MAT1856/APM466: Mathematical Finance	**Winter 2024**

Assignment #1: Yield Curves

Professor: Luis Seco

Note: Please bring any questions about this assignment to your TA's weekly office hour.

1.1 Introduction

Due: Monday, February 5th, 2024 at 11AM ET to be submitted online via Crowdmark.

For each weekday from Jan 8th, 2023, to January 22, 2023 (inclusive, 2 weeks worth, 10 days), collect all historical close prices for all Canadian Government Bonds which have a maturity less than 10 years from January 8, 2024 on the "Frankfurt" Exchange; I.e., all bonds listed via the following two links:

1. `https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=shortterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=&currency=184&rating=&country=19`

2. `https://markets.businessinsider.com/bonds/finder?borrower=71&maturity=midterm&yield=&bondtype=2%2c3%2c4%2c16&coupon=&currency=184&rating=&country=19`

The data surrounding the bonds found on the above two links will be used for calculating the yield curve (ytm curve), spot curve, and forward curve. **Please familiarize yourself with these definitions, as per the links, as those will be the definitions used for this assignment.**

For each bond, you will also need to collect the following information: coupon, ISIN, issue date, maturity date. All this data is available on the "Snapshot" page before clicking through to the "Historical" page.

This assignment is split into 2 parts, the first asks questions about some fundamentals of fixed income and mathematical finance. The second will be an empirical exercise in generating yield curves, and in particular the 1, 2, 3, 4 and 5 year rates, and analyzing these rates through PCA.

1.2 Expectations

1. You may use R, Python, or any programming language (no Excel without approval from TA beforehand) of your choice to answer the "empirical questions".

2. Please have your final report typeset using LaTeX and the following template: template. The website: www.overleaf.com is particularly useful.

3. **Important/New:** Your report must be no longer than 3 pages long in total (excluding references).

4. Each of the "fundamental questions" must be answered in clear and coherent full sentences.

5. At the end of your report you must cite all references and include a link to a GitHub repository with all your code used for the project.

6. You may, and are encouraged, to discuss how to do these questions with your peers. However, your write-up must be done individually, and the sharing of your write-up or code before the deadline is prohibited.

7. A 5% penalty per day past the deadline up until 1 week (35%) will apply for late submissions.

Additional Notes: Marks will be awarded for each question as either full-, half-, or zero-marks according to if the question was answered with a few small mistakes, substantial mistakes but fundamental idea still correct, or fundamental idea wrong / no answer respectively. -10 marks (each) if expectations 2, 3, or 4 not adhered to.

2 Questions

2.1 Fundamental Questions - 25 points

1. *(5 points total)* (One sentence each.)

   (a) *(1 point)* Why do governments issue bonds and not simply print more money?

   (b) *(2 points)* Give a hypothetical example of why the long-term part of a yield curve might flatten.

   (c) *(2 points)* Explain what quantitative easing is and how the (US) Fed has employed this since the beginning of the COVID-19 pandemic.

2. *(10 points)* We asked you to pull data for all bonds, but if you'd like to construct a yield a "0-5 year" yield & spot curves, as the government of Canada issues all of its bonds with a semi-annual coupon, when bootstrapping you'll only need 10 or 11 bonds to perform this task. Ideally, the bonds in any yield curve should be consistent in some way with one another so that yields are easier to compare. Select (list) 10 bonds that you will use to construct the aforementioned curves with an explanation of why you selected those 10 bonds based on the characteristics we asked you to collect for each bond (coupon, issue date, maturity date, etc.).

   (*Note: 1) There is a unique ideal answer, 2) To easily refer to a bond, please use the following convention: "CAN 2.5 Jun 24" refers to the Canadian Government bond with a maturity in June 24 and a coupon of 2.5*).

3. *(10 points)* In a few plain English sentences, in general, if we have several stochastic processes for which each process represents a unique point along a stochastic curve (assume points/processes are evenly distributed along the curve), what do the eigenvalues and eigenvectors associated with the covariance matrix of those stochastic processes tell us?

   (*Hint: This is called Principal Component Analysis*)

2.2 Empirical Questions - 75 points

4. *(40 points total)*

   (a) *(10 points)* First, calculate each of your 10 selected bonds' yield (ytm). Then provide a well-labeled plot with a 5-year yield curve (ytm curve) corresponding to each day of data superimposed on-top of each other. You may use any interpolation technique you deem appropriate provided you include a reasonable explanation for the technique used.

   (b) *(15 points)* Write a pseudo-code (explanation of an algorithm) for how you would derive the spot curve with terms ranging from 1-5 years from your chosen bonds in part 2. (Please recall the day convention simplifications provided in part 2 as well.) Then provide a well-labeled plot with a 5-year spot curve corresponding to each day of data superimposed on-top of each other.

   (c) *(15 points)* Write a pseudo-code for how you would derive the 1-year forward curve with terms ranging from 2-5 years from your chosen bonds in part 2 (I.e., a curve with the first point being the 1yr-1yr forward rate and the last point being the 1yr-4yr rate). Then provide a well-labeled plot with a forward curve corresponding to each day of data superimposed on-top of each other.

5. *(20 points)* Calculate two covariance matrices for the time series of daily log-returns of yield, and forward rates (no spot rates). In other words, first calculate the covariance matrix of the random variables X_i, for $i = 1, \ldots, 5$, where each random variable X_i has a time series $X_{i,j}$ given by:

$$X_{i,j} = \log(r_{i,j+1}/r_{i,j}), \quad j = 1, \ldots, 9$$

   then do the same for the following forward rates - the 1yr-1yr, 1yr-2yr, 1yr-3yr, 1yr-4yr.

6. *(15 points)* Calculate the eigenvalues and eigenvectors of both covariance matrices, and in one sentence, explain what the first (in terms of size) eigenvalue and its associated eigenvector imply.

3 Hints

Dear Students, please review the comments and guidelines below and if you still have any questions or confusion, do not hesitate to email me directly. -Azin azin.sharifi@utoronto.ca.

3.1 Google Colab Template for Bootstrapping

Here is a template I uploaded for you in Google Colab to get started. You do not have to choose Python or this template. You could do your assignment in R, C, or any other language you like. This is just for you to have a rough idea of where to start.

- Template Bootstrapping on Google Colab

3.2 Present Value of a Bond

First, it's important to understand the present value formula for a bond. A bond's present value (PV) is calculated by discounting the future cash flows (both coupon payments and the face value) back to the present time. The formula is:

$$PV = \sum_{t=1}^{n} \frac{C}{(1 + r_t)^t} + \frac{F}{(1 + r_n)^n}$$

Here C is the coupon payment, F is the face value of the bond (paid at maturity), r_t is the spot rate for each period t, n is the number of periods until maturity.

3.3 Bootstrapping Method

The bootstrapping method is used to derive the spot rate curve from the prices of coupon-bearing bonds. The spot rate for a particular maturity is the interest rate that makes the present value of the bond's payments equal to its current market price. The formula for bootstrapping is iterative, starting from the shortest maturity bond. For the first bond (1-year bond), the spot rate r_1 is:

$$r_1 = \left(\frac{F + C}{P} \right)^{\frac{1}{n}} - 1$$

For subsequent bonds, the formula becomes more complex as it involves solving for the spot rate that equates the present value of the bond's cash flows to its current price, considering the spot rates already calculated for shorter maturities. For a bond maturing in n years:

$$P = \sum_{t=1}^{n-1} \frac{C}{(1 + r_t)^t} + \frac{C + F}{(1 + r_n)^n}$$

Here r_n is the spot rate for the n-year bond, which is the unknown in the equation and needs to be solved iteratively.

3.4 Zero-Coupon Bond

For a zero-coupon bond, the formula is simpler as there are no intermediate coupon payments. The spot rate for a zero-coupon bond is directly calculated from its price:

$$r = \left(\frac{F}{P} \right)^{\frac{1}{n}} - 1$$

In this case: F is the bond's face value or maturity value, P is the bond's current price, n is the number of years to maturity, and r is the annual spot rate.

3.5   Forward rates from Spot rates

The formula to calculate a forward rate $F_{t,t+n}$ for the period between time t and $t+n$ is:

$$F_{t,t+n} = \left(\frac{(1 + S_{t+n})^{t+n}}{(1 + S_t)^t} \right)^{\frac{1}{n}} - 1$$

Here: S_{t+n} is the spot rate for the period ending in $t+n$, S_t is the spot rate for the period ending in t, $F_{t,t+n}$ is the forward rate from time t to $t+n$.

The assumption is that the rates are compounded annually. If rates are compounded more frequently (semi-annually, or quarterly), the formula needs to be adjusted to account for the compounding frequency. For instance, for semi-annual compounding, the formula would be modified to:

$$F_{t,t+n} = \left(\frac{(1 + S_{t+n})^{2(t+n)}}{(1 + S_t)^{2t}} \right)^{\frac{1}{2n}} - 1$$

In this modified formula, the exponents are multiplied by the number of compounding periods per year (2 for semi-annual), and the division in the exponent of the final term is by the total number of compounding periods over the forward rate term (2n for semiannual over n years).

The formula to calculate forward rates from spot rates is slightly different from the discrete compounding case. In continuous compounding, interest is effectively compounded an infinite number of times per year.

3.6   Continuous Compounding

The formula to calculate a forward rate $F_{t,t+n}$ for the period between time t and $t+n$ under continuous compounding is:

$$F_{t,t+n} = \frac{S_{t+n} \cdot (t + n) - S_t \cdot t}{n}$$

Here: S_{t+n} is the continuously compounded spot rate for the period ending in $t+n$, S_t is the continuously compounded spot rate for the period ending in t, $F_{t,t+n}$ is the forward rate from time t to $t+n$.

In continuous compounding, the value of an investment grows according to the exponential function, and the spot rates are interpreted as the force of interest. This formula calculates the equivalent annual forward rate for the period between t and $t+n$ based on the spot rates.

3.7   Calculating Covariance Matrices for Daily Log-Returns of Yield and Forward Rates

3.7.1   Time Series of Daily Log-Returns

Question 5 is about calculating covariance matrices for time series of daily log returns. Log-returns are commonly used in finance for their convenient properties, such as time-additivity and approximate normality for small returns.

The log-return for a day j is calculated using the formula $X_{i,j} = \log\left(\frac{r_{i,j+1}}{r_{i,j}}\right)$, where $r_{i,j}$ and $r_{i,j+1}$ are the rates for consecutive days.

3.7.2   Understanding i and j

i: This index represents different variables or instruments being considered. In your case, for the yield rates, $i = 1, \ldots, 5$ implies you have five different yield rates (for example, they could be yields for different maturities).

j: This index represents different time points or days in your time series. In your case, $j = 1, \ldots, 9$ suggests you are looking at a time series of 9 days for each yield rate.

3.7.3 Covariance Matrices

Please note that the first step is to calculate the daily log returns for each yield rate across the 9 days. You'll end up with a series of log returns for each i. Then, calculate the covariance matrix for these log returns. The covariance matrix will show how the returns of these different yield rates co-vary with each other over the period. In a matrix form, each element of the matrix represents the covariance between the log returns of yields i and i'.

3.7.4 Applying the Same to Forward Rates

The same methodology is then applied to forward rates specified (1yr-1yr, 1yr-2yr, $1yr - 3yr, 1yr - 4yr$). These forward rates represent the rate agreed upon now for lending or borrowing over a period starting one year from now and lasting for 1,2, 3, or 4 years, respectively. - You calculate log returns for these forward rates similarly and then compute the covariance matrix for these returns.