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COMP9101 (T2-2020)

Homework 1 – Q5

(a) $f(n) = (\log_2 n)^2$

Using $\log(a^b) = b\log a$, we can obtain that:

$$g(n) = \log_2\left(n^{\log_2 n}\right)^2 = 2\log_2\left(n^{\log_2 n}\right) = 2(\log_2 n)^2$$

It is obvious that $f(n)$ and $g(n)$ have the same asymptotic growth rate. Therefore, $f(n) = \Theta\big(g(n)\big)$.

(b) We assume that $g(n) = O\big(f(n)\big)$, which means we need to prove that $g(n) \le cf(n)$. Since the log function is monotonically increasing, this is equivalent to:

$$\log_2 g(n) \le \log_2\big(c \cdot f(n)\big) = \log_2 c + \log_2 f(n)$$

$$\log_2 2^{\sqrt[10]{n}} \le \log_2 c + \log_2 n^{10}$$

Now if we take $c = 1$ and apply some basic math simplification, then it is enough to show that for sufficiently large n:

$$\sqrt[10]{n} \le 10\log_2 n$$

$$\frac{n^{\frac{1}{10}}}{10\log_2 n} \le 1$$

Then compute the limit using the L'Hopital's rule:

$$\lim_{n\to\infty} \frac{n^{\frac{1}{10}}}{10\log_2 n} = \lim_{n\to\infty} \frac{\left(n^{\frac{1}{10}}\right)'}{(10\log_2 n)'} = \lim_{n\to\infty} \frac{\frac{1}{10}\cdot n^{-\frac{9}{10}}}{10\cdot\frac{1}{n\ln 2}} = \lim_{n\to\infty} \frac{\ln 2}{100}\cdot n^{\frac{1}{10}} = 0$$

Therefore, for sufficiently large number n we will have $\frac{n^{\frac{1}{10}}}{10\log_2 n} \le 1$.

(c) $f(n) = n^{1+(-1)^n} = n^2$, n is even number

$= 1$, n is odd number

As is shown above, if n is an even number, $g(n) = n \leq cf(n)$ for some c and all sufficiently large n. This gives us $g(n) = O\big(f(n)\big)$.

However, if n is an odd number, $f(n) = 1 \leq cg(n)$ for some c and all sufficiently large n. This gives us $f(n) = O\big(g(n)\big)$.

Since that it is impossible for every two functions $f(n)$ and $g(n)$ either $f(n) = O\big(g(n)\big)$ or $g(n) = O\big(f(n)\big)$, for this pair function $f(n)$ and $g(n)$, neither $f(n) = O\big(g(n)\big)$ nor $g(n) = O\big(f(n)\big)$.