Chen Pery – 313283656 | Roi Papo - 316327451

Theoretical questions

1. Let $f(\mathbf{x})$ be the image that would be formed on the image plane if the camera was absolutely still. The actually formed digital image of the k-th frame can be described as

$$g_k[\mathbf{n}] = (f * h_k)(\mathbf{n}),$$

where $h_k(\mathbf{x})$ is the effective point spread function (PSF) corresponding to the k-th frame. Express h_k in terms of the camera motion.

1.

- The integration time T is mentioned as 1msec, which means that each frame in the burst has an exposure time of T. This integration time determines how long the camera sensor collects light during each exposure. So, we calculate each frame during the time interval of T.
- The actually formed digital image of the k-th frame is $g_k[n] = (f * h_k)(n)$
The variable n represents a pixel location.
denote $\tau_{o(t)}$ to be the translation operator depend on t because the motion trajectory of the camera shake movement is given as the location of sensor center o(t) at time t in image plane unit.
- The camera optics is an ideal anti-aliasing filter with band-limited imaged scenes according to the sampling rate of the pixels,
Therefore, the camera optics, has a natural high frequency cut-off to diffraction limit by a box function, following the lecture, we choose the box function - $box(x) = \left[|x| < \frac{1}{2}\right]^2$.

(The camera movement and hand tremor are all limited to a single plane, which is parallel to the image plane, hence the dimension is 2.)

$$h_k(x) = \int_k^{k+T} \tau_{o(t)} box(x) dt$$

2. Express the Fourier transform of h_k.

2.

$$\mathcal{H}_k(\xi) \overset{\substack{\text{by}\\\text{def}}}{=} \mathcal{F}\big(h_k(x)\big)(\xi) = \mathcal{F}\left(\int_k^{k+T} \tau_{o(t)} box(x) dt\right)(\xi)$$

$$\overset{\substack{properties\ of\ \mathcal{F}:\\ linearity}}{=} \int_k^{k+T} \mathcal{F}(\tau_{o(t)} box(x))(\xi) dt$$

$$\overset{\substack{properties\ of\ \mathcal{F}:\\ translation}}{=} \int_k^{k+T} \left(e^{-2\pi i \xi^T o(t)} \cdot \mathcal{F}\big(box(x)\big)(\xi)\right) dt$$

$$\overset{\substack{FT\ of\ box\ is\\ sinc\ as\ seen\\ in\ the\ lecture}}{=} \int_k^{k+T} e^{-2\pi i \xi^T o(t)} \cdot sinc(\xi) dt$$

3. Show that the action of the camera can be expressed fully in the digital domain as

$$G_k[\boldsymbol{\omega}] = F[\boldsymbol{\omega}] \cdot P_k[\boldsymbol{\omega}] \operatorname{sinc}(\boldsymbol{\omega}).$$

Write an expression for the frequency response of the discrete kernel P_k.

3.

The digital domain involves representation and processing of signals in a digital format with discrete values. The digital data, which might be continuous and analog is converted into a digital format.

The action of the camera is represented by $g_k[n]$. we want to express it fully in the digital domain, so we can use the Fourier Transform on it, resulting in a discrete representation. (Discrete domain Fourier transform)

$$G_k[\omega] = \big(\mathcal{F}(f * h_k)\big)[\omega] \overset{\substack{properties\ of\ \mathcal{F}: \\ convolution}}{\cong} (\mathcal{F}f)[\omega](\mathcal{F}h_k)[\omega]$$

$$\overset{q2}{\cong} (\mathcal{F}f)[\omega] \cdot \mathcal{H}_k[\omega] = (\mathcal{F}f)[\omega] \cdot \int_k^{k+T} e^{-2\pi i\omega^T o(t)} \cdot sinc(\omega) dt$$

$$\overset{\substack{sinc(\omega) \\ independent \\ on\ t}}{\cong} (\mathcal{F}f)[\omega] \cdot \int_k^{k+T} e^{-2\pi i\omega^T o(t)} dt \cdot sinc(\omega)$$

$denote\ \ \mathcal{F}[\omega] = (\mathcal{F}f)[\omega]\ \ and\ \ P_k[\omega] = \int_k^{k+T} e^{-2\pi i\omega^T o(t)} dt,$

$$= \mathcal{F}[\omega] \cdot P_k[\omega] sinc(\omega)$$

$$G_k[\omega] = \mathcal{F}[\omega] \cdot P_k[\omega] sinc(\omega)$$

When $P_k[\omega] = \int_k^{k+T} e^{-2\pi i\omega^T o(t)} dt$

4. Write an upper bound on $|P_k[\boldsymbol{\omega}]|$.

4.

$$|P_k[\omega]| = \left|\int_k^{k+T} e^{-2\pi i\omega^T o(t)} dt\right| \overset{\substack{\text{Triangle} \\ \text{inequality}}}{\lesssim} \int_k^{k+T} |e^{-2\pi i\omega^T o(t)}| dt$$

$$\overset{\substack{the\ absolute\ value\ of \\ a\ complex\ number\ e^{i\theta}\ is\ 1}}{\cong} \int_k^{k+T} 1\, dt = T$$

So, the upper bound on $|P_k[\omega]|$ is T.

5. Assume that on the interval $\tau \in [k, k+1)$, the camera moves with constant velocity v in the horizontal direction, $\mathbf{o}(\tau) = (\tau - k - 0.5)\, v\, \mathbf{e}_1$, where $\mathbf{e}_1 = (1,0,0)^{\mathrm{T}}$. Express $P_k[\boldsymbol{\omega}]$ in this case. How does $|P_k[\boldsymbol{\omega}]|$ depend on v?

5.

The camera moves with constant velocity v in the horizonal direction on the interval $\tau \in [k, k+1]$, so we can use $o(t) = (t-k-0.5)ve_1 \overset{e_1=(1,0,0)^T}{\cong} [(t-k-0.5)v, 0, 0]^T$ denote $\omega = [\omega_1, \omega_2, \omega_3]^T$.

$$\#\ \omega^T o(t) = ([\omega_1, \omega_2, \omega_3]^T)^T [(t-k-0.5)v, 0, 0]^T = [(t-k-0.5)v\omega_1, 0, 0]^T$$

$$P_k[\omega] = \int_k^{k+T} e^{-2\pi i \omega^T o(t)} dt = \int_k^{k+T} e^{-2\pi i \omega^T o(t)} dt \overset{(\#)}{\cong} \int_k^{k+T} e^{-2\pi i (t-k-0.5)v\omega_1} dt$$

$$\overset{\substack{independent \\ on\ t}}{\cong} e^{2\pi i \omega_1 v(k+0.5)} \int_k^{k+T} e^{-2\pi i \omega_1 v t} dt \overset{T=1}{\cong} e^{2\pi i \omega_1 v(k+0.5)} \left[\frac{e^{-2\pi i \omega_1 v t}}{-2\pi i \omega_1 v}\right]_k^{k+1}$$

$$e^{2\pi i \omega_1 v(k+0.5)} \left[\frac{e^{-2\pi i \omega_1 v(k+1)}}{-2\pi i \omega_1 v} - \frac{e^{-2\pi i \omega_1 v k}}{-2\pi i \omega_1 v}\right] = \frac{e^{2\pi i \omega_1 v(k+0.5)}}{-2\pi i \omega_1 v} \left[e^{-2\pi i \omega_1 v(k+1)} - e^{-2\pi i \omega_1 v k}\right]$$

$$\frac{e^{2\pi i \omega_1 v(k+0.5)}}{-2\pi i \omega_1 v} \cdot e^{-2\pi i \omega_1 v k}(e^{-2\pi i \omega_1 v} - 1) = \frac{e^{\pi i \omega_1 v}}{-2\pi i \omega_1 v}(e^{-2\pi i \omega_1 v} - 1) = \frac{e^{-\pi i \omega_1 v} - e^{\pi i \omega_1 v}}{-2\pi i \omega_1 v}$$

By Euler's formula

$$\overset{sinx=\frac{e^{ix}-e^{-ix}}{2i}}{\cong} \frac{e^{\pi i \omega_1 v} - e^{-\pi i \omega_1 v}}{2\pi i \omega_1 v} \overset{x=\pi\omega_1 v}{\cong} \frac{\sin(\pi\omega_1 v)}{\pi\omega_1 v}$$

$$\overset{sinc(t)=\frac{\sin(t)}{t}}{\cong} sinc(\pi\omega_1 v)$$

So, $|P_k[\omega]| = |sinc(\pi\omega_1 v)|$

The dependence of $|P_k[\omega]|$ on v is through the function sinc.

- When $v = 0$ The sinc function evaluates to $sinc(0) = 1$, leading to $|P_k[\omega]| = 1$ when $v = 0$
- As v increases the argument $\pi\omega_1 v$ becomes larger, leading to more oscillations in the sinc function.

Furthermore, the upper bound of $|sinc(\pi\omega_1 v)|$ is 1, given that T=1 in the context of this question, this aligns precisely with the result obtained in question 4.

6. Generalize the previous result to the case where on the interval $\tau \in [k, k+1)$, $\mathbf{o}(\tau) = \mathbf{q} + \tau\,\mathbf{v}$.

6.

Now, $o(t) = q + vt$.

$$P_k[\omega] = \int_k^{k+T} e^{-2\pi i \omega^T o(t)} dt \overset{T=1}{\cong} \int_k^{k+1} e^{-2\pi i \omega^T (q+vt)} dt$$

$$= e^{-2\pi i \omega^T q} \int_k^{k+1} e^{-2\pi i \omega^T vt} dt = e^{-2\pi i \omega^T q} \left[\frac{e^{-2\pi i \omega^T vt}}{-2\pi i \omega^T v}\right]_k^{k+1}$$

$$\frac{e^{-2\pi i \omega^T q}}{-2\pi i \omega^T v}\left(e^{-2\pi i \omega^T v(k+1)} - e^{-2\pi i \omega^T vk}\right) = \frac{e^{-2\pi i \omega^T q}}{-2\pi i \omega^T v} \cdot e^{-2\pi i \omega^T vk}\left(e^{-2\pi i \omega^T v} - 1\right)$$

$$= \frac{e^{-2\pi i \omega^T (q+vk)}}{-2\pi i \omega^T v}\left(e^{-2\pi i \omega^T v} - 1\right) = \frac{e^{-2\pi i \omega^T (q+vk)}}{-2\pi i \omega^T v}\left(e^{-2\pi i \omega^T v} - 1\right)$$

$$= \frac{e^{-2\pi i \omega^T (q+vk)}}{-2\pi i \omega^T v}\left(e^{-2\pi i \omega^T v} - \frac{e^{\pi i \omega^T v}}{e^{\pi i \omega^T v}}\right) = \frac{e^{-2\pi i \omega^T (q+vk)}}{-2\pi i \omega^T v}\left(\frac{e^{-2\pi i \omega^T v} \cdot e^{\pi i \omega^T v}}{e^{\pi i \omega^T v}} - \frac{e^{\pi i \omega^T v}}{e^{\pi i \omega^T v}}\right) =$$

By Euler's formula

$$= \frac{e^{-2\pi i \omega^T (q+vk)}}{\pi \omega^T v} \cdot \frac{1}{e^{\pi i \omega^T v}}\left(\frac{e^{-\pi i \omega^T v} - e^{\pi i \omega^T v}}{-2i}\right) \overset{sinx=\frac{e^{ix}-e^{-ix}}{2i}}{\cong} \frac{e^{-2\pi i \omega^T (q+vk)}}{\pi \omega^T v} \cdot \frac{1}{e^{\pi i \omega^T v}}\left(\frac{e^{\pi i \omega^T v} - e^{-\pi i \omega^T v}}{2i}\right)$$

$$\overset{x=\pi\omega^T v}{\cong} \frac{e^{-2\pi i \omega^T (q+vk)}}{\pi \omega^T v} \cdot \frac{1}{e^{\pi i \omega^T v}} \cdot \sin(\pi \omega^T v) = \frac{e^{-2\pi i \omega^T (q+vk)}}{e^{\pi i \omega^T v}} \cdot \frac{\sin(\pi \omega^T v)}{\pi \omega^T v}$$

$$\overset{sinc(t)=\frac{\sin(t)}{t}}{\cong} e^{-2\pi i \omega^T (q+vk)} \cdot e^{-\pi i \omega^T v} \cdot sinc(\pi \omega^T v) = e^{-2\pi i \omega^T (q+vk) - \pi i \omega^T v} \cdot sinc(\pi \omega^T v)$$

$$= e^{-\pi i \omega^T (2(q+vk)+v)} \cdot sinc(\pi \omega^T v) = e^{-\pi i \omega^T (2q+2vk+v)} \cdot sinc(\pi \omega^T v)$$

7. Suppose that N frames have been acquired such that in the k-th frame the camera was moving with some *unknown* but constant velocity $\mathbf{v}_k$. Assume the frames $g_1, \ldots, g_N$ have been pre-aligned such that $o(k+0.5) = 0$ for each k, meaning that the camera is located at the origin in the middle of its movement during which the frame was captured. Suggest a way to estimate f, when h_k and P_k are unknown. This problem is known as *blind deblurring*.

Referring to the findings in "Removing Camera Shake via Weighted Fourier Burst Accumulation" by Delbracio and Sapiro,
"If the photographer takes a burst of images, a modality available in virtually all modern digital cameras, we show that it is possible to combine them to get a clean sharp version. This is done without explicitly solving any blur estimation and subsequent inverse problem."

we will use the weighted average of the Fourier transform of the images burst, but with weights depending on the Fourier spectrum magnitude.
Due to camera shake, each image in the burst exhibits distinct blurring patterns, resulting in mostly different blurring kernels ($P_k[\omega]$). Consequently, each Fourier frequency will be differently attenuated on each frame of the burst.
Drawing from the paper's insight that 'the strongest frequency values represent the least attenuated,' we capitalize on this observation by selecting frequencies with lower attenuation from each burst image and we assign them greater weight in the Fourier transform of the reconstructed image.
This strategic weighting contributes to the generation of a final image characterized by sharpness.

we begin with a burst images as our input: $\{g_1[n], \ldots, g_N[n]\}$, by employing the method discussed in question 3, we compute the discrete domain Fourier transform to obtain: $\{G_1[\omega], \ldots, G_N[\omega]\}$.

$$G_k[\omega] = \mathcal{F}[\omega] \cdot P_k[\omega] sinc(\omega)$$

$$\mathcal{F}[\omega] = \frac{1}{N} \sum_{i=1}^{N} \frac{G_k[\omega]}{P_k[\omega] sinc(\omega)}$$

For that, We use the equality in q6 :
$P_k[\omega] = e^{-\pi i \omega^T (2q+2vk+v)} \cdot sinc(\pi \omega^T v)$
Instead use v, we use v_k because "the k-th frame the camera was moving with some unknown but constant velocity v_k."
In addition, "assume the frames $g_1, \ldots, g_N$ $have\ been\ pre-alined\ such\ that$ $(k+0.5) = 0$ for each k" so we use, $q_k = -(v_k k + 0.5 v_k)$, because:

$o(t) = q + vt$, and $o(k+0.5) = 0 \Rightarrow q + v(k+0.5) = 0 \Rightarrow q = -vk - 0.5v$

$$\Rightarrow q_k = -(v_k k + 0.5 v_k)$$

Therefore,
$P_k[\omega] = e^{-\pi i \omega^T (2q_k + 2v_K k + v_K)} \cdot sinc(\pi \omega^T v_K) = e^{-\pi i \omega^T (-2(v_k k + 0.5 v_k) + 2 v_K k + v_K)} \cdot sinc(\pi \omega^T v_K)$
$= e^{-\pi i \omega^T (-2(v_k k + 0.5 v_k) + 2 v_K k + v_K)} \cdot sinc(\pi \omega^T v_K) = e^{-\pi i \omega^T \cdot 0} \cdot sinc(\pi \omega^T v_K) = sinc(\pi \omega^T v_K)$

in this outcome, we observe a resemblance to the findings discussed in question 5 .

The exact value of v_k is unknown.
The term $P_k[\omega]$, represented by $sinc(\pi\omega^T v_K)$, is responsible for the blurring in the image
Another representation is possible, as mentioned earlier.
we determine the wight assigned to each frequency in every image by assessing its magnitude relative to the other images in the burst.

We use hyperparameter p (regularization weights) to achieve a balance -
let p be a non-negative integer.

$$W_k[\omega] = \frac{|G_k[\omega]|^p}{\sum_{i=1}^{N} |G_k[\omega]|^p}$$

The integer p controls the aggregation of the images in the Fourier domain. If p = 0, the restored image is just the average of the burst, while if p → ∞, each reconstructed frequency takes the maximum value of that frequency along the burst. (for simplicity We will use p=1)

$$\Rightarrow \quad \mathcal{F}[\omega] = \frac{1}{N}\sum_{i=1}^{N} \frac{G_k[\omega]}{P_k[\omega]sinc(\omega)} = \frac{1}{N}\sum_{i=1}^{N} \frac{G_k[\omega]}{sinc(\pi\omega^T v_K)sinc(\omega)}$$

$$= \frac{1}{N}\sum_{i=1}^{N} \frac{G_k[\omega]}{sinc(\omega)} \cdot \frac{1}{sinc(\pi\omega^T v_K)}$$

$$let\ wighted\ \mathcal{F}[\omega], \qquad \tilde{\mathcal{F}}[\omega] = \sum_{i=1}^{N} W_k[\omega] \cdot \frac{G_k[\omega]}{sinc(\omega)}$$

$$f(x) \approx \mathcal{F}^{-1}(\tilde{\mathcal{F}}[\omega])$$