Probability Inequalities

Thm 1 (Gaussian Tail Inequality): Let $X \sim \mathcal{N}(0,1)$. Then

$$\mathbb{P}(|X| > \epsilon) \leq \frac{2}{\epsilon} e^{-\epsilon^2/2} \tag{1}$$

Additionally:

$$\mathbb{P}(|\overline{X}_n| > \epsilon) \leq \frac{1}{\sqrt{n}\epsilon} e^{-n\epsilon^2/2} \tag{2}$$

Proof

Note that $f_X(x) = (2\pi)^{-1/2} e^{-x^2/2}$ (pdf for $\mathcal{N}(0,1)$)
$\implies \mathbb{P}(X > \epsilon) = \int_\epsilon^\infty f_X(s) ds$ (taking upper range w.l.o.g)
$\leq \frac{1}{\epsilon} \int_\epsilon^\infty s f_X(s) ds$ (Since $\frac{s}{\epsilon} > \frac{s}{s}$ in $[\epsilon, \infty)$)
$= \frac{1}{\epsilon} \int_\epsilon^\infty f'_X(s) ds$ (taking derivative)
$= \frac{1}{\epsilon} f_X(\epsilon) \leq \frac{1}{\epsilon} e^{-\epsilon^2/2}$ (f.t.c. and bound pdf)
$\implies \mathbb{P}(|X| > \epsilon) \leq \frac{2}{\epsilon} e^{-\epsilon^2/2}$ (by symmetry of pdf)

Also:

For $X_1, \ldots, X_n \sim \mathcal{N}(0,1)$, $\overline{X}_n = n^{-1} \sum_i X_i \sim \mathcal{N}(0, 1/n)$
$\implies \overline{X}_n \overset{d}{=} n^{-1/2} Z$, where $Z \sim \mathcal{N}(0,1)$ (sample mean identity)
$\implies \mathbb{P}(|\overline{X}_n| > \epsilon) = \mathbb{P}(n^{-1/2}|Z| > \epsilon)$
$= \mathbb{P}(|Z| > n^{1/2}\epsilon) \leq \frac{2}{\sqrt{n}\epsilon} e^{-n\epsilon^2/2}$ (using thm for $X \sim \mathcal{N}(0,1)$)

Thm 2 (Markov Inequality): Let X be a non-negative random variable s.t. $\mathbb{E}(X)$ exists. Then $\forall\ t > 0$

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t} \tag{3}$$

Proof

Since $X > 0$, $\mathbb{E}(X) = \int_0^\infty x f_X(x) dx$ (def of $\mathbb{E}(X)$)
$= \int_0^t x f_X(x) dx + \int_t^\infty x f_X(x) dx$ (split up integral)
$\geq \int_t^\infty x f_X(x) dx$ (keep upper part only)
$\geq t \int_t^\infty f_X(x) dx$ ($t \leq x$ for $x \in [t, \infty)$)
$= t\mathbb{P}(X > t)$

Thm 3 (Chebyshev's Inequality): Let $\mu = \mathbb{E}(X)$ and $\sigma^2 = \text{Var}(X)$. Then

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2} \tag{4}$$

$$\mathbb{P}(|(X - \mu)/\sigma| \geq t) \leq \frac{1}{t^2} \tag{5}$$

Proof

$\mathbb{P}(|X - \mu| \geq t) = \mathbb{P}(|X - \mu|^2 \geq t^2)$ (square both sides)
$\leq \frac{\mathbb{E}[(X-\mu)^2]}{t^2} = \frac{\sigma^2}{t^2}$ (using Markov's inequality)

Also:

The second part follows by setting $t = t\sigma$.