UNIT – I

DIFFERENTIAL EQUATIONS

INTRODUCTION

During the past three decades the development of non-linear analysis, dynamical systems and their applications to Science and Engineering has stimulated renewed enthusiasm for the theory of Ordinary Differential Equations (ODE).

An Ordinary Differential Equation is an equation containing a function of one independent variable and its derivatives.Differential equations have wide application in various engineering and science discipline. In general, modeling variations of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain or concentration of pollutant with the cahnge of time 't' or location, or both would require differential equation . The study of differential equation began in 1675, when Gottfried Willhelm Von Leibnig (1646 - 1716) wrote the equation.

$$\int x\, dt = \frac{x^2}{2}$$

The search for general methods of integrading differential equations began when Isaac Newton (1646 - 1727) classified first order Differential equations

(i) $\dfrac{dy}{dx} = f(x)$

(ii) $\dfrac{dy}{dx} = f(x, y)$

(iii) $x\dfrac{\partial y}{\partial x} + y\dfrac{\partial y}{\partial x} = u$

The first two classes contain only ordinary derivatives of one dependent variable, with respect to single independent variable, and are known today as Differential Equation and third is partial differential equation.

A simple example is Newton's second law of motion, the relationship between the displacement 'x' and time 't' of the object under the force F which leads to the differential equation is

$$m\left(\frac{d^2x(t)}{dt^2}\right) = F[x(t)] \qquad ...(1)$$

for the motion of a particle with constant mass 'm'. In general, F depends on the position x(t) on the particle at time 't' and so the unknown function x(t) appears on both sides of the differential equations as in the notation *F[x(t)].*

Linear Differential equations with Constant Coefficients

The general form on nth order linear differential equation with constant coefficient is

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + ... + a_{n-1}\frac{dy}{dx} + a_n y = X \qquad ...(2)$$

where $a_0 \neq 0, a_1, a_2, \ldots, a_n$ are constants and 'X' is a function of x.

put $\quad D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2} \cdots, D^n = \frac{d^n}{dx^n}$

Eqn (2) becomes

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_{n-1}D + a_n)\, y = X \qquad \ldots(3)$$

The general solution of (3) is

y = Complementary Function + Particular Integral

i.e., y = C.F. + P.I.

To find the Complementary Function

The Auxiliary Equation of (3) is obtained by putting $D = m$ and X = 0.

$\therefore$ The Auxiliary eqn is

$$a_0m^n + a_1m^{n-1} + \ldots + a_n = 0 \qquad \ldots(4)$$

Solving equation (4), we get 'n' roots for 'm'. Say $m_1, m_2, \ldots. m_n$.

Case (i): If all the roots m_1, m_2, &$m_n$are real and different, then the complementary function,

$$C.F. = Ae^{m_1x} + Be^{m_2x} + Ce^{m_3x} + ...$$

Case (ii): If any two roots are equal say $m_1 = m_2 = m$ (say), then the complementary function is given by

$$C.F. = (Ax + B)e^{mx} \ (or) \ (A + Bx)e^{mx}$$

Case (iii): If any three roots are equal say $m_1 = m_2 = m_3 = m$, then the complementary function is given by

$$C.F. = (Ax^2 + Bx + C)e^{mx} \ (or) \ (A + Bx + Cx^2)e^{mx}$$

Case (iv): If the roots are imaginary of the form $(\alpha \pm i\beta)$ then the complementary function is

$$C.F. = e^{\alpha x}[A\cos \beta x + B\sin \beta x]$$

To find particular Integral

When the R.H.S. of the given differential equation is zero, we need not find particular Integral. When R.H.S. of a given differential equation is a function of x say e^{ax}, *sin ax* or *cos ax*, x^n, $e^{ax} f(x)$, $x f(x)$, we have to find particular Integral.

Case (i): If $f(x) = e^{ax}$, then $P.I. = \frac{1}{F(D)} e^{ax}$ Replace D by a in $F(D)$, provided $F(D) \neq 0$.

If $F(a) = 0$, then $P.I. = \frac{x}{F'(D)} e^{ax}$, provided $F'(a) \neq 0$.

If $F'(a) = 0$, then $P.I. = \frac{x^2}{F''(D)} e^{ax}$, provided $F''(a) \neq 0$ and so on.

Case (ii): If $f(x) = \sin ax$ or $\cos ax$ then

$$P.I. = \frac{1}{F(D)} \sin ax \text{ or } \cos ax$$

Replace D^2 by $-a^2$ in $F(D)$, provided $F(D) \neq 0$.

If $F(D) = 0$, when we replace D^2 by $-a^2$ then

$$P.I. = \frac{x}{F'(D)} \sin ax \text{ (or) } \cos ax$$

Again replace D^2 by $-a^2$ in $F''(D)$ provided $F''(D) \neq 0$, then $P.I. = \frac{x^2}{F''(D)} \sin ax$ (or) $\cos ax$ and the process may be repeated if $F(D) = 0$ and so on.

Case (iii): If $f(x) = x^n$, then $P.I. = \frac{1}{F(D)} x^n$

$= [F(D)]^{-1} x^n$

Expand $= [F(D)]^{-1}$ by using Binomial theorem and then operate on x^n.

Case (iv): If $f(x) = e^{ax} X$, where X is $\sin ax$ (or) $\cos ax$, then

$$P.I. = \frac{1}{F(D)} e^{ax} X = e^{ax} \frac{1}{F(D+a)} X$$

Here $\frac{1}{F(D+a)} X$ can be evaluated by using anyone of the first three types.

Case (v): If $f(x) = x^n \sin ax$ or $x^n \cos ax$, then

$$P.I. = \frac{1}{F(D)} x^n \sin ax \ \ (or)\ x^n \cos ax$$

Now, $\frac{1}{F(D)} x^n (\cos ax + i \sin ax)$

$$= \frac{1}{F(D)} x^n e^{iax}$$

$$= e^{iax} \frac{1}{F(D+ia)} x^n$$

$$\therefore \frac{1}{F(D)} x^n \sin ax = \text{Imaginary part of } \ e^{iax} \frac{1}{F(D+ia)} x^n$$

$$\frac{1}{F(D)} x^n \cos ax = \text{Real part of } \ e^{iax} \frac{1}{F(D+ia)} x^n$$

Example

1. Find the complementary function of $\frac{d^2y}{d^2x} + 2\frac{dy}{dx} + y = 0$

Solution:The given equation can be written as $(D^2 + 2D + 1)\, y = 0$

The Auxiliary equation is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = -1, -1$

$C.F. = (A + Bx)\, e^{-x}$

2. Solve the equation $(D^2 + 3D + 2)\, y = 0$

Solution:The Auxiliary equation is $m^2 + 3m + 2 = 0$

$(m + 1)(m + 2) = 0$

$m = -2, -1$

$y = Ae^{-2x} + Be^{-x}$

3. Find the complementary function of $(D^2 + 1)\, y = 0$

Solution:The A.E. is $m^2 + 1 = 0 \Rightarrow m^2 = -1$

$m = \pm i$

$y = e^{0x}(A \cos x + B \sin x)$

$\therefore y = A \cos x + B \sin x$

4. Solve $y'' + 4y' + 20y = 0$

Solution:The A.E. is $m^2 + 4m + 20 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 80}}{2}$$

$$= \frac{-4 \pm \sqrt{-64}}{2} = \frac{-4 \pm 8i}{2}$$

$= -2 \pm 4i$

$\therefore y = e^{-2x} (A \cos 4x + B \sin 4x)$

5. Solve the equation $(D^3 - 3D^2 + 4D - 2)\, y = 0$

Solution:The A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$(m - 1)(m^2 - 2m + 2) = 0$

$m = 1$ or $m = 1 \pm i$

The solution is $y = Ae^x + e^x (B \cos x + C \sin x)$

6. Find the complementary function of $(D^3 + 2D^2 + D)\, y = 0$

Solution:The A.E. is $m^3 + 2m^2 + m = 0$

$m (m^2 + 2m + 1) = 0$

$m = 0, (m + 1)^2 = 0$

$m = 0, -1, -1$

$C.F = A + (Bx + C)\, e^{-x}$

7. Solve the equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Solution:The given equation can be written as

$(D^2 - 5D + 6)\, y = 0$

The auxiliary equation is $m^2 - 5m = 6 = 0$

$(m - 3)(m - 2) = 0$

$m = 2, 3$

$y = Ae^{2x} + Be^{3x}$

8. Solve the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution:The given equation can be written as

$(D^2 + D)\, y = 0$

The auxiliary equation is $m^2 + m = 0$

$m(m + 1) = 0$

$m = 0, -1$

$y = A + Be^{-x}$

9. Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

Solution:The given equation can be written as

$(D^3 + 2D^2 - D - 2)\, y = 0$

The auxiliary equation is $m^3 + 2m^2 - m - 2 = 0$

$m^2 (m + 2) - 1(m + 2) = 0$

$(m^2 - 1)\, (m + 2) = 0$

$m^2 = 1$ or $m = -2$

$m = -1, 1, -2$

$y = Ae^{-2x} + Be^{-x} + Ce^{x}$

10. Find the complementary function of $(D^2 - 2D + 2)\, y = 0$

Solution:The auxiliary equation is $m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$C.F. = e^{x} (A \cos x + \sin x)$

11. Solve $(D^2 - 4D + 13)\, y = 0$

Solution:The auxiliary equation is $m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$y = e^{2x}(A \cos 3x + B \sin 3x)$$

12. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Solution:The auxiliary equation is $m^4 + 8m^2 + 16 = 0$

$$(m^2 + 4)^2 = 0$$

$$m^2 + 4 = 0, \; m^2 + 4 = 0$$

$$m = \pm 2i \qquad m = \pm 2i$$

$$\therefore y = (A_1 + A_2x)\cos 2x + (A_3 + A_4x)\sin 2x$$

13. Solve $(D^2 - 9)\, y = 0$

Solution:The auxiliary equation is $m^2 - 9 = 0$

$$\therefore m^2 = 9$$

$$m = \pm 3$$

$$y = Ae^{-3x} + Be^{3x}$$

14. Solve $(D^2 - 5D + 7)\, y = 0$

Solution:The A.E. is $m^2 - 5m + 7 = 0$

$$m = \frac{5 \pm \sqrt{25-28}}{2}$$

$$= \frac{5 \pm \sqrt{-3}}{2} = \frac{5}{2} \pm \frac{i\sqrt{3}}{2}$$

$$y = e^{\frac{5}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right)$$

15. Solve $\dfrac{d^2y}{dx^2} + a^2y = 0$

Solution:The given equation can be written a

$(D^2 + a^2)\,y = 0$

The A.E. is $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ai$

$\therefore y = A\cos ax + B\sin ax.$

Exercises: (Part A)

(1) Solve $(D^2 + 2D - 15)\,y = 0$

(2) $(2D^2 + 7D + 5)\,y = 0$

(3) Solve $\dfrac{d^2y}{dx^2} - \dfrac{2dy}{dx} + 5y = 0$

(4) $(D^2 + 6D + 9)\,y = 0$

(5) $(D^2 + 8D + 16)\,y = 0$

(6) $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 25y = 0$

(7) $(D^2 + D + 1)\,y = 0$

(8) $\dfrac{d^2y}{dx^2} - \dfrac{2dy}{dx} + 3y = 0$

(9) $(D^2 + 16)\,y = 0$

(10) $\dfrac{d^2y}{dx^2} + y = 0$

(11) $$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$$

(12) $(D^2 + 8D + 16)\, y = 0$

(13) $(D^2 - 7D + 12)\, y = 0$

(14) $(D^3 - 6D^2 + 11D - 6)\, y = 0$

(15) $(D^2 + 7)\, y = 0$

Answers

(1) $y = Ae^{3x} + 13e^{-5x}$

(2) $y = Ae^{-x} + Be^{\frac{-5x}{2}}$

(3) $y = (Ax + B)e^{-4x}$

(4) $y = (Ax + B)e^{-3x}$

(5) $y = (Ax + B)e^{-4x}$

(6) $y = (Ax + B)e^{5x}$

(7) $$y = e^{-\frac{x}{2}}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$$

(8) $y = e^{-x}\left[A\cos\sqrt{2}x + B\sin\sqrt{2}x\right]$

(9) $y = A\cos 4x + B\sin 4x$

(10) $y = A\cos x + B\sin x$

(11) $y = Ae^{2x} + B\cos 2x + C\sin 2x$

(12) $y = (Ax + B)e^{-4x}$

(13) $y = Ae^{3x} + Be^{4x}$

(14) $y = Ae^{x} + Be^{2x} + Ce^{3x}$

(15) $y = A\cos\sqrt{7}x + B\sin\sqrt{7}x$

Problem based on P.I. = $\frac{1}{f(D)}e^{ax}$

1. Find the particulars integral of $(D - 1)\, y = e^x$ [s.u. May '07]

Solution: Particular Integral $= \frac{1}{D-1}e^x$

$= \frac{1}{1-1}e^x$

(replacing D by 1)

$= \frac{e^x}{0} = \frac{xe^x}{1} (\because Dr\ is\ 0)$

$= xe^x$

2. Find the particular integral of $(D^2 - 4D + 13)$ y = e^{2x}

Solution: Particular Integral $= \frac{1}{D^2-4D+13}e^{2x}$

$= \frac{1}{2^2-4x^2+3}e^{2x}$

(replacing D by 2)

$= \frac{1}{4-8+13}e^{2x}$

$= \frac{1}{9}e^{2x}$

3. Find the particular Integral of $(D^2 - 2D + 1)\, y = e^x$ [s.u. May '10]

Solution: Particular Integral $= \frac{1}{D^2-2D+1}e^x$

$= \frac{1}{1^2-2\times1+1}e^x$

(replacing D by 1)

$$= \frac{1}{0} e^x$$

$$= \frac{xe^x}{2D-1} (\because Dr\ is\ 0)$$

$$= \frac{xe^x}{2-2} \text{(replacing D by 1)}$$

$$= \frac{xe^x}{0}$$

$$= \frac{x^2 e^x}{2} (\because Dr\ is\ 0)$$

4. Find the particular Integral of $(D^2 – 4D + 4)\ y = \cos h2x$ [s.u. Dec ‘07]

Solution: Particular Integral $= \frac{1}{(D^2 - 4D + 1)} \cos h2x$

$$= \frac{1}{(D^2 - 4D + 1)} \left(\frac{e^{2x} + e^{-2x}}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 4D + 4} e^{2x} + \frac{1}{D^2 - 4D + 4} e^{-2x} \right]$$

$$= \frac{1}{2} \left[\frac{1}{(2)^2 - 4 \times 2 + 4} e^{2x} + \frac{1}{(-2)^2 - 4 \times (-2) + 4} e^{-2x} \right]$$

$$= \frac{1}{2} \left[\frac{1}{0} e^{2x} + \frac{1}{16} e^{-2x} \right]$$

$$= \frac{1}{2} \left[\frac{xe^{2x}}{2D - 4} + \frac{e^{-2x}}{16} \right]$$

$$= \frac{1}{2} \left[\frac{xe^{2x}}{2 \times 2 - 4} + \frac{e^{-2x}}{16} \right]$$

$$= \frac{1}{2} \left[\frac{x^2 e^{2x}}{2} + \frac{e^{-2x}}{16} \right]$$

5. Solve $(D-2)^2 y = e^{2x}$ where $D = \dfrac{d}{dx}$ [s.u. Dec ‘06]

Solution: Auxiliary Equation: $(m-2)^2 = 0$

$m = 2, 2$

Complimentary Function $(C.F) = (Ax + B)\, e^{2x}$

Particular Integral $= \dfrac{1}{(D-2)^2} e^{2x}$

$= \dfrac{1}{(2-2)^2} e^{2x}$ (replacing D by 2)

$= \dfrac{x}{(2D-4)} e^{2x}$ (Since Dr is 0)

$= \dfrac{xe^{2x}}{2\times 2-4}$

$= \dfrac{xe^{2x}}{0}$

$= \dfrac{x^2 e^{2x}}{2}$

Complete Solution y = Complimentary function + Particular Integral

$$y = (Ax+B)e^{2x} + \frac{x^2 e^{2x}}{2}$$

6. Solve $(D^2 - 1)\, y = e^x$ [s.u Dec ’11]

Solution: Auxiliary Equation $m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

Complimentary Function $= Ae^{x} + Be^{-x}$

Particular Integral $= \frac{1}{(D^2-1)}e^x$

$= \frac{1}{1^2-1}e^x$ (Replace D by 1)

$= \frac{e^x}{0}$

$= \frac{xe^x}{2D}$ $(\because Dr\ is\ Zero)$

$= \frac{xe^x}{2\times 1} = \frac{xe^x}{2}$

Complete Solution: $y = Ae^x + Be^{-x} + \frac{xe^x}{2}$

7. Find the particular integral of $(D^2 + D - 6)\, y = e^{2x}$ [s.u. Dec '12]

Solution: Particular Integral $= \frac{1}{D^2+D-6}e^{2x}$

$= \frac{1}{2^2+2-6}e^{2x}$

$= \frac{1}{0}e^{2x}$

$= \frac{xe^{2x}}{2D+1}$

$= \frac{xe^{2x}}{2\times 2+1}$

$= \frac{xe^{2x}}{5}$

8. Solve $(D^2 - 3D + 2)\, y = e^{4x}$ where $D = \frac{d}{dx}$ [s.u Dec'09]

Solution: AuxiliaryEquation $m^2 - 3m + 2 = 0$

$(m-1)(m-2)=0$

$m = 1, 2$

Complimentary Function = $Ae^x + Be^{2x}$

Particular Integral $= \dfrac{1}{(D^2-3D+2)}e^{4x}$

$= \dfrac{1}{(4^2-3\times 4+2)}e^{4x}$

(replacing D by 4)

$= \dfrac{1}{6}e^{4x}$

Complete Solution: $y = Ae^x + Be^{2x} + \dfrac{e^{4x}}{6}$

9. Solve $(4D^2 - 4D + 1)\, y = 4$

Solution: Auxiliary Equation $4m^{2+} - 4m + 1 = 0$

$4m^2 - 2m - 2m + 1 = 0$

$2m(2m-1) - 1(2m-1) = 0$

$(2m-1)^2 = 0$

$m = \dfrac{1}{2}, \dfrac{1}{2}$

Complimentary Function = $(Ax+B)e^{\frac{1}{2}x}$

Particular Integral $= \dfrac{1}{4D^2-4D+1}4e^{0x}$

$= \dfrac{1}{4\times 0^2 - 4\times 0+1}4e^{0x}$

$= \dfrac{4e^{0x}}{1} = 4$

Complete Solution: $y = (Ax + B)e^{\frac{x}{2}} + 4$

10. Solve $(D^3 - 3D^2 + 4D - 2)\, y = e^x$

Solution: Auxiliary Equation is $m^3 - 3m^2 + 4m - 2 = 0$.

$m = 1$ satisfies the equation

$\therefore m - 1$ is a factor of this equation

To find the other roots, divide the given equation by $m - 1$

$$\begin{array}{r|rrrr} & 1 & -3 & 4 & -2 \\ 1 & & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm \sqrt{4i^2}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore$ The roots of the equation are $m = 1, 1 + i, 1 - i$.

Complimentary Function $= Ae^x + e^x(A \cos x + B \sin x)$

Particular Integral: $$\frac{1}{(D^3 - 3D^2 + 4D - 2)} e^x = \frac{1}{1^3 - 3 \times 1^2 + 4 \times 1 - 2} e^x$$

$$= \frac{e^x}{0}$$

$$= \frac{xe^x}{(3D^2 - 6D + 4)} \text{ (Since the denominator is 0)}$$

$$= \frac{xe^x}{(3 \times 1^2 - 6 \times 1 + 4)} = xe^x$$

Complete Solution: $y = Ae^x + e^x (A \cos x + B \sin x) + xe^x$

11. $(D^2 - 3D + 2)\, y = e^{4x} (\sin hx)$

Solution: Auxiliary Equation $\Rightarrow m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

Complimentary Function = $Ae^x + Be^{2x}$

Particular Integral $= \dfrac{1}{(D^2 - 3D + 2)} e^{4x} (\sin\ hx)$

$$= \frac{1}{(D^2 - 3D + 2)} e^{4x} \left(\frac{e^x - e^{-x}}{2} \right)$$

$$= \frac{1}{(D^2 - 3D + 2)} \left(\frac{e^{5x} - e^{3x}}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{(D^2 - 3D + 2)} e^{5x} - \frac{1}{(D^2 - 3D + 2)} e^{3x} \right]$$

$$= \frac{1}{2} \left[\frac{1}{(5^2 - 3 \times 5 + 2)} e^{5x} - \frac{1}{(3^2 - 3 \times 3 + 2)} e^{3x} \right]$$

$$= \frac{1}{2} \left[\frac{e^{5x}}{12} - \frac{e^{3x}}{2} \right]$$

$$= \frac{e^{5x} - 6e^{3x}}{24}$$

Complete Solution: $y = Ae^x + Be^{2x} + \dfrac{e^{5x} - 6e^{3x}}{24}$

Problems based on $P.I. = \dfrac{1}{f(D)} x^n$

Following formulae are important

$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

12. Find the particular integral of $(D^2 - 9D + 20)\, y = 20x$ [s.u Dec '10]

Solution: Particular Integral $= \dfrac{1}{(D^2 - 9D + 20)} 20x$

$$= \frac{20}{20\left(1+\dfrac{D^2-9D}{20}\right)} x$$

$$= \left(1+\frac{D^2-9D}{20}\right)^{-1} x$$

$$= \left(1-\left(\frac{D^2-9D}{20}\right)+\dots\right)x$$

$$= x+\frac{9}{20}D(x)-D^2(x)\dots$$

$$= x+\frac{9}{20}$$

13. Solve $(D^2 - 1)\, y = x$

Solution: Auxiliary Equation $\Rightarrow m^2 - 1 = 0$

$m^2 = 1$

$m^2 = \pm 1$

Complimentary Function $= Ae^{x} + Be^{-x}$

Particular Integral $= \dfrac{1}{D^2-1} x$

$$= \frac{-1}{1-D^2} x$$

$$= -(1-D^2)^{-1} x$$

$$= -(1+D^2+(D^2)^2\dots)x$$

$$= -(x + D^2(x) + D^4(x)...)$$

$$= -x$$

Complete Solution: $y = Ae^x + Be^{-x} - x$

14. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$

Solution: Auxiliary Equation $\Rightarrow m^2 - 5m + 6 = 0$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

Complimentary Function = $Ae^{3x} + Be^{2x}$

Particular Integral $= \frac{1}{D^2 - 5D + 6}(x^2 + 3)$

$$= \frac{1}{6\left(1 + \frac{D^2 - 5D}{6}\right)}(x^2 + 3)$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2 ...\right](x^2 + 3)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{36} + \frac{25D^2}{36} - \frac{10D^3}{36}...\right](x^2 + 3)$$

$$= \frac{1}{6}\left[(x^2 + 3) - \frac{1}{6}D^2(x^2 + 3) + \frac{5}{6}D(x^2 + 3) + 0 + \frac{25}{36}D^2(x^2 + 3) - 0...\right]$$

$$= \frac{1}{6}\left[(x^2 + 3) - \frac{1}{6}(2) + \frac{5}{6}(2x) + \frac{25}{36}(2)\right]$$

$$= \frac{1}{6}\left[x^2 + 3 - \frac{1}{3} + \frac{5}{6}x + \frac{25}{36}\right]$$

$$= \frac{1}{108}\left[18x^2 + 30x + 73\right]$$

Complete solution: $y = Ae^{3x} + Be^{2x} + \frac{1}{108}[18x^2 + 30x + 73]$

15. Solve $(D^4 - 2D^3 + D^2)\, y = x^3$

Solution: Auxiliary Equation $\Rightarrow m^4 - 2m^3 + m^2 = 0$

$$m^2 (m^2 - 2m + 1) = 0$$

$$m^2 (m - 1)^2 = 0$$

$$m = 0, 0, 1, 1$$

Complimentary Function = $(Ax + B)\, e^{0x} + (Cx + D)\, e^x$

Particular Integral $= \frac{1}{D^4 - 2D^3 + D^2} x^3$

$$= \frac{1}{D^2(D^2 - 2D + 1)} x^3$$

$$= \frac{1}{D^2}[1 + D^2 - 2D]^{-1} x^3$$

$$= \frac{1}{D^2}[1 - (D^2 - 2D) + (D^2 - 2D)^2 - (D^2 - 2D)^3 \ldots]x^3$$

$$= \frac{1}{D^2}[1 + 2D + 3D^2 + 4D^3]x^3$$

(omitting $D^4$and higher powers]

$$= \frac{1}{D^2}[x^3 + 2D(x^3) + 3D^2(x^3) + 4D^3(x^3)]$$

$$= \frac{1}{D^2}[x^3 + 6x^2 + 18x + 24]$$

$$= \frac{1}{D}\left[\frac{x^4}{4} + 6\frac{x^3}{3} + \frac{18x^2}{2} + 24x\right]$$

$$= \frac{x^5}{4\times 5} + \frac{6x^4}{3\times 4} + \frac{18x^3}{2\times 3} + \frac{24x^2}{2}$$

$$= \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x$$

Complete Solution: $y = (A + Bx) + (C + Dx)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$

16. Solve (D^3 + 8) = x^4 + 2x + 1

Solution: Auxiliary Equation $\Rightarrow$ m^3 + 8 = 0

m = –2 satisfies the equation

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 8 \\ -2 & 0 & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

(m+2) (m^2 – 2m + 4) = 0

m = –2 or m = $\dfrac{2 \pm \sqrt{4-16}}{2}$

m = –2 or m = $\dfrac{2 \pm \sqrt{12i}}{2}$

m = –2 m = $\dfrac{2 \pm i2\sqrt{3}}{2}$

m = –2 m = $1 \pm i\sqrt{3}$

Complimentary Function = $Ae^{-2x} + e^x \left\lfloor B\cos\sqrt{3}x + \sin\sqrt{3}x \right\rfloor$

Particular Integral $= \dfrac{1}{(D^3+8)}(x^4 + 2x + 1)$

$$= \frac{1}{8\left(\frac{1+D^3}{8}\right)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+2x+1)$$

$$= \frac{1}{8}\left(1-\frac{D^3}{8}+...\right)(x^4+2x+1)$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{1}{8}D^3\ (x^4+2x+1)\right]$$

$$= \frac{1}{8}\left[x^4+2x+1-3x\right]$$

$$= \frac{1}{8}\left[x^4-x+1\right]$$

Complete Solution: $y = Ae^{-2x} + e^{x}(B\cos\sqrt{3}x + \sin\sqrt{3}x) + \frac{1}{8}(x^4 - x + 1)$

Exercises

Solve:

1. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = e^x$

2. $(D^2+6D+9)y = 5e^{3x}$

3. $(D^2+6)y = 6e^{3x}$

4. $(D^2+2D+2)y = \sin\ hx$

5. $(D^2-4D+4)y = e^{2x}$

6. $(D^3-3D+4D-2)y = e^x$

7. $(D^3+3D^2+3D+1)y = e^{-x}$

8. $(D^2-4)y = x^3$

9. $(D^2+5D+4)y = x^2+7x+9$

10. $(D^2+D-6)y=x$

11. $(D^2-3D+2)y=2x^2+1$

12. $(D^3-1)y=x$

13. $(D^3-13D+12)y=x$

14. $(D^3-2D+D)y=x^2+x$

15. $(D^3-3D^2-6D+8)y=x$

16. $(D^4-2D^3+D^2)y=x^2+e^x$

17. $(D^3+8)y=x^4+2x+1+\cos h2x$

18. $(D^4-4)y=x\sin\, h2x$

19. Find the particular integral of $(D-1)^3y = 2\cos hx$

20. Find the particular integral of $(D^2 + \mathrm{a}^2)\, y = b\cos ax + c\sin ax$

Answer

1. $\left[Ans: y=C_1e^{-x}+C_2e^{-5x}+\frac{1}{21}e^{2x}\right]$

2. $\left[Ans: y=(C_1+C_2x)e^{-3x}+\frac{5e^{3x}}{36}\right]$

3. $\left[Ans: y=C_1\cos 3x+C_2\sin 3x+\frac{e^{3x}}{3}\right]$

4. $\left[Ans: y=e^{-x}(C_1\cos x+C_2\sin\, x)+\frac{e^x}{10}-\frac{e^{-x}}{2}\right]$

5. $\left[Ans: y=(C_1\,\_+C_2x)e^{2x}+\frac{x^2}{2}e^{2x}\right]$

6. $\left[Ans: y=C_1e^x+e^x(c_2\cos x+c_3\sin\, x)+xe^x\right]$

7. $\left[Ans: y = (C_1 + C_2 x + C_3 x^2)e^{-x} + \frac{x^3}{6} e^{-x} \right]$

8. $\left[Ans: y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{8} x(2x^2 + 3) \right]$

9. $\left[Ans: y = C_1 e^{-x} + C_2 e^{-4x} + \frac{1}{4}\left(x^2 + \frac{9}{2} x + \frac{23}{8} \right) \right]$

10. $\left[Ans: y = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{36}(6x + 1) \right]$

11. $\left[Ans: y = C_1 e^{x} + C_2 e^{2x} + x^2 + 3x + 4 \right]$

12. $\left[Ans: y = C_1 + C_2 e^{x} + C_3 e^{-x} \frac{1}{2} x^2 \right]$

13. $\left[Ans: y = C_1 e^{x} + C_2 e^{-4x} + C_3 e^{3x} \frac{1}{144}(12x + 13) \right]$

14. $\left[Ans: y = C_1 + (C_2 = C_3 x)e^{-x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x \right]$

15. $\left[Ans: y = C_1 e^{x} + C_2 e^{-2x} + C_3 e^{4x} + \frac{1}{8}\left(x + \frac{3}{4} \right) \right]$

16. $\left[Ans: y = (C_1 x + C_2) + (C_3 x + C_4)e^{x} + \frac{x^4}{12} + \frac{2x^3}{3} + 3x^2 + \frac{x^2}{2} e^{x} \right]$

17. $\left[Ans: y = Ae^{-2x} + e^{x}\left(B\cos\sqrt{3}x + C\sin\sqrt{3}x \right) + \frac{1}{8}\left(x^4 - x + 1 \right) \frac{+1}{96}\left(3e^{2x} + 4xe^{-2x} \right) \right]$

18. $\left[Ans: y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x}{3} \sin hx + \frac{2}{9} \cos hx \right]$

19. $\left[Ans: y = \frac{x^3}{6} e^{x} - \frac{1}{8} e^{-x} \right]$

20. $\left[Ans: y = \frac{x}{2a}(b\sin ax - c\cos ax) \right]$

1. Solve $(D^2 - 4D + 3)y = \sin 3x$

Auxiliary Equation is $m^2 - 4m = 0$

$(m - 1)\ (m - 3) = 0$

$m = 1, 3$

Complementary function $(C.F) = Ae^{x} + Be^{3x}$

P.I. $= \frac{1}{D^2 - 4D + 3}\sin 3x$

$D^2 \rightarrow -a^2 = -9$

$= \frac{1}{-9 - 4D + 3}\sin 3x$

$= \frac{1}{-6 - 4D}\sin 3x$

$= \frac{1}{-6 - 4D} \times \frac{-6 + 4D}{-6 + 4D}\sin 3x$

$= \frac{(-6 + 4D)}{36 - 16D^2}\sin 3x$

$D^2 \rightarrow -a^2 = -9$

$= \frac{(-6 + 4D)}{36 - 16(-9)}\sin 3x$

$= \frac{1}{36 + 44}[-6\sin 3x + 4D(\sin 3x)]$

$= \frac{1}{180}[-6\sin 3x + 12\cos 3x]$

$= \frac{6}{180}[2\cos 3x - \sin 3x]$

P.I. $= \frac{1}{30}[2\cos 3x - \sin 3x]$

C.S = C.F + P.I

$y = Ae^{x} + Be^{3x} + \frac{1}{30}[2\cos 3x - \sin 3x]$

2. Solve $(D^2 - 2D + 1)y = \cos 2x + x^2$

Solution: Auxiliary Equation is $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = -1$ (twice)

C.F = $(Ax + B)\, e^{-x}$

P.I. $= \frac{1}{D^2 - 2D + 1}(\cos 2x + x^2)$

$= \frac{1}{D^2 + 2D + 1}\cos 2x + \frac{1}{D^2 + 2D + 1}x^2$

P.I. = P.I.$_1$ + P.I.$_2$

P.I.$_1$ $= \frac{1}{D^2 + 2D + 1}\cos 2x$

$D^2 \to -a^2 = -4$

$= \frac{1}{-4 + 2D + 1}\cos 2x$

$= \frac{1}{-3 + 2D} \times \frac{-3 - 2D}{-3 - 2D}\cos 2x$

$= \frac{-3 - 2D}{9 - 4D^2}\cos 2x$

$D^2 \to -a^2 = -4$

$= \frac{-(3 + 2D)}{9 - 4(-4)}\cos 2x$

$$= \frac{-(3+2D)}{9+16}\cos 2x$$

$$= \frac{-1}{25}[3\cos 2x + 2D(\cos 2x)]$$

$$\text{P.I.}_1 = \frac{-1}{25}[3\cos 2x - 4\sin 2x]$$

$$\text{P.I.}_1 = \frac{-1}{25}[3\cos 2x - 4\sin 2x]$$

$$\text{P.I.}_2 = \frac{1}{D^2+2D+1}x^2$$

$$= \frac{1}{1+D^2+2D}x^2$$

$$= [1+(D^2+2D)]^{-1}x^2$$

$$= [1-(D^2+2D)+(D^2+2D)^2+...]x^2$$

$$= [x^2 - D^2 + 2D)x^2 + (D^2+2D)^2x^2 + ...]$$ [neglecting D^3& higher powers]

$$= [x^2 - D^2(x^2) - 2D(x^2) + 4D^2(x^2)]$$

$$= [x^2 - 2 - 4x + 4(2)]$$

$$\text{P.I.}_1 = x^2 - 4x + 6$$

C.S. = C.F. + P.I_1+ P.I_2

$$y = (Ax+B)e^{-x} - \frac{1}{25}[3\cos 2x - 4\sin 2x] + x^2 - 4x + 6$$

3. Solve $(D^2 - 8D + 9)y = 8\sin 5x$

Solution: Auxiliary Eqn is $m^2 - 8m + 9 = 0$

$$m = \frac{8 \pm \sqrt{64-36}}{2}$$

$$m = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$$

$$\text{C.F} = Ae^{(4+\sqrt{7})x} + Be^{(4-\sqrt{7})x}$$

$$\text{P.I} = 8\frac{1}{D^2 - 8D + 1}\sin 5x \quad D^2 \to -a^2 = -25$$

$$= 8\frac{1}{-25 - 8D + 9}\sin 5x$$

$$= 8\frac{1}{-16 - 8D}\sin 5x$$

$$= -\frac{8}{8}\frac{1}{2 + D} \times \frac{2 - D}{2 - D}\sin 5x$$

$$= -\frac{(2 - D)}{4 - D^2}\sin 5x \qquad D^2 \to -a^2 = -25$$

$$= \frac{(2 - D)\sin 5x}{4 - (-25)}$$

$$= \frac{1}{29}[2\sin 5x - D(\sin 5x)]$$

$$\text{P.I} = \frac{1}{29}[2\sin 5x - 5(\cos 5x)]$$

$$\text{C.S} = \text{C.F} + \text{P.I}$$

$$y = Ae^{(4+\sqrt{7})x} + Be^{(4-\sqrt{7})x} - \frac{1}{29}[2\sin 5x - 5\cos 5x]$$

4. Solve $(D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Solution: Auxiliary Eqn is $m^3 + 2m^2 + m = 0$

$m\,(m^2 + 2m + 1) = 0$

$m = 0;\ m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$m = 0$, $m = -1$ (twice)

C.F. = $Ae^{0x} + (Bx + C)\, e^{-x}$

C.F. = $A + (Bx + C)\, e^{-x}$

$$\text{P.I} = \frac{1}{D^3 + 2D^2 + D}[e^{2x} + \sin 2x]$$

$$\text{P.I} = \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$$

P.I. = P.I.$_1$ + P.I.$_2$

$$\text{P.I.}_1 = \frac{1}{D^3 + 2D^2 + D} e^{2x}$$

$$D \rightarrow a = 2$$

$$= \frac{1}{8 + 2(4) + 2} e^{2x}$$

$$\text{P.I.}_1 = \frac{1}{18} e^{2x}$$

$$\text{P.I.}_2 = \frac{1}{D^3 + 2D^2 + D} \sin 2x \qquad D^2 \rightarrow -a^2 = -4$$

$$= \frac{1}{-4D + 2(-4) + D} \sin 2x$$

$$= \frac{1}{-8 - 3D} \sin 2x$$

$$= \frac{1}{-8 - 3D} \times \frac{-8 + 3D}{-8 + 3D} \sin 2x$$

$$= \frac{-8 + 3D}{64 - 9D^2} \sin 2x \qquad D^2 \rightarrow -a^2 = -4$$

$$= \frac{-8 + 3D}{64 - 9(-4)} \sin 2x$$

$$= \frac{-8+3D}{64+36}\sin 2x$$

$$= \frac{1}{100}[-8\sin 2x + 3D(\sin 2x)]$$

$$= \frac{1}{100}[-8\sin 2x + 6\cos 2x]$$

$$= \frac{2}{100}[3\cos 2x - 4\sin 2x]$$

$$\text{P.I.}_2 = \frac{1}{50}[3\cos 2x - 4\sin 2x]$$

C.S $= \text{C.F} + \text{P.I.}_1 + \text{P.I.}_2$

$$y = A + (Bx + C)e^{-x} + \frac{1}{18}e^{2x} + \frac{\;}{50}[3\cos 2x - 4\sin 2x]$$

5. Solve $(D^2 - 4D + 3)y = \cos 3x$

Solution: Auxiliary Eqn is $m^2 - 4m + 3 = 0$

$(m-1)(m-3) = 0$

$m = 1, 3$

C.F $= Ae^{x} + Be^{3x}$

$$\text{P.I} = \frac{1}{D^2 - 4D + 3}\cos 3x \qquad D^2 \to a^2 = -9$$

$$= \frac{1}{-9-4D+3}\cos 3x$$

$$= \frac{1}{-6-4D}\cos 3x$$

$$= \frac{1}{-6-4D} \times \frac{-6+4D}{-6+4D}\cos 3x$$

$$= \frac{-6+4D}{36-16D^2}\cos 3x \qquad D^2 \to -a^2 = -9$$

$$= \frac{-6+4D}{36-16(-9)}\cos 3x$$

$$= \frac{-6+4D}{36+144}\cos 3x$$

$$= \frac{1}{180}[-6\cos 3x + 4D(\cos 3x)]$$

$$= \frac{1}{180}[-6\cos 3x + 12\sin 3x]$$

$$= \frac{-6}{180}[\cos 3x + 2\sin 3x]$$

P.I $= \frac{-1}{30}[\cos 3x + 2\sin 3x]$

C.S = C.F + P.I

$y = Ae^{x} + Be^{3x} - \frac{1}{30}[\cos 3x + 2\sin 3x]$

6. Solve $(D^2 + 3D + 2)y = 2\sin^2 x$

Auxiliary equation is $m^2 + 3m + 2 = 0$

$(m + 1)(m + 2) = 0$

$m = -1, -2$

C.F $= Ae^{-x} + Be^{-2x}$

P.I $= \frac{1}{D^2 + 3D + 2} 2\sin^2 x$

$$= \frac{1}{D^2 + 3D + 2}\cancel{2}\left[\frac{1-\cos 2x}{\cancel{2}}\right]$$

$$= \frac{1}{D^2+3D+2}(1-\cos 2x)$$

$$= \frac{1}{D^2+3D+2} - \frac{1}{D^2+3D+2}\cos 2x$$

$$= \frac{1}{D^2+3D+2}e^{0x} - \frac{1}{D^2+3D+2}\cos 2x$$

$D \to a = 0$, $D^2 \to - a^2 = -4$

$$= \frac{1}{2}e^{0x} - \frac{1}{-4+3D+2}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{-2+3D} \times \frac{-2-3D}{-2-3D}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{4-9D^2}(2+3D)\cos 2x \qquad D^2 \to -a^2 = -4$$

$$= \frac{1}{2} + \frac{1}{4+36}[2\cos 2x + 3D\cos 2x]$$

$$= \frac{1}{2} + \frac{2}{40}[2\cos 2x - 6\sin 2x]$$

$$= \frac{1}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$$

P.I. $= \frac{1}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$

C.S = C.F. + P.I

$y = Ae^{-x} + Be^{-2x} + \frac{1}{2} + \frac{1}{20}[\cos 2x - 3\sin 2x]$

7. Solve $(D^2 + 6D + 8)\, y = \cos^2 x$

Solo A. Eqn is $m^2 + 6m + 8 = 0$

$(m + 2)(m + 4) = 0$

$m = -2, -4$

C.F $= Ae^{-2x} + Be^{-4x}$

P.I $= \dfrac{1}{D^2+6D+8}\cos^2 x$

$= \dfrac{1}{D^2+6D+8}\left(\dfrac{1+\cos 2x}{2}\right)$

$= \dfrac{1}{2}\dfrac{1}{D^2+6D+8}(1) + \dfrac{1}{2}\dfrac{1}{D^2+6D+8}\cos 2x \qquad D^2 \to -a^2 = -4$

$= \dfrac{1}{2}\dfrac{1}{8} + \dfrac{1}{2}\dfrac{1}{(-4+6D+8)}\cos 2x$

$= \dfrac{1}{16} + \dfrac{1}{2}\dfrac{4-6D}{4-6D} \times \dfrac{1}{4+6D}\cos 2x$

$= \dfrac{1}{16} + \dfrac{1}{2}\dfrac{4-6D}{16-36D^2}\cos 2x$

$= \dfrac{1}{16} + \dfrac{1}{2}\dfrac{(4-6D)}{16-36(-4)}\cos 2x$

$= \dfrac{1}{16} + \dfrac{1}{2}\dfrac{1}{16+144}(4-6D)\cos 2x$

$= \dfrac{1}{16} + \dfrac{1}{2}\dfrac{1}{160}[4\cos 2x - 6D(\cos 2x)]$

$= \dfrac{1}{16} + \dfrac{1}{320}[4\cos 2x + 12\sin\, 2x]$

$= \dfrac{1}{16} + \dfrac{4}{320}[\cos 2x + 3\sin\, 2x]$

P.I $= \dfrac{1}{16} + \dfrac{1}{80}[\cos 2x + 3\sin\, 2x]$

C.S = C.F. + P.I

y $= Ae^{-2x} + Be^{-4x} + \dfrac{1}{16} + \dfrac{1}{80}[\cos 2x + 3\sin\, 2x]$

8. Solve $(D^2 + 16)y = e^{-3x} + \cos 4x$

Auxiliary Eqn is $m^2 + 16 = 0$

$m^2 = -16$

$m = \pm 4i$

$\alpha = 0 \quad \beta = 4$

C.F. $= A \cos 4x + B \sin 4x$

$$\text{P.I.}_1 = \frac{1}{D^2+16} e^{-3x}$$

$D \to a = -3$

$$= \frac{1}{9+16} e^{-3x}$$

$$\text{P.I.}_1 = \frac{1}{25} e^{-3x}$$

$$\text{P.I.}_2 = \frac{1}{D^2+16} \cos 4x$$

$D^2 \to -a^2 = -16$

$$= \frac{1}{-16+16} \cos 4x$$

$$= x \frac{1}{2D} \cos 4x$$

$$= \frac{x}{2} \frac{\sin 4x}{4}$$

$$\text{P.I.}_2 = \frac{x \sin 4x}{8}$$

C.S $=$ C.F. $+$ P.I.$_1$ $+$ P.I.$_2$

$$y = A \cos 4x + B \sin 4x + \frac{1}{25} e^{-3x} + \frac{x \sin 4x}{8}$$

9. Solve $(D^2 + 9)\, y = \sin 3x$

Solution: Auxiliary Eqn is $m^2 + 9 = 0$

$m^2 = -9$

$m = \pm 3i$

C.F $= A\cos 3x + B\sin 3x$

P.I. $= \dfrac{1}{D^2+9}\sin 3x \qquad D^2 \to -a^2 = -9$

$= \dfrac{1}{-9+9}\sin 3x$

$= x\dfrac{1}{2D}\sin 3x$

P.I $= -\dfrac{x\cos 3x}{6}$

C.S = C.F + P.I.

$y = A\cos 3x + B\sin 3x - \dfrac{x\cos 3x}{6}$

10. Solve $(D^2 + 1)\, y = \cos x$

Auxiliary Eqn is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

C.F $= A\cos x + B\sin x$

P.I. $= \dfrac{1}{D^2+1}\cos x \qquad D^2 \to -a^2 = -1$

$= \dfrac{1}{-1+1}\cos x$

$= x\dfrac{1}{2D}\cos x$

P.I $= -\dfrac{x\sin x}{2}$

C.S = C.F + P.I.

$y = A\cos x + B\sin x - \dfrac{x\sin x}{2}$

11. Solve $(D^2 - 4D + 3)\, y = \sin 3x \cos 2x$

Auxiliary Eqn is $m^2 - 4m + 3 = 0$

$(m-1)(m-3) = 0$

$m = 1, 3$

C.F $= Ae^{x} + Be^{3x}$

P.I. $= \dfrac{1}{D^2 - 4D + 3}\sin 3x\cos 2x$

$= \dfrac{1}{D^2 - 4D + 3}\left[\dfrac{1}{2}(\sin(3x+2x) + \sin(3x-2x))\right]$

P.I $= \dfrac{1}{2}\dfrac{1}{D^2 - 4D + 3}\sin 5x + \dfrac{1}{2}\dfrac{1}{D^2 - 4D + 3}\sin x$

P.I. = P.I.$_1$ + P.I.$_2$

P.I.$_1$ $= \dfrac{1}{2}\dfrac{1}{D^2 - 4D + 3}\sin 5x \qquad D^2 \to -a^2 = -25$

$= \dfrac{1}{2}\dfrac{1}{-25 - 4D + 3}\sin 5x$

$= \dfrac{1}{2}\dfrac{1}{-22 - 4D}\sin 5x$

$= \dfrac{-1}{4}\dfrac{1}{11 + 2D}\times\dfrac{11 - 2D}{11 - 2D}\sin 5x$

$= \dfrac{-1}{4}\dfrac{(11 - 2D)}{(121 - 4D^2)}\sin 5x \qquad D^2 \to -a^2 = -25$

$$= \frac{-1}{4}\frac{(11-2D)}{(121-49-25)}\sin 5x$$

$$= \frac{-1}{4}\frac{(11-2D)}{121+100}\sin 5x$$

$$= \frac{-1}{4}\frac{1}{221}(11\sin 5x - 2D\sin 5x)$$

$$= \frac{-1}{884}(11\sin 5x - 10\cos 5x)$$

$$\text{P.I.}_1 = \frac{1}{884}[10\cos 5x - 11\sin 5x]$$

$$\text{P.I.}_2 = \frac{1}{2}\frac{1}{D^2-4D+3}\sin x \qquad D^2 \to -a^2 = -1$$

$$= \frac{1}{2}\frac{1}{(-1-4D+3)}\sin x$$

$$= \frac{1}{2}\frac{1}{(2-4D)}\sin x$$

$$= \frac{1}{2}\frac{1}{2-4D}\times\frac{2+4D}{2+4D}\sin x$$

$$= \frac{1}{2}\frac{1}{(4-16D^2)}(2+4D)\sin x \qquad D^2 \to -a^2 = -1$$

$$= \frac{1}{2}\frac{(2+4D)}{(4-16(-1))}\sin x$$

$$= \frac{1}{2}\frac{1}{20}(2\sin x + 4D\sin x)$$

$$= \frac{1}{\cancel{2}}\frac{1}{20}\cancel{2}(\sin x + 2\cos x)$$

$$\text{P.I.}_2 = \frac{1}{20}[\sin x + 2\cos x]$$

C.S = C.F + P.I.$_1$ + P.I.$_2$

$$y = Ae^{x} + Be^{3x} + \frac{1}{884}[10\cos 5x - 11\sin 5x] + \frac{1}{20}[\sin x + 2\cos x]$$

12. Solve $(D^2 + 4)\, y = 2\cos x \cos 3x$

The auxiliary Eqn is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

C.F $= A\cos 2x + B\sin 2x$

$$\text{P.I} = \frac{1}{D^2+4}[2\cos x\cos 3x]$$

$$= \frac{1}{D^2+4}\left[\cancel{2}\frac{1}{\cancel{2}}[\cos(3x+x)+\cos(3x-x)]\right]$$

$$= \frac{1}{D^2+4}[\cos 4x + \cos 2x]$$

$$= \frac{1}{D^2+4}\cos 4x + \frac{1}{D^2+4}\cos 2x$$

$D^2 \to -a^2 = -16$ $\qquad$ $D^2 \to -a^2 = -4$

$$= \frac{1}{-16+4}\cos 4x + \frac{1}{-4+4}\cos 2x$$

$$= \frac{1}{-12}\cos 4x + x.\frac{1}{2D}\cos 2x$$

$$\text{P.I} = -\frac{1}{12}\cos 4x + \frac{x\sin 2x}{4}$$

C.S = C.F + P.I

$$y = A\cos 2x + B\sin 2x - \frac{1}{12}\cos 4x + \frac{x\sin 2x}{4}$$

Exercise

1. Solve $(D^2 - 4D + 4)y = \cos 2x$

2. Solve $(D^2 + 3D + 2)y = \sin 3x$

3. Solve $(D^2 + 1)y = \cos(2x - 1)$

4. Solve $(D^2 - 3D + 2)y = 7\cos x$

5. Solve $(D^2 - 7D + 12)y = e^{5x} + \cos 2x$

6. Solve $(D^2 - 4D + 3)y = \sin 2x$

7. Solve $(D^2 - 4D - 5)y = 3\cos 4x + e^{2x}$

8. Solve $(D^2 - 4D + 4)y = \cos x + e^x$

9. Solve $(D^3 + 2D^2 + D)y = e^{-2x} + \sin 2x$

10. Solve $(D^3 + 4D)y = \sin 2x$

11. Solve $(D^2 + 4)y = \cos 2x$

12. Solve $(D^2 + 6D + 5)y = \sin^2 x$

13. Solve $(D^2 + 4)y = 3\cos^2 x + x^2$

14. Solve $(D^2 + 9)y = \sin^3 x$

15. Solve $(D^2 + 6D + 5)y = \cos^2 x$

16. Solve $(D^3 - 1)y = \cos\frac{x}{2}\sin\frac{x}{2} + 2e^x$

17. Solve $(D^2 + 2D + 1)y = \sin 2x\cos x$

18. Solve $(D^2 + 1)y = \sin 2x\sin x$

19. Solve $(D^2 + 1)y = 2\sin x\cos 3x$

20. Solve $(D^2 + D + 1)y = 2\cos 2x\cos x$

Answer

1. $y=(Ax+B)e^{2x}-\frac{1}{8}\sin 2x$

2. $y=Ae^{-x}+Be^{-2x}+\frac{1}{130}[9\cos 3x=7\sin 3x]$

3. $y=A\cos x+B\sin x-\frac{1}{3}\cos(2x-1)$

4. $y=Ae^{x}+Be^{2x}+\frac{7}{10}[\cos x-3\sin x]$

5. $y=Ae^{3x}+Be^{4x}+\frac{e^{5x}}{2}+\frac{1}{130}[4\cos 2x-7\sin 2x]$

6. $y=Ae^{x}+Be^{3x}+\frac{1}{65}[8\cos 2x-\sin 2x]$

7. $y=Ae^{-x}+Be^{5x}-\frac{1}{9}e^{2x}-\frac{3}{697}[21\cos 4x+16\sin 4x]$

8. $y=(Ax+B)e^{2x}+e^{x}-\frac{1}{8}\sin 2x$

9. $y=A+(Bx+C)e^{-x}-\frac{1}{2}e^{-2x}+\frac{1}{50}[3\cos 2x-4\sin 2x]$

10. $y=A+B\cos 2x+C\sin 2x-\frac{x}{8}\sin 2x$

11. $y=A\cos 2x+B\sin 2x+\frac{x\sin 2x}{4}$

12. $y=Ae^{-5x}+Be^{-x}+\frac{1}{10}-\frac{1}{290}[\cos 2x+12\sin 2x]$

13. $y=A\cos 2x+B\sin 2x+\frac{x^2}{4}-\frac{1}{4}+\frac{3x\sin 2x}{8}$

14. $y=A\cos 3x+B\sin 3x+\frac{3\sin x}{32}+\frac{x\cos 3x}{24}$

15. $y = Ae^{-5x} + Be^{-x} + \frac{1}{10} + \frac{1}{290}[\cos 2x + 12\sin 2x]$

16. $y = Ae^{x} + Be^{-\frac{x}{2}}\left[B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right] + \frac{2xe^{x}}{3} + \frac{1}{4}[\cos x - \sin x]$

17. $y = [Ax + B]e^{-x} - \frac{1}{100}[3\cos 3x + 4\sin 3x] - \frac{\cos x}{4}$

18. $y = A\cos x + B\sin x + \frac{x\sin x}{4} + \frac{1}{16}\cos 3x$

19. $y = A\cos x + B\cos x - \frac{1}{15}\sin 4x + \frac{1}{3}\sin 2x$

20. $y = e^{-\frac{x}{2}}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{145}[8\cos 3x + 3\sin 3x] + \sin x$

Type (4): $f(x) = e^{ax}\sin bx \ (or) \ e^{ax}\cos bx$

Method of finding P.I.

$$\text{P.I} = \frac{1}{\Phi(D)}e^{ax}\sin bx$$

Replace D by $D + a$

$$\text{P.I} = e^{ax}\left[\frac{1}{\Phi(D+a)}\right]\sin bx$$

$\left[\frac{1}{\Phi(D+a)}\right]\sin bx$ is evaluated using type (2)

Note: If $f(x) = xV$ where $V = \sin ax$ or $\cos ax$, then

P.I $= x.\dfrac{V}{\Phi(D)} - \dfrac{\Phi'(D).V}{[\Phi(D)]^2}$

1. Solve: $(D^2+5D+4)Y = e^{-x}\sin 2x$

Solution: Given, $(D^2+5D+4)y = e^{-x}\sin 2x$

(i) To find C.F.

$(D^2+5D+4)y = 0$

The auxiliary equation is

$m^2 + 5m + 4 = 0$

$(m+1)(m+4) = 0$

$m+1 = 0, m+4 = 0$

$m = -1, m = -4$

The roots are real and distinct

$\therefore$ C.F $= Ae^{m_1x} + Be^{m_2x}$

$= Ae^{-x} + Be^{-4x}$

(ii) To find P.I.

P.I $= \dfrac{1}{D^2+5D+4}e^{-x}\sin 2x$

Here, $a = -1$ and $b = 2$

Replace D by $D + a = D - 1$

P.I $= e^{-x}\dfrac{1}{(D-1)^2+5(D-1)+4}\sin 2x$

$= e^{-x}\dfrac{1}{D^2-2D+1+5D-5+4}\sin 2x$

$$= e^{-x}\frac{1}{D^2+5D}\sin 2x$$

Replace D^2 by $-b^2 = -4$

$$\text{P.I} = e^{-x}\frac{1}{-4+3D}\sin 2x$$

$$\text{P.I} = e^{-x}\frac{(3D+4)}{(3D+4)(3D-4)}\sin 2x$$

$$= e^{-x}\frac{(6\cos 2x+4\sin 2x)}{9D^2-16}$$

Replace D^2 by $-b^2 = -4$

$$\text{P.I} = e^{-x}\frac{(6\cos 2x+4\sin 2x)}{9(-4)-16}$$

$$= 2e^{-x}\frac{3\cos 2x+2\sin 2x}{(-52)}$$

$$= \frac{e^{-x}}{26}(3\cos 2x+2\sin 2x)$$

The solution is

$$y = \text{C.F + P.I}$$

$$= Ae^{-x}+Be^{-4x}+\frac{(-e^{-x})}{26}(3\cos 2x+2\sin 2x)$$

$$= Ae^{-x}+Be^{-4x}-\frac{(e^{-x})}{26}(3\cos 2x+2\sin 2x)$$

2. Solve: $(D^2-2D+4)y = e^x \sin x$

Solution:Given, $(D^2-2D+4)y = e^x \sin x$

(i) To find C.F.

$$(D^2-2D+4)y = 0$$

The auxiliary equation is

$m^2 - 2m + 4 = 0$

Here, a = 1, b = -2 and c = 4

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm i2\sqrt{3}}{2}$$

$$m = 1 \pm i\sqrt{3}$$

The roots are imaginary Here α = 1 and ß = $\sqrt{3}$

$$\text{C.F} = e^{ax}(A\cos ßx + B\sin ßx)$$

$$= e^{x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right)$$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4} e^x \sin x$$

Here a = 1, and b = 1

Replace D by D + a = D + 1

$$\text{P.I} = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 4} \sin x$$

P.I $= = e^x \frac{1}{D^2+3} \sin x$

Replace D^2 by $-b^2 = -1$

$\therefore D^2 + 3 \quad = -1 + 3$

$= 2$

$\neq 0$

$\therefore$ P.I $= e^x . \frac{\sin x}{2}$

The solution is

$y \quad$ = C.F + P.I.

$$y = e^x \left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + \frac{1}{2} e^x \sin x$$

3. Solve: $(D^2 - 2D + 5)y = e^{2x} \cos x$

Solution: Given, $(D^2 - 2D + 5)y = e^{2x} \cos x$

(i) To find C.F

$(D^2 - 2D + 5)y = 0$

The auxiliary equation is

$m^2 - 2m + 5 = 0$

Here $a = 1$, $b = -2$ and $c = 5$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$m \quad = 1 \pm 2i$

The roots are imaginary. Here α = 1 and ß = 2

$\therefore$ C.F $= e^{ax}(A\cos ßx + B\sin ßx)$

C.F $= e^{x}(A\cos 2x + B\sin 2x)$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 2D + 5} e^{2x} \cos x$$

Here, a = 2 and b = 1

Replace D By D + a = D + 2

$$\text{P.I} = e^{2x}.\frac{1}{(D+2)^2 - 2(D+2) + 5}\cos x$$

$$= e^{2x}.\frac{1}{D^2 + 4D + 4 - 2D - 4 + 5}\cos x$$

$$= e^{2x}.\frac{1}{D^2 + 2D + 5}\cos x$$

Replace D^2 By $-b^2$ = -1

$$\text{P.I} = e^{2x}.\frac{1}{-1 + 2D + 5}\cos x$$

$$= e^{2x}.\frac{1}{2D + 4}\cos x$$

$$= \frac{e^{x}}{2}.\frac{1}{D+2}\cos x$$

$$= \frac{e^{2x}}{2}.\frac{(D-2)\cos x}{(D-2)(D+2)}$$

$$= \frac{e^{2x}}{2} \cdot \left[\frac{-\sin x - 2\cos x}{D^2 - 4}\right]$$

Replace D^2 by $-b^2 = -1$

$\therefore\ D^2 - 4 = -1 - 4 = -5 \neq 0$

$$\therefore\ \text{P.I} = \frac{e^{2x}}{2}\left[\frac{-\sin x - 2\cos x}{-5}\right]$$

$$= \frac{-e^{2x}}{(-10)}(\sin x + 2\cos x)$$

$$= \frac{e^{2x}}{10}(\sin x + 2\cos x)$$

The solution is y = C.F. + P.I

i.e., $y = e^x(A\cos 2x + B\sin 2x) + \frac{e^{2x}}{10}(\sin x + 2\cos x)$

4. Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \cos x$

Solution:Given, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \cos x$

$$\left(\frac{d^2}{dx^2} - 5\frac{d}{dx} + 6\right) y = e^x \cos x$$

$$(D^2 - 5D + 6)y = e^x \cos x$$

(i) To find C.F

$(D^2 - 5D + 6)\, y = 0$

The auxiliary equation is

$m^2 - 5m + 6 = 0$

$(m - 2)\,(m - 3) = 0$

$\therefore m - 2 = 0, m - 3 = 0$

$m = 2, m = 3$

The roots are real and distinct

$\therefore$ C.F $= Ae^{m_1 x} + Be^{m_2 x}$

$= Ae^{2x} + Be^{3x}$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 5D + 6} e^x \cos x$$

Here $a = 1$ and $b = 1$

Replace D by $D + a = D + 1$

$$\text{P.I} = e^x \frac{1}{(D+1)^2 - 5(D+1) + 6} \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 5D - 5 + 6} \cos x$$

$$= e^x \frac{1}{D^2 - 3D + 2} \cos x$$

Replace D^2 by $-b^2 = -1$

$$\text{P.I} = e^x \frac{1}{-1 - 3D + 2} \cos x$$

$$= e^x \frac{1}{-3D + 1} \cos x$$

$$= -e^x \frac{1}{3D - 1} \cos x$$

$$= -e^x \frac{(3D+1)\cos x}{(3D+1)(3D-1)}$$

$$= e^x \frac{[-3\sin x + \cos x]}{9D^2 - 1}$$

Replace D^2 by $-b^2 = -1$

P.I $= \dfrac{-e^x(-3\sin x + \cos x)}{9(-1) - 1}$

$$= \frac{e^x}{-10}[\cos x - 3\sin x]$$

The solution is

y = C.F + P.I

$$= Ae^{2x} + Be^{3x} + \frac{e^x}{10}(\cos x - 3\sin x)$$

5. Solve: $(D^2 + 4D + 3)y = e^{-x}\sin x$

Solution:Given, $(D^2 + 4D + 3)y = e^{-x}\sin x$

(i) To find C.F.

$(D^2 + 4D + 3)\, y = 0$

The auxiliary equation is

$m^2 + 4m + 3 = 0$

$(m + 3)\,(m + 1) = 0$

$m + 3 = 0,\ m + 1 = 0$

$m = -3,\ m = -1$

The roots are real and distinct

$\therefore$ C.F $= Ae^{m_1 x} + Be^{m_2 x}$

$$= Ae^{-3x} + Be^{(-1)x}$$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2+4D+3} e^{-x} \sin x$$

Here $a = -1$; $b = 1$

Replace D by $D + a = D - 1$

$$\text{P.I} = e^{-x} \frac{1}{(D-1)^2+4(D-1)+3} \sin x$$

$$= e^{-x} \frac{1}{D^2-2D+1+4D-4+3} \sin x$$

$$= e^{-x} \frac{1}{D^2+2D} \sin x$$

Replace $D^2$by $-b^2 = -1$

$$\text{P.I} = e^{-x} \frac{1}{-1+2D} \sin x$$

$$= e^{-x} \frac{1}{2D-1} \sin x$$

$$= e^{-x} \frac{(2D+1) \sin x}{(2D-1)(2D+1)}$$

$$= e^{-x} \frac{[2\cos x + \sin x]}{4D^2-1}$$

Replace D^2 by $-b^2 = -1$

$\therefore\ 4D^2 - 1 = 4(-1) - 1 = -5 \neq 0$

$$\text{P.I} = e^{-x} \frac{(2\cos x + \sin x)}{-5}$$

$$= \frac{-e^{-x}}{5}(2\cos x + \sin x)$$

The solution is

$y = \text{C.F} + \text{P.I}$

$$= Ae^{-3x} + Be^{-3} - \frac{e^{-x}}{5}(2\cos x + \sin x)$$

6. Solve: $\frac{d^2y}{dx^2} + 4y = e^x \sin x$

Solution:Given, $\frac{d^2y}{dx^2} + 4y = e^x \sin x$

$$(D^2 + 4)\ y = e^x \sin x$$

(i) To find C.F

$(D^2 + 4)\ y = 0$

The auxiliary equation is

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$= \pm 2i$

$= 0 \pm 2i$

The roots are imaginary

Here, $\alpha = 0$, $ß = 2$

$\therefore$ C.F $= e^{\alpha x}(A\cos ßx + B\sin ßx)$

$= (A\cos 2x + B\sin 2x)$

C.F $= A\cos 2x + B\sin 2x$

(ii) To find P.I

P.I $= \frac{1}{D^2 + 4} e^x \sin x$

Here $a = 1$, $b = 1$

Replace D by $D + a = D + 1$

$$\text{P.I} = e^x \frac{1}{(D+1)^2+4}\sin x$$

$$= e^x \frac{1}{D^2+2D+1+4}\sin x$$

$$= e^x \frac{1}{D^2+2D+5}\sin x$$

Replace D^2 by $-b^2 = -1$

$$\text{P.I} = e^x \frac{1}{-1+2D+5}\sin x$$

$$= e^x \frac{1}{2D+4}\sin x$$

$$= \frac{e^x}{2}\frac{1}{D+2}\sin x$$

$$= \frac{e^x}{2}\frac{(D-2)\sin x}{(D-2)(D+2)}$$

$$= \frac{e^x}{2}\frac{(\cos x - 2\sin x)}{D^2-4}$$

Replace D^2 by $-b^2 = -1$

$\therefore D^2 - 4 = -1 - 4 = -5 \neq 0$

$$\text{P.I} = \frac{e^x}{2}\frac{(\cos x - 2\sin x)}{(-5)}$$

$$= \frac{e^x}{10}(\cos x - 2\sin x)$$

The solution is

$y = \text{C.F} + \text{P.I}$

$$y \quad = A\cos 2x + b\sin 2x + \frac{(-e^{x})}{10}(\cos x - 2\sin x)$$

7. Solve: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-3x}\sin 2x$

Solution:Given, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-3x}\sin 2x$

$$(D^2 + 4D + 4)y = e^{-3x}\sin 2x$$

(i) To find C.F

$(D^2 + 4D + 4)y = 0$

The auxiliary equation is

$m^2 + 4m + 4 \quad = 0$

$(m + 2)(m + 2) \quad = 0$

$m + 2 = 0, m + 2 \quad = 0$

$m = -2, \qquad m = -2$

$\therefore m = -2$ (twice)

The roots are real and equal

$\therefore$ C.F $= e^{mx}(Ax + B)$

$= e^{-2x}(Ax + B)$

(ii) To find P.I

P.I $= \frac{1}{D^2 + 4D + 4}e^{-3x}\sin 2x$

Here $a = -3$ and $b = 2$

Replace D by $D + a = D - 3$

$\therefore$ P.I. $= e^{-2x}.\frac{1}{(D-3)^3 + 4(D-3) + 4}\sin 2x$

$$= e^{-3x} \frac{1}{D^2 - 6D + 9 + 4D - 12 + 4} \sin 2x$$

$$\text{P.I} = e^{-3x} \frac{1}{D^2 - 2D + 1} \sin 2x$$

Replace D^2 by $-b^2 = -4$

$$\text{P.I} = e^{-3x} \frac{1}{-4 - 2D + 1} \sin 2x$$

$$= e^{-3x} \frac{1}{-2D - 3} \sin 2x$$

$$= e^{-3x} \frac{(2D - 3)\sin 2x}{(2D - 3)(2D + 3)}$$

$$= e^{-3x} \frac{[2(2\cos 2x) - 3\sin 2x]}{4D^2 - 9}$$

Replace D^2 by $-b^2 = -4$

$\therefore 4D^2 - 9 = 4(-4) - 9 = 16 - 9 = 25 \neq 0$

$$\therefore \text{ P.I} = -e^{-3x} \frac{(4\cos 2x - 3\sin 2x)}{-25}$$

$$= \frac{e^{-3x}}{25}(4\cos 2x - 3\sin 2x)$$

The solution is

$y = \text{C.F} + \text{P.I}$

$$y = e^{-2x}(Ax + B) + \frac{e^{-3x}}{25}(4\cos 2x - 3\sin 2x)$$

8. Solve: $(D^2 + 2D)y = e^{-x}\cos x$

Solution:Given $(D^2 + 2D)y = e^{-x}\cos x$

$(D^2 + 2D)y = 0$

The auxiliary equation is

$m^2 + 2m \quad = 0$

$m\,(m + 2) \quad = 0$

$m = 0,\ m + 2 \quad = 0$

$m = 0, \qquad m \quad = -2$

$\therefore$ The roots are real and distinct

$$\begin{aligned} \text{C.F} &= Ae^{m_1 x} + Be^{m_2 x} \\ &= Ae^{0x} + Be^{-2x} \\ &= A + Be^{-2x} \end{aligned}$$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 + 2D} e^{-x} \cos x$$

Here a = –1, b = 1

Replace D by $D + a = D - 1$

$$\begin{aligned} \text{P.I} &= e^{-x} \frac{1}{(D-1)^2 + 2(D-1)} \cos x \\ &= e^{-x} \frac{1}{D^2 - 2D + 1 + 2D - 2} \cos x \\ &= e^{-x} \frac{1}{D^2 - 1} \cos x \end{aligned}$$

Replace D^2 by $-b^2 = -1$

$\therefore D^2 - 1 = -1 - 1 = -2 \neq 0$

$$\text{P.I} = e^{-x} \frac{\cos x}{-2}$$

$= -\frac{e^{-x}}{2}\cos x$

The solution is

$y \quad = \text{C.F} + \text{P.I}$

$= A + Be^{-2x} - \frac{e^{-x}}{2}\cos x$

9. Solve: $(D^2 - 4D + 13)y = e^{2x}\cos 3x$

Solution:Given, $(D^2 - 4D + 13)y = e^{2x}\cos 3x$

(i) To find C.F

$(D^2 - 4D + 13)y = 0$

The auxiliary equation is

$m^2 - 4m + 13 = 0$

Here $a = 1$, $b = -4$, $c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm i6}{2}$$

$m = 2 \pm i3$

The roots are imaginary

Here $\alpha = 2$ and $ß = 3$

C.F $= e^{ax}(A\cos ßx + B\sin ßx)$

$= e^{2x}(A\cos 3x + B\sin 3x)$

(ii) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 4D + 13} e^{2x} \cos 3x$$

Here, $a = 2$ and $b = 3$

$$\text{P.I} = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 13} \cos 3x$$

$$\text{P.I} = e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 13} \cos 3x$$

$$= e^{2x} \frac{1}{D^2 + 9} \cos 3x$$

Replacing D^2 by $-b^2 = -9$

$\therefore D^2 + 9 = -9 + 9 = 0$

$$\text{P.I} = e^{2x} x \frac{\cos 3x}{2D}$$

$$= \frac{x}{2} e^{2x} \left(\frac{\cos 3x}{D} \right)$$

$$= \frac{x}{2} e^{2x} \int \cos 3x dx$$

$$= \frac{x}{2} e^{2x} \frac{\sin 3x}{3}$$

$$= \frac{x}{6} e^{2x} \sin 3x$$

The solution is

y = C.F + P.I

$= e^{2x}(A\cos 3x + B\sin 3x) + \frac{x}{6} e^{2x} \sin 3x$

10. Solve: $(D^2 - 2D + 5)y = e^x \cos 2x$

Solution:Given, $(D^2 - 2D + 5)y = e^x \cos 2x$

(i) To find C.F

$(D^2 - 2D + 5)y = 0$

The auxiliary equation is

$m^2 - 2m + 5 = 0$

Here $a = 1$, $b = -2$ and $c = 5$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm i4}{2}$$

$= 1 \pm 2i$

The roots are imaginary. Here $\alpha = 1$ and $ß = 2$.

C.F $= e^{2x}(A\cos ßx + B\sin ßx)$

$= e^{x}(A\cos 2x + B\sin 2x)$

(ii) To find P.I

P.I $= \frac{1}{D^2 - 2D + 5} e^x \cos 2x$

Here $a = 1$, $b = 2$

Replace D by $D + a = D + 1$

P.I $= e^x \dfrac{1}{(D+1)^2 - 2(D+1) + 5} \cos 2x$

$= e^x \dfrac{1}{D^2 + 2D + 1 - 2D - 2 + 5} \cos 2x$

$= e^x \dfrac{1}{D^2 + 4} \cos 2x$

Replace D^2 by $-b^2 = -4$

$\therefore D^2 + 4 = -4 + 4 = 0$

$\therefore$ P.I$= e^x . x \dfrac{\cos 2x}{2D}$

$= \dfrac{xe^x}{2}\left(\dfrac{\cos 2x}{D}\right)$

$= \dfrac{xe}{2}\left(\dfrac{1}{2} \sin 2x\right)$

$= \dfrac{x}{4} e^x \sin 2x$

The solution is

y = C.F + P.I

$= e^x (A \cos 2x + B \sin 2x) + \dfrac{x}{4} e^x \sin 2x$

11. Solve: $(D+1)^2 y = e^{-x} \cos x$

Solution:Given, $(D+1)^2 = e^{-x} \cos x$

(i) To find C.F.

$(D + 1)^2 y = 0$

The auxiliary equation is

$(m + 1)^2 = 0$

$(m + 1)\ (m + 1) = 0$

$(m + 1) = 0,\ (m + 1) = 0$

$m = -1,\ m = -1$

$m = -1$ (twice)

The roots are real and equal

$$\therefore \text{C.F} = e^{mx}(Ax+B)$$

$$= e^{-x}(Ax+B)$$

(ii) To find P.I

$$\text{P.I} = \frac{1}{(D+1)^2} e^{-x} \cos x$$

$$= \frac{1}{D^2+2D+1} e^{-x} \cos x$$

Here $a = -1$ and $b = 1$

Replace D by $D + a = D - 1$

$$\therefore \text{P.I} = e^{-x} \frac{1}{(D-1)^2+2(D-1)+1} \cos x$$

$$\text{P.I} = e^{-x} \frac{1}{D^2-2D+1+2D-2+1} \cos x$$

$$= e^{-x} \frac{\cos x}{D^2}$$

$$= e^{-x} \frac{1}{D}\left(\frac{1}{D} \cos x\right)$$

$$= e^{-x} \frac{1}{D}\left(\int \cos x\, dx\right)$$

$$= e^{-x} \frac{1}{D}(\sin x)$$

$= e^{-x}\int \sin x \, dx$

$= e^{-x}\left(-\cos x\right)$

$= -e^{-x}\cos x$

The solution is

$y \quad = \text{C.F} + \text{P.I}$

$= e^{-x}(Ax+B) - e^{-x}\cos x$

12. Solve: $(D^2 - 2D + 1)y = xe^x \sin x$

Solution:

Given $(D^2 - 2D + 1)y = xe^x \sin x$

(i) To find C.F

$(D^2 - 2D + 1)y = 0$

The auxiliary equation is

$m^2 - 2m + 1 = 0$

$(m - 1)(m - 1) = 0$

$m = 1, m = 1$

$\therefore m = 1$ (twice)

The roots are real and equal

C.F $= e^{mx}(Ax + B)$

$= e^{x}(Ax + B)$

(i) To find P.I

$$\text{P.I} = \frac{1}{D^2 - 2D + 1} xe^x \sin x$$

Here $a = 1$, $b = 1$

Replace D by $D + a = D + 1$

$$\text{P.I} = e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x\sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} x\sin x$$

$$\text{P.I} = e^x \left(\frac{x\sin x}{D^2} \right)$$

$$= e^x \left[x \frac{1}{D^2} \sin x \right] - e^x \frac{2D}{(D^2)^2} \sin x$$

$$= e^x \left[x \frac{1}{D^2} \sin x \right] - e^x \frac{2\cos x}{(D^2)^2}$$

Replace D^2 by $-b^2 = -1$

$$\text{P.I} = e^x \left(x \frac{\sin x}{(-1)} \right) - e^x \frac{2\cos x}{(-1)^2}$$

$$= -xe^x \sin x - 2e^x \cos x$$

$$= -e^x (x\sin x - 2\cos x)$$

The solution is

$$y = \text{C.F} + \text{P.I}$$

$$= e^x (Ax + B) - e^x (x\sin x - 2\cos x)$$

Exercises

Solve the following differential equations

1. $(D^2 - 4D + 13)y = e^{2x} \cos 3x$

2. $(D^2 + 4D + 3)y = e^x \sin x$

3. $(D^2 - 2D + 5)y = e^{2x} \sin x$

4. $(D+1)^2 y = e^{-x} \cos x$

5. $(D^2 - 5D + 6)y = e^x \cos 2x$

6. $(D^2 - 4D + 3)y = 3e^x \cos 2x$

7. $(D^2 - 6D + 13)y = 8e^{3x} \sin 4x$

8. $(D^2 - 4D + 13)y = e^{2x} \cos 3x$

9. $\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 4y = e^{-x} \sin 2x$

10. $\dfrac{d^2 x}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \cos x$

Answer

1. $y = e^{2x}[A\cos 3x + B\sin 3x] + \dfrac{1}{6} xe^{2x} \sin 3x$

2. $y = Ae^{-x} + Be^{-x} \dfrac{e^x}{85}(6\cos x - 7\sin x)$

3. $y = e^x(A\cos 2x + B\sin 2x) - \dfrac{e^{2x}}{20}(2\cos x - 4\sin x)$

4. $y = (Ax + B)e^{-x} - e^{-x} \cos x$

5. $y = Ae^{2x} + Be^{3x} = \dfrac{e^x}{20}(\cos 2x + 3\sin 2x)$

6. $y = Ae^x + Be^{2x} - \dfrac{3}{8} e^x[\sin 2x + \cos 2x]$

7. $y = e^{3x}(A\cos 2x + B\sin 2x) - xe^{3x} \cos 4x$

8. $y = e^{2x}[A\cos 3x + B\sin 3x] + \dfrac{xe^{2x}}{6} \sin 3x$

9 $y = e^{-2x}(Ax+B) - \frac{1}{25}e^{-x}(4\cos 2x + 3\sin 2x)$

10. $Y = (Ax+B)\,e^{x} - e^{x}(x\sin x + 2\cos x)$

Problem based on $f(x) = x^n e^{ax}$

1. Solve $(D^2 - D - 6)y = xe^{-2x}$

Solution: Given $(D^2 - D - 6)y = xe^{-2x}$

The Auxiliary Equation is

$m^2 - m - 6 = 0$

$m = 3, -2$

C.F = $Ae^{-2x} + Be^{3x}$

P.I $= \frac{1}{D^2 - D - 6} x^{e-2x}$

$= e^{-2x}\frac{1}{(D-2)^2 - (D-2) - 6}x$

$= e^{-2x}\frac{1}{D^2 - 4D + 4 - D + 2 - 6}x$

$= e^{-2x}\frac{1}{D^2 - 5D}x$

$= e^{-2x}\frac{1}{5D\left[\frac{D}{5} - 1\right]}x$

$= e^{-2x}\frac{1}{-5D\left[1 - \frac{D}{5}\right]}x$

$= e^{-2x}\frac{1}{-5D}\left[1 - \frac{D}{5}\right]^{-1}x$

$$= \frac{e^{-2x}}{-5D}\left[1+\frac{D}{5}+\left(\frac{D}{5}\right)^2+\ldots\right]x \qquad \left[\begin{array}{l} \because D(x)=1 \\ D^2(x)=0 \\ \vdots \\ D^n(x)=0 \\ \forall n \geq R \end{array}\right]$$

$$= \frac{e^{-2x}}{-5D}\left[x+\frac{D}{5}(x)+\frac{D^2(x)}{25}+\ldots\right]$$

$$= \frac{e^{-2x}}{-5D}\left[x+\frac{1}{5}\right]$$

$$= \frac{e^{-2x}}{-5}\int\left(x+\frac{1}{5}\right)dx$$

$$= \frac{e^{-2x}}{-5}\left[\frac{x^2}{2}+\frac{1}{5}x\right]$$

$$= \frac{xe^{-2x}}{-5}\left[\frac{x}{2}+\frac{1}{5}\right]$$

$$= \frac{xe^{-2x}}{-5}\left[\frac{5x+2}{10}\right]$$

$$= \frac{xe^{-2x}}{-50}[5x+2]$$

y = C.F + P.I

$$= Ae^{-2x}+Be^{3x}-\frac{xe^{-2x}}{50}(5x+2)$$

2. Solve $(D^2-4D+4)y = x^2e^{2x}$

Solution: Given $(D^2-4D+4)y = x^2e^{2x}$

The Auxiliary Equation is

$m^2 - 4m + 4 = 0$

$m = 2, 2$

C.F $= (A + Bx)\, e^{2x}$

P.I $= \frac{1}{D^2 - 4D + 4} x^2 e^{2x}$

$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2$

$= e^{2x} \frac{1}{D^2 + 4D + 4 - 4D - 8 + 4} x^2$

$= e^{2x} \frac{1}{D^2 + 4D - 4D + 8 - 8} x^2$

$= e^{2x} \frac{1}{D} \int x^2 dx$

$= e^{2x} \frac{1}{D} \left[\frac{x^3}{3} \right]$

$= \frac{e^{2x}}{3} \int x^3 dx$

$= \frac{e^{2x}}{3} \left(\frac{x^4}{4} \right)$

$= \frac{e^{2x} x^4}{12}$

y = C.F + P.I

$= (A + Bx)e^{2x} + \frac{x^4 e^{2x}}{12}$

3. Solve: $(D^3 - D)y = e^x x$

Solution: Given $(D^3 - D)y = e^x x$

The Auxiliary Equation is

$m^3 - 1 = 0$

$m\,(m^2 - 1) = 0$

$\Rightarrow m = 0,\ m = \pm 1$

C.F = $Ae^{0x} + Be^{-x} + Ce^{x}$

$= A + Be^{-x} + Ce^{x}$

P.I $= \dfrac{1}{(D^3 - D)} e^{x} x$

$$= e^{x} \frac{1}{(D+1)^3 - (D+1)} x$$

$$= e^{x} \frac{1}{D^3 + 3D^2 + 3D + 1 - D - 1} x \quad \left[\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\right]$$

$$= e^{x} \frac{1}{D^3 + 3D^2 + 2D} x$$

$$= e^{x} \frac{1}{2D\left[\frac{D^2}{2} + \frac{3D}{2} + 1\right]} x$$

$$= e^{x} \frac{1}{2D\left[\frac{D^2 + 3D}{2} + 1\right]} x$$

$$= \frac{e^{x}}{2D}\left(1 + \frac{D^2 + 3D}{2}\right)^{-1} x$$

$$= \frac{e^{x}}{2D}\left[1 - \left(\frac{D^2 + 3D}{2}\right) + \left(\frac{D^2 + 3D}{2}\right)^2 + \right] x$$

$$= \frac{e^{x}}{2D}\left[1 - \frac{D^2}{2} - \frac{3D}{2} + \ldots\right] x$$

$$= \frac{e^x}{2D}\left[x - \frac{D^2}{2}(x) - \frac{3D}{2}(x)\right]$$

$$= \frac{e^x}{2D}\left[x - (0) - \frac{3}{2}(1)\right] \qquad \left[\because \begin{array}{c} D(x) = 1 \\ D^2(x) = 0 \end{array}\right]$$

$$= \frac{e^x}{2D}\left[x - \frac{3}{2}\right]$$

$$= \frac{e^x}{2}\int \left(x - \frac{3}{2}\right)dx$$

$$= \frac{e^x}{2}\left[\frac{x^2}{2} - \frac{3}{2}x\right]$$

$$= \frac{xe^x}{4}(x-3)$$

$\therefore$ y = C.F + P.I

$$= A + Be^{-x} + Ce^x + \frac{xe^x}{4}(x-3)$$

4. Solve: $(D^2 - 2D + 1)y = e^x(3x^2 - 1)$

Solution: Given $(D^2 - 2D + 1)y = e^x(3x^2 - 1)$

The Auxiliary Equation is

$m^2 - 2m + 1 = 0$

$m = 1, 1$

P.I $= \dfrac{1}{(D^2 - 2D + 1)} e^x(3x^2 - 1)$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1}(3x^2 - 1)$$

$$= e^x \frac{1}{D^2+2D+1-2D-2+1}(3x^2-1)$$

$$= e^x \frac{1}{D^2}(3x^2-1)$$

$$= e^x \frac{1}{D}\int (3x^2-1)dx$$

$$= e^x \frac{1}{D}\left[\frac{3x^3}{3}-x\right]$$

$$= e^x \frac{1}{D}\left[x^3-x\right]$$

$$= e^x \int (x^3-x)dx$$

$$= e^x\left[\frac{x^4}{4}-\frac{x^2}{2}\right]$$

$\therefore$ y = C.F + P.I

$$= (A+Bx)e^x + \frac{e^x x^4}{4} - \frac{e^x x^2}{2}$$

ALITER

Given $(D^2-2D+1)y = e^x(3x^2-1)$

$$= 3e^x x^2 - e^x$$

C.F = $(A + Bx)$ e^x

P.I$_1$ $= \frac{1}{(D^2-2D+1)} 3e^x x^2$

$$= 3e^x \frac{1}{(D+1)^2-2(D+1)+1} x^2$$

$$= 3e^x \frac{1}{D^2+2D+1-2D-2+1} x^2$$

$$= 3e^x \frac{1}{D^2}(x^2)$$

$$= 3e^x \frac{1}{D}\int x^2 dx$$

$$= 3e^x \frac{1}{D}\left(\frac{x^3}{3}\right)$$

$$= e^x \frac{1}{D}(x^2) = e^x \int x^3 dx$$

P.I$_1$ $= e^x \frac{x^4}{4}$

P.I$_2$ $= \frac{1}{D^2 - 2D + 1} e^x$

$$= \frac{1}{1 - 2 + 1}$$

$$= \frac{1}{0} e^x$$

$$= x \frac{1}{2D - 2} e^x$$

$$= x \frac{1}{2 - 2} e^x$$

$$= x . \frac{1}{0} e^x$$

$$= x^2 \frac{1}{2} e^x$$

$$= \frac{x^2 e^x}{2}$$

$\therefore$ P.I = P.I$_1$ – P.I$_2$

$$= \frac{x^4 e^x}{4} - \frac{x^2}{2}$$

$y \quad = \text{C.F} + \text{P.I}$

$$= (A+Bx)\, e^x + \frac{x^4 e^x}{4} - \frac{x^2}{2} e^x$$

5. Solve $(D^2+2D-1)y = (x+e^x)2$

$= x^2 + 2xe^x + e^{2x}$

The Auxiliary Equation is $m^2 + 2m - 1 = 0$

(i.e) $\quad m = \dfrac{-2 \pm \sqrt{4+4}}{2}$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

C.F $= Ae^{(-1-\sqrt{2})} + Be^{(-1+\sqrt{2})x}$

P.I$_1$ $= \dfrac{1}{D^2+2D-1} x^2$

$$= \frac{-1}{[1-(D^2+2D)]} x^2$$

$$= -[1-(D^2+2D)]^{-1} x^2$$

$$= -[1+(D^2+2D)+(D^2+2D)^2+\ldots]x^2$$

$$= -[1+D^2+2D+D^4+4D^3+4D^2+\ldots]x^2$$

$$= -[x^2+D^2(x^2)+2D(x^2)+D^4(x^2)+4D^3(x^2)+4D^2(x^2)]$$

$$= -[x^2 + 2 + 2(2x) + 0 + 0 + 4(2)]$$

$$= -[x^2 + 2 + 4x + 8]$$

$$= -[x^2 + 4x + 10]$$

$$\text{P.I}_2 = \frac{1}{D^2 + 2D - 1} 2xe^x$$

$$= 2e^x \frac{1}{(D+1)^2 + 2(D+1) - 1}$$

$$= 2e^x \frac{1}{D^2 + 2D + 1 + 2D + 2 - 1} x$$

$$= 2e^x \frac{1}{D^2 + 4D + 2} x$$

$$= 2e^x \frac{1}{\left(\frac{D^2}{2} + \frac{4D}{2} + 1\right)} x$$

$$= 2e^x \frac{1}{2\left(1 + \left(\frac{D^2 + 4D}{2}\right)\right)} x$$

$$= e^x \left(1 + \frac{D^2 + 4D}{2}\right)^{-1} x$$

$$= e^x \left[1 - \left(\frac{D^2 + 4D}{2}\right) + \left(\frac{D^2 + 4D}{2}\right)^2 - \ldots\right] x$$

$$= e^x \left(1 - \frac{D^2}{2} - \frac{4D}{2} + \ldots.\right) x \qquad \left[\begin{array}{l} \because D(x) = 1 \\ D^2(x) = 0 \end{array}\right.$$

$$= e^x \left[x - \frac{D^2(x)}{2} - \frac{4D(x)}{2}\right]$$

$$= e^x [x - (0) - 2]$$

$= e^x(x-2)$

P.I$_3$ $= \frac{1}{D^2+2D-1}e^{2x}$

$= \frac{1}{4+4-1}e^{2x}$ [$\because$ Replace D by 2]

$= \frac{1}{7}e^{2x}$

$\therefore$ P.I = P.I$_1$ + P.I$_2$ + P.I$_3$

$= -(x^2+4x+10)+e^x(x-2)+\frac{1}{7}e^{2x}$

y = C.F + P.I

$= Ae^{(-1-\sqrt{2})x}+Be^{(1+\sqrt{2})x}-(x^2+4x+10)+e^x(x-2)+\frac{1}{7}e^{2x}$

6. Solve $(D^2-2D+2)y = e^x x^2+5+e^{-2x}$

Solution:Given $(D^2-2D+2)y = e^x x^2+5+e^{-2x}$

The Auxiliary Equation is

$m^2 - 2m + 2 = 0$

$m = 1 \pm i$

C.F $= e^x[A\cos x + B\sin x]$

P.I$_1$ $= \frac{1}{D^2-2D+2}e^x x^2$

$= e^x \frac{1}{(D+1)^2-2(D+1)+2}x^2$

$$= e^x \frac{1}{D^2+2D+1-2D-2+2} x^2$$

$$= e^x \frac{1}{D^2+1} x^2$$

$$= e^x (1+D^2)^{-1} x^2$$

$$= e^x [1-D^2+D^4-\ldots] x^2$$

$$= e^x [x^2 - D^2(x^2) + D^4(x^2) - \ldots]$$

$$= e^x [x^2 - 2]$$

$$P.I_2 = \frac{1}{D^2-2D+2} 5e^{0x}$$

$$= 5\frac{1}{(0)-2(0)+2} e^{0x}$$

$$= \frac{5}{2}$$

$$P.I_3 = \frac{1}{D^2-2D+2} e^{-2x}$$

$$= \frac{1}{(-2)^2-2(-2)+2} e^{-2x}$$

$$= \frac{1}{4+4+2} e^{-2x}$$

$$= \frac{1}{10} e^{-2x}$$

$$P.I = P.I_1 + P.I_2 + P.I_3$$

$$= e^x (x^2-2) + \frac{5}{2} + \frac{1}{10} e^{-2x}$$

$$y = C.F + P.I$$

$$= e^{x}(A\cos x + B\sin x) + e^{x}(x^2 - 2) + \frac{5}{2} + \frac{1}{10}e^{-2x}$$

7. Solve $(D^3 - 7D - 6)y = (1 + x)e^{2x}$

Solution: $(D^3 - 7D - 6)y = e^{2x} + xe^{2x}$

The Auxiliary Equation is $m^3 - 7m - 6 = 0$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & 0 & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \lfloor 0 \end{array}$$

$(m + 1)(m^2 - m - 6) = 0$

$m = -1, -2, 3$

C.F $= Ae^{-2x} + Be^{-x} + Ce^{3x}$

$$\text{P.I}_2 = \frac{1}{D^3 - 7D - 6} xe^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^3 - 7(D+2) - 6} x$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 12D + 8 - 7D - 14 - 6} x$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 5D - 12} x$$

$$= e^{2x} \frac{1}{12\left[\left(\frac{D^3 + 6D^2 + 5D}{12}\right) - 1\right]} x$$

$$= e^{2x} \frac{1}{-12\left[1 - \left(\frac{D^3 - 6D^2 + 5D}{12}\right)\right]} x$$

$$= \frac{-e^{2x}}{12}\left[1 - \frac{(D^3 + 6D^2 + 5D)}{12}\right]^{-1} x$$

$$= \frac{-e^{2x}}{12}\left[1 - \frac{(D^3 + 6D^2 + 5D) + \ldots}{12}\right]x$$

$$= \frac{-e^{2x}}{12}\left[x - \frac{(D^3 + 6D^2 + 5D)(x)}{12} + \ldots\right]$$

$$= \frac{-e^{2x}}{12}\left[x - \frac{5D(x)}{12}\right]$$

$$= \frac{-e^{2x}}{12}\left[x - \frac{5}{12}\right]$$

$$\text{P.I}_1 = \frac{1}{(D^3 - 7D - 6)}e^{2x}$$

$$= \frac{1}{8 - 14 - 6}e^{2x}$$

$$= \frac{-e^{2x}}{12}$$

$\therefore$ P.I = P.I$_1$+ P.I$_2$

$$= \frac{-e^{2x}}{12} - \frac{e^{2x}}{12}\left(x - \frac{5}{12}\right)$$

$$= \frac{-e^{2x}}{12}\left[1 + \left(x - \frac{5}{12}\right)\right]$$

$$= \frac{-e^{2x}}{12}\left(x + \frac{7}{12}\right)$$

$\therefore$ y = C.F + P.I

$$= Ae^{-2x} + Be^{-x} + Ce^{3} - \frac{e^{2x}}{12}\left(x + \frac{7}{12}\right)$$

8. Solve: $(D^2 - 2D + 1)y = xe^x \sin x$

Solution: Given $(D^2 - 2D + 1)y = xe^x \sin x$

The Auxiliary Equation is

$m^2 - 2m + 1 = 0$

C.F $= (A + Bx)e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} xe^x \sin x$$

$$= e^x \left[\frac{1}{(D+1)^2 - 2(D+1) + 1}\right] x \sin x$$

$$= e^x \left[\frac{1}{D^2 + 2D + 1 - 2D - 2 + 1}\right] x \sin x$$

$$= e^x \left[\frac{1}{D^2 + 2D - 2D - 2 + 1}\right] x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D} \int (x \sin x)\, dx$$

$$= e^x \frac{1}{D} [-x \cos x - (-\sin x)]$$

$$= e^x \left[- \int x \cos dx + \int \sin x dx\right]$$

$$= e^x [-(x \sin x + \cos x) - \cos x]$$

$u = x \quad : v = \cos x$

$u' = 1 \quad : v_1 = \sin x$

$u'' = 0 \quad : v_2 = -\cos x$

$$= e^x [-x \sin x - \cos x - \cos x]$$

$$= -e^x [x \sin x + 2 \cos x]$$

y = C.F + P.I

$$= (A + Bx)e^x - e^x (x \sin x + 2 \cos x)$$

9. Solve: $(D^2 - 4D + 4y) = 8x^2 e^{2x} \sin 2x$

Solution:Given $(D^2 - 4D + 4y) = 8x^2 e^{2x} \sin 2x$

The Auxiliary Equation is

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$ (twice)

C.F $= (A + Bx)e^{2x}$

$$\text{P.I} = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{D} \left[\int x^2 \sin 2x dx \right. \qquad \left[\begin{array}{l} By\ Berroulli\ \ formula \\ \int uv dx = uv_1 - u'v_2 + u''v_3 \end{array} \right.$$

$$= 8e^{2x} \frac{1}{D} \left[\frac{-x^2 \cos 2x}{2} + \frac{2x \sin 2x}{4} + \frac{2\cos 2x}{8} \right]$$

$$= 8e^{2x} \frac{1}{D} \left[\frac{-x^2 \cos 2x}{2} + \frac{x}{2} \sin 2x + \frac{\cos 2x}{4} \right]$$

$$= 8e^{2x} \frac{1}{D} \left[-\frac{1}{2} \int x^2 \cos 2x dx + \frac{1}{2} \int x \sin 2x dx + \frac{1}{4} \int \cos 2x dx \right]$$

$$= 8e^{2x} \left[\frac{-1}{2} \left(\frac{x^2 \sin 2x}{2} + \frac{2x \cos 2x}{4} - \frac{2 \sin 2x}{8} + \frac{1}{2} \left[\frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} \right] + \frac{1}{8} \sin 2x \right) \right]$$

$$= 8e^{2x} \left[\frac{-x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + \frac{1}{8} \sin 2x \right]$$

$$= 8e^{2x}\left[\frac{-x^2 \sin 2x}{4} - \frac{2x\cos 2x}{4} + \frac{3\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\frac{-x^2 \sin 2x}{3} - \frac{x\cos 2x}{4} + \frac{3\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\frac{-2x^2 \sin 2x}{8} - \frac{4x\cos 2x}{8} + \frac{3\sin 2x}{8}\right]$$

$$= 8e^{2x}[-2x^2 \sin 2x - 4x\cos 2x + 3\sin 2x]$$

$$= 8e^{2x}[(3-2x^2)\sin 2x - 4x\cos 2x]$$

10. Solve: $\frac{d^2y}{dx^2} - 4y = x\sin hx$

Solution:The Auxiliary Equation is

$m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F $= Ae^{2x} + Be^{-2x}$

P.I $= \frac{1}{D^2-4} x\sin hx$

$$= \frac{1}{D^2-4} x\left[\frac{e^x - e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}e^x x - \frac{1}{D^2-4}e^{-x}x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D+1-4}x - e^{-x}\frac{1}{D^2-2D+1-4}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{D^{2}+2D-3}x-e^{-x}\frac{1}{D^{2}-2D-3}x\right]$$

$$= \frac{1}{2}\left[e^{x}\frac{1}{3\left(\frac{D^{2}}{3}+\frac{2D}{3}-1\right)}x-e^{-x}\frac{1}{3\left(\frac{D^{2}}{3}-\frac{2D}{3}-1\right)}x\right]$$

$$= \frac{1}{6}\left[e^{x}\frac{1}{-\left[1-\left(\frac{D^{2}}{3}+\frac{2D}{3}\right)\right]}x-e^{-x}\frac{1}{-\left[1-\left(\frac{D^{2}}{3}-\frac{2D}{3}\right)\right]}x\right]$$

$$= -\frac{1}{6}\left\{e^{x}\left[1-\left(\frac{D^{2}}{3}+\frac{2D}{3}\right)\right]^{-1}x-e^{-x}\left[1-\left(\frac{D^{2}}{3}-\frac{2D}{3}\right)\right]^{-1}x\right\}$$

$$= -\frac{1}{6}\left\{e^{x}\left[1+\frac{D^{2}}{3}+\frac{2D}{3}+\ldots\right]x-e^{-x}\left[1+\frac{D^{2}}{3}-\frac{2D}{3}+\ldots\right]x\right\}$$

$$= \frac{-1}{6}\left\{e^{x}\left[x+\frac{D^{2}}{3}(x)+\frac{2D}{3}(x)+\ldots\right]-e^{-x}\left[x+\frac{D^{2}}{3}(x)-\frac{2D}{3}(x)+\ldots\right]\right\}$$

$$= \frac{-1}{6}\left\{e^{x}\left[x+\frac{2}{3}\right]-e^{-x}\left[x-\frac{2}{3}\right]\right\}$$

$$= \frac{-1}{6}\left\{x(e^{x}-e^{-x})+\frac{2}{3}(e^{x}+e^{-x})\right\}$$

$$= \frac{-1}{6}\left\{2x\sin\ hx+\frac{2}{3}(2\cos hx\right\}$$

$$= \frac{-1}{6}\left[2x\sin\ hx+\frac{4}{3}\cos hx\right]$$

$$= -\frac{x}{3}\sin\ hx-\frac{2}{9}\cos hx$$

$\therefore$ y $=$ C.F + P.I

$$= \quad Ae^{2x} + Be^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

11. Solve: $(D^3 - 3D^2 + 3D - 1)y = e^{-3}x^3$

Solution: Given $(D^3 - 3D^2 + 3D - 1)y = e^{-x}x^3$

The Auxiliary Equation is

$m^3 - 3m^2 + 3m - 1 = 0$

$(m - 1)^3 = 0$

$m = 1, 1, 1$ (thrice)

C.F $= (A + Bx + Cx^2)e^x$

$$\text{P.I} \quad = \frac{1}{(D-1)^3} e^{-x}x^3$$

$$= e^{-x} \frac{1}{(D-1-1)^3} x^3$$

$$= e^{-x} \frac{1}{(D-2)^3} x^3$$

$$= e^{-x} \frac{1}{2^3\left[\frac{D}{2} - 1\right]^3} x^3$$

$$= \frac{e^{-x}}{-8\left[1 - \frac{D}{2}\right]^3} x^3$$

$$= \frac{e^{-x}}{-8}\left[1 - \frac{D}{2}\right]^{-3} x^3$$

$$= \frac{e^{-x}}{-8}\left[1 + 3\frac{D}{2} + \frac{3(3+1)}{2!}\frac{D^2}{4} + \frac{3(3+1)(3+2)}{3!}\frac{D^3}{8} + \ldots\right]$$

$$\left[\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+2)(n+2)}{3!}x^3 + \ldots\right.$$

$$= \frac{e^{-3}}{-8}\left[x^3 + \frac{3}{2}D(x^3) + \frac{3.4}{2}\frac{D^2(x^2)}{4} + \frac{3.4(5)}{8}\frac{D^3}{8}(x^3) + \ldots.\right]$$

$$= \frac{e^{-3}}{-8}\left[x^3 + \frac{3}{2}(3x^2) + \frac{3}{2}(6x)\frac{5}{4}(6)\right]$$

$$= \frac{e^{-3}}{-8}\left[x^3 + \frac{9}{2}x^2 + 9x + 15\right]$$

P.I $= \dfrac{e^{-x}}{-16}\left[2x^3 + 9x^2 + 18x + 15\right]$

y = C.F + P.I

$$= (A + Bx + Cx^2)e^x - \frac{e^{-x}}{16}[2x^3 + 9x^2 + 18x + 15]$$

12. Solve: $(D^2 - 4)y = x\sin h2x$

Solution: Given $(D^2 - 4)y = x\sin h2x$

$$= x\left[\frac{e^{2x} - e^{-2x}}{2}\right]$$

$$= \frac{xe^{2x}}{2} - \frac{xe^{-2x}}{2}$$

The Auxiliary Equation is

$m^2 - 4 = 0$

$\Rightarrow m^2 = 4$

$\therefore m = \pm 2$

C.F. $= Ae^{-2x} + Be^{2x}$

$$\text{P.I.} = \frac{e^{2x}}{8}\frac{1}{D}\left[x-\frac{D}{4}(x)+\frac{D^2}{16}(x)....\right]$$

$$= \frac{e^{2x}}{8}\frac{1}{D}\left[x-\frac{1}{4}\right]$$

$$= \frac{e^{2x}}{8}\left[\frac{x^2}{2}-\frac{x}{4}\right]$$

$$= \frac{xe^{2x}}{8}\left[\frac{2x-1}{4}\right]$$

$$= \frac{xe^{2x}}{32}[2x-1]$$

$$\text{P.I}_1 = \frac{1}{(D^2-4)}\frac{x}{2}e^{-2x}$$

$$= \frac{e^{2x}}{2}\frac{1}{(D-2)^2-4}x$$

$$= \frac{e^{-2x}}{2}\frac{1}{D^2+4D+4-4}x$$

$$= \frac{e^{-2x}}{2}\frac{1}{(D^2-4D)}x$$

$$= \frac{e^{-2x}}{8}\frac{1}{-4D\left[1-\frac{D}{4}\right]}x$$

$$= \frac{e^{-2x}}{-8}\frac{1}{D}\left[1-\frac{D}{4}\right]^{-1}x$$

$$= \frac{e^{-2x}}{-8D}\left[1+\frac{D}{4}+\left(\frac{D}{4}\right)^2+...\right]x$$

$$= \frac{e^{-2x}}{-8D}\left[x + \frac{D}{4}(x) + \frac{D^2}{16}(x) + ...\right]$$

$$= \frac{e^{-2x}}{-8D}\left[x + \frac{1}{4}\right]$$

$$= \frac{-e^{-2x}}{8D}\int \left(x + \frac{1}{4}\right)dx = \frac{-e^{-2x}}{8}\left[\frac{x^2}{2} + \frac{x}{4}\right]$$

$$= \frac{-xe^{-2x}}{32}[2x+1]$$

P.I $= P.I_1 + P.I_2$

$$= \frac{xe^{2x}}{32}(2x-1) - \frac{xe^{2x}}{32}(2x+1)$$

$$y = Ae^{-2x} + Be^{2x} + \frac{xe^{2x}}{32}(2x-1) - \frac{xe^{2x}}{32}(2x+1)$$

Exercise Problems

1. $(D^2 - 2D + 2)y = x^2e^{3x}$
2. $(D^2 + 4D + 4)y = e^{-x}x^2$
3. $(D^2 - 4D - 5)y = e^{-2x}(x+1)$
4. $(D^2 + 8D + 15)y = e^{3x}x$
5. $(D^2 + 9)y = (x^2 + 1)e^{3x}$
6. $(D^2 - 2D + 1)y = x^2 3x$
7. $(D^2 - 2D + 1)y = x^2e^{x}$
8. $(D^2 + 2D + 1)y = \frac{e^{-x}}{x^2}$
9. $(D^2 - 4)y = x^2 \cos h2x$

Answers

1. $y = e^{x}(A\cos x + B\sin\ x) + \frac{1}{125}e^{3x}(25x^{2} - 40x + 22)$

2. $y = (Ax + B)e^{-2x} + e^{-x}(x^{2} - 4x + 6)$

3. $y = Ae^{5x} + Be^{-x} + \frac{e^{-2x}}{7}\left[x + \frac{15}{7}\right]$

4. $y = Ae^{-3x} + Be^{-5x} + \frac{e^{3x}}{48}\left[x + \frac{7}{24}\right]$

5. $y = A\cos 3x + B\sin\ 3x + \frac{e^{3x}}{18}\left[x^{2} - \frac{2x}{3} + \frac{10}{9}\right]$

6. $y = (A + Bx)e^{x} + \frac{e^{3x}}{4}\left[x^{2} - 2x + \frac{3}{2}\right]$

7. $y = (A + Bx)e^{x} + \frac{x^{4}e^{x}}{12}$

8. $y = (A + Bx)e^{-x} - e^{-x}\log x$

9. $y = Ae^{-2x} + Be^{2x} + \frac{x}{96}(8x^{2}\sinh\ 2x - 6x\cosh 2x + 3\sinh\ 2x)$

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLES CO-EFFICIENTS

I - Cauchy Euler Type

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}x\frac{dy}{dx} + a_n y = X \qquad \ldots(1)$$

Where $a_0, a_1, ..., a_n$ are constants and X is a function of x is called Euler's homogeneous linear differential equation.

Equation (1) can be reduce to constant co-efficient by means of transformation

$x = e^x$(or) $z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$$

$$= \frac{1}{x}\frac{dy}{dz}[\because x = e^z \Rightarrow z = \log x]$$

$$x\frac{dy}{dx} = D'y \qquad \text{where } D' = d/dz \qquad (2)$$

Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x}\frac{d^2y}{dz^2} - \frac{d}{x}\frac{-1}{x^2}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$= (D'^2 - D')\,y \qquad \text{where } D' = d/dz$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)y \qquad \ldots(3)$$

Similarly $x^3\dfrac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$ …(4)

and so on. Substituting (2), (3), (4) and so on in (1), differential equation of variables co-efficients reduced to constant co-efficients and can be solved by any one of the known methods.

II – Legendre's Type

An equation of the form

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n y = X \qquad \ldots(1)$$

where $a_0, a_1, \ldots, a_n$ are constants and X is a function of x is called Legendre's linear differential equation.

Equation (1) can be reduced to linear differential equation with constant co-efficient by the substitution.

$ax + b = e^z \Rightarrow z = \log(az + b)$

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dz}$$

$(ax + b) D = aD'$, where $D' = d / dz$

Similarly $(ax + b)^2 D^2 = a^2 D(D'-1)$ and so on.

Examples

1. Solve $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = 0$ [S.U May' 09, Dec' 10, May' 12]

Solution: Given $[x^2D^2 - xD + 1]\, y = 0$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$ where $D' = d / dz$

Equation (1) reduces to

$(D'(D'-1) - D'+1)\, y = 0$

$(D'^2 - 2D' + 1)\, y = 0$

A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

C.F $= y = (Az + B)\, e^z$

$y = (A \log x + B)\, x$

2. Solve $xy'' + y' + \dfrac{y}{x} = 0$ [S.U May' 07, May' 11]

Solution: Given $\left[xD^2 + D + \dfrac{1}{x} \right] y = 0$

[Multiply by x]

$[x^2D^2 + xD + 1]\, y = 0$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$, where $D' = d/dz$

(1) reduces to $[D'(D' - 1) + D' + 1]\, y = 0$

$[D'^2 + 1]\, y = 0$

A.E : $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

C.F $= y = A \cos z + B \sin z$

$y = A \cos(\log x) + B \sin(\log x)$

3. Solve $(x^2D^2 - 3xD + 4)\, y = 0$ [S.U Dec' 07]

Solution: Given $(x^2D^2 - 3xD + 4)\, y = 0$ ...(1)

$x^2D^2 = D'(D' - 1)$

$xD = D'$, Where $D' = d/dz$

(1) reduces to $[D'(D'-1) - 3D' + 4]\, y = 0$

$(D'^2 - 4D' + 4)\, y = 0$

A.E : $m^2 - 4m + 4 = 0$

$m = 2, 2$ (repeated roots)

C.F = $y = (Az + B)\, e^{2z}$

$y = (A \log x + B)\, e^{2\log x}$

$= (A \log x + B)\, x^2$

4. Solve $xy'' + y' = 0$ [S.U Dec '08]

Solution: Given $xy'' + y' = 0$

multiply by $x \Rightarrow x^2y'' + xy' = 0$ …(1)

$(x^2D^2 + xD)\, y = 0$

A.E : $(D'(D' - 1) + D')\, y = 0$ $[\because x^2D^2 = D'(D' - 1)\; xD = D']$

$D'^2 y = 0$

A.E: $m^2 = 0$

$m = 0, 0$

y = C.F. = $(Az + B)e^{0z}$

y = $A \log x + B$

5. Convert the Euler equation $(x^2D^2 - 7xD + 12)\, y = x^2$ into a differential equation with constant co-efficients [S.U. Dec '09]

Solution: Given $(x^2D^2 - 7xD + 12)\, y = x^2$ ...(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$, where $D' = \dfrac{dy}{dz}$

Equation (1) reduces to

$(D'(D' - 1) - 7D' + 12)\, y = (e^z)^2$

$(D'^2 - 8D' + 12)\, y = e^{2z}$ …(2)

Equation (2) is a linear differential equation with constant co-efficients.

6. Transform $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x$ into linear differential equation with constant co-efficients [S.U. Dec '11]

Solution: Given $[x^2D^2 - xD + 2]y = x$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$

Equation (1) reduces to

$[D'(D' - 1) - D' + 2]y = e^z$

$[D'^2 - 2D' + 2]y = e^z$ …(2)

Equation (2) is a linear differential equation with constant co-efficients.

7. Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + \dfrac{2y}{x} = 0$ [S.U.Dec' 11]

Solution: Given $\left[xD^2 + 4D + \dfrac{2}{x}\right]y = 0$

Multiply by x

$$[x^2D^2 + 4xD + 2]y = 0$$

Put $x = e^z \Rightarrow z = \log x$

$$[D'(D'-1) + 4D' + 2]y = 0$$

$$[D'^2 + 3D' + 2]y = 0$$

A.E: $m^2 + 3m + 2 = 0$

$(m + 2)(m + 1) = 0$

$m = -1, -2$

C.F $= y = Ae^{-z} + Be^{-2z}$

$$y = \frac{A}{x} + \frac{B}{x^2} \quad [\because x = e^z]$$

8. Solve $(x^2D^2 - 2xD - 4)y = 32 (\log x)^2$ [S.U May' 09, May' 11]

Solution: Given $(x^2D^2 - 2xD - 4)y = 32 (\log x)^2$

Put $x = e^z \Rightarrow z = \log x$

$$x^2D^2 = D'(D'-1)$$

$$xD = D', \quad D' = \frac{d}{dz}$$

$$[D'(D'-1) - 2D' - 4]y = 32z^2$$

$$[D'^2 + 3D' - 4]y = 32z^2 \qquad \ldots(2)$$

Equation (2) is a linear differential equation with constant co-efficients.

A.E: $m^2 - 3m - 4 = 0$

$(m - 4)(m + 1) = 0$

$m = 4, -1$

C.F $= Ae^{4Z} + Be^{-z}$

$$\text{P.I.} = \frac{1}{D'^2 - 3D' - 4} 32z^2$$

$$= 32 \frac{1}{-4\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]} z^2$$

$$= -8\left[1 - \left(\frac{D'^2 - 3D'}{4}\right)\right]^{-1} z^2$$

$$= -8\left[1 + \left(\frac{D'^2 - 3D'}{4}\right) + \left(\frac{D'^2 - 3D'}{4}\right)^2 + \ldots\right] z^2 \qquad [\because (1-x)^{-1} = 1 + x + x^2 + \ldots]$$

$$= -8\left[z^2 + \frac{D'^2}{4}(z^2) - \frac{3}{4}D'(z^2) + \frac{9D'^2}{16}(z^2)\right]$$

$$\{\because D'(z^2) = 2z, \quad D'^2(z^2) = 2, \quad D'^2(z^2) = 0\}$$

$$= -8\left[z^2 + \frac{2}{4} - \frac{3}{4}(2z) + \frac{9}{16}(2)\right]$$

$$\text{P.I} = -8\left[z^2 - \frac{3}{2}z + \frac{13}{8}\right]$$

$\therefore$ General solution y = C.F + P.I

$$= Ae^{4z} + Be^{-z} - 8\left[z^2 - \frac{3z}{2} + \frac{13}{8}\right]$$

$$y = Ax^4 + \frac{B}{x} - [8(\log x)^2 - 12(\log x) + 13]$$

9. Solve $(x^2D^2 + xD + 1)y = \log x \sin\,(\log x)$ [S.U.Dec '07]

Solution: Given $(x^2D^2 + xD + 1)y = \log x \sin\,(\log x)$ …(1)

Put $x = e^z \Rightarrow z = \log x$

$x^2D^2 = D'(D'-1)$

$xD = D'$

(1) reduces to

$(D'(D'-1) + D' + 1)y = z \sin z$

$(D'^2 + 1)y = z \sin z$

A.E: $m^2 + 1 = 0$

$m = \pm i$

C.F $= A \cos z + B \sin z$

P.I $= \dfrac{1}{D'^2 + 1} z \sin z$

$= \dfrac{1}{D'^2 + 1}$ I.P. of $e^{iz} z$

$= I.P.\ of\ e^{iz} \dfrac{1}{(D'+i)^2 + 1} z$

$= I.P.\ of\ e^{iz} \dfrac{1}{D'^2 + 2iD'} z$

$= I.P.\ of\ e^{iz} \dfrac{1}{2iD'\left(1 + \dfrac{D'}{2i}\right)} z$

$= I.P.\ of\ e^{iz} \dfrac{-i}{2D'} \left(1 - \dfrac{iD'}{2}\right)^{-1} z$

$= I.P.\ of\ e^{iz} \dfrac{-i}{2D^1} \left[1 - \dfrac{iD'}{2} + \left(\dfrac{iD'}{2}\right)^2 + \ldots.\right] z$

$= I.P.\ of\ e^{iz} \dfrac{-i}{2D'} \left[z + \dfrac{i}{2}\right] \quad \{\because D'(z) = 1 \quad D'^2(z) = 0\}$

$$= I.P.\ of\ e^{iz}\frac{-1}{2}\left[\frac{z^2}{2}+\frac{iz}{2}\right] \qquad \{\because \frac{1}{D'} = Int.w.r.t.z\}$$

$$= I.P.\ of (\cos z + \sin z)\left(\frac{-iz^2}{4}+\frac{z}{4}\right)$$

P.I $= \dfrac{z^2}{4}\cos z + \dfrac{z}{4}\sin z$

y = C.F + P.I

General solution: $y = A\cos z + B\sin z - \dfrac{z^2}{4}\cos z + \dfrac{z}{4}\sin z$

$$y = A\cos(\log x) + B\sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

10. Solve $(2x+3)^2\dfrac{d^2y}{dx^2} - 2(2x+3)\dfrac{dy}{dx} - 12y = 6x$ [S.U. Dec '09]

Solution: The given eqn is a Legendre's linear differential equation

Put $(2x+3) = e^z \Rightarrow z = \log(2x+3)$

$$(2x+3)\frac{dy}{dx} = 2\frac{dy}{dz} = 2D'y$$

$$(2x+3)^2\frac{d^2y}{dx^2} = 4D'(D'-1)y$$

Equation reduces to

$$[4D'(D'-1) - 2(2D') - 12]y = 6\left[\frac{e^z-3}{2}\right] \quad [\because 2x+3 = e^z]$$

$$[4D'^2 - 8D' - 12]y = 3[e^z - 3]$$

$$[D'^2 - 2D' - 3]y = \frac{3}{4}(e^z - 3) \qquad \ldots(1)$$

Equation (1) is a linear differential equation with constant co-efficients

A.E: $m^2 - 2m - 3 = 0$

$(m - 3)(m + 1) = 0$

$m = 3, -1$

$\therefore$ C.F $= Ae^{3z} + Be^{-z}$

$$\text{P.I} = \frac{1}{D'^2 - 2D' - 3}\frac{3}{4}(e^z - 3)$$

$$= \frac{3}{4}\frac{1}{D'^2 - 2D' - 3}(e^z - 3e^{0z})$$

$$= \frac{3}{4}\left[\frac{-1}{4}e^z + 1\right] \quad \text{[subst. D}' = 1 \text{ in I}^{\text{st}} \text{ and D}' = 0 \text{ in II}^{\text{nd}}\text{]}$$

y = C.F + P.I

General Solution

$$y = Ae^{3z} + Be^{-z} - \frac{3}{16}e^z + \frac{3}{4}$$

$$y = A(2x+3)^3 + B(2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4}$$

11. Solve $(x^2D^2 + 4xD + 2)y = x\log x$ [S.U. Dec' 06, May' 11]

Solution: Given $(x^2D^2 + 4xD + 2)y = x\log x$ …(1)

Put $x = e^Z \Rightarrow z = \log x$

$x^2D^2 = D'(D' - 1)$

$xD = D'$

$(1) \Rightarrow (D'(D'-1)+4D'+2)y = ze^z$

$(D'^2+3D'+2)y = ze^z$

A.E: $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -1, -2$

C.F $= Ae^{-z} + Be^{-2z}$

$$\text{P.I.} = \frac{1}{D'^2+3D'+2} ze^z$$

$$= e^z \frac{1}{(D'+1)^2+3(D'+1)+2} z$$

$$= e^z \frac{1}{D'^2+5D'+6} z$$

$$= \frac{e^z}{6} \frac{1}{\left(1+\frac{D'^2+5D'}{6}\right)} z$$

$$= \frac{e^z}{6}\left[1+\frac{D'^2+5D'}{6}\right]^{-1} z$$

$$= \frac{e^z}{6}\left[1-\left(\frac{D'^2+5D'}{6}\right)+\left(\frac{D'^2+5D'}{6}\right)-\dots\right]z$$

$$\text{P.I} = \frac{e^z}{6}\left[z-\frac{5}{6}\right] \qquad [\because \quad D'(z)=1, D'^2(z)=0]$$

y $=$ C.F + P.I

General Solution

$$y = Ae^{-z}+Be^{-2z}+\frac{e^z}{6}\left[z-\frac{5}{6}\right]$$

$$y = \frac{A}{x} + \frac{B}{x^2} + \frac{x}{6}\left(\log x - \frac{5}{6}\right)$$

12. Solve $(x^2D^2 - xD + 4)y = x^2 \sin(\log\ x)$ [S.U May' 10]

Solution: Given

$$(x^2D^2 - xD + 4)y = x^2 \sin(\log\ x) \qquad \ldots(1)$$

Put $x = e^z, z = \log x$

$x^2D^2 = D'(D'-1)$

$x^2D^2 = D'(D'-1)$

Equation (1) reduces to

$(D'(D-1) - D' + 4)y = e^{2Z} \sin\ z$

$(D'^2 - 2D' + 4)y = e^{2Z} \sin\ z$

A.E: $m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{2 \pm 2i\sqrt{3}}{2}$$

$m = 1 \pm i\sqrt{3}$

C.F $= e^z(A\cos\sqrt{3}z + B\sin\sqrt{3}z)$

$$\text{P.I} = \frac{1}{D'^2 - 2D' + 4} e^{2z} \sin\ z$$

$$= I.P.\ of\ \frac{1}{D'^2 - 2D' + 4} e^{2z} e^{iz}$$

$$= I.P.\ of\ \frac{1}{D'^2 - 2D' + 4} e^{(2+i)z}$$

$$= I.P.\ of\ \frac{1}{(2+i)^2 - 2(2+i) + 4} e^{(2+i)z} \quad \text{Replace } D' = 2 + i$$

$$= I.P.\ of\ \frac{1}{3+2i} e^{2z} e^{iz}$$

$$= I.P.\ of\ \frac{(3-2i)}{13} e^{2z} (\cos z + i \sin z)$$

P.I $= \dfrac{e^{2z}}{13}(3\sin z - 2\cos z)$

y = C.F + P.I

$$= e^{z}(A\cos\sqrt{3}z + B\sin\sqrt{3}z) + \frac{e^{2z}}{13}(3\sin z - 2\cos z)$$

General solution

$$y = x\left(A\cos\left(\sqrt{3}\log x\right) + B\sin\left(\sqrt{3}\log x\right)\right) + \frac{x^2}{13}(3\sin(\log x) - 2\cos(\log x))$$

13. Solve $x^2 \dfrac{d^2 y}{dx^2} + 3x\dfrac{dy}{dx} + 5y = x\cos(\log x)$ [S.U May' 08]

Solution: Put $e^{z} = x \Rightarrow z = \log x$

$$x^2 D^2 = D'(D'-1)$$

$$xD = D',\ D' = \frac{d}{dz}$$

Then the given equation becomes

$$[D'(D'-1) + 3D' + 5]y = e^{z}\cos z$$

$$xD = D',\ D' = \frac{d}{dz}$$

A.E: $m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$m = -1 \pm 2i$$

$$\text{C.F} = e^{-z}(A\cos 2z + B\sin 2z)$$

$$\text{P.I} = \frac{1}{D'^2 + 2D'^2 + 5} e^z \cos z$$

$$= R.P \frac{1}{D'^2 + 2D'^2 + 5} e^z e^{iz}$$

$$= R.P \frac{1}{D'^2 + 2D' + 5} e^{(1+i)z}$$

$$= R.P \frac{1}{(1+i)^2 + 2(1+i) + 5} e^{(1+i)z}$$

$$= R.P \frac{1}{7+4i} e^z e^{iz}$$

$$= R.P \frac{(7-4i)}{65} e^z (\cos z + i\sin z)$$

$$= \frac{e^z}{65}(7\cos z + 4\sin z)$$

ALITER

$$\text{P.I} = \frac{1}{D'^2 + 2D' + 5} e^z \cos z$$

$$= e^z \frac{1}{(D'+1)^2 + (D'+1) + 5} \cos z$$

$$= e^z \frac{1}{D'^2 + 4D' + 8} \cos z \quad \text{Replace } D'^2 = -1$$

$$= e^{z}\,\frac{(4D'-7)}{(4D'+7)(4D'-7)}\cos z$$

$$= e^{z}\,\frac{[4(-\sin z)-7\cos z]}{(4D')^{2}-7^{2}}$$

$$= \frac{e^{z}}{65}(4\sin z+7\cos z)$$

y = C.F + P.I

$$= e^{-z}(A\cos 2z+B\sin 2z)+\frac{e^{z}}{65}(7\cos z+4\sin z)$$

$$y = \frac{1}{x}(A\cos(2(\log x))+B\sin(2(\log x))+\frac{x}{65}[4\sin(\log x)+7\cos(\log x)]$$

14. Solve $(x^2D^2-xD+2)y = x\log x$ [S.U. Dec 11]

Solution: Given $(x^2D^2-xD+2)y = x\log x$ …(1)

Put $x = e^{z} \Rightarrow z = \log x$

$x^2D^2 = D'(D'-1)$

$xD = D'$

Equation (1) reduces to

$(D'(D'-1)-D'+2)y = ze^{z}$

$(D'^2-2D'+2)y = ze^{z}$

A.E: $m^2 - 2m + 2 = 0$

$$m = \frac{2\pm\sqrt{4-8}}{2}$$

$$= \frac{2\pm 2i}{2}$$

$m \quad = 1 \pm i$

C.F. $= e^z (A \cos z + B \sin z)$

$$\text{P.I.} = \frac{1}{D'^2 - 2D' + 2} ze^z$$

$$= e^z \frac{1}{(D'+1)^2 - 2(D'+1) + 2} z$$

$$= e^z \frac{1}{D'^2 + 1} z$$

$$= e^z (1 + D'^2)^{-1} z$$

$$= e^z (1 - D'^2 + (D'^2)^2 - \ldots) z$$

$$\text{P.I} = ze^z [\Rightarrow D'(z) = 1, D'^2(z) = 0]$$

$y \quad = \text{C.F} + \text{P.I}$

$$= e^z (A\cos + B\sin z) + ze^z$$

$$y = x(A\cos(\log x) + B\sin(\log x) + x\log x$$

15. Solve $x^2 y'' + xy' + y = \cos(2\log x)$ [S.U Dec 08]

Solution: Given $(x^2 D^2 + xD + 1)y = \cos(2\log x)$

Put $x = e^z \Rightarrow z = \log x$

$$x^2 D^2 = D'(D' - 1)$$

$$xD = D', \quad D' \frac{d}{dz}$$

$$(D'(D' - 1) + D' + 1)y = \cos 2z$$

$$(D'^2 + 1)y = \cos 2z$$

A.E. $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

C.F $= A\cos z + B\sin z$

P.I. $= \dfrac{1}{D'^2+1}\cos 2z$

Replace $D'^2 = -4$

$= \dfrac{-1}{3}\cos 2z$

$y = \text{C.F} + \text{P.I}$

$= A\cos z + B\sin z - \dfrac{1}{3}\cos 2z$

$y = A\cos(\log x) + B\sin(\log x) - \dfrac{1}{3}\cos(2\log x)$

Exercise

1. Convert the equation $xy'' - 3y' + x^{-1}y = x^2$ as a linear equation with constant co-efficients.

2. Solve: $x^3y''' + 3x^2y' + xy' + y = 0$

3. Solve: $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}$

4. Solve: $(x^2D^2 - xD + 1)y = \left(\dfrac{\log x}{x}\right)^2$

5. Solve: $(x^2D^2 + xD + 1)y = \sin(2\log x)\sin(\log x)$

6. Solve: $(x+1)^2\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} + y = 4\cos\log(x+1)$

7. Solve: $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

8. Solve: $(x^2D^2 + xD - 9)y = \sin^3(\log x)$

9. Solve: $(x^2D^2 - 3xD - 5)y = \sin(\log x)$

10. Solve: $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + 4y = x^2 + 2\log x$

11. Solve: $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x^2+1)^2$

12. Solve: $x^2\dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} + 25y = (\log x)^2$

13. Solve: $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2\log x$

14. Solve: $\left(D^2 - \dfrac{D}{x} + \dfrac{1}{x^2}\right)y = \dfrac{2\log x}{x^4}$

15. Find Particular integral for $(x^3D^3 + 3x^2D^2 + xD + 1)y = \sin(\log x)$

16. Solve: $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2$

17. Solve: $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^2 + \sin(5\log x)$

18. Solve: $(x^2D^2 + 4xD + 2)y = x = \dfrac{1}{x}$

19. Solve: $x^3y''' + 2x^2y'' - xy' + y = \log x$

20. Solve: $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = \dfrac{5}{x^2}$

Answer

1. $(D'^2 - 4D' + 1)y = e^{3z}$

2. $y = \frac{A}{x} + \sqrt{x}\left[Bos\left(\frac{\sqrt{3}}{2}\log x\right) + C\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$

3. $y = \frac{A}{x} + \frac{B}{x^2} + \frac{x^2}{12} - \frac{\log x}{x^2}$

4. $y = (A\log x + B)x + \frac{1}{9x^2}\left((\log x)^2 + \frac{4}{3}(\log x) + \frac{2}{3}\right)$

5. $y = A\cos(\log x) + B\sin(\log x) - \frac{1}{16}\sin(3\log x) - \frac{1}{4}\log x\cos(\log x)$

6. $y = A\cos(\log(x+1)) + B\sin(\log(x+1)) + 2\log(x+1)\sin\log(x+1)$

7. $y = A(3x+2)^2 + \frac{B}{(3x+2)^2} + \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$

8. $y = Ax^3 + \frac{B}{x^3} - \frac{3}{40}(\sin(\log x)) + \frac{1}{72}\sin(3\log x)$

9. $y = Ax^5 + \frac{B}{x} + \frac{2\cos(\log x) - 3\sin(\log x)}{26}$

10. $y = \frac{1}{\sqrt{x}}\left[A\cos\left(\frac{\sqrt{15}}{2}\log x\right) + B\sin\left(\frac{\sqrt{15}}{2}\log x\right)\right] + \frac{x^2}{10} + \frac{1}{2}\left(\log x - \frac{1}{4}\right)$

11. $y = Ax^4 + \frac{B}{x^5} + \frac{x^4\log x}{9} - \frac{x^2}{7} - \frac{1}{20}$

12. $y = \frac{1}{x^4}[A\cos(3\log x) + B\sin(3\log x)] + \frac{1}{25}\left[(\log x)^2 - \frac{16}{25}\log x + \frac{78}{625}\right]$

13. $y = \frac{A}{x^3} + Bx^3 - \frac{x^2}{2}\left(\log x + \frac{2}{3}\right)$

14. $y = x(A\log x + B) + \frac{2}{9x^2}\left(\log x + \frac{2}{3}\right)$

15. $P.I = \frac{1}{2}(\sin(\log\ x) + \cos(\log\ x))$

16. $y = x^2(\log x + B) + x^2(\log x)^2$

17. $y = Ax + Bx^2 + x^2 \log x + \frac{15\cos(5\log x) - 23\sin(5\log x)}{754}$

18. $y = \frac{A}{x} + \frac{B}{x^2} + \frac{x}{6} + \frac{\log x}{x}$

19. $y = \frac{A}{x} + (B\log x + C) + \log x + 1$

20. $y = Ax^3 + \frac{B}{x^3} - \frac{1}{x^2}$

Simultaneous linear Differential Equations with constant co-efficients

Simultaneous linear Differential Equations:

The system of differential equations which consist of one independent and two or more dependent variables.

To solve such system completely, we must have as many simultaneous equations as the number of dependent variables.

Here we consider only the 1st order simultaneous linear differential equations.

Let *x*, *y* be the two dependent variables and'*t*' be the independent variable, then consider the system like

$$f_1(D)x + g_1(D)y = \Phi(t) \qquad \ldots(1)$$

$$f_2(D)x + g_2(D)y = \Phi(t) \qquad \ldots(2)$$

$$D = \frac{d}{dt}$$

Where f_1, f_2, g_1, g_2 are polynominals in D.

Methods to solve the simultaneous equations

Solving simultaneous differential equation is based on the process of elimination of the variables which is applied in solving the simple algebraic equations.

Method 1

The method of solution is to eliminate the dependent variables *x* or *y* between the two given equations. First getting an equation in the dependent variable and then solve the equations by the methods as used already.

After getting the solution either for x or y, substitute the solution either in (1) or in (2) to get the solution for the other dependent variable.

Note (1)

The number of arbitrary constants in the solution of the system

$$f_1(D)x + g_1(D)y = \varphi_1(t)$$

$$f_2(D)x + g_2(D)y = \varphi_2(t)$$

is equal to the degree of D in the determinant $\begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix}$ (i.e.) The number of arbitrary constants in the solution of the system is equal to the number of dependent variables appeared in the system.

Example:

Consider the system $\dfrac{dx}{dt} + y = \sin t; \dfrac{dy}{dt} + x = \cos t.$ Here the number of arbitraryconstants is equal to the number of dependent variables in the system

Let $D = \dfrac{d}{dt}$, then the above system is

$$Dx + y = \sin t \qquad \ldots(1)$$

$$Dx + x = \cos t \qquad \ldots(2)$$

Now $\begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} = \begin{vmatrix} D & 0 \\ 0 & D \end{vmatrix} = D^2,$ Here the degree is 2

$\therefore$ We have two arbitrary constants in the solution of the system.

Method 2

Elimination of *x* in the equations (1) and (2) gives the solution of *y*, in which we have two arbitrary constants.

Likewise elimination of *y* in the equations (1) and (2) gives the solution.

But according to the rule (in Note), we must have only two arbitrary constants. Therefore we can write the relation between the arbitrary constants. (i.e) one constant can be expressed interms of other.

Note (2)

Also, we can eliminate first *y* and then using *y* we may find out the solution of *x*.

Problems

1. Solve: $\frac{dx}{dt}+y-1=\sin t; \frac{dy}{dt}+x=\cos t$ (S.U 2006)

Solution:Given $\frac{dx}{dt}+y-1=\sin t; \frac{dy}{dt}+x=\cos t$

Let $D=\frac{d}{dt}$, then the above equations are $Dx+y-1=\sin t$

(i.e) $Dx+y=\sin t+1$ …(1)

And $Dy+x=\cos t$ …(2)

First we can eliminate *x* from (1) and (2)

$(2)\times D \Rightarrow D^2 y+Dx=D(\cos t)=-\sin t$ …(3)

(i.e) $Dx+D^2 y=-\sin t$

$$\begin{array}{ll} & \cancel{D}x+y=\sin t+1 \\ (1)-(3) & \underline{\cancel{D}x+D^2 y=-\sin t} \\ & (-D^2+1)y=2\sin t+1 \end{array}$$

(i.e) $-(D^2-1)y = 2\sin t + 1$

(i.e) $(D^2-1)y = -2\sin t - 1$ which is 2^{nd} order differential equation in y with constant coefficients.

$\therefore$ The solution of $y(t) = C.F. + P.I.$

Now the Auxiliary equation is

$m^2 - 1 = 0$

(i.e) $m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. $= Ae^t + Be^{-t}$

P.I. $= \dfrac{1}{D^2-1}(-2\sin t - 1)$

$= \dfrac{1}{D^2-1}(-2\sin t) - \dfrac{1}{D^2-1}e^{0t}$

$= \dfrac{-2}{-1-1}\sin t - \dfrac{1}{0-1}e^{0t}$

$= \dfrac{-2}{-2}\sin t + \dfrac{1}{1}e^{0t}$

$= \sin t + 1$

$\therefore y = Ae^t + Be^{-t} + \sin t + 1$

To find the solution of x, first we can find Dy.

$Dy = D[Ae^t + Be^{-t} + \sin t + 1]$

$= Ae^t - Be^{-t} + \cos t$

Consider the equation (2)

$Dy + x = \cos t$

$\Rightarrow x = \cos t - Dy$

$x = \cos t - (Ae^{t} - Be^{t} + \cos t)$

$= \cancel{\cos t} - Ae^{t} + Be^{t} - \cancel{\cos t}$

$x = Be^{-t} - Ae^{t}$

$\therefore$ The solutions are

$x = Be^{-t} - Ae^{t}$

$y = Ae^{-t} + Be^{t} + \sin t + 1$

where A and B are arbitrary constants.

2. Solve $\frac{dx}{dt} + y = \sin t; \frac{dy}{dt} + x = \cos t$ where $x(0) = 2$ and $y(0) = 5$ (S.U 2007)

Solution:Given $\frac{dx}{dt} + y = \sin t; \frac{dy}{dt} + x = \cos t$

(i.e)

$Dx + y = \sin t;$ …(1)

$Dy + x = \cos t$ …(2)

First we can eliminate x from (1) and (2)

$(2) \times D \Rightarrow Dx + D^2 y = D\cos t = -\sin t$ …(3)

$$\begin{array}{r} Dx \cancel{+} y = \sin t \\ \underline{D\cancel{x} + D^2 y = -\sin t} \end{array}$$

$(1) - (3) \Rightarrow (-D^2 - 1)y = 2\sin t$

(i.e) $(D^2 - 1)y = 2\sin t$, this is a 2nd order differential equation in y with constant coefficients.

y = C.F. + P.I.

Now the auxiliary equation is

$m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m^2 = \pm 1$

$\Rightarrow$ C.F. $= Ae^t + Be^{-t}$

P.I. $= \dfrac{1}{D^2 - 1}(-2\sin t) = \dfrac{-2}{-1-1}\sin t$ (Replace $D^2$by – 1)

$= \dfrac{2}{2}\sin t = \sin t$

$\therefore y \;=\; Ae^t + Be^{-t} + \sin t$ …(4)

To find the solution of x, first let us find Dy and substitute the value in (2)

$Dy \;=\; D[Ae^t + Be^{-t} \sin t] = Ae^t - Be^{-t} + \cos t$

Now (2) $\Rightarrow Dy + x = \cos t$

$\Rightarrow x \;= \cos t - Dy$

$= \cos t - (Ae^t - Be^{-t} + \cos t)$

$= \cancel{\cos} t - Ae^t + Be^{-t} - \cancel{\cos} t$

$x \;= Be^{-t} - Ae^t$ …(5)

To find the value of A and B, We can use the given conditions

Given $x(0) = 2$, $y(0) = 0$

(4) $\Rightarrow y(0) = Ae^0 + Be^{-0} + \sin(0)$ Since $t = 0$

$0 = A + B$

$\Rightarrow B \;= -A$

(5) $\Rightarrow x(0) = Be^{-o} - Ae^o$ [$t = 0$]

$2 = B - A$

$2 = -A - A = -2A$

$\therefore A = -1$

$\Rightarrow B = 1$

$\therefore$ The solutions are

$x = e^{-t} + e^{t} = 2\cosh t \qquad \because \cos ht = \dfrac{e^{t} + e^{-t}}{2}$

$y = e^{-t} - e^{t} + \sin t$

$= -(e^{t} - e^{-t}) + \sin t \qquad \because \sin\ ht = \dfrac{e^{t} - e^{-t}}{2}$

$y = -2\sinh\ t + \sin t$

3. Solve $Dx + y = \cos t\ ; x + Dy = 2\sin t.$

Solution:Given

$Dx + y = \cos t;$ …(1)

$x + Dy = 2\sin t$ …(2)

where $D = \dfrac{d}{dt}$

First we can eliminate y from (1) & (2)

$(1) \times D \Rightarrow D^2 x + Dy = D(\cos t) = -\sin t$

(i.e)

$$\begin{array}{r} D^2 x + \cancel{Dy} = -\sin t \text{ ....(3)} \\ x + \cancel{Dy} = 2\sin t \qquad\qquad \\ \hline (3) - (2) \Rightarrow (D^2 - 1)x = -3\sin t \text{ ....}(I) \end{array}$$

which is a 2nd order differential equation in x with constant coefficients the solution of

$x(t) = C.F. + P.I.$

Now Auxiliary equation is

$m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. $= Ae^t + Be^{-t}$

$$\text{P.I.} = \frac{1}{D^2 - 1}(-3\sin t)$$

$$= \frac{-3}{-1-1}\sin t \qquad \text{Replace } D^2 \text{ by } -1$$

$$= \frac{3}{2}\sin t$$

$$\therefore x(t) = Ae^t + Be^{-t} + \frac{3}{2}\sin t \qquad \ldots(4)$$

Now to find y we can find Dx and then substitute Dx in (1).

$$Dx = D\left[Ae^t + Be^{-t} + \frac{3}{2}\sin t\right]$$

$$= Ae^t - Be^{-t} + \frac{3}{2}\cos t$$

(1) $\Rightarrow Dx + y = \cos t$

$\therefore y = \cos t - Dx$

$$= \cos t - \left[Ae^t - Be^{-t} + \frac{3}{2}\cos t\right]$$

$$= \cos t - Ae^t - Be^{-t} - \frac{3}{2}\cos t$$

$$= \frac{\cos t}{2} - Ae^{t} + Be^{-t}$$

$$y \quad = Be^{-t} - \frac{\cos t}{2} - Ae^{t}$$

Aliter

To find y we can eliminate x from (1) & (2)

$$(2) \times D \Rightarrow Dx + D^2 y = D(2\sin t) = 2\cos t$$

$$(6) - (1) \Rightarrow \cancel{Dx} + D^2 y = 2\cos t \text{ ....(6)}$$

$$\cancel{Dx} + y = \cos t$$

$$(D^2 - 1)y = \cos t \text{ ....(II)}$$

$\therefore\ y = CF + PI$

Auxiliary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

C.F. = $Ce^{t} + De^{-t}$

[Since already A, B are introduced in x]

$$\text{P.I.} \quad = \frac{1}{D^2 - 1}\cos t$$

$$= \frac{1}{-1-1}\cos t \quad \text{Replace } D^2 \text{ by } -1$$

$$= \frac{-1}{2}\cos t$$

$$\therefore\ y(t) = Ce^{t} + De^{-t} - \frac{1}{2}\cos t \qquad \text{...(7)}$$

Here the solutions of x and y contains four constants. But the solutions for x and y should contain as the order of equations (I) and (II).

Hence the values of C and D should be expressed interms of A and B as explained below.

Inserting the values of x and y in (1)

(i.e) $Dx + y = \cos t \quad \left[Here\ Dx = \frac{d}{dt}(x) \right]$

$$D\left[Ae^t + Be^{-t} + \frac{3}{2}\sin t \right] + Ce^t + De^{-t} - \frac{1}{2}\cos t = \cos t$$

(i.e) $\frac{d}{dt}\left[Ae^t + Be^{-t} + \frac{3}{2}\sin t \right] + Ce^t + De^{-t} - \frac{1}{2}\cos t = \cos t$

$$Ae^t - Be^{-t} + \frac{3}{2}\cos t + Ce^t + De^{-t} - \frac{1}{2}\cos t - \cos t = 0$$

$$\Rightarrow Ae^t + Be^{-t} + \cancel{\frac{3}{2}}\cos t + Ce^t + De^{-t} - \cancel{\frac{3}{2}}\cos t = 0$$

$$\Rightarrow Ae^t - Be^{-t} + Ce^t + De^{-t} = 0$$

(i.e) $(A + C)e^t + (D - B)e^{-t} = 0.$

$$A + C = 0, D - B = 0$$

(i.e) $-A = +C; D = B$ ($\therefore e^t$, e^{-t} will not be zero)

$\therefore$ The solutions are $x = Ae^t + Be^{-t} + \frac{3}{2}\sin t$

In (7), But $C = -A, D = B \quad y = Ae^t + Be^{-t} - \frac{1}{2}\cos t$

Where A and B are arbitrary constants.

4. Solve $\frac{dx}{dt} - y = t; \quad \frac{dy}{dt} = t^2 - x$ (S.U 2008, 2012)

Solution: Given $Dx - y = t; \quad Dy = t^2 - x \qquad where \ \frac{d}{dt} = D$

(ie) $\therefore CF = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right)$

First we eliminate x from (1) and (2)

$$Dx + D^2 y = 2\cancel{t} \rightarrow (3)$$

$$(3) - (1) \Rightarrow D\cancel{x} - y = t \rightarrow (1)$$

$$\overline{\qquad (D^2 + 1)y = t \qquad}$$

is a differential equation in y with constant coefficients.

$\therefore y = CF + PI$

Now the auxiliary equation is

$m^2 + 1 = 0$

(ie) $m^2 = -1$

$m = \pm i$

$\therefore$C.F. = $A \cos t + B \sin t$

P.I. $= \frac{1}{D^2 + 1}(t) = [1 + D^2]^{-1}(t)$

$= [1 - D^2 + D^4 - ...](t)$

$= t - D^2(t) + D^4(t) - ...$

$= t$ $\qquad [\therefore D^2(t) = 0, D^4(t) = 0\ldots$

Neglecting D^2 and higher powers of D]

$\therefore y = A \cos t + B \sin t + t \qquad [(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots, |x|<1.]$

To find x, find Dy and substitute in equation (2)

$$Dy = \frac{d}{dt}(y) = -A\sin t + B\cos t + 1$$

$$(2) \Rightarrow x + Dy = t^2$$

$$\Rightarrow x = t^2 - Dy$$

$$= t^2 - [-A\sin t + B\cos t + 1]$$

$$= t^2 + A\sin t - B\cos t - 1$$

$$x(t) = A\sin t - B\cos t + t^2 - 1$$

$\therefore$ The solutions are

$$x(t) = A\sin t - B\cos t + t^2 - 1$$

$$y(t) = A\cos t - B\sin t + t$$

Where A and B are arbitrary constants.

5. Solve $(D + 2)\, x - 3y = t$; $(D + 2)\, y - 3x = e^{2t}$ (S.U 2009)

Solution: Given

$$(D + 2)\, x - 3y = t \quad \rightarrow (1)$$

$$(D + 2)\, y - 3x = e^{2t} \quad \rightarrow (2)$$

First we can eliminate x from (1) & (2)

$$(1) \times 3 \Rightarrow 3(D+2)x - 3(3y) = 3t$$

$$(D+2)x - 9y = 3t \rightarrow (3)$$

$$(2) \times (D+2) \Rightarrow -3(D+2)x + (D+2)^2 y = (D+2)e^{2t}$$

$$\Rightarrow -3(D+2)x + (D+2)^2 y = (D+2)e^{2t}$$

$$= 2e^{2t} + 2e^{2t}$$

$$\Rightarrow -3(D+2)x + (D+2)^2 y = 4e^{2t} \rightarrow (4)$$

$$(3)+(4) \qquad \cancel{3(D+2)x} - \quad 9y \quad = 3t$$

$$\cancel{-3(D+2)x} + (D+2)^2 y = 4e^{2t}$$

$$D^2 + 4D + 4 - 9)y = 4e^{2t} + 3t$$

(ie) $(D^2 + 4D - 5)\, y = 4e^{2t} + 3t$ is differential equation in y with arbitrary constants.

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is

$m^2 + 4m - 5 = 0$

(ie) $(m + 5)\,(m - 1) = 0$

$m = -5, \qquad m = 1$

$\therefore$ C.F. $= Ae^{t} + Be^{-5t}$

$$\text{P.I.} \quad = \frac{1}{D^2+4D-5}(4e^{2t}+3t)$$

$$= \frac{4}{D^2+4D-5}e^{2t} + \frac{3}{D^2+4D-5}(t)$$

$$= \frac{4}{2^2+4(2)-5}e^{2t} + \frac{3}{-5\left[1+\dfrac{D^2+4D}{-5}\right]}(t)$$

$$= \frac{4}{4+8-5}e^{2t} - \frac{3}{5}\left[1-\frac{D^2+4D}{5}\right]^{-1}(t)$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[1+\left(\frac{D^2}{5}+\frac{4D}{5}\right)+\left(\frac{D^2}{5}+\frac{4D}{5}\right)^2+\ldots.\right](t)$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[1+\frac{4D}{5}\right](t)$$ [neglecting D^2 and higher power of D, since $D^2(t) = 0$]

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left[t+\frac{4}{5}D(t)\right]$$

$$= \frac{4}{7}e^{2t} - \frac{3}{5}\left(t + \frac{4}{5}\right)$$

$$= \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)$$

$$\therefore y = Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)$$

Now consider $(D+2)y - 3x = e^{2t}$

(ie) $Dy + 2y - 3x = e^{2t}$

(ie) $\frac{d}{dt}\left[Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)\right]$

$$+2\left[Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)\right] - e^{2t} = 3x$$

(ie) $3x = Ae^{t} + 5Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}(3)$

$$+2Ae^{t} + 2Be^{-5t} + \frac{8}{5}e^{2t} - \frac{2}{5}(3t) - \frac{2}{5}\frac{12}{6} - e^{2t}$$

$$3x = 3Ae^{t} - 3Be^{-5t} + \frac{16}{7}e^{2t} - e^{2t} - \frac{3}{5} - \frac{24}{25} - \frac{2}{5}(3t)$$

$$= 3Ae^{t} - 3Be^{-5t} + \frac{9}{7}e^{2t} - \frac{6}{5}t - \frac{39}{25}$$

$$\therefore x = \frac{1}{3}\left[3Ae^{t} - 3Be^{-5t} + \frac{9}{7}e^{2t} - \frac{6}{5}t - \frac{39}{25}\right]$$

$$x = Ae^{t} - Be^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25}$$

$\therefore$ The Solutions are $y = Ae^{t} + Be^{-5t} + \frac{4}{7}e^{2t} - \frac{1}{5}\left(3t + \frac{12}{5}\right)$

$x = Ae^{t} - Be^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{5}$, where A and B are arbitrary constants.

6. Solve: $\frac{dx}{dy} - 7x + y = 0; \quad \frac{dy}{dt} - 2x - 5y = 0$ (S.U 2010)

Solution:Given $Dx - 7x + y = 0; \quad Dy - 2x - 5y = 0$ where $D = \frac{d}{dy}$

(ie) $(D-7)x + y = 0 \rightarrow (1)$

$-2x + (D-5)y = 0 \rightarrow (2)$

First we can eliminate $x$from (1) and (2)

$(1) \times 2 \Rightarrow \quad \cancel{2(D-7)\,x} + 2y = 0 \ldots\ldots\ldots..(3)$

$(2) \times (D-7) \Rightarrow \cancel{-2(D-7)\,x} + (D-5)(D-7)\,y = 0 \ldots\ldots\ldots..(4)$

$(3) \times 4 \Rightarrow \quad (D^2 - 5D - 7D + 35 + 2)\,y = 0$

(ie) $(D^2 - 12D + 37)y = 0$ is a differential equation is y with constant coefficients.

$\therefore$ y = C.F. + P.I.

Now the Auxiliary equation is $m^2 - 12m + 37 = 0$

$$m = \frac{12 \pm \sqrt{144-148}}{2} = \frac{12 \pm \sqrt{-4}}{2} = \frac{12 \pm 2i}{2}$$

$m = 6 \pm i$

$\therefore$ C.F. $= e^{6t}(A\cos t + B\sin t)$

P.I. $= 0$

$\therefore y = e^{6t}(A\cos t + B\sin t)$

Now $(2) \Rightarrow (D-5)\,y - 2x = 0$

$2x = (D-5)\,y$

$= Dy - 5y$

$$= \frac{d}{dt}[e^{6t}(A\cos t + B\sin t)] - 5[e^{6t}(A\cos t + B\sin t)]$$

$$= e^{6t}\frac{d}{dt}[(A\cos t + B\sin t)] + \frac{d}{dt}(e^{6t})[(A\cos t + B\sin t)] - 5e^{6t}(A\cos t + B\sin t)$$

$$= e^{6t}[(-A\sin t + B\cos t)] + 6e^{6t}[(A\cos t + B\sin t)] - 5e^{6t}[(A\cos t + B\sin t)]$$

$$= e^{6t}[B\cos t - A\sin t] + e^{6t}(A\cos t + B\sin t)$$

$$2x = e^{6t}[\cos t(B + A) + \sin t(B - A)]$$

$$\therefore \quad x = \frac{e^{6t}}{2}[\cos t(B + A) + \sin t(B - A)]$$

The solutions are $x = \dfrac{e^{6t}}{2}[\cos t(B + A) + \sin t(B - A)]$

$y = e^{6t}(A\cot t + B\sin t)$ where A and B are arbitrary constants

7. Solve: $\dfrac{dx}{dt} + y = 0; \quad \dfrac{dy}{dt} + x = 2\cos t$

Solution:Given $\dfrac{dx}{dt} + y = 0; \quad \dfrac{dy}{dt} + x = 2\cos t$

(ie) $Dx + y = 0 \to (1) \qquad Dy + x = 2\cos t \to (2)$

First eliminate x from (1) and (2)

$(2)\times D \Rightarrow D^2y + Dx = D(2\cos t)$

(ie) $Dx + D^2y = -2\sin t \to (3)$

$(1) \Rightarrow \qquad Dx + 2y \qquad = 0$

$(3) \Rightarrow \quad Dx + D^2y \qquad = -2\sin t$

$(1) - (3) \Rightarrow \qquad -D^2y + y \qquad = 2\sin t$

$-(D^2 - 1)\,y = 2\sin t$

(ie) $(D^2 - 1)\,y = 2 \sin t$ is a differential equation in y with constant co-efficient.

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is $m^2 - 1 = 0$

(ie) $m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore$ C.F. $= Ae^t + Be^{-t}$

$$\text{P.I.} \quad = \frac{1}{D^2 - 1}(-2\sin t)$$

$$= \frac{-2}{-1-1}\sin t \qquad \text{Replace } D^2 \text{ by } -1$$

$$= \sin t$$

$\therefore y \quad = Ae^t + Be^{-t} + \sin t$

To find the solution of x, first find $\frac{d}{dt}(y)$ and put this value in (2)

$$Dy \quad = \frac{d}{dy}(y) = D[Ae^t + Be^{-t} + \sin t]$$

$$= Ae^t - Be^{-t} + \cos t$$

(2) $\Rightarrow Dy + x = 2\cos t$

(ie) $x = 2\cos t - Dy$

$= 2\cos t - (Ae^t - Be^{-t} + \cos t)$

$= 2\cos t - Ae^t + Be^{-t} - \cos t$

$x = Be^{-t} - Ae^t + \cos t$

$\therefore$ The solutions are $x = Be^{-t} - Ae^t + \cos t$

$y = Ae^t + Be^{-t} + \sin t$

Here A and B are arbitrary constants.

8. Solve: $\frac{dx}{dt} - y = t; \quad \frac{dy}{dt} + x = 1$

Solution:Given $\frac{dx}{dt} - y = t; \quad \frac{dy}{dt} + x = 1$

(ie) $Dx - y = t \rightarrow (1); \qquad Dy + x = 1 \rightarrow (2)$

First eliminate x from (1) and (2)

$(2)\times D \Rightarrow D^2y + Dx = D(1) = 0$

$$Dx + D^2y = 0 \ldots\ldots\ldots\ldots\ldots(3)$$

$$-Dx - y = t$$

$-(1) + (3) \Rightarrow \quad (D^2 + 1)\, y = -t$

is a 2nd order differential equation in y with constant coefficients

$\therefore y = \text{C.F.} + \text{P.I.}$

Now the auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

$\therefore$ C.F. $= A\cos t + B\sin t$

P.I. $= \frac{1}{(D^2+1)}(-t)$

$= (1 + D^2)^{-1}(-t)$

$= [1 - D^2 + (D^2)^2 - \ldots](-t) \quad [\therefore (1+x)^{-1} = 1 - x + x^2 - \ldots]$

$= -t + D^2(t^2) + \ldots$

$= -t$ [Neglecting D^2 and higher powers of $D \because D^2(t) = 0$]

$\therefore y \quad = A\cos t + B\sin t - t$

To find x, find Dy and substitute this in (2)

$Dy \quad = \dfrac{dy}{dt} = \dfrac{d}{dt}[A\cos t + B\sin t - t]$

$= -A \sin t + B \cos t - 1$

(2) reduces $Dy = 1 + A \sin t - B \cos t + 1$

$x = 2 + A \sin t - B \cos t$

The solutions are $x = 2 + A \sin t - B \cos t$

$y = A \cos t - B \sin t - t$ where A, B are arbitrary constants,

9. Solve: $\dfrac{dx}{dt} + y = \sin 2t; \quad \dfrac{dy}{dt} - x = \cos 2t;$ (S.U 2011)

Solution:Given $Dx + y = \sin 2t \rightarrow (1); \quad Dy - x = \cos 2t \left[\because D = \dfrac{d}{dt}\right] \rightarrow (2)$

First we can eliminate x between (1) & (2)

D×(2) $\Rightarrow D^2y - Dx = D(\cos 2t) = -2 \sin 2t$

$-Dx + D^2y = -2 \sin 2t$……………………(3)

(ie) $Dx + y = \sin 2t$……………………………(1)

(1) +(3)$\Rightarrow$ $(D^2 + 1)\, y = -\sin 2t$

is a 2nd order differential equation in y with constant coefficients

$\therefore y$ = C.F. + P.I.

The auxiliary equation is $m^2 + 1 = 0$

(ie) $m^2 = -1$

(ie) $m = \pm i$

$\therefore$ C.F. $= A\cos t + B\sin t$

P.I. $= \dfrac{1}{D^2+1}(-\sin 2t) = \dfrac{1}{-2^2+1}\sin 2t$

$$= \frac{-1}{-4+1}\sin 2t = \frac{1}{-3}\sin 2t = \frac{1}{3}\sin 2t$$

$$\therefore y \quad = A\cos t + B\sin t + \frac{1}{3}\sin 2t$$

Now let to find *Dy*, get *x*, we can use *Dy* in (2)

$$Dy \quad = \frac{d}{dt}(y) = \frac{d}{dt} A\cos t + B\sin t + \frac{1}{3}\sin 2t$$

$$= A\sin t + B\cos t + \frac{2}{3}\cos 2t$$

$$(2) \Rightarrow Dy - x = \cos 2t$$

$$x = Dy - \cos 2t$$

$$= -A\sin t + B\cos t + \frac{2}{3}\cos 2t - \cos 2t$$

$$x \quad = -A\sin t + B\cos t - \frac{1}{3}\cos 2t$$

$\therefore$ The solutions are $x = -A\sin t + B\cos t - \frac{1}{3}\cos 2t$

$y = A\cos t + B\sin t + \frac{1}{3}\sin 2t$ where A and B are arbitrary constants.

10. Solve: $(D+4)x + 3y = t; \quad 2x + (D+5)y = e^{2t}$

Solution:

Given $(D+4)x + 3y = t \rightarrow (1); \quad 2x + (D+5)y = e^{2t} \rightarrow (2)$

First we can eliminate *x* from (1) and (2)

$(1) \times 2 \Rightarrow 2 \qquad 2(D + 4)\, x + \qquad 6y \quad = 2t$

$(2) \times (D + 4) \Rightarrow 2(D + 4)\, x + (D + 5)(D + 4)\, y = (D + 4)e^{2t}$.......(4)

$(3) - (4) \Rightarrow \qquad 6y - (D + 5)(D + 4)\, y = 2t - D(e^{2t}) - 4e^{2t}$

(ie) $\qquad 6y - (D^2 + 5D + 4D + 20)\, y = 2t - 2e^{2t} - 4e^{2t}$

$[-(D^2 + 9D + 20) + 6]\, y = 2t - 6e^{2t}$

$-[D^2 + 9D + 20 - 6]\, y = 2t - 6e^{2t}$

$(D^2 + 9D + 14)\, y = -(2t - 6e^{2t})$

$(D^2 + 9D + 14)\, y = 6e^{2t} - 2t$, is a 2nd order differential equations in y with constant co-efficients.

$\therefore y$ = C.F. + P.I.

The auxiliary equation is $m^2 + 9m + 14 = 0$

$(m + 7)(m + 2) = 0$

$m = -7, m = -2$

$\therefore$ C.F. $= A^{-2t} + Be^{-7t}$

Now P.I. $= \dfrac{1}{D^2 + 9D + 14}(6e^{2t} - 2t)$

$$= \frac{1}{D^2 + 9D + 14} 6e^{2t} - \frac{2}{D^2 + 9D + 14} t$$

$$= \frac{6}{2^2 + 9(2) + 14} e^{2t} - \frac{2}{14\left(1 + \dfrac{D^2 + 9D}{14}\right)} t$$

$$= \frac{6}{4 + 18 + 14} e^{2t} - \frac{2}{14}\left[1 + \left(\frac{D^2}{14} + \frac{9D}{14}\right)\right]^{-1}(t)$$

$$\frac{6}{36} e^{2t} - \frac{1}{7}\left[1 - \left(\frac{D^2}{14} + \frac{9D}{14}\right) + \left(\frac{D^2}{14} + \frac{9D}{14}\right)^2 - ....\right](t)$$

$$= \frac{1}{6}e^{2t} - \frac{1}{7}\left[t - \frac{9}{14}D(t)\right]$$ [Neglecting D^2 and higher powers of D $\because D^2(t) = 0$]

$$= \frac{1}{6}e^{2t} - \frac{1}{7}\left(t - \frac{9}{14}\right)$$

$$= \frac{1}{6}e^{2t} - \frac{t}{7} + \frac{9}{98}$$

$$\therefore y \quad = \; Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}$$

To find x, First find $\frac{d}{dt}(y)$ and substitute $\frac{d}{dt}(y)$ in (2)

$$Dy \quad = \; D\left[Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}\right]$$

$$= -2Ae^{-2t} - 7Be^{-7t} + \frac{2e^{2t}}{6} - \frac{1}{7}$$

$$= -2Ae^{-2t} - 7Be^{-7t} + \frac{e^{2t}}{3} - \frac{1}{7}$$

Now $(2) \Rightarrow 2x + (D+5)y = e^{2t}$

(ie) $2x = e^{2t} - (D+5)y = e^{2t} - Dy - 5y$

$$2x = e^{2t} - \left(-2Ae^{-2t} - 7Be^{-7t} + \frac{1}{3}e^{2t} - \frac{1}{7}\right)$$

$$-5\left(Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}\right)$$

$$= e^{2t} + 2Ae^{-2t} + 7Be^{-7t} - \frac{1}{3}e^{2t} + \frac{1}{7}$$

$$-5Ae^{-2t} + 5Be^{-7t} - \frac{5e^{2t}}{6} + \frac{5t}{7} - \frac{45}{98}$$

$$= e^{2t} - \frac{1}{3}e^{2t} - \frac{5}{6}e^{2t} - 3Ae^{-2t} + 2Be^{-7t} + \frac{5t}{7} - \frac{45}{98} + \frac{1}{7}$$

$$= \left(\frac{6-2-5}{6}\right)e^{2t} - 3Ae^{-2t} + 2Be^{-7t} + \frac{5t}{7} + \left(\frac{-45+14}{98}\right)$$

$$2x = -\frac{1}{6}e^{2t} - 3Ae^{-2t} + 2Be^{-7t} + \frac{5t}{7} - \frac{31}{98}$$

$$\therefore x = -\frac{1}{12}e^{2t} - \frac{3}{2}Ae^{-2t} + Be^{-7t} + \frac{5t}{14} - \frac{31}{196}$$

$\therefore$ The solutions are $x = Be^{-7t} - \frac{3}{2}Ae^{-2t} - \frac{1}{12}e^{2t} + \frac{5t}{14} - \frac{31}{136}$,

$y = Ae^{-2t} = Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}$ where A and B are arbitrary constants.

11. Solve $\frac{dx}{dt} + 2x - 2y = e^t;\quad \frac{dy}{dt} + 2x + y = 0$

Solution: Given $Dx + 2x - 2y = e^t$

(i.e) $(D+2)x - 2y = e^t$ …(1)

$Dy + 2x + y = 0$

(i.e) $(D+1)y + 2x = 0$ …(2)

First eliminate x from (1) and (2)

(1) × 2 $\cancel{2(D+2)\,x}$ - 22 y = 2e^t……………(3)

(2) ×(D + 2)⟹2(D + 2) $\cancel{x}$ + (D + 1)(D +2) y = (D + 2)(0)…….(4)

(3) –(4)⟹ -4y - (D^2 + D + 2D+ 2) y = 2e^{2t}

(ie) –[D^2 + 3D + 2 + 4]y = 2e^t

(D^2 + 3D + 6)y = 2e^t is an differential equation with constant co-efficients in y.

$\therefore y$ = C.F. + P.I.

Now the auxiliary equation is

$m^2+3m + 6 = 0$

$$\therefore \text{ C.F.} = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right)$$

$$\text{P.I.} = \frac{1}{D^2+3D+6}(-2e^t) = -\frac{2}{1+3(1)+6}e^t$$

$$= -\frac{2}{10}e^t = -\frac{1}{5}e^t$$

$$\therefore y = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) - \frac{1}{5}e^t$$

$$Dy = D\left[e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) - \frac{1}{5}e^t\right]$$

Substitute Dy in (2)

$(2) \Rightarrow (D+1)y + 2x = 0$

$Dy + y + 2x = 0 \Rightarrow 2x = -Dy - y.$

$$2x = -\left[e^{\frac{-3}{2}t}D\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) + D(e^{\frac{-3}{2}1})\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right)\right]$$

$$+\frac{1}{5}D(e^t) - \left[e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2} + B\sin\frac{\sqrt{15}}{2}t\right) - \frac{1}{5}e^t\right]$$

$$= e^{\frac{-3}{2}t}\left(-\frac{\sqrt{15}}{2}A\sin\frac{\sqrt{15}}{2}t + \frac{\sqrt{15}}{2}B\cos\frac{\sqrt{15}}{2}t\right)$$

$$+\frac{3}{2}e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t + B\sin\frac{\sqrt{15}}{2}t\right) + \frac{1}{5}e^t$$

$$-e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)+\frac{1}{5}e^{t}$$

$$= e^{\frac{-3}{2}t}\frac{\sqrt{15}}{2}\left(B\cos\frac{\sqrt{15}}{2}t-A\sin\frac{\sqrt{15}}{2}t\right)+\frac{1}{2}e^{\frac{-3}{2}e^{t}}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)+\frac{2}{5}e^{t}$$

$$= e^{\frac{-3}{2}t}\left[\cos\frac{\sqrt{15}}{2}t\left(\frac{\sqrt{15}}{2}B+\frac{A}{2}\right)+\sin\frac{\sqrt{15}}{2}t\left(\frac{B}{2}-\frac{A\sqrt{15}}{2}\right)\right]+\frac{2}{5}e^{t}$$

$$\therefore x = \frac{1}{2}e^{\frac{-3}{2}t}\left[\cos\left(\frac{\sqrt{15}}{2}t\right)\left(\frac{A+B\sqrt{15}}{2}\right)+\sin\left(\frac{\sqrt{15}}{2}t\right)\left(\frac{B-A\sqrt{15}}{2}\right)\right]+\frac{1}{5}e^{t}$$

$\therefore$ The solution are

$$x = \frac{1}{2}e^{\frac{-3}{2}t}\left[\cos\left(\frac{\sqrt{15}t}{2}\right)\left(\frac{A+B\sqrt{15}}{2}\right)+\sin\left(\frac{\sqrt{15}t}{2}\right)\left(\frac{B-A\sqrt{15}}{2}\right)\right]+\frac{1}{5}e^{t}$$

$y = e^{\frac{-3}{2}t}\left(A\cos\frac{\sqrt{15}}{2}t+B\sin\frac{\sqrt{15}}{2}t\right)-\frac{1}{5}e^{t}$, where A and B are arbitrary constants.

12. Solve $D^2x + y = \sin t$; $D^2y + x = \cos t$

Solution: Given $D^2x + y = \sin t \rightarrow (1)$

$x + D^2y = \cos t \rightarrow (2)$

First we can be eliminate x from (1) and (2)

(2) $\times D^2 \Rightarrow$ $D^2x + D^4y = D^2(\cos t) = D(-\sin t) = -\cos t$

$\cancel{D^2x} + D^4y = -\cos t$…………………. (3)

$\cancel{D^2x} + y = \sin t$………………………. (1)

(3) – (1) $\Rightarrow$ $(D^4 - 1)\, y = \cos t - \sin t$

is a differential equation in y with constant co-efficients.

$\therefore y =$ C.F. + P.I.

Now the auxiliary equation is

$m^4 - 1 = 0$

(i.e) $(m^2 + 1)(m^2 - 1) = 0$

$\Rightarrow m^2 = -1, m^2 = 1$

$\Rightarrow m = \pm i, \ m = \pm 1.$

$\therefore$ C.F. $= C_1e^t + C_2e^{-t} + C_3 \cos t + C_4 \sin t$

$$\text{P.I.} = \frac{1}{D^4 - 1}(-\cos t - \sin t)$$

$$= \frac{-1}{D^4 - 1}\cos t - \frac{1}{D^4 - 1}(\sin t)$$

$$= \frac{t}{4D^3}\cos t - \frac{1}{4D^3}(\sin t)$$

$$= \frac{t}{4}\int \left\{\int \left(\int \cos t dt\right)dt\right\}dt - \frac{t}{4}\int \left\{\int \left(\int \sin t dt\right)dt\right\}dt$$

$$= \frac{t}{4}\int \left\{\int \sin t dt\right\}dt - \frac{t}{4}\int \left\{\int -\cos t dt\right\}dt$$

$$= -\frac{t}{4}\int -\cos t dt - \frac{t}{4}\int -\sin t dt$$

$$= +\frac{t}{4}\sin t - \frac{t}{4}\cos t = \frac{t}{4}(\sin t - \cos t)$$

$$\therefore y = C_1e^t + C_2e^{-t} + C_3\cos t + C_4 \sin t + \frac{t}{4}(\sin t - \cos t)$$

First we can find $Dy = \frac{d}{dt}(y)$

$$Dy = C_1e^t - C_2e^{-t} + C_3 \sin t - C_4\cos t + \frac{t}{4}(-\cos t - \sin t) + \frac{1}{4}(\sin t - \cos t)$$

$$\therefore D^2y = C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t + \frac{t}{4}(-\sin t - \cos t)$$

$$+\frac{1}{4}(-\cos t - \sin t) + \frac{1}{4}(-\cos t - \sin t)$$

Now (2) $\Rightarrow x = \cos t - D^2 y$

$$x = \cos t - \left[C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t + \frac{t}{4}(\cos t - \sin t) - \frac{2}{4}(\cos t + \sin t)\right]$$

$$= \cos t - C_1e^t - C_2e^{-1} + C_3\cos t + C_4\sin t - \frac{t}{4}(\cos t - \sin t) + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$= \frac{3}{2}\cos t + \frac{1}{2}\sin t - C_1e^t - C_2e^{-t} + C_3\cos t + C_4\sin t - \frac{t}{4}(\cos t - \sin t)$$

$\therefore$ The solutions are

$$x = -\frac{3}{2}\cos t + \frac{1}{2}\sin t - C_1e^t - C_2e^{-t} + C_3\cos t + C_4\sin t - \frac{t}{4}(\cos t - \sin t)$$

$$y = C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t + \frac{t}{4}(\sin t - \cos t)$$

Where C_1, C_2, C_3 and C_4 are arbitrary constants.

Exercise

1. Solve $\frac{dx}{dt} + 2dy = -\sin t; \quad \frac{dy}{dt} - 2x = \cos t$
2. Solve $Dx - (D-2)y = \cos t; (D-2)x + Dy = \sin 2t$
3. Solve $(2D+1)x + (3D+1)y = e^t; (D+5)x + (D+7)y = 2e^t$
4. Solve $\frac{dx}{dt} + 2x + 3y = 0; \quad 3x + \frac{dy}{dt} + 2y = 2e^{2t}$
5. Solve $\frac{dx}{dt} + 2x + 3y = 2e^{2t}; \quad \frac{dy}{dt} + 3x + 2y = 0$

6. Solve $\frac{dx}{dt}+y=-\sin t; \quad \frac{dy}{dt}+x=\cos t$

7. Solve $\frac{dx}{dt}+2x-3y=5t; \quad \frac{dy}{dt}-3x+2y=2e^{2t}$

8. Solve $\frac{dx}{dt}+2y=\sin 2t; \quad \frac{dy}{dt}-2x=\cos 2t$

9. Solve $\frac{dx}{dt}+2y=-5e^{t}; \quad \frac{dy}{dt}-2x=5e^{t}$

10. Solve $Dx+3y=\cos t; \quad 2x+5Dy=\sin t$

11. Solve $\frac{dx}{dt}+5x-2y=t; \quad \frac{dy}{dt}+2x+y=0$ *given* $x=y=0$ *when* $t=0$

12. Solve $\frac{dx}{dt}+2y=5e^{t}; \quad \frac{dy}{dt}-2x=5e^{t}$, *given* $x=y=0$ *when* $t=0$

13. Solve $(D+4)x+3y=0; \; 2x+(D+5)y=0$

14. Solve $\frac{dx}{dt}+2x-3y=t; \quad \frac{dy}{dx}-3x+2y$

15. Solve $\frac{dx}{dt}+5x+y=e^{t}; \quad \frac{dy}{dt}+3y-x=e^{2t}$

16. Solve $\frac{dx}{dt}-2x+Dy=\sin 2t; \; Dx-(D-2)y=\cos 2t$

Answers

1. $x=A\cos 2t+B\sin 2t-\cos t; \;\; y=A\sin 2t-B\cos 2t-\sin t$

2. $x=e^{t}(A\sin t-B\cos t)-\frac{1}{2}\sin 2t$

$y=e^{t}(A\sin t-B\cos t)-\frac{1}{2}\sin 2t$

3. $x=Ae^{-2t}+Be^{-7t}+\frac{5}{14}t-\frac{31}{196}-\frac{1}{2}e^{2t}$

$$y=-\frac{2}{3}Ae^{-2t}+Be^{-7t}-\frac{1}{7}t+\frac{9}{98}+\frac{1}{6}e^{2t}$$

4. $x=Ae^{t}+Be^{-5t}+\frac{6}{7}e^{2t};\ y=Be^{-5t}-Ae^{t}+\frac{8}{7}e^{2t}$

5. $x=Ae^{-5t}+Be^{t}+\frac{8}{7}e^{2t};\ y=Ae^{-5t}-Be^{t}-\frac{6}{7}e^{2t}$

6. $x=Ae^{t}+Be^{-t};+y=Be^{-t}-Ae^{t}+\sin t$

7. $x=Ae^{t}-Be^{-5t}+\frac{6}{7}e^{2t}-2t-\frac{13}{5}$

$$y=Ae^{t}-Be^{-5t}+\frac{8}{7}e^{2t}-3t-\frac{12}{5}$$

8. $x=-A\sin 2t+B\cos 2t-\frac{1}{2}\cos 2t$

$$y=A\cos 2t+B\sin 2t$$

9. $x=B\cos 2t-A\sin 2t-e^{t}$

$$y=A\cos 2t-B\sin 2t-3e^{t}$$

10. $x=\left[\frac{1}{2}\sin t-5\left(A\frac{\sqrt{6}}{5}e^{\frac{\sqrt{6}}{5}t}-B\frac{\sqrt{6}}{5}e^{\frac{\sqrt{6}}{5}t}-\frac{1}{11}\sin t\right)\right]$

$$y=Ae^{\frac{\sqrt{6}}{5}t}+Be^{\frac{\sqrt{6}}{5}t}+\frac{1}{11}\cos t$$

11. $x=-\frac{1}{27}(1+6t)e^{-3t}+\frac{1}{27}(1+3t)$

$$y=-\frac{2}{27}(2+3t)e^{-3t}+\frac{2}{27}(2-3t)$$

12. $x=-3\cos 2t-\sin 2t+3e^{t}$

$$y = \cos 2t + 3\sin 2t - e^{t}$$

13. $$x = Ae^{-2t} + Be^{-7t}$$

$$y = \frac{1}{3}\left[3Be^{-7t} - 2Ae^{-2t}\right]$$

14. $$x = Ae^{-5t} + Be^{t} + \frac{1}{5} - \frac{2}{25}[5t-4] + \frac{1}{6}te^{t}$$

$$y = \frac{1}{3}\left[-3Ae^{-5t} + 3Be^{t} + \frac{1}{2}te^{t} - \frac{4}{25}(5t-4) - t\right]$$

15. $$x = (At+B)e^{-4t} + \frac{4}{25}e^{t} - \frac{e^{2t}}{36}$$

$$y = -At^{-4t} - e^{-4t}(At+B) + \frac{e^{t}}{25} + \frac{7}{36}e^{2t}$$

16. $$x = e^{t}(A\cos t + B\sin t) - \frac{1}{2}\cos 2t$$

$$y = e^{4}(A\sin t - B\cos t) - \frac{1}{2}\sin 2t$$

17. $$x = -\frac{3}{2}Ae^{-2t} + Be^{-7t} - \frac{1}{12}e^{2t}$$

$$y = Ae^{-2t} + Be^{-7t} + \frac{1}{6}e^{2t}$$

18. $$x = (A+Bt)e^{t} + Ce^{-\frac{3t}{2}} - \frac{t}{2}$$

$$y = (-2A+6B-2Bt)e^{t} - \frac{c}{3}e^{\frac{3t}{2}} - \frac{1}{3}$$

19. $$x = (At+B)\cos t + (ct+D)\sin t + \frac{1}{25}e^{t}(4\sin t - 3\cos t)$$

$$y = -(At+B)\sin t + (ct+D)\cos t - \frac{1}{25}e^{t}(3\sin t - 4\cos t)$$

20. $x = Ae^{2t} + Be^{-2t} + c\cos 2t + D\sin 2t - \frac{4}{15}\sin t + \frac{3}{16}t$

$$y = \frac{A}{3}e^{2t} + \frac{B}{3}e^{-2t} + 3c\cos 2t + 3D\sin 2t - \frac{1}{5}\sin t + \frac{5}{16}t$$

METHOD OF VARIATION OF PARAMETERS

This method is very useful in finding the general solution of the second order equation.

$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X$ where a_1, a_2 are constants and 'X' is a functionof x .

The complementary function is

C.F $= Af_1 + Bf_2$

Where A, B are constants and $f_1$and $f_2$are functions of x . Thenparticular Integral,

$P.I. = Pf_1 + Qf_2$

Where $P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx, \quad Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} dx$

The complete solution is

$y = Af_1 + Bf_2 + P.I.$

1. Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by using method of variation of parameters.

Solution:Given $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

(i.e) $(D^2+1)y = \operatorname{cosec} x$

The Auxiliary Equation is

$m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm i$ [imaginary roots]

$\therefore$ C.F. $= A\cos x + B\sin x$

Here $f_1 = \cos x$; $f_2 = \sin x$;

$f_1' = -\sin x$; $f_2' = \cos x$

$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$

P.I. $= P f_1 + Q f_2$

where

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\sin x \operatorname{cosec} x}{1} dx \qquad [\because x = \operatorname{cosec} x]$$

$$= -\int \sin x . \frac{1}{\sin x} dx$$

$$= -\int dx$$

$$= -x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x . \operatorname{cosec} x ds}{1}$$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \int \cot x dx$$

$$= \log(\sin x)$$

$\therefore P.I = Pf1 + Qf2$

$= -x \cos x + \sin x \log(\sin x)$

$\therefore y = C.F + P.I$

$= A \cos x + B \sin x - x \cos x + \sin x \log(\sin x)$

2. Solve $\frac{d^2y}{dx^2} + y = \tan x$ by the method of variation of parameters.

Solution:Given $(D^2+1)y = \tan x$

The Auxiliary eqn is

$m^2 + 1 = 0 \Rightarrow m^2 = -1$

m = ± i

C.F = $A \cos x + B \sin x$

$f_1 = \cos x$; $f_2 = \sin x$;

$f_1' = -\sin x$; $f_2' = \cos x$;

$f_1 f_2' - f_1' f_2 = 1$

$P.I. = Pf_1 + Qf_2$

$$P = \int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx = -\int \frac{\sin x . \tan x}{1} dx$$

$$= -\int \frac{\sin x . \sin x}{\cos x} dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} dx$$

$$= -\int \frac{1}{\cos x} dx + \int \frac{\cos^2 x}{\cos x} dx$$

$$= \int -\sec x + \int \cos x dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \tan x}{1} dx$$

$$= \int \cos x \frac{\sin x}{\cos x} dx$$

$$= \int \sin x \, dx$$

$$= -\cos x$$

P.I $= Pf_1 + Qf_2$

$= [-\log(\sec x + \tan x) + \sin x] \cos x + (-\cos x) \sin x$

$= -\cos x \log(\sec x + \tan x) + \sin x \cos x - \sin x \cos x$

$\therefore$ P.I $= -\cos x \log (\sec x + \tan x)$

$\therefore y$ = C.F + P.I

$= A \cos x + B \sin x - \cos x \log (\sec x + \tan x)$

3. Solve $(D^2 + 4) y = \tan 2x$ using the method of variation of parameters.

Solution:Given $(D^2 + 4) y = \tan 2x$

The Auxiliary Equation is $m^2 + 4 = 0$

$m = \pm 2i$

C.F = $A \cos 2x + B \sin 2x$

$f_1 = \cos x$; $f_2 = \sin x$;

$f_1' = -2 \sin x$; $f_2' = 2\cos x$;

$f_1 f_2' - f_1' f_2 = 2 \cos^2 2x + 2 \sin^2 2x = 2$

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1^1 f_2} dx$$

$$= -\int \frac{\sin 2x . \tan 2x}{2} dx$$

$$= \frac{1}{2}\int \sin 2x \times \frac{\sin 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{(1-\cos^2 2x)}{\cos 2x} dx$$

$$= -\frac{1}{2}\int \frac{1}{\cos 2x} dx + \frac{1}{2}\int \frac{\cos^2 2x dx}{\cos 2x} dx$$

$$= -\frac{1}{2}\left[\frac{1}{2}\log(\sec 2x + \tan 2x)\right] + \frac{1}{2}.\frac{1}{2}\sin 2x$$

$$= -\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x$$

$$Q = \int \frac{f_1 X}{f_1 f' - f f_2} dx$$

$$= \int \frac{\cos 2x}{2} \tan 2x dx$$

$$= \frac{1}{2}\int \cos 2x \frac{\sin 2x}{\cos 2x} dx$$

$$= \frac{1}{2}\int \sin 2x dx$$

$$= \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)$$

$$= -\frac{1}{4}\cos 2x$$

$$\text{P.I} = Pf_1 + Qf_2$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x) + \frac{1}{4}\cos 2x \sin 2x$$

$$\frac{1}{4}\sin 2x\cos 2x$$

$$= -\frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)$$

y = C.F. + P.I

$$= A\cos 2x + B\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)$$

4. Solve $\frac{d^2y}{dx^2}+4y=4\tan 2x$ using method of variation of parameters.

Solution:Given $\frac{d^2y}{dx^2}+4y=4\tan 2x$

i.e $(D^2+4)y = 4\tan 2x$

$m^2 + 4 = 0$

$m = \pm 2i = 0 \pm 2i$ [imaginary root $\alpha = 0$, $ß = 2$]

C.F = e^{0x} (A cos 2x + B sin 2x)

Now C.F = A cos 2x + B sin 2x

Here f_1 = cos 2x; f_2 = sin 2x;

f' = –2 sin 2x; f' = 2cos 2x;

$f_1f_2' - f'f_2$ = 2 cos2$x$cos2x+ 2 sin 2x sin 2x

= 2 (cos^2 2x + sin^2 2x) = 2

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2X}{f_1f_2' - f_1'f_2}dx$$

$$= -\int \frac{\sin 2x 4\tan 2x}{2}dx \qquad \because x = 4\tan 2x$$

$$= -2 \frac{\sin^2 2x}{\cos 2x} dx$$

$$= -2\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= 2\left[\int \frac{1}{\cos 2x} dx - \int \frac{\cos^2 2x}{\cos 2x} dx\right]$$

$$= -2\left[\int \sec 2x dx - \int \cos 2x dx\right]$$

$$= -2\left[\frac{1}{2} \log(\sec 2x + \tan 2x\right] + 2\frac{\sin 2x}{2}$$

$$= -\log(\sec 2x + \tan 2x) + \sin 2x$$

$$= \sin 2x - \log(\sec 2x + \tan 2x)$$

$$Q = \int \frac{f_1 X}{f_1 f'_2 - f'_1 f_2} dx$$

$$= \int \frac{\cos 2x 4 \tan 2x}{2} dx$$

$$= 2\int \cos 2x \frac{\sin 2x}{\cos 2x} dx$$

$$= 2\int \sin 2x dx$$

$$= 2\left(\frac{-\cos 2x}{2}\right)$$

$$= -\cos 2x$$

$$\text{P.I} = Pf_1 + Qf_2$$

$$= \cos 2x[\sin 2x - \log(\sec 2x + \tan 2x)]$$

$$= -\cos 2x \sin 2x$$

$$= -\cos 2x \log(\sec 2x + \tan 2x)]$$

$\therefore y \quad = \text{C.F} + \text{P.I}$

$= (A\cos 2x + B\sin 2x) - \cos 2x \log(\sec 2x + \tan 2x)$

5. Solve $(D^2 + 2D + 5)y = e^{-x}\tan x$, by method of variation of parameters.

Solution:Given $(D^2 + 2D + 5)y = e^{-x}\tan x$,

The Auxiliary Equation is

$m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

C.F $= e^{-x}(A\cos 2x + B\sin 2x)$

Here $f_1 = e^{-x}\cos 2x$

$f'_1 = -2e^{-x}\sin 2x - e^{-x}\cos 2x$

$f_2 = e^{-x}\sin 2x$

$f'_2 = 2e^{-x}\cos 2x - e^{-x}\sin 2x$

$\therefore f_1 f'_2 - f_2 f_1' = 2e^{-2x}\cos^2 2x - e^{-x}\sin 2x\cos 2x$

$+ 2e^{-2x}\sin^2 2x + e^{-2x}\sin 2x\cos 2x$

$= 2e^{-2x}(\cos^2 x + \sin^2 x)$

$= 2e^{-2x}$

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f'_2 - f'_1 f_2}\,dx$$

$$= -\int \frac{e^{-x}\sin 2x}{2e^{-2x}} e^{-x} \tan x dx \qquad [\because x = e^{-x} \tan z]$$

$$= -\frac{1}{2}\int 2\sin x \cos x \frac{\sin x}{\cos x}$$

$$= -\int \sin^2 x dx$$

$$= -\int \frac{1-\cos 2x}{2} dx$$

$$= -\frac{1}{2}\left[x - \frac{1\sin 2x}{2}\right]$$

$$= -\frac{1}{2}x + \frac{\sin 2x}{4}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{e^{-x}\cos 2x}{2e^{-2x}} e^{-x} \tan x dx$$

$$= \frac{1}{2}\int \cos 2x \tan x dx$$

$$= \frac{1}{2}\int (2\cos^2 x - 1)\frac{\sin x}{\cos x} dx \qquad \because \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\int 2\cos^2 x \frac{\sin x}{\cos x} - \frac{1}{2}\int \frac{\sin x}{\cos x} dx \qquad 2\cos^2 x - 1 = \cos 2x$$

$$= \frac{1}{2}\int 2\sin x \cos x dx - \frac{1}{2}\int \frac{\sin x}{\cos x} dx$$

$$= \frac{1}{2}\int \sin 2x dx - \frac{1}{2}\int \frac{\sin x}{\cos x} dx$$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2}\right] + \frac{1}{2}\log \cos x$$

$$= -\frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)$$

P.I $= Pf_1 + Qf_2$

$$= -\frac{x}{2} + \frac{\sin 2x}{4} - \frac{\cos 2x}{3} + \frac{1}{2}\log(\cos x)$$

y $= C.F + P.I$

$$= Ae^{-x}\cos 2x + Be^{-x}\sin 2x - \frac{x}{2} + \frac{\sin 2x}{4} - \frac{\cos 2x}{4} + \frac{1}{2}\log(\cos x)$$

6. Solve $y'' - 2y' + 2y = e^x \tan x$, using method of variation of parameters.

Solution:Given $(D^2 - 2D + 2)y = e^{-x}\tan x$,

The A.E is $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{4-8}}{2}$$

The A.E is $= \frac{2 \pm \sqrt{-4}}{2}$

$$= \frac{2 \pm \sqrt{2i}}{2}$$

$= 1 \pm i$

C.F $= e^x (A\cos x + B\sin x)$

$= Ae^x \cos x + Be^x \sin x$

f_1 $= e^{-x}\cos x - e^x \sin x$

f'_2 $= e^x \sin x + e^{-x}\cos x$

$f_1 f'_2 - f'_1 f_2$ $= e^{2x}\cos x \sin x + e^{2x}\cos^2 x$

$-e^{2x}\cos x \sin x + e^{2x}\sin^2 x$

$= e^{2x}(\cos^2 x + \sin^2 x)$

$= e^{2x}$

$P.I. = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f'_2 - f_1' f_2} dx$$

$$= -\int \frac{e^x \sin x}{e^{2x}} e^x \tan x dx$$

$$= -\int \sin x \frac{\sin x}{\cos x} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{(1 - \cos^2 x)}{\cos x} dx$$

$$= -\int \frac{1}{\cos x} dx + \int \frac{\cos^2 x}{\cos x} dx$$

= -log(secx +tanx) +sinx

7. Solve $(D^2 - 4D + 4)y = e^{2x}$, by the method of variation of

Solution:Given $(D^2 - 4D + 4)y = e^{2x}$

The A.E. is $m^2 - 4m + 4 = 0$

$(m - 2)^2 = 0$

$m = 2, 2$ (Equal roots)

C.F. $= (Ax + B)e^{2x}$

$= Axe^{2x} + Be^{2x}$

$= Af_1 + Bf_2$

Here

$f_1 = xe^{2x}$ $f_2 = e^{2x}$

$f'_1 = 2xe^{2x} + e^{2x}$ $\qquad\qquad$ $f'_2 = 2e^{2x}$

$f_1f'_2 - f_1'f_2 = 2xe^{4x} - 2xe^{4x} - e^{4x}$

$= -e^{4x}$

P.I. $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$$

$$= -\int \frac{e^{2x}.e^{2x}}{-e^{4x}}\,dx$$

$$= -\int dx$$

$$= x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx$$

$$= \int \frac{xe^{2x}e^{2x}}{-e^{4x}}\,dx$$

$$= -\int xdx$$

$$= -\frac{x^2}{2}$$

$$\text{P.I} = x(xe^{2x}) + \frac{-x^2}{2}e^{2x}$$

$$= x^2xe^{2x} - \frac{x^2}{2}e^{2x}$$

$$= \frac{x^2}{2}e^{2x}$$

y = C.F + P.I

$$= (Ax + B)e^{2x} + \frac{x^2}{2}e^{2x}$$

8. Solve $(D^2+4)y = \sec 2x$ by using method of variation of parameters.

Solution:

Given $(D^2+4)y = \sec 2x$

The Auxiliary Equation is $m^2 + 4 = 0$

$m = \pm 2i$

C.F. $= e^{ex}(A\cos 2x + B\sin 2x)$

$= A\cos 2x + B\sin 2x$

Here

$f_1 = \cos 2x$ $f_2 = \sin 2x$

$f'_1 = 2\sin 2x$ $f'_2 = 2\cos 2x$

$f_1 f'_2 - f'_1 f_2 = 2$

P.I. $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f'_2 - f'_1 f_2}\,dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2}\,dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2}\,dx$$

$$= -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x}\,dx$$

Put $t = \cos 2x$

$dt = -2\sin 2x\,dx$

$$\therefore p = -\frac{1}{2}\int \frac{dt}{2t}$$

$$= -\frac{1}{4}\int \log t$$

$$p = -\frac{1}{4}\log(\cos 2x)$$

$$= \frac{1}{4}\log(\cos 2x)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos 2x \sec 2x}{2} dx$$

$$= \frac{1}{2}\int dx$$

$$= \frac{1}{2}x$$

P.I. $= Pf_1 + Qf_2$

$$= \frac{1}{4}\cos 2x \log(\cos 2x) + \frac{1}{2}x \sin 2x$$

$\therefore y$ = C.F + P.I

$$= A\cos 2x + B\sin 2x + \frac{1}{4}\cos 2x \log(\cos 2x) + \frac{1}{2}x\sin 2x$$

9. Solve $(D^2+1)y = \sec x$ by the method of variation of parameters.

Solution:Given $y'' + y = \sec x$

(ie) $(D^2+1)y = \sec x$

The Auxiliary Equation is $m^2 + 1 = 0$

$m = \pm i$

C.F. $= A\cos x + B\sin x$

$= Af_1 + Bf_2$

Here f_1= cos$x$ $\quad$ f_2= sin x

f_1'= - sin x $\quad$ f_2'= cos x

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x \sec x}{1} dx$$

$$= -\int \frac{\sin x}{\cos x} dx$$

= log (cos x)

put t = cos x

P = log t

P.I $= Pf_1 + Qf_2$

$$\therefore P = -\int -\frac{dt}{t}$$

= log t

= log (cos x)

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \sec 2x}{1} dx$$

$\therefore$P.I = cos x log (cos x) + x sin x

y = C.F + P.I

$$= A\cos x + B\sin x + \cos x \log(\cos x) + x\sin x$$

10. Solve $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

Solution:Given $(D^2 + a^2)y = \tan ax$

The Auxiliary Equation is $m^2 + a^2 = 0$

$\Rightarrow m = \pm ai$

C.F. $= e^{ax} (A \cos ax + B \sin ax)$

$f_1 = \cos ax$ $\qquad$ $f_2 = \cos ax$

$f'_1 = - a \sin ax$ $\qquad$ $f'_2 = a \cos ax$

$f_1 f'_2 - f_1' f_2 = a \cos^2 ax = a \sin^2 ax = a$

P.I $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin ax \tan ax}{a} dx$$

$$= -\frac{1}{a}\int \sin ax \times \frac{\sin ax}{\cos ax} dx$$

$$= -\frac{1}{a}\int \frac{(1-\cos^2 ax)}{\cos ax} dx$$

$$= -\frac{1}{a}\int \left[\frac{1}{\cos ax} - \frac{\cos^2 ax}{\cos ax}\right] dx$$

$$= -\frac{1}{a}\int [\sec ax - \cos ax] dx$$

$$= -\frac{1}{a}\int \sec ax dx \times \frac{1}{a}\int \cos ax dx$$

$$= -\frac{1}{a}\left(\frac{1}{a}\log(\sec ax + \tan ax)\right) + \frac{1}{a}\left[\frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a^2}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos ax \tan ax}{a} dx$$

$$= \frac{1}{a} \int \sin ax\, dx$$

$$= \frac{1}{a} - \left[\frac{\cos ax}{a} \right] = -\frac{1}{a^2} \cos ax$$

P.I $= Pf_1 + Qf_2$

$$= \frac{1}{a^2} \cos ax \log(\sec ax + \tan ax) + \frac{1}{a^2} \sin ax \cos ax - \frac{1}{a^2} \sin ax \cos ax$$

$$= \frac{1}{a^2} [\cos ax \log(\sec ax + \tan ax)]$$

y = C.F + P.I

$$= A \cos ax + B \sin ax - \frac{1}{a^2} \cos ax \log(\sec ax + \tan ax)$$

11. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = x \sin x$

Solution:Given $\frac{d^2y}{dx^2} + y = x \sin x$

(ie) $(D^2 + 1)y = x \sin x$

The Auxiliary Equation is $m^2 + 1 = 0$

$$m = \pm i$$

C.F. = $A \cos x + B \sin x$

$f_1 = \cos x$ $\qquad$ $f_2 = \sin x$

$f'_1 = -\sin x$ $\qquad$ $f'_2 = \cos x$

$f_1 f'_2 - f_1' f_2 = 1$

$$P \quad = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x(x \sin x)}{1} dx$$

$$= -\int x \sin^2 x \, dx$$

$$\because u = x \qquad d_v = \cos 2x$$

$$du = dx \qquad v = \frac{\sin 2x}{2}$$

$$= -\int x \sin^2 x \, dx$$

$$= -\int \frac{x}{2} dx \int \frac{x \cos 2x}{2} dx$$

$$= -\frac{x^2}{4} + \frac{1}{2}\left[x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx\right]$$

$$= -\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}$$

$$Q \quad = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x(x \sin x)}{1} dx$$

$$= \int x \frac{\sin 2x}{2} dx$$

$$\therefore u = x \qquad dv = \sin 2x$$

$$du = dx \qquad v = -\frac{\cos 2x}{2}$$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2} + \int \frac{\cos 2x dx}{2}\right]$$

$$= \frac{-x\cos 2x}{4} + \frac{1}{8}\sin 2x$$

P.I $= Pf_1 + Qf_2$

$$= \left[\frac{-x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}\right]\cos x + \left[\frac{-x\cos 2x}{4} + \frac{1}{8}\sin 2x\right]\sin x$$

$$= \frac{-x^2}{4}\cos x + \frac{x}{4}\sin 2x\cos x + \frac{\cos x\cos 2x}{8}$$

$$\frac{-x\cos 2x\sin x}{4} + \frac{1}{8}\sin 2x\sin x$$

$$= \frac{-x^2}{4}\cos x + \frac{x}{4}[\sin 2x\cos x - \cos 2x\sin x]$$

$$+ \frac{1}{8}[\cos 2x\cos x + \sin 2x\sin x]$$

$$= \frac{-x^2}{4}\cos x + \frac{x}{4}\sin(2x - x) + \frac{1}{8}\cos(2x - x)$$

$$= \frac{-x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

$\therefore$ y $=$ C.F + P.I

$$= A\cos x + B\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

$$= \left(A + \frac{1}{8}\right)\cos x + B\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

$$= c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

where $c_1 = A + \frac{1}{8} c_2 = B$

12. Solve $\frac{d^2y}{dx^2} + y = \cos ec\ x \cot x$ using the method of variation of parameters.

Solution:Given $(D^2+1)y = \cos ec\ x \cot x$

The Auxiliary Equation is $m^2 + 1 = 0$

$$m = \pm i$$

C.F = $A \cos x + B \sin x$

$f_1 = \cos x$ $\qquad\qquad f_2 = \sin x$

$f_1 f'_2 - f'_1 f_2 = \cos^2 x + \sin^2 x = 1$

P.I $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f'_2 - f'_1 f_2} dx$$

$$= -\int \frac{\sin x \cos ecx \cot x}{1} dx$$

$$= -\int \sin x \frac{1}{\sin x} \frac{\cos x}{\sin x} dx$$

$$= \log(\sin x)$$

$$Q = \int \frac{f_1 X}{f_1 f'_2 - f'_1 f_2} dx$$

$$= \int \cos x \cos ecx \cot x dx$$

$$= \int \cos x \frac{1}{\sin x} \frac{\cos x}{\sin x} dx$$

$$= \int \frac{1 - \sin^2 x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx - \int dx$$

$$= \int \cos ec^2 x dx - x$$

$= -\cot x - x$

P.I $= Pf_1 + Qf_2$

$= -\cos x \log(\sin x) - \sin x \cot x - x \sin x$

$= -\cos x \log(\sin x) - [\cot x + x] \sin x$

y $= \text{C.F} + \text{P.I}$

$= A \cos x + B \sin x - \cos x \log(\sin x) - [\cot x + x] \sin x$

Exercie Problems

1. Using method of variation of parameters solve. $(D^2 + 9)y = \sec 3x$
2. Solve $(D^2 + 1)y = \cot x$
3. Solve $(D^2 + 25)y = \tan 5x$
4. Solve $(D^2 + 16)y = \operatorname{cosec} 4x$
5. Solve $(D^2 + 25)y = \sec 5x$
6. Solve $(D^2 + 9)y = \cot 3x$
7. Solve $(D^2 - 6D + 9)y = \dfrac{e^{3x}}{x}$
8. Solve $(D^2 + 36)y = \cos ec 6x$
9. Solve $(D^2 - 1y = \dfrac{1}{1+e^x}$
10. Solve $(D^2 - 1)y = \dfrac{2}{1+e^x}$
11. Solve $y'' + y = x$
12. Solve $y'' - 3y' + 2y = x^2$
13. Solve $(2D^2 - D - 3)y = 25e^{-x}$

Answers

1. $y = A\cos 3x + B\sin 3x + \frac{1}{9}\cos 3x\log(\cos 3x) + \frac{1}{3}x\sin 3x$

2. $y = A\cos x + B\sin x - \sin x\log(\cos ecx + \cot x)$

3. $y = A\cos 5x + B\sin 5x - \frac{1}{25}[\log(\sec 5x + \tan 5x])) - \sin 5x]\cos 5x - \frac{\cos 5x}{25}\sin 5x$

4. $y = A\cos 4x + B\sin 4x - \frac{1}{9}x\cos 4x + \frac{1}{16}[\log(\sin 4x)]\sin 4x$

5. $y = A\cos 5x + B\sin 5x - \frac{1}{25}\cos 5x\log(\sec 5x) + \frac{1}{5}x\sin 5x$

6. $y = A\cos 3x + B\sin 3x - \frac{\sin 3x\cos 3x}{3} + \frac{\cos 3x\sin 3x}{3}$

7. $y = (Ax + B)e^{3x} + \log x(xe^{3x}) - xe^{3x}$

8. $y = A\cos 6x + B\sin 6x - \frac{1}{6}x\cos 6x + \frac{1}{36}\sin 6x.\log(\sin 6x)$

9. $y = Ae^{x} + Be^{-x} + \frac{e^{x}}{2}[-e^{-x} + \log(1 + e^{x}) - x] - \frac{1}{2}e^{-x}\log(1 + e^{x})$

10. $y = Ae^{x} + Be^{-x} - 1 + e^{x}\log(1 + e^{-x}) - e^{-x}\log(1 + e^{x})$

11. $y = A\cos x + B\sin x + x$

12. $y = Ae^{x} + Be^{-x} + \frac{1}{2}x^{2} - \frac{3}{2}x + \frac{7}{4}$

13. $y = Ae^{\frac{3}{2}x} + Be^{-x} - 2e^{-x} - 5xe^{-x}$