Random Variables

A **random variable** X is a map $X : \Omega \to \mathbb{R}$. For $A \subset \mathbb{R}$ we write

$$\mathbb{P}(X \in A) = \mathbb{P}(\{w \in \Omega : X(w) \in A\}) \tag{1}$$

The **cdf** F_X of X is

$$F_X(x) = \mathbb{P}(X \le x) \tag{2}$$

For continuous X, the **pdf** f_X is a function satisfying

$$\int_A f_X(x)dx = \mathbb{P}(X \in A) \tag{3}$$

Note that $f_X = F_X'$.

Transformations

Let $Y = g(X)$, $\mathcal{X} = \{x : f_X(x) > 0\}$, and $\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$ ($\mathcal{X}$ and $\mathcal{Y}$ called the *support* of X and Y). Then $\forall\ A \subset \mathcal{Y}$

$$\mathbb{P}(Y \in A) = \mathbb{P}(X \in \{x : g(x) \in A\}) \tag{4}$$

For the cdf F_Y

$$F_Y(y) = \mathbb{P}(X \in \{x : g(x) \le y\}) = \int_{\{x:g(x)\le y\}} f_X(x)dx \tag{5}$$

For g monotonic

$$F_Y(y) = \begin{cases} F_X(g^{-1}(y)) & \text{if } g \text{ increasing} \\ 1 - F_X(g^{-1}(y)) & \text{if } g \text{ decreasing} \end{cases} \tag{6}$$

Additionally, for g monotonic, if $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| \qquad \text{for } y \in \mathcal{Y} \tag{7}$$

Expected Values

The **mean** or **expected value** of $g(X)$ is

$$\mathbb{E}(g(X)) = \int g(x)dF(x) = \int g(x)dP(x) \tag{8}$$

Related properties and definitions:

$$(a)\ \ \mu = \mathbb{E}(X) \tag{9}$$
$$(b)\ \ \mathbb{E}(\sum_i c_i g_i(X_i)) = \sum_i c_i \mathbb{E}(g_i(X_i)) \tag{10}$$
$$(c)\ \ \mathbb{E}\left(\prod_i X_i\right) = \prod_i \mathbb{E}(X_i), \quad X_1, \dots, X_n \text{ indep't} \tag{11}$$
$$(d)\ \ Var(X) = \sigma^2 = \mathbb{E}((X-\mu)^2) \quad \text{is the } \textbf{variance} \text{ of X} \tag{12}$$
$$(e)\ \ Var(X) = \mathbb{E}(X^2) - \mu^2 \tag{13}$$
$$(f)\ \ Var\left(\sum_i a_i X_i\right) = \sum_i a_i^2 Var(X_i), \quad X_1, \dots, X_n \text{ indep't} \tag{14}$$
$$(g)\ \ Cov(X,Y) = \mathbb{E}((X-\mu_X)(Y-\mu_Y)) \ \text{ is the } \textbf{covariance} \tag{15}$$
$$(h)\ \ Cov(X,Y) = \mathbb{E}(XY) - \mu_x\mu_Y \tag{16}$$
$$(i)\ \ \rho(X,Y) = Cov(X,Y)/\sigma_x\sigma_y, \quad -1 \le \rho(X,Y) \le 1 \tag{17}$$

The **conditional expectation** of Y given X is the random variable $g(X) = \mathbb{E}(Y|X)$, where

$$\mathbb{E}(Y|X = x) = \int yf(y|x)dy \tag{18}$$
$$\text{and } \ f(y|x) = f_{X,Y}(x,y)/f_X(x) \tag{19}$$

The *Law of Total/Iterated Expectation* is

$$\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y|X)] \tag{20}$$

The *Law of Total Variance* is

$$Var(Y) = Var[\mathbb{E}(Y|X)] + \mathbb{E}[Var(Y|X)] \tag{21}$$

The *Law of Total Covariance* is

$$Cov(X,Y) = \mathbb{E}(Cov(X,Y|Z)) + Cov(\mathbb{E}(X|Z), \mathbb{E}(Y|Z)) \tag{22}$$

Moment Generating Function

The **mgf** of X is

$$M_X(t) = \mathbb{E}(e^{tX}) \tag{23}$$

Properties:

$$(a)\ \ M_X^{(n)}(t)|_{t=0} = \mathbb{E}(X^n) \quad \text{is the } \mathbf{n^{th}} \textbf{ moment} \text{ of X} \tag{24}$$
$$(b)\ \ M_x(t) = M_Y(t) \ \ \forall t \ \text{ around } 0 \implies X \stackrel{d}{=} Y \tag{25}$$
$$(c)\ \ M_{aX+b}(t) = e^{bt} M_X(at) \tag{26}$$
$$(d)\ \ M_{\sum_i X_i}(t) = \prod_i M_{X_i}, \quad X_1, \dots, X_n \text{ indep't} \tag{27}$$

Independence

Random variables X and Y are **independent**, written $X \perp\!\!\!\perp Y$, iff

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B) \tag{28}$$

If (X,Y) is a random vector with pdf $f_{X,Y}$, then

$$X \perp\!\!\!\perp Y \iff f_{X,Y}(x,y) = f_X(x)f_Y(y) \tag{29}$$

Distributions

Discrete distributions:

$$(a)\ \ \text{Bernoulli } \ f(x|p) = p^x(1-p)^{1-x}, \quad x \in \{0,1\} \tag{30}$$
$$(b)\ \ \text{Binomial } \ f(x|n,p) = \binom{n}{x} p^x(1-p)^{n-x}, \ \ x \in \{0,1,\dots,n\} \tag{31}$$
$$(c)\ \ \text{Poisson } \ f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \in \{0,1,2,\dots\} \tag{32}$$

Continuous distributions:

$$(a)\ \ \text{Uniform } \ f(x|a,b) = \frac{1}{b-a}, \quad x \in [a,b] \tag{33}$$
$$(b)\ \ \text{Normal } \ f(x|\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}, \ \ x \in \mathbb{R} \tag{34}$$
$$(c)\ \ \text{Gamma } \ f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \ \ x \in \mathbb{R}_+, \alpha\ \beta > 0 \tag{35}$$

Miscellaneous

To include: MGFs and/or CDFs for above distributions. More distributions. Misc. useful things: L'Hôpital's rule, Taylor approximation, definitions of e, Leibnitz's Rule.