A Simple Level-Set Method for Solving Stefan Problems

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Level-Set Methods
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Agendas

- Abstract
- Classical Stefan problem formulation
- New Stefan problem formulation using level-set Methods
- Brief outlines of Discretization
- Numerical results

What is Stefan problem?

Classical Stefan Problem

- Stefan problems are problems that deal with change of phases
- Gas, liquid and solid
- One of the most common examples is ice melting in water
- In classical case, we often model Stefan problems from solid to liquid
- The new Stefan problem can model change of phase from solid to liquid

Classical Stefan Problem Formulation

Change of phases involves losing or gaining heat/energy of a material.

Latent Heat

The heat and energy gaining and losing in a matter

Classical Stefan Problem

When a solid material melts it forms an interface between solid and liquid part of the material. And let $s(t)$ be the position function for the interface

$$s(t) = \begin{cases} s(t)+, & \text{interface position of solid} \\ s(t)-, & \text{interface position of liquid} \end{cases} \tag{1}$$

Also, at the interface the temperature is the melting temperature of the material. Let it be T_m.

Classical Stefan Problem

So at the interface, the temperature function can be written as

$$T(s(t)+,t) = T(s(t)-,t) = T_m \tag{2}$$

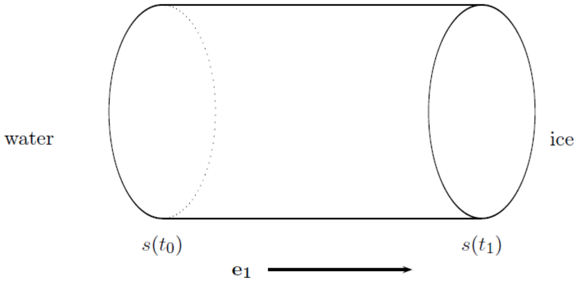
Classical Stefan Problem

In physics, we learn that heat and energy contributes to change of temperature through heat diffusion. Let $D(t)$ be the finite region boundary that the interface can reach the farthest and k_i be the diffusion coefficients. We get

$$HeatFlux = -k_i D(t) T(t) \tag{3}$$

Classical Stefan Problem

Now, consider a piece of ice in cylinder form



In order to find the total heat gain and loss we need to take integral of heat flux

Classical Stefan Problem

We set up an integral in cylindrical coordinates based on equation (2).

$$\int_{t_0}^{t_1} \int_{D(t)} [-k_1 D(s(t)-, t) \cdot e_1 - k_2 D(s(t)+, t) \cdot (-e_1)] dx\ dy\ dt$$
$$= A(D) \int_{t_0}^{t_1} [-k_1(s(t)-, t) + k_2(s(t)+, t)]\ dt$$

Classical Stefan Problem

Work out the integral above we get our Stefan condition

$$-k_1 T(s(t)-, t) + k_2 T(s(t)+, t) = LV \tag{4}$$

where L is latent heat and V is the derivative of interface position function which is interface moving velocity.

Level-Set Methods Implementation

Now, consider D as a square region. This time, instead of melting, we think about the solidification of a material. Imagine you drop a supercooled object into a liquid and the liquid starts to freeze.

Level-Set Methods Implementation

Since the material undergoes solidification process, the interface moves outward from solid to liquid. Let **n** be normal vector pointing outward at each point. And based on classical Stefan problem condition (4) we can get temperature rate of change.

$$\frac{\partial T}{\partial t} = \begin{cases} c_1 \dfrac{\partial T_{solid}}{\partial t} = \nabla \cdot (k_1 \nabla T), & \mathbf{x} \in \Omega \\ c_2 \dfrac{\partial T_{liquid}}{\partial t} = \nabla \cdot (k_2 \nabla T), & \mathbf{x} \in \Omega^c \end{cases} \tag{5}$$

Level-Set Methods Implementation

We get Stefan condition for the new Stefan problem

$$LV = -\left[k_2 \frac{\partial T_{liq}}{\partial n} - k_1 \frac{\partial T_{sol}}{\partial n}\right] \tag{6}$$

Level-Set Methods Implementation

Since we are considering solidification process, we have to consider **surface tension** amd **molecular kinetic coefficient**. Such relation is given by **Gibbs Thomson relation**.

Gibbs Thomson Relation

$$T(\mathbf{x}, t) = -\epsilon_c \kappa - \epsilon_v V \tag{7}$$

Note taht κ is curvature.

Level-Set Methods Implementation

For simplicity, the temperature function at the interface is

$$\frac{\partial T}{\partial t} = \nabla \cdot \nabla T = \Delta T \tag{8}$$

Level-Set Methods Implementation

So equation (5) becomes

$$\frac{\partial T}{\partial n} = \begin{cases} c_1 \dfrac{\partial T_{solid}}{\partial t} = \nabla \cdot (k_1 \nabla T), & \mathbf{x} \in \Omega \\ \Delta T, & \mathbf{x} \in \Gamma \\ c_2 \dfrac{\partial T_{liquid}}{\partial t} = \nabla \cdot (k_2 \nabla T), & \mathbf{x} \in \Omega^c \end{cases} \tag{9}$$

Let latent heat to be one, and Stefan condition reduces to

$$V = -\left[\frac{\partial T}{\partial n}\right], \qquad \mathbf{x} \in \Gamma \tag{10}$$

at the interface

Level-Set Methods Implementation

So the problem is simplified to finding T and $\Gamma(t)$

$$\frac{\partial T}{\partial t} = \Delta T \tag{11}$$

$$V = -\left[\frac{\partial T}{\partial n}\right] \tag{12}$$

at the interface.

Level-Set Methods Implementation

We construct a level set function ϕ such that at any point time t, the front is equal to the zero level set of ϕ.

$$\Gamma(t) = \{\mathbf{x} \in D : \phi(\mathbf{x}, t) = 0\} \tag{13}$$

Level-Set Methods Implementation

First we need to construct signed distance function

$$\phi(\mathbf{x}, 0) = \begin{cases} +d, & \text{outside the solid} \\ 0, & \text{at the interface} \\ -d, & \text{within the solid} \end{cases} \tag{14}$$

Level-Set Methods Implementation

The idea behind the level set method is to move ϕ with the correct speed V at the front and then update temperature; with new position stored in temperature function

Level-Set Methods Implementation

Given the normal speed V we need to construct an advection eqaution

$$\phi_t = F|\nabla\phi| \tag{15}$$

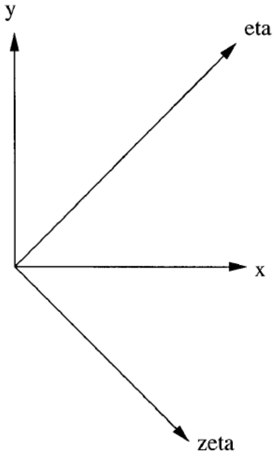
Along with other necessary components to solve this Stefan problem

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} \tag{16}$$

$$\kappa = \nabla \cdot \mathbf{n} \tag{17}$$

Level-Set Methods Implementation

In the new Stefan problem, we compute ∇T at every grid point. And F is the extension of V off the interface. Add extra η and ζ coordinates.



Level-Set Methods Implementation

$$\left[\frac{\partial T}{\partial x}\right]_t + S(\phi\phi_x)\left[\frac{\partial T}{\partial x}\right]_x = 0 \tag{18}$$

$$\left[\frac{\partial T}{\partial y}\right]_t + S(\phi\phi_y)\left[\frac{\partial T}{\partial y}\right]_y = 0 \tag{19}$$

$$\left[\frac{\partial T}{\partial \eta}\right]_t + S(\phi\phi_\eta)\left[\frac{\partial T}{\partial \eta}\right]_\eta = 0 \tag{20}$$

$$\left[\frac{\partial T}{\partial \zeta}\right]_t + S(\phi\phi_\zeta)\left[\frac{\partial T}{\partial \zeta}\right]_\zeta = 0 \tag{21}$$

Reinitialization of ϕ

$$\phi_t = S(\phi_0)(1 - |\nabla \phi|) \tag{22}$$

where $\phi_0 = \phi(\mathbf{x}, 0)$ is the zero level-set function. We can smooth the sign function S by the equation

$$S_\epsilon(\phi_0) = \frac{\phi_0}{\sqrt{\phi_0^2 + \epsilon^2}} \tag{23}$$

Growing Frank Sphere

For demonstration we will use a two dimension Stefan problem called *Growing Frank Spheres*. The solid region is a cylinder of radius $R = St^{1/2}$ and the temperature field is $T(r,t)$ is given by

$$T(r,t) = T(s) = \begin{cases} T_\infty \left(1 - \frac{F(s)}{F(S)}\right), & s > S \\ 0, & s < S \end{cases} \tag{24}$$

where $r = \sqrt{x^2+y^2}$, $s = r/t^{1/2}$ and T_∞ is a constant.

Growing Frank Sphere