236201- HW3-Wet part:

[redacted]

a.

Original Image I

Deteriorated Image I(1)

Deteriorated Image I(2)

Deteriorated Image I(12)

b.

We will use here $\tilde{\imath} = \sqrt{-1}$ as the complex number i to avoid mistakes.

The elements of row number i of the degraded image can be expressed as:

$$I_{ij}^{noisy} = I_{ij} + A_i \cos(2\pi fj + \varphi_i) \text{ where } \frac{n}{\frac{1}{f}} = q, q \in \mathbb{N} \Rightarrow f = \frac{q}{n}$$

So, we can express row number I in the following way:

$$r_i = [I_{i0} + A_i \cos(2\pi f0 + \varphi_i),\ \ I_{i1} + A_i \cos(2\pi f1 + \varphi_i), \dots, I_{i(n-1)} + A_i \cos(2\pi f(n-1) + \varphi_i)]$$

To calculate the DFT representation of row number i (changing to DFT^* base), we need to multiply DFT and r_i^T:

$$\text{DFT } r_i^T =$$

$$\frac{1}{\sqrt{n}} \begin{bmatrix} W_n^{*0\cdot 0} & W_n^{*1\cdot 0} & \cdots & W_n^{*(n-1)\cdot 0} \\ W_n^{*0\cdot 1} & W_n^{*1\cdot 1} & \cdots & W_n^{*(n-1)\cdot 1} \\ \vdots & \ddots & & \vdots \\ W_n^{*0\cdot(n-1)} & W_n^{*1\cdot(n-1)} & \cdots & W_n^{*(n-1)\cdot(n-1)} \end{bmatrix} \begin{bmatrix} I_{i0} + A_i \cos(2\pi f\cdot 0 + \varphi_i) \\ I_{i1} + A_i \cos(2\pi f\cdot 1 + \varphi_i) \\ \vdots \\ I_{i(n-1)} + A_i \cos(2\pi f\cdot (n-1) + \varphi_i) \end{bmatrix} =$$

$$\frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{k=0}^{n-1} W_n^{*k\cdot 0} \left(I_{ik} + A_i \cos(2\pi f\cdot k + \varphi_i)\right) \\ \sum_{k=0}^{n-1} W_n^{*k\cdot 1} \left(I_{ik} + A_i \cos(2\pi f\cdot k + \varphi_i)\right) \\ \vdots \\ \sum_{k=0}^{n-1} W_n^{*k\cdot(n-1)} \left(I_{ik} + A_i \cos(2\pi f\cdot k + \varphi_i)\right) \end{bmatrix}$$

It holds that for $0 \le m \le n-1$:

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W_n^{*k\cdot m} \left(I_{ik} + A_i \cos(2\pi f\cdot k + \varphi_i)\right) =$$

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} W_n^{*k\cdot m} \left(I_{ik} + A_i \cos(2\pi f\cdot k + \varphi_i)\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + A_i W_n^{*k\cdot m}\cos(2\pi f\cdot k+\varphi_i) \underbrace{=}_{cosx=\frac{e^{ix}+e^{-ix}}{2}}$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + A_i W_n^{*k\cdot m}\left(\frac{e^{\tilde{i}(2\pi f\cdot k+\varphi_i)}+e^{-\tilde{i}(2\pi f\cdot k+\varphi_i)}}{2}\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + A_i W_n^{*k\cdot m}\left(\frac{e^{\tilde{i}\left(\frac{2\pi\cdot qk}{n}+\varphi_i\right)}+e^{-\tilde{i}\left(\frac{2\pi\cdot qk}{n}+\varphi_i\right)}}{2}\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + A_i W_n^{*k\cdot m}\left(\frac{e^{\tilde{i}\left(\frac{2\pi\cdot qk}{n}\right)\cdot} e^{\tilde{i}\varphi_i}+e^{-\tilde{i}\left(\frac{2\pi\cdot qk}{n}\right)}e^{-\tilde{i}\varphi_i}}{2}\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + A_i W_n^{*k\cdot m}\left(\frac{W_n^{qk}e^{\tilde{i}\varphi_i}+W_n^{*qk}e^{-\tilde{i}\varphi_i}}{2}\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + \frac{A_i W_n^{*k\cdot m}W_n^{qk}e^{\tilde{i}\varphi_i}}{2} + \frac{A_i W_n^{*k\cdot m}W_n^{*qk}e^{-\tilde{i}\varphi_i}}{2} =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + \frac{A_i W_n^{k(-m+q)}e^{\tilde{i}\varphi_i}}{2} + \frac{A_i W_n^{*k\cdot(m+q)}e^{-\tilde{i}\varphi_i}}{2} =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + \frac{A_i e^{\tilde{i}\varphi_i}}{2\sqrt{n}}\sum_{k=0}^{n-1} A_i W_n^{k(-m+q)} + \frac{A_i e^{-\tilde{i}\varphi_i}}{2\sqrt{n}}\sum_{k=0}^{n-1} A_i W_n^{-k(m+q)} =$$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik} + \frac{A_i e^{\tilde{i}\varphi_i}}{2\sqrt{n}}\sum_{k=0}^{n-1} (W_n^{(q-m)})^k + \frac{A_i e^{-\tilde{i}\varphi_i}}{2\sqrt{n}}\sum_{k=0}^{n-1} (W_n^{(-q-m)})^k \underbrace{=}_{(*)}$$

$$\mathrm{DFT}(I_i)_m + \frac{A_i e^{\tilde{i}\varphi_i}}{2\sqrt{n}}\, n\cdot\delta_{q\ (\mathrm{mod}\ n),m} + \frac{A_i e^{-\tilde{i}\varphi_i}}{2\sqrt{n}}\, n\cdot\delta_{-q\ (\mathrm{mod}\ n),m}$$

(*)

- $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} W_n^{*k\cdot m}\, I_{ik}$= $\mathrm{DFT}(I_i)_m$ by the definition of multiplication between row m of the DFT matrix and the transpose row i from the original image.

-$\sum_{k=0}^{n-1}(W_n^{(q-m)})^k$ is the sum of a geometric sequence.

If $(q-m)\mathrm{mod}\ n \equiv 0$, we will $\sum_{k=0}^{n-1}(1)^k = n$.

If $q \bmod n \neq m$ (m is already smaller than n), we will get:

$\sum_{k=0}^{n-1} (W_n^{(q-m)})^k = \frac{1 \cdot \left(\left(W_n^{(q-m)} \right)^n - 1 \right)}{W_n^{(q-m)}} = 0.$

Therefore, we can say $\sum_{k=0}^{n-1} (W_n^{(q-m)})^k = n \cdot \delta_{q(\mathrm{mod}\ n),m}$

- $\sum_{k=0}^{n-1} (W_n^{(-q-m)})^k$ is also the sum of geometric sequence. We can apply a similar logic as in the previous bullet and get that $\sum_{k=0}^{n-1} (W_n^{(q-m)})^k = n \cdot \delta_{-q(\mathrm{mod}\ n),m}$

Hence, we get:

$$\mathrm{DFT} \cdot\ r_i^T =$$

$$\frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{k=0}^{n-1} W_n^{*k \cdot 0} \left(I_{ik} + A_i \cos(2\pi f \cdot k + \varphi_i \right) \\ \sum_{k=0}^{n-1} W_n^{*k \cdot 1} \left(I_{ik} + A_i \cos(2\pi f \cdot k + \varphi_i \right) \\ \vdots \\ \sum_{k=0}^{n-1} W_n^{*k \cdot (n-1)} \left(I_{ik} + A_i \cos(2\pi f \cdot k + \varphi_i \right) \end{bmatrix} =$$

$$\mathrm{DFT} \cdot I_i^T + \begin{bmatrix} \frac{A_i e^{\tilde{\imath}\varphi_i}}{2\sqrt{n}} n \cdot \delta_{q\ (\mathrm{mod}\ n),0} + \frac{A_i e^{-\tilde{\imath}\varphi_i}}{2\sqrt{n}} n \cdot \delta_{-q\ (\mathrm{mod}\ n),0} \\ \frac{A_i e^{\tilde{\imath}\varphi_i}}{2\sqrt{n}} n \cdot \delta_{q\ (\mathrm{mod}\ n),1} + \frac{A_i e^{-\tilde{\imath}\varphi_i}}{2\sqrt{n}} n \cdot \delta_{-q\ (\mathrm{mod}\ n),1} \\ \vdots \\ \frac{A_i e^{\tilde{\imath}\varphi_i}}{2\sqrt{n}} n \cdot \delta_{q\ (\mathrm{mod}\ n),n-1} + \frac{A_i e^{-\tilde{\imath}\varphi_i}}{2\sqrt{n}} n \cdot \delta_{-q\ (\mathrm{mod}\ n),n-1} \end{bmatrix} =$$

$$\mathrm{DFT}\cdot \mathrm{I}_i^T + \begin{bmatrix} \frac{A_i\sqrt{n}}{2}\left(e^{\tilde{\imath}\varphi_i}\delta_{q\ (\mathrm{mod}\ n),0} + e^{-\tilde{\imath}\varphi_i}\delta_{-q\ (\mathrm{mod}\ n),0}\right) \\ \frac{A_i\sqrt{n}}{2}\left(e^{\tilde{\imath}\varphi_i}\delta_{q\ (\mathrm{mod}\ n),1} + e^{-\tilde{\imath}\varphi_i}\delta_{-q\ (\mathrm{mod}\ n),1}\right) \\ \vdots \\ \frac{A_i\sqrt{n}}{2}\left(e^{\tilde{\imath}\varphi_i}\delta_{q\ (\mathrm{mod}\ n),n-1} + e^{-\tilde{\imath}\varphi_i}\delta_{-q\ (\mathrm{mod}\ n),n-1}\right) \end{bmatrix} =$$

$$\mathrm{DFT}\cdot \mathrm{I}_i^T + \frac{A_i\sqrt{n}}{2}\begin{bmatrix} \left(e^{\tilde{\imath}\varphi_i}\delta_{q\ (\mathrm{mod}\ n),0} + e^{-\tilde{\imath}\varphi_i}\delta_{-q\ (\mathrm{mod}\ n),0}\right) \\ \left(e^{\tilde{\imath}\varphi_i}\delta_{q\ (\mathrm{mod}\ n),1} + e^{-\tilde{\imath}\varphi_i}\delta_{-q\ (\mathrm{mod}\ n),1}\right) \\ \vdots \\ \left(e^{\tilde{\imath}\varphi_i}\delta_{q\ (\mathrm{mod}\ n),n-1} + e^{-\tilde{\imath}\varphi_i}\delta_{-q\ (\mathrm{mod}\ n),n-1}\right) \end{bmatrix} =$$

$$\mathrm{DFT}\cdot \mathrm{I}_i^T + \frac{A_i\sqrt{n}}{2}\left(e^{\tilde{\imath}\varphi_i}\, e^s_{q\ mod\ n} + e^{-\tilde{\imath}\varphi_i}\, e^s_{-q\ mod\ n}\right) =$$

$$\mathrm{DFT}\cdot \mathrm{I}_i^T + \frac{A_i\sqrt{n}}{2}\left(e^{\tilde{\imath}\varphi_i}\, e^s_{q\ mod\ n} + e^{-\tilde{\imath}\varphi_i}\, e^s_{(n-q)\ mod\ n}\right)$$

- $e^s_k = (0, 0, \ldots, 1, \ \ldots, 0)^T (standard\ basis\ vector\ where\ the\ k^{th} element\ is\ 1)$

c.

We will use here $\tilde{\imath} = \sqrt{-1}$ as the complex number i to avoid mistakes.

The elements of row number i of the degraded image by weighted average noise can be expressed as:

$$\mathrm{I}_{ij}^{\mathrm{avg\ noisy}} = \mathrm{I}_{ij} + \frac{w_1(A_i^{(1)}\cos\left(2\pi f_1 j + \varphi_i^{(1)}\right)) + w_2(A_i^{(2)}\cos\left(2\pi f_2 j + \varphi_i^{(2)}\right))}{w_1 + w_2} =$$

$$\mathrm{I}_{ij}^{\mathrm{avg\ noisy}} = \mathrm{I}_{ij} + \frac{w_1(A_i^{(1)}\cos\left(2\pi f_1 j + \varphi_i^{(1)}\right))}{w_1 + w_2} + \frac{w_2(A_i^{(2)}\cos\left(2\pi f_2 j + \varphi_i^{(2)}\right))}{w_1 + w_2}$$

$$\text{where } n\cdot f_1 = q_1, n\cdot f_2 = q_2, \qquad q_1, q_2 \in \mathbb{N}$$

We observe we can write the i^{th} row of avg noisy in the following way:

$$\left(r_i^{avg-noisy}\right)^T = I_i^T + \frac{w_1}{w_1+w_2}\, \underline{v_{noise_i}^{(1)}} + \frac{w_2}{w_1+w_2}\, \underline{v_{noise_i}^{(2)}}$$

Where I_i^T is the i^{th} row of the original image I and

$$\underline{v_{noise_i}^{(t)}} = \begin{bmatrix} \mathrm{A_i^{(t)}} \cos\left(2\pi \mathrm{f_t} \cdot 0 + \varphi_\mathrm{i}^{(1)}\right) \\ \mathrm{A_i^{(t)}} \cos\left(2\pi \mathrm{f_t} \cdot 1 + \varphi_\mathrm{i}^{(2)}\right) \\ \vdots \\ \mathrm{A_i^{(t)}} \cos\left(2\pi \mathrm{f_t} \cdot (\mathrm{n}-1) + \varphi_\mathrm{i}^{(1)}\right) \end{bmatrix}$$ is a noise added to row number i.

By Applying the DFT matrix on $\left(r_i^{avg-noisy}\right)^T$ we get:

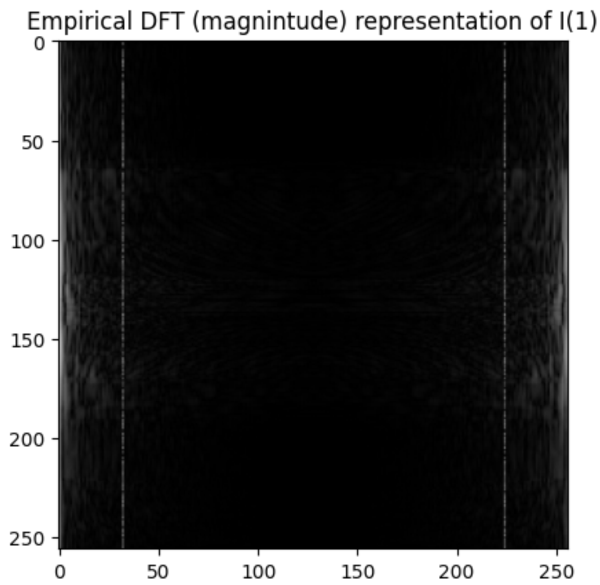
$$DFT \cdot \left(r_i^{avg-noisy}\right)^T = DFT \cdot \left(I_i^T + \frac{w_1}{w_1+w_2}\, \underline{v_{noise_i}^{(1)}} + \frac{w_2}{w_1+w_2}\, \underline{v_{noise_i}^{(2)}}\right) =$$

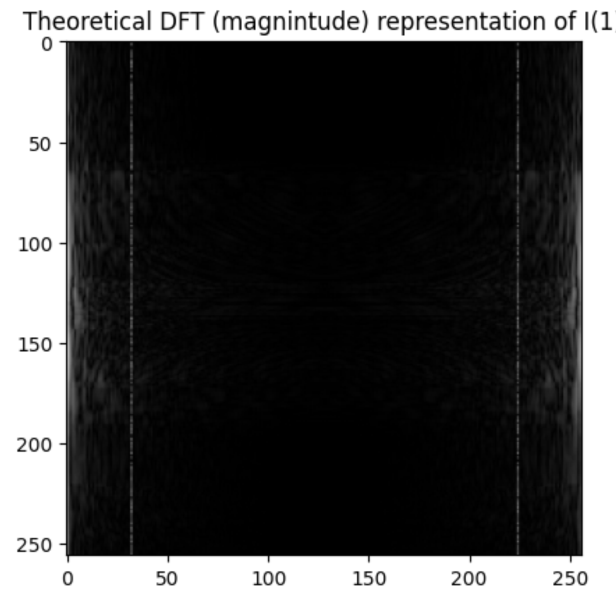
$$DFT \cdot I + \frac{w_1}{w_1+w_2} DFT \cdot\, \underline{v_{noise_i}^{(1)}} + \frac{w_2}{w_1+w_2}\, DFT \cdot \underline{v_{noise_i}^{(2)}}$$

Based on what we got on section b where we applied DFT on the noise we get:

$$\begin{aligned} DFT \cdot \left(r_i^{avg-noisy}\right)^T \\ &= DFT \cdot I + \frac{w_1}{w_1+w_2} \frac{\mathrm{A_i^{(1)}}\sqrt{\mathrm{n}}}{2} \left(\mathrm{e}^{\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(1)}}\, e_{q_1\, mod\, n}^{s} + \mathrm{e}^{-\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(1)}}\, e_{(n-q_1) mod\, n}^{s}\right) \\ &+ \frac{w_2}{w_1+w_2} \frac{\mathrm{A_i^{(2)}}\sqrt{\mathrm{n}}}{2} \left(\mathrm{e}^{\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(2)}}\, e_{q_2\, mod\, n}^{s} + \mathrm{e}^{-\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(2)}}\, e_{(n-q_2) mod\, n}^{s}\right) = \end{aligned}$$

d.



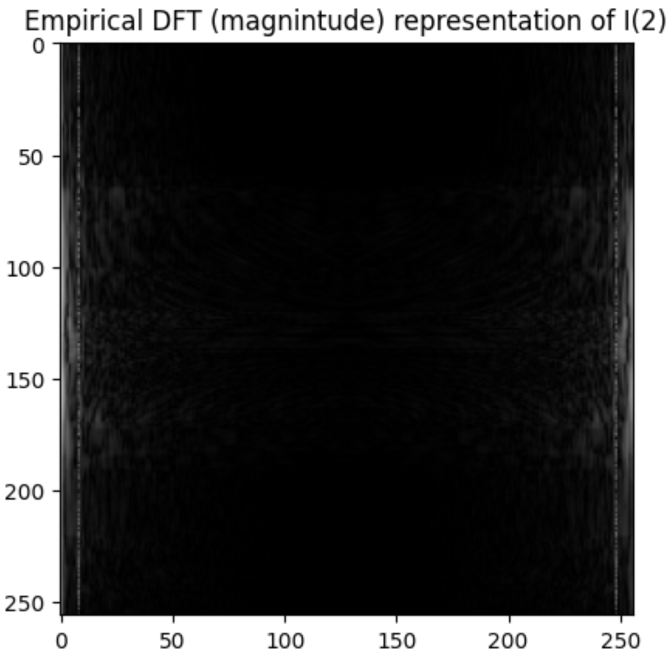


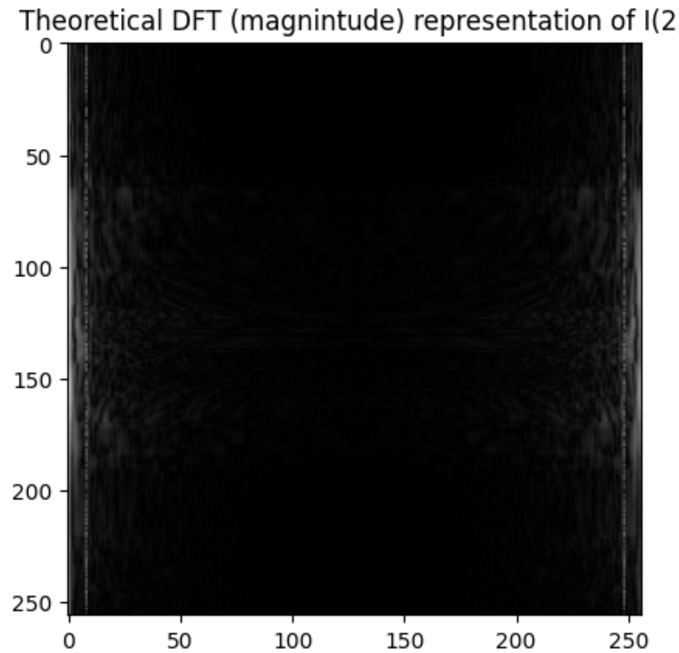
Based on theory we should get for $f=\frac{1}{8}, \mathrm{n}=256, \mathrm{q}=\frac{1}{8}\cdot 256=32$ the transformation of each row can be represented as::

$$\mathrm{DFT}\cdot \mathrm{I}_{\mathrm{i}}^{T}+\frac{\mathrm{A}_{\mathrm{i}}\sqrt{256}}{2}\left(\mathrm{e}^{\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{32}^{s}+\mathrm{e}^{-\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{256-32}^{s}\right)=\mathrm{DFT}\cdot \mathrm{I}_{\mathrm{i}}^{T}+8\mathrm{A}_{\mathrm{i}}\left(\mathrm{e}^{\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{32}^{s}+\mathrm{e}^{-\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{224}^{s}\right)$$

Hence, theory says after applying the Discrete Fourier Transform (DFT) to the noise, it will be expressed only at frequencies 32 and 224.

This empirical dft is consistent with our theoretical prediction. In the images, the magnitude of the coefficients in columns 32 and 224 is greater than zero, compared to the magnitudes of other coefficients corresponding to different frequencies. This result supports our theoretical analysis. Although we also transformed the original image to the Fourier domain and added the transformed noise, there aren't clear frequencies for the image (they are more spread out in the lower frequencies), so the noise is much more emphasized in the Fourier domain.



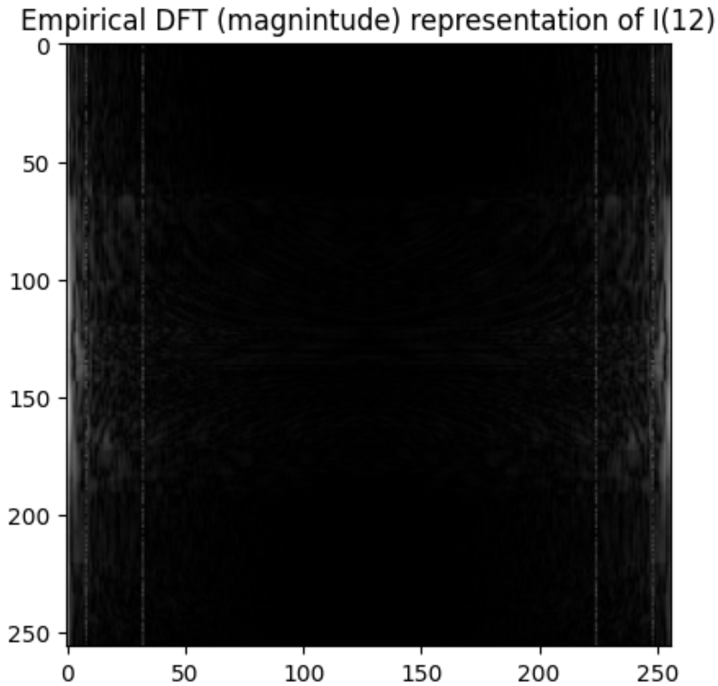


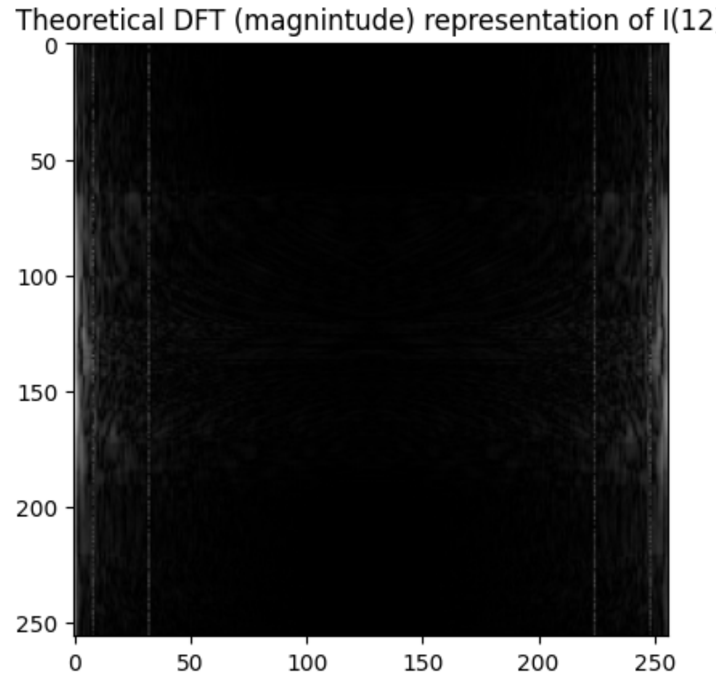
Based on theory we should get for $f = \frac{1}{32}, \mathrm{n} = 256, \mathrm{q} = \frac{1}{32} \cdot 256 = 8$ the transformation of each row can be represented as::

$$\mathrm{DFT} \cdot \mathrm{I}_{\mathrm{i}}^{T} + \frac{\mathrm{A_i}\sqrt{256}}{2}\left(\mathrm{e}^{\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{8}^{s} + \mathrm{e}^{-\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{256-8}^{s}\right) = \mathrm{DFT} \cdot \mathrm{I}_{\mathrm{i}}^{T} + 8\mathrm{A_i}\left(\mathrm{e}^{\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{8}^{s} + \mathrm{e}^{-\tilde{\imath}\varphi_{\mathrm{i}}}\, e_{248}^{s}\right)$$

According to this theoretical prediction, after applying the Discrete Fourier Transform (DFT) to the noise, it will be represented only at frequencies 8 and 248. Empirical observations confirm this, as the noise is indeed expressed at these frequencies.

Additionally, using a smaller frequency for the noise results in the noise being expressed at lower frequencies. This means the noise appears smoother and oscillates more slowly, as lower frequencies correspond to less rapid changes in the signal.





Based on theory we should get for $f_1 = \frac{1}{8}, \mathrm{n} = 256, \mathrm{q}_1 = \frac{1}{8} \cdot 256 = 32$,

$f_2 = \frac{1}{32}, \mathrm{q}_2 = \frac{1}{32} \cdot 256 = 8$ the transformation of each row can be represented as:

$$
\begin{aligned}
DFT \cdot \left(r_i^{avg-noisy}\right)^T \\
&= DFT \cdot I + \frac{1}{2}\frac{\mathrm{A}_\mathrm{i}^{(1)}\sqrt{256}}{2}\left(\mathrm{e}^{\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(1)}}\, e_{32}^{s} + \mathrm{e}^{-\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(1)}}\, e_{224}^{s}\right) \\
&+ \frac{1}{2}\frac{\mathrm{A}_\mathrm{i}^{(2)}\sqrt{256}}{2}\left(\mathrm{e}^{\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(2)}}\, e_{8}^{s} + \mathrm{e}^{-\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(2)}}\, e_{248}^{s}\right) =
\end{aligned}
$$

$$
= DFT \cdot I + 4\mathrm{A}_\mathrm{i}^{(1)}\left(\mathrm{e}^{\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(1)}}\, e_{32}^{s} + \mathrm{e}^{-\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(1)}}\, e_{224}^{s}\right) + 4\mathrm{A}_\mathrm{i}^{(2)}\left(\mathrm{e}^{\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(2)}}\, e_{8}^{s} + \mathrm{e}^{-\tilde{\mathrm{i}}\varphi_\mathrm{i}^{(2)}}\, e_{248}^{s}\right)
$$

Hence, the theory predicts that after applying the Discrete Fourier Transform (DFT) to the average of the noises, they will be expressed only at frequencies 32, 224, 8, and 248. This means that the impact of each noise component appears only at its respective frequencies. Since these frequencies do not overlap, the average noise is spread out across more frequencies.

This empirical observation of the DFT is consistent with our theoretical prediction e.

Reconstructed Image I(1)

```
mse I1: 5.05101174972814e-05
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Reconstructed Image I(2)

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mse I2: 0.0006590660366675701
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Reconstructed Image I(12)

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mse I12: 0.0007096245425748356
```

We observed that overall, the reconstruction worked well but was not perfect. The best reconstruction was performed on I1 (f=1/8). The worst reconstruction was performed on I12. This can be explained by the fact that the photo itself had lower frequencies, as seen in the plot in an earlier section. Therefore, when we zeroed out the frequencies at 32 and 224, we didn't lose much information. In contrast, for I2 (f=1/32), when we zeroed out frequencies at 8 and 248, we might have also lost significant information. For the averaged noise, we zeroed out four frequencies, which likely resulted in more information loss compared to the other two. This is why the reconstruction has a higher mean squared error (MSE).