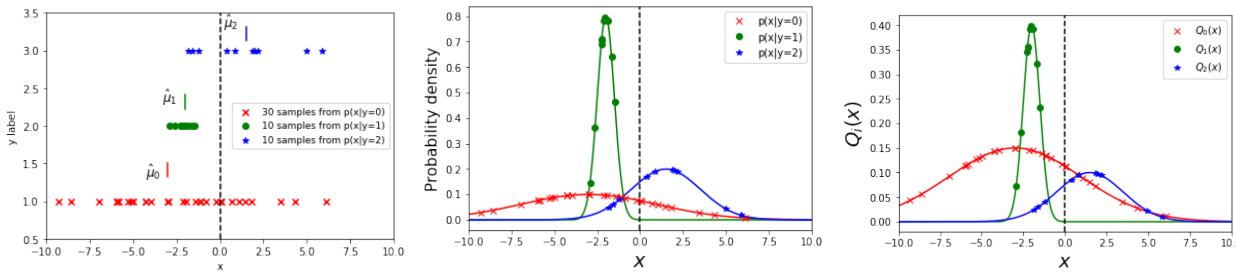
1 Gaussian Discriminant Analysis

We have N iid samples $\{(X_n, Y_n)\}_{n=1}^N$ with values $\{(x_n, y_n)\}_{n=1}^N$, where $x_n \in \mathbb{R}$ is an observable and $y_n \in \{0, 1, 2\}$ is the class to which the sample belongs. We'll denote by N_i the number of samples that belong to class $Y = i$. We have plotted the samples in the figure to the left. You want to build a classifier such that you can predict the class of new unlabeled samples $X = x$. You have been told that the conditional probabilities $p(x|y)$ are Gaussian distributions.



(a) How would you use Maximum Likelihood Estimation (MLE) to estimate the probabilities $p(X|Y)$ and $\pi_i = p(Y = i)$ from the samples?

(b) How would you use these probabilities to derive the Bayes decision rule? What equations are satisfied by the points in the decision boundary $r^*(x)$? Leave the solution in terms of $Q_0(x), Q_1(x), Q_2(x)$, where

$$e^{Q_i(x)} = \sqrt{(2\pi)}p(X = x|Y = i)p(Y = i) = \frac{\pi_i}{\sigma_i} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}$$

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(c) What do you observe about the region of values of X where the label $Y = 0$ is assigned? Could you express this region with a set of inequalities?

(d) You receive a new unlabeled sample $X = 0$, what class would you assign to it? Is it the class which mean is closest?

(e) What would have happen if you used Linear Discriminant Analysis and assumed uniform priors?

(f) *Bonus question: Is it possible that there is a certain class $y = i$ for which there is no x such that the Bayes decision rule picks this class i*

2 Maximum Likelihood Estimation for reliability testing

Suppose we are reliability testing n units taken randomly from a population of identical appliances. We want to estimate the mean failure time of the population. We assume the failure times come from an exponential distribution with parameter $\lambda > 0$, whose probability density function is $f(t) = \lambda e^{-\lambda t}$ (on the domain $t \geq 0$).

(a) In an ideal (but impractical) scenario, we run the units until they all fail. The failure time $T_1, T_2, \ldots, T_n$ for units $1, 2, \ldots, n$ are observed to be $t_1, t_2, \ldots, t_n$.

Formulate the likelihood function $\mathcal{L}(\lambda; t_1, \ldots, t_n)$ for our data. Then find the maximum likelihood estimate $\hat{\lambda}$ for the distribution's parameter. (Remember that it's equivalent, and usually easier, to optimize the log-likelihood)

(b) In a more realistic scenario, we run the units for a fixed time h. The failure time for $T_1, T_2, \ldots, T_r$ are observed to be $t_1, t_2, \ldots, t_r$, where $0 \leq r \leq n$. The remaining $n - r$ units survive the entire time h without failing. Let's find the maximum likelihood estimate $\hat{\lambda}$ for our model distribution parameters! To do so:

   (a) What is the probability that a unit will not fail during time h?

   (b) Write the new likelihood function $\mathcal{L}(\lambda; h, n, r, t_1, \ldots, t_r)$.

   (c) Optimize to find the MLE estimate, and give it a physical interpretation.