Short HW1 - Preparing for the course

Useful python libraries, Probability, and Linear algebera

Instructions

General

- **First, don't panic!**
  - This assignment seems longer than it actually is.
  - In the first part, you are mostly required to run *existing* code and complete short python commands here and there.
  - In the two other parts you need to answer overall 4 analytic questions.
  - **Note**: The other 4 *short* assignments will be shorter and will *not* require programming.
- **Individually or in pairs?** Individually only.
- **Where to ask?** In the Piazza forum.
- **How to submit?** In the webcourse.
- **What to submit?** A pdf file with the completed jupyter notebook (including the code, plots and other outputs) and the answers to the probability/algebra questions (Hebrew or English are both fine).
  Or two separate pdf files in a zip file. All submitted files should contain **your ID number** in their names.
- **When to submit?** Monday 07.11.2022 at 23:59.
  - **Late submissions**: we will recieve late submissions for additional 24 hours, but deduce 5 points from the grade.

Specific

- First part: get familiar with popular python libraries useful for machine learning and data science. We will use these libraries heavily throughout the major programming assignments.
  - You should read the instructions and run the code blocks sequentially.
    In 10 places you are reqired to complete missing python commands or answer short questions (look for the **TODO** comments, or notations like **(T3)** etc.). Try to understand the flow of this document and the code you run.
  - Start by loading the provided jupyter notebook file (*Short_HW1.ipynb*) to Google Colab, which is a very convenient online tool for running python scripts combined with text, visual plots, and more.
  - Alternatively, you can install jupyter locally on your computer and run the provided notebook there.
- Second and third parts: questions on probability and linear algebra to refresh your memory and prepare for the rest of this course.
  The questions are mostly analytic but also require completing and running simple code blocks in the jupyter notebook.
  - Forgot your linear algebra? Try watching Essence of LA or reading The Matrix Cookbook.
  - Forgot your probability? Try reading Probability Theory Review for Machine Learning.
    - Correction: In 3.2 it says that $X \perp Y \implies \mathrm{Var}(X+Y) = \mathrm{Var}(X)\mathrm{Var}(Y)$ but it should say $X \perp Y \implies \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

Important: How to submit the notebook's output?

You should only submit PDF file(s). In the print dialog of your browser, you can choose to `Save as PDF`. However, notice that some of the outputs may be cropped (become invisible), which can harm your grade.

To prevent this from happening, tune the "scale" of the printed file, to fit in the *entire* output. For instance, in Chrome you should lower the value in `More settings->Scale->Custom` to contain the entire output (50%~ often work well).

Good luck!

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What is pandas?

Python library for Data manipulation and Analysis

- Provide expressive data structures designed to make working with "relational" or "labeled" data both easy and intuitive.
- Aims to be the fundamental high-level building block for doing practical, real world data analysis in Python.
- Built on top of NumPy and is intended to integrate well within a scientific computing.
- Inspired by R and Excel.

Pandas is well suited for many different kinds of data:

- **Tabular data** with heterogeneously-typed columns, as in an SQL table or Excel spreadsheet
- Ordered and unordered (not necessarily fixed-frequency) **time series data**.
- **Arbitrary matrix data** (homogeneously typed or heterogeneous) with row and column labels
- Any other form of observational / statistical data sets (can be unlabeled)

Two primary data structures

- **Series** (1-dimensional) – Similar to a column in Excel's spreadsheet
- **Data Frame** (2-dimensional) – Similar to R's data frame

A few of the things that Pandas does well

- Easy handling of **missing data** (represented as NaN)
- Automatic and explicit **data alignment**
- Read and Analyze **CSV** , Excel Sheets Easily
- Operations
- Filtering, Group By, Merging, Slicing and Dicing, Pivoting and Reshaping
- Plotting graphs

Pandas is very useful for interactive data exploration at the data preparation stage of a project

The offical guide to Pandas can be found here

Pandas Objects

```
import pandas as pd
import numpy as np
```

Series is like a column in a spreadsheet.

```
s = pd.Series([1,3.2,np.nan,'string'])
s
```

```
0         1
1       3.2
2       NaN
3    string
dtype: object
```

DataFrame is like a spreadsheet – a dictionary of Series objects

```
data = [['ABC', -3.5, 0.01], ['ABC', -2.3, 0.12], ['DEF', 1.8, 0.03],
['DEF', 3.7, 0.01], ['GHI', 0.04, 0.43], ['GHI', -0.1, 0.67]]

df = pd.DataFrame(data, columns=['gene', 'log2FC', 'pval'])

df
```

	gene	log2FC	pval
0	ABC	-3.50	0.01
1	ABC	-2.30	0.12
2	DEF	1.80	0.03
3	DEF	3.70	0.01
4	GHI	0.04	0.43
5	GHI	-0.10	0.67

Input and Output

How do you get data into and out of Pandas as spreadsheets?

- Pandas can work with XLS or XLSX files.
- Can also work with CSV (comma separated values) file
- CSV stores plain text in a tabular form
- CSV files may have a header
- You can use a variety of different field delimiters (rather than a 'comma'). Check which delimiter your file is using before import!

Import to Pandas

> `df = pd.read_csv('data.csv', sep='\t', header=0)`

For Excel files, it's the same thing but with read_excel

Export to text file

> `df.to_csv('data.csv', sep='\t', header=True, index=False)`

The values of header and index depend on if you want to print the column and/or row names

Case Study – Analyzing Titanic Passengers Data

```
import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np
import pandas as pd
import os
```

```
#set your working_dir
working_dir = os.path.join(os.getcwd(), 'titanic')

url_base = 'https://github.com/Currie32/Titanic-Kaggle-Competition/raw/master/{}.csv'
train_url = url_base.format('train')
test_url = url_base.format('test')

# For .read_csv, always use header=0 when you know row 0 is the header row
train = pd.read_csv(train_url, header=0)
test = pd.read_csv(test_url, header=0)
# You can also load a csv file from a local file rather than a URL
```

(T1) Use `pandas.DataFrame.head` to display the top 6 rows of the `train` table

```
# TODO: print the top 6 rows of the table
train.head(6)
```

	PassengerId	Survived	Pclass	Name	Sex	Age	SibSp	Parch	Ticket	Fare	Cabin	Embarked
0	1	0	3	Braund, Mr. Owen Harris	male	22.0	1	0	A/5 21171	7.2500	NaN	S
1	2	1	1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1	0	PC 17599	71.2833	C85	C
2	3	1	3	Heikkinen, Miss. Laina	female	26.0	0	0	STON/O2. 3101282	7.9250	NaN	S
3	4	1	1	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1	0	113803	53.1000	C123	S
4	5	0	3	Allen, Mr. William Henry	male	35.0	0	0	373450	8.0500	NaN	S
5	6	0	3	Moran, Mr. James	male	NaN	0	0	330877	8.4583	NaN	Q

VARIABLE DESCRIPTIONS:

Survived - 0 = No; 1 = Yes
Age - Passenger's age
Pclass - Passenger Class (1 = 1st; 2 = 2nd; 3 = 3rd)
SibSp - Number of Siblings/Spouses Aboard
Parch - Number of Parents/Children Aboard
Ticket - Ticket Number
Fare - Passenger Fare
Cabin - Cabin ID
Embarked - Port of Embarkation (C = Cherbourg; Q = Queenstown; S = Southampton)

```
train.columns
```

```
Index(['PassengerId', 'Survived', 'Pclass', 'Name', 'Sex', 'Age', 'SibSp',
       'Parch', 'Ticket', 'Fare', 'Cabin', 'Embarked'],
      dtype='object')
```

Understanding the data (Summarizations)

```
train.info()
```

```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 891 entries, 0 to 890
Data columns (total 12 columns):
 #   Column       Non-Null Count  Dtype
---  ------       --------------  -----
 0   PassengerId  891 non-null    int64
 1   Survived     891 non-null    int64
 2   Pclass       891 non-null    int64
```

```
 3   Name        891 non-null    object
 4   Sex         891 non-null    object
 5   Age         714 non-null    float64
 6   SibSp       891 non-null    int64
 7   Parch       891 non-null    int64
 8   Ticket      891 non-null    object
 9   Fare        891 non-null    float64
 10  Cabin       204 non-null    object
 11  Embarked    889 non-null    object
dtypes: float64(2), int64(5), object(5)
memory usage: 83.7+ KB
```

```
train.shape
```

```
(891, 12)
```

```
# Count values of 'Survived'
train.Survived.value_counts()
```

```
0    549
1    342
Name: Survived, dtype: int64
```

```
# Calculate the mean fare price
train.Fare.mean()
```

```
32.204207968574636
```

```
# General statistics of the dataframe
train.describe()
```

	PassengerId	Survived	Pclass	Age	SibSp	Parch	Fare
count	891.000000	891.000000	891.000000	714.000000	891.000000	891.000000	891.000000
mean	446.000000	0.383838	2.308642	29.699118	0.523008	0.381594	32.204208
std	257.353842	0.486592	0.836071	14.526497	1.102743	0.806057	49.693429
min	1.000000	0.000000	1.000000	0.420000	0.000000	0.000000	0.000000
25%	223.500000	0.000000	2.000000	20.125000	0.000000	0.000000	7.910400
50%	446.000000	0.000000	3.000000	28.000000	0.000000	0.000000	14.454200
75%	668.500000	1.000000	3.000000	38.000000	1.000000	0.000000	31.000000
max	891.000000	1.000000	3.000000	80.000000	8.000000	6.000000	512.329200

Selection examples

Selecting columns

```
# Selection is very similar to standard Python selection
df1 = train[["Name", "Sex", "Age", "Survived"]]
df1.head()
```

	Name	Sex	Age	Survived
0	Braund, Mr. Owen Harris	male	22.0	0
1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1
2	Heikkinen, Miss. Laina	female	26.0	1
3	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1
4	Allen, Mr. William Henry	male	35.0	0

Selecting rows

```
df1[10:15]
```

	Name	Sex	Age	Survived
10	Sandstrom, Miss. Marguerite Rut	female	4.0	1
11	Bonnell, Miss. Elizabeth	female	58.0	1
12	Saundercock, Mr. William Henry	male	20.0	0
13	Andersson, Mr. Anders Johan	male	39.0	0
14	Vestrom, Miss. Hulda Amanda Adolfina	female	14.0	0

Filtering Examples

Filtering with one condition

```
# Filtering allows you to create masks given some conditions
df1.Sex == 'female'
```

```
0      False
1       True
2       True
3       True
4      False
       ...
886    False
887     True
888     True
889    False
890    False
Name: Sex, Length: 891, dtype: bool
```

```
onlyFemale = df1[df1.Sex == 'female']
onlyFemale.head()
```

	Name	Sex	Age	Survived
1	Cumings, Mrs. John Bradley (Florence Briggs Th...	female	38.0	1
2	Heikkinen, Miss. Laina	female	26.0	1
3	Futrelle, Mrs. Jacques Heath (Lily May Peel)	female	35.0	1
8	Johnson, Mrs. Oscar W (Elisabeth Vilhelmina Berg)	female	27.0	1
9	Nasser, Mrs. Nicholas (Adele Achem)	female	14.0	1

Filtering with multiple conditions

(T2) Alter the following command so `adultFemales` will contain only females whose age is 18 and above.
You need to filter using a **single** mask with multiple conditions (google it!), i.e., without creating any temporary dataframes.

Additionally, update the `survivalRate` variable to show the correct rate.

```
# TODO: update the mask
adultFemales = df1[(df1.Sex == 'female') & (df1.Age >= 18)]
# TODO: Update the survival rate
survivalRate = adultFemales.Survived.mean()
print("The survival rate of adult females was: {:.2f}%".format(survivalRate * 100))
```

```
The survival rate of adult females was: 77.18%
```

Aggregating

Pandas allows you to aggregate and display different views of your data.

```
df2 = train.groupby(['Pclass', 'Sex']).Fare.agg(np.mean)
df2
```

```
Pclass  Sex
1       female    106.125798
        male       67.226127
2       female     21.970121
        male       19.741782
3       female     16.118810
        male       12.661633
Name: Fare, dtype: float64
```

```
pd.pivot_table(train, index=['Pclass'], values=['Survived'], aggfunc='count')
```

Pclass	Survived
1	216
2	184
3	491

The following table shows the survival rates for each combination of passenger class and sex.
(T3) Add a column showing the mean **age** for such a combination.

```
# TODO: Also show the mean age per group
pd.pivot_table(train, index=['Pclass', 'Sex'], values=['Survived', "Age"], aggfunc='mean')
```

Pclass	Sex	Age	Survived
1	female	34.611765	0.968085
	male	41.281386	0.368852
2	female	28.722973	0.921053
	male	30.740707	0.157407
3	female	21.750000	0.500000
	male	26.507589	0.135447

(T4) Use this question on stackoverflow, to find the mean survival rate for ages 0-10, 10-20, etc.).

```
# TODO: find the mean survival rate per age group
ageGroups = np.arange(0, 81, 10)
survivalPerAgeGroup = train.groupby(pd.cut(train.Age, ageGroups))[["Age", "Survived"]].mean()
survivalPerAgeGroup
```

Age	Age	Survived
(0, 10]	4.268281	0.593750
(10, 20]	17.317391	0.382609
(20, 30]	25.423913	0.365217
(30, 40]	35.051613	0.445161
(40, 50]	45.372093	0.383721
(50, 60]	54.892857	0.404762
(60, 70]	63.882353	0.235294
(70, 80]	73.300000	0.200000

```
type(train.groupby(pd.cut(train.Age, ageGroups)).Survived.mean())
```

```
pandas.core.series.Series
```

Filling missing data (data imputation)

Note that some passenger do not have age data.

```
print("{} out of {} passengers do not have a recorded age".format(df1[df1.Age.isna()].shape[0], df1.shape[0]))
```

```
177 out of 891 passengers do not have a recorded age
```

```
df1[df1.Age.isna()].head()
```

	Name	Sex	Age	Survived
5	Moran, Mr. James	male	NaN	0
17	Williams, Mr. Charles Eugene	male	NaN	1
19	Masselmani, Mrs. Fatima	female	NaN	1
26	Emir, Mr. Farred Chehab	male	NaN	0
28	O'Dwyer, Miss. Ellen "Nellie"	female	NaN	1

Let's see the statistics of the column **before** the imputation.

```
df1.Age.describe()
```

```
count    714.000000
mean      29.699118
std       14.526497
min        0.420000
25%       20.125000
50%       28.000000
75%       38.000000
max       80.000000
Name: Age, dtype: float64
```

Read about `pandas.Series.fillna`.

(T5) Replace the missing ages `df1` with the general age *median*, and insert the result into variable `filledDf` (the original `df1` should be left unchanged).

```
# TODO : Fill the missing values
filledDf = df1.fillna(value={'Age': df1.Age.median()})
```

```
print("{} out of {} passengers do not have a recorded age".format(filledDf[filledDf.Age.isna()].shape[0], filledDf.shape[0]))
```

```
0 out of 891 passengers do not have a recorded age
```

Let's see the statistics of the column **after** the imputation.

```
filledDf.Age.describe()
```

```
count    891.000000
mean      29.361582
std       13.019697
min        0.420000
25%       22.000000
50%       28.000000
75%       35.000000
max       80.000000
Name: Age, dtype: float64
```

(T6) Answer below: which statistics changed, and which did not? Why? (explain briefly, no need to be very formal.)

Answer:

Changes:

count - because there are more entries with valid Age value (non NaN)

std - there is new entries with values close to the mean, and therefore std will get smaller.

mean - each of the median values that were added is smaller then the mean, and therefore it made the new mean smaller .

25% - there are new valid entries, and therefore the 25% point is from bigger number of entries, and all the entries at 25% are not bigger then the median, and therefore there are more (maybe) bigger values, and therefore the 25% value will be (maybe) bigger (as it actually turns out to be).

75% - there are new valid entries, and therefore the 75% point is from bigger number of entries, and all the entries at 75% are not smaller then the median, and therefore there are more (maybe) smaller values, and therefore the 75% value will be (maybe) smaller (as it actually turns out to be).

Note - used the term "maybe" because for example, if all values, contain the median (or at least large enough number of entries), then the 25% will remain the same value, but in our case it does not remain the same, because this is not the situation.

Not Changes:

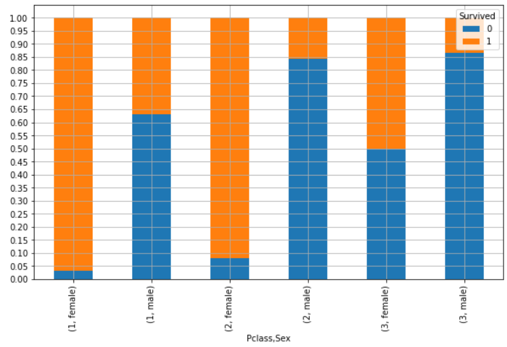
min/max - the new valid entries contain the median, the minimum/maximum ,respectively, value is still exists, and it is smaller/bigger (or equal), respectively, to the median, and therefore it stays the same after the change.

50% - this is exactly the median, we added more entries with this value, and therefore the median remain the same (amount of bigger/smaller values from median remained the same)

Plotting

Basic plotting in pandas is pretty straightforward

```
new_plot = pd.crosstab([train.Pclass, train.Sex], train.Survived, normalize="index")
new_plot.plot(kind='bar', stacked=True, grid=False, figsize=(10,6))
plt.yticks(np.linspace(0,1,21))
plt.grid()
```



(T7) Answer below: which group (class $\times$ sex) had the best survival rate? Which had the worst?

Answer: TODO

Best - (1, female)

Worst - (3, male)

What is Matplotlib

A 2D plotting library which produces publication quality figures.

- Can be used in python scripts, the python and IPython shell, web application servers, and more …
- Can be used to generate plots, histograms, power spectra, bar charts, errorcharts, scatterplots, etc.
- For simple plotting, pyplot provides a MATLAB-like interface
- For power users, a full control via OO interface or via a set of functions

There are several Matplotlib add-on toolkits

- Projection and mapping toolkits basemap and cartopy.
- Interactive plots in web browsers using Bokeh.
- Higher level interface with updated visualizations Seaborn.

Matplotlib is available at www.matplotlib.org

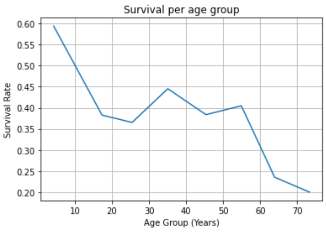
```
import matplotlib.pyplot as plt
import numpy as np
```

Line Plots

The following code plots the survival rate per age group (computed above, before the imputation).

(T8) Use the matplotlib documentation to add a grid and suitable axis labels to the following plot.

```
plt.plot(survivalPerAgeGroup.Age, survivalPerAgeGroup.Survived)
_ = plt.title("Survival per age group")
# TODO : Update the plot as required.
_ = plt.xlabel("Age Group (Years)")
_ = plt.ylabel("Survival Rate")
plt.grid(True)
```




```
survivalPerAgeGroup
```

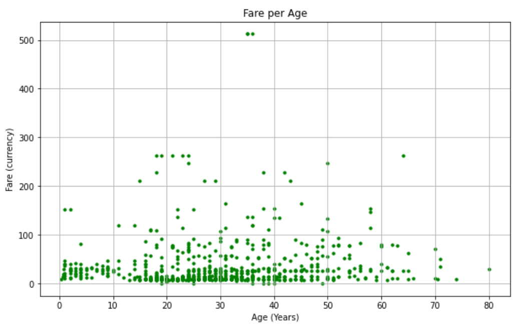
Age	Age	Survived
(0, 10]	4.268281	0.593750
(10, 20]	17.317391	0.382609
(20, 30]	25.423913	0.365217
(30, 40]	35.051613	0.445161
(40, 50]	45.372093	0.383721
(50, 60]	54.892857	0.404762
(60, 70]	63.882353	0.235294
(70, 80]	73.300000	0.200000

Scatter plots

(T9) Alter the matplotlib.pyplot.scatter command, so that the scattered dots will be `green`, and their size will be `10`.

Also, add a grid and suitable axis labels.

```
# TODO : Update the plot as required.
plt.figure(figsize=(10,6))
plt.scatter(train.Age, train.Fare, s=10, c="green")
plt.grid(True)
_ = plt.title('Fare per Age')
_ = plt.xlabel("Age (Years)")
_ = plt.ylabel("Fare (currency)")
```



(T10) Answer below: approximately how old are the two highest paying passengers?

Answer: According to the scatter plot above, the two highest paying passengers are approximately 36-37 years old

---

Probability refresher

Q1 - Variance of empirical mean

Let $X_1, \ldots, X_m$ be i.i.d random variables with mean $\mathbb{E}[X_i] = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2$.
We would like to "guess", or more formally, estimate (לשערך), the mean μ from the observations $x_1, \ldots, x_m$.
We use the empirical mean $\overline{X} = \frac{1}{m}\sum_i X_i$ as an estimator for the unknown mean μ. Notice that $\overline{X}$ is itself a random variable.
Note: The instantiation of $\overline{X}$ is usually denoted by $\hat{\mu} = \frac{1}{m}\sum_i x_i$, but this is currently out of scope.

1. Express analytically the expectation of $\overline{X}$.
   Answer: $\mathbb{E}\left[\overline{X}\right] = \mathbb{E}\left[1/m \sum_i X_i\right] = 1/m \sum_i E[X_i] = 1/m \sum_i \mu = (m/m) * \mu = \mu.$

2. Express analytically the variance of $\overline{X}$.
   Answer: $\mathrm{Var}\left[\overline{X}\right] = \mathrm{Var}\left[\frac{1}{m}\sum_i X_i\right] = \frac{1}{m^2}\mathrm{Var}\left[\sum_i X_i\right] = \frac{1}{m^2}\sum_i \mathrm{Var}[X_i] = \frac{m\sigma^2}{m^2} = \frac{\sigma^2}{m}$

   The above is using the fact that $Var[X+Y] = Var[X] + Var[Y]$ for independent X, Y variables and the property $Var[\alpha Y] = \alpha^2 Var[Y]$.

You will now verify the expression you wrote for the variance.
We assume $\forall i : X_i \sim \mathcal{N}(0, 1)$.
We compute the empirical mean's variances for sample sizes $m = 1, \ldots, 30$.
For each sample size m, we sample m normal variables and compute their empirical mean. We repeat this step 100 times, and compute the variance of the empirical means (for each m).

3. Complete the code blocks below according to the instructions and verify that your analytic function of the empirical mean's variance against as a function of m suits the empirical findings.

```
all_sample_sizes = range(1, 31)
repeats_per_size = 100

allVariances = []

for m in all_sample_sizes:
  empiricalMeans = []

  for _ in range(repeats_per_size):
    # Random m examples and compute their empirical mean
    X = np.random.randn(m)
    empiricalMeans.append(np.mean(X))

  # TODO: Using numpy, compute the variance of the empirical means that are in
  # the `empiricalMeans` list (you can google the numpy function for variance)
  variance = np.var(empiricalMeans)

  allVariances.append(variance)
```

Complete the following computation of the anayltic variance (according to the your answers above). You can try to use simple arithmetic operations between an `np.array` and a scalar, and see what happens! (for instance, `2 * np.array(all_sample_sizes)`.)

```
# TODO: compute the analytic variance
# (the current command wrongfully sets the variance of an empirical mean
#  of a sample with m variables simply as 2*m)
np_size_array_asfloat = np.array(all_sample_sizes).astype(float)
sigma = 1
analyticVariance = sigma**2 / np_size_array_asfloat
```

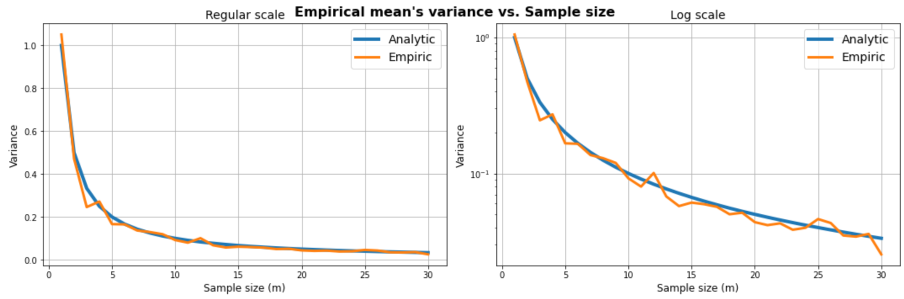
The following code plots the results from the above code. **Do not** edit it, only run it.

```
fig, axes = plt.subplots(1,2, figsize=(15,5))
axes[0].plot(all_sample_sizes, analyticVariance, label="Analytic", linewidth=4)
axes[0].plot(all_sample_sizes, allVariances, label="Empiric", linewidth=3)
axes[0].grid()
axes[0].legend(fontsize=14)
axes[0].set_title("Regular scale", fontsize=14)
axes[0].set_xlabel("Sample size (m)", fontsize=12)
axes[0].set_ylabel("Variance", fontsize=12)

axes[1].semilogy(all_sample_sizes, analyticVariance, label="Analytic", linewidth=4)
axes[1].semilogy(all_sample_sizes, allVariances, label="Empiric", linewidth=3)
axes[1].grid()
axes[1].legend(fontsize=14)
axes[1].set_title("Log scale", fontsize=14)
axes[1].set_xlabel("Sample size (m)", fontsize=12)
axes[1].set_ylabel("Variance", fontsize=12)

_ = plt.suptitle("Empirical mean's variance vs. Sample size",
            fontsize=16, fontweight="bold")

plt.tight_layout()
```



Reminder - Hoeffding's Inequality

Let $\theta_1, \ldots, \theta_m$ be i.i.d random variables with mean $\mathbb{E}[\theta_i] = \mu$.

Additionally, assume all variables are bound in $[a, b]$ such that $\Pr[a \leq \theta_i \leq b] = 1$.

Then, for any $\epsilon > 0$, the empirical mean $\bar{\theta}(m) = \frac{1}{m}\sum_i \theta_i$ holds:

$$\Pr\left[\left|\bar{\theta}(m) - \mu\right| > \epsilon\right] \leq 2\exp\left\{-\frac{2m\epsilon^2}{(b-a)^2}\right\}.$$

Q2 - Identical coins and the Hoeffding bound

We toss $m \in \mathbb{N}$ identical coins, each coin 100 times.
All coins have the same *unknown* probability of showing "heads", denoted by $p \in (0, 1)$.
Let θ_i be the (observed) number of times the i-th coin showed "heads".

1. What is the distribution of each θ_i?
   Answer: $\theta_i \sim \mathrm{Bin}(100, p)$.
2. What is the mean $\mu = \mathbb{E}[\theta_i]$?
   Answer: $\mathbb{E}[\theta_i] = 100p$.
3. We would like to use the empirical mean defined above as an estimator $\bar{\theta}(m)$ for μ.
   Use Hoeffding's inequality to compute the *smallest* sample size $m \in \mathbb{N}$ that can guarantee an error of $\epsilon = 1$ with confidence 0.9 (notice that we wish to estimate μ, not p).
   That is, find the smallest m that holds $\Pr\left[\left|\bar{\theta}(m) - \mu\right| > 1\right] \leq 0.1$.

Answer:
$\Pr\left[\left|\bar{\theta}(m) - \mu\right| > 1\right] \leq 2\exp\left\{-\frac{2m1^2}{(b-a)^2}\right\} = 2\exp\left\{-\frac{2m}{(b-a)^2}\right\} \leq 0.1$

$\exp\left\{-\frac{2m}{(b-a)^2}\right\} \leq \frac{1}{20}$

$-\frac{2m}{(b-a)^2} \leq \ln\frac{1}{20} = -\ln 20$

$\frac{2m}{(b-a)^2} \geq \ln 20$

When $\Pr[0 \leq \theta_i \leq 100] = 1$ And this is the best range, because theta_i may be 0 and it may also be a 100

$\frac{2m}{(100-0)^2} = \frac{2m}{100^2} = \frac{m}{5000} \geq \ln 20$

$m \geq 5000 \ln 20$

Therefore, the smallest m will be $\lceil 5000 \ln 20 \rceil = 14979$

4. The following code simulates tossing $m = 10^4$ coins, each 100 times. For each coin, we use the empirical mean as the estimator and save it in the `all_estimators` array. The (unknown) probability of each coin is 0.7.
Complete the missing part so that for each coin, an array of 100 binary `observations` will be randomized according to the probability p.

```
m = 10**4
tosses = 100
p = 0.7
all_estimators = []

# Repeat for n coins
for coin in range(m):
  # TODO: Use Google to find a suitable numpy.random function that creates
  # a binary array of size (tosses,), where each element is 1
  # with probability p, and 0 with probability (1-p).
  observations = np.random.choice([1,0], size=(tosses,), p=[p, 1-p])

  # Compute and save the empirical mean
  estimator = np.mean(observations)
  all_estimators.append(estimator)
```

Double-click (or enter) to edit

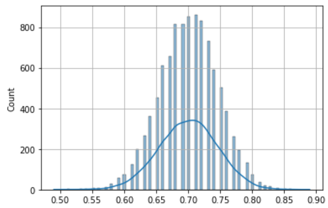
5. The following code plots the histogram of the estimators (empirical means). Run it. What type of distribution is obtained (no need to specify the exact paramters of the distribution)? Explain **briefly** what theorem from probability explains this behavior (and why).

Answer: Normal distribution. according to Central Limit Theorem, given n i.i.d random variables, $X_i, i \in [1, \ldots, n]$, the random variable $\overline{X_n}$ which is defined to be the mean of the n random variables defined above, will approach normal distribution as n →∞.

In our case we have m=10**4 samples of the mean of 100 toses (each tose will be denoted as X_i), each tose is independed and bernoulli ditributed, and therefore the m samples of $\overline{X_n}$ will be normally distributed.

This approximation considered a good approximation whenever n is big enough and $np \geq 5$ and $n(1-p) = nq \geq 5$, in our situation, $100 * 0.7 = 70 \geq 5$ and $100 * (1 - 0.7) = 30 \geq 5$, and n is 10000 which is big (enough), therefore it is indeed a good approximation.

```
import seaborn as sns
sns.histplot(all_estimators, bins=tosses, kde=True)
plt.grid()
```



Numerical linear algebra refresher

Reminder - Positive semi-definite matrices

A symmetric real matrix $A \in \mathbb{R}^{n \times n}$ is called positive semi-definite (PSD) iff:

$$\forall x \in \mathbb{R}^n \setminus \{0_n\} : x^\top Ax \geq 0.$$

If the matrix holds the above inequality *strictly*, the matrix is called positive definite (PD).

Q3 - PSD matrices

1. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric PSD matrix. Recall that all eigenvalues of real symmetric matrices are real.
Prove that all the eigenvalues of A are non-negative.
Answer:
A is symmetric, and therefore can be expressed using a spectral decomposition, $A = Q^T KQ$ for K diagonal matrix that holds As eigenvalues and Q is orthogonal matrix.

A is PSD, therefore,

$\forall x \in \mathbb{R}^n, 0 \leq x^T Ax = x^T Q^T KQx = (Qx)^T K(Qx)$

$z = Qx$

Q orthogonal, and therefore z(x)=Qx is a bijection, and therefor each x corrisponds to only single z value (and also the other way around), therefore we can say that,

$\forall z \in \mathbb{R}^n, \exists x \in \mathbb{R}^n, z = Qx, 0 \leq x^T Ax = z^T Kz = \sum_{i=1}^n \lambda_i z_i^2$

Therefore, each eigenvalue of A (and also of K) $(K)_{t,t}$, could be extracted using z vector which is 0 at all indexes except index t, in the t index it will have value of 1.

It will result in,

$0 \leq \sum_{i=1}^n \lambda_i z_i^2 = \lambda_t * 1^2 = \lambda_t$

And therfore all eigenvalues of A are non negative.

---

2. Let $A, B \in \mathbb{R}^{n \times n}$ be a symmetric PSD matrices and a symmetric PD matrix.
Prove or refute: the matrix $A + B$ is positive definite.
Answer:
True.

First of all, lets show that (A+B) is symmetric:

$(A + B)^\top = A^\top + B^\top = A + B$

A is PSD, and therefore $\forall x \in \mathbb{R}^n s.t. x \neq 0, x^T Ax \geq 0$,

B is PD and therfore $\forall x \in \mathbb{R}^n s.t. x \neq 0, x^T Bx > 0$

$\forall x \in \mathbb{R}^n, s.t. x \neq 0, x^T (A + B)x = x^T Ax + x^T Bx \geq x^T Bx > 0$

And therefore, $A + B$ is PD

Double-click (or enter) to edit

Q4 - Gradients

Define $f : \mathbb{R}^d \to \mathbb{R}$, where $f(w) = w^\top x + b$, for some vector $w \in \mathbb{R}^d$ and a scalar $b \in \mathbb{R}$.

Recall: the gradient vector is defined as $\nabla_w f = \left[\frac{\partial f}{\partial w_1}, \ldots, \frac{\partial f}{\partial w_d}\right]^\top \in \mathbb{R}^d$.

1. Prove that $\nabla_w f = x$.

Recall/read the definition of the Hessian matrix $\nabla_w^2 f \in \mathbb{R}^{d\times d}$.

2. Find the Hessian matrix $\nabla_w^2 f$ of the function f defined in this question.
3. Is the matrix you found positive semi-definite? Explain.

Now, define $g:\mathbb{R}^d \to \mathbb{R}$, where $g(w)=\frac{1}{2}\|w\|^2$.

4. Find the gradient vector $\nabla_w g$.
5. Find the Hessian matrix $\nabla_w^2 g$.
6. Is the matrix you found positive semi-definite? is it positive definite? Explain.

Answers:

1.

$$\nabla_w f = \left[\frac{\partial f}{\partial w_1},\ldots,\frac{\partial f}{\partial w_d}\right]^\top = [\frac{\partial(w_1x_1+\ldots+w_dx_d+b_1)}{\partial w_1},\ldots,\frac{\partial(w_1x_1+\ldots+w_dx_d+b_d)}{\partial w_d}]^\top = [x_1,\ldots,x_d]^\top = x$$

2.

$$\forall w\in\mathbb{R}^d,\forall i,j\in\mathbb{R},\ \frac{\partial^2 f}{\partial w_i\partial w_j} = \frac{\partial}{\partial w_i}(\frac{\partial f}{\partial w_j}) = \frac{\partial}{\partial w_i}(\frac{\partial(w_1x_1+\ldots+w_dx_d+b_1)}{\partial w_j}) = \frac{\partial}{\partial w_i}(x_j) = \frac{\partial x_j}{\partial w_i} = 0$$

And therefore, Hessian matrix will be

$$\nabla_w^2 f = \mathbf{0}\in\mathbb{R}^{d\times d}$$

3.

Yes.

$$\forall x\in\mathbb{R}^d, x^\top(\nabla_w^2 f)x = x^\top\mathbf{0}\,x = x^\top\mathbf{0} = 0 \geq 0$$

In addition, the zero matrix is symmetric.

Therefore, $\nabla_w^2 f$ is PSD.

4.

$$\frac{1}{2}\|w\|^2 = \frac{1}{2}(w_1^2+\cdots+w_d^2)$$

$$\nabla_w g = [\frac{\partial g}{\partial w_1},\ldots,\frac{\partial g}{\partial w_d}]^\top = [\frac{\partial\frac{1}{2}(w_1^2+\cdots+w_d^2)}{\partial w_1},\ldots,\frac{\partial\frac{1}{2}(w_1^2+\cdots+w_d^2)}{\partial w_d}]^\top = \frac{1}{2}[2w_1,\ldots,2w_d]^\top = [w_1,\ldots,w_d]^\top = w$$

5.

for row i in the Hessian matrix, we will have the following row,

$$[\frac{\partial^2 g}{\partial w_i\partial w_1},\ldots,\frac{\partial^2 g}{\partial w_i\partial w_d}] = [\frac{\partial}{\partial w_i}(\frac{\partial g}{\partial w_1}),\ldots,\frac{\partial}{\partial w_i}(\frac{\partial g}{\partial w_d})] = [\frac{\partial}{\partial w_i}(w_1),\ldots,\frac{\partial}{\partial w_i}(w_d)] = [\frac{\partial w_1}{\partial w_i},\ldots,\frac{\partial w_d}{\partial w_i}] = [0,\ldots,0,1,0,\ldots,0]$$

s.t. the index of the value 1 in the above array is i.

Therefore, each row i at the Hessian matrix will have all 0s except of the value in the i'th index(which will be 1), and therefore, the result Hessian matrix will be the Identity Matrix I,

$$\nabla_w^2 g = I\in\mathbb{R}^{d\times d}$$

6.

Yes, it is PD (and therefore also PSD).

$$\forall x\in\mathbb{R}^d s.t.\, x\neq 0, x^\top(\nabla_w^2 g)x = x^\top Ix = x^\top x = \|x\|^2 > 0$$

$\|x\|^2 = 0 \iff x=0$ and because in the above situation $x\neq 0$ then $\|x\|^2\neq 0$ and because $\|y\|^2\geq 0$ for every $y\in\mathbb{R}^d$, then it holds in the above situation that $\|x\|^2>0$

Therefore, $\nabla_w^2 g$ is PD, and also PSD.