COURSE : ENGINEERING MATHEMATICS I
COURSE CODE : SMTA1101

UNIT 3

FUNCTIONS OF SEVERAL VARIABLES

Jacobians

Changing variable is something we come across very often in Integration. There are many reasons for changing variables but the main reason for changing variables is to convert the integrand into something simpler and also to transform the region into another region which is easy to work with. When we convert into a new set of variables it is not always easy to find the limits. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables. In order to change variables in an integration we will need the **Jacobian** of the transformation.

If $f_1, f_2, \ldots, f_n$ are n differentiable functions of n variables $x_1, x_2, \ldots, x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is defined as the Jacobian of $f_1, f_2, \ldots, f_n$ with respect to the n variables $x_1, x_2, \ldots, x_n$ and is denoted by $\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$.

If u and v are functions of x and y, then $J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$

If u, v and w are functions of x, y and z, then J(u,v,w)= $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties of the Jacobian

1. Chain Rule for Jacobians: If u and v are functions of independent variables r and s and each of r and s are functions of the variables x and y, then u and v are functions of x and y. Further the jacobians satisfy the chain rule $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$
2. If u and v are functions of x and y, then x and y can be solved in terms of u and v. Then $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$.
3. If u, v and w are functions of x, y and z and if u, v, w are functionally related or dependent then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Problems

1. If u = xyz, v = $x^2 + y^2 + z^2$, w = x + y + z, find J (x, y, z)

   Solution J (x, y, z) = $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

   Therefore $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$

2. If u = x + y + z, u^2v = y + z, u^3 w = z, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}$

   Solution: $u_x = u_y = u_z = 1$

$v_x = \frac{-2(y+z)}{u^3}$, $v_y = \frac{u-2(y+z)}{u^3}$, $v_z = \frac{u-2(y+z)}{u^3}$

$w_x = \frac{-3z}{u^4}$, $w_y = \frac{-3z}{u^4}$, $w_z = \frac{1}{u^3} - \frac{3z}{u^4}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{-2(y+z)}{u^3} & \frac{u-2(y+z)}{u^3} & \frac{u-2(y+z)}{u^3} \\ \frac{-3z}{u^4} & \frac{-3z}{u^4} & \frac{u-3z}{u^4} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{-2(y+z)}{u^3} & \frac{1}{u^2} & \frac{1}{u^2} \\ \frac{-3z}{u^4} & 0 & \frac{1}{u^3} \end{vmatrix} \quad c_2 \to c_2 - c_1, c_3 \to c_3 - c_1$$

$$= \frac{1}{u^5}$$

3. If $u = \frac{x+y}{1-xy}$ $and\, v = \tan^{-1} x + \tan^{-1} y$, show that $\frac{\partial(u,v)}{\partial(x,y)} = 0$

Solution: Let x = tanθ and y = tanφ,

then $u = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta \tan\varphi} = \tan(\theta+\varphi)$

And v = θ + φ

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\theta,\varphi)} \cdot \frac{\partial(\theta,\varphi)}{\partial(x,y)}$$

$u_\theta = \sec^2(\theta+\varphi)$, $u_\varphi = \sec^2(\theta+\varphi)$, $v_\theta = v_\varphi = 1$

$$\frac{\partial(u,v)}{\partial(\theta,\varphi)} = \begin{vmatrix} \sec^2(\theta+\varphi) & \sec^2(\theta+\varphi) \\ 1 & 1 \end{vmatrix} = 0$$

$$Thus\ \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

4. Show that $u = x^3 - y^3z^3$, $v = x^2 + y^2z^2 + xyz$ and $w = x - yz$ are functionally dependent and also find the relation.

Solution: If u, v, w are functionally dependent then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3x^2 & -3y^2z^3 & -3y^3z^2 \\ 2x+yz & 2yz^2+xz & 2y^2z+xy \\ 1 & -z & -y \end{vmatrix}$$

$$= yz\begin{vmatrix} 3x^2 & -3y^2z^2 & -3y^2z^2 \\ 2x+yz & 2yz\ \ +x & 2yz+x \\ 1 & -1 & -1 \end{vmatrix} \text{ taking z common from } c_2 \text{ and y from } c_3$$

$$= 0 \ \ (\text{Two columns are identical})$$

Since $x^3 - y^3z^3 = (x - yz)(x^2 + xyz + y^2z^2)$

u = v w is the relation between the three variables.

5. If $u = e^x\cos y$, $v = e^x\sin y$, where $x = lr + sm$, $y = mr - sl$, verify if

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(r,s)}$$

6. $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^{2x}$$

$$\frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} l & m \\ m & -l \end{vmatrix} = -l^2 - m^2$$

$u = e^{lr+sm}\cos(mr-sl)$, $v = e^{lr+sm}\sin(mr-sl)$

$u_r = le^{lr+sm}\cos(mr-sl) - m\,e^{lr+sm}\sin(mr-sl)$

$u_s = me^{lr+sm}\cos(mr-sl) + l\,e^{lr+sm}\sin(mr-sl)$

$v_r = le^{lr+sm}\sin(mr-sl) + m\,e^{lr+sm}\cos(mr-sl)$

$v_s = me^{lr+sm}\sin(mr-sl) - l\,e^{lr+sm}\cos(mr-sl)$

$$\frac{\partial(u,v)}{\partial(r,s)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} = -e^{2x}(l^2+m^2) = \frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(r,s)}$$

Useful Links for this topic

1. http://mathwiki.ucdavis.edu/Calculus/Vector_Calculus/Multiple_Integrals/Jacobians
2. http://www-astro.physics.ox.ac.uk/~sr/lectures/multiples/Lecture5reallynew.pdf
3. http://math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/vcalc/jacpol/jacpol.html
4. http://www.google.co.in/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=0CCMQFjAB&url=http%3A%2F%2Fwww.tcc.edu%2FVML%2FMth163%2Fdocuments%2FJacobians.pptx&ei=1DKSVZfbG867uAS_t4CABg&usg=AFQjCNHQDmFpTK-pU16sC61WTkwouEvUFA&bvm=bv.96783405,d.c2E
5. http://math.etsu.edu/multicalc/prealpha/Chap3/Chap3-3/printversion.pdf

Taylor's Series

Statement : Let $f(x,y)$ be a function of two variables x, y which possess continuous partial derivatives at all points (x,y). Then

$$f(x+h,y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f + \frac{1}{4!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^4 f + \cdots$$

Another form of Taylor series :

$$\begin{aligned} f(x,y) = f(a,b) &+ \frac{1}{1!}[\,(x-a)f_x(a,b) + (y-b)f_y(a,b)] \\ &+ \frac{1}{2!}[\,(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] \\ &+ \frac{1}{3!}[\,(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) \\ &+ (y-b)^3 f_{yy}(a,b)] + \ldots \end{aligned}$$

Maclaurin's series :

The Taylor series expansion of $f(x,y)$ about the point (0, 0) is called Maclaurin's series.

$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \frac{1}{4!}\left[x^4 f_{xxxx}(0,0) + 4x^3 y f_{xxxy}(0,0) + 6x^2y^2 f_{xxyy}(0,0) + 4xy^3 f_{xyyy}(0,0) + y^4 f_{yyyy}(0,0)\right] + \ldots$

Problems

1. Find the Taylor series expansion of $\cos(x-y)$ upto second degree terms.

Solution:

Taylor series expansion of $f(x,y)$ upto second degree term is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

Let $f(x,y) = \cos(x-y)$ $\qquad f(0,0) = \cos 0 = 1$

$f_x(x,y) = -\sin(x-y)$ $\qquad f_x(0,0) = -\sin 0 = 0$

$f_y(x,y) = \sin(x-y)$ $\qquad f_y(0,0) = \sin 0 = 0$

$f_{xx}(x,y) = -\cos(x-y)$ $\qquad f_{xx}(0,0) = -\cos 0 = -1$

$f_{xy}(x,y) = \cos(x-y)$ $\qquad f_{xy}(0,0) = \cos 0 = 1$

$f_{yy}(x,y) = -\cos(x-y)$ $\qquad f_{yy}(0,0) = -\cos 0 = -1$

Taylor series expansion of $f(x,y) = \cos(x-y)$ upto second degree terms

$$\cos(x-y) = 1 + \frac{1}{1!}\left[x(0) + y(0)\right] + \frac{1}{2!}\left[x^2(-1) + 2xy(1) + y^2(-1)\right]$$

i.e., $\cos(x-y) = 1 - \frac{1}{2!}\left[x^2 - 2xy + y^2\right]$

2. Expand e^{x+y} in powers of $(x-1)$ and $(y+1)$ upto and including second degree term.

Solution:

Taylor series expansion of $f(x,y)$ about the point (a,b) i.e., in powers of $(x-a)$ and $(y-b)$ upto second degree term is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$

Let $f(x,y) = e^{x+y}$ $\qquad f(1,-1) = e^{1-1} = e^0 = 1$

$f_x(x,y) = e^{x+y}$ $\qquad f_x(1,-1) = e^{1-1} = e^0 = 1$

$f_y(x,y) = e^{x+y}$ $\qquad f_y(1,-1) = e^{1-1} = e^0 = 1$

$f_{xx}(x,y) = e^{x+y}$ $\qquad f_{xx}(1,-1) = e^{1-1} = e^0 = 1$

$f_{xy}(x,y) = e^{x+y}$ $\qquad f_{xy}(1,-1) = e^{1-1} = e^0 = 1$

$f_{yy}(x,y) = e^{x+y}$ $\qquad f_{yy}(1,-1) = e^{1-1} = e^0 = 1$

Taylor series expansion of $f(x,y) = e^{x+y}$ in powers of $(x-1)$ and $(y+1)$ upto second degree term is given by

$$f(x,y) = f(1,-1) + \frac{1}{1!}[(x-1)f_x(1,-1) + (y+1)f_y(1,-1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,-1) + 2(x-1)(y+1)f_{xy}(1,-1) + (y+1)^2 f_{yy}(1,-1)]$$

i.e.,

$$e^{x+y} = 1 + \frac{1}{1!}[(x-1)(1) + (y+1)(1)] + \frac{1}{2!}[(x-1)^2(1) + 2(x-1)(y+1)(1) + (y-1)^2(1)]$$

i.e., $e^{x+y} = 1 + \frac{1}{1!}[(x-1) + (y+1)] + \frac{1}{2!}[(x-1)^2 + 2(x-1)(y+1) + (y-1)^2]$

3. Expand $tan^{-1}\frac{y}{x}$ as a Taylor series in the neighbourhood of (1, 1) upto second degree term

Solution:

Taylor series expansion of $f(x,y)$ in the neighbourhood of (a,b) upto second degree term is given by

$$f(x,y) = f(a,b) + \frac{1}{1!}[(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)]$$

Let $f(x,y) = tan^{-1}\frac{y}{x}$ $\qquad f(1,\ 1) = tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

$f_x(x,y) = \frac{1}{1+(\frac{y}{x})^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$ $\qquad f_x(1,1) = -\frac{1}{2}$

$f_y(x,y) = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$ $\qquad f_y(1,1) = \frac{1}{2}$

$f_{xx}(x,y) = -\frac{(x^2+y^2)(0)-y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$ $\quad f_{xx}(1,1) = \frac{1}{2}$

$f_{xy}(x,y) = -\frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad f_{xy}(1,1) = 0$

$f_{yy}(x,y) = \frac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$ $\quad f_{yy}(1,1) = -\frac{1}{2}$

Taylor series expansion of $f(x,y) = tan^{-1}\frac{y}{x}$ in the neighbourhood of (1, 1) upto second degree term is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}[(x-1)f_x(1,1) + (y-1)f_y(1,1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)]$$

i.e, $tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \frac{1}{1!}\left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right] + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + \right.$

$\left. 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right]$

i.e, $tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}[(x-1) - (y-1)] + \frac{1}{4}[(x-1)^2 - (y-1)^2]$

4. Using Taylor series show that $\log(1+x+y) = \frac{(x+y)}{1} - \frac{(x+y)^2}{2} + \frac{(x+y)^3}{3} - \ldots$

Solution:

Taylor series expansion of $f(x,y)$ upto third degree term is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}[xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$$

Let $f(x,y) = \log(1+x+y)$ $\quad f(0,\ 0) = \log 1 = 0$

$f_x(x,y) = \frac{1}{1+x+y}$ $\quad f_x(0,0) = 1$

$f_y(x,y) = \frac{1}{1+x+y}$ $\quad f_y(0,0) = 1$

$f_{xx}(x,y) = -\frac{1}{(1+x+y)^2}$ $\quad f_{xx}(0,0) = -1$

$f_{xy}(x,y) = -\frac{1}{(1+x+y)^2}$ $\quad f_{xy}(0,0) = -1$

$f_{yy}(x,y) = -\frac{1}{(1+x+y)^2}$ $\quad f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = \frac{2}{(1+x+y)^3}$ $\quad f_{xxx}(0,0) = 2$

$f_{xxy}(x,y) = \frac{2}{(1+x+y)^3}$ $\quad f_{xxy}(0,0) = 2$

$f_{xyy}(x,y) = \frac{2}{(1+x+y)^3}$ $\quad f_{xyy}(0,0) = 2$

$f_{yyy}(x,y) = \frac{2}{(1+x+y)^3}$ $\quad f_{yyy}(0,0) = 2$

Taylor series expansion of $f(x,y) = \log(1 + x + y)$ upto third degree term is given by

$$\log(1 + x + y) = 0 + \frac{1}{1!}[x(1) + y(1)] + \frac{1}{2!}[x^2(-1) + 2xy(-1) + y^2(-1)] + \frac{1}{3!}[x^3(2) + 3x^2y(2) + 3xy^2(2) + y^3(2)]$$

i.e., $\log(1 + x + y) = (x + y) - \frac{1}{2!}[x^2 + 2xy + y^2] + \frac{1}{3}[x^3 + 3x^2y + 3xy^2 + y^3]$

i.e., $\log(1 + x + y) = \frac{(x+y)}{1} - \frac{(x+y)^2}{2} + \frac{(x+y)^3}{3} - \ldots$

5. Expand $\cos x \; \cos y$ in powers of $x,\; y$ upto fourth degree terms.

Solution:

Taylor series expansion of $f(x,y)$ upto third degree term is given by

$$f(x,y) = f(0,0) + \frac{1}{1!}[xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)] + \frac{1}{3!}[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)] + \frac{1}{4!}[x^4f_{xxxx}(0,0) + 4x^3yf_{xxxy}(0,0) + 6x^2y^2f_{xxyy}(0,0) + 4xy^3f_{xyyy}(0,0) + y^4f_{yyyy}(0,0)]$$

Let $f(x,y) = \cos x \cos y$ $\quad f(0,\ 0) = 1$

$f_x(x,y) = -\sin x \cos y$ $\quad f_x(0,0) = 0$

$f_y(x,y) = -\cos x \sin y$ $\quad f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos x \cos y$ $\quad f_{xx}(0,0) = -1$

$f_{xy}(x,y) = sinx \sin y$ $\quad f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -\cos x \cos y \qquad f_{yy}(0,0) = -1$

$f_{xxx}(x,y) = \sin x \cos y \qquad f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = \cos x \sin y \qquad f_{xxy}(0,0) = 0$

$f_{xyy}(x,y) = \sin x \cos y \qquad f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = \cos x \sin y \qquad f_{yyy}(0,0) = 0$

$f_{xxxx}(x,y) = \cos x \cos y \qquad f_{xxxx}(0,0) = 1$

$f_{xxxy}(x,y) = -\sin x \sin y \qquad f_{xxxy}(0,0) = 0$

$f_{xxyy}(x,y) = \cos x \cos y \qquad f_{xxyy}(0,0) = 1$

$f_{xyyy}(x,y) = -\sin x \sin y \qquad f_{xyyy}(0,0) = 0$

$f_{yyyy}(x,y) = \cos x \cos y \qquad f_{yyyy}(0,0) = 1$

Taylor series expansion of $f(x,y) = \cos x \cos y$ in powers of $x,\ y$ upto fourth degree terms

$$\begin{aligned}\cos x \cos y = {} & 1 + \frac{1}{1!}[\,x(0) + y(0)] + \frac{1}{2!}[\,x^2(-1) + 2xy(0) + y^2(-1)] \\ & + \frac{1}{3!}[\,x^3(0) + 3x^2y(0) + 3xy^2(0) + y^3(0)] \\ & + \frac{1}{4!}[\,x^4(1) + 4x^3y(0) + 6x^2y^2(1) + 4xy^3(0) + y^4(1)]\end{aligned}$$

i.e., $\cos x \cos y = 1 - \frac{1}{2}(x^2 + y^2) + \frac{1}{24}(x^4 + 6x^2y^2 + y^4)$

Maxima and Minima of functions of two variables

The problem of determining the maximum or minimum of a function is encountered in geometry, mechanics, physics, and other fields, and was one of the motivating factors in the development of the calculus in the seventeenth century.

A function of two variables can be written in the form z = f(x, y). A critical point is a point (a, b) such that the two partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero at the point (a, b). A relative maximum or a relative minimum occurs at a critical point.

A critical point is a maximum if the value of f at that point is greater than its value at all its sufficiently close neighboring points.

A critical point is a minimum if the value of f at that point is less than its value at all its sufficiently close neighboring points.

A critical point is a saddle point if the value of f at that point is greater than its value at some neighboring point and if the value of f at that point is less than its value at some other neighboring point. Saddle point is a point which is neither a maximum nor a minimum.

Working rule for identifying critical points of the function z = f(x,y) and to classify them

Step 1: Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Solving $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ gives the critical points (a,b) at which a maxima or minima may exist.

Step 2: Find the value of $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$ at all the points (a,b) got in step 1.

Step 3:

i. If $r < 0$ and $rt - s^2 > 0$ the f(x,y) has a maximum point at (a,b) and the corresponding maximum value is f(a,b).
ii. If $r > 0$ and $rt - s^2 > 0$ the f(x,y) has a minimum point at (a,b) and the corresponding minimum value is f(a,b).
iii. If $rt - s^2 < 0$ the f(x,y) has neither a maximum nor a minimum point at (a,b) and the point is called a saddle point.
iv. If $rt - s^2 = 0$ the further investigation is required to classify the point

Problems

1. Find the maxima and minima of the function, if any, for the function $f(x,y) = y^2 + 4xy + 3x^2 + x^3$

   Solution: $f_x = 4y + 6x + 3x^2$, $f_y = 2y + 4x$. Equate f_x and f_y to zero

   $4y + 6x + 3x^2 = 0$ ......(1)
   $2y + 4x = 0$ ......(2)
   Solving equations (1) and (2) we get the critical points (0,0) and (2/3,-4/3)
   $r = f_{xx} = 6 + 6x$, $t = f_{yy} = 2$, $s = f_{xy} = 4$

Critical Point	r	$rt - s^2$	Classification
(0,0)	6	-4	Saddle point
(2/3,-4/3)	10	4	Minimum point

The point (2/3, -4/3) is a minimum point of the function and the minimum value F(2/3,-4/3) = -4/27

2. Find the maxima and minima of the function $f(x,y) = xy\,(a - x - y)$

Solution: $f_x = ay - 2xy - y^2$, $f_y = ax - x^2 - 2xy$. Equate f_x and f_y to zero

$y(a - 2x - y) = 0$ ……(1)
$x(a - x - 2y) = 0$ ……(2)
Solving equations (1) and (2) we get the critical points (0,0), (a,0), (0,a) and (a/3, a/3).

$r = f_{xx} = -2y$, $t = f_{yy} = -2x$, $s = f_{xy} = a - 2x - 2y$

Critical Point	r	$rt - s^2$	Classification
(0,0)	0	$-a^2$	Saddle point
(a,0)	0	$-a^2$	Saddle point
(0,a)	-2a	$-a^2$	Saddle point
(a/3, a/3)	-2a/3	$a^2/3$	Maximum or minimum point

(a/3, a/3) is the only point which could be either be a maximum or a minimum.
r depends on the value of 'a'.
$r = -2a/3 < 0$ if 'a' is positive
$r = -2a/3 > 0$ if 'a' is negative
Hence f(x,y) has a maximum at (a/3, a/3) if 'a' is positive and has a minimum at (a/3, a/3) if 'a' is negative.

The value is $f(a/3, a/3) = \frac{a^3}{27}$

3. Examine the function $f(x,y) = x^3 + 3xy^2 - 15x^2 + 72x - 15y^2$ for extreme values

Solution: $f_x = 3x^2 + 3y^2 - 30x + 72$, $f_y = 6xy - 30y$. Equate f_x and f_y to zero

$3x^2 + 3y^2 - 30x + 72 = 0$ ......(1)
$6xy - 30y = 0$ ......(2)
Solving equations (1) and (2) we get the critical points (5,1), (5,-1), (4,0) and (6,0).

$r = f_{xx} = 6x - 30$, $t = f_{yy} = 6x - 30$, $s = f_{xy} = 6y$

Critical Point	r	$rt - s^2$	Classification
(5,1)	0	-36	Saddle point
(5,-1)	0	-36	Saddle point
(4,0)	-6	36	Maximum point
(6, 0)	6	36	Minimum point

(4, 0) is a maximum point and the maximum value is f(4,0) = 112
(6, 0) is a minimum point and the minimum value is f(6,0) = 108

4. Find the extreme values of the function $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

Solution: $u_x = 2xy^2 - 10x - 8y$, $u_y = 2x^2y - 8x - 10y$. Equate f_x and f_y to zero

$2xy^2 - 10x - 8y = 0$ ......(1)
$2x^2y - 8x - 10y = 0$ ......(2)
Since (0,0) satisfies both (1) and (2), (0, 0) is a critical point. To get the other points rewrite equation (1)

From (1) we get $x = \dfrac{8y}{2y^2 - 10}$ ....(3).

Substitute this in (2)

$$2\left(\frac{8y}{2y^2 - 10}\right)^2 y - 8\left(\frac{8y}{2y^2 - 10}\right) - 10y = 0 \text{.......(4)}$$

Solving (4) we get y = 3, -3, 1, -1 Substitute these values in (3) we get the critical points (0, 0), (1,-1), (-1,1), (3,3), (-3,-3).

$r = f_{xx} = 2y^2 - 10$, $t = f_{yy} = 2x^2 - 10$, $s = f_{xy} = 4xy - 8$

Critical Point	r	$rt - s^2$	Classification
(0,0)	-10	36	Maximum point
(1,-1)	-8	-80	Saddle point
(-1,1)	-8	-80	Saddle point
(3, 3)	8	-720	Saddle point
(-3, -3)	8	-720	Saddle point

(0,0) is a maximum point and the maximum value is f(0,0) = 0.

5. Show that x = a/2, y = a/3 makes the function $u = ax^3 y^2 - x^4 y^2 - x^3y^3$ a maximum.

Solution:

$u_x = 3ax^2y^2 - 4x^3y^2 - 3x^2y^3$........(1)

$u_y = 2ax^3y - 2x^4y - 3x^3y^2$............(2)

Put x = a/2 and y =a/3 in both equations (1) and (2)

Since both u_x and u_y are zero at (a/2,a/3), it is a critical point.

$u_{xx} = 6axy^2 - 12x^2y^2 - 6xy^3$

$u_{yy} = 2ax^3 - 2x^4 - 6x^3y$

$u_{xy} = 6ax^2y - 8x^3y - 9x^2y^2$

r at (a/2, a/3) $= -\dfrac{a^4}{9}$ which is negative for any value of 'a'.

t at (a/2, a/3) $= -\dfrac{a^4}{8}$

s at (a/2, a/3) $= -\dfrac{a^4}{12}$

$rt - s^2$ at (a/2, a/3) = $\dfrac{a^8}{144}$ which is positive for any value of 'a'.

Since r is negative and $rt - s^2$ is positive the point (a/2, a/3) is a maximum point

Useful Links for this topic

1. http://personal.maths.surrey.ac.uk/st/S.Zelik/teach/calculus/max_min_2var.pdf
2. http://www.maths.manchester.ac.uk/~mheil/Lectures/2M1/Material/Chapter2.pdf
3. http://tutorial.math.lamar.edu/Classes/CalcIII/RelativeExtrema.aspx
4. http://www.ccs.neu.edu/home/lieber/courses/algorithms/cs4800/f10/lectures/11.4.Maximizing.pdf
5. http://www.maths.manchester.ac.uk/~ngray/MATH19662/Section%204%20-%20Functions%20of%20Two%20Variables.pdf

Constrained Maxima and Minima

Sometimes we may require to find the extreme values of a function of three (or more) variables say $f(x,y,z)$ which are not independent, but are connected by a relation say $g(x,y,z) = 0$. The extreme values of a function in such a situation is called constrained extreme values.

In such situations, we use $g(x,y,z) = 0$ to eliminate one of the variables, say z from the given function, thus converting the function of three variables as a function of only two variables. Then we find the unconstrained maxima and minima of the converted function.

When this procedure cannot be used, we use Lagrange's method.

Lagrange's Multiplier Method

Sometimes we may require to find the maximum and minimum values of a function $f(x,y,z)$ where x, y, z subject to the constraint $g(x,y,z) = 0$.

(1)

We define a function $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

where λ is the Lagrange's multiplier independent of x, y, z.

The neccessary condition for a maximum or minimum are

$\frac{\partial F}{\partial x} = 0$

(2)

$\frac{\partial F}{\partial y} = 0$

(3)

$\frac{\partial F}{\partial z} = 0$

(4)

Solving the four equations (1), (2), (3) and (4) we get the values of x, y, z, λ which give the extreme values of $f(x,y,z)$

Problems

1. Prove that the stationary values of $a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ occur at $x = \frac{a+b+c}{a}$, $y = \frac{a+b+c}{b}$, $z = \frac{a+b+c}{c}$

Solution:

Let $f = a^3x^2 + b^3y^2 + c^3z^2$

$g = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$

and $F = f + \lambda g = a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$

$\frac{\partial F}{\partial x} = 0$ implies $2a^3x - \frac{\lambda}{x^2} = 0 \Rightarrow a^3x^3 = \frac{\lambda}{2}$

(1)

$\frac{\partial F}{\partial y} = 0$ implies $2b^3y - \frac{\lambda}{y^2} = 0 \Rightarrow b^3y^3 = \frac{\lambda}{2}$

(2)

$\frac{\partial F}{\partial z} = 0$ implies $2c^3x - \frac{\lambda}{z^2} = 0 \Rightarrow c^3z^3 = \frac{\lambda}{2}$

(3)

From (1), (2) and (3) we get $a^3x^3 = b^3y^3 = c^3z^3$

i.e., $ax = by = cz$

i.e., $\frac{a}{\frac{1}{x}} = \frac{b}{\frac{1}{y}} = \frac{c}{\frac{1}{z}} = \frac{a+b+c}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = \frac{a+b+c}{1}$

consider $\frac{a}{\frac{1}{x}} = \frac{a+b+c}{1} \Rightarrow x = \frac{a+b+c}{a}$

consider $\frac{b}{\frac{1}{y}} = \frac{a+b+c}{1} \Rightarrow y = \frac{a+b+c}{b}$

consider $\frac{c}{\frac{1}{z}} = \frac{a+b+c}{1} \Rightarrow z = \frac{a+b+c}{c}$

Thus f is stationary at this point $= \frac{a+b+c}{a}$, $y = \frac{a+b+c}{b}$, $z = \frac{a+b+c}{c}$.

2. Find three positive constants such that their sum is a constant and their product is maximum.

Solution:

Let the three positive constants be x, y, z such that $x + y + z = a$.

Let $f = xyz$

and $g = x + y + z - a$ (1)

We have to maximize $f = xyz$ subject to the constraint $g = x + y + z - a$

Let $F = f + \lambda g = xyz + \lambda(x + y + z - a)$

$\frac{\partial F}{\partial x} = 0$ implies $yz + \lambda(1) = 0 \Rightarrow yz = -\lambda$

(2)

$\frac{\partial F}{\partial y} = 0$ implies $xz + \lambda(1) = 0 \Rightarrow xz = -\lambda$

(3)

$\frac{\partial F}{\partial z} = 0$ implies $xy + \lambda(1) = 0 \Rightarrow xy = -\lambda$

(4)

From (2), (3) and (4) we get $yz = xz = xy$

Consider $yz = xz \Rightarrow y = x$

Consider $xz = xy \Rightarrow z = y$

Therefore $x = y = z$

Substituting in (1), we get $x + x + x = a \Rightarrow 3x = a \Rightarrow x = \frac{a}{3}$

Therefore $= \frac{a}{3}$, $z = \frac{a}{3}$.

Hence the three numbers are $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}$.

3. Split 24 into three parts such that continued product of the first , square of the second and cube of the third may be minimum.

Solution:

Let the three parts be x, y, z such that $x + y + z = 24$.

Let $f = xy^2z^3$ (1)

We have to minimize $f = xy^2z^3$ subject to the constraint $g = x + y + z - 24$

Let $F = f + \lambda g = xy^2z^3 + \lambda(x + y + z - 24)$

$\frac{\partial F}{\partial x} = 0$ implies $y^2z^3 + \lambda(1) = 0 \Rightarrow y^2z^3 = -\lambda$

(2)

$\frac{\partial F}{\partial y} = 0$ implies $2xyz^3 + \lambda(1) = 0 \Rightarrow 2xyz^3 = -\lambda$

(3)

$\frac{\partial F}{\partial z} = 0$ implies $3xy^2z^2 + \lambda(1) = 0 \Rightarrow 3xy^2z^2 = -\lambda$

(4)

From (2), (3) and (4) we get $y^2z^3 = 2xyz^3 = 3xy^2z^2$

Consider $y^2z^3 = 2xyz^3 \Rightarrow y = 2x$

Consider $y^2z^3 = 3xy^2z^2 \Rightarrow z = 3x$

Substituting in (1), we get $x + 2x + 3x = 24 \Rightarrow 6x = 24 \Rightarrow x = 4$

Therefore $= 8$, $z = 12$.

Hence the three parts of 24 are $4, 8, 12$.

4. Find the shortest distance of the point (2, 1, -3) from the plane 2x + y = 2z + 4

Solution:

Let the foot of the perpendicular from the point (2, 1, -3) to the plane be $2x + y = 2z + 4$ be $P(x, y, z)$.

Shortest distance from (2, 1, -3) to the point $P(x, y, z)$ on the plane is the perpendicular distance $d = \sqrt{(x-2)^2 + (y-1)^2 + (z+3)^2}$

$\therefore d^2 = (x-2)^2 + (y-1)^2 + (z+3)^2$

We have to find the minimum distance d equivalently d^2 subject to the constraint $2x + y = 2z + 4$.

Let $f = (x-2)^2 + (y-1)^2 + (z+3)^2$

and $g = 2x + y - 2z - 4$ (1)

Let $F = f + \lambda g = (x-2)^2 + (y-1)^2 + (z+3)^2 + \lambda(2x + y - 2z - 4)$

$\frac{\partial F}{\partial x} = 0$ implies $2(x-2) + \lambda(2) = 0 \Rightarrow x - 2 = -\lambda$ (2)

$\frac{\partial F}{\partial y} = 0$ implies $2(y-1) + \lambda(1) = 0 \Rightarrow 2(y-1) = -\lambda$ (3)

$\frac{\partial F}{\partial z} = 0$ implies $2(z+3) + \lambda(-2) = 0 \Rightarrow -(z+3) = -\lambda$

(4)

From (2), (3) and (4) we get $x - 2 = 2(y-1) = -(z+3)$

Consider $x - 2 = 2(y-1) \Rightarrow x = 2y \Rightarrow y = \frac{x}{2}$

Consider $x - 2 = -z - 3 \Rightarrow$ x $= -z - 1 \Rightarrow z = -1 - x$

Substituting this in (1) we get $2x + \frac{x}{2} + 2x + 2 = 4 \Rightarrow x = \frac{4}{9}$

$\Rightarrow y = \frac{4}{18}$ i.e., $y = \frac{2}{9}$ and $z = -\frac{13}{9}$

Shortest distance from (2, 1, -3) to the point $P\left(\frac{4}{9}, \frac{2}{9}, -\frac{13}{9}\right)$ on the plane is given by

$\sqrt{(\frac{4}{9}-2)^2 + (\frac{2}{9}-1)^2 + (-\frac{13}{9}+3)^2} = \frac{7}{3}$

5. Find the points on the surface $z^2 = xy + 1$ nearest to the origin

Solution:

Let the point on the surface $z^2 = xy + 1$, which is nearest to the origin be $P(x, y, z)$.

Distance from this point $P(x, y, z)$ to the origin is $d = \sqrt{x^2 + y^2 + z^2}$

$\therefore f(x, y, z) = x^2 + y^2 + z^2$ (1)

But $z^2 = xy + 1$ (2)

Using (2) in (1), we get $f(x, y, z) = x^2 + y^2 + xy + 1$

Now $\frac{\partial f}{\partial x} = 2x + y$ $\qquad \frac{\partial f}{\partial y} = 2y + x$

$r = \frac{\partial^2 f}{\partial x^2} = 2$ $\quad$ $t = \frac{\partial^2 f}{\partial y^2} = 2$ $\quad$ $s = \frac{\partial^2 f}{\partial x \partial y} = 1$

To find the point at which maximum and minimum occurs we equate

$\frac{\partial f}{\partial x} = 0 \;\Rightarrow\; 2x + y = 0$ (3)

$\frac{\partial f}{\partial y} = 0 \;\Rightarrow\; 2y + x = 0$ (4)

Solving (3) and (4) we get $\Rightarrow$ $x = 0,\; y = 0$

Substituting for $x = 0,\; y = 0$ in (2) we get $z^2 = 1 \;\Rightarrow\; z = \pm 1$

Therefore stationary points are (0, 0 , 1) and 0, 0, -1).

At the stationary point (0, 0, 1) $rt - s^2 = 3 > 0$ $\Rightarrow$ the function has a minimum at (0, 0, 1)

At the stationary point (0, 0, -1) $rt - s^2 = 3 > 0$ $\Rightarrow$ the function has a minimum at (0, 0,-1)

Hence the points on the surface nearest to the origin are (0, 0 , 1) and 0, 0, -1).