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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – IV – DISCRETE MATHEMATICS – SMTA 1302

UNIT IV: COMBINATORICS

COURSE CONTENT: Mathematical induction – The basics of counting – The pigeonhole principle – Permutations and combinations – Recurrence relations – Solving linear recurrence relations – Generating functions – Inclusion and exclusion principle and its applications

MATHEMATICAL INDUCTION

Example 1: Show that $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let P(n) : $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)}$

1.P(1): $\frac{1}{1.2} = \frac{1}{1(1+1)}$ is true.

2.ASSUME

P(k): $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{k(k+1)}$

$= \frac{k}{k+1}$ is true. -> (1)

CLAIM : P(k+1) is true.

P(k+1) = $\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} \quad \frac{1}{(k+1)(k+2)}$ using (1)

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{(k.k)+2k+1}{(k+1)(k+2)}$

$= \frac{(k+1)(k+1)}{(k+1)(k+2)}$

$= \frac{(k+1)}{(k+2)}$

$= \frac{k+1}{(k+1)+1}$

= P(k+1) is true.

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$ Is true for all n .

EXAMPLE 2 : Using mathematical induction prove that if

n>=1, then 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! - 1

SOLUTION:

Let p(n) : 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! - 1

1.P(1) : 1.1! = (1+1)! – 1 is true

2 . ASSUME p(k) : 1.1! + 2.2! +3.3! +……+k.k!

= (k+1)! – 1 is true

CLAIM : p(k+1) is true.

P(k+1) = 1.1! + 2.2! +3.3! +……+k.k! +(k+1)(k+1)!

= (k+1)!-1 +(k+1)(k+1)!

= (k+1)! [(1+k+1)] – 1

= (k+1)! (k+2) – 1

= (k+2) ! – 1

= [(k+1) + 1]! - 1

P(k+1)is true.

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION,

P(n) : 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! – 1 , n>=1

EXAMPLE 3 : Use mathematical induction , prove that $\sum^{n}_{m=0} 3^{m} = \frac{(3\wedge n+1)-1}{2}$

SOLUTION:

Let p(n): $3^0 + 3^1 + \ldots\ldots\ 3^n = \frac{(3\wedge n+1)-1}{2}$

1.p(0) : $3^0 = \frac{(3\wedge 0+1)-1}{2} = \frac{2}{2} = 1$ is true .

2.ASSUME

P(k):): $3^0 + 3^1 + \ldots\ldots.. 3^n = \frac{(3\wedge k+1)-1}{2}$ is true.

CLAIM : p(k+1)is true.

P (k+1) :): $3^0 + 3^1 + 3^2 + \ldots\ldots.. + 3^k + 3^{k+1}$

$= \frac{(3\wedge k+1)-1}{2} + 3^{k+1}$ using (1)

$= \frac{(3\wedge k+1)+2.(3\wedge k+1)-1}{2}$

$= \frac{3(3\wedge k+1)-1}{2}$

$= \frac{(3\wedge k+2)-1}{2}$

$= \frac{(3\wedge(k+1)+1)-1}{2}$

P(k+1)is true.

By the principle of mathematical induction.

P(n): $\sum^{n}_{m=0} 3^m = \frac{(3\wedge n+1)-1}{2}$ is true for n>=0

EXAMPLE 4 :Use mathematical induction , prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$, n>=2

SOLUTION:

Let p(n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$, n>=2

1.p(2): that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$ = (1.707) > $\sqrt{2}$ + (1.414) is true

2.ASSUME

P(k): that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$ is true -> (1)

CLAIM : p(k+1) is true.

P(k+1) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\sqrt{k} + \frac{1}{\sqrt{k+1}}$ using (1)

$$\frac{\sqrt{k}\ \sqrt{k+1}\ +1}{\sqrt{k+1}}$$

$$\frac{\sqrt{k(k+1)}\ +1}{\sqrt{k+1}}$$

$$> \frac{\sqrt{k.k}\ +1}{\sqrt{k+1}}$$

$$> \frac{k+1}{\sqrt{k+1}}$$

$$> \sqrt{k+1}$$

P(k+1) $> \sqrt{k+1}$

P(K+1) is true

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION.

that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n+1}$

EXAMPLE 5: Using mathematical induction ,prove that $1^2 + 3^2 + 5^2 + \dots\dots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

SOLUTION :

Let p(n): $1^2 + 3^2 + 5^2 + \dots\dots (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$

1.p(1): $1^2 = \frac{1}{3} 1(2-1)(2+1) = \frac{1}{3} . 3$

=1 is true.

2.ASSUME p(k)is true.

$1^2 + 3^2 + 5^2 + \dots\dots (2k-1)^2 = \frac{1}{3} n(2k-1)(2k+1)$ -> (1) Is true.

CLAIM : p(k+1) is true.

P (k+1) $= \frac{1}{3} k (2k-1) (2k+1) +(2k+1)^2$ using (1)

$= \frac{1}{3} (2k+1) [k(2k-1) +3(2k+1)]$

$= \frac{1}{3} (2k+1) (2k2+5k+3)$

$= \frac{1}{3} (2k+1)(2k+3)(k+1)$

$= \frac{1}{3} (k+1) [2(k+1)-1][2(k+1)+1]$

P(k+1) is true .

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION,

$P(n) = 1^2 + 3^2 + 5^2 + \ldots\ldots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

EXAMPLE 6:Use mathematical induction to show that $n^3 - n$ is divisible by 3. For $n \in Z^+$

SOLUTION:

Let p(n): $n^n - n$ is divisible by 3.

1. p(1): $1^3 - 1$ is divisible by 3,is true.
2. ASSUME p(k): $k^3 - k$ is divisible by 3. -> (1)

CLAIM : p(k+1) is true .

P (k+1): $(k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + 1 - k - 1$

$= (k^3 - k) + 3(k^2 + k)$ ->(2)

(1) => $k^3 - k$ Is divisible by 3 and $3(k^2 + k)$ is divisible by 3 ,we have equation (2) is divisibleby 3

Therefore P(k+1) is true.

By the principle of mathematical induction , $n^3 - n$ is divisible by 3.

Strong Induction:

There is another form of mathematics induction that is often useful in proofs.In this form we use the basis step as before, but we use a different inductive step. We assume that p(j)is true for j=1…,k and show that p(k+1)must also be true based on this assumption . This is called strong Induction (and sometimes also known as the second principles of mathematical induction).

We summarize the two steps used to show that p(n)is true for all positive integers n.

Basis Step : The proposition P(1) is shown to be true

Inductive Step: It is shown that

$[P(1) \wedge P(2) \wedge \ldots\ldots \wedge P(k)] \rightarrow P(k+1)$

NOTE:

The two forms of mathematical induction are equivalent that is, each can be shown to be valid proof technique by assuming the other

EXAMPLE 1: Show that if n is an integer greater than 1, then n can be written as the product of primes.

SOLUTION:

Let P(n) be the proportion that n can be written as the product of primes

Basis Step : P(2) is true , since 2 can be written as the product of one prime

Inductive Step: Assume that P(j) is positive for all integer j with j<=k. To complete the Inductive Step, it must be shown that P(k+1) is trueunder the assumption.

There are two cases to consider namely

i) When (k+1) is prime
ii) When (k+1) is composite

Case 1 : If (k+1) is prime, we immediately see that P(k+1) is true.

Case 2: If (k+1) is composite

Then it can be written as the product of two positive integers a and b with 2<=a<b<=k+1. By the Innduction hypothesis, both a and b can be written as the product of primes, namely those primes in the factorization of a and those in the factorization of b .

WELL ORDERING PROPERTY

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non-empty set of non negative integers has a least element.

The well-ordering property can often be used directly in the proof.

PERMUTATION AND COMBINATION

PERMUTATION

A Permutation is an arrangement of set of n objects in a definite order taken some or all at a time.

Example: 1.Three letters a,b,c can be arranged

abc, acb, bac, bca, cab, cba. We have taken all the three for arrangement.

2. Using the three letters a,b,c the total no. of arrangements or permutation taking two at a time.

ab, bc, ac, ba, cb, ca.

The no. of permutation of n objects taken r at a time is denoted by P(n,r) or nP_r and is defined as

$$nP_r = \frac{n!}{(n-r)!} \text{ where } r \leq n.$$

Corollary

If r= n,

$$nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Permutation with repetition

Let $P(n; n_1,n_2,\ldots,n_r)$ denote the no. of permutation of n objects of which n_1 are alike n_2 are alike …n_r are alike then the formula is given by

$$P(n; n_1, n_2, n_3, \ldots n_r) = \frac{n!}{n_1! n_{2!} n_{3!} \ldots . n_{r!}}$$

Circular Permutation

Arrangement of objects in a circle is called Circular Permutation. A circular Permutation of n different objects is (n-1)!

Solved Problems

1. Find the value of n if $nP_5 = 42nP_3$ where n>4

Solution

$$\frac{n!}{(n-5)!} = 42 \frac{n!}{(n-3)!}$$

$$\frac{1}{(n-5)!} = 42 \frac{1}{(n-3)(n-4)(n-5)!}$$

$(n-3)(n-4) = 42$

$n^2-7n-30 = 0$

$(n-10)(n+3) = 0$

n = 10, -3

Since n is positive, n = 10

2. How many four digit nos. can be formed by using the digits 1 to 9. If repetition of digits are not allowed.

Solution

$$9P_4 = \frac{9!}{5!} = \frac{9\times8\times7\times6\times5!}{5!}$$

= 3024.

3. Find the no. of permutations of the letters of the word ALLAHABAD.

Solution

There are 9 letters in this word. To form different words containing all these 9 letters is

$$= \frac{9!}{4!2!}$$

4. (i) A committee of 3 is to be chosen out of 5 English, 4 French, 3 Indians and the committee to contain one each. In how many ways can this be done? (ii) In how many arrangements one particular Indian can be chosen?

Solution

(i)One English member can be chosen in 5 ways

One French member can be chosen in 4 ways

One Indian member can be chosen in 3 ways

No of ways the committee can be formed = 5x4x3 = 60 ways.

(ii)Since the Indian member is fixed, we have to fill the remaining two places choosing one from English and French each. This can be done in 5x4 = 20 ways.

5. There are 5 trains from Chennai to Delhi and back to Chennai. In how many ways a person go Chennai to Delhi and return to Chennai.

Solution

5 x 4 = 20.

6. There is a letter lock with three rings, each ring with 5 letters and the password is unknown. How many different useless attempts are made to open the lock.

Solution

Total no. of attempts = 5 x 5 x 5 = 5^3

Only one will unlock , so the total no. of useless attempts is (5^3-1) = 125 -1

= 124.

7. (i) Find the no. of arrangements of the letters of the word ELEVEN,(ii) How many of them begin and end with E. (iii) How many of them have three E's together. (iv) How many begin with E and end with N.

Solution

(i) $\frac{6!}{3!} = 6 \times 5 \times 4 = 120$ ways.

(ii) First and last places are fixed, the remaining 4 places are done in 4! ways.

(iii) Treat the 3 E's as a single element.

Therefore, this single element along with L,V,N can be arranged in 4! ways.

(iv) $\frac{4!}{2!} = 4 \times 3 = 12.$

8. There are 6 different books on Physics, 3 on Chemistry, 2 on Mathematics. In how many ways can they be arranged on a shelf if the books of the same subject are always together?

Solution

Considering Physics books, Chemistry books, Mathematics books as three elements, three elements can be arranged in 3! ways. Also
Physics books can themselves be arranged in 6! Ways
Chemistry books can themselves be arranged in 3! Ways
Mathematics books can themselves be arranged in 2! Ways
No.of arrangements = 3! 6! 3! 2!

9. Find the no. of arrangements in which 6 boys and 4girls can be arranged in a line such that all the girls sit together and all the boys sit together.

Solution

The no. of arrangement with all the girls sit together and all the boys sit together is 2! 4! 6! ways.

10. Find the no. of ways in which 10 exam papers can be arranged so that 2 particular papers may not come together.

Solution

2 particular papers should not come together. The remaining 8 papers can be arranged in 8! ways.The 2 papers can be filled in 9 gaps in between these 8 papers in $9P_2$ ways.

11. In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution

The 5 lions should be arranged in the 5 places marked 'L'.
This can be done in 5! ways.
The 4 tigers should be in the 4 places marked 'T'.
This can be done in 4! ways.
Therefore, the lions and the tigers can be arranged in 5!*4!= 2880 ways

12. In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?

Solution

5 boys can be seated in a row in 5! ways.
Also the girls can be seated in 3! ways
The 3 girls can be filled in the 6 gaps between the boys in $6P_3$ ways.
Total no of arrangements = 5! x 3! x $6P_3$ = 1440

13. There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairy tales, novels and plays.

Solution

There are 4 books on fairy tales and they have to be put together.
They can be arranged in 4! ways.

Similarly, there are 5 novels.
They can be arranged in 5! ways.

And there are 3 plays.
They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in 4!*5!*3!= 17280 ways

13. Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 5.3. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution

First, we consider the books on fairy tales, novels and plays as single objects.
These three objects can be arranged in 3!=6 ways.
Let us fix one of these 6 arrangements.
This may give us a specific order, say, novels -> fairy tales -> plays.

Given this order, the books on the same subject can be arranged as follows.
The 4 books on fairy tales can be arranged among themselves in 4!=24 ways.
The 5 novels can be arranged in 5!=120 ways.
The 3 plays can be arranged in 3!=6 ways.

For a given order, the books can be arranged in 24*120*6=17280 ways.
Therefore, for all the 6 possible orders the books can be arranged in 6*17280= 103680 ways.

14. In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution

Let 4 girls be one unit and now there are 6 units in all.
They can be arranged in 6! ways.In each of these arrangements 4 girls can be arranged in 4! ways.
=> Total number of arrangements in which girls are always together =6!*4!=720*24= 17280.

15. How many arrangements of the letters of the word ‘BENGALI’ can be made

(i) If the vowels are never together.
(ii) If the vowels are to occupy only odd places.

Solution

There are 7 letters in the word ‘Bengali; of these 3 are vowels and 4 consonants.

(i) Considering vowels a, e, i as one letter, we can arrange 4+1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.
=> Total number of words =5!*3!
=120*6=720
So there are total of 720 ways in which vowels are ALWAYS TOGEGHER.

Now,
Since there are no repeated letters, the total number of ways in which the letters of the word ‘BENGALI’ cab be arranged:
=7!=5040

So,
Total no. of arrangements in which vowels are never together:
=ALL the arrangements possible – arrangements in which vowels are ALWAYS TOGETHER
=5040–720=4320

ii) There are 4 odd places and 3 even places. 3 vowels can occupy 4 odd places in 4P3 ways and 4 constants can be arranged in 4P4 ways.
=> Number of words $=4P_3*4P_4= 576$.

16. In how many ways 5 gentlemen and 3 ladies can be arranged along a round table so that no 2 ladies are together.

Solution:

The 5 gentlemen can be arranged in a round table in (5-1)! = 4! ways.

Since no 2 ladies are together, they can occupy the 5 gaps in between the gentlemen in $5P_3$ ways.

Therefore, total no. of arrangements = $5P_3$ x 4!

COMBINATION

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S1,S2,S3,S4. Then you can choose two pairs in the following ways:

1. {S1,S2} 2. {S1,S3} 3. {S1,S4}
4. {S2,S3} 5. {S2,S4} 6. {S3,S4}

[Observe that {S1,S2} is the same as {S2,S1}. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know that it can be done in 4P4=12 ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below:

1. {S1,S2} -> S1 on Monday and S2 on Tuesday or S2 on Monday and S1 on Tuesday
2. {S1,S3} -> S1 on Monday and S3 on Tuesday or S3 on Monday and S1 on Tuesday
3. {S1,S4} -> S1 on Monday and S4 on Tuesday or S4 on Monday and S1 on Tuesday
4. {S2,S3} -> S2 on Monday and S3 on Tuesday or S3 on Monday and S2 on Tuesday
5. {**S2**,S4} -> S2 on Monday and S4 on Tuesday or S4 on Monday and S2 on Tuesday
6. {S3,S4} -> S3 on Monday and S4 on Tuesday or S4 on Monday and S3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.
This argument holds good in general as we can see from the following theorem.

Theorem

Let n≥1 be an integer and r≤n. Let us denote the number of ways of choosing r objects out of n objects by nCr. Then
nCr=$\frac{nP_r}{r!}$.

Example: Find the number of subsets of the set {1,2,3,4,5,6,7,8,9,10,11} having 4 elements.

Solution

Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 4 elements of this set which has 11 elements.

$11C_4 = \frac{11\times10\times9\times8}{1\times2\times3\times4} = 330$

Example: 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution

For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $12C_4$=495.

Therefore, we can draw 495 quadrilaterals.

Example: In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution

Number of ways of choosing 2 black pens from 5 black pens $=5C_2=\frac{5P_2}{2!}=\frac{5\times4}{1\times2}=10$

Number of ways of choosing 2 white pens from 3 white pens $=3C_2=\frac{3P_2}{2!}=\frac{3\times2}{1\times2}=3$

Number of ways of choosing 2 red pens from 4 red pens $=4C_2=\frac{4P_2}{2!}=\frac{4\times3}{1\times2}=6$

=> By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in 10*3*6= 180ways.

Example: A question paper consists of 10 questions divided into two parts A and B. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part A and at least 2 from part B. In how many ways can the candidate select the questions if he can answer all questions equally well?

Solution

The candidate has to select six questions in all of which at least two should be from Part A and two should be from Part B. He can select questions in any of the following ways:

	Part A	**Part B**
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is: $5C_2*5C_4$=10*5=50

If the candidate follows choice (ii), the number of ways in which he can do so is: $5C_3*5C_3=10*10=100$

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is:$5C_4*5C_2=5*10=50$

Therefore, the candidate can select the question in 50+100+50= 200 ways**.**

Example: A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when:**(i)** At least 2 women are included?**(ii)** At most 2 women are included?

Solution

(i) When at least 2 women are included.
The committee may consist of

3 women, 2 men: It can be done in $4C*6C_2$ ways
Or, 4 women, 1 man: It can be done in $4C_4*6C_1$ ways
or, 2 women, 3 men: It can be done in $4C_2*6C_3$ ways

=> Total number of ways of forming the committee:
$=4C_3*6C_2+4C_4*6C_1+4C_2*6C_3= 186$ ways

(ii) When at most 2 women are included
The committee may consist of

2 women, 3 men: It can be done in $4C_2*6C_3$ ways
Or, 1 women, 4 men: It can be done in $4C_1*6C_4$ ways
Or, 5 men: It can be done in $6C_5$ ways

=> Total number of ways of forming the committee:
$=4C_2*6C_3+4C_1*6C_4+6C_5= 186$ ways

Example: The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and at least 4 bowlers?

Solution

We are to choose 11 players including 1 wicket keeper and 4 bowlers
or, 1 wicket keeper and 5 bowlers.

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players
$=2C_1*5C_4*9C_6=840$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players
$=2C_1*5C_5*9C_5=252$

=> Total number of ways of selecting the team:
=840+252= 1092

Example: There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf?

Solution

4 novels can be selected out of 5 in $5C_4$ ways.
2 biographies can be selected out of4 in $4C_2$ ways.
Number of ways of arranging novels and biographies:
$=5C_4*4C_2=30$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6!=720 ways.
By the Counting Principle, the total number of arrangements =30*720= **21600**

Example: From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels?

Solution

From 5 consonants, 3 consonants can be selected in $5C_3$ ways.
From 4 vowels, 2 vowels can be selected in $4C_2$ ways.
Now with every selection, number of ways of arranging 5 letters is $5P_5$

Total number of words $=5C_3*4C_2*5P_5= 7200$.

Binomial Theorem

$(a+b)^n = nC_0a^n+ nC_1a^{n-1}b+\ldots+ nC_nb^n$

Example: Find the coefficient of the independent term of x in expansion of $(3x - (2/x^2))^{15}$.

Solution

The general term of $(3x - (2/x^2))^{15}$ is written, as $T_{r+1} = {}^{15}C_r\, (3x)^{15-r}\, (-2/x^2)^r$. It is independent of x if,

$15 - r - 2r = 0 \Rightarrow r = 5$

$\therefore \quad T_6 = {}^{15}C_5(3)^{10}(-2)^5 = -\, {}^{16}C_5\, 3^{10}\, 2^5.$

Example: Find the value of the greatest term in the expansion of $\sqrt{3}(1+(1/\sqrt{3}))^{20}$.

Solution

Let T_{r+1} be the greatest term, then $T_r < T_{r+1} > T_{r+2}$

Consider : $T_{r+1} > T_r$

$\Rightarrow {}^{20}C_r \quad (1/\sqrt{3})^r > 20C_{r-1}(1/\sqrt{3})^{r-1}$

$\Rightarrow ((20)!/(20-r)!r!)\ (1/(\sqrt{3})^r) \ > \ ((20)!/(21-r)!(r-1)!)\ (1/(\sqrt{3})^{r-1})$

$\Rightarrow r < 21/(\sqrt{3}+1)$

$\Rightarrow r < 7.686$ .... (i)

Similarly, considering $T_{r+1} > T_{r+2}$

$\Rightarrow r > 6.69$ .... (ii)

From (i) and (ii), we get

$r = 7$

Hence greatest term = $T_8 = 25840/9$

RECURRENCE RELATIONS

Definition

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence , namely $a_0,a_{1,\ldots\ldots,}a_{n-1}$,for all integers n with n>=0,where n_0 is a non –ve integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a recurrence relation satisfies a recurrence relation , then the sequence is called a solution of the recurrence relation.

For example ,we consider the famous Fibonacci sequence

0,1,1,2,3,5,8,13,21,.....,

which can be represented by the recurrence relation.

$F_n=F_{n-1}+F_{n-2}$,n>=2

& $F_0=0,F_1=1$. Here $F_0=0$ & $F_1=1$ are called initial conditions.

It is a second order recurrence relation.

Solving Linear Homogenous Recurrence Relations with Constants Coefficients.

Step 1: Write down the characteristics equation of the given recurrence relation .Here ,the degree of character equation is 1 less than the number of terms in recurrence relations.

Step 2: By solving the characteristics equation first out the characteristics roots.

Step 3: Depends upon the nature of roots ,find out the solution a_n as follows:

Case 1: Let the roots be real and distinct say $r_1, r_2, r_3, \ldots, r_n$ then

$$A_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \ldots\ldots + \alpha_n r_n^n,$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case 2: Let the roots be real and equal say $r_1 = r_2 = r_3 = r_n$ then

$$A_n = \alpha_1 r_1^n + n\alpha_2 r_2^n + n^2 \alpha_3 r_3^n + \ldots\ldots + n^2 \alpha_n r_n^n,$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case 3: When the roots are complex conjugate, then

$$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$$

Case 4: Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation.

$S(k)=2[S(k-1)]^2-kS(k-3)$ cannot be solved.

Example If sequence $a_n=3.2^n$,n>=1, then find the recurrence relation.

Solution:

For n>=1

$a_n=3.2^n$,

now, $a_{n-1}=3.2^{n-1,}$

$=3.2^n / 2$

$a_{n-1}=a^n/2$

$a_n = 2(a_n-1)$

$a_n = 2a_n-1$, for $n\geq 1$ with $a_n=3$

Example

Find the recurrence relation for S(n) = 6(-5),$n\geq 0$

Sol :

Given $S(n) = 6(-5)^n$

$S(n-1) = 6(-5)^{n-1}$

$=6(-5)^n / -5$

$S(n-1) = S(n) / -5$

$S_n = -5.5 (n-1)$, $n\geq 0$ with s(0) =6

Example Find the relation from $Y_k =A.2^k + B.3^k$

Sol :

Given $Y_k =A.2^k + B.3^k$ --------------→(1)

$Y_{k+1} =A.2^{k+1} + B.3^{k+1}$

$=A.2^k .2 + B3^k .3$

$Y_{k+1} =2A.2^k + 3B.3^k$ ------------------------→(2)

$Y_{k+2} =4A.2^k + 9B.3^k$ ------------------------→(3)

(3) – 5(2) + 6(1)

$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 4A.2^k + 9B.3^k - 10A.2^k - 15B.3^k + 6A.2^k + 6B.3^k$

$=0$

$\therefore Y_{k+1} - 5y_{k+1} + 6y_k = 0$ in the required recurrence relation.

Example

Solve the recurrence relation defind by $S_o = 100$ and S_k (1.08) S_{k-1} for $k \geq 1$

Sol ;

Given $S_0 = 100$

$S_k = (1.08)\ S_{k-1}$, $k \geq 1$

$S_1 = (1.08)\ S_0 = (1.08)100$

$S_2 = (1.08)\ S_1 = (1.08)(1.08)100$

$=(1.08)^2\ 100$

$S_3 = (1.08)\ S_2 = (1.08)(1.08)^2 100$

$= =(1.08)^3\ 100$

$S_k = (1.08)S_{k-1} = (1.08)^k 100$

Example Find an explicit formula for the Fibonacci sequence .

Sol ;

Fibonacci sequence 0,1,2,3,4........ satisify the recurrence relation

$fn= f_{n-1} + f_{n-2}$

$fn- f_{n-1} - f_{n-2} = 0$

& also satisfies the initial condition $f_0=0,f_1=1$

Now , the characteristic equation is

$r_2-r-1 =0$

Solving we get $r=1 \pm 1+4 / 2$

$= 1 \pm 5 / 2$

$fn = \alpha_1\ (1+5/2)^n + \alpha_2\ (1-5/2)^n$ ----→(A)

given f_0 =0 put n=0 in (A) we get

$f0 = \alpha_1\ (1+5/2)^0 + \alpha_2\ (1-5/2)^0$

(A)➔ $\alpha 1 + \alpha 2 = 0$ ------------------------→(1)

given f_1 =1 put n=1 in (A) we get

$f_1 = \alpha_1\ (1+5/2)^1 + \alpha_2\ (1-5/2)^1$

(A)➔$(1+5/2)^n + \alpha_2\ (1-5/2)^n\ \alpha_2 = 1$ -------→(2)

To solve(1) and (2)

(1) X(1+ 5 / 2) => $(1+5/2)\ \alpha_1 + (1+5/2)\ \alpha_2 = 0$ -----→(3)

$(1+5/2)\ \alpha_1 + (1+5/2)\ \alpha_2 = 1$------→(2)

(-) (-) (-)

$1/2\ \alpha_2 + 5/2\ \alpha_2 - 1/2\ \alpha_2 + 5/2\ \alpha_2 = -1$

$2\ 5\ \ d_2 = -1$

$\alpha_2 = -1/5$

Put $\alpha_2 = -1/5$ in eqn (1) we get $\alpha_1\ 1/5$

Substituting these values in (A) we get

Solution $fn=1/5\ (1+5/2)^n - 1/5\ (1+5/2)^n$

Example

Solve the recurrence equation

$a_n = 2a_{n-1} - 2a_{n-2}$, $n \geq 2$ & $a_0 = 1$ & $a_1 = 2$

Sol :

The recurrence relation can be written as

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is

r2 – 2r -2 =0

Roots are r= $2 \pm 2i / 2$

$= 1 \pm i$

LINEAR NON HOMOGENEOUS RECRRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation of the form

$a_n = c_1\ a_{n-1}+c_2\ a_{n-2}+$………..$c_k\ a_{n-k}$+F(n)……………. (A)

Where c_1 , c_2 ,…. c_k are real numbers and F(n) is a function not identically zero depending only on n,is called a non-homogeneous recurrence relation with constant coefficient.

Here ,the recurrence relation

$a_n = c_1\ a_{n-1}+c_2\ a_{n-2}+$………..$c_k\ a_{n-k}$+F(n)……………. (B)

Is called Associated homogeneous recurrence relation.

NOTE:

(B) is obtained from (A) by omitting F(n) for example ,the recurrence relation

$a_n = 3\ a_{n-1}+2_n$ is an example of non-homogeneous recurrence relation .Its associated

Homogeneous linear equation is

$a_n = 3\ a_{n-1}$ [By omitting F(n) = 2n]

PROCEDURE TO SOLVE NON-HOMOGENEOUS RECURRENCE RELATIONS:

The solution of non-homogeneous recurrence relations is the sum of two solutions.

1.solution of Associated homogeneous recurrence relation (By considering RHS=0).

2.Particular solution depending on the RHS of the given recurrence relation

STEP1:

a) if the RHS of the recurrence relation is

$a_0 + a_1$ n .... a_r n^r, then substitute

c_0 $+c_1$ $n + c_2$ n^2+..........c_r $(n-1)^r$ in place of $a_n - 1$ ....and so on ,in the LHS of the given recurrence relation

(b) if the RHS is a^n then we have

Case1:if the base a of the RHS is the characteristric root,then the solution is of the can^n .therefore substitute ca^n in place of a_n ,ca^{n-1} in place of c(n-1) a_{n-1} etc..

Case2: if the base a of RHS is not a root , then solution is of the form ca^n therefore substitute ca^n in place of a_n , ca^{n-1} in place of a_{n-1} etc..

STEP2:

At the end of step-1, we get a polynomial in 'n' with coefficient c_0,c_1……on LHS

Now, equating the LHS and compare the coefficients find the constants c_0,c_1,….

Example

Solve $a_n = 3\ a_{n-1}$ +2n with a_1 =3

Solution:

Give the non-homogeneous recurrence relation is

$a_n - 3\ a_{n-1}$ -2n=0

It's associated homogeneous equation is

$a_n - 3\ a_{n-1}$ =0 [omitting f(n) =2n]

It's characteristic equation is

r-3=0 => r=3

now, the solution of associated homogeneous equation is

$a_n\ (n) = \propto, 3^n$

To find particular solution

Since F(n) =2n is a polynomial of degree one,then the solution is of the from

$a_n = c_n$ +d (say) where c and d are constant

Now, the equation

$a_n = 3\ a_{n-1}$ +2n becomes

c_n +d =3(c(n-1)+d)+2n

[replace a_n by c_n +d a_{n-1} by c(n-1)+d]

- ⇨ c_n +d =3cn-3c+3d+2n
- ⇨ 2cn+2n-3c+2d=0
- ⇨ (2+2c)n+(2d-3c)=0
- ⇨ 2+2c=0 and 2d-3c=0
- ⇨ Saving we get c=-1 and d=-3/2 therefore cn+d is a solution if c=-1 and d=-3/2

a_n (p)=-n-3/2

Is a particular solution.

General solution

a_n=a_n(n)+ a_n(p)

$a_n = \propto 3^n$ -n-3/2 ………………..(A)

Given $a_1 = 3$ put n=1 in (A) we get

$a_1 = \propto\ 1(3)^1$ -1-3/2

3=3$\propto_1$-5/2

$3 \propto_1 = 11/2$

$\propto_1 = 11/6$

Substituting $\propto_1 = 11/6$ in (A) we get

General solution

$a_n = -n - 3/2 + (11/6)3^n$

Example:

Solve s(k)-5s(k-1)+6s(k-2)=2

With s(0)=1 ,s(1)=-1

Solution:

Given non-homogeneous equation can be written as

$a_n = 5\, a_{n-1} + 6\, a_{n-2} - 2 = 0$

The characteristic equation is

$r^2 - 5r + 6 = 0$

roots are r=2,3

the general solution is

$3_n(n) = \propto_1 (2)^n + \propto_2 (3)^n$

To find particular solution

As RHS of the recurrence relation is constant ,the solution is of the form C , where C is a constant

Therefore the equation

$a_n - 5\, a_{n-1} - 6\, a_{n-2} - 2 = 2$

$c - 5c + 6c = 2$

$2c=2$

$c=2$

the particular solution is

$s_n(p)=1$

the general solution is

$s_n = s_n(n) + s_n(p)$

$s_n = \propto_1 (2)^n + \propto_2 (3)^n + 1$……….(A)

$s_n = \propto_1 (2)^n + \propto_2 (3)^n + 1$……….(A)

Given $s_0=1$ put n=0 in (A) we get

$s_0 = \propto_1 (2)^0 + \propto_2 (3)^0 + 1$

$s_0 = \propto_1 + \propto_2 + 1$

(A) => $s_0 = 1 = \propto_1 + \propto_2 + 1$

$\propto_1 + \propto_2 = 0$……………(1)

Given a_1=-1 put n=1 in(A)

⇨ $S_1 = \propto_1 (2)^1 + \propto_2 (3)^1 + 1$
⇨ (A) $-1 = \propto_1 (2) + \propto_2 (3) + 1$
⇨ $2 \propto_1 + 3 \propto_2 = -2$……….(1)

$\propto_1 + \propto_2 = 0$

$2 \propto_1 + 3 \propto_2 = -2$……………….(2)

By solving (1) and (2)

$\propto_1 = 2, \propto_2 = -2$

Substituting $\propto_1 = 2, \propto_2 = -2$ in (A) we get

Solution is

⇨ $S_{(n)} = 2.(2)^n - 2.(3)^n + 1$

Example

Solve $a_n - 4\,a_{n-1} + 4\,a_{n-2} = 3n + 2^n$

$a_0 = a_1 = 1$

Solution:

The given recurrence relation is non-homogeneous

Now, its associated homogeneous equation is,

$a_n - 4\,a_{n-1} + 4\,a_{n-2} = 0$

Its characteristic equation is

$r^2 - 4r + 4 = 0$

$r = 2, 2$

solution , $a_n(n) = \propto_1 (2)^n + n \propto_2 (2)^n$

$a_n(n) = (\propto_1 + n \propto_2) 2^n$

To find particular solution

The first term in RHS of the given recurrence relation is 3n.therefore ,the solution is of the form $c_1 + c_2 n$

Replace a_n by $c_1 + c_2 n$, a_{n-1} by $c_1 + c_2(n-1)$

And a_{n-2} by $c_1 + c_2(n-2)$ we get

$(c_1 + c_2 n) - 4(c_1 + c_2(n-1)) + 4(c_1 + c_2(n-2)) = 3n$

⇨ $c_1 - 4c_1 + 4c_1 + c_2 n - 4c_2 n + 4c_2 n + 4c_2 - 8c_2 = 3n$

⇨ $c_1 + c_2 n - 4c_2 = 3n$

Generating function:

The generating function for the sequence 'S' with terms $a_0, a_1, \ldots\ldots a_n$

Of real numbers is the infinite sum.

Equating the corresponding coefficient we have

c_1-4c_2=0 and c_2=3

c_1=12 and c_2=3

Given a_0=1 using in (2)

(2) => $\propto_1$+12=1

Given a_1=1 using in (2)

(2)=> ($\propto_1$+$\propto_2$)2+12+3+1/2 .2=1

=> (2 $\propto_1$+2 $\propto_2$)+16=1……………..(14)

(3) $\propto_1$=-11

Using in (4) we have $\propto_2$=7/2

Solution a_n=(-11+7/2n)2^n+12+3n+1/2$n^2 2^n$

G(x)=G(s,x)= $a_0+a_1x+,\ldots\ldots a_nx^n+\ldots\ldots = \sum_{n=0}^{\infty} a^n x^n$

For example,

i) the generating function for the sequence 'S' with the terms 1,1,1,1…..i.s given by,

G(x)=G(s,x)= $\sum_{n=0}^{\infty} x^n$=1/1-x

ii)the generation function for the sequence 'S' with terms 1,2,3,4…..is given by

G(x)=G(s,x)= $\sum_{n=0}^{\infty}(n+1)x^n$

=1+2x+3x^2+……..

=$(1-x)^{-2}$=1/$(1-x)^2$

2.Solution of recurrence relation using generating function

Procedure:

Step1:rewrite the given recurrence relation as an equation with 0 as RHS

Step2:multiply the equation obtained in step(1) by x^n and summing if form 1 to ∞ (or 0 to ∞) or (2 to ∞).

Step3:put G(x)= $\sum_{n=0}^{\infty} a^n x^n$ and write G(x) as a function of x

Step 4:decompose G(x) into partial fraction

Step5:express G(x) as a sum of familiar series

Step6:Express a_n as the coefficient of x^n in G(x)

The following table represent some sequence and their generating functions

step1	sequence	generating function
1	1	1/1-z
2	$(-1)^n$	1/1+z
3	a^n	1/1-az
4	$(-a)^n$	1/1+az
5	n+1	$1/1-(z)^2$
6	n	$1/(1-z)^2$
7	n^2	$z(1+z)/(1-z)^3$
8	na^n	$az/(1-az)^2$

Eg:use method of generating function to solve the recurrence relation

$a_n=3a_{n-1}+1$; $n\geq1$ given that $a_0=1$

solution:

let the generating function of $\{a_n\}$ be

$$G(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$a_n=3a_{n-1}+1$$

multiplying by x^n and summing from 1 to ∞,

$$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty}(a_{n-1}x^n)+\sum_{n=1}^{\infty}(x^n)$$

$$\sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty}(a_{n-1}x^{n-1})+\sum_{n=1}^{\infty}(x^n)$$

$$G(x)-a_0=3xG(x)+x/1-x$$

$$G(x)(1-3x)=a_0+x/1-x$$

$$=1+x/1-x$$

G(x)(1-3x)=1=x+x/1-x

G(x)=1/(1-x)(1-3x)

By applying partial fraction

G(x)=-1/2/1-x+3/2/1-3x

$G(x)=-1/2(1-x)^{-1}+3/2(1-3x)^{-1}$

$G(x)[1-x-x^2]=a_0-a_1x-a_0x$

$G(x)[1-x-x^2]= a_0-a_0x+a_1x$

$G(x)=1/1-x-x^2 \quad [a_0=1,\ a_1=1]$

$$=\frac{1}{(1-1+\sqrt{5}\ \ x/2)(1-1-\sqrt{5}\ \ x/2)}$$

$$=\frac{A}{(1-(\frac{1+\sqrt{5}}{2})x)}+\frac{B}{(1-(\frac{1-\sqrt{5}}{2})x)}$$

Now,

$$1/1-x-x^2 =\frac{A}{(1-(\frac{1+\sqrt{5}}{2})x)}+\frac{B}{(1-(\frac{1-\sqrt{5}}{2})x)}\ldots\ldots\ldots\ldots(1)$$

$$1=A[1-(\frac{1+\sqrt{5}}{2})x)]+B[1-(\frac{1-\sqrt{5}}{2})x)]\ldots\ldots(2)$$

Put x=0 in (2)

(2)=> A+B=1

Put $x= 2/1-\sqrt{5}$ in (2)

$$(2)\Rightarrow\quad 1=B[1-\frac{1+\sqrt{5}}{1-\sqrt{5}}]$$

$$1=B[\frac{1-\sqrt{5}-1-\sqrt{5}}{1-\sqrt{5}}\ \ ]$$

$$1=B[\frac{-2\sqrt{5}}{1-\sqrt{5}}]$$

$B=\frac{1-\sqrt{5}}{-2\sqrt{5}}$

(3) => $A=\frac{1+\sqrt{5}}{2\sqrt{5}}$

Sub A and B in (1)

$G(x)=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)[1-(\frac{1+\sqrt{5}}{2})x]^{-1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)[1-(\frac{1-\sqrt{5}}{2})x]^{-1}$

$=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)[1+(\frac{1+\sqrt{5}}{2})x+(\frac{1-\sqrt{5}}{2}x)]^{2}+\ldots\ldots$

$=\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)[1+(\frac{1-\sqrt{5}}{2})x+(\frac{1-\sqrt{5}}{2}x)]^{2}+\ldots\ldots$

a_n=coefficient of x^n in G(x)

solving we get

$a_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$

Pigeon Hole Principle :

If $(n + 1)$ pigeon occupies 'n' holes then atleast one hole has more than 1 pigeon.

Proof :

Assume $(n + 1)$ pigeon occupies 'n' holes.

Claim : Atleast one hole has more than one pigeon.

Suppose not, *i.e.,* Atleast one hole has not more than one pigeon.

Therefore, each and every hole has exactly one pigeon.

Since, there are 'n' holes, which implies, we have totally 'n' pigeon.

which is a $\Rightarrow\Leftarrow$ to our assumption that there are $(n + 1)$ pigeon.

Therefore, atleast one hole has more than 1 pigeon.

THE PIGEONHOLE PRINCIPLE

If n pigeonholes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon. Generalized pigeonhole principle is: - If n pigeonholes are occupied by kn+1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.

Example1: Find the minimum number of students in a class to be sure that three of them are born in the same month.

Solution: Here n = 12 months are the Pigeonholes
And k + 1 = 3
K = 2

Example2: Show that at least two people must have their birthday in the same month if 13 people are assembled in a room.

Solution: We assigned each person the month of the year on which he was born. Since there are 12 months in a year.

So, according to the pigeonhole principle, there must be at least two people assigned to the same month.

THE PRINCIPLE OF INCLUSION –EXCLUSION

Assume two tasksT_1 and T_2 that can be done at the same time(simultaneously) now to find the number of ways to do one of the two tasks $T_1$and T_2, if we add number ways to do each task then it leads to an over count. since the ways to do both tasks are counted twice. To correctly count the number of ways to do each of the two tasks and then number of ways to do both tasks

i.e $\wedge(T_1 v T_2)=\wedge(T_1)+\wedge(T_2)-\wedge(T_1\wedge T_2)$

this technique is called the principle of Inclusion –exclusion

FORMULA:$_4$

1) $|A_1 v A_2 v A_3|=|A_1|+|A_2|+|A_3|-|A_1\wedge A_2|-|A_1\wedge A_3|-|A_2\wedge A_3|+|A_1\wedge A_2\wedge A_3|$

2) $|A_1 v A_2 v A_3 v A_4|=|A_1|+|A_2|+|A_3|+|A_4|-|A_1\wedge A_2|-|A_1\wedge A_3|-|A_1\wedge A_4|-|A_2\wedge A_3|-|A_2\wedge A_4|-|A_3\wedge A_4|+|A_1\wedge A_2\wedge A_3|+|A_1\wedge A_2\wedge A_4|+|A_1\wedge A_3\wedge A_4|+|A_2\wedge A_3\wedge A_4|+|A_1\wedge A_2\wedge A_3\wedge A_4|$

Example

A survey of 500 from a school produced the following information.200 play volleyball,120 play hockey,60 play both volleyball and hockey. How many are not playing either volleyball or hockey?

Solution:

Let A denote the students who volleyball

Let B denote the students who play hockey

It is given that

$n=500$

$|A|=200$

$|B|=120$

$|A \wedge B|=60$

Bt the principle of inclusion-exclusion, the number of students playing either volleyball or hockey

$|A \vee B|=|A|+|B|-|A \wedge B|$

$|A \vee B|=200+120-60=260$

The number of students not playing either volleyball or hockey=500-260

=240

Example

In a survey of 100 students it was found that 30 studied mathematics,54 studied statistics,25 studied operation research,1 studied all the three subjects.20 studied mathematics and statistic,3 studied mathematics and operation research And 15 studied statistics and operation research

1.how many students studied none of these subjects?

2.how many students studied only mathematics?

Solution:

1) Let A denote the students who studied mathematics

Let B denote the students who studied statistics

Let C denote the student who studied operation research

Thus $|A|=30$,$|B|=54$,$|C|=25$,$|A\wedge B|=20$,$|A\wedge C|=3$,$|B\wedge C|=15$,and $|A\wedge B\wedge C|=1$

By the principle of inclusion-exclusion students who studied any one of the subject is

$$|A\vee B\vee C|=|A|+|B|+|C|=|A\wedge B|-|A\wedge C|-|B\wedge C|+|A\wedge B\wedge C|$$

$$=30+54+25-20-3-15+1$$

$$=110-38=72$$

Students who studied none of these 3 subjects=100-72=28

2) now ,

The number of students studied only mathematics and statistics=$n(A\wedge B)-n(A\wedge B\wedge C)$

$=20-1=19$

The number of students studied only mathematics and operation research=$n(A\wedge C)-n(A\wedge B\wedge C)$

$=3-1=2$

Then The number of students studied only mathematics =30-19-2=9

Example

How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution:

Let A denote the set of positive integers not exceeding 1000 are divisible by 7

Let B denote the set of positive integers not exceeding 1000 that are divisible by 11.

Then $|A|=[1000/7]=[142.8]=142$

$|B|=[1000/11]=[90.9]=90$

$|A\wedge B|=[1000/7*11]=[12.9]=12$

The number of positive integers not exceeding 1000 that are divisible either 7 or 11 is |AvB|

By the principle of inclusion –exclusion

|AvB|=|A|+|B|-|A^B|

=142+90-12=220

There are 220 positive integers not exceeding 1000 divisible by either 7 or 11
