Engineering Mathematics- I
SMTA1101

UNIT 4

INTEGRAL CALCULUS I

- Definite integrals
- Properties of definite integrals and problems
- Beta and Gamma integrals
- Relation between them
- Properties of Beta and Gamma integrals with proofs
- Evaluation of definite integrals in terms of Beta and Gamma function.

Definite Integrals

Property 1: $$\int_a^b f(x)dx = \int_a^b f(z)dz$$

Property 2: $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property 3: $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Property 4: $$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Property 5: $$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is even} \end{cases}$$

Problems based on definite Integrals

PROBLEM (1) Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x)dx \qquad (1)$$

By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)dx \qquad (2)$$

Adding (1) & (2)

$$2I = \int_0^{\frac{\pi}{2}} \log \sin x dx + \int_0^{\frac{\pi}{2}} \log \cos x dx \quad (\text{Since } \because \log a + \log b = \log ab)$$

$$= \int_0^{\frac{\pi}{2}} \log[\sin x \cos x] dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx \qquad \left(\because \sin x \cos x = \frac{\sin 2x}{2}\right)$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx \qquad (3)$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \log\sin y dy$$

$$= \frac{1}{2}(2)\int_0^{\pi/2} \log\sin y dy$$

$$= \int_0^{\pi/2} \log\sin y dy$$

$$= \int_0^{\pi/2} \log\sin x dx \quad \_ \qquad (4)$$

To evaluate $\int_0^{\frac{\pi}{2}} \log(\sin 2x)dx$

Put $\quad 2x = y, 2dx = dy$

when $\quad x = 0, y = 0$

$\qquad x = \frac{\pi}{2}, y = \pi$

sub (4) in (3)

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = \frac{-\pi}{2}\log 2$$

PROBLEM (2) Evaluate $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

$$\text{let } I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta \qquad (1)$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan\theta}\right]d\theta \qquad (2)$$

(1) + (2) $\Rightarrow$

$$2I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta + \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log\left[(1+\tan\theta)\left(\frac{2}{1+\tan\theta}\right)\right]d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log 2 d\theta = \log 2\int_0^{\frac{\pi}{4}} d\theta$$

$$2I = \log 2[\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\log 2$$

$$\therefore 2I = \frac{\pi}{4}\log 2$$

$$\therefore I = \frac{\pi}{8}\log 2$$

BETA AND GAMMA FUNCTIONS

Gamma Functions:

Gamma function is defined as $\int_0^\infty e^{-x} x^{n-1} dx; n > 0$ and it is denoted by Γn

(i.e) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx,\ n > 0$

Beta function:

Beta function is defined as $\int_0^1 x^{m-1}.(1-x)^{n-1} dx, m > 0, n > 0$ and it in denoted by β(m, n)

(i.e) $\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}.dx; m > 0, n > 0$

Result : 1 Recurrence formula for Γn

$$\Gamma(n+1) = n\Gamma n$$

Result : 2 $\Gamma 1 = 1$

Result 3: when 'n' is a positive integer, then $\Gamma n+1 = n!$

Properties of Beta function:

1) **Symmertric Property:** β(m, n) = β(n, m)

2) Transformation of Beta function:

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{n+m}}.dy$$

3) **Trigonometric form of Beta function:**

$$\beta(m,n) = 2\int_0^{\pi/2} \sin 2^{m-1}\theta.\cos^{2n-1}\theta d\theta$$

Relation between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Proof: W.K.T $\Gamma n = \int_0^\infty e^{-x}\cdot x^{n-1}dx$

Put $\quad x = y^2$

$$dx = 2y\,dy$$

$$\Gamma n = \int^\infty e^{-y2}\cdot(y^2)^{n-1}2y\cdot dy$$

$$= 2\int_0^\infty e^{-y2}.y^{2x-2}.y^1dy$$

$$\Gamma n = 2\int_0^\infty e^{-y2}\cdot y^{2x-1}dy$$

Similarly $\quad \Gamma(m) = 2\int_0^\infty e^{-x^2}\cdot x^{2m-1}\cdot dx$

$$\therefore \Gamma(m)\cdot\Gamma(n) = 2\int_0^\infty e^{-x^2}\cdot x^{2m-1}dx\cdot 2\int_0^\infty e^{-y^2}y^{2x-1}\cdot dy$$

$$= 4.\int_0^\infty \int_0^\infty e^{-(x^2-y^2)}x^{2m-1}.y^{2n-1}.dx.dy$$

Put $\quad x = r\cos\theta; \quad y = r\sin\theta$

Hence |J| = r, by change of variables (jacobian)

$$dxdy = r.dr.d\theta, where\, r = |J|\,(ie)r^2 = x^2 + y^2$$

The region of integration is the complete first quadrant.

In which r varies from 0 to ∞

θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \overline{|(m)}\cdot\overline{|(n)} = 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n-2}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}\cdot|r|\cdot dr\cdot d\theta$$

$$= 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2m+2n-1}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}.dr.d\theta$$

$$= 4\int_0^{\infty} e^{-r^2} r^{2m+2n-1}dr.\int_0^{\frac{\pi}{2}} (\cos\theta)^{2m-1}(\sin\theta)^{2n-1}.d\theta$$

$$=4\int_0^{\infty} e^{-r^2}[r^2]^{m+n-1}\frac{1}{2}d(r)^2.\int_0^{\frac{\pi}{2}} (\cos\theta)^{2m-1}.(\sin\theta)^{2n-1}d\theta$$

$$\therefore \Gamma(m)\cdot\Gamma(n) = 4\left[\frac{1}{2}\Gamma(m+n)\right]\cdot\left[\frac{1}{2}\cdot\beta(m,n)\right]$$

Using Beta & Gamma Properties.

$$= \frac{4}{4}\left[\Gamma(m+n)\right]\cdot\beta(m,n)$$

$$= \Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Result : $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: W.K.T $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}.(\cos\theta)^{2n-1}\, d\theta$

Put $m = n = \frac{1}{2}$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2\times\frac{1}{2}-1}.(\cos\theta)^{2\times\frac{1}{2}-1}\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1.d\theta$$

$$= 2[\theta]_0^{\frac{\pi}{2}} = 2\times\frac{\pi}{2} = \pi \qquad (1)$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$$

W.K.T $\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \quad \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

By (1) $\pi = \frac{\left[\Gamma\frac{1}{2}\right]^2}{\Gamma(1)} = \frac{\left[\Gamma\frac{1}{2}\right]^2}{1}$

$$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

Hence proved

PROBLEM (3)

Evaluate $\int_0^\infty e^{-x^2} dx$

Solution

Put $x^2 = t; \; 2x dx = dt$

$$\therefore \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{2}\left\lceil \frac{1}{2} \right.$$

$$= \frac{1}{2}\sqrt{\pi}$$

PROBLEM (4) Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x dx$ using Gamma functions

Solution:

we know that

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

Here p = 6, q = 7

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6 x \cos^7 x dx = \frac{1}{2} \frac{\Gamma\frac{7}{2} \times \Gamma\left(\frac{8}{2}\right)}{\Gamma\frac{15}{2}} = \frac{\frac{1}{2}\Gamma\left(\frac{7}{2}\right) \times 3!}{\frac{13}{2} \times \frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{1}{2} \times \frac{6 \times 2^4}{13 \times 11 \times 9 \times 7}$$

$$= \frac{16}{3003}$$

Property 1: $$\int_a^b f(x)dx = \int_a^b f(z)dz$$

$$\text{Proof : L.H.S} = \int_a^b f(x)dx = [F(x)]_a^b$$

$$= F[b] - F[a]$$

$$R.H.S = \int_a^b f(z)dz = [F(Z)]_a^b$$

$$= F[b] - F[a]$$

$$\text{L.H.S} = \text{R.H.S}$$

Property 2: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof : L.H.S $= \int_a^b f(x)dx = [F(x)]_a^b = F[b] - F[a]$

R.H.S $= -\int_b^a f(x)dx = -[F(x)]_b^a$

$= -\left[F(a) - F(b)\right]$

$= \left[F(b) - F(a)\right]$

L.H.S $=$ R.H.S

Property 3: $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_a^b f(x)dx$$

$$\text{Proof : L.H.S} = \int_a^b f(x)dx$$

$$= [F(x)]_a^b = F(b) - F(a)$$

$$\text{R.H.S} = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= [F(x)]_a^c + [F(x)]_c^b$$

$$= F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a)$$

Hence L.H.S = R.H.S

Property 4: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof : Consider, LHS

Put $x = a - z$

$dx = -dz$

If $x = 0 \Rightarrow z = a$

$x = a \Rightarrow z = 0$

$$\int_0^a f(x)dx = \int_a^0 f(a-z)(-dz)$$

$$= -\int_a^0 f(a-z)dz$$

$$= \int_0^a f(a-z)dz \qquad \text{[by property 2]}$$

$$= \int_0^a f(a-x)dx \qquad \text{[by property 1]}$$

$$= \text{R.H.S}$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Property 5: $\Gamma 1 = 1$

we know that $\Gamma n = \int_0^\infty e^{-x} x^{n-1}.dx$

Put n = 1 $\quad \Gamma 1 = \int_0^\infty e^{-x} dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty .dx$

$$= \left(\frac{e^{-\infty}}{-1}\right) - \left(\frac{e^{-0}}{-1}\right) = 0 + 1 = 1$$

$$\Gamma 1 = 1$$

Property 6: $\beta(m, n) = \beta(n, m)$

Proof : W.K.T $\beta(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}.dx$

W.K.T $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\therefore \beta(m,n) = \int_0^1 (1-x)^{m-1}.[1-(1-x)]^{n-1}.dx$$

$$= \int_0^1 (1-x)^{m-1}.x^{n-1}.dx$$

$$= \int_0^1 x^{n-1}.(1-x)^{m-1}.dx$$

$\beta(m,n) = \beta(n,m)$, by definition of Beta function.

Thank You