finite difference schemes on nonuniform grid

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Assume the grid is arbitrary, we consider the 4th order schemes such that 4 points are involved.

1 Biased central schemes

These schemes were derived by Jianchun et al. (1995). Suppose there are 4 points $(x_{-2}, x_{-1}, x_0, x_1)$ and define $d_{-2} = x_0 - x_{-2}$, $d_{-1} = x_0 - x_{-1}$, and $d_1 = x_1 - x_0$. One can derive the scheme for the 1st-order derivative is

$$f_0' \approx c_{-2}f_{-2} + c_{-1}f_{-1} + c_0 f_0 + c_1 f_1 \tag{1}$$

where

$$c_{-2} = \frac{d_{-1}d_1}{d_{-2}(d_{-2}+d_1)(d_{-2}-d_{-1})} \tag{2a}$$

$$c_{-1} = -\frac{d_{-2}d_1}{d_{-1}(d_{-2}-d_{-1})(d_{-1}+d_1)} \tag{2b}$$

$$c_1 = \frac{d_{-2}d_{-1}}{d_1(d_{-1}+d_1)(d_{-2}+d_1)} \tag{2c}$$

and $c_0 = -(c_{-2} + c_{-1} + c_1)$. The scheme for the 2nd-order derivative is

$$f_0'' \approx c_{-2}f_{-2} + c_{-1}f_{-1} + c_0 f_0 + c_1 f_1 \tag{3}$$

where

$$c_{-2} = \frac{2(d_1 - d_{-1})}{d_{-2}(d_{-2}+d_1)(d_{-2}-d_{-1})} \tag{4a}$$

$$c_{-1} = \frac{2(d_{-2} - d_1)}{d_{-1}(d_{-2}-d_{-1})(d_{-1}+d_1)} \tag{4b}$$

$$c_1 = \frac{2(d_{-2} + d_{-1})}{d_1(d_{-1}+d_1)(d_{-2}+d_1)} \tag{4c}$$

and $c_0 = -(c_{-2} + c_{-1} + c_1)$.

2 Right scheme

Assume the four points are (x_0, x_1, x_2, x_3) and define $H_0 = x_1 - x_0$, $H_1 = x_2 - x_0$, $H_2 = x_3 - x_0$. One can derive that the scheme for 1st order derivative is

$$f_0' = c_0 f_0 + c_1 f_1 + c_2 f_2 + c_3 f_3 \tag{5}$$

where

$$c_1 = \frac{1}{H_0}\left(1 + \frac{H_0}{H_1} \cdot \frac{H_0 - H_2}{H_2 - H_1} + \frac{H_0}{H_2} \cdot \frac{H_1 - H_0}{H_2 - H_1}\right)^{-1} \tag{6a}$$

$$c_2 = \frac{1}{H_1}\left(1 + \frac{H_1}{H_0} \cdot \frac{H_2 - H_1}{H_0 - H_2} + \frac{H_1}{H_2} \cdot \frac{H_1 - H_0}{H_0 - H_2}\right)^{-1} \tag{6b}$$

$$c_3 = \frac{1}{H_2}\left(1 + \frac{H_2}{H_1} \cdot \frac{H_0 - H_2}{H_1 - H_0} + \frac{H_2}{H_0} \cdot \frac{H_2 - H_1}{H_1 - H_0}\right)^{-1} \tag{6c}$$

and $c_0 = -(c_1 + c_2 + c_3)$. The scheme for 2nd order derivative is

$$f_0'' = c_0 f_0 + c_1 f_1 + c_2 f_2 + c_3 f_3 \tag{7}$$

where

$$c_1 = \frac{2}{H_0^2}\left(1 - \frac{H_1}{H_0} \cdot \frac{H_0^2 - H_2^2}{H_1^2 - H_2^2} - \frac{H_2}{H_0} \cdot \frac{H_1^2 - H_0^2}{H_1^2 - H_2^2}\right)^{-1} \tag{8a}$$

$$c_2 = \frac{2}{H_1^2}\left(1 - \frac{H_2}{H_1} \cdot \frac{H_1^2 - H_0^2}{H_2^2 - H_0^2} - \frac{H_0}{H_1} \cdot \frac{H_2^2 - H_1^2}{H_2^2 - H_0^2}\right)^{-1} \tag{8b}$$

$$c_3 = \frac{2}{H_2^2}\left(1 - \frac{H_1}{H_2} \cdot \frac{H_2^2 - H_0^2}{H_1^2 - H_0^2} - \frac{H_0}{H_2} \cdot \frac{H_1^2 - H_2^2}{H_1^2 - H_0^2}\right)^{-1} \tag{8c}$$

and $c_0 = -(c_1 + c_2 + c_3)$.

References

Liu Jianchun, Gary A Pope, and Kamy Sepehrnoori. A high-resolution finite-difference scheme for nonuniform grids. *Applied mathematical modelling*, 19(3):162–172, 1995.