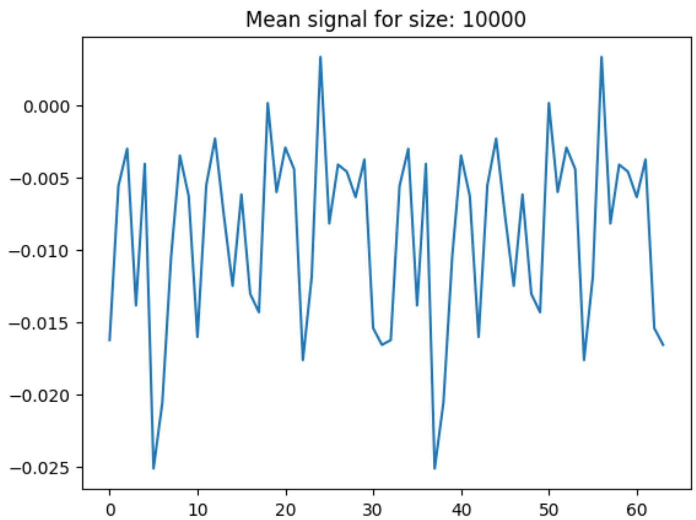
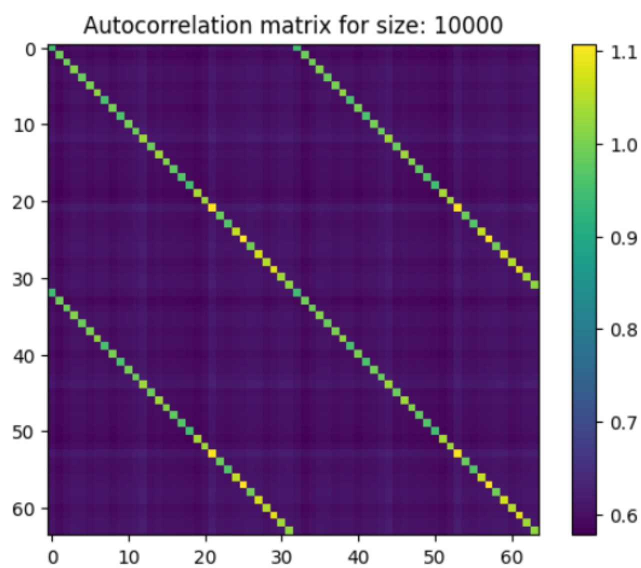
236201- HW4-Wet part:

[redacted]

a. The empirical approximation of the mean of the signal:



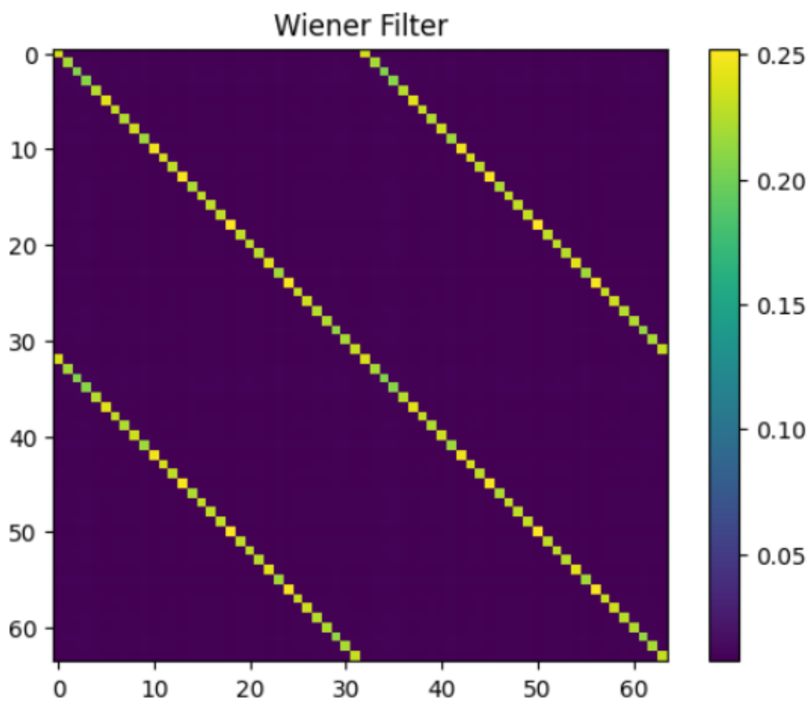
The empirical approximation of the autocorrelation of the signal:



As we can see from the plots, the empirical mean and empirical autocorrelation closely approximate the analytical results. The values of the mean entries are very close to zero, indicating a good match with the theoretical expectation. Additionally, the empirical autocorrelation matrix resembles the theoretical one, confirming its circulant structure. Most of the entries are approximately equal to $c = 0.6$ while the main diagonal is nearly 1. There are also two additional diagonals with values close to 1, corresponding to the indices $i = j + N/2 -$ and $j = i + N/2 - 1$.

We used 10,000 realizations for our analysis. This number of samples provided a good approximation of the analytical autocorrelation matrix. When using a smaller number of samples, the results were less stable. As the number of samples increases, we expect the empirical mean and covariance to converge more closely to the true mean and covariance.

b. The Weiner filter matrix we got:

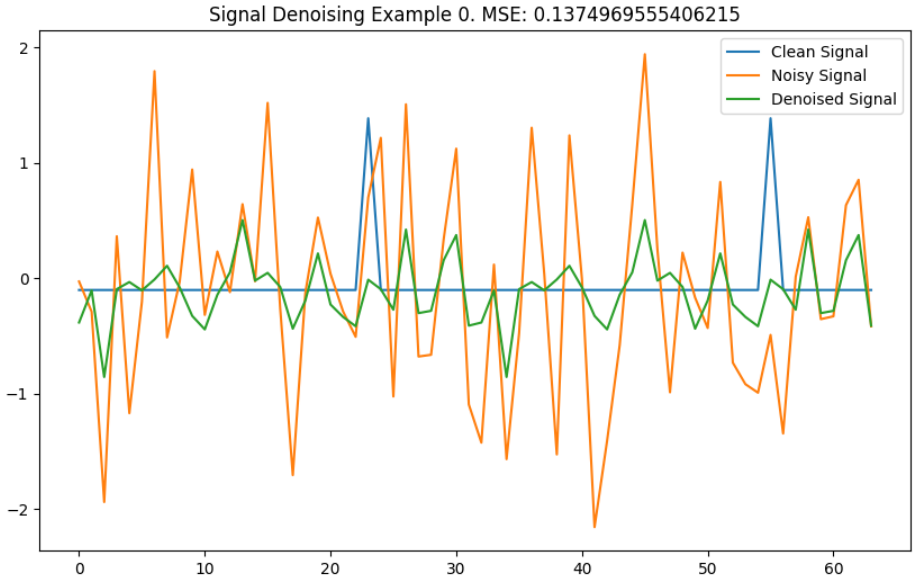


The structure of the matrix is circulant, as expected, since:

$$W = R_{\varphi} H^{*} \left(H R_{\varphi} H^{*} + \sigma_n^2 I \right)^{-1}$$

In our case, R_{φ} is circulant as we as we observed in Exercise 2 and in the previous question. Additionally, $H = I$ and $\sigma_n = 1$.

Thus, $W = R_{\varphi}\left(R_{\varphi} + I\right)^{-1}$. $R_{\varphi} + I$ is invertible is invertible because it is the sum of a symmetric PSD matrix (R_{φ}) and a PD matrix (I). Moreover, the sum of circulant matrices results in a circulant matrix. The inverse of a circulant matrix is also circulant, and the product of circulant matrices is circulant.Therefore, W should should indeed be circulant in this case.



The total MSE we got (by comparing the original signal and the denoised one) is :

```
Average MSE: 0.22972125125375178
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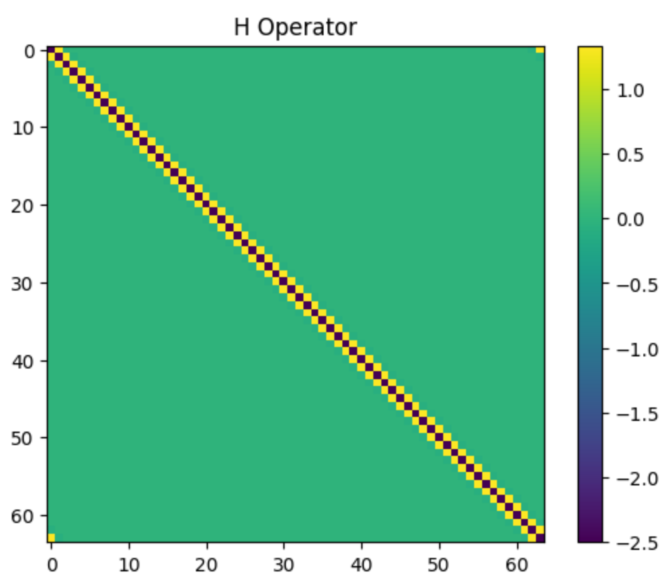
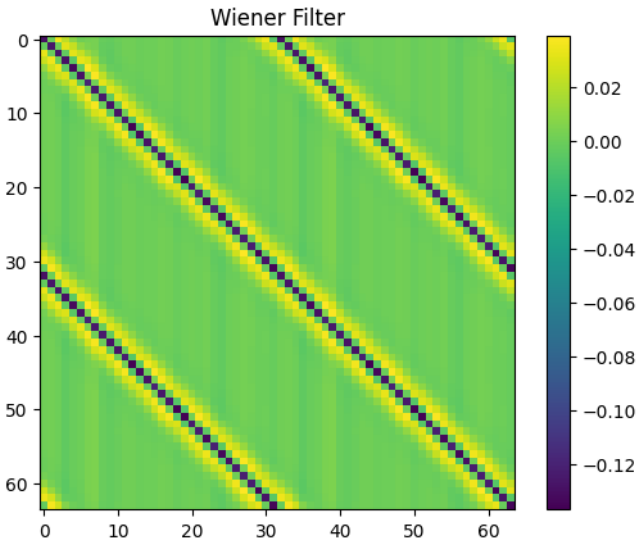
We can see from the plots that the Wiener filter was able to reduce a significant part of the noise. However, the noise reduction is not perfect, as indicated by the MSE and the plots. Specifically, we can see that when the noise was negative, the Wiener filter treated the random entries where the random variable is M+L as noise and reduced the L component even further.

c. H operator:



The Weiner filter is:



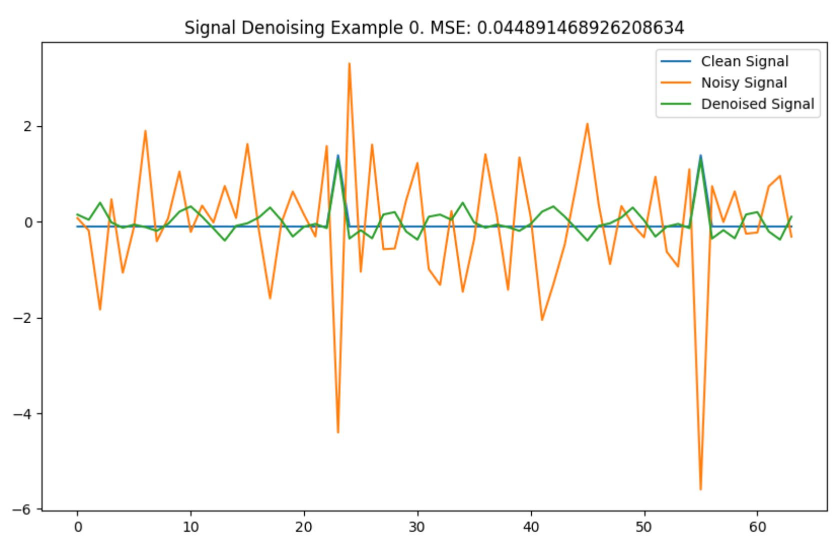
The structure of the matrix is circulant, as expected, since:

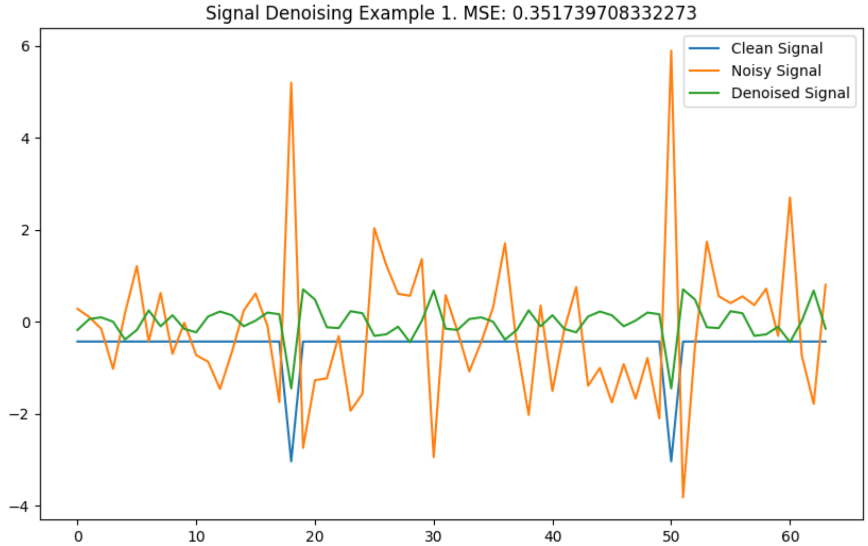
$$W = R_{\varphi}H^{*}\left(HR_{\varphi}H^{*} + \sigma_{n}^{2}I\right)^{-1}$$

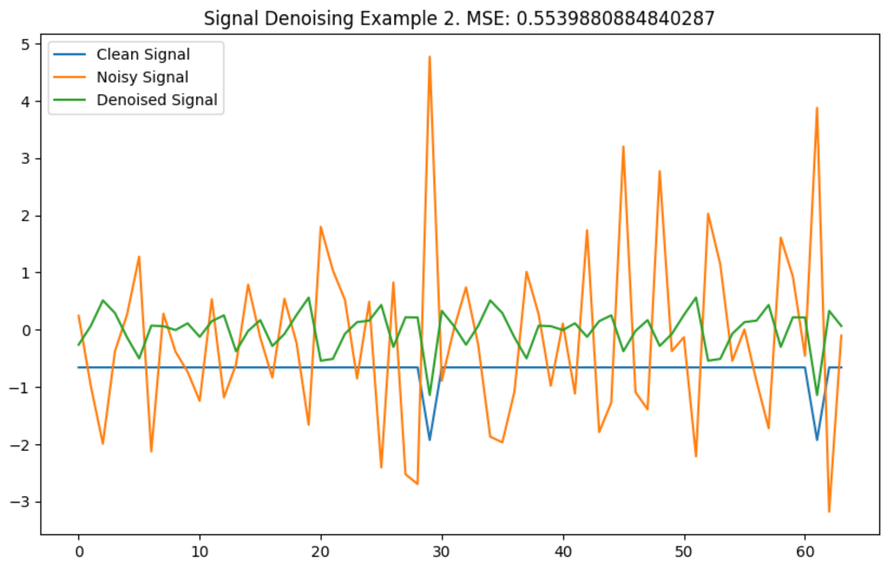
In our case, R_{φ} is circulant as we as we observed in Exercise 2 and in the previous question. Additionally, H is the operator from question 1 and $\sigma_{n} = 1$.

$HR_{\varphi}H^{*}$ is PSD matrix since R_{φ} is symmetric PSD, and therefore the product $HR_{\varphi}H^{*}$ will be PSD as shown in numerical algorithms.
Thus, $W = R_{\varphi}\left(HR_{\varphi}H^{*} + I\right)^{-1}$. $HR_{\varphi}H^{*} + I$ is invertible is invertible because it is the sum of a symmetric PSD matrix ($HR_{\varphi}H^{*}$) and a PD matrix (I). Moreover, the sum of circulant matrices results in a circulant matrix. The inverse of a circulant matrix is also circulant, and the product of circulant matrices is circulant.Therefore, W should should indeed be circulant in this case.

Since H is not a full rank matrix as shown in Question 1, there will be some parts of the original signal which we will not be able to restore, so we expect a higher MSE.







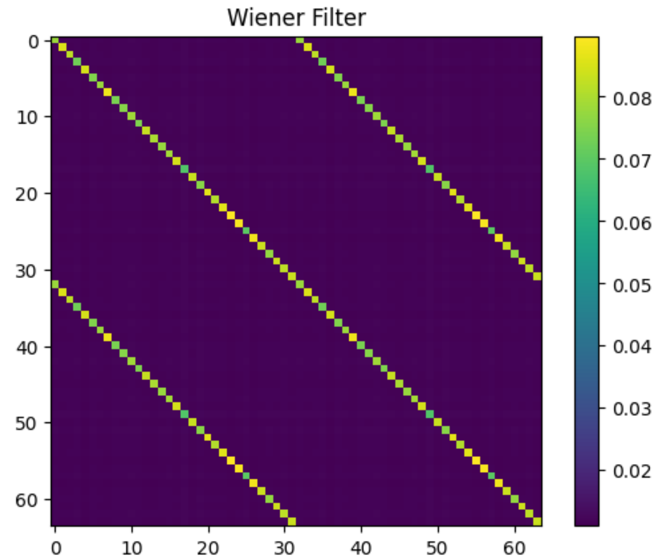
The total MSE we got (by comparing the original signal and the denoised one) is :

```
Average MSE: 0.7558188601165552
```

We can see from the plots that the Wiener filter was able to reduce a significant portion of the noise. However, we observed a higher MSE compared to the previous section, since H is not invertible. As a result, we lose part of the original signal—the portion that belongs to the null space of H. This loss leads to a higher reconstruction error.

d.

Repeating b with $\sigma_n^2 = 5$

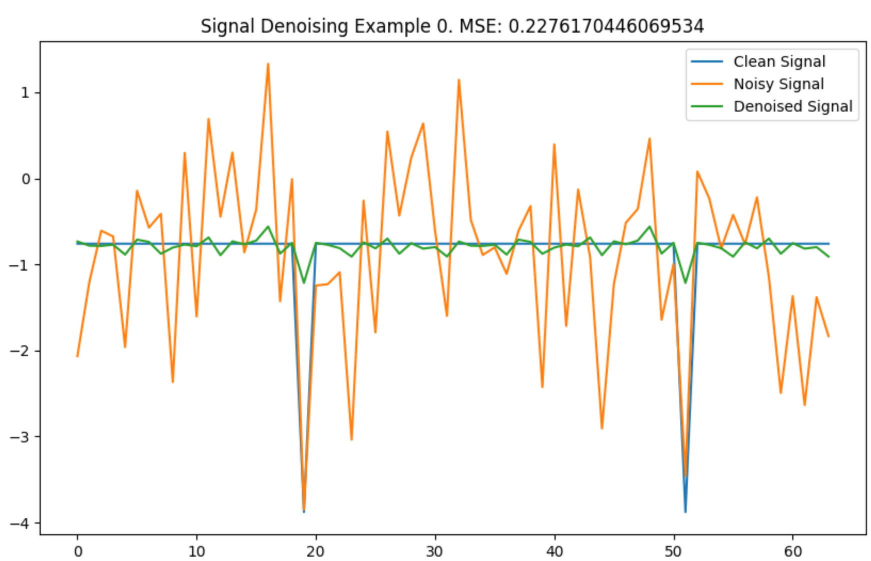


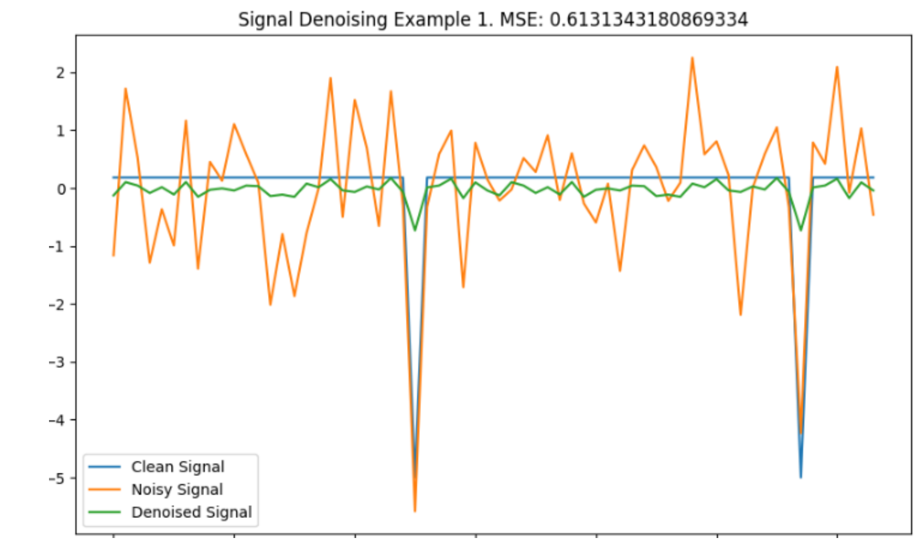
The structure of the matrix is circulant, as expected, since:

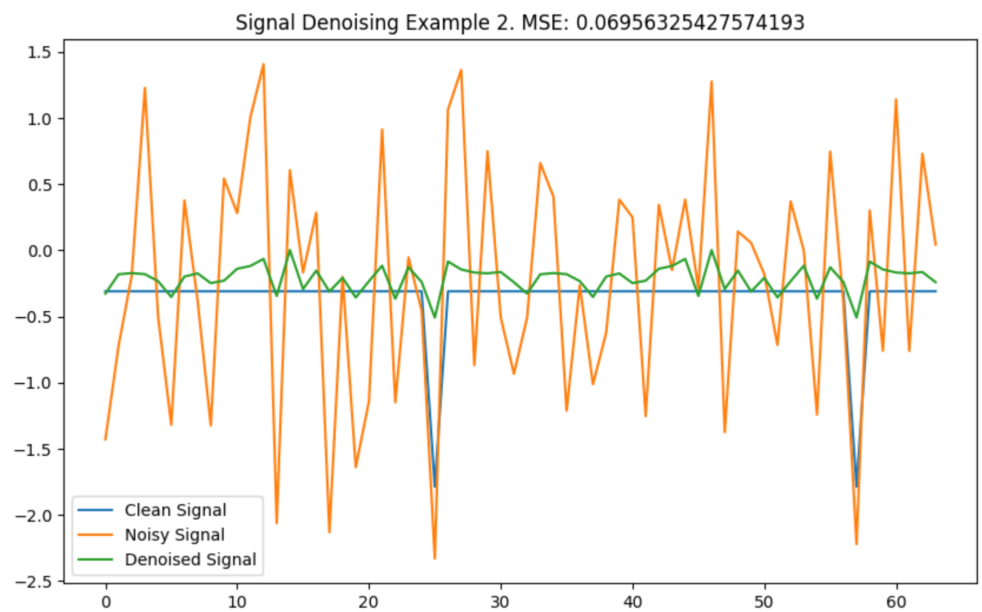
$$W = R_\varphi H^* \left(H R_\varphi H^* + \sigma_n^2 I\right)^{-1}$$

In our case, R_φ is circulant as we as we observed in Exercise 2 and in the previous question. Additionally, $H = I$ and $\sigma_n^2 = 5$.

Thus, $W = R_\varphi \left(R_\varphi + 5I\right)^{-1}$ which is still circulant as explained in section b.
However, we can see that when we enlarged the variance of the gaussian noise, we got that the Weiner filter values have been decreased.





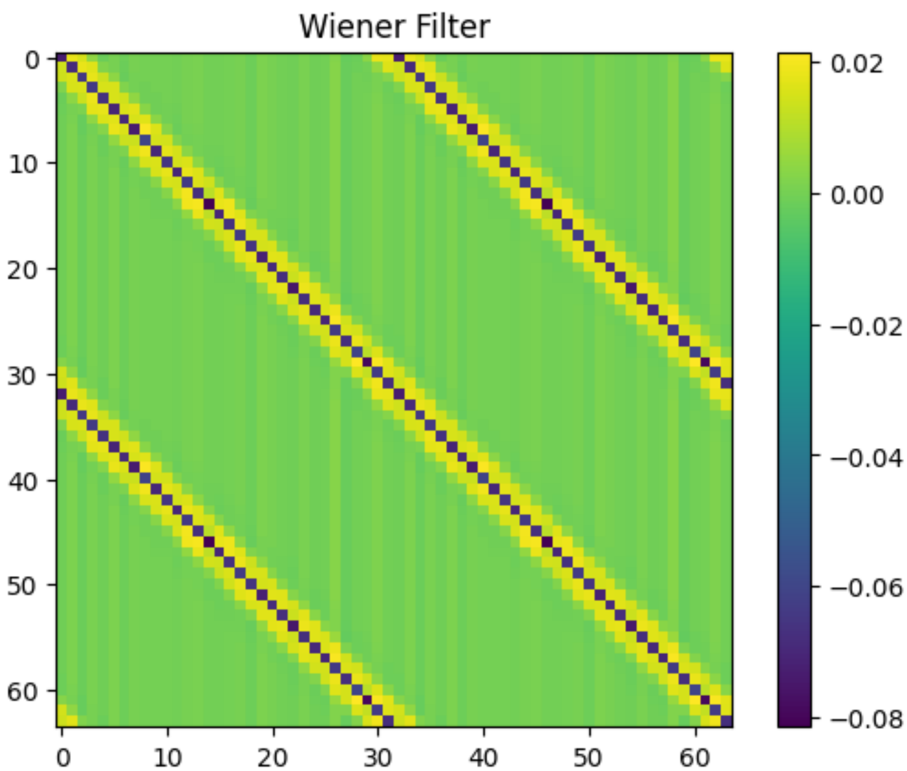


The total MSE we got (by comparing the original signal and the denoised one) is :

Average MSE: 0.3219747845230624

We can see from the plots that as the variance of the white Gaussian noise increases, the reconstructed signal remains a good approximation. However, we do observe that the MSE is slightly higher compared to the case when $\sigma_n^2 = 1$. is indicates that as the noise level increases, it also limits the filter's ability to accurately reconstruct the signal.

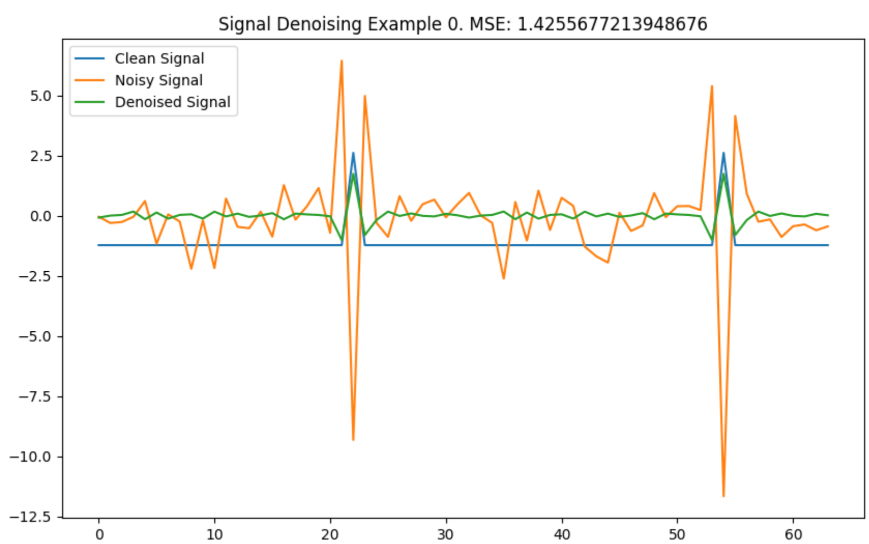
Repeating c with $\sigma_n^2 = 5$

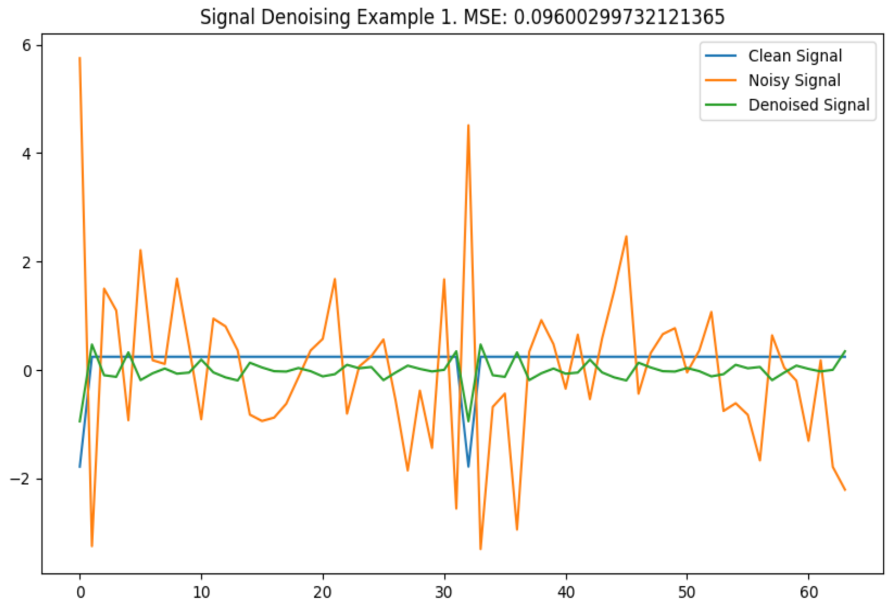


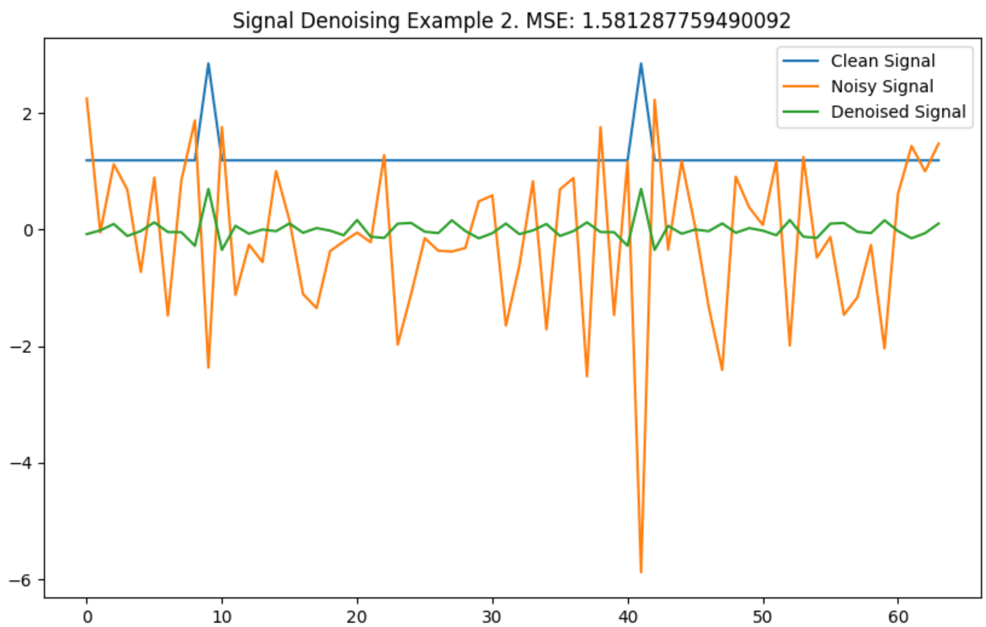
Tl

In our case, R_φ is circulant as we as we observed in Exercise 2 and in the previous question. Additionally, H is the operator from question 1 and $\sigma_n^2 = 5$.

Thus, W which is still circulant as explained in section c.
We did got different circulant matrix compared to section c since this part changed: $\sigma_n^2 I$.







The total MSE we got (by comparing the original signal and the denoised one) is :

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Average MSE: 0.7769489222289114
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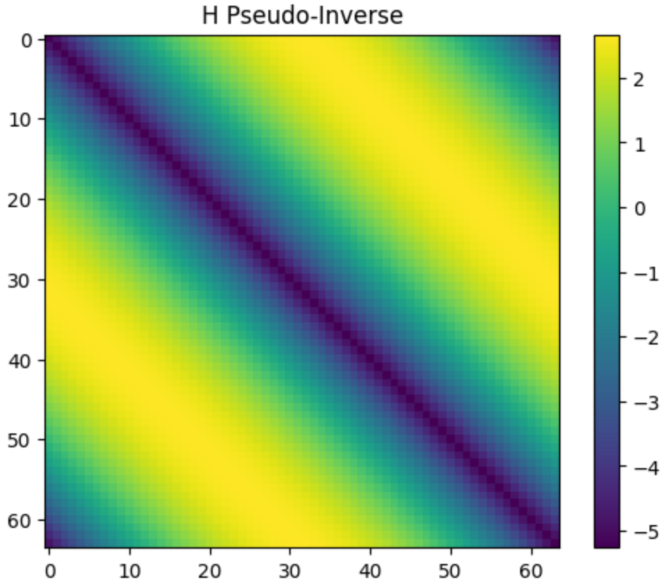
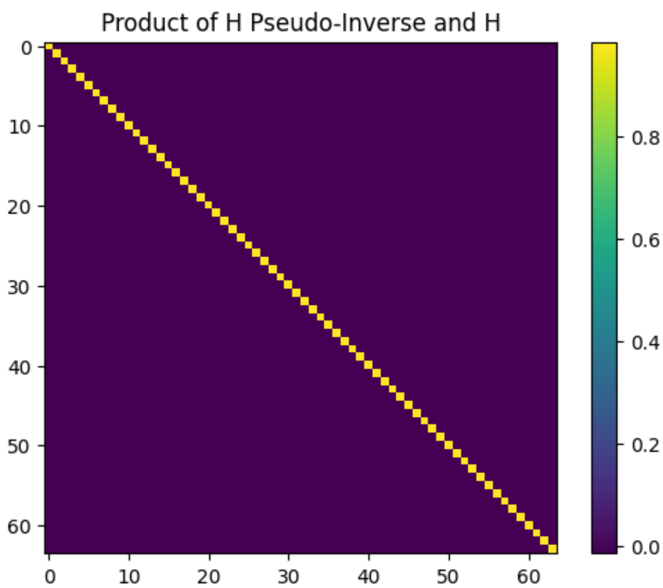
We can see once again that the Wiener filter significantly reduced a large portion of the noise. However, the approximation is not as good in some cases because the original signal is both distorted by H, which is not invertible, and affected by noise that could be significant. As a result, we achieved a higher reconstruction error.

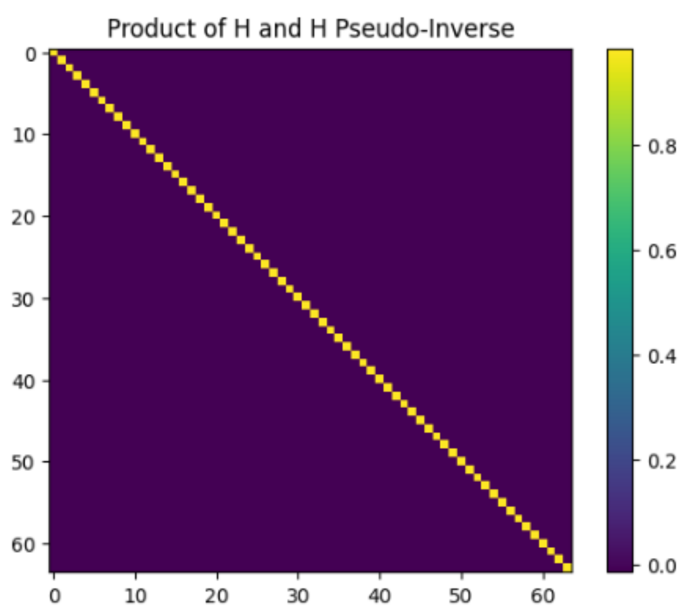
e.



H Pseudo-Inverse
0
10
20
30
40
50
60
2
1
0
−1
−2
−3
−4
−5



Product of H Pseudo-Inverse and H
0
10
20
30
40
50
60
0.8
0.6
0.4
0.2
0.0



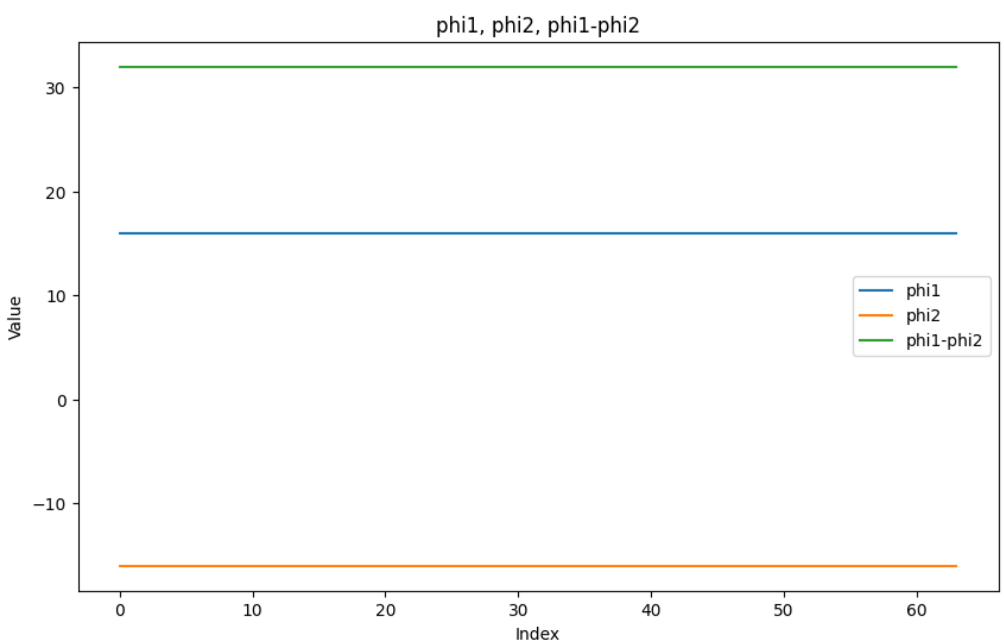
Product of H and H Pseudo-Inverse
0
10
20
30
40
50
60
0.8
0.6
0.4
0.2
0.0

The first plot shows that $H^{\dagger}$ is also circulant, which is expected since we are computing the pseudo-inverse using the DFT matrix.
We can also see that the products $H^{\dagger}H, HH^{\dagger}$ are close to identity matrices, with diagonal elements near 1 and off-diagonal elements close to 0. This indicates that the pseudo-inverse effectively recovers the action of H within its relevant subspaces. However, since H is doesn't have full rank, the products don't result in perfect identity matrices(it's hard to see from the plots but we verified the rank of the each one of the product is 63 as expected).

Yes, we can find two signals $\phi_1, \phi_2 \ such \ that \ \|\phi_1 - \phi_2\|_2 \geq 256 \ staysfing \ \|H^{\dagger}\phi_1 - H^{\dagger}\phi_2\|$.
In exercise one we observed that the null space of $Null(H)$ is spanned by the first column of DFT^* which is $(1, 1, \dots, 1)^T$. The pseudo-inverse is constructed in a way that preserves the same null space . The reason is that if $\lambda_i^H = 0 \ => \ \lambda_i^{H^{\dagger}} = 0$ as well, and the corresponding eigenvectors of $H^{\dagger}$ is the same as those of H.
Therefore, $(1, 1, \dots, 1)^T$ also span Null $(H^{\dagger})$. So, we can choose any two signals ϕ_1, ϕ_1 that are spanned by $(1, 1, \dots, 1)^T$ such that $\|\phi_1 - \phi_2\|_2 \geq 256$.
The two signals we chose are $\phi_1 = \ 16(1, 1, \dots, 1)^T, -16(1, 1, \dots, 1)^T$.

We get that $\|\phi_1 - \phi_2\|_2 = \|32(1, 1, \dots, 1)\|_2 = 32\|(1, 1, \dots, 1)\|_2 = 32\sqrt{64} = 256$.

Empirical results:
The empirical $\|\phi_1 - \phi_2\|_2$ and the plot of $\phi_1 - \phi_2, \phi_1, \phi_2$:

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Euclidean Norm phi1- phi2: 256.0
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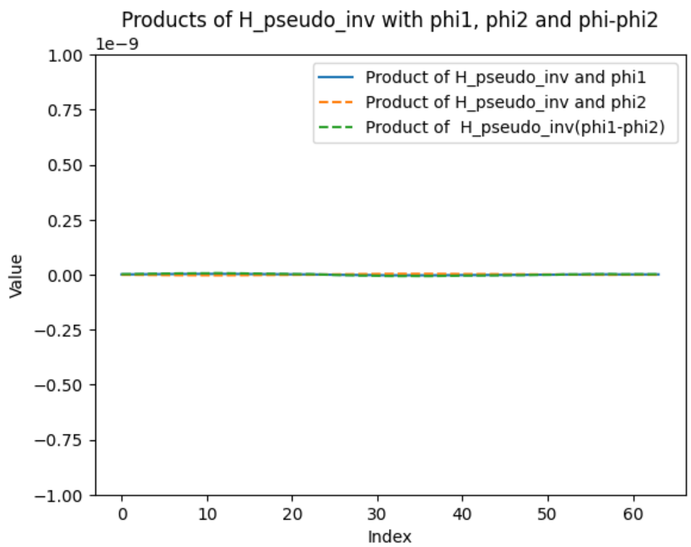
The empirical $\left\|H^{\dagger}\phi_1 - H^{\dagger}\phi_2\right\|_2$ and the plot of $H^{\dagger}\phi_1 - H^{\dagger}\phi_2, H^{\dagger}\phi_1, H^{\dagger}\phi_2$:




```
Euclidean Norm phi1- phi2: 3.204732678619319e-11
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The empirical results align well with our analytical analysis. We did get that $\left\|H^{\dagger}\phi_1 - H^{\dagger}\phi_2\right\|_2 \approx 0, \|\phi_1 - \phi_2\|_2 = 256$,as expected.
Additionally, we confirmed that $H^{\dagger}\phi_1 = H^{\dagger}\phi_2 \approx 0$ which is consistent with the fact that ϕ_1, ϕ_2 were chosen to lie in the null space of $H^{\dagger}$.