Propagation of Alfvén waves in a 1D radial solar wind

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1 Equation set

We solve the linearized incompressible MHD equation:

$$\frac{\partial \boldsymbol{z}^{\pm}}{\partial t} + (\boldsymbol{U} \pm \boldsymbol{V_A}) \cdot \nabla \boldsymbol{z}^{\pm} + \boldsymbol{z}^{\mp} \cdot \nabla (\boldsymbol{U} \mp \boldsymbol{V_A}) + \frac{1}{2} \left(\boldsymbol{z}^{-} - \boldsymbol{z}^{+} \right) \nabla \cdot \left(\boldsymbol{V_A} \pm \frac{1}{2} \boldsymbol{U} \right) = \boldsymbol{NL}^{\pm} \quad (1)$$

where $\boldsymbol{U}$ and $\boldsymbol{V_A}$ are the background velocity and Alfvén velocity,

$$\boldsymbol{z}^{\pm} = \boldsymbol{u} \mp \boldsymbol{v_A} \quad (2)$$

are the two Elsässer variables (1st-order), $\boldsymbol{NL}^{\pm}$ are the nonlinear terms, which are set to be zero.

We assume that the background fields are radial and the wave polarization is along the transverse direction. After Fourier transform in time, the above equation becomes

$$(U \pm V_A) \frac{dz^{\pm}}{dr} - i\omega z^{\pm} + \frac{U \mp V_A}{r} z^{\mp} + \frac{1}{2r^2} \frac{d}{dr} \left[r^2 \left(V_A \mp \frac{1}{2} U \right) \right] \left(z^{-} - z^{+} \right) = 0 \quad (3)$$

2 Solution

Equation (3 is a linear ODE equation set with a singular point (Alfvén point r_A where $U = V_A$). We can arbitrarily set the amplitude of z^+ at the Alfvén point and then calculate z^- at the Alfvén point as

$$z^{-}(\omega, r_A) = \frac{G(r_A) - \frac{U+V_A}{r}}{G(r_A) - i\omega} \times z^{+}(\omega, r_A) \quad (4)$$

where

$$G(r) = \frac{1}{2r^2} \frac{d}{dr} \left[r^2 \left(V_A + \frac{1}{2} U \right) \right] \quad (5)$$

Then we can integrate equation (3) from the Alfvén point either inward or outward.

The difficulty here is to evaluate dz^-/dr right at the Alfvén point, at which both the numerator and the denominator are zero. Using L'Hospital's rule, we can re-write dz^-/dr at r_A as

$$\left. \frac{dz^-}{dr} \right|_{r_A} = \frac{-\left(\frac{U+V_A}{r}\right)' z^+ + \left(\frac{V_A'}{2} + \frac{U'}{4} - \frac{U}{2r} \right) \frac{dz^+}{dr} - \left(\frac{V_A''}{2} + \frac{U''}{4} + \frac{V_A' + U'/2}{r} - \frac{V_A + U/2}{r^2} \right) (z^- - z^+)}{(U' - V_A') - i\omega + G(r_A)} \quad (6)$$