Engineering Mathematics- I
SMTA1101

UNIT 5
INTEGRAL CALCULUS II

- Double integrals in Cartesian and polar co-ordinates
- Change the order of integration
- Change of variables from Cartesian to polar co-ordinates
- Area of plane curves using double integrals
- Triple integrals
- Volume using triple integrals in Cartesian co-ordinates

Problem (1) Evaluate $\int_2^a \int_2^b \frac{dxdy}{xy}$

$$\int_2^a \int_2^b \frac{dxdy}{xy} = \int_2^a \frac{1}{y}[\log x]_{x=2}^{x=b}\, dy$$

$$= \int_2^a \frac{1}{y}[\log b - \log 2]\, dy$$

$$= \int_2^a \frac{1}{y}[\log\left(\frac{b}{2}\right)]\, dy$$

$$= \log\left(\frac{b}{2}\right) \int_2^a \frac{1}{y}\, dy$$

$$= \log\left(\frac{b}{2}\right) [\log y]_{y=2}^{y=a}$$

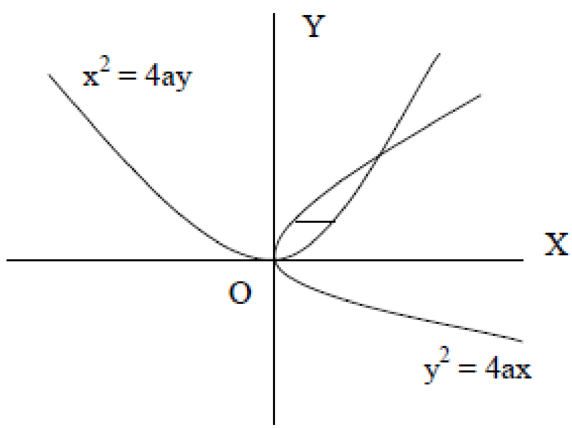
$$= \log\left(\frac{b}{2}\right)\log\left(\frac{a}{2}\right)$$

Problem (2)

Change the order of integration and hence evaluate $\int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$

$$\text{Let } I = \int_{0}^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$$

The limits for y varies from $y=\dfrac{x^2}{4a}$ to $y=2\sqrt{ax}$ and the limits for x varies from x = 0 to x = 4a. The region of integration is enclosed between the curves (parabolas) $x^2 = 4ay$ and $y^2 = 4ax$ and the lines x = 0 and x = 4a. The two parabolas intersect at (0, 0) and (4a, 4a).



To change the order of integration, first integrate w.r.t x and then w.r.t y. Since first integration is w.r.t x, we consider a horizontal strip. The limits for x varies from $x = y^2/4a$ to $x = 2\sqrt{ay}$ and then y varies from y = 0 to y = 4a.

Hence,

$$I = \int_o^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\, dx\, dy = \int_o^{4a} \left[\frac{x^2}{2} . y \right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_o^{4a} \left[\frac{4ay}{2} . y - \frac{y^4}{32a^2} . y \right] dy$$

$$= \left[2a . \frac{y^3}{3} - \frac{1}{32a^2} \frac{y^6}{6} \right]_0^{4a}$$

$$= \left[2a . \frac{64a^3}{3} - \frac{1}{32a^2} \frac{(4a)^6}{6} \right]$$

$$= \frac{128a^4}{3} - \frac{64a^4}{3}$$

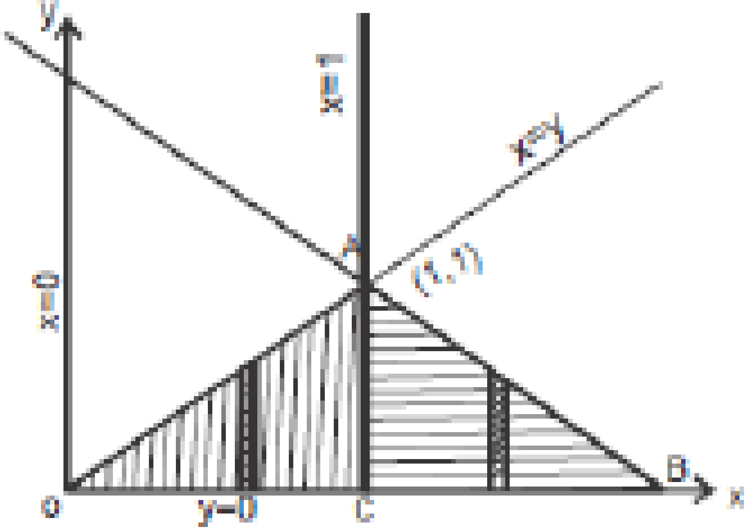
$$= \frac{64a^4}{3}$$

Problem (3)

Change the order of integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ and hence evaluate it.

Solution:

The region of integration is bounded by $x = y$, $x = 2 - y$, $y = 0$ and $y = 1$ which is shown in the figure.



When we change the order of integration, we first integrate with respect to y keeping x as constant. When the region of integration is covered by vertical strip, it does not intersect the region of integration in the same fashion. Hence the region ΔOAB is splitted into two subregions ΔOAC and ΔCAB. Hence

$$\iint_{OAB} xydxy = \iint_{OAC} xydydx + + \iint_{CAB} xydydx$$

$$\int_0^1 \int_y^{2-y} xydxdy = \int_0^1 \int_0^x xydydx + \int_1^2 \int_0^{2-x} xydydx$$

$$= \int_0^1 \left(\frac{xy^2}{2}\right)_0^x dx + \int_1^2 \left(\frac{xy^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3 dx + \int_1^2 \frac{x(2-x)^2}{2} dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1 + \int_1^2 \frac{x(4+x^2-4x)}{2} dx$$

$$=\frac{1}{2}\left(\frac{1}{4}-0\right)+\frac{1}{2}\left(4\frac{x^2}{2}+\frac{x^4}{4}-4\frac{x^3}{3}\right)_1^2$$

$$=\frac{1}{2}\left[\frac{1}{4}+\left(8+\frac{16}{4}-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}+\left(12-\frac{32}{3}\right)-\left(\frac{24+3-16}{12}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}+\frac{5}{12}\right]$$

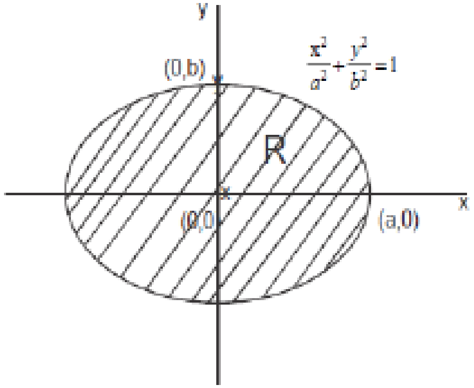
$$=\frac{8}{12\times 2}$$

$$=\frac{1}{3}$$

Area Using Double Integral

Problem (4) Find the Area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Solution:



From the equation of the ellipse, we have

$$\frac{y}{b}=\pm\sqrt{1-\frac{x^2}{a^2}}$$

So, the region of integration R can be considered as the area bounded by

$$x=-a \; and \; x=a, y\frac{-b}{a}\sqrt{a^2-x^2} \; and \; y=\frac{b}{a}\sqrt{a^2-x^2}$$

$$\text{Area} \quad =\iint_R dydx = 4\times \text{ Area in first quadrant}$$

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dydx$$

$$= 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[(0-0)+\frac{a^2}{2}\left(\frac{\pi}{2}-0\right)\right]$$

$= \pi ab$ square units.

Problem (5)

Find the area of the circle $x^2 + y^2 = r^2$ lies in the positive quadrant

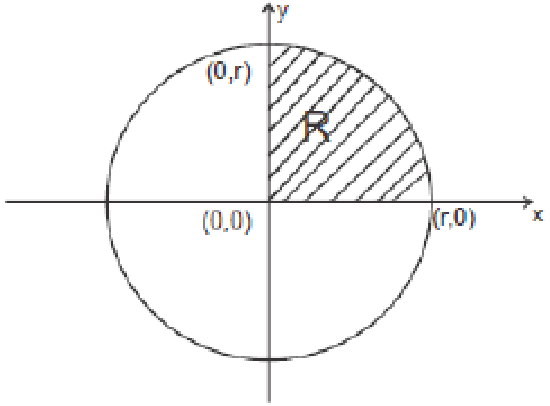
Solution

The circle lies in the first quadrant is bounded by $x = 0, y = 0, x^2 + y^2 = r^2$

Therefore, the region of integration R can be considered as the area bounded by $x = 0, x = r, y = 0 \, and \; y = \sqrt{r^2 - x^2}$

$$\text{Area} = \iint_R dxdy$$

$$= \int_{x=0}^{r} \left[\int_{y=0}^{\sqrt{r^2-x^2}} dy \right] dx$$



$$\text{Area} = \int_{x=0}^{r} [y]_0^{\sqrt{r^2-x^2}}\, dx$$

$$= \int_0^r \sqrt{r^2-x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{r^2-x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r$$

$$= 0 + \frac{r^2}{2}\left(\frac{\pi}{2} - 0\right)$$

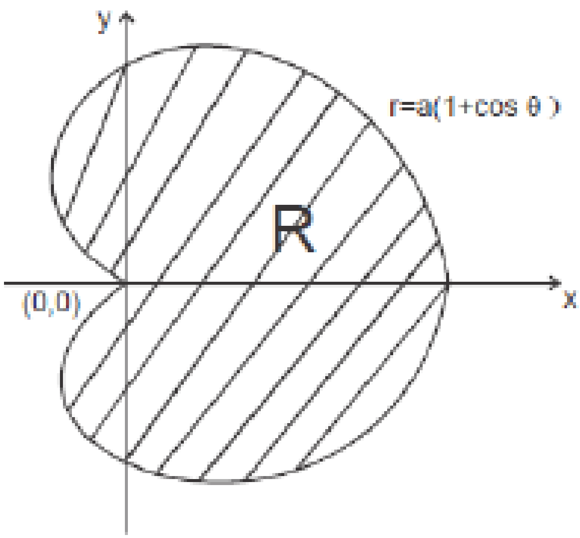
$$= \frac{\pi r^2}{4} \text{ square units.}$$

Problem (6) Find the area of the cardioids $r = a(1+\cos\theta)$

Solution

Area $= \iint_R dxdy$

$= \iint r dr d\theta$



Given $r = a(1+\cos\theta)$

Limits

$r : 0 \to a(1+\cos\theta)$

$\theta . 0 \to 2\pi$

$$\text{Area} = \int_0^{2\pi}\int_0^{a(1+\cos\theta)} r dr d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^2}{2} 2\int_0^{\pi} (1+\cos\theta)^2 d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= a^2 \int_0^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= a^2 \left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= a^2 \left[(\pi - 0) + 2(0-0) + \frac{1}{2}\left[(\pi - 0) + \frac{1}{2}(0-0)\right]\right]$$

$$= a^2 \left(\pi + \frac{1}{2}\pi\right) = \frac{3\pi}{2} a^2 \qquad \text{Square units.}$$

Volume using Triple Integrals

Problem (7)

Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ without transformation.

Solution

V = 8 x volume in the first octant

z varies from $z = 0$ to $z = \sqrt{a^2 - x^2 - y^2}$

$y$varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$

$x$varies from $x = 0$ to $x =$ a

$$\therefore V = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\, dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy dx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2} \sin^{-1}\left(\frac{y}{\sqrt{a^2 - x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left[\frac{a^2 - x^2}{2} \sin^{-1}\left(\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}} \right) + \frac{\sqrt{a^2 - x^2}}{2} \sqrt{a^2 - x^2 - (a^2 - x^2)} \right] dx$$

$$= 8\int_0^a \left[\frac{a^2 - x^2}{2} \sin^{-1}(1) + 0 \right] dx$$

$$= 8\int_0^a \left(\frac{a^2 - x^2}{2} . \frac{\pi}{2} \right) dx$$

$$= 8 \times \frac{\pi}{4} \int_0^a \left(a^2 - x^2\right) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^2 \times a - \frac{a^3}{3} - 0 \right]$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right] = 2\left[\frac{3a^3 - a^3}{3} \right] = \frac{4\pi a^3}{3} \, cubic\ units$$

Problem (8)

Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the first octant using triple integration.

Solution

Given $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (1)$$

Volume $= \iiint dz\,dy\,dx$

To find x limit put $y=0$ *and* $z=0$ we get (line integral)

$$(1) \Rightarrow \frac{x^2}{a^2}=1 \Rightarrow x^2=a^2 \Rightarrow x=\pm a$$

ie, $x=0$ to $x=a$ ($\because$ first octant area)

To find y limit put $z=0$ we get (surface integral)

$$(1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\Rightarrow y = \pm b\sqrt{1 - {x^2}/{a^2}}$$

$$\Rightarrow y = b\sqrt{1 - {x^2}/{a^2}}$$

ie, $y = 0, y = b\sqrt{1 - \dfrac{x^2}{a^2}}$ ($\because$ first octant area)

To find z limit [volume integral]

$$(1) \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$\Rightarrow z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$\Rightarrow z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$z = 0 \; to \; z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\text{volume} = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= \int_{b}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[z\right]_{0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= \int_{0}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-0\right] dydx$$

$$= c\int_{0}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dydx$$

$$= c\int_{0}^{a} \int_{0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dydx$$

$$= \frac{c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left[b^2\left(1-\frac{x^2}{a^2}\right)\right]-y^2}\,dy\,dx$$

$$= \frac{c}{b}\int_0^a \left[\frac{y\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}}{2}+\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{y}{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}}\right)\right]_0^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dx$$

$$= \frac{c}{b}\int_0^a \left[0+\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\left(\frac{\pi}{2}\right)\right]dx$$

$$= \frac{\pi c b^2}{4b} \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{\pi b^2 c}{4b} \left(x - \frac{x^3}{a^2 \times 3}\right)_0^a$$

$$= \frac{\pi bc}{4} \left(a - \frac{a^3}{3a^2}\right)$$

$$= \frac{\pi bc}{4} \left(a - \frac{a}{3}\right)$$

$$= \frac{\pi bc}{4} \frac{2a}{3} = \frac{\pi abc}{6}$$

Hence the volume of the ellipsoid

$$V = 8 \times \frac{\pi abc}{6} = \frac{4}{3} \pi abc \text{ cubic units.}$$

Thank You