203201- HW2-Wet part:

[redacted]

Numerical and Practical Bit Allocation for Two Dimensional Signals

a.

$$\frac{\partial}{\partial x}[\phi(x,y)] = A\sin(2\pi\omega_y y)\,(-\sin(2\pi\omega_x x)\,2\pi\omega_x) = -2A\pi\omega_x \sin(2\pi\omega_x x)\sin(2\pi\omega_y y)$$

$$\frac{\partial}{\partial y}[\phi(x,y)] = A\cos(2\pi\omega_x x)\cos(2\pi\omega_y y)\,2\pi\omega_y = 2A\pi\omega_y \cos(2\pi\omega_x x)\cos(2\pi\omega_y y)$$

The energy of the partial derivatives are:

$$E_x = \int\int_{[0,1]^2}\left(\frac{\partial\phi}{\partial x}(x,y)\right)^2 \text{dxdy} = \int\int_{[0,1]^2}\left(-2A\pi\omega_x\sin(2\pi\omega_x x)\sin(2\pi\omega_y y)\right)^2 dxdy$$

$$= 2^2A^2\pi^2\omega_x^2\int_0^1\int_0^1 \sin^2(2\pi\omega_x x)\sin^2(2\pi\omega_y y)\,dx\,dy$$

$$= 4A^2\pi^2\omega_x^2\left(\int_0^1 \sin^2(2\pi\omega_x x)\;dx\right)\left(\int_0^1 \sin^2(2\pi\omega_y y)\;dy\right)$$

Using $\sin^2(x) = \frac{1-\cos(2x)}{2}$ *we get:*

$$\int\int_{[0,1]^2}\left(\frac{\partial\phi}{\partial x}(x,y)\right)^2 \text{dxdy} = 4A^2\pi^2\omega_x^2\left(\int_0^1 \frac{1-\cos(4\pi\omega_x x)}{2}\,dx\right)\left(\int_0^1 \frac{1-\cos(4\pi\omega_y y)}{2}\,dy\right)$$

$$= 4A^2\pi^2\omega_x^2\,\frac{1}{2}\left[x - \frac{\sin(4\pi\omega_x x)}{4\pi\omega_x}\right]_0^1 \frac{1}{2}\left[y - \frac{\sin(4\pi\omega_y y)}{4\pi\omega_y}\right]_0^1$$

$$= 4A^2\pi^2\omega_x^2 \frac{1}{4} \cdot 1 \cdot 1 = A^2\pi^2\omega_x^2$$

$$E_y = \int\int_{[0,1]^2} \left(\frac{\partial \phi}{\partial y}(x,y)\right)^2 \text{dxdy} = \int\int_{[0,1]^2} \big(2A\pi\omega_y \cos(2\pi\omega_x x)\cos(2\pi\omega_y y)\big)^2 \, dxdy$$

$$= 2^2A^2\pi^2\omega_y^2 \int_0^1 \int_0^1 \cos^2(2\pi\omega_x x)\cos^2(2\pi\omega_y y) \, dy \, dx$$

$$= 4A^2\pi^2\omega_y^2 \left(\int_0^1 \cos^2(2\pi\omega_x x) \, dx\right)\left(\int_0^1 \cos^2(2\pi\omega_y y) \, dy\right)$$

Using $\cos^2(x) = \frac{1+\cos(2x)}{2}$ we get :

$$\int\int_{[0,1]^2} \left(\frac{\partial \phi}{\partial y}(x,y)\right)^2 \text{dxdy} = 4A^2\pi^2\omega_y^2 \left(\int_0^1 \frac{1+\cos(4\pi\omega_x x)}{2} \, dx\right)\left(\int_0^1 \frac{1+\cos(4\pi\omega_y y)}{2} \, dy\right)$$

$$= 4A^2\pi^2\omega_y^2 \frac{1}{2}\left[x + \frac{\sin(4\pi\omega_x x)}{4\pi\omega_x}\right]_0^1 \frac{1}{2}\left[y + \frac{\sin(4\pi\omega_y y)}{4\pi\omega_y}\right]_0^1$$

$$= 4A^2\pi^2\omega_y^2 \frac{1}{4} \cdot 1 \cdot 1 = A^2\pi^2\omega_y^2$$

The value Range of the function $\phi(x, y) = \phi(x, y) = A\cos(2\pi\omega_x x)\sin(2\pi\omega_y y)$ is [-A,A].

The reason is that $1 \leq \cos(2\pi\omega_x x) \leq 1$ and $-1 \leq \sin(2\pi\omega_y y) \leq 1$

Therefore, their product is bounded: $1 \leq \cos(2\pi\omega_x x)\sin(2\pi\omega_y y) \leq 1$

Multiplying by A we get that the possible range of values the function can get is [-A,A].

Thus, we get that $Range(\phi) = 2A$

Given that $A = 2500, \omega_x = 2, \omega_y = 7$ we get:

-Energy $\left(\frac{\partial\phi(x,y)}{\partial x}\right) = E_x = A^2\pi^2\omega_x^2 = 2500^2 \cdot 4 \cdot \pi^2$

-Energy$\left(\frac{\partial\phi(x,y)}{\partial y}\right) = E_y = A^2\pi^2\omega_y^2 = 2500^2 \cdot 49 \cdot \pi^2$

-Value range = 2A= 5000

b.

The signal:

c.

We got numeric value range 4999.995 vs analytical 5000.

We got numeric E_x= 246740109.70157394 vs analytical E_x= 246740110

We got numeric E_y= 3022566299.0979996 vs analytical E_y= 3022566348

We got close numerical and analytical energies, and the value ranges are close.

The results will be closer when the resolution gets higher, as we will approximate the integral and the derivative better.

d.

We approximate the solution via two methods:

1) Lagrange multipliers
2) Iterative method

1) **Lagrange multipliers:**

Given:

$$E_x = \int\int_{[0,1]^2} \left(\frac{\partial \phi}{\partial x}(x,y)\right)^2 \text{dxdy}$$

$$E_y = \int\int_{[0,1]^2} \left(\frac{\partial \phi}{\partial y}(x,y)\right)^2 \text{dxdy}$$

$$= Range(\phi) = \phi_H - \phi_L$$

$$B : Total\ bit\ budget$$

We want to minimize the Mean Squared Error (MSE) which we approximated to be:

$$MSE_{N_x,N_y,b} \approx \frac{1}{12N_x^2}E_x + \frac{1}{12N_y^2}E_y + \frac{1}{12}\frac{(\phi_H - \phi_L)^2}{2^{2b}}$$

Constraint:

$$N_x \cdot N_y \cdot b \leq B$$

We assume the optimal solution will use the full budget (B), allowing us to use Lagrange multipliers to solve the problem.

The Lagrangian for this problem combines the objective function and the constraint using a Lagrange multiplier $lambda$:

$$\mathcal{L}(N_x, N_y, b, \lambda) = \frac{1}{12N_x^2}E_x + \frac{1}{12N_y^2}E_y + \frac{1}{12}\frac{(\phi_H - \phi_L)^2}{2^{2b}} + \lambda(N_x \cdot N_y \cdot b - B)$$

To find the optimal values, we need to take the partial derivatives of the Lagrangian with respect to N_x, N_y, b λ, and set them to zero:

$$\frac{\partial \mathcal{L}}{\partial N_x} = -\frac{E_x}{6N_x^3} + \lambda N_y b = 0$$

$$\frac{\partial \mathcal{L}}{\partial N_y} = -\frac{E_y}{6N_y^3} + \lambda N_x b = 0$$

$$\frac{\partial \mathcal{L}}{\partial b} = -\frac{(\phi_H - \phi_L)^2 \ln(2)}{6 \cdot 2^{2b}} + \lambda N_x N_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = N_x \cdot N_y \cdot b - B = 0$$

From the first two equations, we can express λ in terms of N_x, N_y, b:

$$\lambda = \frac{E_x}{6N_x^3 N_y b}$$

$$\lambda = \frac{E_y}{6N_y^3 N_x b}$$

Setting the two expressions for λ equal we get:

$$\frac{E_x}{6N_x^3 N_y b} = \frac{E_y}{6N_y^3 N_x b}$$

We can assume that N_x, N_y, b are not equal to zero since all are needed to obtain a valid solution for the bit allocation problem. Thus, we get:

$$E_x N_y^2 = E_y N_x^2$$

Since E_X and E_y represent norms in the space of squared integral functions L^2, both energies will be positive. We also want a solution where both $N_x > 0$ and $N_y > 0$, so we can apply the square root function to both sides of the equation and ignore the negative solution:

$$\frac{N_y}{N_x} = \sqrt{\frac{E_y}{E_x}}$$

Using the relationship between N_y and N_x, and substitute back into the constraint:

$$N_x \cdot N_y \cdot b = B$$

$$N_x \cdot \left(N_x \sqrt{\frac{E_y}{E_x}} \right) \cdot b = B$$

$$N_x^2 \sqrt{\frac{E_y}{E_x}} \cdot b = B$$

$$N_x = \left(\frac{B}{b \sqrt{\frac{E_y}{E_x}}} \right)^{1/2}$$

Then:

$$N_y = N_x \sqrt{\frac{E_y}{E_x}} = \left(\frac{B}{b \sqrt{\frac{E_y}{E_x}}} \right)^{1/2} \cdot \sqrt{\frac{E_y}{E_x}} = \left(\frac{B \cdot \sqrt{E_y}}{b \cdot \sqrt{E_x}} \right)^{1/2}$$

So we managed to express both N_x, N_y, using b. Now we need to find the optimal b using the third equation:

$$-\frac{(\phi_H - \phi_L)^2 \ln(2)}{6 \cdot 2^{2b}} + \lambda N_x N_y = 0$$

Substitute λ:

$$-\frac{(\phi_H - \phi_L)^2 \ln(2)}{6 \cdot 2^{2b}} + \frac{E_x}{6 N_x^3 N_y b} N_x N_y = 0 \backslash \frac{(\phi_H - \phi_L)^2 \ln(2)}{6 \cdot 2^{2b}} = \frac{E_x}{6 N_x^2 b}$$

$$Using\ N_x^2 = \frac{B}{b\sqrt{\frac{E_y}{E_x}}} we\ \ get:$$

$$\frac{(\phi_H - \phi_L)^2 \ln(2)}{6 \cdot 2^{2b}} = \frac{E_x}{6\frac{B}{b\sqrt{\frac{E_y}{E_x}}}b}$$

$$(\phi_H - \phi_L)^2 \ln(2)\, 6\frac{B}{\sqrt{\frac{E_y}{E_x}}} = 6 \cdot 2^{2b} \cdot E_x \backslash \frac{(\phi_H - \phi_L)^2 ln(2)B}{\sqrt{(E_y)\sqrt{E_x}}} = 2^{2b}$$

$$\Rightarrow b = \frac{\log_2\left(\frac{(\phi_H - \phi_L)^2 ln(2)B}{\sqrt{(E_y)\sqrt{E_x}}}\right)}{2}$$

and we also have:

$$N_x = \left(\frac{B}{b\sqrt{\frac{E_y}{E_x}}}\right)^{1/2}$$

$$N_y = N_x\sqrt{\frac{E_y}{E_x}} = \frac{B}{N_x b}$$

We implemented this in our code by evaluating b using the approximated energies and value range. Then we calculated Nx, Ny based on b. (We checked upper/lower b and for each b we also calculated upper Nx, lower Nx and we chose the one that minimized the MSE).

2) Iterative method:

The second method we implemented is an iterative algorithm. This approach leverages the 1D solution we previously developed in class. The algorithm iteratively fixes either $N_x \; or$ and then solves for the remaining N and b, similar to the Lloyd-Max algorithm.

The steps of the algorithm are as follows:

1. Initialization:

   1.1 Start with an initial guess for N_y.
   1.2 $prev\_mse \leftarrow \text{maxint}$

2. Repeat until convergence or maximum iterations are reached:

   2.1 $B_1 \leftarrow \dfrac{B}{N_y}$

   2.2 $b \leftarrow$ Solve bit allocation for(B1) using lambert W function

   2.3 $b \leftarrow \lfloor b \rfloor \; if \; mse(\lfloor b \rfloor) < mse(\lceil b \rceil) \; else \; \lceil b \rceil$

   2.4. $N_x \leftarrow \left\lfloor \frac{B1}{b} \right\rfloor$

   2.5. $B_2 \leftarrow \dfrac{B}{N_x}$

   2.6 $b \leftarrow$ Solve bit allocation for(B2) using lambert W function

   2.7 $b \leftarrow \lfloor b \rfloor \; if \; mse(\lfloor b \rfloor) < mse(\lceil b \rceil) \; else \; \lceil b \rceil$

   2.8. $N_y \leftarrow \left\lfloor \frac{B2}{b} \right\rfloor$

   2.9 $b \leftarrow \left\lfloor \frac{B}{\text{NxNy}} \right\rfloor$

   2.10 if (|mse(b,Ny,Nx)-prev_mse|< ϵ) break;

   2.11 $\text{prev\_mse} \leftarrow \text{mse(b, Ny, Nx)}$

3. Return N_x, N_y, b

e.

We got for **B = 5000**:

Lagrange: $b = 3, N_x = 21, N_y = 79$

Iterative: $b = 4, N_x = 25, N_y = 50$

B = 50000

Lagrange: $b = 5, N_x = 54, N_y = 185$

Iterative: $b = 6, N_x = 46, N_y = 181$

g.

B = 5000:

Grid search: $b = 3, N_x = 21, N_y = 79$

Lagrange: $b = 3, N_x = 21, N_y = 79$

Iterative: $b = 4, N_x = 25, N_y = 50$

B = 50000

Grid search: $b = 5, N_x = 54, N_y = 185.$

Lagrange: $b = 5, N_x = 54, N_y = 185$

Iterative: $b = 6, N_x = 46, N_y = 181$

Firstly, since we are looking for optimal solutions which are integers, the grid search solution is like the brute force solution and therefore represent us the best solution we can find for this bit budget (we run on all possibilities). We should still remember that it's the best solution for the formulation that was presented in class (which is also an approximation). However, this approach is more computationally expensive.

We can see that Lagrange multipliers method resulted us the same optimal solution as grid search. This method is less expensive because we computed the optimal solution manually given B, value-range, and energies. We also can notice that the iterative method is more accurate as the budget of bits grow. It also very sensitive to the starting point.

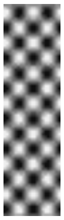
The iterative solution was also close to the grid search solution, but it did not result us the exact optimal solution. However, it also did not require us to solve the problem manually and it also converged fast (under 20 iterations).

We also observed that for a higher bit budget, N_y received most of the new budget.

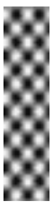
This makes sense because if we fix x, we can see our signal $\phi(x, y) = A\cos(2\pi 2x)\sin(2\pi 7y)$

will have shorter period (higher frequency) in the y direction, necessitating more samples in the y direction to capture the fluctuations for a given x.

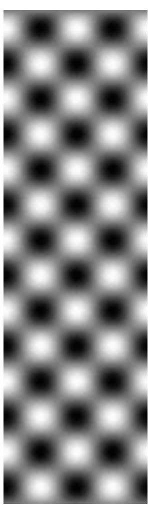
Reconstruct The image for B =5000 (Lagrange and search) $\boldsymbol{b = 3, N_x = 21, N_y = 79}$



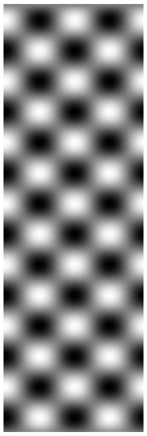
Reconstruct The image B =5000 (Iterative) $\boldsymbol{b = 4, N_x = 17, N_y = 73}$



Reconstruct The image (Lagrange and search) $\boldsymbol{b = 5, N_x = 54, N_y = 185}$



Reconstruct The image (iterative) $\boldsymbol{b} = 6, \boldsymbol{N_x} = \mathbf{5}2, \boldsymbol{N_y} = \mathbf{1}60$



h.

a.

Based on the calculations we showed in the previous a sections (we presented formulas for general parameters) we get:

Given that $A = 2500, \omega_x = 7, \omega_y = 2$

we get:

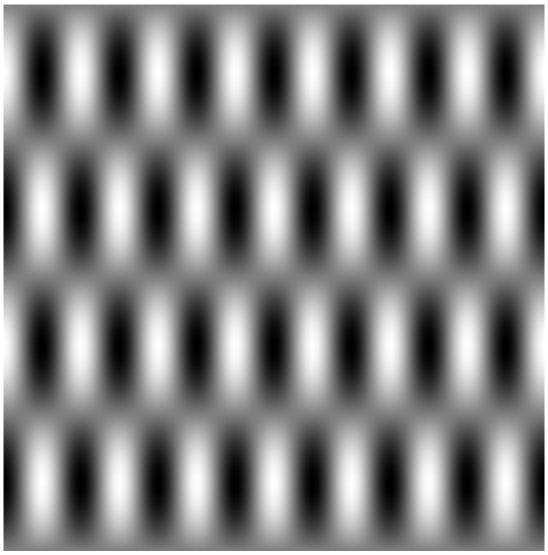
$$E_x = Energy\left(\frac{\partial \phi(x,y)}{\partial x}\right) = A^2\pi^2\omega_x^2 = 2500^2 \cdot 49 \cdot \pi^2$$

$$E_y = Energy\left(\frac{\partial \phi(x,y)}{\partial y}\right) = A^2\pi^2\omega_y^2 = 2500^2 \cdot 4 \cdot \pi^2$$

$$Value\ range\ =\ 2A =\ 5000$$

b. We will present the signal using uniform sampling with N= 500 on both axes (500

and high number of quantization levels (256)



C.

Comparing results

We got numeric value range 4999.995 vs analytical 5000.

We got numeric E_x= 3022566347.833 vs analytical E_x= 3022566348

We got numeric E_y= 246740110.0272 vs analytical E_y= 246740110

We got close numerical and analytical energies, and the value ranges are pretty close.

The results will be closer when the resolution gets higher, as we will approximate the integral and the derivative better.

d+e

We got for **B = 5000**:

Lagrange: $b = 3, N_x = 77, N_y = 21$

Iterative: $b = 4, N_x = 78, N_y = 16$

B = 50000

Lagrange: $b = 5, N_x = 188, N_y = 53$

Iterative: $b=6, N_x=181, N_y=46$

f.

We got:

B = 5000:

- Lagrange: $b=3, N_x=77, N_y=21$
- Iterative: $b=4, N_x=78, N_y=16$
- Grid search: : $b=3, N_x=79, N_y=21$

B = 50000:

- Lagrange: $b=5, N_x=188, N_y=53$
- Iterative: $b=6, N_x=181, N_y=46$
- Grid search: : $b=5, N_x=185, N_y=54$

Firstly, we can observe that we got a symmetric result of the grid search compared to the case where $\omega_x=2, \omega_y=7$. This is what we expected since, given the energies:

$$E_X = \text{Energy}\left(\frac{\partial\phi(x,y)}{\partial x}\right) = A^2\pi^2\omega_x^2$$

$$E_Y = \text{Energy}\left(\frac{\partial\phi(x,y)}{\partial y}\right) = A^2\pi^2\omega_y^2$$

and the MSE loss is:

$$MSE_{N_x,N_y,b} \approx \frac{1}{12N_x^2}E_x + \frac{1}{12N_y^2}E_y + \frac{1}{12}\frac{(\phi_H-\phi_L)^2}{2^{2b}}$$

We can see that the roles of N_x and N_y are the same in the MSE, and the energy calculations are the same up to the decision of ω. Thus, changing ω_x. and ω_y is symmetric in a sense, and we should have gotten opposite results for N_x and N_y as we did.

We did not obtain the same results for numeric and grid search as before. The Lagrange method results are close to the grid search result.

However,we should notice that earlier the ratio $\sqrt{\frac{E_y}{E_x}} > 1$ since $E_y > E_x$, and when we changed the roles of x and y, we get $\sqrt{\frac{E_y}{E_x}} < 1$. This can cause numeric instability and lead to different results for N_x (our method first calculate b, than N_x than N_y based on b, N_x).

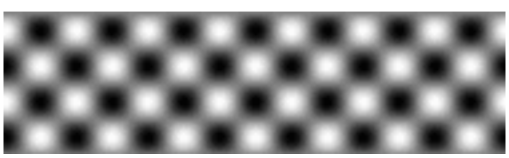
For the iterative search, we again see that as the budget is larger the approximation is closer to the grid search solution.

Reconstruct the image (Lagrange) for $\boldsymbol{b = 3, N_x = 77, N_y = 21}$

Reconstruct the image (iterative) for $\boldsymbol{b} = 4, \boldsymbol{N_x} = \mathbf{7}8, \boldsymbol{N_y} = 16$

Reconstruct the image (grid search) for $\boldsymbol{b} = 3, \boldsymbol{N_x} = \mathbf{7}9, \boldsymbol{N_y} = 21$

Reconstruct image (Numeric) for $\boldsymbol{b = 5, N_x = 188, N_y = 53}$



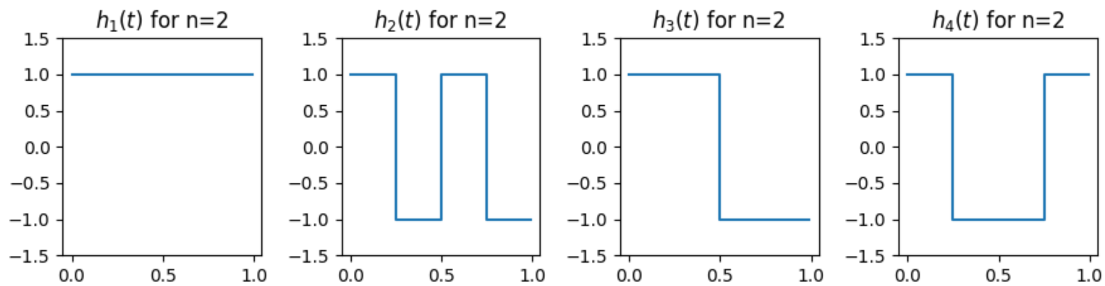
Reconstruct image (iterative) for $\boldsymbol{b} = 6, \boldsymbol{N_x} = \mathbf{18}1, \boldsymbol{N_y} = 46$

Reconstruct image (grid) for $\boldsymbol{b} = 6, \boldsymbol{N_x} = \mathbf{18}5, \boldsymbol{N_y} = 54$

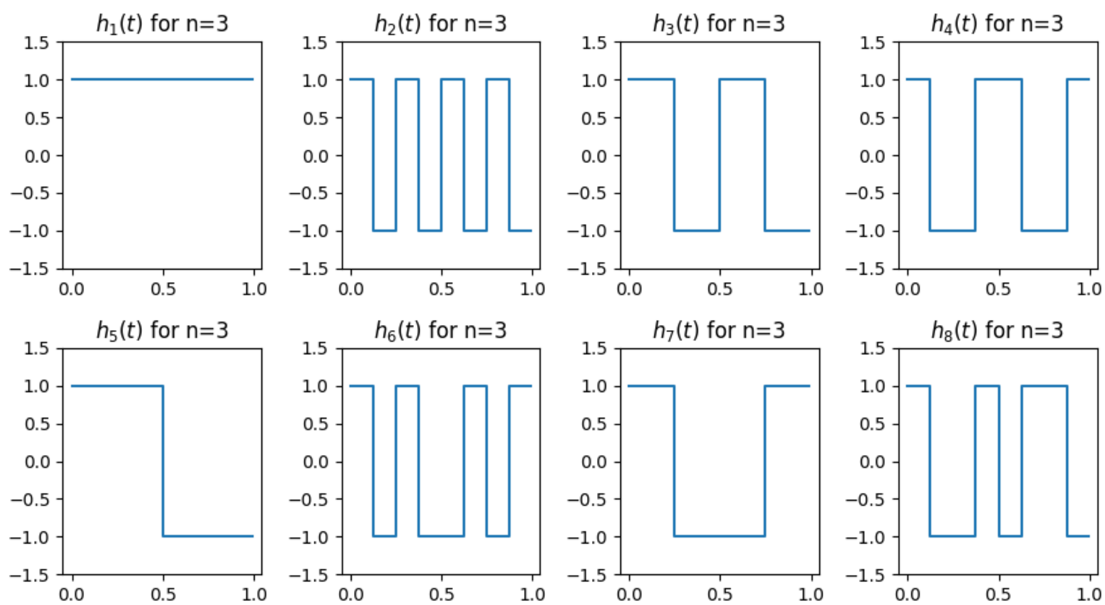
Hadamard, Hadamard-Walsh, and Haar matrices

b. Hamdard basis:

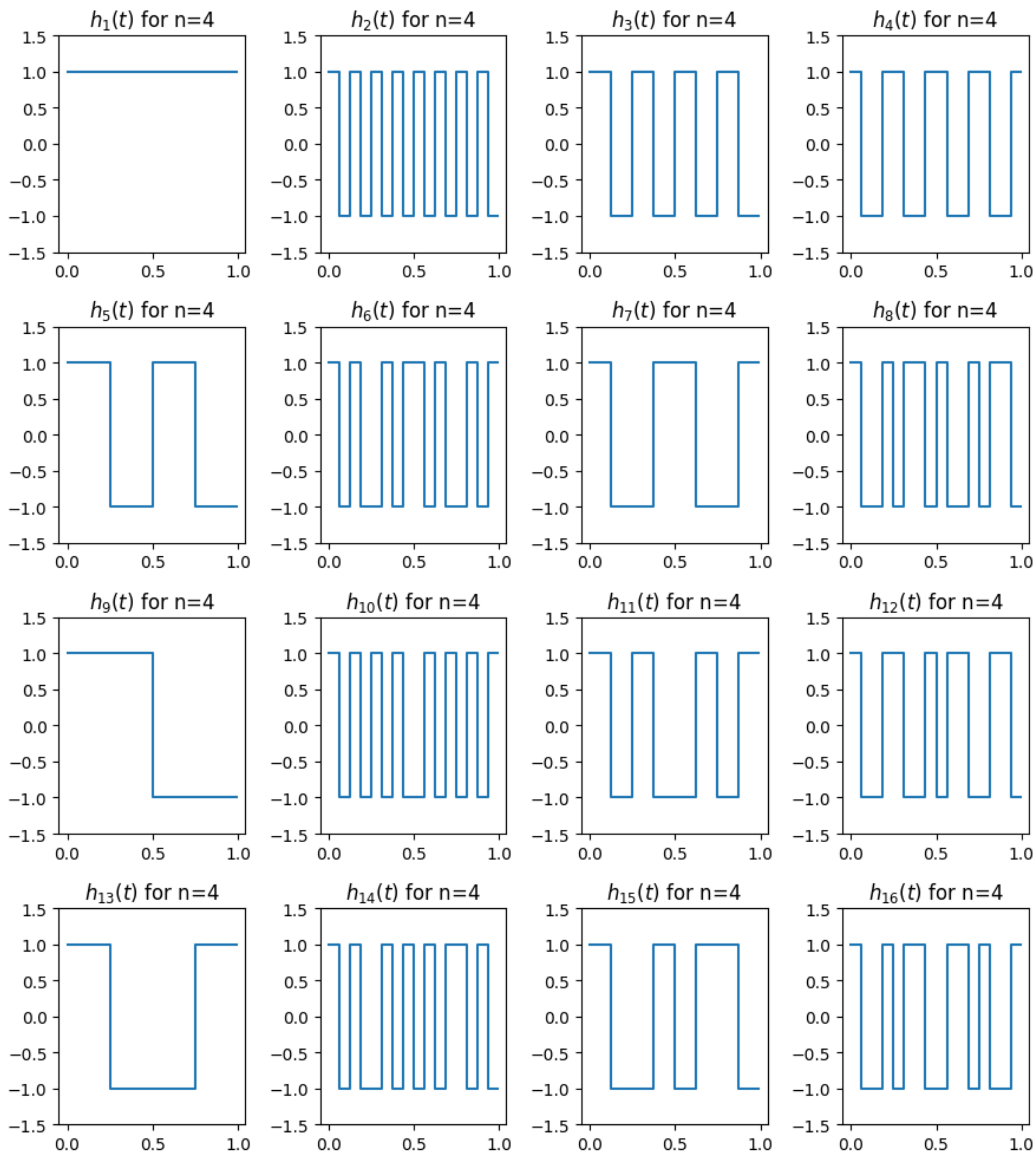
n= 2:



n=3:



n=4:


$h_1(t)$ for n=4
$h_2(t)$ for n=4
$h_3(t)$ for n=4
$h_4(t)$ for n=4
$h_5(t)$ for n=4
$h_6(t)$ for n=4
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$h_8(t)$ for n=4
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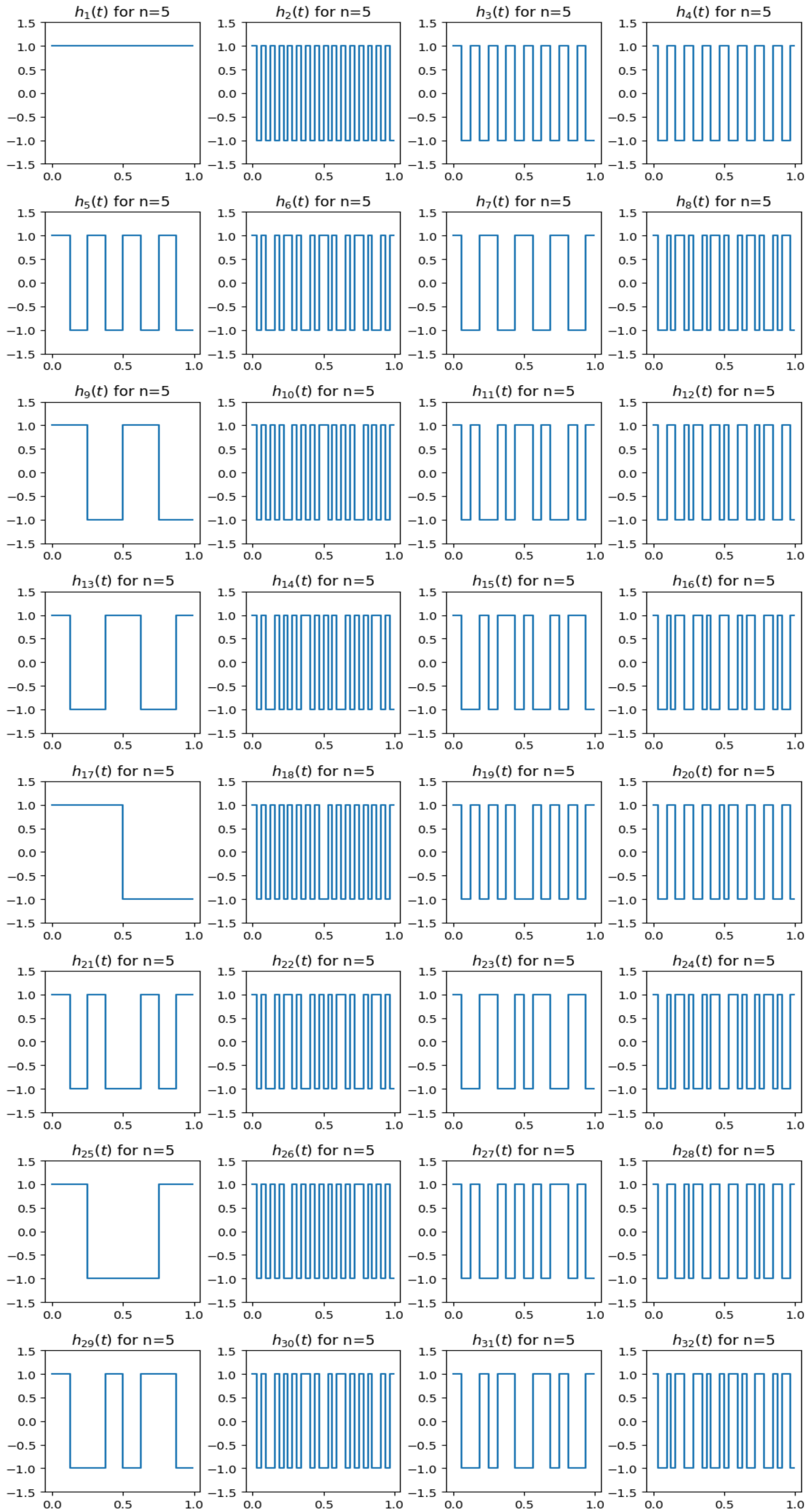
$h_{15}(t)$ for n=4
$h_{16}(t)$ for n=4

n=5



$h_1(t)$ for n=5
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$h_{29}(t)$ for n=5
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$h_{31}(t)$ for n=5
$h_{32}(t)$ for n=5
1.5
1.0
0.5
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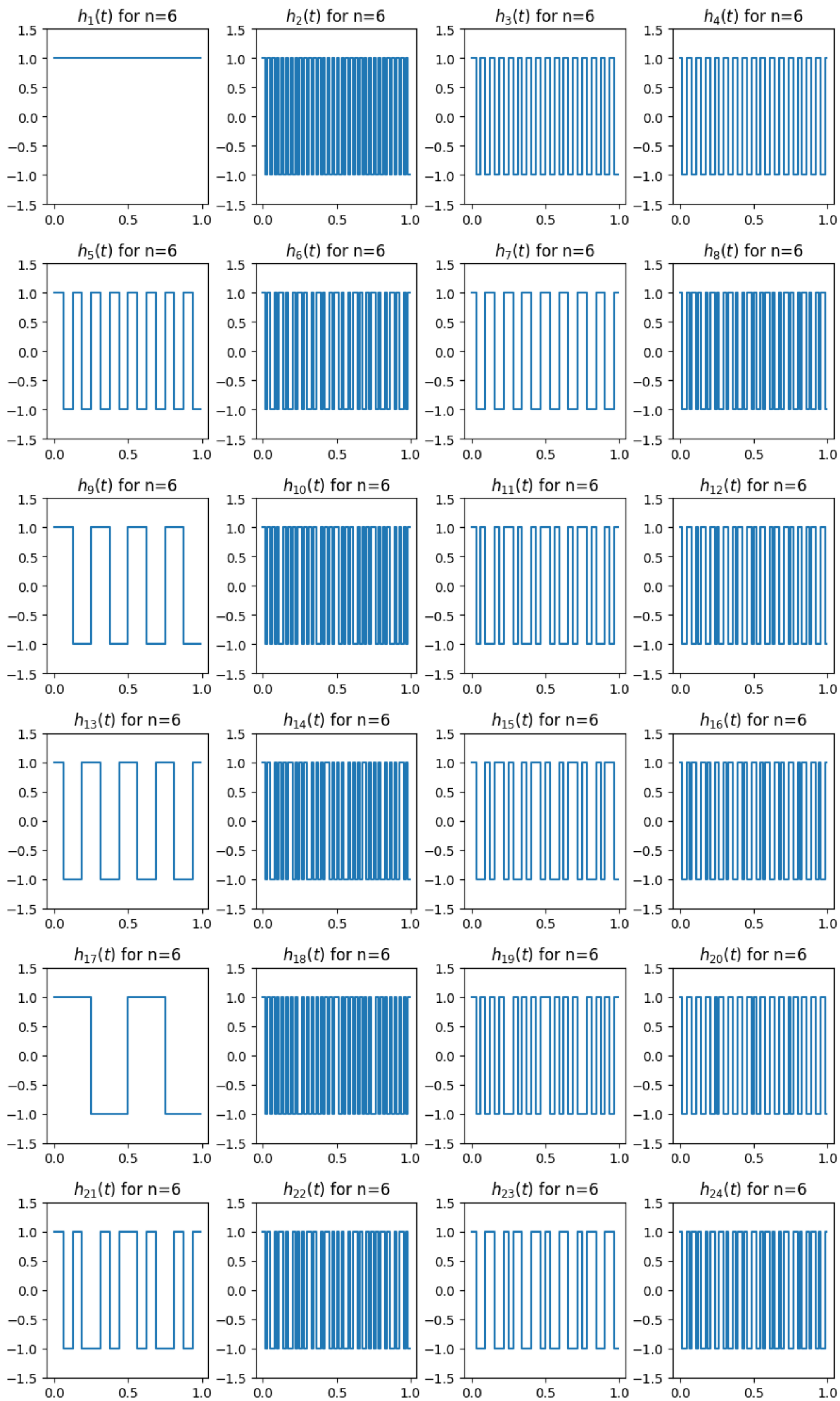
−0.5
−1.0
−1.5
0.0
0.5
1.0

n=6


$h_{1}(t)$ for n=6
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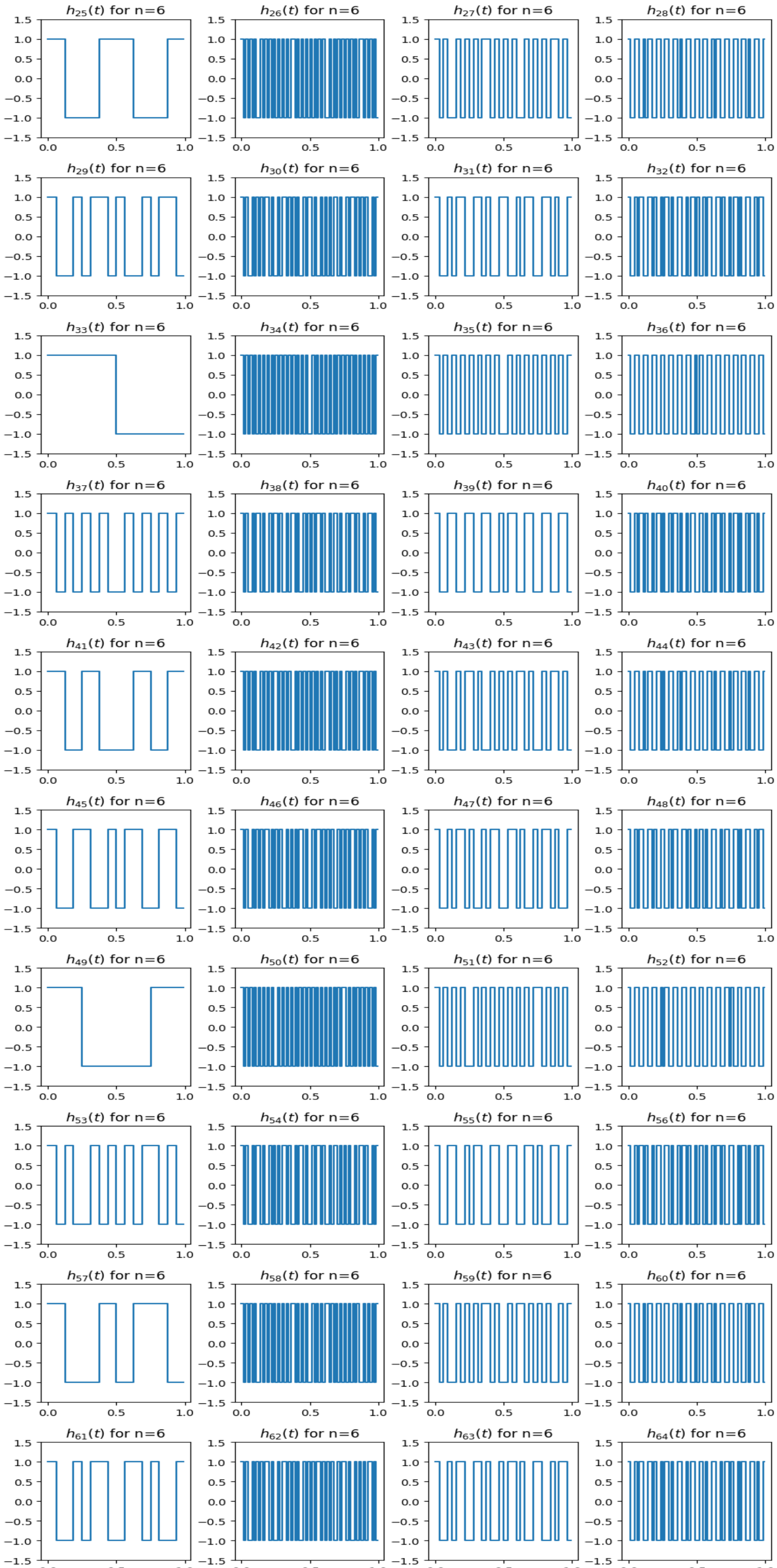
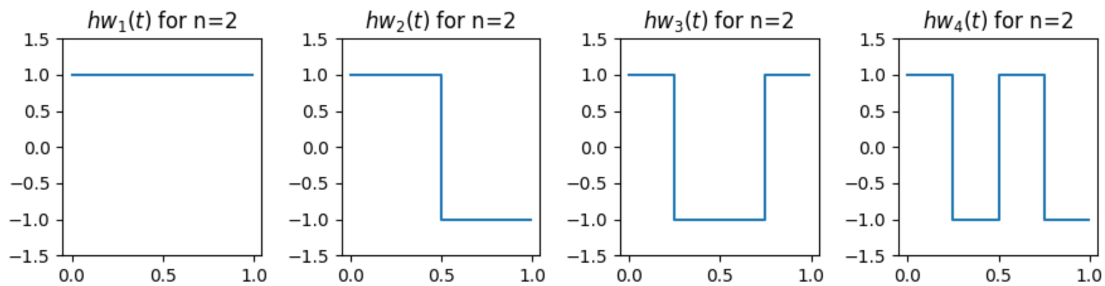
$h_{24}(t)$ for n=6

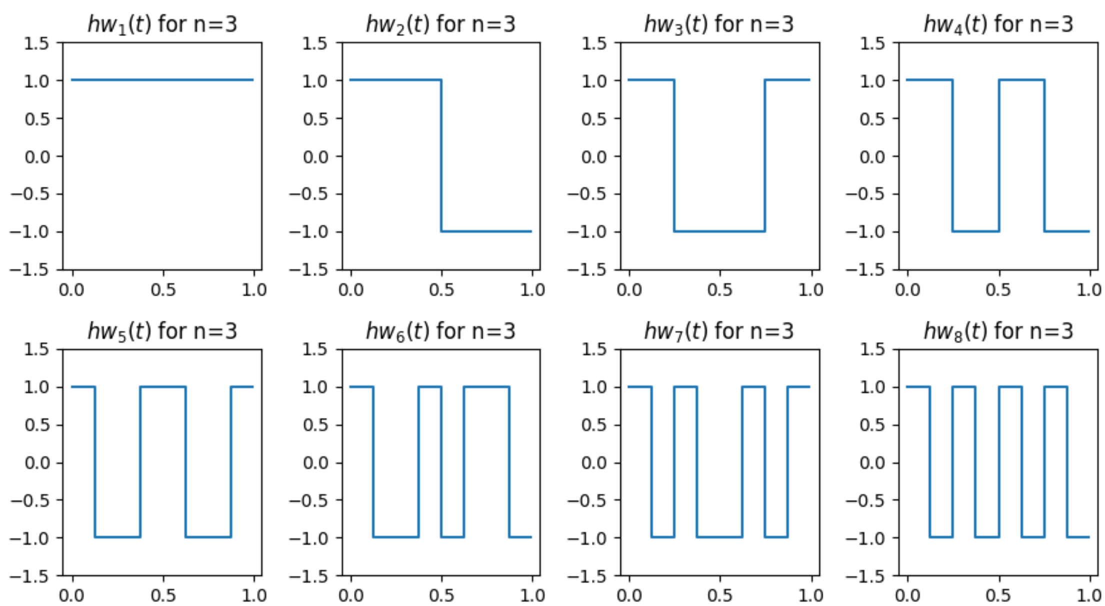

$h_{25}(t)$ for n=6
$h_{26}(t)$ for n=6
$h_{27}(t)$ for n=6
$h_{28}(t)$ for n=6
$h_{29}(t)$ for n=6
$h_{30}(t)$ for n=6
$h_{31}(t)$ for n=6
$h_{32}(t)$ for n=6
$h_{33}(t)$ for n=6
$h_{34}(t)$ for n=6
$h_{35}(t)$ for n=6
$h_{36}(t)$ for n=6
$h_{37}(t)$ for n=6
$h_{38}(t)$ for n=6
$h_{39}(t)$ for n=6
$h_{40}(t)$ for n=6
$h_{41}(t)$ for n=6
$h_{42}(t)$ for n=6
$h_{43}(t)$ for n=6
$h_{44}(t)$ for n=6
$h_{45}(t)$ for n=6
$h_{46}(t)$ for n=6
$h_{47}(t)$ for n=6
$h_{48}(t)$ for n=6
$h_{49}(t)$ for n=6
$h_{50}(t)$ for n=6
$h_{51}(t)$ for n=6
$h_{52}(t)$ for n=6
$h_{53}(t)$ for n=6
$h_{54}(t)$ for n=6
$h_{55}(t)$ for n=6
$h_{56}(t)$ for n=6
$h_{57}(t)$ for n=6
$h_{58}(t)$ for n=6
$h_{59}(t)$ for n=6
$h_{60}(t)$ for n=6
$h_{61}(t)$ for n=6
$h_{62}(t)$ for n=6
$h_{63}(t)$ for n=6
$h_{64}(t)$ for n=6

d. Walsh Hamdard basis:

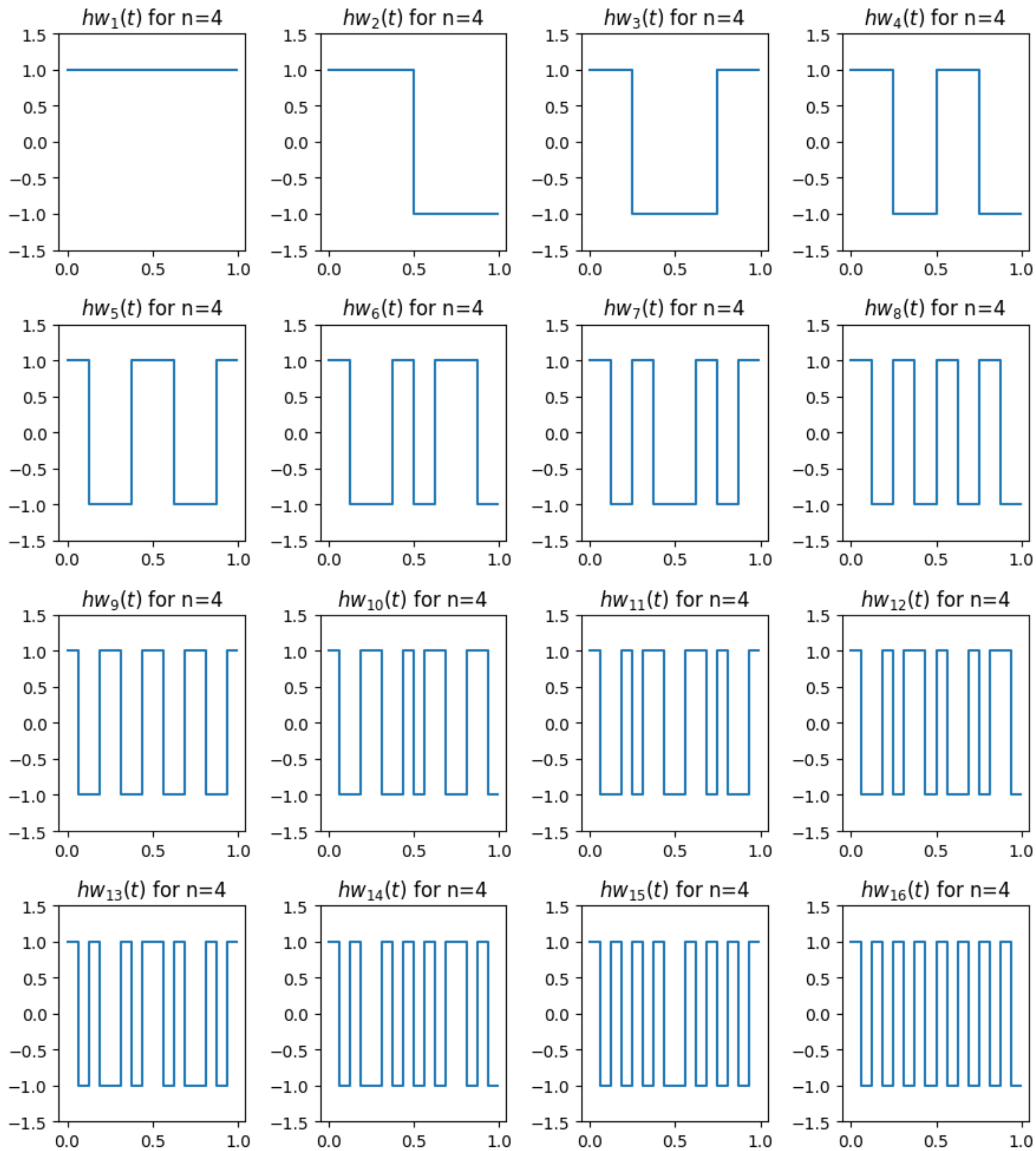
n = 2



n = 3

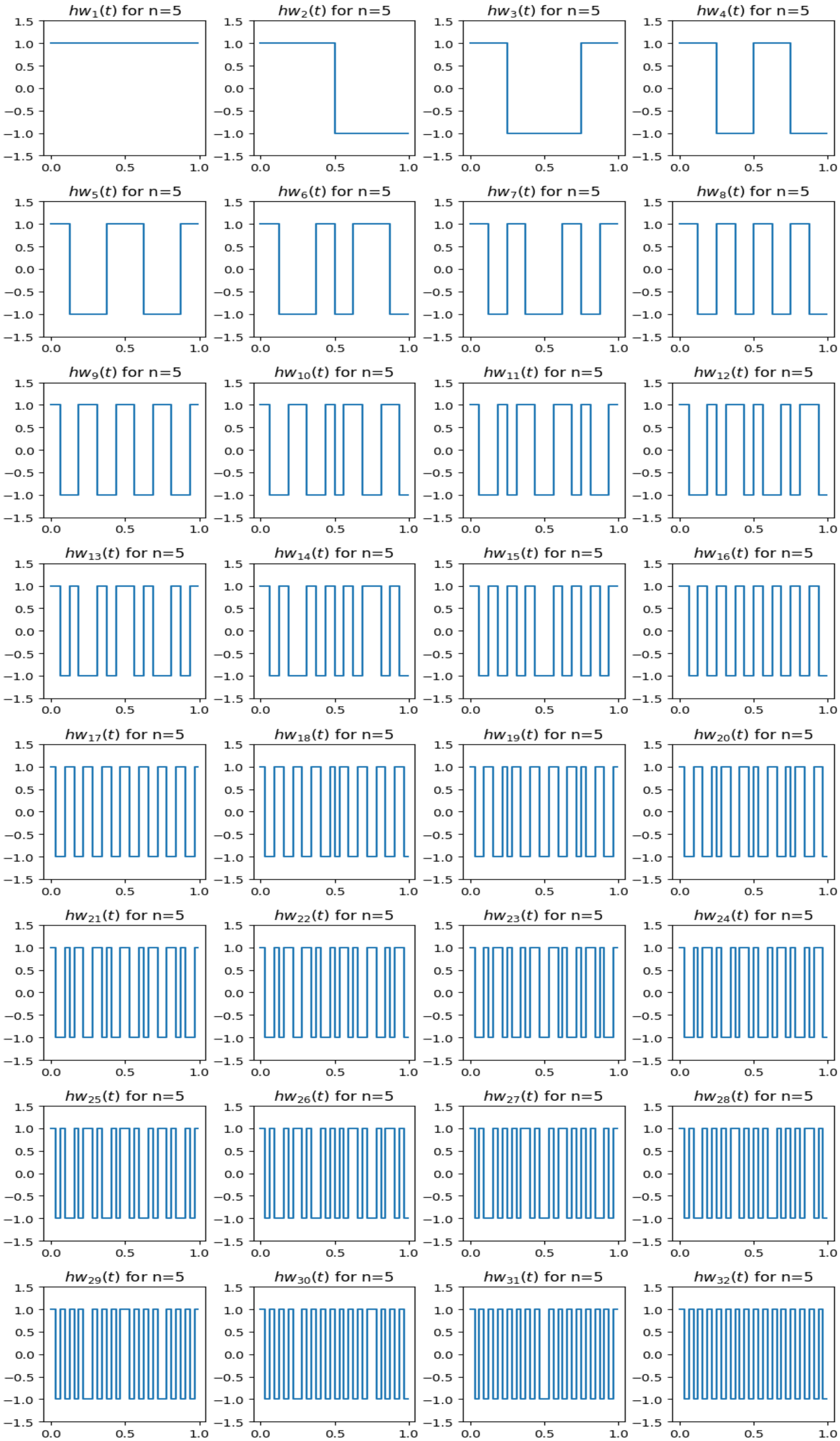


n = 4



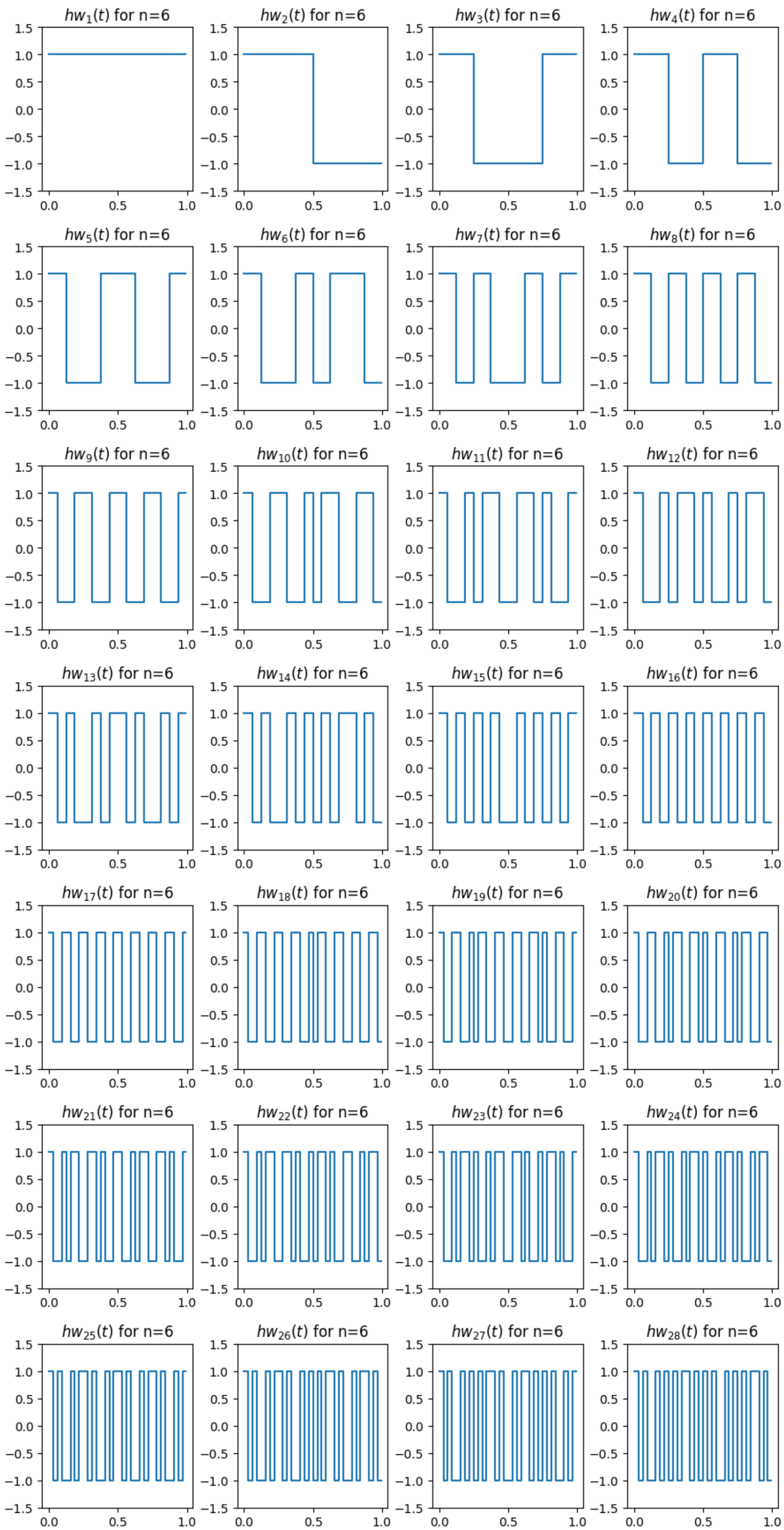
$hw_1(t)$ for n=4
$hw_2(t)$ for n=4
$hw_3(t)$ for n=4
$hw_4(t)$ for n=4
$hw_5(t)$ for n=4
$hw_6(t)$ for n=4
$hw_7(t)$ for n=4
$hw_8(t)$ for n=4
$hw_9(t)$ for n=4
$hw_{10}(t)$ for n=4
$hw_{11}(t)$ for n=4
$hw_{12}(t)$ for n=4
$hw_{13}(t)$ for n=4
$hw_{14}(t)$ for n=4
$hw_{15}(t)$ for n=4
$hw_{16}(t)$ for n=4
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.5
1.0

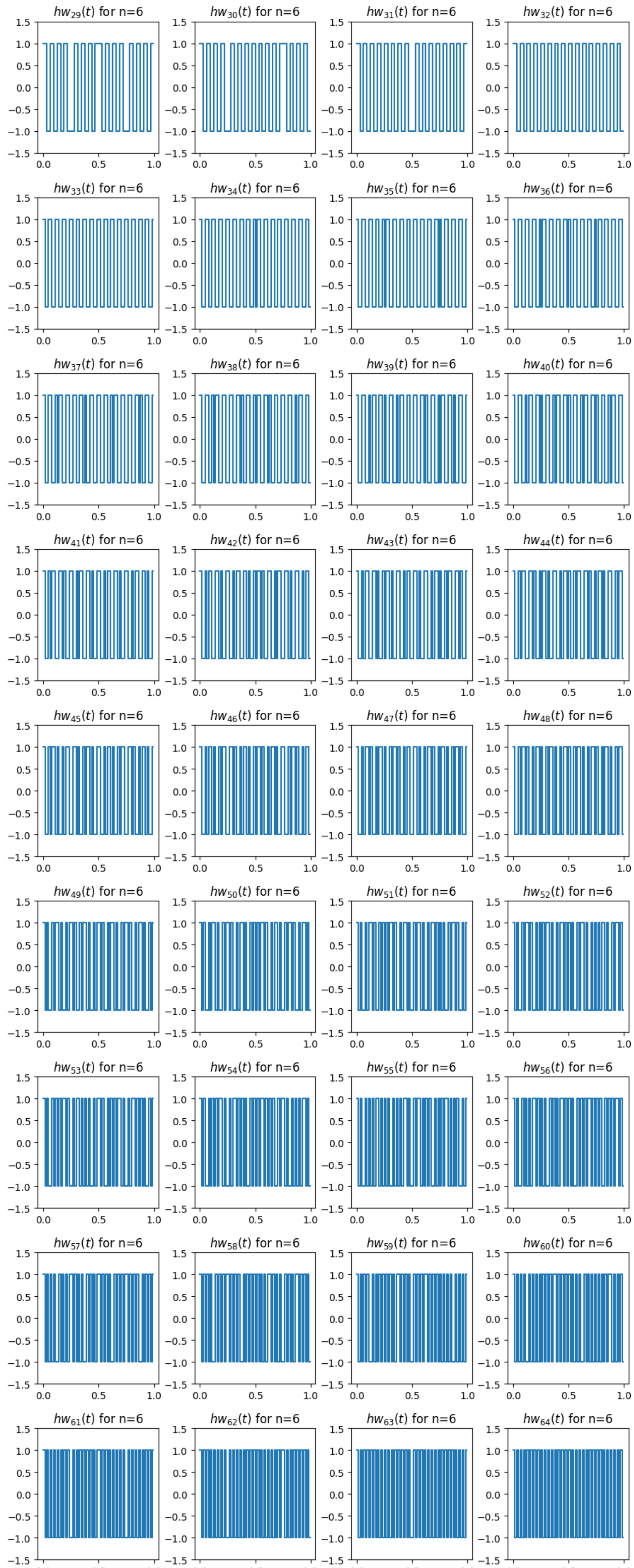
n=5



$hw_1(t)$ for n=5
$hw_2(t)$ for n=5
$hw_3(t)$ for n=5
$hw_4(t)$ for n=5
$hw_5(t)$ for n=5
$hw_6(t)$ for n=5
$hw_7(t)$ for n=5
$hw_8(t)$ for n=5
$hw_9(t)$ for n=5
$hw_{10}(t)$ for n=5
$hw_{11}(t)$ for n=5
$hw_{12}(t)$ for n=5
$hw_{13}(t)$ for n=5
$hw_{14}(t)$ for n=5
$hw_{15}(t)$ for n=5
$hw_{16}(t)$ for n=5
$hw_{17}(t)$ for n=5
$hw_{18}(t)$ for n=5
$hw_{19}(t)$ for n=5
$hw_{20}(t)$ for n=5
$hw_{21}(t)$ for n=5
$hw_{22}(t)$ for n=5
$hw_{23}(t)$ for n=5
$hw_{24}(t)$ for n=5
$hw_{25}(t)$ for n=5
$hw_{26}(t)$ for n=5
$hw_{27}(t)$ for n=5
$hw_{28}(t)$ for n=5
$hw_{29}(t)$ for n=5
$hw_{30}(t)$ for n=5
$hw_{31}(t)$ for n=5
$hw_{32}(t)$ for n=5

n=6:

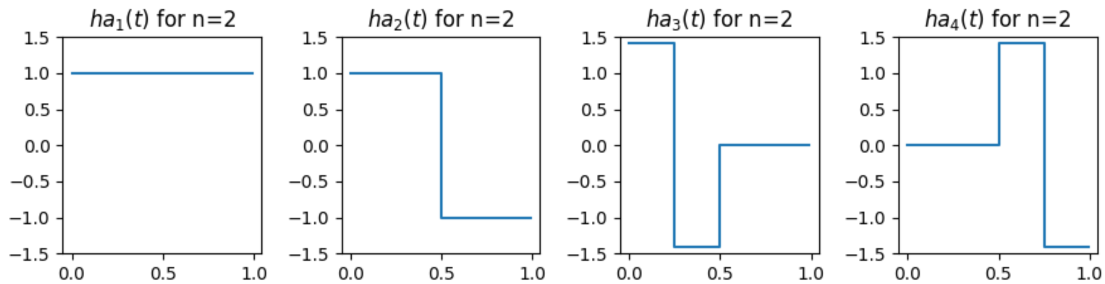

$hw_1(t)$ for n=6
$hw_2(t)$ for n=6
$hw_3(t)$ for n=6
$hw_4(t)$ for n=6
$hw_5(t)$ for n=6
$hw_6(t)$ for n=6
$hw_7(t)$ for n=6
$hw_8(t)$ for n=6
$hw_9(t)$ for n=6
$hw_{10}(t)$ for n=6
$hw_{11}(t)$ for n=6
$hw_{12}(t)$ for n=6
$hw_{13}(t)$ for n=6
$hw_{14}(t)$ for n=6
$hw_{15}(t)$ for n=6
$hw_{16}(t)$ for n=6
$hw_{17}(t)$ for n=6
$hw_{18}(t)$ for n=6
$hw_{19}(t)$ for n=6
$hw_{20}(t)$ for n=6
$hw_{21}(t)$ for n=6
$hw_{22}(t)$ for n=6
$hw_{23}(t)$ for n=6
$hw_{24}(t)$ for n=6
$hw_{25}(t)$ for n=6
$hw_{26}(t)$ for n=6
$hw_{27}(t)$ for n=6
$hw_{28}(t)$ for n=6



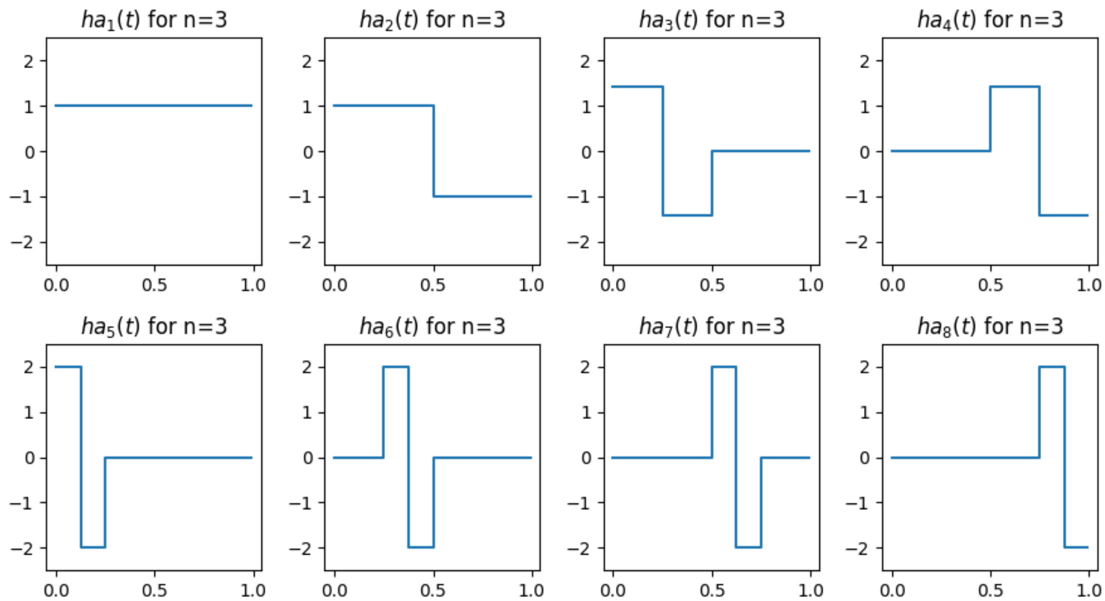
$hw_{29}(t)$ for n=6
$hw_{30}(t)$ for n=6
$hw_{31}(t)$ for n=6
$hw_{32}(t)$ for n=6
$hw_{33}(t)$ for n=6
$hw_{34}(t)$ for n=6
$hw_{35}(t)$ for n=6
$hw_{36}(t)$ for n=6
$hw_{37}(t)$ for n=6
$hw_{38}(t)$ for n=6
$hw_{39}(t)$ for n=6
$hw_{40}(t)$ for n=6
$hw_{41}(t)$ for n=6
$hw_{42}(t)$ for n=6
$hw_{43}(t)$ for n=6
$hw_{44}(t)$ for n=6
$hw_{45}(t)$ for n=6
$hw_{46}(t)$ for n=6
$hw_{47}(t)$ for n=6
$hw_{48}(t)$ for n=6
$hw_{49}(t)$ for n=6
$hw_{50}(t)$ for n=6
$hw_{51}(t)$ for n=6
$hw_{52}(t)$ for n=6
$hw_{53}(t)$ for n=6
$hw_{54}(t)$ for n=6
$hw_{55}(t)$ for n=6
$hw_{56}(t)$ for n=6
$hw_{57}(t)$ for n=6
$hw_{58}(t)$ for n=6
$hw_{59}(t)$ for n=6
$hw_{60}(t)$ for n=6
$hw_{61}(t)$ for n=6
$hw_{62}(t)$ for n=6
$hw_{63}(t)$ for n=6
$hw_{64}(t)$ for n=6

f. Haar basis

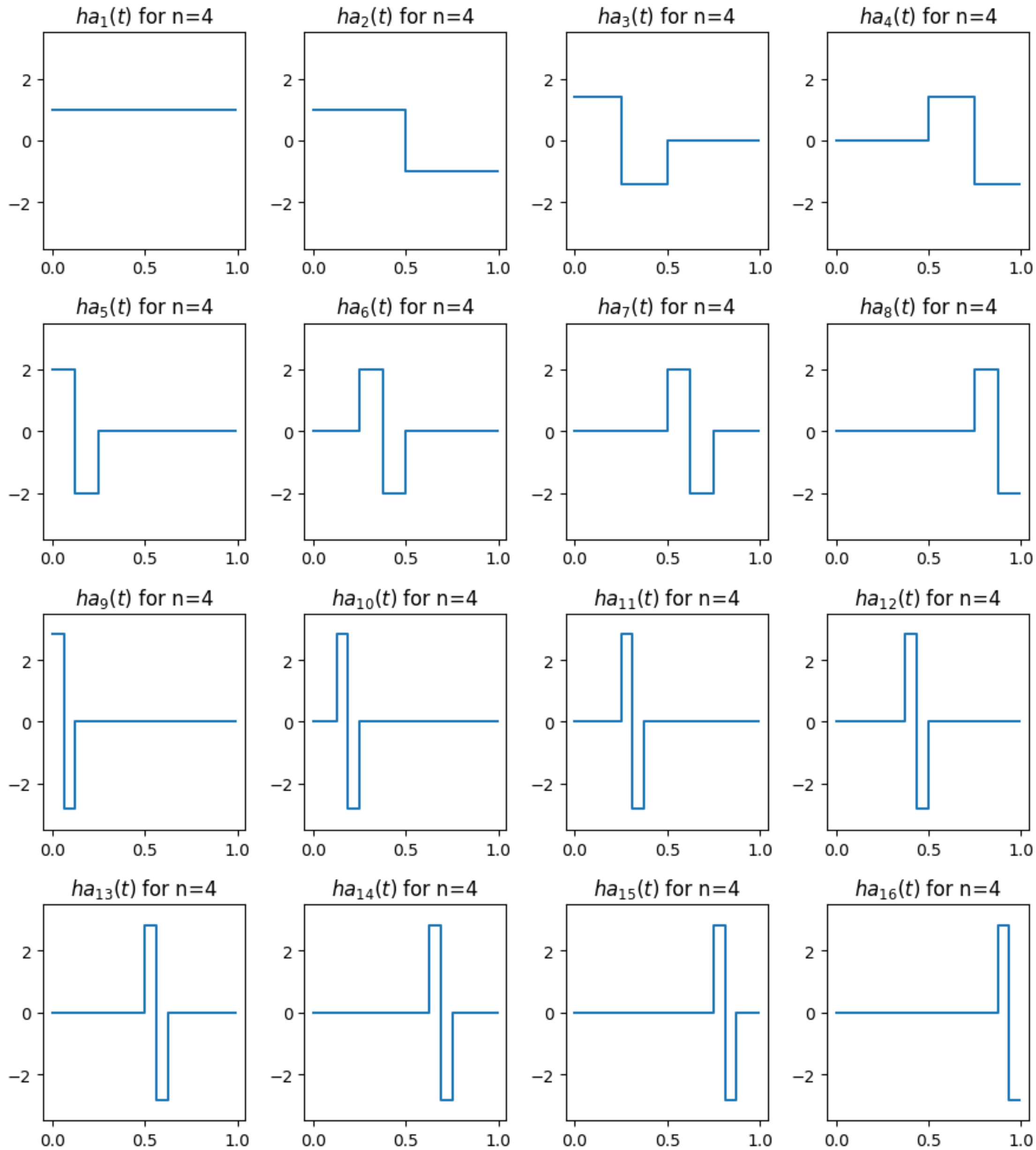
n=2



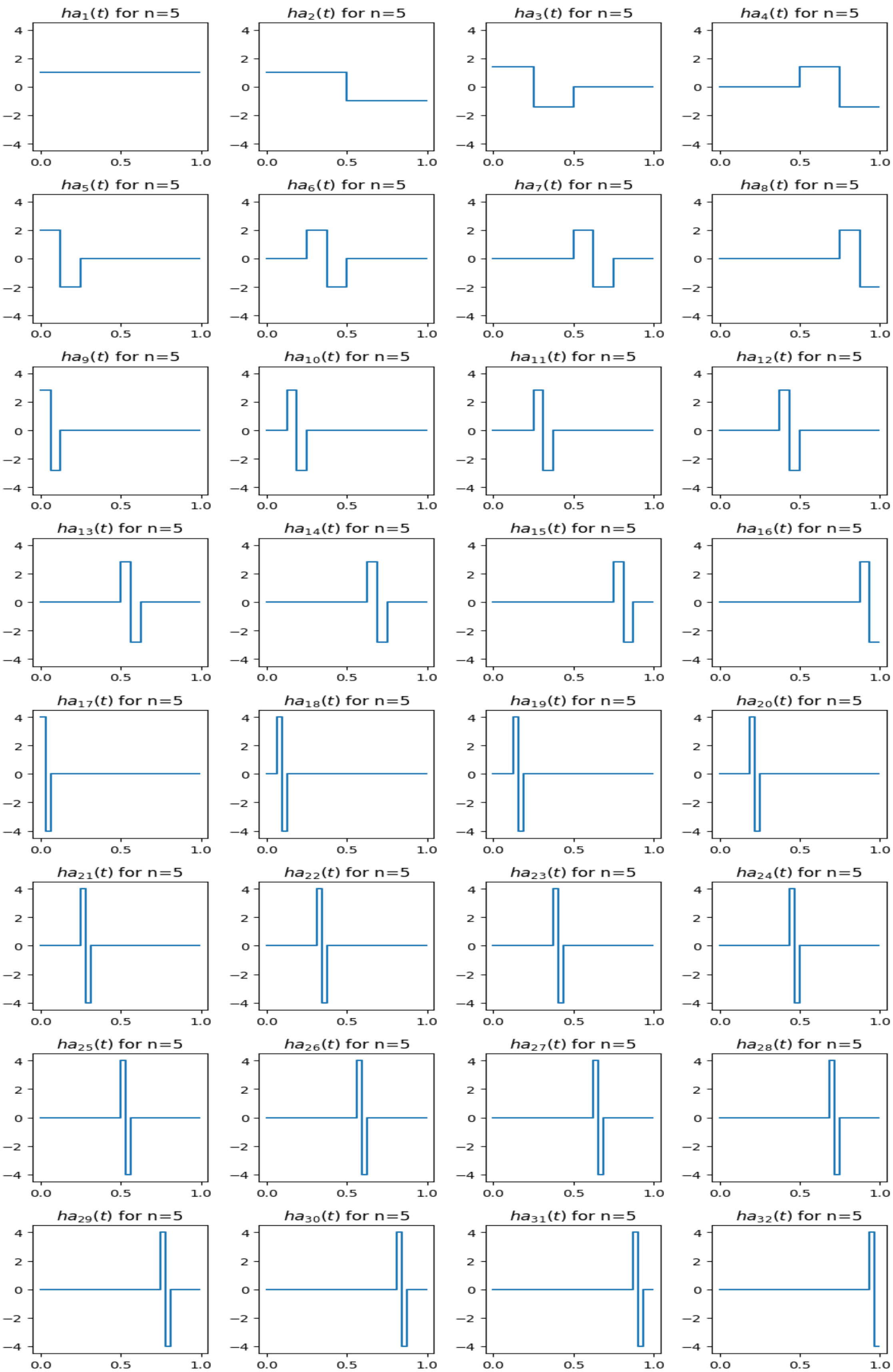
n=3



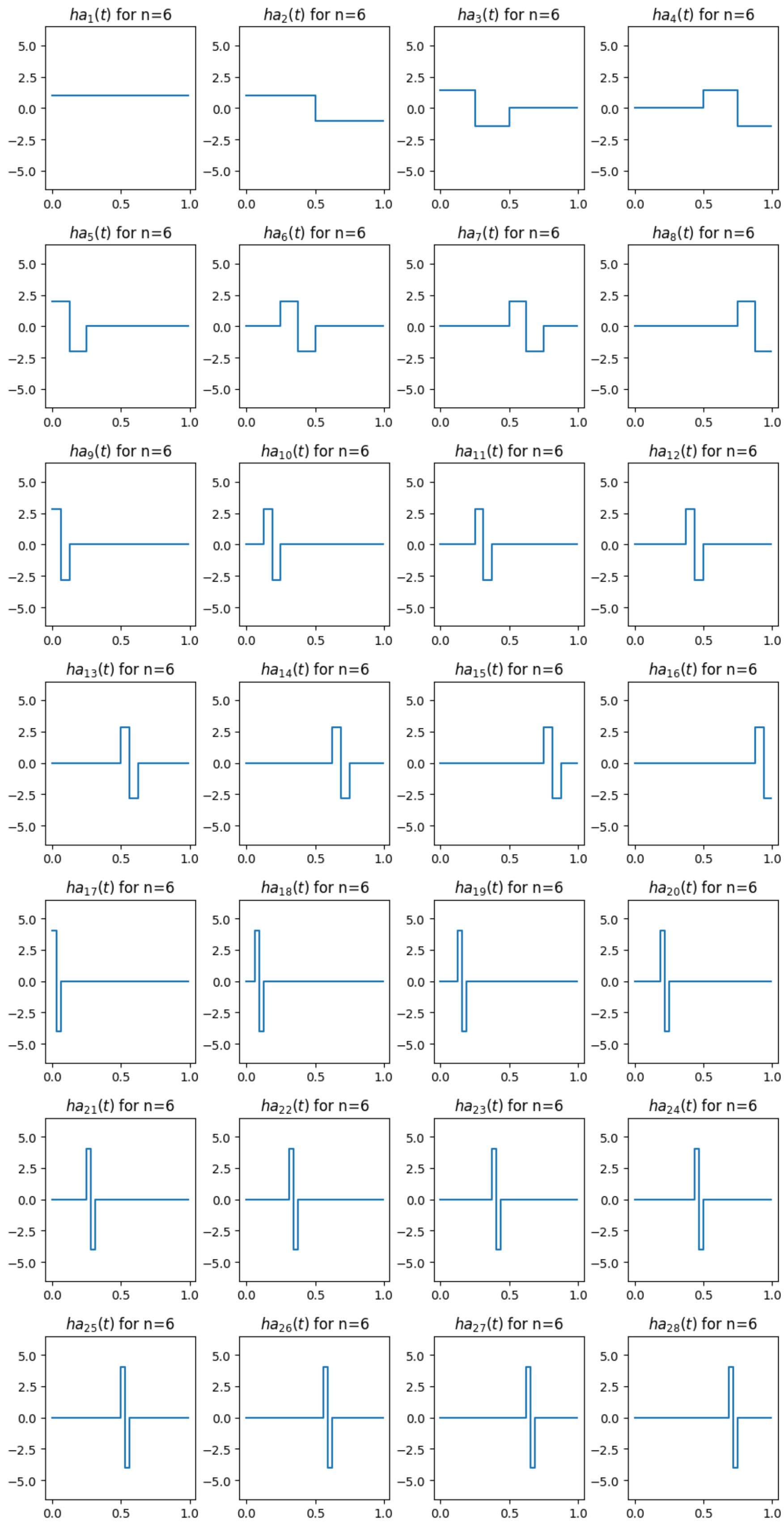
n=4

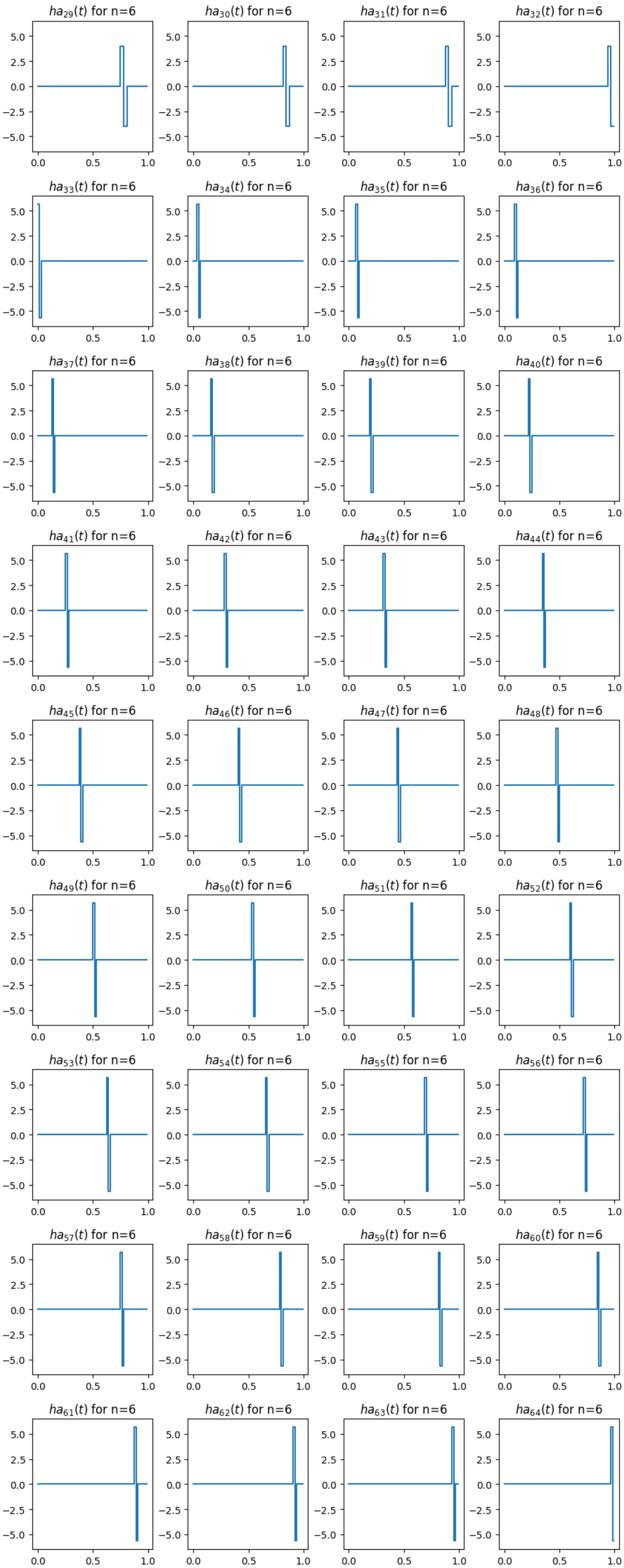

$ha_1(t)$ for n=4
$ha_2(t)$ for n=4
$ha_3(t)$ for n=4
$ha_4(t)$ for n=4
$ha_5(t)$ for n=4
$ha_6(t)$ for n=4
$ha_7(t)$ for n=4
$ha_8(t)$ for n=4
$ha_9(t)$ for n=4
$ha_{10}(t)$ for n=4
$ha_{11}(t)$ for n=4
$ha_{12}(t)$ for n=4
$ha_{13}(t)$ for n=4
$ha_{14}(t)$ for n=4
$ha_{15}(t)$ for n=4
$ha_{16}(t)$ for n=4
2
0
−2
0.0
0.5
1.0

n=5:


$ha_1(t)$ for n=5
$ha_2(t)$ for n=5
$ha_3(t)$ for n=5
$ha_4(t)$ for n=5
$ha_5(t)$ for n=5
$ha_6(t)$ for n=5
$ha_7(t)$ for n=5
$ha_8(t)$ for n=5
$ha_9(t)$ for n=5
$ha_{10}(t)$ for n=5
$ha_{11}(t)$ for n=5
$ha_{12}(t)$ for n=5
$ha_{13}(t)$ for n=5
$ha_{14}(t)$ for n=5
$ha_{15}(t)$ for n=5
$ha_{16}(t)$ for n=5
$ha_{17}(t)$ for n=5
$ha_{18}(t)$ for n=5
$ha_{19}(t)$ for n=5
$ha_{20}(t)$ for n=5
$ha_{21}(t)$ for n=5
$ha_{22}(t)$ for n=5
$ha_{23}(t)$ for n=5
$ha_{24}(t)$ for n=5
$ha_{25}(t)$ for n=5
$ha_{26}(t)$ for n=5
$ha_{27}(t)$ for n=5
$ha_{28}(t)$ for n=5
$ha_{29}(t)$ for n=5
$ha_{30}(t)$ for n=5
$ha_{31}(t)$ for n=5
$ha_{32}(t)$ for n=5

n=6:


$ha_1(t)$ for n=6
$ha_2(t)$ for n=6
$ha_3(t)$ for n=6
$ha_4(t)$ for n=6
$ha_5(t)$ for n=6
$ha_6(t)$ for n=6
$ha_7(t)$ for n=6
$ha_8(t)$ for n=6
$ha_9(t)$ for n=6
$ha_{10}(t)$ for n=6
$ha_{11}(t)$ for n=6
$ha_{12}(t)$ for n=6
$ha_{13}(t)$ for n=6
$ha_{14}(t)$ for n=6
$ha_{15}(t)$ for n=6
$ha_{16}(t)$ for n=6
$ha_{17}(t)$ for n=6
$ha_{18}(t)$ for n=6
$ha_{19}(t)$ for n=6
$ha_{20}(t)$ for n=6
$ha_{21}(t)$ for n=6
$ha_{22}(t)$ for n=6
$ha_{23}(t)$ for n=6
$ha_{24}(t)$ for n=6
$ha_{25}(t)$ for n=6
$ha_{26}(t)$ for n=6
$ha_{27}(t)$ for n=6
$ha_{28}(t)$ for n=6
5.0
2.5
0.0
−2.5
−5.0
0.0
0.5
1.0


$ha_{29}(t)$ for n=6
$ha_{30}(t)$ for n=6
$ha_{31}(t)$ for n=6
$ha_{32}(t)$ for n=6
$ha_{33}(t)$ for n=6
$ha_{34}(t)$ for n=6
$ha_{35}(t)$ for n=6
$ha_{36}(t)$ for n=6
$ha_{37}(t)$ for n=6
$ha_{38}(t)$ for n=6
$ha_{39}(t)$ for n=6
$ha_{40}(t)$ for n=6
$ha_{41}(t)$ for n=6
$ha_{42}(t)$ for n=6
$ha_{43}(t)$ for n=6
$ha_{44}(t)$ for n=6
$ha_{45}(t)$ for n=6
$ha_{46}(t)$ for n=6
$ha_{47}(t)$ for n=6
$ha_{48}(t)$ for n=6
$ha_{49}(t)$ for n=6
$ha_{50}(t)$ for n=6
$ha_{51}(t)$ for n=6
$ha_{52}(t)$ for n=6
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$ha_{54}(t)$ for n=6
$ha_{55}(t)$ for n=6
$ha_{56}(t)$ for n=6
$ha_{57}(t)$ for n=6
$ha_{58}(t)$ for n=6
$ha_{59}(t)$ for n=6
$ha_{60}(t)$ for n=6
$ha_{61}(t)$ for n=6
$ha_{62}(t)$ for n=6
$ha_{63}(t)$ for n=6
$ha_{64}(t)$ for n=6

g.

For a given basis of orthonormal functions $\{\beta_i\}_{i=1}^{N}$,the best approximation is

$$\Sigma_{i=1}^{N} < \beta_i(t), \phi(t) > \beta_i(t)$$

We can create new orthonormal families from the standard basis by applying a unitary base change using a unitary matrix:

$$\begin{pmatrix} b1(t) \\ b2(t) \\ \vdots \\ b_{2^n}(t) \end{pmatrix} = U^\top \begin{pmatrix} \sqrt{\frac{1}{|\Delta_1|}}\,\Delta_1(t) \\ \sqrt{\frac{1}{|\Delta_2|}}\,\Delta_2(t) \\ \vdots \\ \sqrt{\frac{1}{|\Delta_N|}}\,\Delta_{2^n}(t) \end{pmatrix}$$

We can achieve the new coefficients representing the signal in the new basis by transforming the coefficients of the standard basis:

$$\begin{pmatrix} \Psi_1^b \\ \Psi_2^b \\ \vdots \\ \Psi_N^b(t) \end{pmatrix} = U^\top \begin{pmatrix} \Psi_1^s \\ \Psi_2^s \\ \vdots \\ \Psi_N^s \end{pmatrix}$$

Using this logic, it's sufficient to calculate the optimal coefficients of the signal φ(t) = t exp(t) using the standard basis, and then use these coefficients to approximate the optimal coefficients of other bases.

Given that n = 2, $t \in [-4,5)$, uniform sampling will create us the following standard basis:

$\Delta = \frac{5-(-4)}{2^2} = \frac{9}{4}$

$\Delta_i = [-4 + i\Delta, -4 + (i+1)\Delta] = [l_i, h_i]\ \ for\ i = 0,1,2,3$

$\{\beta^s\}_{i=1}^{4} = \{\sqrt{\frac{4}{9}}\mathbb{1}_{[-4,-1.75]}, \sqrt{\frac{4}{9}}\mathbb{1}_{[-1.75,0.5]}, \sqrt{\frac{4}{9}}\mathbb{1}_{[0.5,2.75]}, \sqrt{\frac{4}{9}}\mathbb{1}_{[2.75,5]}\}$

$<g(t),f(t)> = \int_{-4}^{5} g(t)f(t)\,dt$

Thus, we can calculate the optimal coefficients when representing the signal in the standard basis as follows:

$$\int exp(t)\ t\ dt\ =\ exp(t)t\ -\ \int exp(t)\backslash,dt\ =\ exp(t)t\ -\ exp(t)\ =\ exp(t)(t-1)$$

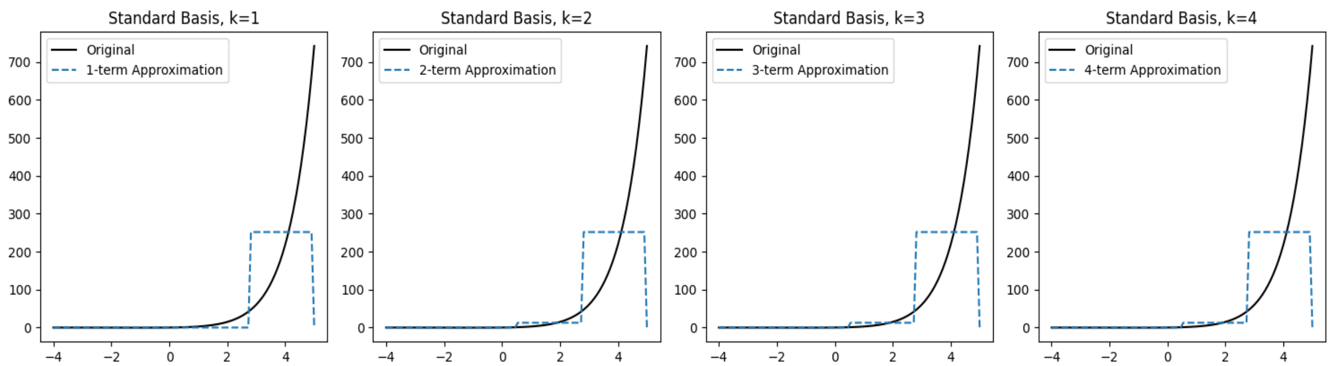
$$\Psi_i^s < \phi(t), \beta_i^s > = \int_{\Delta_i} exp(t)\sqrt{\frac{4}{9}}t\ dt = \sqrt{\frac{4}{9}}[exp(t)(t-1)]_{t=l_i}^{t=h_i}$$

We will use this closed formula to calculate the coefficients of the standard basis and then we will transform them using unitary bases to the requested bases (Hadamard, Haar, Walsh-Hadamard).

Results in next page.

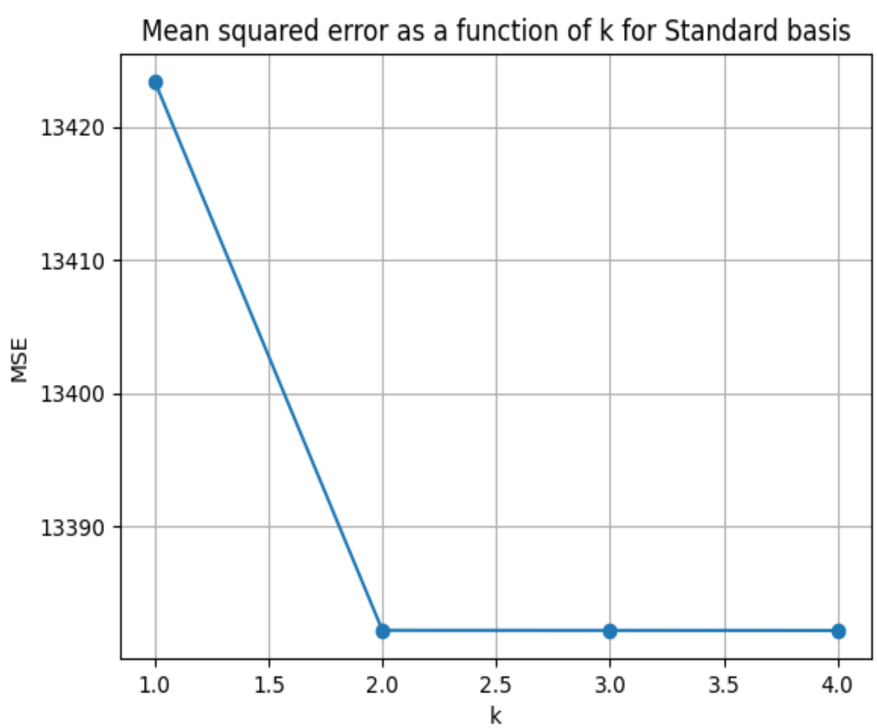
Plotting k-term best approximations for standard basis

The non-sorted coefficients for Standard basis are ['-0.258', '-0.231', '18.799', '377.519']



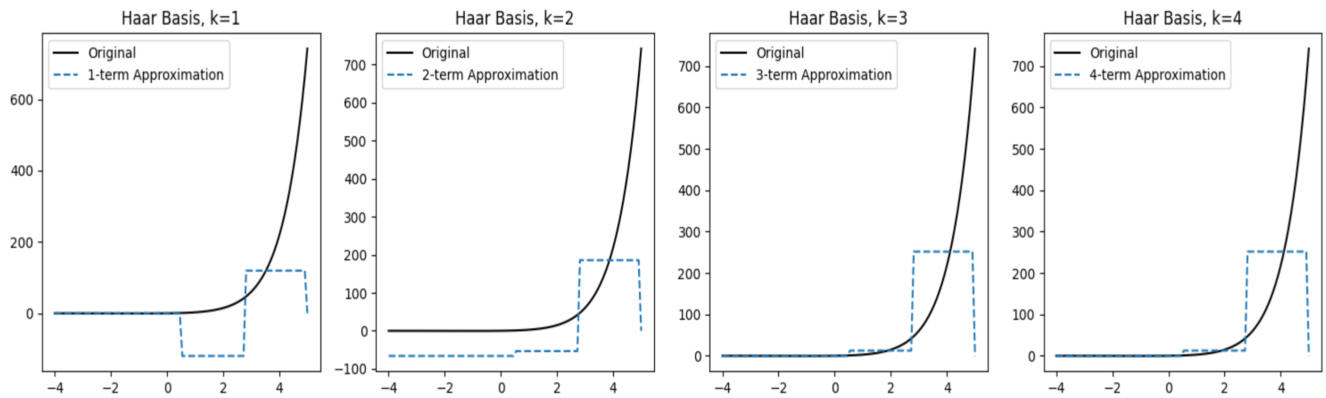
<u>The mse values of the k approximations are:</u>

k=1, MSE=13423.38786112699
k=2, MSE=13382.2221230829
k=3, MSE=13382.215038329885
k=4, MSE=13382.208965131673



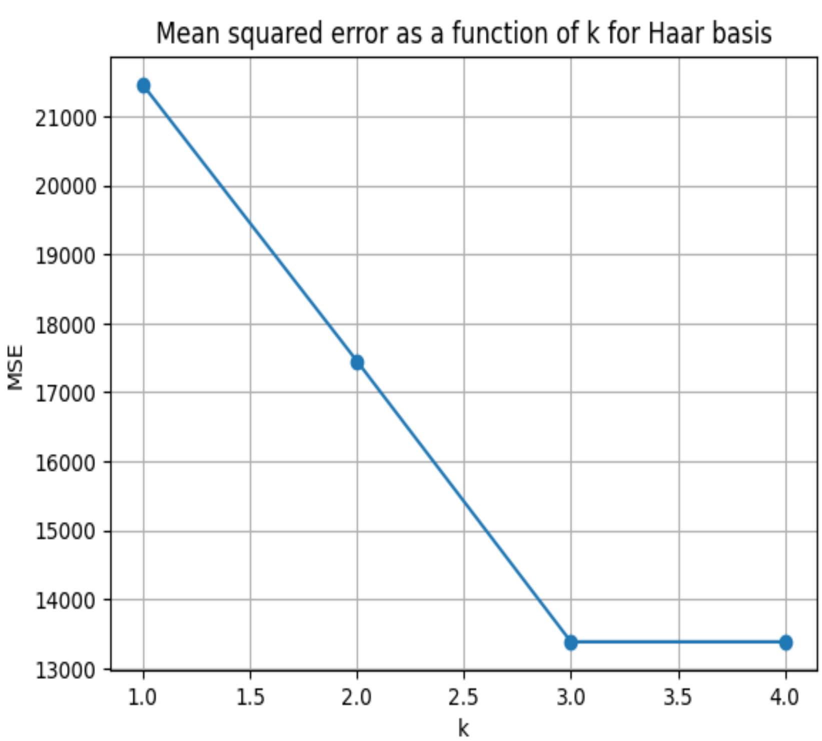
Plotting k-term best approximations for Haar basis

The non-sorted coefficients for Haar basis are ['197.915', '-198.403', '-0.019', '-253.653']



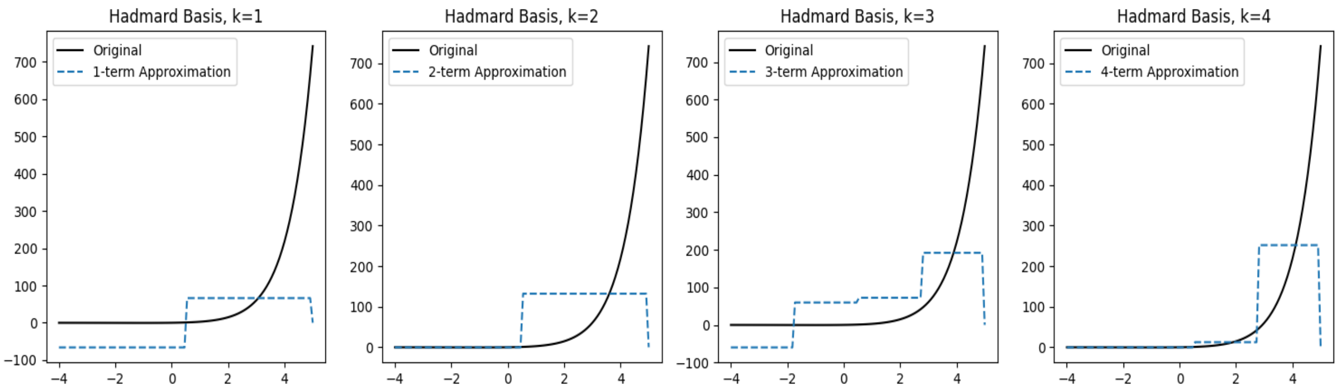
The mse values of the k approximations are:

k=1, MSE=21461.678108608532
k=2, MSE=17455.263190391146
k=3, MSE=13382.2089812949
k=4, MSE=13382.208965131675



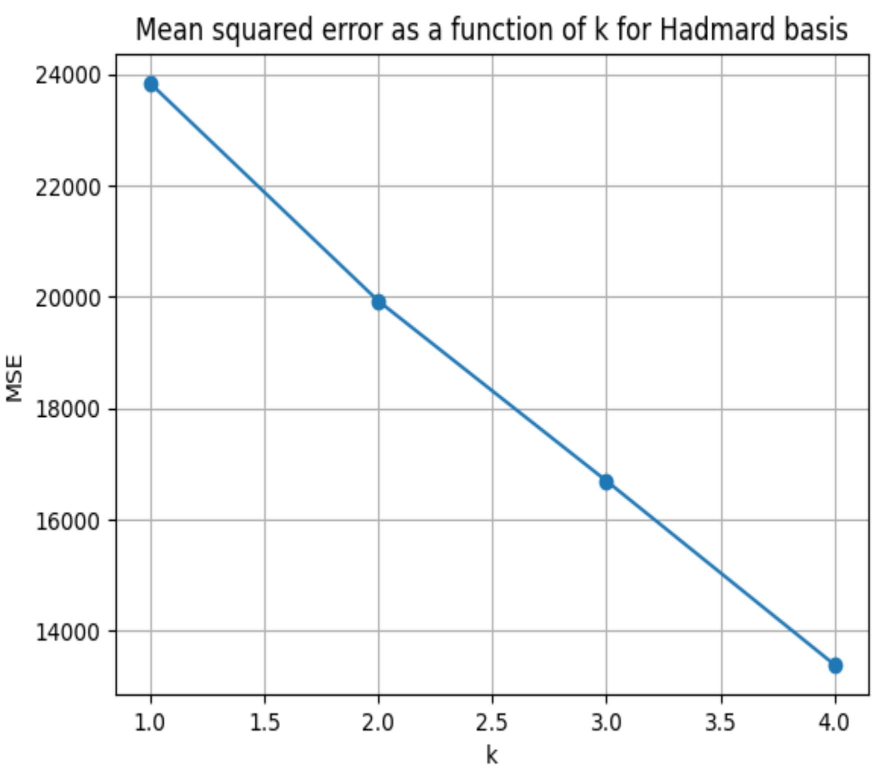
Plotting k-term best approximations for Hadamard basis

The non-sorted coefficients for Hadmard basis are ['197.915', '-179.373', '-198.403', '179.346']



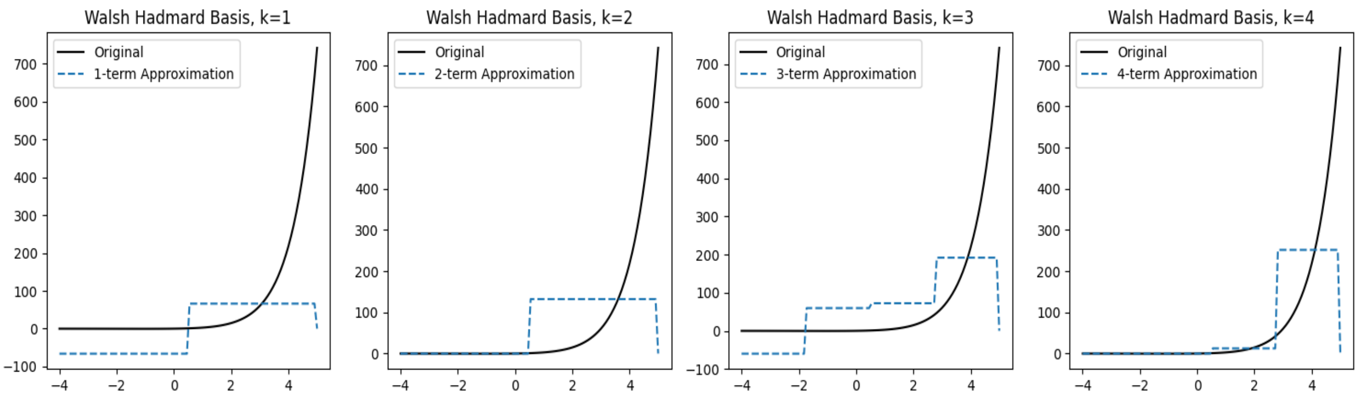
The mse values of the k approximations are:

k=1, MSE=23839.935193700217
k=2, MSE=19924.649543298772
k=3, MSE=16688.822303498433
k=4, MSE=13382.208965131675



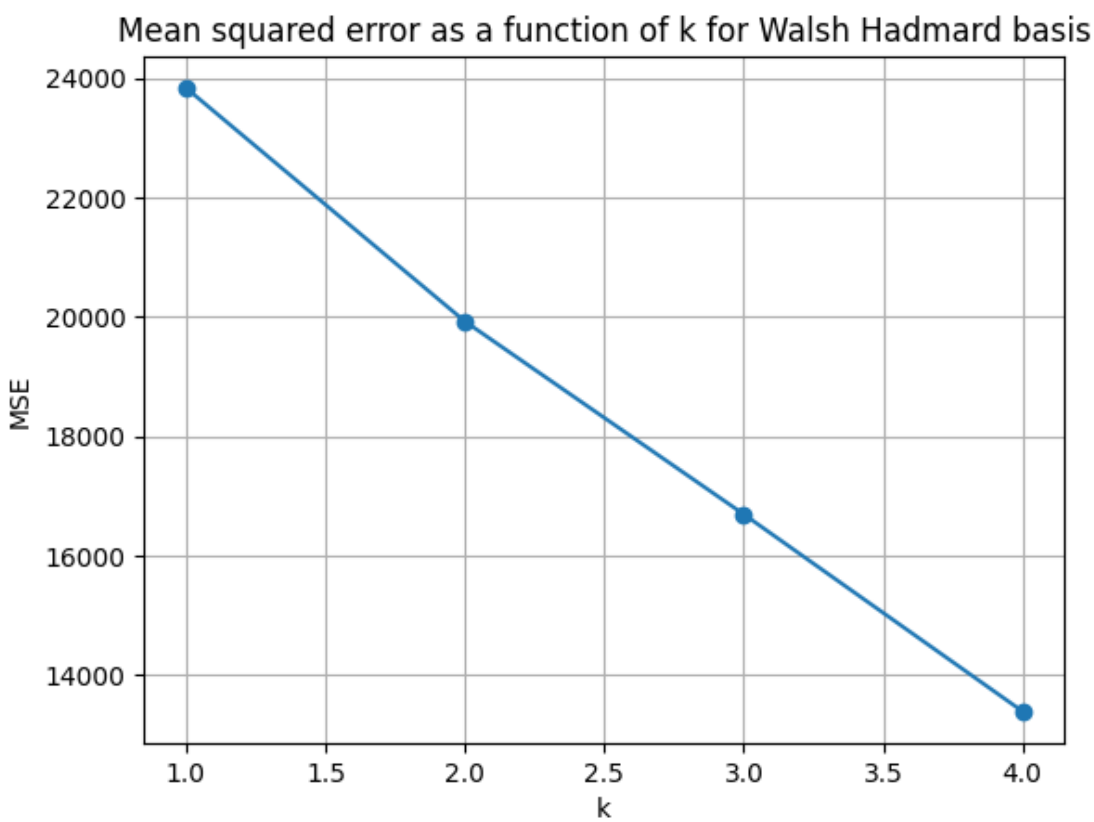
Plotting k-term best approximations for Walsh Hadamard basis

The non-sorted coefficients for Walsh Hadamard basis are ['197.915', '-198.403', '179.346', '-179.373']



The mse values of the k approximations are:

k=1, MSE=23839.935193700217
k=2, MSE=19924.649543298772
k=3, MSE=16688.822303498433
k=4, MSE=13382.208965131675



We observed that the standard basis is the most effective for representing the signal. This can be attributed to the function exp(x)x having a high rate of change within the interval [2.75, 5] (indicated by the largest coefficient in that range). Consequently, using this function alone captures the "change" in the signal, even though other intervals are ignored.

The second-best basis in terms of Mean Squared Error (MSE) is the Haar basis.

As expected, the Hadamard and Walsh-Hadamard bases yield identical approximations for each k, as they represent the same basis but ordered differently. Since we prioritize the basis by the largest coefficients, this reordering does not affect the result. Additionally, for k=4, all four bases (standard, Haar, Hadamard, and Walsh-Hadamard) result in the same error. This outcome is anticipated because all four bases span the same subspace for k=4.