Now, $F_1 = 3x^2 - 8y^2, F_2 = 4y - 6xy$

$$\frac{\partial F_1}{\partial y} = 1 - 16y, \quad \frac{\partial F_2}{\partial x} = -6y$$

$$\because \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} (-6y + 16y) dxdy$$

$$= \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} 10y dxdy$$

$$= 10 \int_{y=0}^{1} y(x)_{x=y^2}^{\sqrt{y}} dy$$

$$= 10 \int_{y=0}^{1} y\left(\sqrt{y} - y^2\right) dy$$

$$= 10 \int_{y=0}^{1} \left(y^{\frac{3}{2}} - y^3\right) dy$$

$$= 10 \left[\frac{y^{5/2}}{\frac{5}{2}} - \frac{y^4}{4}\right]_{y=0}^{1}$$

$$= 10\left[\frac{2}{5} - \frac{1}{4}\right]$$

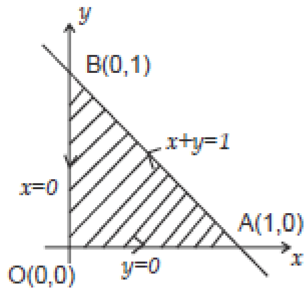
$$= 10\left(\frac{3}{20}\right) = \frac{3}{2} \qquad \ldots (3)$$

From (2) and (3), we see that

$$\int_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

(i.e) Green's theorem is verified.

(b) C is $x = 0$, $y = 0$, $x + y = 1$



$$\int_C F_1 dx + F_2 dy = \int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{OA} [3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$+\int_{AB}[(3x^2-8y^2)dx+(4y-6xy)dy+\int_{BO}[(3x^2-8y^2)dx+(4y-6xy)dy$$

$$=I_1+I_2+I_3 \text{ (say)} \qquad \ldots (1)$$

$$I_1=\int_{OA}(3x^2-8y^2)dx+(4y-6xy)dy$$

Along OA, $y = 0$

$dy = 0$

x varies from 0 to 1

$$I_1=\int_0^1 3x^2dx=3\left(\frac{x^3}{3}\right)_0^1=1$$

$$I_2=\int_{AB}(3x^2-8y^2)dx+(4y-6xy)dy$$

Along AB, $x + y = 1$

$y = 1 - x$

$dy = -dx$

x varies from 1 to 0

$$\therefore I_2=\int_1^0 [3x^2-8(1-x)^2]dx+[4(1-x)-6x(1-x)](-dx)$$

$$=\int_1^0(-11x^2+26x-12)dx$$

$$=\left[-11\left(\frac{x^3}{3}\right)+26\left(\frac{x^2}{2}\right)-12(x)\right]_1^0$$

$$=0-\left[\frac{-11}{3}+13-12\right]$$

$$=\frac{8}{3}$$

$$I_3=\int_{BO}(3x^2-8y^2)dx+(4y-6xy)dy$$

Along BO, $x = 0$

$dx = 0$

y varies from 1 to 0

$$\therefore I_3 = \int_1^0 4y dy = \left(\frac{y^2}{2}\right)_1^0$$

$$= 2(0-1) = -2$$

From (1),

$$\int_C (F_1 dx + F_2 dy) = I_1 + I_2 + I_3$$

$$= 1 + \frac{8}{3} - 2$$

$$= \frac{5}{3} \qquad \dots (2)$$

Now, $F_1 = 3x^2 - 8y^2, \qquad F_2 = 4y - 6xy$

$$\frac{\partial F_1}{\partial y} = -16y, \quad \frac{\partial F_2}{\partial x} = -6y$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy = \int_{y=0}^{1} \int_{x=0}^{1-y} (-6y + 16y) dx dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1-y} 10y dx dy$$

$$= 10 \int_{y=0}^{1} y(x)_{x=0}^{1-y} dy$$

$$= 10 \int_{y=0}^{1} y(1-y) dy$$

$$= 10 \int_{y=0}^{1} y - y^2 dy$$

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{1}$$

$$= 10\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 10\left(\frac{1}{6}\right) = \frac{5}{3} \qquad \ldots (3)$$

From (2) and (3), we see that

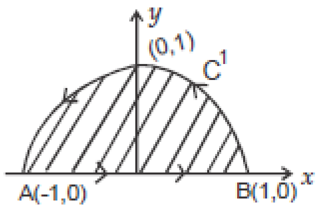
$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$$

(ie) Green's theorem is verified.

2) Verify Green's theorem in a plane with respect to $\oint_C (2x^2 - y^2)dx + (x^2 + y^2)dy$, where C is the boundary of the region in the xoy-plane enclosed by the x-axis and the upper half of the circle $x^2 + y^2 + 1$.

Solution:

The Green's theorem is,



$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$$

Here $F_1 = 2x^2 - y^2$, $F_2 = x^2 + y^2$

$$\int_C (F_1 dx + F_2 dy) = \int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$$

$$= \int_{AB} (2x^2 - y^2)dx + (x^2 + y^2)dy + \int_{C^1} (2x^2 - y^2)dx + (x^2 + y^2)dy$$

$$= I_1 + I_2 \text{ (say)} \qquad \ldots (1)$$

Along AB, $y = 0$

$dy = 0$

x varies from -1 to 1

$$\therefore I_1 = \int_{x=-1}^{1} (2x^2)dx$$

$$= 2\left(\frac{x^3}{3}\right)_{-1}^{1}$$

$$= \frac{2}{3}(1+1) = \frac{4}{3}$$

$$I_2 = \int_{c^1} (2x^2 - y^2)dx + (x^2 + y^2)dy$$

Along the upper half of the circle $C^1 : x^2 + y^2 = 1$,

The parametric equations are,

$x = \cos\theta, y = \sin\theta$

$dx = -\sin\theta d\theta, dy = \cos\theta d\theta$

θ varies from 0 to π

$$\therefore I_2 = \int_0^{\pi} (2\cos^2\theta - \sin^2\theta)(-\sin\theta d\theta) + (\cos^2\theta + \sin^2\theta)\cos\theta d\theta$$

$$= \int_0^{\pi} (-2\cos^2\theta\sin\theta + \sin^3\theta + \cos\theta)d\theta$$

$$= \int_0^{\pi} (-2(1-\sin^2\theta)(\sin\theta) + \sin^3\theta + \cos\theta)d\theta$$

$$= \int_0^{\pi} (-2\sin\theta + \sin^3\theta + \cos\theta)d\theta$$

$$= \int_0^{\pi} (-2\sin\theta + \frac{3}{4}(3\sin\theta - \sin 3\theta) + \cos\theta)d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{4}\sin\theta - \frac{3}{4}\sin 3\theta + \cos\theta\right]d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{4}(-\cos\theta) - \frac{3}{4}\left(\frac{-\cos 3\theta)}{3}\right) + \sin\theta\right]_0^{\pi}$$

$$= \left(\frac{1}{4} - \frac{1}{4} + 0\right) - \left(-\frac{1}{4} + \frac{1}{4} + 0\right)$$

$$= 0.$$

From (1),

$$\int_C F_1 dx + F_2 dy = I_2 + I_2$$

$$= \frac{4}{3} + 0$$

$$= \frac{4}{3} \qquad \ldots (2)$$

Now, $F_1 = 2x^2 - y^2 \quad F_2 = x^2 + y^2$

$$\frac{\partial F_1}{\partial y} = -2y, \qquad \frac{\partial F_2}{\partial x} = 2x$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{y=0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} [2x - (-2y)] dxdy$$

$$= 2 \int_{y=0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y) dxdy$$

$$= 2 \int_{y=0}^{1} \left[\frac{x^2}{2} + yx \right]_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= 2 \int_{y=0}^{1} \left[\frac{1-y^2}{2} + y\sqrt{1-y^2} \right] - \left[\frac{1-y^2}{2} - y\sqrt{1-y^2} \right] dy$$

$$= 2 \int_{y=0}^{1} \left[\frac{1}{2} - \frac{y^2}{2} + y\sqrt{1-y^2} - \frac{1}{2} + \frac{y^2}{2} + y\sqrt{1-y^2} \right] dy$$

$$= 2 \int_{y=0}^{1} 2y\sqrt{1-y^2}\, dy$$

Put $1 - y^2 = t$ when $y = 0, \quad t = 1$

$-2ydy = dt$ $\qquad y = 1, \quad t = 0$

$2ydy = -dt$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = 2 \int_{t=1}^{1} \sqrt{t}\,(-dt)$$

$$= -2 \left[\frac{t^{3/2}}{3/2} \right]_1^0$$

$$-2\left(\frac{2}{3}\right)[0-1]=\frac{4}{3} \qquad \ldots (3)$$

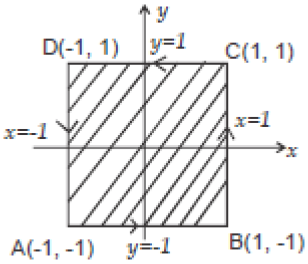
From (2) and (3), we see that

$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

(ie) Green's theorem is verified.

3. Verify Green's theorem in a plane to evaluate $\int_C x^2(1+y)dx+(x^3+y^3)dy$, where C is the square formed by $x = \pm 1$ and $y = \pm 1$

Solution



The Green's theorem is

$$\int_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

Here $F_1 = x^2(1+y),\ F_2 = x^3 + y^3$

$$\int_C (F_1 dx + F_2 dy) = \int_C x^2(1+y)dx+(x^3+y^3)dy$$

$$= \int_{AB} x^2(1+y)dx+(x^3+y^3)dy + \int_{BC} x^2(1+y)dx+(x^3+y^3)dy$$

$$+ \int_{CD} x^2(1+y)dx+(x^3+y^3)dy + \int_{DA} x^2(1+y)dx+(x^3+y^3)dy$$

$$= I_1 + I_2 + I_3 + I_4 \quad \text{(say)} \qquad \ldots (1)$$

$$I_1 = \int_{AB} x^2(1+y)dx+(x^3+y^3)dy$$

Along AB, $y = -1$

$dy = 0$

x varies from -1 to 1

$$I_1 = \int_{-1}^{1} [x^2(1-1)dx+0]$$

$$=0$$

$$I_2 = \int_{BC} x^2(1+y)dx + (x^3+y^3)dy$$

Along *BC*, $x = 1$

$dx = 0$

y varies from -1 to 1

$$\therefore I_2 = \int_{-1}^{1} (1+y^3)dy$$

$$= \left(y + \frac{y^4}{4} \right)_{-1}^{1}$$

$$= \left(1+\frac{1}{4}\right) - \left(-1+\frac{1}{4}\right)$$

$$= 1+\frac{1}{4}+1-\frac{1}{4} = 2$$

$$I_3 = \int_{CD} x^2(1+y)dx + (x^3+y^3)dy$$

Along *CD*, $y = 1$

$dy = 0$

x varies from 1 to -1

$$I_3 = \int_{1}^{-1} [x^2(1+1)dx + 0]$$

$$= 2\int_{1}^{-1} x^2 dx$$

$$= 2\left(\frac{x^3}{3}\right)_{1}^{-1}$$

$$= \frac{2}{3}(-1-1) = \frac{-4}{3}$$

$$I_4 = \int_{DA} x^2(1+y)dx + (x^3+y^3)dy$$

Along *DA*, $x = -1$

$dx = 0$

y varies from 1 to -1

$$I_4 = \int_{1}^{-1} [0(-1+y^3)dy]$$

$$= \left[-y + \frac{y^4}{4}\right]_1^{-1}$$

$$= \left(1+\frac{1}{4}\right) - \left(-1+\frac{1}{4}\right)$$

$$= 1 + \frac{1}{4} + 1 - \frac{1}{4} = 2$$

From (1).

$$\int_C F_1 dx + F_2 dy = I_1 + I_2 I_3 + I_4$$

$$= 0 + 2 - \frac{4}{3} + 2$$

$$= \frac{8}{3} \qquad \ldots (2)$$

Now $F_1 = x^2(1+y), \quad F_2 = x^3 + y^3$

$$\frac{\partial F_1}{\partial y} = x^2, \qquad \frac{\partial F_2}{\partial x} = 3x^2$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = 4 \int_{y=0}^{1} \int_{x=0}^{1} (3x^2 - x^2) dxdy$$

$$= 4 \int_{y=0}^{1} \int_{x=0}^{1} 2x^2 dxdy$$

$$= 8 \int_{y=0}^{1} \left(\frac{x^3}{3}\right)_{x=0}^{1} dy$$

$$= \frac{8}{3} \int_{y=0}^{1} dy$$

$$=\frac{8}{3}(y)_0^1$$

$$=\frac{8}{3}(1-0)$$

$$=\frac{8}{3} \qquad \ldots (3)$$

From (2) and (3), we se that

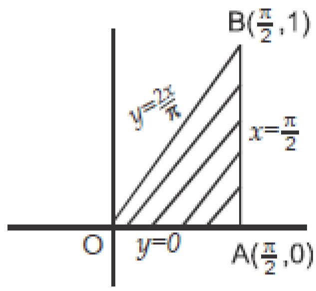
$$\int_C (F_1dx+F_2dy)=\iint_R \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy \qquad \text{(ie) Green's theorem is verified.}$$

4. Evaluate using Green theorem $\int_C [(y-\sin x)dx+\cos x dy]$, where C is the triangle formed by $y=0, x=\frac{\pi}{2}, y=\frac{2x}{\pi}$.

Solution:

Here $F_1 = y - \sin x$, $F_2 = \cos x$

$$\frac{\partial F_1}{\partial y}=1, \frac{\partial F_2}{\partial x}=-\sin x$$



Using Green's theorem,

$$\int_C (F_1dx+F_2dy)=\iint_R \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy$$

(ie) $$\int_C (y-\sin x)dx+\cos x dy=\int_{y=0}^{1}\int_{x=\pi y/2}^{\pi/2}(-\sin x-1)dxdy$$

$$=-\int_{y=0}^{1}\int_{x=\pi y/2}^{\pi/2}(\sin x+1)dxdy$$

$$=-\int_{y=0}^{1}(-\cos x+x)_{x=\pi y/2}^{\pi/2}dy$$

$$=\int_{y=0}^{1}\left(0+\frac{\pi}{2}\right)-\left(-\cos\left(\frac{\pi v}{2}\right)dy+\frac{\pi y}{2}\right)dy$$

$$= -\int_{y=0}^{1}\left(\frac{\pi}{2} + \cos\left(\frac{\pi v}{2}\right) - \frac{\pi y}{2}\right)dy$$

$$= \left[-\frac{\pi y}{2} + \frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}} - \frac{\pi}{2}\frac{y^2}{2} \right]_{y=0}^{1}$$

$$= -\left(\frac{\pi}{2} + \frac{2}{\pi} - \frac{\pi}{4}\right)$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

5. By the use of Green's theorem, show that area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (xdy - ydx)$. Hence find the area of an ellipse.

Solution:

By Green's theorem is planes,

$$\int_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy$$

Put $F_1 = -y$ and $F_2 = x$

$$\frac{\partial F_1}{\partial y} = -1 \; and \; \frac{\partial F_2}{\partial x} = 1$$

Hence from (1),

$$\int_C -ydx + xdy = \iint_R (1+1)dxdy$$

$$= 2\iint_R dxdy$$

$$= 2A$$

Where A is the required area.

$$\therefore A = \frac{1}{2}\int_C (xdy - ydx)$$

Any point (x,y) on the ellipse is given by

$x = a\cos\theta,\quad y = b\sin\theta\quad$ where θ is the parameter.

$dx = -a\sin\theta d\theta\qquad dy = b\cos\theta d\theta$

$$\therefore \text{Area of the ellipse} = \frac{1}{2}\int_C (xdy - ydx)$$

$$= \frac{1}{2}\int_0^{2\pi} a\cos\theta(b\cos\theta d\theta) - b\sin\theta(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)d\theta$$

$$= \frac{1}{2}(ab)\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} 2\theta$$

$$= \frac{ab}{2}(\theta)_0^{2\pi} = \pi ab$$

Problems for practice

1. Verify Gauss divergence theorem for the function $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$ over the cube bounded by $x = 0$, $y = 0$, $x = a$, $y = a$, $z = 0$, $z = a$.

2. Verify Gauss divergence theorem for $F = y\vec{i} + x\vec{j} + z^2\vec{k}$ for the cylindrical region S given by $x^2 + y^2 = a^2$, $z = 0$, $z = h$.

3. Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by planers $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

4. Verify Gauss divergence theorem for $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$.

5. Using divergence theorem, evaluate $\iint_S \vec{F}.\hat{n}ds$ and S is the closed surface enclosing a volume V and $\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$.

6. Find $\iint_S \vec{F}.\hat{n}ds$ where $\vec{F} = (2x + 3z)\vec{i} - (xz + y)\vec{j} + (y^2 + 2z)\vec{k}$ and S is the surface of the sphere having centre at (3,-1,2) and radius 3.

7. Verify Stoke's theorem for $\vec{F} = (xy + y^2)\vec{i} - x^2\vec{j}$ the region in the xoy plane bounded by $y = x$ and $y = x^2$.

8. Verify Stoke's theorem for the function $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by $x = \pm a$, $y = 0$, $y = b$.

9. Verify Stoke's theorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$ integrated around the square in the plane $z = 0$ whose sides are along the lines $x = 0$, $x = a$, $y = 0$, $y = a$.

10. Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ where S is the open surface of the cube formed by $x = 0$, $x = 2$, $y = 0$, $y = 2$ and $z = 2$.

11. Verify Stoke's theorem for $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half of the sphare $x^2 + y^2 + z^2 = 1$ and C is the circular boundary in the xoy plane.

12. Evaluate $\int_C \vec{F}.\overrightarrow{dr}$ by Stoke's theorem, where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x + z)\vec{k}$ and C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

13. If C is a simple closed curve and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ prove that $\int_C \vec{r}.\overrightarrow{dr} = 0$ by using Stoke's theorem.

14. Use Stoke's theorem to find $\int_C \vec{F}.\overrightarrow{dr}$ when $\vec{F} = (xy - x^2)\vec{i} + x^2\vec{j}$ and C is the boundary of the triangle in the xoy plane formed by $x = 1$, $y = 0$ and $y = x$.

15. Verify Green's theorem is a plane to find the finite area enclosed by parabola $y^2 = 4ax$ and $x^2 = 4ay$.

16. Verify Green's theorem in a plane for $\int_C ((xy + y^2)dx + x^2dy)$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

17. Verify Green's theorem in a plane for $\int_C [(x^2 + 2xy)dx + (y^2 + x^3y)dy]$ where C is a square with vertices A(0,0), B(1,0), D(1,1) and E(0,1).

18. Verify Green's theorem for the integral $\int_C [(x^2 - 2xy)dx + (x^2y + 3)dy]$ along the boundary of the region defined be $y^2 = 8x$ and $x = 2$.

19. Verify Green's theorem for $\int_C [e^{-x}\sin y dx + e^{-x}\cos y dy]$ where C is the rectangle with vertices $(0,0), (\pi,0), \left(\pi, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$

20. Show that the integral $\int_{(0,0)}^{(1,1)} [(x^2 + y^2)dx + 2xydy]$ is independent of the path of integration.

21. Evaluate by Green theorem in the plate $\int_C [(\cos x \sin y - xy)dx + \sin x \cos y dy]$ which C is the circle $x^2 + y^2 = 1$.

22. Using Green's theorem, evaluate $\int_C [x^2 y dx + x^2 dy]$ where C is the boundary of the triangle with vertices (0,0), (1,0), (1,1)

23. Evaluate by Green's theorem $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$ which C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$, $(0,1)$.

24. A vector field $\vec{F}$ is given $\vec{F} = \sin y \vec{i} + x(1 + \cos y)\vec{j}$ Evaluate the integral $\int_C \vec{F}.\overrightarrow{dr}$ where C is the circular path given by $x^2 + y^2 = a^2$.

25. Find the area of the loop of folium of Descartes $x^3 + y^3 = 3axy$, $a > 0$.

[Hint: Area $= \frac{1}{2}\int_C xdy - ydx$

$$= \frac{1}{2}\int_C x^2 d\left(\frac{y}{x}\right) = \frac{1}{2}\int_C x^2 dt; x = \frac{3at}{1+t^3}, y = tx$$ Limits of t are 0 to 1.]