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SCHOOL OF SCIENCE AND HUMANITIES

DEPARTMENT OF MATHEMATICS

UNIT – II – DISCRETE MATHEMATICS – SMTA 1302

UNIT II SET THEORY

Basic concepts of Set theory - Laws of Set theory - Partition of set, Relations - Types of Relations: Equivalence relation, Partial ordering relation - Graphs of relation - Hasse diagram, Functions: Injective, Surjective, Bijective functions, Compositions of functions, Identity and Inverse functions.

The concept of a set is used in various disciplines and particularly in computers.

Basic Definition:

1. “A collection of well defined objects is called a set”.

The capitals letters are used to denote sets and small letters are used for denote objects of the set. Any object in the set is called element or member of the set. If *x* is an element of the set *X*, then we write $x \in X$, to be read as ‘x *belongs to X’* , and if *x* is not an element of *X*, the we write $x \notin X$ to be read as *‘ x does not belongs to X’*.

2. The number of elements in the set A is called *cardinality* of the set A, denoted by |*A*| or *n(A)* . We note that in any set the elements are distinct. The collection of sets is also a set.

$S = \{P_1, \{P_2, P_3\}, P_4, P_5\}$

Here $\{P_2, P_3\}$ itself one set and it is one element of *S* and |*S*|=4.

3. Let A and B be any two sets. If every element of A is an element of B, then A is called a *subset* of B is denote by $'A \subseteq B'$.

We can say that A *contained (included)* in B, (or) B *contains (includes)* A.

Symbolically, $A \subseteq B$ (or) $B \supseteq A$

Logically, $A \subseteq B = (x\forall)\{x \in A \rightarrow x \in B\}$

Let $A = \{1,2,3,4,5\},\ B = \{1,2,4\},\ C = \{1,5\},\ D = \{2\},\ E = \{1,4,2\}$

Then $B \subseteq A,\ C \subseteq A,\ D \subseteq A, D \subseteq B$

$C \nsubseteq B$, since $5 \in C \Rightarrow 5 \notin B,\ E \subseteq B$ and $B \subseteq E$.

Some of the important properties of set inclusion.

For any sets A, B and C

$A \subseteq A$ (Reflexive)

$(A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C)$ (Transitive)

Note that $A \subseteq B$ does not imply $B \subseteq A$ except for the following case.

4. Two sets A and B are said to be *equal* if and only if $A \subseteq B$ and $B \subseteq A$,

$i.e., \qquad A = B \Leftrightarrow (A \subseteq B \ and\ B \subseteq C)$

Example $\{1,2,4\} = \{4,1,2\}$ and $P = \{\{1,2\},4\},\ Q = \{1,2,4\}$ then $P \neq Q$

Since $\{1,2\} \in P$ and $\{1,2\} \notin Q$ eventhough $1,2 \in Q$.

The equality of sets is reflexive, symmetric, and transitive.

5. A set A is said to be a *proper subset* of a set B if $A \subseteq B$ and $A \neq B$. Symbolically it is written as $A \subset B. i.e.,\ A \subset B \Leftrightarrow (A \subseteq B \ \wedge\ A \neq B)$

$\subset$ is also called a *proper inclusion.*

6. A set is said to be *universal set* if it includes every set under our discussion. A universal set is denoted by ∪ or E.

In other words, if *p(x)* is a predicate.$E = \{x|p(x) \vee \neg p(x)\}$

One can observe that universal set contains all the sets.

7. A set is said to be *empty set* or *null set* if it does not contain any element, which id denoted by $\emptyset$.

In other words, if *p(x)* is a predicate.$\emptyset = \{x | p(x) \vee \neg p(x)\}$

One can observe that null set is a subset for all sets.

8. For a set A, the set of all subsets of A is called the *power set* of A. The power set of A is denoted by$\rho(A)$ or $2^{\wedge}$ $i.e.,\ \rho(A) = \{S \mid S \subseteq A\}$

Example, Let $A = \{a, b, c\}$

Then $\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\},\ A\}$

Then set $\emptyset$ and A are called *improper subsets* of A and the remaining sets are called *proper subsets* of A.

One can easily note that the number of elements of $\rho(A)$ is $2^{|A|}.i.e.,\ |\rho(A)| = 2^{|A|}$

SOME OPERATIONS ON SETS

1. Intersection of sets

Definition:

Let A and B be any two sets, the *intersection* of A and B is written as $A \cap B$ is the set of all elements which belong to both A and B.

Symbolically

$A \cap B = \{\, x \mid x \in A \; and \; x \in B\}$

Example$A = \{1,2,3,4,5,6\},\ B = \{2,4,6,8\}$ then $A \cap B = \{2,4,6\}$. From the definition of intersection it follows that for any sets A,B,C and universal set E.

$A \cap A = A$ $\quad A \cap B = B \cap A$ $\quad A \cap (B \cap C) = (A \cap B) \cap C$

$A \cap E = A$ $\quad A \cap \emptyset = \emptyset$

2. Disjoint sets

Definition:

Two set A and B are called *disjoint* if and only if $A \cap B = \emptyset$, that is, A and B have no element in common.

Example $A = \{1,2,3\}$ $B = \{5,7,9\}$ $C = \{3,4\}$

$A \cap B = \emptyset,\ A \cap C = \{3\},\ B \cap C = \emptyset$

A and B are disjoint and B and C also, but A and C are not disjoint.

3. Mutually disjoint sets

Definition:

A collection of sets is called a *disjoint collection*, if for every pair of sets in the collection, are disjoint. The elements of a disjoint collection are said to be *mutually disjoint.*

Let $A = \{A_i\}_{i \in I}$ be an indexed set, A is mutually disjoint if and only if

$A_i \cap A_j = \emptyset$ for all $i, j \in I, i \neq j$.

Example

$A_1 = \{\{1,2\},\{3\}\}$, $A_2 = \{\{1\},\{2,3\}\}$, $A_3 = \{\{1,2,3\}\}$

Then $A = \{A_1, A_2, A_3\}$ is a disjoint collection of sets.

$A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$

4. Unions of sets

Definition:

The *union* of two sets A and B, written as $A \cup B$, is the set of all elements which are elements of A or the elements of B or both.

Symbolically $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Example Let $A = \{1,2,3,4,5,6\}$ $B = \{2,4,6,8\}$ then $A \cup B = \{1,2,3,4,5,,6,8\}$

From the union, it is clear that, for any sets A, B,C, and universal set E.

$A \cup A = A$ $\quad A \cup B = B \cup A$ $\quad A \cup (B \cup C) = (A \cup B) \cup C$

$A \cup E = E$ $\quad A \cup \emptyset = A$

5. Relative complement of a set

Definition:

Let A and B are any two sets. The *relative complement* of B in A, written $A - B$, is the set of elements of A which are not elements of B.

Symbolically $A - B = \{ x \mid x \in A \; or \; x \notin B\}$

Note that $A - B = A \cap \bar{B}$.

Example Let $A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A - B = \{1,3,5\}$

$B - A = \{8\}$

It is clear from the definition that, for any set A and B.

$A - B = \emptyset$

$A - B \neq B - A$

$A - \emptyset = A$

6. Complement of a set

Definition:

Let A be any set, and E be universal. The relative complement of A in E is called *absolute complement or complement* of A. The complement of A is denoted by $\bar{A}$ (or A^C or $\backsim A$)

Symbolically

$E - A = \bar{A} = \{ x \mid x \in E \text{ and } x \notin A\}$

Example Let $E = \{1,2,3,4,\dots\}$ be universal set and

$A = \{2,4,6,8,\dots\}$ be any set in E.

Then

$\bar{A} = \{1,3,5,7,\dots\}$

From the definition, for any sets A$\bar{\bar{A}} = A \quad \bar{\emptyset} = E$

$\bar{E} = \emptyset \quad A \cup \bar{A} = E A \cap \bar{A} = \emptyset$

7. Boolean sum of sets

Definition:

Let A and B are any two sets. The *symmetric difference or Boolean sum* of A and B is the set A+B defined by

$A + B = (A - B) \cup (B - A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$

(or) $A + B = \{ x \mid x \in A \text{ and } x \notin B\} \cup \{ x \mid x \in B \text{ and } x \notin A\}$

Example Let

$A = \{1,2,3,4,5,6\}$

$B = \{2,4,6,8\}$ then

$A + B = \{1,3,5,8\}$ From the definition, for any sets A and B.

$A + A = \emptyset, \; A + \emptyset = A$

$A + E = \bar{A}, \; A + B = B + A$ and

$A + (B + C) = (A + B) + C$

8. The principle of duality

If we interchange the symbols $\cap$, $\cup$, E and $\emptyset$, $\subseteq$ and $\supseteq$, $\subset$ and $\supset$, in a set equation or expression. We obtain a new equation or expression is said to be *dual* of the original on *(primal).*

" If T is any theorem expressed in terms of $\cap$,$\cup$ and $-$ deducible from the given basic laws, then the dual of T is also a theorem".

Note that, the theorem T is proved in *m* steps, then dual of T also proved in *m* step.

Example The dual of $A \cap \overline{A} = \emptyset$ is given by $A \cup \overline{A} = E$.

Remark: Dual (Dual T) =T.

Identities on sets

$A \cup A = A$	Idempotent laws
$A \cap A = A$	
$A \cup B = B \cup A$	Commutative laws
$A \cap B = B \cap A$	
$(A \cup B) \cup C = A \cup (B \cup C)$	Associative laws
$(A \cap B) \cap C = A \cap (B \cap C)$	
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
$A \cup (A \cap B) = A$	Absorption laws
$A \cap (A \cup B) = A$	
$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$	De Morgan's laws

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

$A \cup \emptyset = A$ $\quad$ $A \cap \emptyset = \emptyset$

$A \cup E = E$ $\quad$ $A \cap E = A$

$A \cup \bar{A} = E$ $\quad$ $A \cap \bar{A} = \emptyset$

$\bar{\emptyset} = E$ $\quad$ $\bar{E} = \emptyset$ $\quad$ $\bar{\bar{A}} = A$

PROBLEMS

1.$S = \{a, b, p, q\}$, $Q = \{a, p, t\}$. Find $S \cup Q$ and $S \cap Q$?

Solution:

$S \cup Q = \{a, b, p, q, t\}$

$S \cap Q = \{a, p\}$

2. If $A = \{a, b, c\}$. Find $\rho(A)$?

Solution:

$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$ and

$|A| = 3$

$|\rho(A)| = 2^3 = 8$

3. Write all proper subsets of $A = \{a, b, c\}$.

Solution:

The proper subsets are

$\rho(A) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$

4. Show that $A \subseteq B \Leftrightarrow A \cap B = A$.

Solution:

If $A \subseteq B$, then $\forall\, x \in A \Longrightarrow x \in B$

Now , let

$x \in A \Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow\; x \in A \cap B$

$A = A \cap B$

If $A \cap B = A$, then

Let $x \in A, \quad x \in A \cap B \;\Longrightarrow x \in B$

Therefore $A \subseteq B$.

5. If $A = \{2,5,6,7\}, B = \{1,2,3,4\},\; C = \{1,3,5,7\}$. Find $A - B, A - C, C - B$ and $B - C$.

Solution:

$A - B = \{5,6,7\}$

$A - C = \{2,6\}$

$C - B = \{5,7\}$

$B - C = \{2,4\}$

6. If $A = \{2,3,4\}, B = \{1,2\},\; C = \{4,5,6\}$. Find $A + B, B + C, A + C, A + B + C$ and $(A + B) + (B + C)$.

Solution:

$A + B = \{1,3,4\}$

$B + C = \{1,2,4,5,6\}$

$A + C = \{2,3,5,6\}$

$A + B + C = \{1,3,5,6\}$

$(A + B) + (B + C) = \{2,3,5,6\}$

Note that

$A + (B + B) + C = A + (\emptyset) + C = A + C = \{2,3,5,6\}$

7. Show that $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Solution:

Let

$x \in A \Rightarrow x \in A\ (or)\ x \in B$

$\Rightarrow x \in A \cup B$

$\Rightarrow A \subseteq A \cup B$

Now let $x \in A \cap B \Rightarrow x \in A\ and\ x \in B$

$\Rightarrow x \in A$

$A \cap B \subseteq A$

Hence $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Remark: $B \subseteq A \cup B, A \cap B \subseteq B$ and $A \cap B \subseteq A \cup B$.

8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.

Solution:

$x \in A - (A \cap B) \Leftrightarrow x \in A\ and\ x \notin (A \cap B)$

$\Leftrightarrow x \in A\ and\ \{x \notin A\ or\ x \notin B\}$

$\Leftrightarrow \{x \in A \; and \; x \notin A\}(or) \{x \in A \; and \; x \notin B\}$

$\Leftrightarrow \; \emptyset \; (or)\{x \in A \; and \; x \notin B\}$

$\Leftrightarrow x \in A \; and \; x \notin B$

$A - (A \cap B) \subseteq A - B$ and $A - B \subseteq A - (A \cap B)$

Therefore $A - (A \cap B) = A - B$.

9. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution:

$x \in A \cup (B \cap C) \Leftrightarrow x \in A \; or \; x \in B \cap C$

$\Leftrightarrow x \in A \; or \; \{x \in B \; and \; x \in C\}$

$\Leftrightarrow \{x \in A \; or \; x \in B\} \; and \; \{x \in A \; or \; x \in C\}$

$\Leftrightarrow \{x \in A \cup B\} \; and \; \{x \in A \cup C\}$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

10. Show that $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

Solution:

Let $x \in \overline{(A \cup B)} \Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \; and \; x \notin B$

$\Leftrightarrow x \in \bar{A} \; and \; x \in \bar{B}$

$\Leftrightarrow x \in \bar{A} \cap \bar{B}$

Therefore $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

11. Show that $(A - B) - C = A - (B \cup C)$.

Solution:

$(A-B)-C=(A-B)\cap\bar{C}$ $\quad\quad$ $(P-Q=P\cap\bar{Q})$

$=(A\cap\bar{B})\cap\bar{C}$

$=A\cap(\bar{B}\cap\bar{C})$ $\quad\quad$ (Associative)

$=A\cap(\overline{B\cup C})$ $\quad\quad$ (De Morgan's law)

12. Show that $A\cap(B-C)=(A\cap B)-(A\cap C)$

Solution:

Let $(A\cap B)-(A\cap C)$

$=(A\cap B)\cap(\overline{A\cap C})$

$=(A\cap B)\cap(\bar{A}\cup\bar{C})$

$=(A\cap B\cap\bar{A})\cup(A\cap B\cap\bar{C})$

$=((A\cap\bar{A})\cap B)\cup(A\cap B\cap\bar{C})$

$=(\emptyset\cap B)\cup(A\cap B\cap\bar{C})$

$=\emptyset\cup(A\cap B\cap\bar{C})$

$=A\cap(B\cap\bar{C})$

$=A\cap(B-C)$

ASSIGNMENT PROBLEMS

Part –A

1. Define a set
2. Define subset of a set. What is mean by proper subset?

(i) Find all subset of $A = \{1,2,3\}$

(ii)Find all proper subsets of A.

3. Define power set.
4. Define disjoint sets with example?
5. If $A = \{1,2,3,4,5\}$ and $B = \{2,4,6,8,10\}$. Find $A \cup B, A \cap B, a - B, B - A,$

   $A + B,$ and $B + A$?

6. Which of the following sets are empty?
7. $\{x \mid x \in R, x + 6 = 6\}$
8. $\{x \mid x\, is\, a\, real\, integer\, such\, that\, x^2 + 1 = 0\}$
9. $\{x \mid x\, is\, a\, real\, integer\, and\, x^2 - 4 = 0\}$
10. State duality principle in set theory.
11. Define cardinality of a set.
12. If a set A has *n* elements, then the number of elements of power set of A is……..
13. Find the intersection of the following sets

    (i) $\{x \mid x^2 - 1 = 0\}, \{x \mid x^2 + 2x + 1 = 0\}$
14. Write the dual of $A \cap \bar{A} = \emptyset$.
15. Let A, B and C sets, such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, can we conclude that B=C.
16. State De Morgan's Laws.
17. Whether the union of sets is commutative or not?

PART –B

1. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
2. Verify the De Morgan's laws

   (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$, (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
3. Show that the intersection of sets is associative.
4. Show that $A - (B - C) = (A - B) \cup (A \cap C)$.
5. Show that $A \cap (B - C) = (A \cap B) - (A \cap C)$
6. Let $A_i = \{1,2,3, \dots\}$ for $i = 1,2,3, \dots$ find (a) $\cup_{i=1}^{n} A_i$ (b) $\cap_{i=1}^{n} A_i$
7. Prove that $A - (A - B) \subset B$.
8. Show that for any two sets A and B, $A - (A \cap B) = A - B$.
9. Prove that $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$.
10. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that B=C.(cancelation law)
11. Show that $A - (B \cup C) = (A - B) \cap (A - C)$.
12. Show that $A + A = \emptyset$, where + is the symmetric difference of sets.
13. Show that $(R \subseteq S)$ and $(S \subset Q)$ imply $R \subset Q$.
14. Given that $A \cap C \subseteq B \cap C$ and $A \cap \bar{C} \subseteq B \cap \bar{C}$. Show that $A \subseteq B$.

CARTESIAN PRODUCT OF SETS

The *Cartesian product* of the sets A and B, is written an $A \times B$, is the set of all ordered pairs in which the first elements are in A and the second elements are in B.

$i.e.\ A \times B = \{\langle x, y\rangle | x \in A\ and\ x \in B\}$

For example

Let $A = \{1,2\}, B = \{a, b, c\},\ c = \{\alpha, \beta\}$

Now

$A \times B = \{\langle 1,a\rangle,\langle 1,b\rangle,\langle 1,c\rangle\langle 2,a\rangle,\langle 2,b\rangle,\langle 3,c\rangle\}$

$A \times C = \{\langle 1,\alpha\rangle,\langle 1,\beta\rangle,\langle 2,\alpha\rangle,\langle 2,\beta\rangle\}$

$A \times B = \{\langle \alpha,a\rangle,\langle \alpha,b\rangle,\langle \alpha,c\rangle\langle \beta,a\rangle,\langle \beta,b\rangle,\langle \beta,c\rangle\}$

It is clear from the definition

$A \times B \neq B \times A$ and $\langle\langle a,b\rangle,c\rangle \in (A \times B) \times C$, is an ordered triple then $\langle a,b\rangle \in A \times B$ and $c \in C$.

Now , $A \times (B \times C) = \{\langle a,\langle b,c\rangle\rangle \,|a \in A \; and \; \langle b,c\rangle \in \langle B,C\rangle\}$

Note that $\langle a,\langle b,c\rangle\rangle$ is not an ordered triple.

This fact show that $(A \times B) \times C \neq A \times (B \times C)$

i.e. Cartesian product is not associative.

Now

$A \times A = A^2 = \{\langle x,y\rangle, \forall x,y \in A\}$ and $A^n = A^{n-1} \times A$.

Note that if A has *n* elements and B has *m* elements $A \times B$ has *nm* elements.

PROBLEMS

1.If $A = \{1,2,3\}$, $B = \{a,b\}$. Find $A \times B, B \times A$ and $A \times A$ and $A^2 \times B$

Solution :

$A \times B = \{\langle 1,a\rangle,\langle 1,b\rangle,\langle 2,a\rangle,\langle 2,b\rangle,\langle 3,a\rangle,\langle 3,b\rangle\}$

$B \times A = \{\langle a,1\rangle,\langle a,2\rangle,\langle a,3\rangle,\langle b,1\rangle,\langle b,2\rangle,\langle b,3\rangle\}$

$A^2 = A \times A = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 1,3\rangle,\langle 2,1\rangle,\langle 2,2\rangle,\langle 2,3\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 3,3\rangle\}$

$A^2 \times B = \{\langle 1,1,a\rangle, \langle 1,1,b\rangle, \langle 1,2,a\rangle, \langle 1,2,b\rangle, \langle 1,3,a\rangle, \langle 1,3,b\rangle, \langle 2,1,a\rangle, \langle 2,1,b\rangle,$

$\langle 2,2,a\rangle, \langle 2,2,b\rangle, \langle 2,3,a\rangle, \langle 2,3,b\rangle, \langle 3,1,a\rangle, \langle 3,1,b\rangle, \langle 3,2,a\rangle, \langle 3,2,b\rangle, \langle 3,3,a\rangle, \langle 3,3,b\rangle\}$

2.Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (B \cap C) \Leftrightarrow x \in A \, and \, y \in B \cap C$

$\Leftrightarrow x \in A \, and \, \{y \in B \, and \, y \in C\}$

$\Leftrightarrow \{x \in A \, and \, y \in B\} \, and \, \{y \in B \, and \, y \in C\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times B\} \, and \, \{\langle x, y\rangle \in A \times C\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times B) \cap (A \times C)\}$

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

3.Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

Solution: For any $\langle x, y\rangle$,

$\langle x, y\rangle \times (A \cap B) \times (C \cap D) \Leftrightarrow x \in (A \cap B) \, and \, y \in (C \cap D)$

$\Leftrightarrow \{x \in A \, and \, x \in B\} \, and \, \{y \in C \, and \, y \in D\}$

$\Leftrightarrow \{x \in A \, and \, y \in C\} \, and \, \{x \in B \, and \, y \in D\}$

$\Leftrightarrow \{\langle x, y\rangle \in A \times C\} \, and \, \{\langle x, y\rangle \in B \times D\}$

$\Leftrightarrow \{\langle x, y\rangle (A \times C) \cap (B \times D)\}$.

ASSIGNMENT PROBLEMS

Part A

1. Define Cartesian product of sets? Given an example?
2. If $A = \{0,1\}$, find A^2.
3. If $A = \{1,2,3\}$ and $B = \{a, b\}$, find $A \times B, B \times A, A^2$.
4. True or False
   I. If $A = \{1,3,5,7,9\}$, the $\{\forall x \in A, x + 2\ is\ a\ prime\ number\}$
   II. If $A = \{1,2,3,4,5\}$, the $\{\ni x \in A, x + 3 = 10\}$
5. If $A \times B = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$

Part B

6. If A,B and C are sets, prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
7. Prove that $(A \times C) - (B \times C) = (A - B) \times C$.
8. If $A = \{a, b\}$ and $B = \{1,2\}$,and $C = \{2,3\}$, find
   I. $A \times (B \cup C)$
   II. $(A \times B) \cup (A \times C)$
   III. $A \times (B \cap C)$
   IV. $(A \times B) \cap (A \times C)$
9. Show that the Cartesian product is not commutative? It is commutative only for equality of sets?

RELATIONS

Binary relation

Any set of ordered pairs defines a *binary relation.*

If *x* and *y* are binary related, under the relation R, the we write $\langle x, y\rangle \in R$ or xRy. If not the case we write $\langle x, y\rangle \notin R$.

1. Example $F = \{\langle x, y\rangle \,|x\, is\, the\, father\, of\, y\}$

$L = \{\langle x, y\rangle \,|x\, and\, y\, are\, real\, number\, and\, x < y\}$

Then F, L are binary relations.

2.Example Let A and B be any two sets, then any non empty subset R of $A \times B$ is called a *binary relation.*

Now

$A = \{1,2,3\}$

$B = \{a, b\}$ then

$A \times B = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

Let

$R_1 = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle, \langle 3, b\rangle\}$

$R_2 = \{\langle 1, b\rangle, \langle 3, a\rangle\}$

$R_3 = \{\langle 2, a\rangle\}$

Then R_1, R_2 and R_3 are binary relations A to B.

Let S be any binary relation. The *domain* of S is the set of all elements *x* such that for some $y, \langle x, y\rangle \in S$.

$D(S) = \{x\, |\langle x, y\rangle \in S, for\ some\ y\, \}$

Similarly, the *range* of S is the set of all elements *y* such that, for some $x, \langle x, y\rangle \in S$

$i.e.$ $\quad R(S) = \{y\, |\langle x, y\rangle \in S, for\ some\ x\, \}$

Let

$S = \{\langle 1,a\rangle,\langle 1,b\rangle,\langle 2,b\rangle,\langle 3,a\rangle\}$

$D(S) = \{1,2,3\}$

$R(S) = \{a,b\}$

If $S \subseteq X \times Y$, then clearly $D(S) \subseteq X$ and $R(S) \subseteq Y$.

In case of $X = Y$, then the relation defined on $X \times X$ is called *an universal relation* in X.

If $X = \emptyset$, then a relation on $X \times X$ is called *void relation* in X.

Since relations are sets, then we can have their union and intersection and so on.

$R \cup S = \{\langle x,y\rangle \mid xRy \text{ or } xSy\}$

$R \cap S = \{\langle x,y\rangle \mid xRy \text{ and } xSy\}$

$R - S = \{\langle x,y\rangle \mid xRy \text{ and } \langle x,y\rangle \notin S\}$

$R + S = \{\langle x,y\rangle \mid \langle x,y\rangle \text{ is either in } R \text{ or in } S \text{ but not in both}\}$

Properties of Binary relations

1. Reflexive

Let R be a binary relation defined on X.

Then R is *reflexive* if, for every $x \in X, \langle x,y\rangle \in R$.

Example:

Let

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 2,3\rangle\}$ and

$S = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 3,3\rangle\}$ are defined on X.

Then R is reflexive, but S is not reflexive. Since $\langle 2,2\rangle \notin S$ and $2 \in X$.

2. Symmetric

A relation R from X to Y is *symmetric* if every $x \in X$ and $y \in Y$, whenever $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.

That is, if $xRy \Rightarrow yRx$, then R is symmetric

Example:

Let

$X = \{1,2\}$

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle\}$ are defined on X.

Then R is symmetric, but S is not symmetric. Since $\langle 1,2\rangle \in S$ but $\langle 1,2\rangle \notin S$.

3. Transitive

A relation R is *transitive* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.

That is, if $xRy \wedge yRz$, then R is transitive.

Example:

Let

$R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 2,1\rangle\}$ and

$S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 3,3\rangle, \langle 2,1\rangle\}$

Then R is transitive, but S is not transitive. Since $\langle 2,1\rangle \in S$ and $\langle 1,2\rangle \in S$ but $\langle 2,2\rangle \notin S$.

4.Irreflexive

A relation R in a set X is *irreflexive* if, for every $x \in X, \langle x, x\rangle \notin R$.

Example:

Let

$A = \{1,2,3\}$

$R = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$ and

$S = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 2,2\rangle, \langle 1,3\rangle\}$

Then R is irreflexive, but S is not reflexive. Since $\langle 3,3\rangle \notin S$ and $\langle 1,1\rangle \in S$.

5. Antisymmetric

A relation R in a set X is *antisymmetric* if, whenever $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then$x = y$.

That is, if $xRy \wedge yRx \Rightarrow x = y$, then R is antisymmetric.

Example:

Let

X be the set of all subsets of E.

R be the inclusion relation $(\subseteq)$ defined on X.

$A \subseteq B \wedge B \subseteq A \Rightarrow A = B$

Therefore R is antisymmetric in X.

6. Relation matrix

Let $X = \{x_1, x_2, \dots x_m\}$, $Y = \{y_1, y_2, \dots y_m\}$ are ordered sets, R be a relation defined from X to Y, then the *relation matrix* of R, is defined as

$M_R = (r_{ij})\ i : 1 \to m, j: 1 \to n$

Example 1:

Let $X = \{1,2,3\}\ Y = \{a, b\}$

$R = \{\langle 1,a\rangle,\langle 1,b\rangle,\langle 2,a\rangle,\langle 3,b\rangle\}$ be a relation from X to Y. Then $M_R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example 2: Let

$R = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 1,3\rangle,\langle 2,2\rangle,\langle 3,1\rangle,\langle 3,2\rangle\}$ be a relation on $X = \{1,2,3\}$. Then $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

7. Composition of Binary Relations

The concept of composition of relation is different from union and intersection of two relations.

Definition:

Let R be a relation from X to Y and S be a relation from Y to Z. Then the composite $R \circ S$ is a relation from X to Z defined by

The operation $\circ$ in $R \circ S$ is called *" composition of relations".*

Example.

Let

$R = \{\langle 1,2\rangle,\langle 2,3\rangle,\langle 3,4\rangle,\langle 2,2\rangle\}$

$S = \{\langle 2,3\rangle,\langle 4,1\rangle,\langle 4,3\rangle,\langle 2,1\rangle\}$. Then

$R \circ S = \{\langle 1,3\rangle,\langle 1,1\rangle,\langle 3,1\rangle,\langle 3,3\rangle,\langle 2,3\rangle,\langle 2,1\rangle\}$

$S \circ R = \{\langle 2,4\rangle,\langle 4,2\rangle,\langle 4,4\rangle,\langle 2,2\rangle\}$

Note that

$R \circ R = R^2$

$R \circ R \circ R = R^2 \circ R = R^3$

$R^{n-1} \circ R = R^n$ etc.,

Definition:

The relation matrix for $R \circ S$ is given by $M_{R \circ S} = M_R \odot M_S$ where $\odot$ is defined as follows.

$M_R \odot M_S = \langle m_{ij} \rangle$ where $m_{ij}(\langle i,j \rangle th\ \ element)$ is 1 if and only if row *i* of M_R and column *j* of M_S have a 1 in the same relative position *k*, for some *k*.

Example:

Let

$R = \{\langle 1,2 \rangle, \langle 1,5 \rangle, \langle 2,2 \rangle, \langle 3,4 \rangle, \langle 5,1 \rangle, \langle 5,5 \rangle\}$

$S = \{\langle 1,3 \rangle, \langle 2,5 \rangle, \langle 3,1 \rangle, \langle 4,2 \rangle, \langle 4,4 \rangle, \langle 5,2 \rangle, \langle 5,3 \rangle\}$. Then

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
M_S &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
M_{R\circ S} &= M_R \odot M_S \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \\
\text{and} & \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$R^2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,5\rangle, \langle 2,2\rangle, \langle 5,1\rangle, \langle 5,2\rangle, \langle 5,5\rangle\}$$

Definition

Let R be a relation from X to Y. The *converse* of R, is written as $\tilde{R}$, is a relation from Y to X such that$xRy \Leftrightarrow x\tilde{R}y$.

Example:

If $R = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 2, a\rangle, \langle b, 3\rangle$

$\tilde{R} = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle a, 2\rangle, \langle b, 3\rangle$

Also it is clear that

1. $R = S \Leftrightarrow \tilde{R} = \tilde{S}$
2. $R \subseteq S \Leftrightarrow \tilde{R} \subseteq \tilde{S}$
3. $\widetilde{R \cup S} = \tilde{R} \cup \tilde{S}$

Result: The relation matrix $M_{\tilde{R}}$ is the transpose of the relation M_R.

$i.e. M_{\tilde{R}} = transpose\ of\ M_R$

Example:

Let

$R = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle$

$\tilde{R} = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,2\rangle, \langle 1,3\rangle, \langle 3,3\rangle$

We have

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{\tilde{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[M_R]^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{\tilde{R}}$$

EQUIVALENCE RELATION

Definition:

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 1:

Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 4,1\rangle, \langle 4,4\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$ is an equivalence relation on X.

Example 2:

Equality of subsets on a universal set is an equivance relation.

Example 3:

Let

$X = \{1,2,3, \dots 7\}$

$R = \{\langle x, y\rangle \mid x - y \text{ is divisible by } 3\}$

Now, $\forall x \in X, x - x = 0$ is divisible by 3.

Therefore $\forall x \in X, \langle x, x\rangle \in R$ (reflexive)

For any $x, y \in X$

Let $\langle x, x\rangle \in R \Rightarrow x - y$ is divisible by 3 we have $-(x - y) = y - x$ is also divisible by 3.

$\langle y, x\rangle \in R$ (symmetric)

Let $\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R$

$\Rightarrow x - y$ is divisible by 3 and $y - z$ is divisible by 3.

$\Rightarrow (x - y)+ (y - z)$ is divisible by 3.

$\Rightarrow x - z$ is divisible by 3.

Therefore $\langle x, y\rangle \in R$ (Transitive)

Therefore R is an equivalence relation on X.

EQUIVALENCE CLASSES

Definition:

Let R be an equivalence relation on a set X. For any $x \in X$, the set $[x]_R \subseteq X$ given by

$[x]_R = \{y \mid xRy \; for \; y \in X\}$

is called an R-*equivalence class* generated by $x \in X$.

Therefore, an equivalence class $[x]_R$ of $x \in X$ is the set of all elements which are related to *x* by an equivalence relation R on X.

Example:

Let Z be the set of all integers and R be the relation called *"congruence modulo 4"* defined by

$R = \{\langle x, y\rangle \mid (x - y) \; is \; divisible \; by \; 4, for \; x \; and \; y \in Z\}$ (or $x \equiv y(mod\ 4)$)
Now, we determine the equivalence classes generated by R.

$[0]_R = \{\ldots - 8, -4,0,4,8 \ldots\}$

$[1]_R = \{\ldots - 7, -3,1,5,9 \ldots\}$

$[2]_R = \{\ldots - 6, -2,2,6,10 \ldots\}$

$[3]_R = \{\ldots - 5, -1,3,7,11 \ldots\}$

Note that

$[0]_R = [4]_R, [1]_R = [5]_R, \dots etc.$

Therefore $\frac{Z}{R} = \{[0]_R, [1]_R, [2]_R, [3]_R\}$

In a similar manner, we get the equivalence classed generated by the relation *"congruence modulo m"* for any integer *m.*

Therefore, an equivalence relation R on X, will divide the set X into an *equivalence classes,* and they are called *portion* of X.

PARTIAL ORDERED RELATION

A relation R on a set X is said to be a partial ordered relation, if R satisfies reflexive, antisymmetric, and transitive.

Example:

Let $\rho(A)$ be the power set of a set A.

Define a subset relation $(\subseteq)$ on $\rho\ (A)$, then $\subseteq$ is a partial ordered relation.

Usually we denote the partial ordered relations as $' \leq '$ is said to be *partially ordered set* (or) *poset,* which is denoted by $\langle X, \leq \rangle$. We will study more about posets in the subsequent sections.

1. Closures of a relation

Let R be a relation on the set X.

2. Reflexive closure

We have the relation R is reflexive if and only if the relation.

$R = \{\langle x, y \rangle \mid \forall\ x \in X\}$ is contained in R.

i.e. R is reflexive $\Leftrightarrow I \subset R$.

Definition:

Let R be a relation on X, then the smallest reflexive relation on X, containing R, is called *reflexive closure* of R.

Therefore $R_1 = R \cup I$ is the reflexive closure of R.

3. Symmetric closure

We have, the relation R is symmetric if $\langle x, y\rangle \in R \Leftrightarrow \langle y, x\rangle \in \tilde{R}$

$i.e. \tilde{R} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$

Definition:

Let R be a relation X, then smallest symmetric relation on X, containing R, is called the *symmetric closure* of R.

Therefore $R \cup \tilde{R}$ is the symmetric of R.

4. Transitive closure

We have, the relation R is transitive, if $\langle x, y\rangle \in R$ *and* $\langle y, z\rangle \in R$ then $\langle x, z\rangle \in R$.

Definition:

A relation R^+ is said to be the *transitive closure* of the relation R on X if R^+ is the smallest transitive relation on X, containing R,

i.e R^+ is the transitive closure of R, if

I. $R \subseteq R^+$
II. R^+ is transitive on X
III. There is no transitive relation R_1 on X, such that $R \subset R_1 \subset R^+$

Remarks:

1. The transitive closure of R can be obtained by

$$R^{+} = R \cup R^{2} \cup R^{3} \cup \dots = \bigcup_{i=1}^{\propto} R^{i}$$

2. We know that $\langle x, z\rangle \in R^{2}$ if and only if there is an element y such that $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$.

   Therefore $\langle a, b\rangle \in R^{n}$ if and only if we can find a sequence $x_1, x_2, \dots . x_{n-1}$ in X such that $\langle a, x_1\rangle, \langle x_1, x_2\rangle, \dots \langle x_{n-1}, b\rangle$ are all in R.

   The sequence $a, x_1, x_2, \dots . x_{n-1}, b$ is said to be a *chain* of length *n* from a to b in R. Here $x_1, x_2, \dots . x_{n-1}$ are called interval vertices of the chain in R. Note that the interval vertices need not be distinct.

PROBLEMS

1. If $P = \{\langle 1,2\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$, $Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$

Find (i) $P \cup Q, P \cap Q, \tilde{P}, \tilde{P} \cup Q$ (ii) domains of $P, P \cup Q, P \cap Q$ and (iii) ranges of $Q, P \cup Q, P \cap Q$.

Solution:

$P \cup Q = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,4\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$

$P \cap Q = \{\langle 2,4\rangle\}$

$\tilde{P} = \{\langle 2,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

$\tilde{P} \cup Q = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 4,2\rangle, \langle 2,1\rangle, \langle 4,3\rangle\}$

Domain of $P = \{1,2,3\}$

Domain of $(P \cup Q) = D(P \cup Q) = \{1,2,3,4\}$

Domain of $(P \cap Q) = D(P \cap Q) = \{2\}$

Range of $Q = R(Q) = \{2,3,4\}$

Range of $(P \cup Q) = R(P \cup Q) = \{2,3,4\}$

Range of $(P \cap Q) = R(P \cap Q) = \{4\}$

It is clear that

$D(P \cup Q) = D(P) \cup D(Q)$ and

$R(P \cap Q) \subseteq R(P) \cap R(Q)$

In general $D(P) = R(\tilde{P})$ and $R(P) = D(\tilde{P})$.

2. Let $X = \{1,2,3,4\}$ and $R = \{\langle x, y\rangle \mid x, y \in X \text{ and } (x - y) \text{ is an integeral non zeromultiple of } 2\}$ $S = \{\langle x, y\rangle \mid x, y \in X \text{ and } (x - y) \text{ is an integeral non zeromultiple of } 3\}$. Find $R \cup S$ and $R \cap S$?

Solution:

Given that $R = \{\langle 1,3\rangle, \langle 3,1\rangle, \langle 2,4\rangle, \langle 4,2\rangle\}$ and

$S = \{\langle 1,4\rangle, \langle 4,1\rangle\}$ $R \cup S = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 4,1\rangle, \langle 4,2\rangle\}$

$R \cap S = \emptyset$

Remarks:

$D(R) = \{1,2,3,4\}$

$R(R) = \{1,2,3,4\}$

$D(S) = \{1,4\}$

$R(S) = \{1,4\}$

3. Let $S = \{\langle x, x^2\rangle \mid x \in N\}$ and $T = \{\langle x, 2x\rangle \mid x \in N\}$, where $= \{0,1,2, \ldots.\}$. Find the range of S and T, find $S \cup T$ and $S \cap T$?

Solution:

$S = \{\langle x, x^2 \rangle \mid x \in N\}$

$= \{\langle 0,0 \rangle, \langle 1,1 \rangle, \langle 2,4 \rangle, \langle 3,9 \rangle, \langle 4,16 \rangle, \dots\dots\ \}$ and

$T = \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle 0,0 \rangle, \langle 1,2 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle, \langle 4,8 \rangle, \dots\dots\ \}$

$R(S) = \{x^2 \mid x \in N\}$

$= \{0,1,4,9,16,25 \dots\dots\ \}$

$R(T) = \{2x \mid x \in N\}$

$= \{0,2,4,6,8,10, \dots\dots\ \}$

$S \cup T = \{\langle x, x^2 \rangle \mid x \in N\} \cup \{\langle x, 2x \rangle \mid x \in N\}$

$= \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = x^2\ (or) 2x\}$

$= \{\langle 0,0 \rangle, \langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,4 \rangle, \langle 3,6 \rangle, \langle 3,9 \rangle, \dots\dots\ \}$

$S \cap T = \{\langle x, y \rangle \mid x, y \in N, such\ that\ y = 2x\ and\ y = x^2\}$

(Now $y = 2x$ and $y = x^2 \Rightarrow 2x = x^2\ i.e.\ x = 0\ or\ x = 2$

$x = 0\ y = 0\ and\ x = 2 \Rightarrow y = 4$)

$S \cap T = \{\langle 0,0 \rangle, \langle 2,4 \rangle\}$

4. Given an example which is neither reflexive nor irreflexive?

Solution:

Let $X = \{1,2,3,4\}$ and

$R = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle, \langle 4,1 \rangle, \langle 4,4 \rangle\}$

Then R is not reflexive, since $\langle 2,2 \rangle \notin R$, for $2 \in X$ and R is not irreflexive, since $1 \in X$, and $\langle 1,1 \rangle \in R$.

5. Test whether the following relations are transitive or not on

$X = \{1,2,3\}$

$R = \{\langle 1,1\rangle, \langle 2,2\rangle\}$

$S = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$

$T = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle\}$.

Solution: The relation R and T are transitive.

Since, in R, we have $\langle 1,1\rangle \in R$, then check any other pair starting with $\langle 1,z\rangle \in R$, then we must have $1R1 \wedge 1Rz \Rightarrow 1Rz$ i.e., $\langle 1,z\rangle \in R$, but there is no pair staring with 1. So, pass on to next pair $\langle 2,2\rangle$ then we check any other pair starting with 2, and so on.

In T, we have $\langle 1,1\rangle \in T$, then there are two pairs $\langle 1,2\rangle$ and $\langle 1,3\rangle$ must be the transitive of $\langle 1,1\rangle \in T$, then we must have $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in T. Then pass to $\langle 1,2\rangle \in T$ the transitive pairs are $\langle 2,1\rangle, \langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must have the pairs $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle$ in T.

Then pass to $\langle 1,3\rangle \in T$, find the transitive pairs of $\langle 1,3\rangle$ and so on, for all pairs in T. Hence T is a transitive relation.

The relation S is not transitive, since for $\langle 1,2\rangle \in S$, the transitive pairs are $\langle 2,2\rangle$ and $\langle 2,3\rangle$ then we must $\langle 1,2\rangle$ and $\langle 1,3\rangle$ in S but $\langle 1,3\rangle \notin S$.

6. Let R denotes a relation on the set of pairs of positive $N \times N$ integers such that $\langle x, y\rangle R \langle u, v\rangle$ if and only if $xv = yu$. Show that R is an equivalence relations.

Solution:

Let

$P = \{\langle x, y\rangle \mid x \textit{ and } y \textit{ are positive integer}\}$

Now R is a relation defined on P as

$\langle x, y\rangle R\, \langle u, v\rangle \;\Leftrightarrow\; xv = yu$ for $\langle x, y\rangle, \langle u, v\rangle \in P$.

Let $\langle x, y\rangle, \langle u, v\rangle$ and $\langle m, n\rangle \in P$.

I. R is reflexive:

We have

$\langle x, y\rangle R\, \langle x, y\rangle \;\Leftrightarrow\; xy = yx$ (RHS) is true.

II. R is symmetric:

Let $\langle x, y\rangle R\, \langle u, v\rangle \;\Leftrightarrow\; xv = yu$

$\Leftrightarrow\; yu = xv$

$\Leftrightarrow\; uy = vx$

$\Leftrightarrow \langle u, v\rangle R\, \langle x, y\rangle$

III. R is transitive:

Let $\langle x, y\rangle R\, \langle u, v\rangle$ and $\langle u, v\rangle R\, \langle m, n\rangle$

$\Leftrightarrow\; (xv = yu)$ and $(un = vm)$

$\Leftrightarrow\; (xv = yu)$ and $(u = \frac{vm}{n})$

$\Leftrightarrow\; xv = y(\frac{vm}{n})$

$\Leftrightarrow\; xn = ym$

$\Leftrightarrow\; \langle u, v\rangle R\, \langle m, n\rangle$

Therefore R is reflexive, symmetric , and transitive.

Hence R is an equivalence relation.

7. Let R and S are equivalence relations on X, show that $R \cap S$ also equivalent? Whether $R \cup S$ is also an equivalent relation. If not given an example.

Solution:

Given let R and S are equivalence relations on X.

Let x, y and $z \in X$.

(i) We have $\langle x, x\rangle \in R$ and $\langle x, x\rangle \in S \Rightarrow \langle x, x\rangle \in R \cap S,\ \forall x \in X$.

Therefore $R \cap S$ is reflexive.

(ii)Let $\langle x, y\rangle \in R \cap S \Rightarrow \langle x, y\rangle \in R$ and $\langle x, y\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R$ and $\langle y, x\rangle \in S$

$\Rightarrow \langle y, x\rangle \in R \cap S$

Therefore $R \cap S$ is symmetric.

(iii) Let $\langle x, y\rangle \in R \cap S$ and $\langle y, z\rangle \in R \cap S$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle x, y\rangle \in S)$ and $(\langle y, z\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow (\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$ and $(\langle x, y\rangle \in R$ and $\langle y, z\rangle \in S)$

$\Rightarrow \langle x, y\rangle \in R$ and $\langle x, z\rangle \in S$

$\Rightarrow \langle x, z\rangle \in R \cap S$

Therefore $R \cap S$ is transitive.

Hence $R \cap S$ is equivalence.

8. Prove that the relation *"congruence modulo m"* over the set of positive integers is an equivalence relation?

Show also that if $x_1 = y_1$ and $x_2 = y_2$ then $(x_1 + x_2) = (y_1 + y_2)$.

Solution:

Let N be the set of all positive integers we have *"congruence modulo m"* relation on N as $x \equiv y (mod\, m) \Leftrightarrow m|\ x - y$, for $x, y \in N$.

Let $x, y, z \in N$

(i) We have

$x - x = 0 = 0m$

Therefore $x \equiv x \ (mod\ m)$ for $x \in N$.

"Congruence modulo m" is reflexive.

(ii)Let

$x \equiv y(mod\ m)$

$\Rightarrow\ m|\ x - y$

$\Rightarrow x - y = km$, for some integer $k \in Z$

$\Rightarrow y - x = (-k)m$, for some integer $-k \in Z$

$\Rightarrow y \equiv x\ (mod\ m)$

"congruence modulo m" is symmetric on N.

(iii) Let

$x \equiv y(mod\ m)$ and $y \equiv z\ (mod\ m)$

$\Rightarrow x - y = k_1 m$, and $y - x = k_2 m$ for some integer $k_1, k_2 \in Z$

$\Rightarrow (x - y) + (y - z) = (k_1 + k_2)m$

$\Rightarrow x - z = (k_1 + k_2)m$ for some integer $k_1 + k_2$

$\Rightarrow x \equiv z\ (mod\ m)$

"Congruence modulo m" is transitive on N.

Hence *"congruence modulo m"* is an equivalence relation.

Let $x_1 \equiv y_1(mod\ m)$ and $x_2 \equiv y_2(mod\ m)$.

Then $m|\ x_1 - y_1$ and $m|\ x_2 - y_2$

i.e., $x_1 - y_1 = k_1 m$ and $x_2 - y_2 = k_2 m$

Now

$(x_1 - y_1) + (x_2 - y_2) = k_1 m + k_2 m$

$(x_1 + x_2) - (y_1 + y_2) = (k_1 + k_2)m$

$\Rightarrow m|(x_1 + x_2) - (y_1 + y_2)$

$(x_1 + x_2) \equiv (y_1 + y_2)(mod\, m)$

9. Let

$X = \{1,2,3,4\}$ and

$R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle\}$ be a relation defined on A. Find the transitive closure of R?

Solution:

The matrix of the relation R is given by

$$
\begin{aligned}
M_R &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
M_{R^2} &= M_R \odot M_R \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
\text{and} \\
M_{R^3} &= M_{R^2} \odot M_R \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_{R^4} &= M_{R^3} \odot M_R \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

As $|A| = 4$, we get

$$
\begin{aligned}
M_{R^+} &= M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

Hence

$R^+ = \{\langle 1,2\rangle,\langle 1,3\rangle,\langle 1,4\rangle,\langle 2,2\rangle,\langle 2,3\rangle,\langle 2,4\rangle,\langle 3,2\rangle,\langle 3,3\rangle,\langle 3,4\rangle,\langle 4,2\rangle,\langle 4,3\rangle,\langle 4,4\rangle\}$

ASSIGNMENT PROBLEMS

Part -A

1. If $R = \{\langle 1,1\rangle,\langle 1,2\rangle,\langle 2,1\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 2,2\rangle\}$ and $S = \{\langle 1,2\rangle,\langle 2,3\rangle,\langle 3,1\rangle,\langle 1,3\rangle,\langle 3,3\rangle\}$ be any relations on $X = \{1,2,3\}$. Find $R \cup S, R \cap S, \tilde{R}, R(R), R(\tilde{S}), D(R \cup S), R(R \cap S)$.
2. Give an example for reflexive, symmetric, transitive and irreflexive relations.
3. Give an example of a relation which is neither reflexive nor irreflexive.
4. Give an example of a relation which is neither symmetric not antisymmetric?
5. Find the graph of the relation $R = \{\langle 1,2\rangle,\langle 1,3\rangle,\langle 2,1\rangle,\langle 2,2\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 3,3\rangle\}$

6. Find the relation matrix of
   $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$
7. If $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$ and
   $= \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$. Find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$,
   $R \circ R \circ S$ and $S \circ S \circ S$?
8. Define equivalence relation and equivalence classes?
9. Define Poset?
10. Define reflexive closure?
11. Define transitive closure of the relation R?
12. Let $R = \{\langle 1,2\rangle, \langle 3,5\rangle, \langle 6,1\rangle, \langle 6,3\rangle, \langle 6,4\rangle\}$ be a relation $A = \{1,2,3,4,5,6\}$.
    Identify the root of the tree of R.
13. Determine whether the relation R is a partial ordered on the set Z, where Z is set of positive integer, and aRb if and only if a=2b.
14. The following relations are on $\{1,3,5\}$. Let R be a relation, xRy if and only if $y = x + 2$, and let S be a relation, xSy if and only if $x \leq y$. Find $R \circ S$ and $S \circ R$?
15. True or False: The relation $<$ on Z^+ is not a partial order since it is not reflexive.

Part B

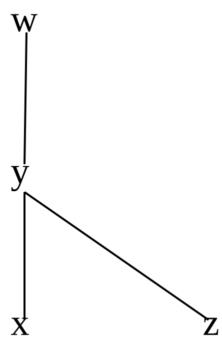
1. Show that the intersection of equivalence relations is an equivalence relation.
2. Determine whether the relations represented by the following zero-one matrices are equivalence relations.

$$a)\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad b)\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. If R and S are symmetric, show that $R \cup S$ and $R \cup S$ are symmetric.
4. Let L be set of all straight lines in the Euclidean plane and R be the relation in L defined by $xRy \Leftrightarrow x$ is perpendicular to y. Is R is Reflexive? Symmetric? Antisymmetric? Transive?
5. Consider the subsets $A = \{1,7,8\}$, $B = \{1,6,9,10\}$ and $C = \{1,9,10\}$ where $E = \{1,2,3 \dots \dots 10\}$ is an universal set. List the non empty minsets generated by A,B and C . Do they form a partition on E?
6. Let $X = \{1,2,3, \dots . .20\}$ and $R = \{\langle x, y\rangle \,|x - y \text{ is divisible by } 5\}$ be a relation on X. Show that R is an equivalent relation and find the partition of X induced by R.
7. If R is an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes constitute a partition on A.
8. Given the relation matrix M_R and M_S. Explain how to find $M_{R\circ S}$, $M_{S\circ R}$and M_{R^2}?
9. Let A be s set of books. Let R be a relation on A such that $\langle a, b\rangle \in R$ if ' book a' with cost more and contains fever pages then ' book b'. In general, is R reflexive? Symmetric? Antisymmetric? Transitive?
10. Let R be a binary relation on the set of all positive integers such that $R = \{\langle a, b\rangle \,|a = b^2\}$. Is R reflexive? Symmetric? Antisymmetric? Transitive? An equivalence relation?

HASSE DIAGRAM

A partial ordering $\leq$ on a finite set P can be represented in a plane by means of a diagram called *Hasse diagram* or a *partially ordered set set diagram* of $\langle P, \leq \rangle$. If $x \ll y$, then we place y above x, and draw a line (edge) between them. The upward direction indicates successor and downward direction indicates the predecessor. And the incomparable elements are in the same horizontal line.



y is immediate successor of x (or) x is immediate predecessor of y.

z is immediate predecessor of y, and x and y are incomparable.

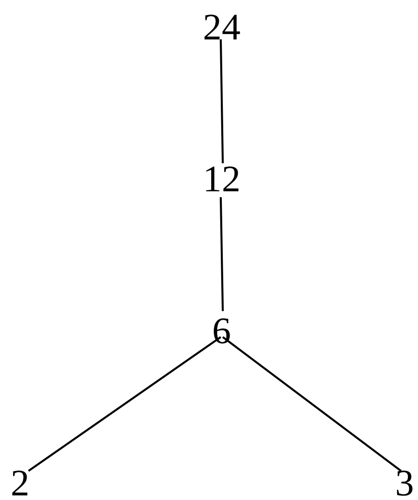
x is predecessor of w but not immediate predecessor.

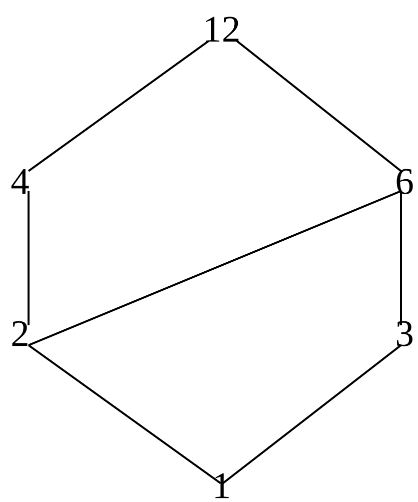
PROBLEMS

1.Let

$P_1 = \{2,3,6,12,24\}$

$P_2 = \{1,2,3,4,6,12\}$ and $\leq$ be a relation such that $x \leq y$ if and only if $x|y$.

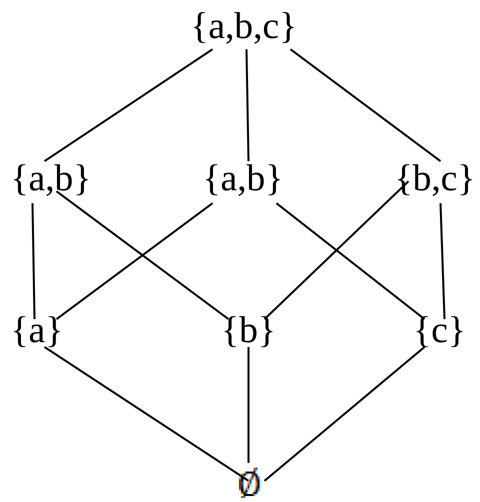

24
12
6
2
3


12
4
6
2
3
1

2.Let

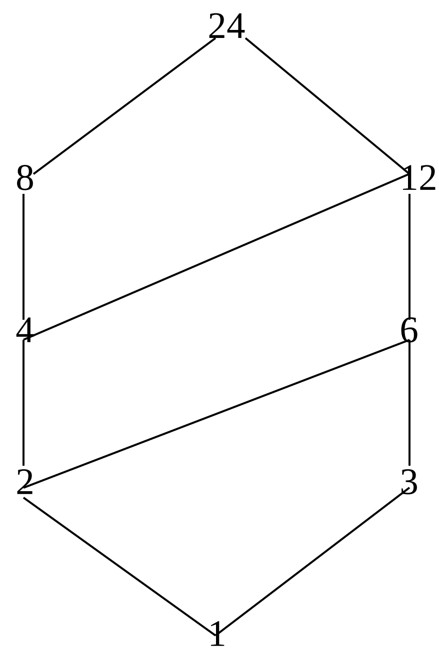
$\rho(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\},\}$ be the power set of $\{a,b,c\}$.

Consider the inclusion $(\subseteq)$ relation as the partial ordering on $\rho(A)$, then the Hasse diagram of $\langle \rho(A), \subseteq \rangle$ is



3.Let us consider the set of all divisor of 24, then it is a poset which is denoted by D_{24}

That is $D_{24} = \{1,2,3,4,6,8,12,24\}$ and let the divisor relation be partial ordering.

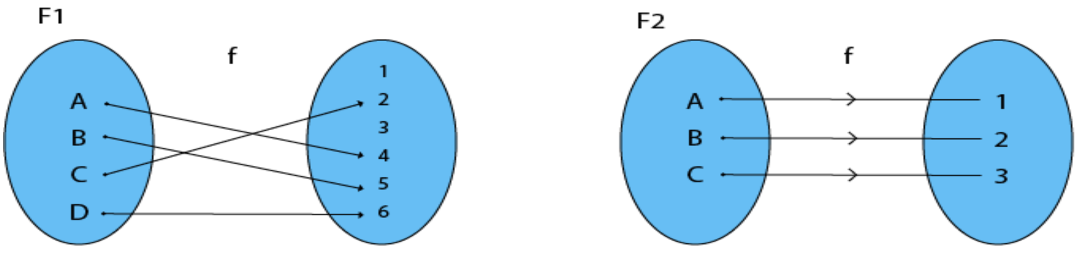

24
8
12
4
6
2
3
1

FUNCTIONS

A function in set theory world is simply a mapping of some (or all) elements from Set A to some (or all) elements in Set B. In the example above, the collection of all the possible elements in A is known as the **domain**; while the elements in A that act as inputs are specially named **arguments**. On the right, the collection of all possible outputs (also known as “range” in other branches), is referred to as the **codomain**; while the collection of actual output elements in B mapped from A is known as the **image.**

Types of Functions

1. Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

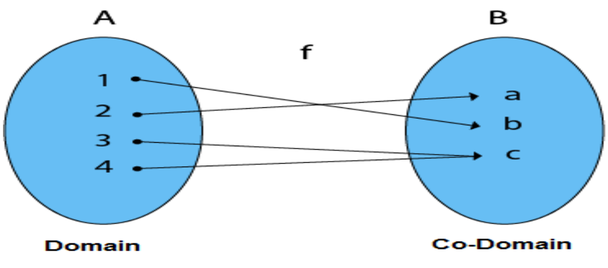


F1 and F2 show one to one Function

2. Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.

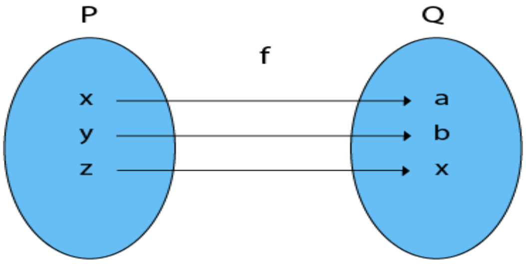
Example: Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.

It is a Surjective Function, as every element of B is the image of some A



Note: In an Onto Function, Range is equal to Co-Domain.

3. Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



Example:

1. Consider P = {x, y, z}
2. Q = {a, b, c}
3. and f: P ⟶ Q such that
4. f = {(x, a), (y, b), (z, c)}

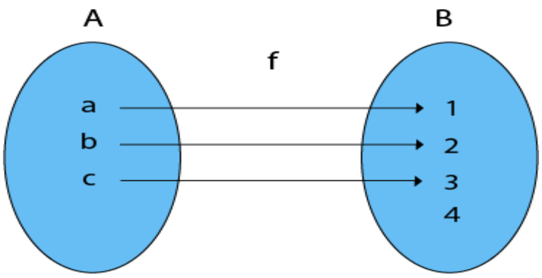
The f is a one-to-one function and also it is onto. So it is a bijective function.

4. Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.

Example:

1. Consider, A = {a, b, c}
2. B = {1, 2, 3, 4} and f: A ⟶ B such that
3. f = {(a, 1), (b, 2), (c, 3)}
4. In the function f, the range i.e., {1, 2, 3} ≠ co-domain of Y i.e., {1, 2, 3, 4}

Therefore, it is an into function



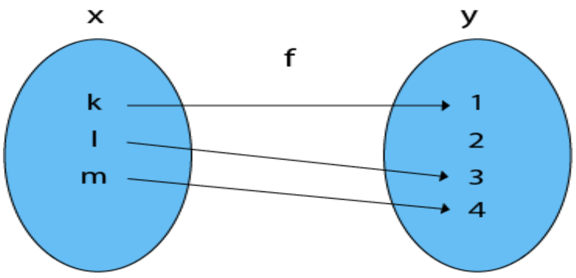
5. One-One Into Functions: Let f: X ⟶ Y. The function f is called one-one into function if different elements of X have different unique images of Y.

Example:

1. Consider, X = {k, l, m}
2. Y = {1, 2, 3, 4} and f: X ⟶ Y such that

3. $f = \{(k, 1), (l, 3), (m, 4)\}$

The function f is a one-one into function

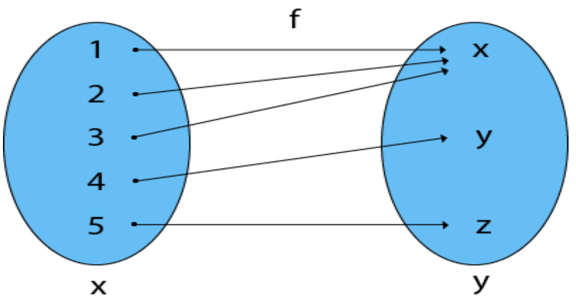


6. Many-One Functions: Let $f: X \rightarrow Y$. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.

Example:

1. Consider $X = \{1, 2, 3, 4, 5\}$
2. $Y = \{x, y, z\}$ and $f: X \rightarrow Y$ such that
3. $f = \{(1, x), (2, x), (3, x), (4, y), (5, z)\}$

The function f is a many-one function



Example 1: Test whether the function $f: R \rightarrow R$, $f(x) = |x| + x$ is one-one onto function

<u>Solution:</u>

(1) Given $f(x) = |x| + x$

$f(3) = |3|+3 = 6$

$f(-3) = |-3|+(-3) = 0$

$f(2) = |2|+2= 4$

$f(-2) = |-2|+(-2) = 0$

$f(-3) = f(-2) = 0$

0 has more than one pre-image. Thus f(x) is not 1-1 function

(2) The range of f is the set of non-negative real numbers.

$\therefore$ f is not onto function

Example 2: Let $S = \{x, x^2 / x \in N\}$ and $T = \{(x,2x)/x \in N\}$ where $N = \{1,2....\}$. Find the range of S and T. Find $S \cup T$ and $S \cap T$

Solution:

$S = \{x, x^2 / x \in N\}$

$S = \{(1,1), (2,4), (3,9), (4,16), \ldots\ldots\ldots\}$

$T = \{(x,2x) / x \in N\}$

$S = \{(1,2), (2,4), (3,6), (4,8), \ldots\ldots\ldots\}$

Range of $S = \{1, 4, 9, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$

Range of $T = \{1, 4, 6, 8, \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\}$

$S \cup T = \{(1,1), (2,4), (3,9), (4,16), (1,2), (3,6), (4,8), \ldots\ldots\ldots\}$

$S \cap T = \{(2,4)\}$

Example 3: If $f: R \rightarrow R$, $g: R \rightarrow R$ are defined by $f(x) = x^2-2$, $g(x) = x+4$, find (fog) and (gof) and check whether these functions are injective, surjective and bijective

Solution:

$fog(x) = f[g(x)] = f(x+4) = (x+4)^2-2 = x^2+8x+14$-----------------(1)

$gof(x) = g[f(x)] = g(x^2-2) = x^2+2$---------------------------------(2)

Given $f: R \rightarrow R$ $g: R \rightarrow R$

$f(x) = x^2-2$

(1) $f(1) = 1^1-2 = -1$

$f(-1) = (-1)^2-2 = -1$

i.e., $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$

Hence f is not 1-1 function

(2) Let $f: R \rightarrow R$

Let $y \in R$. Suppose $x \in R$ such that $f(x) = y$

$x^2-2 = y$

$x^2 = y+2$

$x = \sqrt{y+2}$

$f(\sqrt{y+2}) = (\sqrt{y+2})^2-2 = y+2-2 = y$

for any $y \in R$ There exist at least one element $\sqrt{y+2} \in R$ such that $f(\sqrt{y+2}) = y$

$\therefore$ f is on to function

$g(x) = x+4$

(1) $g(x_1) = g(x_2)$

$x_1+4 = x_2+4$

$x_1 = x_2$

g is 1-1 function

(2) g: R→R

Let $y \in R$. Suppose $x \in R$ such that $f(x) = y$

$x = y-4$ for any $y \in R$

There exist at least one element $y-4 \in R$ such that

$g(y-4) = y$

$\therefore$ g is on to function

As f is not 1-1 but onto, f is not bijective

As g is 1-1 and onto, g is bijective

Theorem 1 : A function f:A→B has an inverse if and only if it is bijective.

Proof.
Suppose g is an inverse for f (we are proving the implication ⇒). Since $g \circ f = I_A$, $g \circ f = I_A$ is injective, so is f . Since $f \circ g = i_B$, $f \circ g = i_B$ is surjective, so is f. Therefore f is injective and surjective, that is, bijective.

Conversely, suppose f is bijective. Let g:B→A, g:B→A be a pseudo-inverse to f. since ff is surjective, $f \circ g = i_{B,}$ $f \circ g = i_B$, and since f is injective, $g \circ f = i_{A,}$ $g \circ f = i_A$.

Theorem 2: Let A and B be nonempty sets, and suppose f : A → B is invertible. Then f −1 : B → A is also invertible, and $(f^{-1})^{-1} = f$.

Proof. f^{-1} is invertible if there is a function g : A → B that satisfies $g \circ f^{-1} = I_B$ and $f^{-1} \circ g = I_A$; and in that case the function g is the unique inverse of f^{-1} . Since g = f is such a function, it follows that f^{-1} is invertible and f is its inverse.

Theorem 3: If f:A→B has an inverse function then the inverse is unique.

Proof.
Suppose g_1 and g_2 are both inverses to f. Then

$$g_1=g_1 \circ I_B=g_1 \circ (f \circ g_2)=(g_1 \circ f) \circ g_2=I_A \circ g_2=g_2, g_1=g_1 \circ I_B=g_1 \circ (f \circ g_2)=(g_1 \circ f) \circ g_2=I_A \circ g_2=g_2,$$

proving the theorem

Theorem 4 : If f : A → B and g : B → C are one-one, then gof : A → C is also one-one.

Proof:
A function f : A → B is defined to be one-one, if the images of distinct elements

of A under f are distinct, i.e. for every $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Given that f: $A \rightarrow B$ and g: $B \rightarrow C$ are one-one.

For any $x_1, x_2 \in A$
$f(x_1)=f(x_2) \Rightarrow x_1=x_2$ …(i)
$g(x_1)=g(x_2) \Rightarrow x_1=x_2$ …(ii)
To show: If $gof(x_1) = gof(x_2)$, then $x_1 = x_2$
Let $gof(x_1) = gof(x_2)$
$\Rightarrow g[f(x_1)] = g[f(x_2)]$
$\Rightarrow f(x_1) = f(x_2)$ …from (i)
$\Rightarrow x_1 = x_2$ …from (ii)
Hence, the functions gof: $A \rightarrow C$ are one-one.

Theorem 5: If $f : A \rightarrow B$ and $g : B \rightarrow C$ are onto, then $gof : A \rightarrow C$ is also onto.

Proof:
Let us consider an arbitrary element $z \in C$
'.' g is onto ∃ a pre-image y of z under the function g such that g (y) = z
………(i)
Also, f is onto, and hence, for y Î B, there exists an element $x \in A$ such that f (x) = y ……(ii)
Therefore, gof (x) = g (f (x)) = g (y) from (ii)
= z from (i)
Thus, corresponding to any element $z \in C$, there exists an element $x \in A$ such that gof (x) = z.
Hence, gof is onto.
Note: In general, if gof is one-one, then f is one-one. Similarly, if gof is onto, then g is onto.
The composition of functions can be considered for n number of functions.

Theorem 6: If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow S$ are functions, then ho(gof) = (hog) o f.

Proof: Let $x \in A$
LHS: ho(gof) (x)
= h(gof(x))
= h(g(f(x))), $\forall$ x in X
RHS: (hog) of f(x)
= hog(f(x))
= h(g(f (x))), $\forall$ x in X.
LHS = RHS
Hence, ho(gof) = (hog)of.

The composition of functions satisfies the associative property.

Theorem 7: Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$

Proof:

Given that $f: A \to B$ and $g: B \to C$ are bijective.
Then $gof : A \to C$ is also bijective. Therefore $(gof)^{-1} : C \to A$ exists.

Also $f^{-1}: B \to A$ and $g^{-1}: C \to B$ exists. Therefore $f^{-1}og^{-1} : C \to A$ exists.

Also we know that

$f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$
$g \circ g^{-1} = I_c$ and $g^{-1} \circ g = I_B$

Consider

$$
\begin{aligned}
(f^{-1}og^{-1}) \text{ o } (gof) &= (f^{-1}o(g^{-1}og)of) \\
&= (f^{-1}o(I_B of)) \\
&= I_A
\end{aligned}
$$

Also

$$
\begin{aligned}
(gof) \text{ o} (f^{-1}og^{-1}) &= (go(f \text{ of}^{-1})og^{-1}) \\
&= (go(I_B og^{-1})) \\
&= I_A
\end{aligned}
$$

Hence, $(gof)^{-1} = f^{-1}og^{-1}$.