Probability Inequalities

Thm 1 (Gaussian Tail Inequality): Let $X \sim \mathcal{N}(0,1)$. Then

$$\mathbb{P}(|X| > \epsilon) \leq \frac{2}{\epsilon} e^{-\epsilon^2/2} \tag{1}$$

Additionally:

$$\mathbb{P}(|\overline{X}_n| > \epsilon) \leq \frac{1}{\sqrt{n}\epsilon} e^{-n\epsilon^2/2} \tag{2}$$

Thm 2 (Markov Inequality): Let X be a non-negative random variable s.t. $\mathbb{E}(X)$ exists. Then $\forall\ t > 0$

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t} \tag{3}$$

Thm 3 (Chebyshev's Inequality): Let $\mu = \mathbb{E}(X)$ and $\sigma^2 = \text{Var}(X)$. Then

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2} \tag{4}$$

$$\mathbb{P}(|(X - \mu)/\sigma| \geq t) \leq \frac{1}{t^2} \tag{5}$$

Lemma 4: Let $\mathbb{E}(X) = 0$ and $a \leq X \leq b$. Then

$$\mathbb{E}(e^{tX}) \leq e^{t^2(b-a)^2/8} \tag{6}$$

Lemma 5 Let X be any random variable. Then

$$\mathbb{P}(X > \epsilon) \leq \inf_{t \geq 0} e^{-t\epsilon} \mathbb{E}(e^{tX}) \tag{7}$$

Thm 6 (Hoeffding's Inequality): $X_1, \ldots, X_n$ iid, $\mathbb{E}(X_i) = \mu$, $a \leq X_i \leq b$. Then $\forall \epsilon > 0$

$$\mathbb{P}(|\overline{X} - \mu| \geq \epsilon) \leq 2e^{-2n\epsilon^2/(b-a)^2} \tag{8}$$