# Pure Exploration of Combinatorial Bandits

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April 15, 2014

## 1 Preliminaries

### 1.1 Problems

Let $n$ be the number of base arms. Let $\mathcal{M} \subseteq 2^{[n]}$ be the set of super arms.

In this note, we consider the following cases of $\mathcal{M}$.

**Example 1** (Explore-$m$). $\mathcal{M}_{\mathsf{TOP}m}(n) = \{M \subseteq [n] \mid |M| = m\}$. *This corresponds to finding the top $m$ arms from $[n]$.*

**Example 2** (Explore-$m$-bandits). *Suppose $n = mk$. Then $\mathcal{M}_{\mathsf{BANDIT}m}(n)$ contains all subsets $M \subseteq [n]$ with size $m$, such that*
$$M \cap \{ik+1, \ldots, (i+1)k\} = 1, \quad \text{for all } i \in \{0, \ldots, m-1\}.$$
*This corresponds to finding the top arms from $m$ bandits, where each bandit has $k$ arms.*

**Example 3** (Perfect Matching). *Let $G = (V, E)$ be a bipartite graph and $|E| = n$. For simplicity, let each edge $e \in E$ corresponds to a unique integer $i \in [n]$, and vice versa. Then $\mathcal{M}_{\mathsf{MATCH}}(n, G)$ contains all subsets $M \subseteq [n]$ such that $M$ corresponds to a perfect matching in $G$.*

### 1.2 Diff-Sets

**Definition 1** (Diff-set). *An $n$-diff-set (or diff-set in short) is a pair of sets $c = (c_+, c_-)$, where $c_+ \subseteq [n]$, $c_- \subseteq [n]$ and $c_+ \cap c_- = \emptyset$.*

**Definition 2** (Difference of sets). *Given any $M_1 \subseteq [n], M_2 \subseteq [n]$. We define $M_1 \ominus M_2 \triangleq C$, where $C = (C_+, C_-)$ is a diff-set and $C_+ = M_1 \backslash M_2$ and $C_- = M_2 \backslash M_1$.*

**Definition 3.** *Denote $\mathsf{diff}[n]$ be the set of all possible $n$-diff-sets.*

**Definition 4** (Set operations of diff-sets). *Let $C = (C_+, C_-), D = (D_+, D_-)$ be two diff-sets. We define $C \cap D \triangleq (C_+ \cap D_+, C_- \cap D_-)$ and $C \backslash D \triangleq (C_+ \backslash D_+, C_- \backslash D_-)$.*

*Further, for all $e \in [n]$, $e \in C \Leftrightarrow (e \in C_+) \vee (e \in C_-)$. And $|C| \triangleq |C_+| + |C_-|$.*

**Definition 5** (Valid diff-set). *Given a set $M \subseteq [n]$ and a diff-set $C = (C_+, C_-)$, we call $C$ a valid diff-set for $M$, iff $C_+ \cap M = \emptyset$ and $C_- \subseteq M$. In this case, we denote $C \prec M$.*

**Definition 6** (Negative diff-set). *Given a diff-set $A = (A_+, A_-)$, we define $\neg A = (A_-, A_+)$.*

#### 1.2.1 diff-set operations

**Definition 7** (Operators $\oplus$ and $\ominus$). *Given any $M \subseteq [n]$ and $C \in \mathsf{diff}[n]$. If $C \prec M$, we define operator $\oplus$ such that $M \oplus C \triangleq M \backslash C_- \cup C_+$. On the other hand if $\neg C \prec M$, we define operator $\ominus$ such that $M \ominus C \triangleq M \oplus (\neg C) = M \backslash C_+ \cup C_-$.*

**Definition 8.** *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. We denote $B \prec A$, if and only if $B_+ \cap A_+ = \emptyset$ and $A_+ \cap A_- = \emptyset$.*

**Definition 9.** *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. If $B \prec A$, we define $A \oplus B = ((A_+ \cup B_+) \backslash (A_- \cup B_-), (A_- \cup B_-) \backslash (A_+ \cup B_+))$.*

**Lemma 1.** *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. If $B \prec A$, then $A \oplus B$ is a diff-set.*

*Proof.* Let $C = A \oplus B$. By definition, we have $C_+ = (A_+ \cup B_+) \backslash (A_- \cup B_-)$ and $C_- = (A_- \cup B_-) \backslash (A_+ \cup B_+)$. We only need to show that $C_+ \cap C_- = \emptyset$.

$$\begin{aligned}
C_+ \cap C_- &= \big((A_+ \cup B_+) \backslash (A_- \cup B_-)\big) \cap \big((A_- \cup B_-) \backslash (A_+ \cup B_+)\big) \\
&= (A_+ \cup B_+) \cap \big((A_- \cup B_-) \backslash (A_+ \cup B_+) \backslash (A_- \cup B_-)\big) \\
&= \emptyset.
\end{aligned}$$

$\square$

**Lemma 2.** *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. If there exists $M \subseteq [n]$ such that $A \prec M$, and $B \prec (M \oplus A)$, then $B \prec A$ and $(M \oplus A \oplus B) \ominus M = A \oplus B$.*

*Proof.* We first show that $B \prec A$. Since $B \prec (M \oplus A)$, we know that $B_+ \cap (M \backslash A_- \cup A_+) = \emptyset$. Therefore, we have

$$\begin{aligned}
\emptyset &= B_+ \cap (M \backslash A_- \cup A_+) \\
&= (B_+ \cap (M \backslash A_-)) \cup (B_+ \cap A_+)
\end{aligned}$$

We see that $B_+ \cap A_+ = \emptyset$.

On the other hand, we have $B_- \subseteq (M \backslash A_- \cup A_+)$, therefore

$$\begin{aligned}
B_- \cap A_- &\subseteq (M \backslash A_- \cup A_+) \cap A_- \\
&= (M \backslash A_- \cap A_-) \cup (A_+ \cap A_-) \\
&= \emptyset.
\end{aligned}$$

Hence we proved that $B \prec A$.

Define $D = (M \oplus A \oplus B) \ominus M$ and write $D = (D_+, D_-)$. Then,

$$\begin{aligned}
D_+ &= (M \oplus A \oplus B) \backslash M \\
&= (M \backslash A_- \cup A_+ \backslash B_- \cup B_+) \backslash M \\
&= (A_+ \cup B_+) \backslash (A_- \cup B_-).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
D_- &= M \backslash (M \oplus A \oplus B) \\
&= M \backslash (M \backslash A_- \cup A_+ \backslash B_- \cup B_+) \\
&= (A_- \cup B_-) \backslash (A_+ \cup B_+).
\end{aligned}$$

$\square$

### 1.2.2 Diff-set class

**Definition 10** (Decomposition of diff-set)**.** *Given $\mathcal{B} \subseteq \mathsf{diff}[n]$ and $D \in \mathsf{diff}[n]$, a decomposition of $D$ on $\mathcal{B}$ is a set $\{b_1, \ldots, b_k\} \subseteq \mathcal{B}$ satisfying the following*

1. *For all $i \in [k]$ and $j \in [k]$, we write $b_i = (b_i^+, b_i^-)$ and $b_j = (b_j^+, b_j^-)$. Then, the following holds $b_i^+ \cap b_j^+ = \emptyset$, $b_i^+ \cap b_j^- = \emptyset$, $b_i^- \cap b_j^+ = \emptyset$ and $b_i^- \cap b_j^- = \emptyset$.*

2. *$D = b_1 \oplus b_2 \oplus \ldots b_k$.*

**Lemma 3.** *Given $\mathcal{B} \subseteq \mathsf{diff}[n]$ and $D \in \mathsf{diff}[n]$. Let $\{b_1, \ldots, b_k\} \subseteq \mathcal{B}$ be a decomposition of $D$ on $\mathcal{B}$. Then,*

1. *Let $D = (D_+, D_-)$ and for all $i \in [k]$, we write $b_i = (b_i^+, b_i^-)$. Then $D_+ = b_1^+ \cup \ldots \cup b_k^+$ and $D_- = b_1^- \cup \ldots \cup b_k^-$.*

2. *For all $M \subseteq [n]$, if $D \prec M$, then, for all $i \in [k]$, we have $b_i \prec M$.*

*Proof.* We prove (1) by induction. Let $D_i = b_1 \oplus \ldots \oplus b_i$ and write $D_i = (D_i^+, D_i^-)$. We show that $D_i^+ = \bigcup_{j=1}^{i} b_i^+$ and $D_{i-} = \bigcup_{j=1}^{i} b_i^-$ for all $i \in [k]$. For $i = 1$, this is trivially true. Then, assume that this is true for some $i > 1$. By definition $D_{i+1} = D_i \oplus b_{i+1}$, hence $D_{i+1}^+ = (D_i^+ \cup b_{i+1}^+) \backslash (D_i^- \cup b_{i+1}^-)$. Note that

$$(D_i^- \cup b_{i+1}^-) \cap (D_i^+ \cup b_{i+1}^+) = (D_i^- \cap D_i^+) \cup (D_i^- \cap b_{i+1}^+) \cup (b_{i+1}^- \cap D_i^+) \cup (b_{i+1}^- \cap b_{i+1}^+)$$
$$= (D_i^- \cap b_{i+1}^+) \cup (b_{i+1}^- \cap D_i^+)$$
$$= \left( \left( \bigcup_{j=1}^{i} b_j^- \right) \cap b_{i+1}^+ \right) \cup \left( \left( \bigcup_{j=1}^{i} b_j^+ \right) \cap b_{i+1}^- \right)$$
$$= \emptyset.$$

Hence $D_{i+1}^+ = D_i^+ \cup b_{i+1}^+$. We can use the same method to show that $D_{i+1}^- = D_i^- \cup b_{i+1}^-$.

Next, we prove (2) using (1). To show that $b_i \prec M$, we only need to show that $b_i^+ \cap M = \emptyset$ and $b_i^- \subseteq M$. Since $D \prec M$, we know that $D_+ \cap M = \emptyset$ and $D_- \subseteq M$. By (1), we see that $b_i^+ \subseteq D_+$ and $b_i^- \subseteq D_-$. Therefore, we have $(b_i^+ \cap M) \subseteq (D_+ \cap M) = \emptyset$ and $b_i^- \subseteq D_- \subseteq M$. $\qquad \square$

**Definition 11** (diff-set class). *Given $\mathcal{M} \subseteq 2^{[n]}$. $\mathcal{B} \subseteq \mathsf{diff}[n]$ is a diff-set class for $\mathcal{M}$, if the following hold.*

1. *$(\emptyset, \emptyset) \notin \mathcal{B}$.*

2. *For all $M \in \mathcal{M}$ and for all $b \in \mathcal{B}$, if $b \prec M$, then $M \oplus b \in \mathcal{M}$.*

3. *For all $M_1 \in \mathcal{M}$ and $M_2 \in \mathcal{M}$, where $M_1 \neq M_2$. Let $D = M_1 \ominus M_2$. Then, there exists a decomposition of $D$ on $\mathcal{B}$.*

**Definition 12** (Rank of diff-set class). *Let $\mathcal{B} \subseteq [n]$ be a diff-set class for some $\mathcal{M}$. We define*

$$\mathrm{rank}(\mathcal{B}) \triangleq \max_{b \in \mathcal{B}} |b|.$$

**Example 4** (diff-set class for Explore-$m$). *One diff-set class $\mathcal{B}$ for $\mathcal{M}_{\mathsf{TOP}m}(n)$ is given by*

$$\mathcal{B} = \{(\{b_1\}, \{b_2\}) \mid b_1 \neq b_2, b_1 \in [n], b_2 \in [n]\}.$$

*Proof omitted. Further, we see that $\mathrm{rank}(\mathcal{B}) = 2$.*

**Example 5** (diff-set class for Explore-$m$-badit). *Let $n = mk$. One diff-set class $\mathcal{B}$ for $\mathcal{M}_{\mathsf{BANDIT}m}(n)$ is given by*

$$\mathcal{B} = \{(\{b_1\}, \{b_2\}) \mid b_1 \neq b_2, \exists i \in \{0, \ldots, k-1\}, b_1 \in \{ik+1, \ldots, (i+1)k\}, b_2 \in \{ik+1, \ldots, (i+1)k\}\}.$$

*Proof omitted. Further, we see that $\mathrm{rank}(\mathcal{B}) = 2$.*

**Example 6** (diff-set class for Perfect Matching). *One diff-set class $\mathcal{B}$ for $\mathcal{M}_{\mathsf{MATCH}}(n, G)$ is the set of all augmenting cycles of $G$. More specifically,*

$$\mathcal{B} = \{(b_+, b_-) | b_+ \cup b_- \text{ is a cycle of } G\}.$$

*Note* $\mathrm{rank}(\mathcal{B}) \leq n$.

## 1.3    Weights and confidence bounds

**Definition 13** (Weight functions). *Define function $w : [n] \to \mathbb{R}^+$ which represents the weight of each base arm. Further, we slight abuse the notations, and extend the definition of $w$ to diff-sets and sets as follows.*

     *1. For all $M \subseteq [n]$, we denote $w(M) = \sum_{e \in M} w(e)$.*

     *2. For all $b = (b_+, b_-) \in \mathsf{diff}[n]$, we denote $w(b) = \sum_{e \in b_+} w(e) - \sum_{e \in b_-} w(e)$.*

**Lemma 4.** *Let $c \in \mathsf{diff}[n], d \in \mathsf{diff}[n]$. Let $w$ be a weight function. Then,*

$$w(c \cup d) = w(c) + w(d) - w(c \cap d). \tag{1}$$

*Proof.* Let $c = (c_+, c_-)$ and $d = (d_+, d_-)$. We have

$$w(c \cup d) = w(c_+ \cup d_+) - w(c_- \cup d_-) \tag{2}$$
$$= w(c_+) + w(d_+) - w(c_+ \cap d_+) - w(c_-) - w(d_-) + w(c_- \cap d_-) \tag{3}$$
$$= w(c) + w(d) - (w(c_+ \cap d_+) - w(c_- \cap d_-)) \tag{4}$$
$$= w(c) + w(d) - w(c \cap d). \tag{5}$$

$\square$

**Definition 14** (Mean weight $\bar{w}_t$, sample size $n_t$). *Given $t > 0$. Define $\bar{w}_t$ be a weight function such that, for all $e \in [n]$, $\bar{w}_t(e)$ equals to the empirical mean of $e$ up to round $t$. Let $n_t : [n] \to \mathbb{N}$, such that $n_t(e)$ equals to number of plays of base arm $e$ up to round $t$.*

**Definition 15** (Confidence radius $\mathrm{rad}_t$). *Given $n$ and $t > 0$. Define $\mathrm{rad}_t : [n] \to \mathbb{R}^+$ satisfying, for all $e \in [n]$,*

$$\mathrm{rad}_t(e) = c_{\mathrm{rad}} \log \left( \frac{c_\delta n t^2}{\delta} \right) \frac{1}{\sqrt{n_t(e)}}, \tag{6}$$

*where $c_{\mathrm{rad}} > 0$ and $c_\delta > 0$ are some universal constants (specify later) and $\delta > 0$ is a parameter.*

     *We extend the notation of $\mathrm{rad}_t$ to diff-sets and sets as follows.*

     *1. For all $M \subseteq [n]$, $\mathrm{rad}_t(M) \triangleq \sum_{e \in M} \mathrm{rad}_t(e)$.*

     *2. For all $b = (b_+, b_-) \in \mathsf{diff}[n]$, $\mathrm{rad}_t(b) \triangleq \mathrm{rad}_t(b_+) + \mathrm{rad}_t(b_-)$.*

**Definition 16** (UCB $w_t^+$). *Define $w_t^+ : [n] \to \mathbb{R}^+$, s.t., for all $e \in [n]$,*

$$w_t^+(e) = \bar{w}_t(e) + \mathrm{rad}_t(e).$$

     *We extend the notation of $w_t^+$ to diff-sets and sets as follows.*

     *1. For all $M \subseteq [n]$, $w_t^+(M) \triangleq \bar{w}_t(M) + \mathrm{rad}_t(M)$.*

     *2. For all $b = (b_+, b_-) \in \mathsf{diff}[n]$, $w_t^+(b) \triangleq \bar{w}_t(b) + \mathrm{rad}_t(b)$.*

**Lemma 5.** *Define random event*

$$\xi = \left\{\forall e \in [n] \; \forall t > 0, |\bar{w}_t(e) - w(e)| \le \mathrm{rad}_t(e)\right\}.$$

*Then, there exist constants* $c_{\mathrm{rad}}$ *and* $c_\delta$,

$$\Pr[\xi] \ge 1 - \delta.$$

*Proof.* Hoeffding inequality and union bound. $\qquad\square$

**Corollary 1.**

$$\xi \implies \forall t, \forall e \in [n] \; w_t^+(e) \ge w(e).$$

$$\xi \implies \forall t, \forall M \subseteq [n], \; w_t^+(M) \ge w(M).$$

$$\xi \implies \forall t, \forall b \in \mathsf{diff}[n] \; w_t^+(b) \ge w(b).$$

## 1.4  Properties of $\mathrm{rad}_t$

**Lemma 6.** *Let* $c \in \mathsf{diff}[n], d \in \mathsf{diff}[n]$. *Then*

$$\mathrm{rad}_t(c\backslash d) = \mathrm{rad}_t(c) - \mathrm{rad}_t(c \cap d). \tag{7}$$

*Proof.* Let $c = (c_+, c_-)$ and $d = (d_+, d_-)$. We have

$$
\begin{aligned}
\mathrm{rad}_t(c\backslash d) &= \mathrm{rad}_t(c_+\backslash d_+) + \mathrm{rad}_t(c_-\backslash d_-) \\
&= \mathrm{rad}_t(c_+) - \mathrm{rad}_t(c_+ \cap d_+) + \mathrm{rad}_t(c_-) - \mathrm{rad}_t(c_- \cap d_-) \\
&= \mathrm{rad}_t(c) - \mathrm{rad}_t(c \cap d).
\end{aligned}
$$

$\square$

**Lemma 7.** *Let* $C = (C_+, C_-)$ *and* $D = (D_+, D_-)$ *be two diff-sets. If* $D \prec C$, *then*

$$\mathrm{rad}_t(C \oplus D) = \mathrm{rad}_t(C) + \mathrm{rad}_t(D) - 2\,\mathrm{rad}_t(C_+ \cap D_-) - 2\,\mathrm{rad}_t(C_- \cap D_+).$$

*In addition, if* $\neg D \prec C$, *then*

$$\mathrm{rad}_t(C \ominus D) = \mathrm{rad}_t(C) + \mathrm{rad}_t(D) - 2\,\mathrm{rad}_t(C_+ \cap D_+) - 2\,\mathrm{rad}_t(C_- \cap D_-).$$

*Proof.* We prove the first part of the lemma. The second part follows from the first part and the definition of $\neg D$.

By definition, we have $C \oplus D = ((C_+ \cup D_+)\backslash(C_- \cup D_-), (C_- \cup D_-)\backslash(C_+ \cup D_+))$. Hence, we have

$$
\begin{aligned}
\mathrm{rad}_t((C_+ \cup D_+)\backslash(C_- \cup D_-)) &= \mathrm{rad}_t(C_+ \cup D_+) - \mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)) \tag{8} \\
&= \mathrm{rad}_t(C_+) + \mathrm{rad}_t(D_+) - \mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)), \tag{9}
\end{aligned}
$$

where the second equality holds due to $C_+ \cap D_+ = \emptyset$ by the definition of $D \prec C$.

Similarly, we have

$$\mathrm{rad}_t((C_- \cup D_-)\backslash(C_+ \cup D_+)) = \mathrm{rad}_t(C_-) + \mathrm{rad}_t(D_-) - \mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)).$$

Combine both equalities, we have

$$\mathrm{rad}_t(C \oplus D) = \mathrm{rad}_t((C_+ \cup D_+)\backslash(C_- \cup D_)) + \mathrm{rad}_t((C_- \cup D_-)\backslash(C_- \cap D_+)) \tag{10}$$

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$$= \operatorname{rad}_t(C_+) + \operatorname{rad}_t(D_+) + \operatorname{rad}_t(C_-) + \operatorname{rad}_t(D_-) - 2\operatorname{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)) \tag{11}$$

$$= \operatorname{rad}_t(C) + \operatorname{rad}_t(D) - 2\operatorname{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)). \tag{12}$$

$\square$

# 2   Algorithm and Main Results

## 2.1   Algorithm

1. Input Parameter: $\delta \in (0,1)$.

2. For $t = 1, \ldots,$

3. Maintain $\bar{w}_t$ and $\operatorname{rad}_t$.

4. Let $M_t = \arg\max_{M \in \mathcal{M}} \bar{w}_t(M)$.

5. Let $D = \arg\max_{C \in \mathsf{diff}[n], C \prec M_t} w_t^+(C)$.

6. If $w_t^+(D) \leq 0$. Then stop and return $M_t$.

7. Otherwise, find $p_t = \arg\min_{e \in D} \operatorname{rad}_t(e)$.

8. Play $p_t$ and observe outcome $x_t$.

9. Go back to step 2.

The step 5 of above procedure can be implemented by:

1. Let $M_t^+ = \arg\max_{M \in \mathcal{M}} \tilde{w}_t(M)$, where $\tilde{w}_t$ is a weight function defined by:

   (a) $\forall e \in M_t$, $\tilde{w}_t(e) = \bar{w}_t(e) - \operatorname{rad}_t(e)$.
   (b) $\forall e \notin M_t$, $\tilde{w}_t(e) = \bar{w}_t(e) + \operatorname{rad}_t(e)$.

2. $D = M_t^+ \ominus M_t$

## 2.2   Main result

**Definition 17** (Optimal diff-sets)**.** *Given a diff-set class $\mathcal{B}$ and the optimal set $M_*$. We define $\mathcal{B}_{\mathsf{opt}}$ as a subset of $\mathcal{B}$, and for all $b \in \mathcal{B}$, $b \in \mathcal{B}_{\mathsf{opt}}$ if and only if, there exists $M \neq M_*$ and $M_* \ominus M$ can be decomposed as $b, b_1, \ldots, b_k$ on $\mathcal{B}$.*

**Definition 18** (Hardness $\Delta_e$ of base arm $e$)**.** *For each $e \in [n]$, we define its hardness $\Delta_e$ as follows*

$$\Delta_e = \min_{b \in \mathcal{B}_{\mathsf{opt}}, e \in b} \frac{1}{\operatorname{rank}(\mathcal{B})} w(b).$$

**Definition 19** (Sufficient exploration)**.** *For all $t > 0$, we define $E_t^3 \subseteq [n]$, such that, for all $e \in [n]$ $e \in E_t^3$ if and only if $\operatorname{rad}_t(e) < \frac{1}{3}\Delta_e$.*

**Corollary 2.** *For all $t > 0$ and $e \in [n]$*

$$n_t(e) \geq O(\frac{1}{\Delta_e^2} \log(\Delta_e n/\delta)) \implies e \in E_t^3.$$

**Theorem 1.** *With probability at least $1 - \delta$, the algorithm returns $M_*$, and the number of samples used by the algorithm are at most*

$$\sum_{e \in [n]} \Delta_e^{-2} \log(\Delta_e n/\delta).$$

# 3  Proof of Main Results

Unless specified, we shall assume the random event $\xi$ (defined in Lemma 5) holds in all the following proofs.

**Lemma 8.** *For any $t > 0$, if the algorithm terminates on round $t$, then $M_t = M_*$.*

*Proof.* Suppose $M_t \neq M_*$. Then $w(M_*) > w(M_t)$. Then, there exists $b \in \mathcal{B}$ such that $b \prec M_t$ and $w(b) > 0$. On the other hand, by Corollary 1, we have $w_t^+(b) > w(b)$. Hence $w_t^+(b) > 0$. This contradicts to the stopping condition of our algorithm. □

**Lemma 9.** *For any $t > 0$. If $e \in E_t^3$, then $p_t \neq e$.*

*Proof.* Suppose that $p_t = e$. Let $D = M_t^+ \ominus M_t$. Let $c, c_1, \ldots, c_k$ be decomposition of $D$ on $\mathcal{B}$. And since $\mathcal{B}$ is a diff-set class, such decomposition exists. Assume, without loss of generality, that $e \in c$.

By Lemma Y, we know that

$$D_+ = c_+ \cup c_1^+ \cup \ldots \cup c_k^+ \quad \text{and} \quad D_- = c_- \cup c_1^- \cup \ldots c_k^-. \tag{13}$$

We also denote $K = \text{rank}(\mathcal{B})$.

**Case (1).** Suppose that $c \in \mathcal{B}_{\mathsf{opt}}$. Then $w(c) > 0$. Since $e \in E_t^3$, we have $\text{rad}_t(e) \leq \frac{1}{3}\Delta_e \leq \frac{1}{3K}w(c)$. In addition, $\forall g \in c_t, g \neq e$, $\text{rad}_t(g) \leq \text{rad}_t(e) \leq \frac{1}{3K}w(c)$. Hence, $\text{rad}_t(c) = \sum_{g \in c_t} \text{rad}_t(g) \leq \frac{|c_t|}{3K}w(c) \leq \frac{1}{3}w(c)$.

Hence, $\bar{w}_t(c) \geq w(c) - \text{rad}_t(c) \geq \frac{2}{3}w(c) > 0$. This means that $\bar{w}_t(M_t \oplus c) = \bar{w}_t(M_t) + \bar{w}_t(c) > \bar{w}_t(M_t)$. Therefore, $M_t \neq \max_{M \in \mathcal{M}} \bar{w}_t(M)$. This contradicts to the definition of $M_t$.

**Case (2).** Suppose that $c_t \notin \mathcal{B}_{\mathsf{opt}}$. Then, one of the following mutually exclusive cases must hold.

**Case (2.1).** $(e \in M_* \wedge e \in c_+)$ or $(e \notin M_* \wedge e \in c_-)$.

Let the decomposition of $M_* \ominus (M_t \oplus D \ominus c)$ on $\mathcal{B}$ be $b, b_1, \ldots, b_l$, which exists due to $\mathcal{B}$ is a diff-set class. Assume wlog that $e \in b$. We write $b = (b_+, b_-)$. It is easy to see that $b \in \mathcal{B}_{\mathsf{opt}}$.

Define $\tilde{D} = (M_t \oplus D \ominus c) \ominus M_t$ and $D' = (M_t \oplus \tilde{D} \oplus b) \ominus M_t$. By Lemma 2, we know that $\tilde{D} = D \ominus c$ and $D' = \tilde{D} \oplus b$. We also write $\tilde{D} = (\tilde{D}_+, \tilde{D}_-)$ and $D' = (D'_+, D'_-)$. By definition, we have

$$
\begin{aligned}
\tilde{D}_+ &= (D_+ \cup c_-) \backslash (D_- \cup c_+) \\
&= (D_+ \cup c_- \backslash D_-) \cap (D_+ \cup c_- \backslash c_+) \\
&= D_+ \cap (D_+ \backslash c_-) \\
&= D_+ \backslash c_+.
\end{aligned}
$$

By the same method, we are able to show that $\tilde{D}_- = D_- \backslash c_-$. Therefore we have

$$\tilde{D}_+ \subseteq D_+ \quad \text{and} \quad \tilde{D}_- \subseteq D_-. \tag{14}$$

First, we show that $\text{rad}_t(c) \leq \frac{1}{3}w(b)$. Since $e \in E_t^3$, $e \in b$ and $b \in \mathcal{B}_{\mathsf{opt}}$, we have $\text{rad}_t(e) \leq \frac{1}{3}\Delta_e \leq \frac{1}{3K}w(b)$. In addition, $\forall g \in c, g \neq e$, $\text{rad}_t(g) \leq \text{rad}_t(e) \leq \frac{1}{3K}w(b)$. Hence,

$$
\begin{aligned}
\text{rad}_t(c) &= \sum_{g \in c} \text{rad}_t(g) \\
&\leq \frac{|c|}{3K}w(b) \\
&\leq \frac{1}{3}w(b). \tag{15}
\end{aligned}
$$

Now, we show that $\text{rad}_t(\tilde{D}_+ \cap b_-) + \text{rad}_t(\tilde{D}_- \cap b_+) + \leq \frac{1}{3}w(b)$. Since Eq. (14), we have $\forall g \in (\tilde{D}_+ \cap b_-) \cup$

$(\tilde{D}_- \cap b_+), g \neq e$, $\mathrm{rad}_t(g) \leq \mathrm{rad}_t(e) \leq \frac{1}{3K} w(b)$. Note that $|\tilde{D}_+ \cap b_-| + |\tilde{D}_- \cap b_+| \leq |b_+| + |b_-| \leq K$. Hence,

$$\mathrm{rad}_t(\tilde{D}_+ \cap b_-) + \mathrm{rad}_t(\tilde{D}_- \cap b_+) = \sum_{g \in (\tilde{D}_+ \cap b_-) \cup (\tilde{D}_- \cap b_+)} \mathrm{rad}_t(g)$$

$$\leq \frac{K}{3K} w(b)$$

$$\leq \frac{1}{3} w(b). \tag{16}$$

Then, we have

$$\mathrm{rad}_t(D') - \mathrm{rad}_t(D) = \mathrm{rad}_t(\tilde{D} \oplus b) - \mathrm{rad}_t(D) \tag{17}$$

$$= \mathrm{rad}_t(\tilde{D}) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) \tag{18}$$

$$= \mathrm{rad}_t(D \ominus c) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) \tag{19}$$

$$= \mathrm{rad}_t(D) + \mathrm{rad}_t(c) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(D_+ \cap c_+) - 2\,\mathrm{rad}_t(D_- \cap c_-)$$
$$- 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) \tag{20}$$

$$= \mathrm{rad}_t(D) + \mathrm{rad}_t(c) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(c_+) - 2\,\mathrm{rad}_t(c_-)$$
$$- 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) \tag{21}$$

$$= \mathrm{rad}_t(b) - \mathrm{rad}_t(c) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+), \tag{22}$$

where Eq. (18) and Eq. (20) follow from Lemma 7, and Eq. (21) follows from Eq. (13).

By the definition of $D$, we have that $w_t^+(D) \geq w_t^+(D')$. This means that

$$\bar{w}_t(D) + \mathrm{rad}_t(D) \geq \bar{w}_t(D') + \mathrm{rad}_t(D') \tag{23}$$

$$= \bar{w}_t(D) - \bar{w}_t(c) + \bar{w}_t(b) + \mathrm{rad}_t(D'). \tag{24}$$

By regrouping the above inequality, we have

$$\bar{w}_t(c) \geq \bar{w}_t(b) + \mathrm{rad}_t(D') - \mathrm{rad}_t(D) \tag{25}$$

$$= \bar{w}_t(b) + \mathrm{rad}_t(b) - \mathrm{rad}_t(c) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) \tag{26}$$

$$\geq w(b) - \mathrm{rad}_t(c) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) \tag{27}$$

$$> w(b) - \frac{1}{3} w(b) - \frac{2}{3} w(b) \tag{28}$$

$$= 0, \tag{29}$$

where Eq. (28) follows from Eq. (15) and Eq. (16).

This contradicts to the definition of $M_t$.

**Case (2.2).** $(e \in M_* \wedge e \in c_-)$ or $(e \notin M_* \wedge e \in c_+)$.

Let the decomposition of $M_* \ominus (M_t \oplus D)$ on $\mathcal{B}$ be $b, b_1, \ldots, b_l$. Assume wlog that $e \in b$. We write that $b = (b_+, b_-)$. Note that $b \in \mathcal{B}_{\mathrm{opt}}$ and hence $w(b) > 0$.

Define $D' = (M_t \oplus D \oplus b) \ominus M_t$. By Lemma 2, we know that $D' = D \oplus b$.

First, we show that $|D \backslash D'| \leq |b|$. Let $C = D \backslash D'$ and write $C = (C_+, C_-)$. We can bound $|C_+|$ as follows.

$$C_+ = D_+ \backslash D'_+$$
$$= D_+ \backslash ((D_+ \cup b_+) \backslash (D_- \cup b_-))$$
$$= (D_+ \cap (D_- \cup b_-)) \cup (D_+ \backslash (D_+ \cup b_+))$$
$$= D_+ \cap b_-.$$

Hence, we have $|C_+| \leq |b_-|$. Then, we move to bounding $|C_-|$

$$
\begin{aligned}
C_- &= D_- \backslash D'_- \\
&= D_- \backslash ((D_- \cup b_-) \backslash (D_+ \cup b_+)) \\
&= (D_- \cap (D_+ \cup b_+)) \cup (D_- \backslash (D_- \cup b_-)) \\
&= D_- \cap b_+.
\end{aligned}
$$

Thus $|C_-| \leq |b_+|$ and we proved that $|D \backslash D'| \leq |b|$.

Next, we show that $\mathrm{rad}_t(D \backslash D') \leq \frac{1}{3} w(b)$. Since $e \in E_t^3$, $e \in b$ and $b \in \mathcal{B}_{\mathsf{opt}}$, we have $\mathrm{rad}_t(e) \leq \frac{1}{3} \Delta_e \leq \frac{1}{3K} w(b)$. In addition, $\forall g \in (D \backslash D'), g \neq e$, $\mathrm{rad}_t(g) \leq \mathrm{rad}_t(e) \leq \frac{1}{3K} w(b)$. Note that $|D \backslash D'| \leq |b| \leq K$. Hence, $\mathrm{rad}_t(D \backslash D') = \sum_{g \in (D \backslash D')} \mathrm{rad}_t(g) \leq \frac{K}{3K} w(b) \leq \frac{1}{3} w(b)$.

We also note that

$$
w(D' \backslash D) - w(D \backslash D') = w(D' \backslash D) + w(D' \cap D) - w(D \cap D') - w(D \backslash D') \tag{30}
$$

$$
= w(D') - w(D) \tag{31}
$$

$$
= w(b), \tag{32}
$$

where we have repeatedly applied Lemma 4.

Then, we show that $w_t^+(D') > w_t^+(D)$.

$$
w_t^+(D') - w_t^+(D) = \bar{w}_t(D') - \bar{w}_t(D) + \mathrm{rad}_t(D') - \mathrm{rad}_t(D) \tag{33}
$$

$$
= \bar{w}_t(D' \backslash D) - \bar{w}_t(D \backslash D') + \mathrm{rad}_t(D' \backslash D) - \mathrm{rad}_t(D \backslash D') \tag{34}
$$

$$
\geq w(D' \backslash D) - w(D \backslash D') - 2\,\mathrm{rad}_t(D \backslash D') \tag{35}
$$

$$
= w(b) - 2\,\mathrm{rad}_t(D \backslash D') \tag{36}
$$

$$
> w(b) - \frac{2}{3} w(b) \tag{37}
$$

$$
= \frac{1}{3} w(b) > 0, \tag{38}
$$

where Eq. (34) follows from Lemma 6 and Eq. (35) follows from the fact that $\bar{w}_t(D' \backslash D) + \mathrm{rad}_t(D' \backslash D) \geq w(D' \backslash D)$ and that $\bar{w}_t(D \backslash D') + \mathrm{rad}_t(D \backslash D') \geq w(D \backslash D')$, under the random event $\xi$.

This contradicts to the fact that $D$ is chosen on round $t$. $\qquad\square$

# 4 Lower Bounds

**Definition 20** (Hardness of arm). *Given $\mathcal{M}$, $M_*$ and $w$. For any $e \in [n]$, we define its hardness $\Delta_e$ as follows*

$$
\Delta_e = \begin{cases} \min_{M: e \in M, M \in \mathcal{M}} w(M_*) - w(M) & \text{if } e \notin M_*, \\ \min_{M: e \notin M, M \in \mathcal{M}} w(M_*) - w(M) & \text{if } e \in M_*. \end{cases}
$$

**Lemma 10.**

$$
\Delta_e = \min_{b: e \in b, b \in \mathcal{B}_{\mathsf{opt}}} w(b).
$$

**Theorem 2.** *For each arm $i \in [n]$, assume that its reward distribution of arm $i$ is a Gaussian distribution with mean $p_i$ and variance 1. Then, for any $\delta \in (0, e^{-16}/4)$ and any $\delta$-correct algorithm $\mathbb{A}$. Let $T$ denote the number of total samples used by algorithm $\mathbb{A}$. We have*

$$
\mathbb{E}[T] \geq \sum_e \frac{1}{16 \Delta_e^2} \log(1/4\delta).
$$

---

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*Proof.* Fix $\delta > 0$, $p_i$ for all $i \in [n]$ and a $\delta$-correct policy $\mathbb{A}$. For each $i \in [n]$, assume that the reward distribution of arm $i$ a Gaussian distribution with mean $p_i$ and variance 1. Then, for any $e \in [n]$, let $T_e$ denote the number of trials of arm $e$ used by algorithm $\mathbb{A}$. In the rest of the proof, we will prove that for any $e \in [n]$, the number of trials of arm $e$ is lower-bounded by

$$\mathbb{E}[T_e] \geq \frac{1}{16\Delta_e^2} \log(1/4\delta). \tag{39}$$

Notice that the theorem will follow immediately by summing up the above bounds for all $e \in [n]$ and setting $c = 16$.

Fix an arm $e \in [n]$. We now focus on proving Eq. (39). Consider two hypothesis $H_0$ and $H_1$. Under each hypothesis, the reward distributions of every arm are still Gaussian distributions with unit variance, but the mean rewards of some arms might be altered. Under hypothesis $H_0$, the mean reward of each arm is

$$H_0 : q_l = p_l, \quad \text{for all } l \in [n].$$

And under hypothesis $H_1$, the mean reward of each arm is

$$H_1 : q_e = \begin{cases} p_e - 2\Delta_e & \text{if } e \in M_* \\ p_e + 2\Delta_e & \text{if } e \notin M_* \end{cases} \quad \text{and } q_l = p_l \quad \text{for all } l \neq e.$$

Define $M_e$ be the "next-to-optimal" set as follows

$$M_e = \begin{cases} \arg\max_{M \in \mathcal{M}: e \in M} w(M) & \text{if } e \notin M_*, \\ \arg\max_{M \in \mathcal{M}: e \notin M} w(M) & \text{if } e \in M_*. \end{cases}$$

By definition, we know that $w(M_*) - w(M_e) = \Delta_e$.

Let $w_0, w_1$ be the weighting functions under $H_0, H_1$ respectively. Notice that $w_0(M_*) - w_0(M_e) = \Delta_e > 0$. On the other hand, $w_1(M_*) - w_1(M_e) = -\Delta < 0$. This means that under $H_1$, $M_*$ is not the optimal set. For $l \in \{0, 1\}$, we use $\mathbb{E}_l$ and $\Pr_l$ to denote the expectation and probability, respectively, under the hypothesis $H_l$.

Define $\theta = 4\delta$. Define

$$t_e^* = \frac{1}{16\Delta_e^2} \log\left(\frac{1}{\theta}\right). \tag{40}$$

Recall that $T_e$ denotes the total number of samples of arm $e$. Define the event $\mathcal{A} = \{T_e \leq 4t_e^*\}$.

First, we show that $\Pr_0[\mathcal{A}] \geq 3/4$. This can be proved by Markov inequality as follows.

$$\Pr_0[T_e > 4t_e^*] \leq \frac{\mathbb{E}_0[T_e]}{4t_e^*}$$
$$= \frac{t_e^*}{4t_e^*} = \frac{1}{4}.$$

Let $X_1, \ldots, X_{T_e}$ denote the sequence of reward outcomes of arm $e$. We define $K_t(e)$ as the sum of outcomes of arm $e$ up to round $t$, i.e. $K_t(e) = \sum_{i \in [t]} X_i$. Next, we define the event

$$\mathcal{C} = \left\{ \max_{1 \leq t \leq 4t_e^*} |K_t(e) - p_e t| < \sqrt{t_e^* \log(1/\theta)} \right\}.$$

We now show that $\Pr_0[\mathcal{C}] \geq 3/4$. First, notice that $K_t(e) - p_e t$ is a martingale under $H_0$. Then, by

Kolmogorov's inequality, we have

$$
\begin{aligned}
\Pr_0\left[\max_{1\le t\le 4t_e^*}|K_t(e)-p_e t|\ge\sqrt{t_e^*\log(1/\theta)}\right] &\le \frac{\mathbb{E}_0[(K_{4t_e^*}(e)-4p_e t_e^*)^2]}{t_e^*\log(1/\theta)}\\
&= \frac{4t_e^*}{t_e^*\log(1/\theta)}\\
&< \frac{1}{4},
\end{aligned}
$$

where the second inequality follows from the fact that $\mathbb{E}_0[(K_{4t_e^*}(e)-4p_e t_e^*)^2]=4t_e^*$; the last inequality follows since $\theta<e^{-16}$.

Then, we define the event $\mathcal{B}$ as the event that the algorithm eventually returns $M_*$, i.e.

$$
\mathcal{B}=\{\mathsf{Out}=M_*\}.
$$

Since the probability of error of the algorithm is smaller than $\delta<1/4$, we have $\Pr_0[\mathcal{B}]\ge 3/4$. Define $\mathcal{S}$ be $\mathcal{S}=\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}$. Then, by union bound, we have $\Pr_0[\mathcal{S}]\ge 1/4$.

Now, we show that if $\mathbb{E}_0[T_e]\le t_e^*$, then $\Pr_1[\mathcal{B}]\ge\delta$. Let $W$ be the history of the sampling process until the algorithm stops (including the sequence of arms chosen at each time and the sequence of observed outcomes). Define the likelihood function $L_l$ as

$$
L_l(w)=p_l(W=w),
$$

where $p_l$ is the probability density function under hypothesis $H_l$. Let $K$ be the shorthand of $K_e(T_e)$.

Assume that the event $\mathcal{S}$ occurred. We will bound the likelihood ratio $L_1(W)/L_0(W)$ under this assumption. To do this, we divide our analysis into two different cases.

**Case (1):** $e\notin M_*$. In this case, the reward distribution of arm $e$ under $H_1$ is a Gaussian distribution with mean $p_e+2\Delta_e$ and variance 1. Recall that the probability density function of a Gaussian distribution with mean $\mu$ and variance $\sigma^2$ is given by $f(x,\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$. Hence, we have

$$
\begin{aligned}
\frac{L_1(W)}{L_0(W)} &= \prod_{i=1}^{T_e}\exp\left(\frac{-(X_i-p_e-2\Delta_e)^2+(X_i-p_e)^2}{2}\right)\\
&= \prod_{i=1}^{T_e}\exp\left(\Delta_e(2X_i-2p_e)-2\Delta_e^2\right)\\
&= \exp\left(\Delta_e(2K-2p_e T_e)-2\Delta_e^2 T_e\right)\\
&= \exp\left(\Delta_e(2K-2p_e T_e)\right)\exp(-2\Delta_e^2 T_e).
\end{aligned}\tag{41}
$$

Next, we bound each individual term on the right-hand side of Eq. (41). We begin with bounding the second term of Eq. (41).

$$
\exp(-2\Delta_e^2 T_e) \ge \exp(-8\Delta_e^2 t_e^*)\tag{42}
$$

$$
= \exp\left(-\frac{8}{16}\log(1/\theta)\right)\tag{43}
$$

$$
= \theta^{1/2},\tag{44}
$$

where Eq. (42) follows from the assumption that event $\mathcal{S}$ occurred, which implies that event $\mathcal{A}$ occurred and therefore $T_e\le 4t_e^*$; Eq. (43) follows from the definition of $t_e^*$.

Then, we bound the first term on the right-hand side of Eq. (41) as follows

$$
\exp\left(\Delta_e(2K-2p_e T_e)\right) \ge \exp\left(-2\Delta_e\sqrt{t_e^*\log(1/\theta)}\right)\tag{45}
$$

$$= \exp\left(-\frac{2}{\sqrt{4}}\log(1/\theta)\right) \tag{46}$$

$$= \theta^{1/2}, \tag{47}$$

where Eq. (45) follows from the assumption that event $\mathcal{S}$ occurred, which implies that event $\mathcal{C}$ and therefore $|2K - 2p_e T_e| \le \sqrt{t_e^* \log(1/\theta)}$; Eq. (46) follows from the definition of $t_e^*$.

Combining Eq. (44) and Eq. (47), we can bound $L_1(W)/L_0(W)$ for this case as follows

$$\frac{L_1(W)}{L_0(W)} \ge \theta. \tag{48}$$

*(End of Case (1).)*

**Case (2):** $e \in M_*$. In this case, we know that the mean reward of arm $e$ under $H_1$ is $p_e - 2\Delta$. Therefore, the likelihood ratio $L_1(W)/L_0(W)$ is given by

$$\begin{aligned}
\frac{L_1(W)}{L_0(W)} &= \prod_{i=1}^{T_e} \exp\left(\frac{-(X_i - p_e + 2\Delta_e)^2 + (X_i - p_e)^2}{2}\right) \\
&= \prod_{i=1}^{T_e} \exp\left(\Delta_e(2p_e - 2X_i) - 2\Delta_e^2\right) \\
&= \exp\left(\Delta_e(2p_e T_e - 2K)\right)\exp(-2\Delta_e^2 T_e).
\end{aligned} \tag{49}$$

Notice that the right-hand side of Eq. (49) differs from Eq. (41) only in its first term. Now, we bound the first term as follows

$$\exp\left(\Delta_e(2K - 2p_e T_e)\right) \ge \exp\left(-2\Delta_e\sqrt{t_e^* \log(1/\theta)}\right) \tag{50}$$

$$= \exp\left(-\frac{2}{4}\log(1/\theta)\right) \tag{51}$$

$$= \theta^{1/2}, \tag{52}$$

where the inequalities hold due to reasons similar to Case (1): Eq. (50) follows from the assumption that event $\mathcal{S}$ occurred, which implies that event $\mathcal{C}$ and therefore $|2K - 2p_e T_e| \le \sqrt{t_e^* \log(1/\theta)}$; Eq. (51) follows from the definition of $t_e^*$.

Combining Eq. (44) and Eq. (47), we can obtain the same bound of $L_1(W)/L_0(W)$ as in Eq. (48), i.e. $L_1(W)/L_0(W) \ge \theta$.

*(End of Case (2).)*

At this point, we have proved that, if the event $\mathcal{S}$ occurred, then the bound of likelihood ratio Eq. (48) holds, i.e. $\frac{L_1(W)}{L_0(W)} \ge \theta$. Hence, we have

$$\begin{aligned}
\frac{L_1(W)}{L_0(W)} &\ge \theta \\
&= 4\delta.
\end{aligned} \tag{53}$$

Define $1_S$ as the indicator variable of event $\mathcal{S}$, i.e. $1_S = 1$ if and only if $\mathcal{S}$ occurs and otherwise $1_S = 0$. Then, we have

$$\frac{L_1(W)}{L_0(W)}1_S \ge 4\delta 1_S$$

holds regardless the occurrence of event $\mathcal{S}$. Therefore, we can obtain

$$
\begin{aligned}
\Pr_1[\mathcal{B}] \geq \Pr_1[\mathcal{S}] &= \mathbb{E}_1[1_S] \\
&= \mathbb{E}_0\left[\frac{L_1(W)}{L_0(W)}1_S\right] \\
&\geq 4\delta\mathbb{E}_0[1_S] \\
&= 4\delta\Pr_0[\mathcal{S}] > \delta.
\end{aligned}
$$

Now we have proved that, if $\mathbb{E}_0[T_e] \leq t_e^*$, then $\Pr_1[\mathcal{B}] > \delta$. This means that, if $\mathbb{E}_0[T_e] \leq t_e^*$, algorithm $\mathbb{A}$ will choose $M_*$ as the output with probability at least $\delta$, under hypothesis $H_1$. However, under $H_1$, we have shown that $M_*$ is not the optimal set since $w_1(M_e) > w_1(M_*)$. Therefore, algorithm $\mathbb{A}$ has a probability of error larger than $\delta$ under $H_1$. This contradicts to the assumption that algorithm $\mathbb{A}$ is a $\delta$-correct algorithm. Hence, we must have $\mathbb{E}_0[T_e] > t_e^* = \frac{1}{16\Delta_e^2}\log(1/4\delta)$.

$\square$

**Theorem 3.** *Assume that, for each arm $i \in [n]$, its reward distribution is a Gaussian distribution with mean $p_i$ and variance 1. Fix any $\delta \in (0, e^{-16}/4)$ and any $\delta$-correct algorithm $\mathbb{A}$.*

*Then, for any $b \in \mathcal{B}_{\mathsf{opt}}$, let $T_b$ denote the number of trials of arms belonging to $b$ by algorithm $\mathbb{A}$. Then,*

$$
\mathbb{E}[T_b] \geq \frac{|b|^2}{32w(b)^2}\log(1/4\delta).
$$

*Proof.* Fix $\delta > 0$, $p_i$ for all $i \in [n]$, diff-set $b = (b_+, b_-)$ and a $\delta$-correct policy $\mathbb{A}$. Assume that the reward distribution of an arm $i \in [n]$ is a Gaussian distribution with mean $p_i$ and variance 1.

We define three hypotheses $H_0$, $H_1$ and $H_2$. Under each of these hypotheses, the reward distribution of each arm is Gaussian with different means. Under hypothesis $H_0$, the mean reward of each arm equals to the original problem instance:

$$H_0 : q_l = p_l, \quad \text{for all } l \in [n].$$

Under hypothesis $H_1$, the mean reward of each arm is given by

$$
H_1 : q_e = \begin{cases} p_e + 2\frac{w(b)}{|b_-|} & \text{if } e \in b_-, \\ p_e & \text{if } e \notin b_-. \end{cases}
$$

And under hypothesis $H_2$, the mean reward of each arm is given by

$$
H_2 : q_e = \begin{cases} p_e - 2\frac{w(b)}{|b_+|} & \text{if } e \in b_+, \\ p_e & \text{if } e \notin b_+. \end{cases}
$$

Since $b \in \mathcal{B}_{\mathsf{opt}}$, it is clear that $\neg b \prec M_*$. Hence we define $M = M_* \ominus b$. Let $w_0, w_1$ and $w_2$ be the weighting functions under $H_0, H_1$ and $H_2$ respectively. It is easy to check that $w_1(M_*) - w_1(M) = -w(b) < 0$ and $w_2(M_*) - w_2(M) = -w(b) < 0$. This means that under $H_1$ or $H_2$, $M_*$ is not the optimal set. Further, for $l \in \{0, 1, 2\}$, we use $\mathbb{E}_l$ and $\Pr_l$ to denote the expectation and probability, respectively, under the hypothesis $H_l$. In addition, let $W$ be the history of the sampling process until algorithm $\mathbb{A}$ stops. Define the likelihood function $L_l$ as

$$L_l(w) = p_l(W = w),$$

where $p_l$ is the probability density function under $H_l$.

Define $\theta = 4\delta$. Let $T_{b_-}$ and $T_{b_+}$ denote the number of trials of arms belonging to $b_-$ and $b_+$, respectively. In the rest of the proof, we will bound $\mathbb{E}_0[T_{b_-}]$ and $\mathbb{E}_0[T_{b_+}]$ individually.

**Part (1): Lower bound of $\mathbb{E}_0[T_{b_-}]$.** In this part, we will show that $\mathbb{E}_0[T_{b_-}] \geq t_{b_-}^*$, where we define $t_{b_-}^* = \frac{|b_-|^2}{16w(b)^2} \log(1/\theta)$.

Consider the complete sequence of sampling process by algorithm $\mathbb{A}$. Formally, let $\tilde{S} = \{(\tilde{I}_1, \tilde{X}_1), \ldots, (\tilde{I}_T, \tilde{X}_T)\}$ be the sequence of all trials by algorithm $\mathbb{A}$, where $\tilde{I}_i$ denotes the arm played in $i$-th trial and $\tilde{X}_i$ be the reward outcome of $i$-th trial. Then, consider the subsequence $S_1$ of $\tilde{S}$ which consists all the trials of arms in $b_-$. Specifically, we write $S_1 = \{(I_1, X_1), \ldots, (I_{T_{b_-}}, X_{T_{b_-}})\}$ such that $S_1$ is a subsequence of $\tilde{S}$ and $I_i \in b_-$ for all $i$.

Next, we define several random events in a way similar to the proof of Theorem 2. Define event $\mathcal{A}_1 = \{T_{b_-} \leq 4t_{b_-}^*\}$. Define event

$$\mathcal{C}_1 = \left\{ \max_{1 \leq t \leq 4t_{b_-}^*} \left| \sum_{i=1}^t X_i - \sum_{i=1}^t p_{I_i} \right| < \sqrt{t_{b_-}^* \log(1/\theta)} \right\}.$$

Define event

$$\mathcal{B} = \{\mathsf{Out} = M_*\}. \tag{54}$$

Define event $\mathcal{S}_1 = \mathcal{A}_1 \cap \mathcal{B} \cap \mathcal{C}_1$. Then, we bound the probability of events $\mathcal{A}_1$, $\mathcal{B}$, $\mathcal{C}_1$ and $\mathcal{S}_1$ under $H_0$ using methods similar to Theorem 2. First, we show that $\mathrm{Pr}_0[\mathcal{A}_1] \geq 3/4$. This can be proved by Markov inequality as follows.

$$\mathrm{Pr}_0[T_{b_-} > 4t_{b_-}^*] \leq \frac{\mathbb{E}_0[T_{b_-}]}{4t_{b_-}^*}$$

$$= \frac{t_{b_-}^*}{4t_{b_-}^*} = \frac{1}{4}.$$

Next, we show that $\mathrm{Pr}_0[\mathcal{C}_1] \geq 3/4$. Notice that the sequence $\left\{ \sum_{i=1}^t X_i - \sum_{i=1}^t p_{I_i} \right\}_{t \in [4t_{b_-}^*]}$ is a martingale. Hence, by Kolmogorov's inequality, we have

$$\mathrm{Pr}_0 \left[ \max_{1 \leq t \leq 4t_{b_-}^*} \left| \sum_{i=1}^t X_i - \sum_{i=1}^t p_{I_i} \right| \geq \sqrt{t_e^* \log(1/\theta)} \right] \leq \frac{\mathbb{E}_0 \left[ \left( \sum_{i=1}^{4t_{b_-}^*} X_i - \sum_{i=1}^{4t_{b_-}^*} p_{I_i} \right)^2 \right]}{t_e^* \log(1/\theta)}$$

$$= \frac{4t_{b_-}^*}{t_{b_-}^* \log(1/\theta)}$$

$$< \frac{1}{4},$$

where the second inequality follows from the fact that all reward distributions have unit variance and hence $\mathbb{E}_0 \left[ \left( \sum_{i=1}^{4t_{b_-}^*} X_i - \sum_{i=1}^{4t_{b_-}^*} p_{I_i} \right)^2 \right] = 4t_{b_-}^*$; the last inequality follows since $\theta < e^{-16}$. Last, since algorithm $\mathbb{A}$ is a $\delta$-correct algorithm with $\delta < 1/4$. Therefore, it is easy to see that $\mathrm{Pr}_0[\mathcal{B}] \geq 3/4$. And by union bound, we have

$$\mathrm{Pr}_0[\mathcal{S}_1] \geq 1/4.$$

Now, we show that if $\mathbb{E}_0[T_{b_-}] \leq t_{b_-}^*$, then $\mathrm{Pr}_1[\mathcal{B}] \geq \delta$. Assume that the event $\mathcal{S}_1$ occurred. We bound the likelihood ratio $L_1(W)/L_0(W)$ under this assumption as follows

$$\frac{L_1(W)}{L_0(W)} = \prod_{i=1}^{T_{b_-}} \exp \left( \frac{-\left( X_i - p_{I_i} - \frac{2w(b)}{|b_-|} \right)^2 + (X_i - p_{I_i})^2}{2} \right)$$

$$= \prod_{i=1}^{T_{b_-}} \exp\left( \frac{w(b)}{|b_-|}(2X_i - 2p_{I_i}) - \frac{2w(b)^2}{|b_-|^2} \right)$$

$$= \exp\left( \frac{w(b)}{|b_-|}\left( \sum_{i=1}^{T_{b_-}} 2X_i - 2p_{I_i} \right) - \frac{2w(b)^2}{|b_-|^2}T_{b_-} \right)$$

$$= \exp\left( \frac{w(b)}{|b_-|}\left( \sum_{i=1}^{T_{b_-}} 2X_i - 2p_{I_i} \right) \right)\exp\left( -\frac{2w(b)^2}{|b_-|^2}T_{b_-} \right). \tag{55}$$

Then, we bound each term on the right-hand side of Eq. (55). First, we bound the second term of Eq. (55).

$$\exp\left( -\frac{2w(b)^2}{|b_-|^2}T_{b_-} \right) \geq \exp\left( -\frac{2w(b)^2}{|b_-|^2}4t_b^* \right) \tag{56}$$

$$= \exp\left( -\frac{8}{16}\log(1/\theta) \right) \tag{57}$$

$$= \theta^{1/2}, \tag{58}$$

where Eq. (56) follows from the assumption that events $\mathcal{S}_1$ and $\mathcal{A}_1$ occurred and therefore $T_{b_-} \leq 4t_{b_-}^*$; Eq. (57) follows from the definition of $t_{b_-}^*$. Next, we bound the first term of Eq. (55) as follows

$$\exp\left( \frac{w(b)}{|b_-|}\left( \sum_{i=1}^{T_{b_-}} 2X_i - 2p_{I_i} \right) \right) \geq \exp\left( -\frac{2w(b)}{|b_-|}\sqrt{t_b^* \log(1/\theta)} \right) \tag{59}$$

$$= \exp\left( -\frac{2}{4}\log(1/\theta) \right) \tag{60}$$

$$= \theta^{1/2}, \tag{61}$$

where Eq. (59) follows since event $\mathcal{S}_1$ and $\mathcal{C}_1$ occurred and therefore $|2K - 2p_e T_e| \leq \sqrt{t_e^* \log(1/\theta)}$; Eq. (60) follows from the definition of $t_{b_-}^*$.

Hence, if event $\mathcal{S}_1$ occurred, we can bound the likelihood ratio as follows

$$\frac{L_1(W)}{L_0(W)} \geq \theta = 4\delta. \tag{62}$$

Let $1_{\mathcal{S}_1}$ denote the indicator variable of event $\mathcal{S}_1$. Then, we have $\frac{L_1(W)}{L_0(W)}1_{\mathcal{S}_1} \geq 4\delta 1_{\mathcal{S}_1}$. Therefore, we can bound $\Pr_1[\mathcal{B}]$ as follows

$$\Pr_1[\mathcal{B}] \geq \Pr_1[\mathcal{S}_1] = \mathbb{E}_1[1_{\mathcal{S}_1}]$$

$$= \mathbb{E}_0\left[ \frac{L_1(W)}{L_0(W)}1_{\mathcal{S}_1} \right]$$

$$\geq 4\delta\mathbb{E}_0[1_{\mathcal{S}_1}]$$

$$= 4\delta\Pr_0[\mathcal{S}_1] > \delta. \tag{63}$$

This means that, if $\mathbb{E}_0[T_{b_-}] \leq t_{b_-}^*$, then, under $H_1$, the probability of algorithm $\mathbb{A}$ returning $M_*$ as output is at least $\delta$. But $M_*$ is not the optimal set under $H_1$. Hence this contradicts to the assumption that $\mathbb{A}$ is a $\delta$-correct algorithm. Hence we have proved that

$$\mathbb{E}_0[T_{b_-}] \geq t_{b_-}^* = \frac{|b_-|^2}{16w(b)^2}\log(1/4\delta). \tag{64}$$

*(End of Part (1).)*

**Part (2): Lower bound of $\mathbb{E}_0[T_{b_+}]$.** In this part, we will show that $\mathbb{E}_0[T_{b_+}] \geq t^*_{b_+}$, where we define $t^*_{b_+} = \frac{|b_+|^2}{16w(b)^2} \log(1/\theta)$. The arguments used in this part are similar to that of Part (1). Hence, we will omit the redundant parts and highlight the differences.

Recall that $\tilde{S}$ is the sequence of all trials by algorithm $\mathbb{A}$. We define $S_2$ be the subsequence of $\tilde{S}$ which contains the trials of arms belonging to $b_+$. We write $S_2 = \{(J_1, Y_1), \ldots, (J_{T_{b_+}}, Y_{T_{b_+}})\}$, where $J_i$ is $i$-th played arm in sequence $S_2$ and $Y_i$ is the associated reward outcome.

We define the random events $\mathcal{A}_2$ and $\mathcal{C}_2$ similar to Part (1). Specifically, we define

$$\mathcal{A}_2 = \{T_{b_+} \leq 4t^*_{b_+}\} \quad \text{and} \quad \mathcal{C}_2 = \left\{ \max_{1 \leq t \leq 4t^*_{b_+}} \left| \sum_{i=1}^{t} Y_i - \sum_{i=1}^{t} p_{J_i} \right| < \sqrt{t^*_{b_+} \log(1/\theta)} \right\}.$$

Using the similar arguments, we can show that $\Pr_0[\mathcal{A}_2] \geq 3/4$ and $\Pr_0[\mathcal{C}_2] \geq 3/4$. Define event $\mathcal{S}_2 = \mathcal{A}_2 \cap \mathcal{B} \cap \mathcal{C}_2$, where $\mathcal{B}$ is defined in Eq. (54). By union bound, we see that

$$\Pr_0[\mathcal{S}_2] \geq 1/4.$$

Then, we show that if $\mathbb{E}_0[T_{b_+}] \leq t^*_{b_+}$, then $\Pr_2[\mathcal{B}] \geq \delta$. We bound likelihood ratio $L_2(W)/L_0(W)$ under the assumption that $\mathcal{S}_2$ occurred as follows

$$
\begin{aligned}
\frac{L_2(W)}{L_0(W)} &= \prod_{i=1}^{T_{b_+}} \exp\left( \frac{-\left(X_i - p_{I_i} + \frac{2w(b)}{|b_-|}\right)^2 + (X_i - p_{I_i})^2}{2} \right) \\
&= \prod_{i=1}^{T_{b_+}} \exp\left( \frac{w(b)}{|b_+|}(2p_{I_i} - 2X_i) - \frac{2w(b)^2}{|b_+|^2} \right) \\
&= \exp\left( \frac{w(b)}{|b_+|} \left( \sum_{i=1}^{T_{b_+}} 2p_{I_i} - 2X_i \right) - \frac{2w(b)^2}{|b_+|^2} T_{b_+} \right) \\
&= \exp\left( \frac{w(b)}{|b_+|} \left( \sum_{i=1}^{T_{b_+}} 2p_{I_i} - 2X_i \right) \right) \exp\left( -\frac{2w(b)^2}{|b_+|^2} T_{b_+} \right) \\
&\geq \theta \\
&= 4\delta,
\end{aligned}
\tag{65}
$$

where Eq. (65) can be obtained using same method as in Part (1) as well as the assumption that $\mathcal{S}_2$ occurred.

Next, similar to the derivation in Eq. (63), we see that

$$\Pr_2[\mathcal{B}] \geq \Pr_2[\mathcal{S}_2] = \mathbb{E}_2[1_{S_2}] = \mathbb{E}_0\left[ \frac{L_2(W)}{L_0(W)} 1_{S_2} \right] \geq 4\delta \mathbb{E}_0[1_{S_2}] > \delta,$$

where $1_{S_2}$ is the indicator variable of event $\mathcal{S}_2$. Therefore, we see that if $\mathbb{E}_0[T_{b_+}] \leq t^*_{b_+}$, then, under $H_2$, the probability of algorithm $\mathbb{A}$ returning $M_*$ as output is at least $\delta$, which is not the optimal set under $H_2$. This contradicts to the assumption that algorithm $\mathbb{A}$ is a $\delta$-correct algorithm. In sum, we have proved that

$$\mathbb{E}_0[T_{b_+}] \geq t^*_{b_+} = \frac{|b_+|^2}{16w(b)^2} \log(1/4\delta). \tag{66}$$

*(End of Part (2))*

Finally, we combine the results from both parts, i.e. Eq. (64) and Eq. (66). We obtain

$$\mathbb{E}_0[T_b] = \mathbb{E}_0[T_{b_-}] + \mathbb{E}_0[T_{b_+}]$$
$$\geq \frac{|b_+|^2 + |b_-|^2}{16w(b)^2} \log(1/4\delta)$$
$$\geq \frac{|b|^2}{32w(b)^2} \log(1/4\delta).$$

$\square$

# 5   PAC Learning

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**Algorithm 1** CombPAC: Fixed confidence pure exploration algorithm
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**Require:** Confidence parameter: $\delta \in (0,1)$; Tolerance parameter: $\epsilon > 0$; Maximization oracle: $\mathrm{Oracle}(w)$.
   **Initialize:** Play each arm $e \in [n]$ once. Initialize empirical means $\bar{w}_n(e)$ and set $T_n(e) \leftarrow 1$ for all $e$.
  1: **for** $t = n, n+1, \dots$ **do**
  2:    $M_t \leftarrow \mathrm{Oracle}(\bar{w}_t)$
  3:    **for** $e \in [n]$ **do**
  4:        **if** $e \in M_t$ **then**
  5:            $\tilde{w}_t(e) \leftarrow \bar{w}_t(e) - \mathrm{rad}_t(e)$
  6:        **else**
  7:            $\tilde{w}_t(e) \leftarrow \bar{w}_t(e) + \mathrm{rad}_t(e)$
  8:        **end if**
  9:    **end for**
10:    $\tilde{M}_t \leftarrow \mathrm{Oracle}(\tilde{w}_t)$
11:    **if** $\tilde{w}_t(\tilde{M}_t) - \tilde{w}_t(M_t) \leq \epsilon$ **then**
12:        $\mathsf{Out} \leftarrow M_t$
13:        **return** $\mathsf{Out}$
14:    **end if**
15:    $D_t \leftarrow \tilde{M}_t \ominus M_t$
16:    $p_t \leftarrow \arg\max_{e \in D_t} \mathrm{rad}_t(e)$
17:    Play $p_t$ and observe the reward
18:    Update $T_{t+1}(p_t)$ and $\bar{w}_{t+1}(p_t)$
19: **end for**
---

**Theorem 4.** *Given confidence parameter $\delta \in (0,1)$, tolerance parameter $\epsilon \geq 0$, number of arms $n$ and a combinatorial problem instance $\mathcal{M} \subseteq 2^{[n]}$. Let oracle $\mathrm{Oracle}(w)$ be a maximization oracle associated with $\mathcal{M}$ such that $\mathrm{Oracle}(w) = \arg\max_{M \in \mathcal{M}} w(M)$, where $w : 2^{[n]} \to R$ is a weight function.*

*Then, with probability at least $1 - \delta$, the output $\mathsf{Out}$ of Algorithm 1 satisfies $w(M_*) - w(\mathsf{Out}) \leq \epsilon$, where $M_* = \arg\max_{M \in \mathcal{M}} w(M)$ is the optimal set. In addition, the number of samples $T$ used by the algorithm satisfy*

$$T \leq \mathbf{H}_\epsilon \log\left(\frac{n}{\delta}\mathbf{H}_\epsilon\right),$$

*where*

$$\mathbf{H}_\epsilon = \sum_{e \in [n]} \min\left\{\frac{\mathrm{rank}(\mathcal{B})^2}{\Delta_e^2}, \frac{n^2}{\epsilon^2}\right\}.$$

**Lemma 11.** *For any arm $e \in [n]$ and any round $t > n$ after initialization, if $\mathrm{rad}_t(e) \leq \max\left\{\frac{\Delta_e}{3\,\mathrm{rank}(\mathcal{B})}, \frac{\epsilon}{n}\right\}$, then $p_t \neq e$.*

*Proof.* If $\mathrm{rad}_t(e) \leq \frac{\Delta_e}{3\,\mathrm{rank}(\mathcal{B})}$, then we can apply Lemma 9 which immediately gives that $p_t \neq e$. Hence, we only need to prove the case that $\mathrm{rad}_t(e) \leq \frac{\epsilon}{n}$. By the definition of $p_t$, we know that for each $i \in D_t$, we have $\mathrm{rad}_t(i) \leq \mathrm{rad}_t(e) \leq \frac{\epsilon}{n}$. Summing up all $i \in D_t$, we obtain

$$\mathrm{rad}_t(D_t) \leq \epsilon. \tag{67}$$

Next, we notice that the definition of $M_t$ gives that $\bar{w}_t(M_t) = \max_{M \in \mathcal{M}} \bar{w}_t(M) \geq \bar{w}_t(M_t \oplus D_t)$. This means that

$$\bar{w}_t(D_t) = \bar{w}_t(M_t \oplus D_t) - \bar{w}_t(M_t) \leq 0. \tag{68}$$

Using the above inequalities, we have.

$$w_t^+(D_t) = \bar{w}_t(D_t) + \mathrm{rad}_t(D_t) \tag{69}$$
$$\leq \bar{w}_t(D_t) + \epsilon \tag{70}$$
$$\leq \epsilon, \tag{71}$$

where Eq. (69) follows from the definition of $w_t^+(\cdot)$; Eq. (70) follows from Eq. (67); Eq. (71) holds since Eq. (68). $\square$

**Lemma 12.** *If Algorithm 1 stops on round $T$, then $w(M_*) - w(\mathsf{Out}) \leq \epsilon$.*

*Proof.* Suppose that $\mathsf{Out} \neq M_*$. Since the algorithm stops on round $T$, we know that $\mathsf{Out} = M_T$. Consider the diff-set $D = M_* \ominus M_T$ and the diff-set $D_T$ defined in Step 15 of Algorithm 1. By Lemma Z, we see that

$$w_T^+(D_T) = \max_{C:C \prec M_T} w_T^+(C) \geq w_T^+(D). \tag{72}$$

On the other hand, the stopping condition of Algorithm 1 gives that

$$\epsilon \geq \tilde{w}_T(\tilde{M}_T) - \tilde{w}_T(M_T)$$
$$= w_t^+(D_T) \geq w_t^+(D) \tag{73}$$
$$\geq w(D) = w(M_*) - w(M_T), \tag{74}$$

where Eq. (73) follows from Eq. (72); Eq. (74) follows from the assumption that event $\xi$ occurred.

$\square$

# 6 Fixed Budget

# 7 Benchmark: uniform algorithm

# References