adversarial cmab

definitions

- $\tilde{c}_t(k) = \tilde{l}_t^T v_k$

observations

- outcomes of all arms $\{l_t(i), \forall v_i(K_t) = 1\}$.

construction of pseudo-observations $\tilde{l}_t$

- requirements
  - $E[\tilde{l}_t] = l_t$
  - $E[\tilde{c}_t(k)] = c_t(k)$ for all $k \in [K]$
- construction

$$\tilde{l}_t(i) = \begin{cases} \frac{l_t(i)}{\sum_{k=1}^K p_{t-1}(k) v_i(k)} & v_i(K_t) = 1, \\ 0 & v_i(K_t) = 0. \end{cases}$$

- properties

$$\begin{aligned} E[\tilde{l}_t(i)] &= \left[\frac{l_t(i)}{\sum_{k=1}^K p_{t-1}(k) v_i(k)} \right] \sum_{k=1}^K p_{t-1}(k) v_i(k) \\ &= l_t(i) \end{aligned}$$

- note

$$\sum_{k=1}^K p_{t-1}(k) v_i(k) = E[v_i(K_t)].$$

  and hence

$$\tilde{l}_t(i) = \frac{l_t(i) v_i(K_t)}{E[v_i(K_t)]}.$$

- $E[v(K_t) v(K_t)^T]_{ij} = E[v_i(K_t) v_j(K_t)]$.
  - hence

$$\begin{aligned} \tilde{l}_t^T E[v(K_t) v(K_t)^T] \tilde{l}_t &= \sum_{ij} \tilde{l}_t(i) \tilde{l}_t(j) E[v(K_t) v(K_t)^T]_{ij} \\ &= \sum_{ij} l_t(i) l_t(j) v_i(K_t) v_j(K_t) \frac{E[v_i(K_t) v_j(K_t)]}{E[v_i(K_t)] E[v_j(K_t)]} \end{aligned}$$

proof: key steps

- cumulative pseudo-loss $\tilde{L}_n(k) = \sum_{t=1}^n \tilde{c}_t(k)$.
- against the best (@upper)

$$\log\left(\frac{W_n}{W_0}\right) \geq -\eta\tilde{L}_n - \log(K).$$

- lower bound (@lower-single)

$$\log\left(\frac{W_t}{W_{t-1}}\right) \leq \frac{-\eta}{1-\gamma}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k) + \frac{\eta\gamma}{1-\gamma}\sum_{k=1}^{K}\tilde{c}_t(k)\mu(k) + \frac{\eta^2}{1-\gamma}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)^2$$

proof missing

- cumulative lower bound (@lower)

$$\log\left(\frac{W_n}{W_0}\right) \leq \frac{-\eta}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k) + \frac{\eta\gamma}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K}\tilde{c}_t(k)\mu(k) + \frac{\eta^2}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)^2$$

- combine (@lower) and (@upper)

$$-\eta\tilde{L}_n - \log(K) \leq \frac{-\eta}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k) + \frac{\eta\gamma}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K}\tilde{c}_t(k)\mu(k) + \frac{\eta^2}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)^2$$

rearrange,

$$\frac{\eta}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k) \leq \eta\tilde{L}_n + \log(K) + \frac{\eta\gamma}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K}\tilde{c}_t(k)\mu(k) + \frac{\eta^2}{1-\gamma}\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)^2$$

divide by $\frac{\eta}{1-\gamma}$,

$$\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k) \leq (1-\gamma)\tilde{L}_n + \frac{1-\gamma}{\eta}\log(K) + \gamma\sum_{t=1}^{n}\sum_{k=1}^{K}\tilde{c}_t(k)\mu(k) + \eta\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)^2$$

- expectations

$$E[\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)] = E[\sum_{t=1}^{n} c_t(K_t)]$$

proof required

- take expectation

$$E[\sum_{t=1}^{n} c_t(K_t)] \leq (1-\gamma)E[\tilde{L}_n] + \frac{1-\gamma}{\eta}\log(K) + \gamma\sum_{t=1}^{n}\sum_{k=1}^{K} E[\tilde{c}_t(k)\mu(k)] + \eta\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)E[\tilde{c}_t(k)^2]$$
$$= (1-\gamma)L_n + \frac{1-\gamma}{\eta}\log(K) + \gamma\sum_{t=1}^{n}\sum_{k=1}^{K} c_t(k)\mu(k) + \eta\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)E[\tilde{c}_t(k)^2]$$

- bound the term $\sum_{t=1}^{n}\sum_{k=1}^{K} p_{t-1}(k)E[\tilde{c}_t(k)^2]$.
  - by [comband], we have

$$\sum_{k=1}^{K} p_{t-1}(k)\tilde{c}_t(k)^2 = \tilde{l}_t^T E[v(K_t)v(K_t)^T]\tilde{l}_t.$$