UNIT – II

GEOMETRICAL APPLICATIONS OF DIFFERENTIAL CALCULUS

Curvature:

At each point on a curve, with equation y=f(x), the tangent line turns at a certain rate. A measure of this rate of turning is the curvature

$$K = \frac{f''(x)}{(1+[f'(x)])^{3/2}}$$

Radius of curvature in Cartesian form:

If the curve is given in Cartesian coordinates as *y(x)*, then the radius of curvature is

$\rho = (1 + [y']^{2})^{(3/2)}/y''$ where $y'=\frac{dy}{dx}$, $y'' = (d^{2}y)/(dx^{2})$.

Radius of curvature in Parametric form:

If the curve is given parametrically by functions *x(t)* and *y(t)*, then the radius of curvature is

$\rho = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y-}$, $x' = \frac{dx}{dt}$, $x=\frac{d^2x}{dt^2}$, $y'=\frac{dy}{dt}$, $y = \frac{d^2y}{dt^2}$

Examples:

1. Find the radius of the curvature at the point $\left(\frac{1}{\frac{4,1}{4}}\right)$ on the curve $\sqrt{x}+\sqrt{y} = 1$.

   Solution: $\sqrt{x}+\sqrt{y} = 1$

   Differentiating w. r. t x ,we get

   $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$. $y' = -\frac{\sqrt{y}}{\sqrt{x}}$

   At $\left(\frac{1}{\frac{4,1}{4}}\right)$, $y' = -1.$

   $y'' = -[(\sqrt{x}\ 1/(2\sqrt{y})\ y' - \sqrt{y}\ 1/(2\sqrt{x}))/x]$

At $\left(\frac{1}{\frac{4,1}{4}}\right)$, $y'' = -[(1/2\ \ 1/(2\ 1/2)\ (-1) - 1/2\ \ 1/(2\ 1/2))/(1/4)]$ = 4.

$$\rho = \frac{(1+1)^{\frac{3}{2}}}{4} = \frac{1}{\sqrt{2}}$$

2. Show that the radius of the curvature at any point of the curve $y = c\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$.

   Solution: $y = c\cosh\left(\frac{x}{c}\right)$

   Differentiating y w. r. t x we get

   $y' = \sinh\left(\frac{x}{c}\right)$

   $y'' = 1/c\ \ \cosh(x/c)$

   $$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{\frac{3}{2}}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = c\cosh^2\left(\frac{x}{c}\right) = \frac{y^2}{c}.$$

3. Find the radius of the curvature of the curve y = x^2(x-3) at the points where the tangent is parallel to the x – axis.

   Solution: y = x^2(x-3)

   Differentiating y w. r. t x we get

   $y' = 3x^2 - 6x$

   $y'' = 6x - 6$

   The points at which the tangent parallel to the x – axis can be found by equating y' to zero.

   i.e., $3x^2 - 6x = 0 \Rightarrow x = 0, x = 2.$

   At x = 0,y" = -6. At x = 2, y" = 6.

   Therefore at x = 0 and x = 2, $\rho = \frac{1}{6}$.

4. Prove that the radius of the curvature of the curve at any point of the cycloid

   $x = a(t + \sin t), y = a(1 + \cos t)$ is $\frac{4a\cos t}{2}$.

Solution: We have $x = a(t + sint), y = a(1 + cost)$.

Therefore $\frac{dx}{dt} = a(1 + cost)$, $\frac{dy}{dt} = asint.$

Now $$\frac{dy}{dt} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{asint}{a(1+cost)} = \frac{\frac{2\sin t}{2}\cos t}{\frac{2}{2cos^2\frac{t}{2}}} = \frac{\tan t}{2}.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\tan t}{2}\right) = \left\{\frac{d}{dt}\left(\frac{\tan t}{2}\right)\right\}\frac{dt}{dx} = \frac{1}{2}sec^2\frac{t}{2}\frac{1}{2\,a(1+cost)} = \frac{1}{4a}sec^4\frac{t}{2}.$$

Hence $$\rho = \frac{\left(1 + tan^2\frac{t}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a}sec^4\frac{t}{2}} = \frac{4\,a\cos t}{2}.$$

Centre and Circle of curvature:

Let the equation of the curve be y = f(x). let P be the given point (x,y) on this curve and Q the point (x+Δx,y+Δy) in the neighborhood of P. let N be the point of intersection of the normals at P and Q. As Q→P, suppose N→C. Then C is the centre of curvature of P. The circle whose centre C and radius ρ is called the circle of curvature. The co-ordinates of the centre of curvature is denoted as $(\bar{x}, \bar{y})$.

where $\bar{x} = x - (y'(1 + y'^2))/y''$, $\bar{y} = y + ((1 + y'^2))/y''$.

Equation of the circle of curvature:

If $(\bar{x}, \bar{y})$ be the coordinates of the centre of curvature and ρ be the radius of curvature at any point (x,y) on a curve, then the equation of the circle of curvature at that point is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

Examples:

1. Find the centre of curvature of the curve $a^2y = x^3$.

   Solution: $a^2y = x^3$

   $\frac{dy}{dx} = \frac{3x^2}{a^2}$ and $\frac{d^2y}{dx^2} = \frac{6x}{a^2}$

$$\overline{x} = x - \frac{x}{2}\left(1 + \frac{9x^4}{a^4}\right) = \frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right]$$

$$\overline{y} = \frac{x^3}{a^2} + \frac{\left[1 + \frac{9x^4}{a^4}\right]}{\frac{6x}{a^2}} = \frac{5x^3}{2a^2} + \frac{a^2}{6x}.$$

Therefore the required centre of curvature is $\left(\frac{x}{2}\left[1 - \frac{9x^4}{a^4}\right], \frac{5x^3}{2a^2} + \frac{a^2}{6x}\right)$.

2. Find the centre of curvature of y = x^2 at $\left(\frac{1}{\frac{2,1}{4}}\right)$.

Solution: y' = 2x, y" = 2.

At $\left(\frac{1}{\frac{2,1}{4}}\right)$, y' = 1, y" = 2.

Therefore $\overline{x} = \frac{1}{2} - \frac{(1+1)}{2} = -\frac{1}{2}, \overline{y} = \frac{1}{4} + 1 = \frac{5}{4}.$

Therefore the required centre of curvature is $\left(-\frac{1}{\frac{2,5}{4}}\right)$.

3. Find the centre of curvature of the curve xy = a^2 at (a,a).

Solution: $y^{\uparrow\prime} = -a^{\uparrow}2/x^{\uparrow}2$, $y'' = 2a^{\uparrow}2\, x^{\uparrow}(-3)$. At (a,a) y' = -1, y" = $\frac{2}{a}$

Therefore $\overline{x} = a + \frac{2}{2/a} = 2a, \overline{y} = a + \frac{2}{2/a} = 2a.$

The required centre of curvature is (2a, 2a).

4. Find the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{\frac{2,3a}{2}}\right)$.

Solution: $x^3 + y^3 = 3axy$

$3x^2 + 3y^2y' = 3a(xy' + y)$

$y' = \frac{ay - x^2}{y^2 - ax}$

y' at $\left(\frac{3a}{\frac{2,3a}{2}}\right)$ is -1.

$y'' = ((y^{\uparrow}2 - ax)(ay^{\uparrow\prime} - 2x) - (ay - x^{\uparrow}2\,)(2yy^{\uparrow\prime} - a))/(y^{\uparrow}2 - ax)^{\uparrow}2$

$y'' at(3a/2,3a/2) = (-32)/3a$

$$\rho = \frac{2\sqrt{2(3a)}}{32}$$

$$\bar{x} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}$$

$$\bar{y} = \frac{3a}{2} - \frac{2}{32/_{3a}} = \frac{21a}{16}.$$

The circle of curvature is $\left(x - \frac{21a}{16}\right)^2 + \left(y - \frac{21a}{16}\right)^2 = \frac{9a^2}{128}$

5. Find the circle of curvature at the point (2,3) on $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

Solution: $\frac{2x}{4} + \frac{2yy'}{9} = 0 \Rightarrow y' = \frac{-9x}{4y} \Rightarrow y'(2,3) = \frac{-3}{2}$

$y'' = (-9(y - xy^{\uparrow\prime}))/(4y^{\uparrow}2\,)$, y'' at $(2,3) = (-3)/2$

$$\rho = \frac{13^{\frac{3}{2}}}{12}, \quad \bar{x} = 2 - \frac{\left(-3/_2\right)\left(1 + 9/_4\right)}{\frac{-3}{2}} = \frac{-5}{4},$$

$$\bar{y} = 3 + \frac{\left(1 + 9/_4\right)}{\frac{-3}{2}} = \frac{5}{6}$$

The circle of curvature is $\left(x + \frac{5}{4}\right)^2 + \left(y - \frac{5}{6}\right)^2 = \frac{13^3}{12^2}$

Evolute and Involute

Evolute: Evolute of the curve is defined as the locus of the centre of curvature for that curve.

Involute : If C' is the evolute of the curve C then C is called the involute of the curve C'.

Procedure to find the evolute:

Let the given curve be f(x,y,a,b) = 0. (1)

Find y' and y" at the point P.

Find the centre of curvature $(\bar{x},\bar{y})$. Using $\bar{x} = x - (y'(1+ y'^2))/y''$,
$\bar{y} = y + ((1+ y'^2))/y''$.   (2)

Eliminate x, y from (1), (2) we get f$(\bar{x}, \bar{y}, a, b) = 0$. (3)

Equation (3) is the required evolute.

Examples:

1. Show that the evolute of the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is another cycloid given by $x = a(\theta - sin\theta), y - 2a = a(1 + cos\theta).$

Solution: $\frac{dx}{d\theta} = a(1 + cos\theta), \frac{dy}{d\theta} = asin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{asin\theta}{a(1+cos\theta)} = \frac{\tan\theta}{2}$$

$y'' = d/d\theta\,(tan\ \theta/2)\ (d\theta)/dx = (sec^4\ \theta/2)/4a$

$$\bar{x} = a(\theta + sin\theta) - \frac{\dfrac{\tan\theta}{2\left(1+tan^2\frac{\theta}{2}\right)}}{sec^4\frac{\theta}{2}/4a} = a(\theta + sin\theta) - 2asin\theta = a(\theta - sin\theta)$$

,

$$\bar{y} = a(1 - cos\theta) + \frac{\left(1+tan^2\frac{\theta}{2}\right)}{sec^4\frac{\theta}{2}/4a} = a(1 - cos\theta) + 4acos^2\frac{\theta}{2} = a(1 + cos\theta) + 2a.$$

$\bar{x} = a(\theta - sin\theta), \bar{y} - 2a = a(1 + cos\theta)$.

The locus of $\bar{x}$ and $\bar{y}$ is $x = a(\theta - sin\theta), y - 2a = a(1 + cos\theta)$.

2. Prove that the evolute of the curve $x = a(cos\theta + \theta sin\theta), y = a(sin\theta - \theta cos\theta)$ is a circle $x^2 + y^2 = a^2.$

Solution: $\frac{dx}{d\theta} = a(-sin\theta + sin\theta + \theta cos\theta) = a\theta cos\theta, \frac{dy}{d\theta} = a\theta sin\theta.$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta cos\theta}{a\theta sin\theta} = tan\theta$$

.

$y'' = 1/(a\theta\ cos^3\ \theta)$,

$$\overline{x} = a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta\left(1+\tan^2\theta\right)}{1/_{a\theta\cos^3\theta}} = a\cos\theta,$$

$$\overline{y} = a(\sin\theta - \theta\cos\theta) + \frac{(1+\tan^2\theta)}{1/_{a\theta\cos^3\theta}} = a\sin\theta.$$

Eliminating , $\overline{x}$ *and* $\overline{y}$ we get $\overline{x}^2 + \overline{y}^2 = a^2$.

The evolute of the given curve is $x^2 + y^2 = a^2$.

ENVELOPE

A curve which touches each member of a given family of curves is called envelope of that family.

Procedure to find envelope for the given family of curves:

Case 1: Envelope of one parameter family of curves

Let us consider y = f(x,α) to be the given family of curves with 'α' as the parameter.

Step 1: Differentiate w.r.t to the parameter α partially, and find the value of the parameter

Step 2: By Substituting the value of parameter α in the given family of curves, we get the required envelope.

Special Case: If the given equation of curve is quadratic in terms of parameter,i.e. $A\alpha^2+B\alpha+c=0$, then envelope is given by **discriminant = 0** i.e. $B^2- 4AC=0$

Case 2: Envelope of two parameter family of curves.

Let us consider y = f(x,α, β) to be the given family of curves, and a relation connecting the two parameters α and β, g(α, β) = 0

Step 1: Consider α as independent variable and β depends α . Differentiate y = f(x,α, β) and g(α, β) = 0, w.r. to the parameter α partially.

Step 2: Eliminating the parameters α, β from the equations resulting from step 1 and g(α, β) = 0, we get the required envelope.

Problems on envelope of one parameter family of curves :

1. Find the envelope of $y = mx + am^p$ where m is the parameter and a, p are constants

Solution : Differentiate $y = mx + am^p$ (1)

with respect to the parameter m, we get,

$$0 = x + pam^{p-1}$$

$$\Rightarrow \quad m = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} \qquad (2)$$

Using (2) eliminate m from (1)

$$y = \left(\frac{-x}{pa}\right)^{\frac{1}{p-1}} x + a\left(\frac{-x}{pa}\right)^{\frac{p}{p-1}}$$

$$\Rightarrow \quad y^{p-1} = \left(\frac{-x}{pa}\right) x^{p-1} + a^{p-1}\left(\frac{-x}{pa}\right)^{p}$$

i.e. $ap^{p} y^{p-1} = -x^{p} p^{p-1} + (-x)^{p}$

which is the required equation of envelope of (1)

2. Determine the envelope of $x\sin\theta - y\cos\theta = a\theta$, where θ being the parameter.

Solution : Differentiate ,

$$x\sin\theta - y\cos\theta = a\theta \qquad (1)$$

with respect to θ , we get,

$$x\cos\theta + y\sin\theta = a \qquad (2)$$

As θ cannot be eliminated between (1) and (2) ,we solve (1) and (2) for x and y in terms of θ.

For this, multiply (2) by sinθ and (1) by cosθ and then subtracting, we get,

$y = a(\sin\theta - \theta\cos\theta)$. Using similar simplification, we get, $x = a(\theta\sin\theta + \cos\theta)$.

3. (Leibnitz's problem) Calculate the envelope of family of circles whose centres lie on the x-axis

and radii are proportional to the abscissa of the centre.

Solution : Let (a,0) be the centre of any one of the member of family of curves with a as the parameter. Then the equation of family of circles with centres on x-axis and radius proportional to the abscissa of the centre is

$(x-a)^2 + y^2 = ka^2$ (1)

where k is the proportionality constant. Differentiating (1) with respect to a, we get,

$-2(x-a) = 2ka$

i.e. $a = \frac{x}{1-k}$.

From (1) , $\left(x - \frac{x}{1-k}\right)^2 + y^2 = \frac{k}{(1-k)^2} x^2$

i.e. $\left(k^2 - k\right)x^2 + (1-k)^2 y^2 = 0, \qquad k \neq 1$

4. Find the envelope of $x\sec^2\theta + y\cos ec^2\theta = a$, where θ is the parameter.

Solution : The given equation is rewritten as , $x\left(1+\tan^2\theta\right) + y\left(1+\cot^2\theta\right) = a$

i.e. $x\tan^4\theta + (x+y-a)\tan^2\theta + y = 0$,

which is a quadratic equation in $t = \tan^2\theta$. Therefore the required envelope is given by the discriminant equation : B^2-4AC = 0

i.e. $(x+y-a)^2 - 4xy = 0$

i.e. $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0$.

Envelope of Two parameter family of curves :

1. Find the envelope of family of straight lines ax+by=1, where a and b are parametersconnected by the relation ab = 1

   Solution :

$ax + by = 1$ (1)

$$ab = 1 \qquad (2)$$

Differentiating (1) with respect to a (considering 'a' as independent variable and 'b' depends on a).

$$x + \frac{db}{da} y = 0$$

i.e. $$\frac{db}{da} = \frac{-x}{y} \qquad (3)$$

Differentiating (2) with respect to a

$$b + a\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b}{a} \qquad (4)$$

From (3) and (4), we have

$$\frac{x}{y} = \frac{b}{a}$$

i.e. $$\frac{ax}{1} = \frac{by}{1} = \frac{ax+by}{2} = \frac{1}{2}$$

$$\therefore \quad a = \frac{1}{2x} \text{ and } b = \frac{1}{2y} \qquad (5)$$

Using (5) in (2), we get the envelope as 4xy = 1

2. Find the envelope of family of straight lines $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$, where a and b are parameters connected by the relation $\sqrt{a} + \sqrt{b} = 1$

Solution :

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1 \qquad (1)$$

$$\sqrt{a}+\sqrt{b}=1 \qquad (2)$$

Differentiating (1) with respect to a

$$\frac{\sqrt{x}}{-2a^{3/2}}+\frac{\sqrt{y}}{-2b^{3/2}}\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-\sqrt{x}}{\sqrt{y}}\frac{b^{3/2}}{a^{3/2}} \qquad (3)$$

Differentiating (2) with respect to a

$$\frac{1}{2\sqrt{a}}+\frac{1}{2\sqrt{b}}\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-\sqrt{b}}{\sqrt{a}} \qquad (4)$$

From (3) and (4), we have

$$\frac{\sqrt{x}}{\sqrt{y}}\frac{b}{a}=1$$

i.e. $$\frac{\sqrt{\frac{x}{a}}}{\sqrt{a}}=\frac{\sqrt{\frac{y}{b}}}{\sqrt{b}}=\frac{\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}}{\sqrt{a}+\sqrt{b}}=\frac{1}{1}$$

$\therefore \quad a=\sqrt{x}$ and $b=\sqrt{y}$ (5)

Using (5) in (2), we get the envelope as $x^{1/4}+y^{1/4}=1$

3. Find the envelope of family of straight lines $\frac{x}{a}+\frac{y}{b}=1$, where a and b are parameters connected by the relation $a^2b^3 = c^5$

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (1)$$

$$a^2b^3 = c^5 \qquad (2)$$

Differentiating (1) with respect to a,

$$\frac{-x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-b^2x}{a^2y} \qquad (3)$$

Differentiating (2) with respect to a

$$2ab^3+3a^2b^2\frac{db}{da}=0$$

i.e. $$\frac{db}{da}=\frac{-2b}{3a} \qquad (4)$$

From (3) and (4), we have

$$\frac{3x}{a}=\frac{2y}{b}$$

i.e. $$\frac{\frac{x}{a}}{3}=\frac{\frac{y}{b}}{2}=\frac{\frac{x}{a}+\frac{y}{b}}{5}=\frac{1}{5}$$

$\therefore$ $a=\dfrac{5x}{3}$ and $b=\dfrac{5y}{2}$ (5)

Using (5) in (2), we get the envelope as $x^2y^3=\dfrac{72}{3125}c^5$

4. Find the envelope of the family of circles whose centres lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and which pass through its centre.

Solution: Let (α,β) be the centre of arbitrary member of family of circles which lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose centre is (0,0). Therefore, equation of the circles passing through origin and having centreat (α,β) is

$$x^2+y^2-2\alpha x-2\beta y=0 \tag{1}$$

with

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \tag{2}$$

Differentiating (1) with respect to α ('α' as independent variable and 'β' depends on α),

$$x+\frac{d\beta}{d\alpha}y=0$$

i.e. $$\frac{d\beta}{d\alpha}=\frac{-x}{y} \tag{3}$$

Differentiating (2) with respect to α

$$\frac{2\alpha}{a^2}+\frac{2\beta}{b^2}\frac{d\beta}{d\alpha}=0$$

i.e. $$\frac{d\beta}{d\alpha}=\frac{-b^2\alpha}{a^2\beta} \tag{4}$$

From (3) and (4), we have

$$\frac{x}{y}=\frac{b^2\alpha}{a^2\beta}$$

i.e. $$\frac{\alpha x}{\frac{\alpha^2}{a^2}}=\frac{\beta y}{\frac{\beta^2}{b^2}}=\frac{\alpha x+\beta y}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}}=\frac{k}{1}$$, where k = αx+βy

$\therefore \qquad \alpha = \dfrac{a^2 x}{k}$ and $\quad \beta = \dfrac{b^2 y}{k}$ (5)

From (1), we have , $x^2 + y^2 = 2k$ (6)

Using (5) and (6) in (2), we get the envelope as $\left(x^2 + y^2\right)^2 = 4\left(a^2 x^2 + b^2 y^2\right)$

5. Determine the equation of the envelope of family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters a and b are connected by the relation $\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1$, *l* and m are non-zero constants.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = 1 \qquad (2)$$

Differentiating (1) with respect to a,

$$\frac{-2x^2}{a^3} - \frac{2y^2}{b^3}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-b^3 x^2}{a^3 y^2} \qquad (3)$$

Differentiating (2) with respect to a

$$\frac{2a}{l^2} + \frac{2b}{m^2}\frac{db}{da} = 0$$

i.e. $$\frac{db}{da} = \frac{-m^2 a}{l^2 b} \qquad (4)$$

From (3) and (4), we have

$$\frac{b^4x^2}{a^4y^2} = \frac{m^2}{l^2}$$

i.e. $$\frac{\frac{x^2}{a^2}}{\frac{a^2}{l^2}} = \frac{\frac{y^2}{b^2}}{\frac{b^2}{m^2}} = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{\frac{a^2}{l^2}+\frac{b^2}{m^2}} = \frac{1}{1}$$

$\Rightarrow$ $a^4 = l^2x^2$ and $b^4 = m^2y^2$

i.e. $a^2 = lx$ and $b^2 = my$ (5)

Using (5) in (2), we get the envelope as $\frac{x}{l} + \frac{y}{m} = 1$

Problems on Evolute as envelope of its normals :

1. Determine the evolute of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by considering it as an envelope of its normal

Solution : Let P (a cosht, b sinht) be any point on the given hyperbola. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cosh t}{a\sinh t} = \frac{b}{a}\coth t$$

Equation of normal line to the hyperbola is

$$(y - b\sinh t) = \frac{-a}{b\cosh t}(x - a\cosh t) \qquad (1)$$

$$\Rightarrow \quad \frac{by}{\sinh t} + \frac{ax}{\cosh t} = a^2 + b^2 \qquad (2)$$

Differentiating (2) partially with respect to t, we have,

$$\frac{-by}{(\sinh t)^2}\cosh t - \frac{ax}{(\cosh t)^2}\sinh t = 0$$

$$\Rightarrow \quad \tanh t = -\left(\frac{by}{ax}\right)^{1/3}$$

$$\Rightarrow \quad \sinh t = \mp\left(\frac{by}{h}\right)^{1/3} \text{ and } \cosh t = \pm\left(\frac{ax}{h}\right)^{1/3} \qquad (3)$$

Where $\qquad h = \sqrt{(ax)^{2/3} - (by)^{2/3}}$

Using (3) in (2) , we get,

$$\frac{by}{-(by)^{1/3}}h + \frac{ax}{(ax)^{1/3}}h = a^2 + b^2$$

i.e. $\left((ax)^{2/3} - (by)^{2/3}\right)\left((ax)^{2/3} - (by)^{2/3}\right)^{1/2} = a^2 + b^2$

i.e. $(ax)^{2/3} - (by)^{2/3} = \left(a^2 + b^2\right)^{2/3}$

2. By considering the evolute of a curve as the envelope of its normal, find the evolute of

$x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

Solution :

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta \, .$$

Equation of normal line to the hyperbola is

$$(y - (\sin\theta - \theta\cos\theta)) = \frac{-1}{\tan\theta}(x - (\cos\theta + \theta\sin\theta))$$

$$\Rightarrow \quad y\sin\theta - \sin^2\theta + \theta\sin\theta\cos\theta = -x\cos\theta + \cos^2\theta + \theta\sin\theta\cos\theta$$

i.e. $\quad y\sin\theta + x\cos\theta = 1 \qquad (1)$

Differentiating (1) with respect to the parameter θ, we have

$$y\cos\theta - x\sin\theta = 0 \qquad (2)$$

Multiplying (1) by cosθ and (2) by sinθ and then subtracting, we have,

$$x = \cos\theta \qquad (3)$$

Similarly we get,

$$y = \sin\theta \qquad (4)$$

Eliminating θ between (3) and (4) we get the required evolute as $x^2 + y^2 = 1$