Pure Exploration of Combinatorial Bandits

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1 Preliminaries

1.1 Problems

Let n be the number of base arms. Let $\mathcal{M} \subseteq 2^{[n]}$ be the set of super arms.

In this note, we consider the following cases of $\mathcal{M}$.

Example 1 (Explore-m)**.** $\mathcal{M}_{\mathsf{TOP}m}(n) = \{M \subseteq [n] \mid |M| = m\}$. *This corresponds to finding the top m arms from $[n]$.*

Example 2 (Explore-m-bandits)**.** *Suppose $n = mk$. Then $\mathcal{M}_{\mathsf{BANDIT}m}(n)$ contains all subsets $M \subseteq [n]$ with size m, such that*

$$M \cap \{ik+1, \dots, (i+1)k\} = 1, \quad \text{for all } i \in \{0, \dots, m-1\}.$$

This corresponds to finding the top arms from m bandits, where each bandit has k arms.

Example 3 (Perfect Matching)**.** *Let $G = (V, E)$ be a bipartite graph and $|E| = n$. For simplicity, let each edge $e \in E$ corresponds to a unique integer $i \in [n]$, and vice versa. Then $\mathcal{M}_{\mathsf{MATCH}}(n, G)$ contains all subsets $M \subseteq [n]$ such that M corresponds to a perfect matching in G.*

1.2 Diff-Sets

Definition 1 (Diff-set)**.** *An n-diff-set (or diff-set in short) is a pair of sets $c = (c_+, c_-)$, where $c_+ \subseteq [n]$, $c_- \subseteq [n]$ and $c_+ \cap c_- = \emptyset$.*

Definition 2 (Difference of sets)**.** *Given any $M_1 \subseteq [n], M_2 \subseteq [n]$. We define $M_1 \ominus M_2 \triangleq C$, where $C = (C_+, C_-)$ is a diff-set and $C_+ = M_1 \backslash M_2$ and $C_- = M_2 \backslash M_1$.*

Definition 3. *Denote $\mathsf{diff}[n]$ be the set of all possible n-diff-sets.*

Definition 4 (Set operations of diff-sets)**.** *Let $C = (C_+, C_-), D = (D_+, D_-)$ be two diff-sets. We define $C \cap D \triangleq (C_+ \cap D_+, C_- \cap D_-)$ and $C \backslash D \triangleq (C_+ \backslash D_+, C_- \backslash D_-)$.*

Further, for all $e \in [n]$, $e \in C \Leftrightarrow (e \in C_+) \vee (e \in C_-)$. And $|C| \triangleq |C_+| + |C_-|$.

Definition 5 (Valid diff-set)**.** *Given a set $M \subseteq [n]$ and a diff-set $C = (C_+, C_-)$, we call C a* valid diff-set *for M, iff $C_+ \cap M = \emptyset$ and $C_- \subseteq M$. In this case, we denote $C \prec M$.*

Definition 6 (Negative diff-set)**.** *Given a diff-set $A = (A_+, A_-)$, we define $\neg A = (A_-, A_+)$.*

1.2.1 diff-set operations

Definition 7 (Operators $\oplus$ and $\ominus$)**.** *Given any $M \subseteq [n]$ and $C \in \mathsf{diff}[n]$. If $C \prec M$, we define operator $\oplus$ such that $M \oplus C \triangleq M \backslash C_- \cup C_+$. On the other hand if $\neg C \prec M$, we define operator $\ominus$ such that $M \ominus C \triangleq M \oplus (\neg C) = M \backslash C_+ \cup C_-$.*

Definition 8. *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. We denote $B \prec A$, if and only if $B_+ \cap A_+ = \emptyset$ and $A_+ \cap A_- = \emptyset$.*

Definition 9. *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. If $B \prec A$, we define $A \oplus B = ((A_+ \cup B_+)\backslash(A_- \cup B_-), (A_- \cup B_-)\backslash(A_+ \cup B_+))$.*

Lemma 1. *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. If $B \prec A$, then $A \oplus B$ is a diff-set.*

Proof. Let $C = A \oplus B$. By definition, we have $C_+ = (A_+ \cup B_+)\backslash(A_- \cup B_-)$ and $C_- = (A_- \cup B_-)\backslash(A_+ \cup B_+)$.

We only need to show that $C_+ \cap C_- = \emptyset$.

$$\begin{aligned} C_+ \cap C_- &= \big((A_+ \cup B_+)\backslash(A_- \cup B_-)\big) \cap \big((A_- \cup B_-)\backslash(A_+ \cup B_+)\big) \\ &= (A_+ \cup B_+) \cap \big((A_- \cup B_-)\backslash(A_+ \cup B_+)\backslash(A_- \cup B_-)\big) \\ &= \emptyset. \end{aligned}$$

□

Lemma 2. *Given two diff-sets $A = (A_+, A_-)$ and $B = (B_+, B_-)$. If there exists $M \subseteq [n]$ such that $A \prec M$, and $B \prec (M \oplus A)$, then $B \prec A$ and $(M \oplus A \oplus B) \ominus M = A \oplus B$.*

Proof. We first show that $B \prec A$. Since $B \prec (M \oplus A)$, we know that $B_+ \cap (M\backslash A_- \cup A_+) = \emptyset$. Therefore, we have

$$\begin{aligned} \emptyset &= B_+ \cap (M\backslash A_- \cup A_+) \\ &= (B_+ \cap (M\backslash A_-)) \cup (B_+ \cap A_+) \end{aligned}$$

We see that $B_+ \cap A_+ = \emptyset$.

On the other hand, we have $B_- \subseteq (M\backslash A_- \cup A_+)$, therefore

$$\begin{aligned} B_- \cap A_- &\subseteq (M\backslash A_- \cup A_+) \cap A_- \\ &= (M\backslash A_- \cap A_-) \cup (A_+ \cap A_-) \\ &= \emptyset. \end{aligned}$$

Hence we proved that $B \prec A$.

Define $D = (M \oplus A \oplus B) \ominus M$ and write $D = (D_+, D_-)$. Then,

$$\begin{aligned} D_+ &= (M \oplus A \oplus B)\backslash M \\ &= (M\backslash A_- \cup A_+\backslash B_- \cup B_+)\backslash M \\ &= (A_+ \cup B_+)\backslash(A_- \cup B_-). \end{aligned}$$

Similarly, we have

$$\begin{aligned} D_- &= M\backslash(M \oplus A \oplus B) \\ &= M\backslash(M\backslash A_- \cup A_+\backslash B_- \cup B_+) \\ &= (A_- \cup B_-)\backslash(A_+ \cup B_+). \end{aligned}$$

□

1.2.2 Diff-set class

Definition 10 (Decomposition of diff-set)**.** *Given $\mathcal{B} \subseteq \mathsf{diff}[n]$ and $D \in \mathsf{diff}[n]$, a decomposition of D on $\mathcal{B}$ is a set $\{b_1, \ldots, b_k\} \subseteq \mathcal{B}$ satisfying the following*

1. *For all* $i \in [k]$ *and* $j \in [k]$*, we write* $b_i = (b_i^+, b_i^-)$ *and* $b_j = (b_j^+, b_j^-)$*. Then, the following holds* $b_i^+ \cap b_j^+ = \emptyset$, $b_i^+ \cap b_j^- = \emptyset$, $b_i^- \cap b_j^+ = \emptyset$ *and* $b_i^- \cap b_j^- = \emptyset$*.*

2. $D = b_1 \oplus b_2 \oplus \ldots b_k$*.*

Lemma 3. *Given* $\mathcal{B} \subseteq \mathsf{diff}[n]$ *and* $D \in \mathsf{diff}[n]$*. Let* $\{b_1, \ldots, b_k\} \subseteq \mathcal{B}$ *be a decomposition of* D *on* $\mathcal{B}$*. Then,*

1. *Let* $D = (D_+, D_-)$ *and for all* $i \in [k]$*, we write* $b_i = (b_i^+, b_i^-)$*. Then* $D_+ = b_1^+ \cup \ldots \cup b_k^+$ *and* $D_- = b_1^- \cup \ldots \cup b_k^-$*.*

2. *For all* $M \subseteq [n]$*, if* $D \prec M$*, then, for all* $i \in [k]$*, we have* $b_i \prec M$*.*

Proof. We prove (1) by induction. Let $D_i = b_1 \oplus \ldots \oplus b_i$ and write $D_i = (D_i^+, D_i^-)$. We show that $D_i^+ = \bigcup_{j=1}^i b_i^+$ and $D_{i-} = \bigcup_{j=1}^i b_i^-$ for all $i \in [k]$. For $i = 1$, this is trivially true. Then, assume that this is true for some $i > 1$. By definition $D_{i+1} = D_i \oplus b_{i+1}$, hence $D_{i+1}^+ = (D_i^+ \cup b_{i+1}^+) \backslash (D_i^- \cup b_{i+1}^-)$. Note that

$$\begin{aligned}
(D_i^- \cup b_{i+1}^-) \cap (D_i^+ \cup b_{i+1}^+) &= (D_i^- \cap D_i^+) \cup (D_i^- \cap b_{i+1}^+) \cup (b_{i+1}^- \cap D_i^+) \cup (b_{i+1}^- \cap b_{i+1}^+) \\
&= (D_i^- \cap b_{i+1}^+) \cup (b_{i+1}^- \cap D_i^+) \\
&= \left(\left(\bigcup_{j=1}^{i} b_j^- \right) \cap b_{i+1}^+ \right) \cup \left(\left(\bigcup_{j=1}^{i} b_j^+ \right) \cap b_{i+1}^- \right) \\
&= \emptyset.
\end{aligned}$$

Hence $D_{i+1}^+ = D_i^+ \cup b_{i+1}^+$. We can use the same method to show that $D_{i+1}^- = D_i^- \cup b_{i+1}^-$.

Next, we prove (2) using (1). To show that $b_i \prec M$, we only need to show that $b_i^+ \cap M = \emptyset$ and $b_i^- \subseteq M$. Since $D \prec M$, we know that $D_+ \cap M = \emptyset$ and $D_- \subseteq M$. By (1), we see that $b_i^+ \subseteq D_+$ and $b_i^- \subseteq D_-$. Therefore, we have $(b_i^+ \cap M) \subseteq (D_+ \cap M) = \emptyset$ and $b_i^- \subseteq D_- \subseteq M$.

□

Definition 11 (diff-set class)**.** *Given* $\mathcal{M} \subseteq 2^{[n]}$*.* $\mathcal{B} \subseteq \mathsf{diff}[n]$ *is a diff-set class for* $\mathcal{M}$*, if the following hold.*

1. $(\emptyset, \emptyset) \notin \mathcal{B}$*.*

2. *For all* $M \in \mathcal{M}$ *and for all* $b \in \mathcal{B}$*, if* $b \prec M$*, then* $M \oplus b \in \mathcal{M}$*.*

3. *For all* $M_1 \in \mathcal{M}$ *and* $M_2 \in \mathcal{M}$*, where* $M_1 \neq M_2$*. Let* $D = M_1 \ominus M_2$*. Then, there exists a decomposition of* D *on* $\mathcal{B}$*.*

Definition 12 (Rank of diff-set class)**.** *Let* $\mathcal{B} \subseteq [n]$ *be a diff-set class for some* $\mathcal{M}$*. We define*

$$\mathrm{rank}(\mathcal{B}) \triangleq \max_{b \in \mathcal{B}} |b|.$$

Example 4 (diff-set class for Explore-m)**.** *One diff-set class* $\mathcal{B}$ *for* $\mathcal{M}_{\mathsf{TOP}m}(n)$ *is given by*

$$\mathcal{B} = \{(\{b_1\}, \{b_2\}) \mid b_1 \neq b_2, b_1 \in [n], b_2 \in [n]\}.$$

Proof omitted. Further, we see that $\mathrm{rank}(\mathcal{B}) = 2$*.*

Example 5 (diff-set class for Explore-m-badit)**.** *Let* $n = mk$*. One diff-set class* $\mathcal{B}$ *for* $\mathcal{M}_{\mathsf{BANDIT}m}(n)$ *is given by*

$$\mathcal{B} = \{(\{b_1\}, \{b_2\}) \mid b_1 \neq b_2, \exists i \in \{0, \ldots, k-1\}, b_1 \in \{ik+1, \ldots, (i+1)k\}, b_2 \in \{ik+1, \ldots, (i+1)k\}\}.$$

Proof omitted. Further, we see that $\mathrm{rank}(\mathcal{B}) = 2$*.*

Example 6 (diff-set class for Perfect Matching)**.** *One diff-set class $\mathcal{B}$ for $\mathcal{M}_{\mathsf{MATCH}}(n, G)$ is the set of all augmenting cycles of G. More specifically,*

$$\mathcal{B} = \{(b_+, b_-) | b_+ \cup b_- \text{ is a cycle of } G\}.$$

Note $\operatorname{rank}(\mathcal{B}) \leq n$.

1.3 Weights and confidence bounds

Definition 13 (Weight functions)**.** *Define function $w : [n] \to \mathbb{R}^+$ which represents the weight of each base arm. Further, we slight abuse the notations, and extend the definition of w to diff-sets and sets as follows.*

1. *For all $M \subseteq [n]$, we denote $w(M) = \sum_{e \in M} w(e)$.*
2. *For all $b = (b_+, b_-) \in \mathsf{diff}[n]$, we denote $w(b) = \sum_{e \in b_+} w(e) - \sum_{e \in b_-} w(e)$.*

Lemma 4. *Let $c \in \mathsf{diff}[n], d \in \mathsf{diff}[n]$. Let w be a weight function. Then,*

$$w(c \cup d) = w(c) + w(d) - w(c \cap d). \tag{1}$$

Proof. Let $c = (c_+, c_-)$ and $d = (d_+, d_-)$. We have

$$\begin{aligned} w(c \cup d) &= w(c_+ \cup d_+) - w(c_- \cup d_-) && (2)\\ &= w(c_+) + w(d_+) - w(c_+ \cap d_+) - w(c_-) - w(d_-) + w(c_- \cap d_-) && (3)\\ &= w(c) + w(d) - (w(c_+ \cap d_+) - w(c_- \cap d_-)) && (4)\\ &= w(c) + w(d) - w(c \cap d). && (5) \end{aligned}$$

□

Definition 14 (Mean weight $\bar{w}_t$, sample size n_t)**.** *Given $t > 0$. Define $\bar{w}_t$ be a weight function such that, for all $e \in [n]$, $\bar{w}_t(e)$ equals to the empirical mean of e up to round t. Let $n_t : [n] \to \mathbb{N}$, such that $n_t(e)$ equals to number of plays of base arm e up to round t.*

Definition 15 (Confidence radius rad_t)**.** *Given n and $t > 0$. Define $\operatorname{rad}_t : [n] \to \mathbb{R}^+$ satisfying, for all $e \in [n]$,*

$$\operatorname{rad}_t(e) = c_{\text{rad}} \log\left(\frac{c_\delta n t^2}{\delta}\right) \frac{1}{\sqrt{n_t(e)}},$$

where $c_{\text{rad}} > 0$ and $c_\delta > 0$ are some universal constants (specify later) and $\delta > 0$ is a parameter.

We extend the notation of rad_t to diff-sets and sets as follows.

1. *For all $M \subseteq [n]$, $\operatorname{rad}_t(M) \triangleq \sum_{e \in M} \operatorname{rad}_t(e)$.*
2. *For all $b = (b_+, b_-) \in \mathsf{diff}[n]$, $\operatorname{rad}_t(b) \triangleq \operatorname{rad}_t(b_+) + \operatorname{rad}_t(b_-)$.*

Definition 16 (UCB w_t^+)**.** *Define $w_t^+ : [n] \to \mathbb{R}^+$, s.t., for all $e \in [n]$,*

$$w_t^+(e) = \bar{w}_t(e) + \operatorname{rad}_t(e).$$

We extend the notation of w_t^+ to diff-sets and sets as follows.

1. *For all $M \subseteq [n]$, $w_t^+(M) \triangleq \bar{w}_t(M) + \operatorname{rad}_t(M)$.*
2. *For all $b = (b_+, b_-) \in \mathsf{diff}[n]$, $w_t^+(b) \triangleq \bar{w}_t(b) + \operatorname{rad}_t(b)$.*

Lemma 5. *Define random event*

$$\xi = \{\forall e \in [n]\ \forall t > 0, |\bar{w}_t(e) - w(e)| \leq \mathrm{rad}_t(e)\}.$$

Then, there exist constants c_{rad} *and* c_δ*,*

$$\Pr[\xi] \geq 1 - \delta.$$

Proof. Hoeffding inequality and union bound. □

Corollary 1.

$$\xi \implies \forall t, \forall e \in [n]\ w_t^+(e) \geq w(e).$$

$$\xi \implies \forall t, \forall M \subseteq [n],\ w_t^+(M) \geq w(M).$$

$$\xi \implies \forall t, \forall b \in \mathsf{diff}[n]\ w_t^+(b) \geq w(b).$$

1.4 Properties of rad_t

Lemma 6. *Let* $c \in \mathsf{diff}[n], d \in \mathsf{diff}[n]$*. Then*

$$\mathrm{rad}_t(c \backslash d) = \mathrm{rad}_t(c) - \mathrm{rad}_t(c \cap d). \tag{6}$$

Proof. Let $c = (c_+, c_-)$ and $d = (d_+, d_-)$. We have

$$\begin{aligned}\mathrm{rad}_t(c \backslash d) &= \mathrm{rad}_t(c_+ \backslash d_+) + \mathrm{rad}_t(c_- \backslash d_-)\\ &= \mathrm{rad}_t(c_+) - \mathrm{rad}_t(c_+ \cap d_+) + \mathrm{rad}_t(c_-) - \mathrm{rad}_t(c_- \cap d_-)\\ &= \mathrm{rad}_t(c) - \mathrm{rad}_t(c \cap d).\end{aligned}$$

□

Lemma 7. *Let* $C = (C_+, C_-)$ *and* $D = (D_+, D_-)$ *be two diff-sets. If* $D \prec C$*, then*

$$\mathrm{rad}_t(C \oplus D) = \mathrm{rad}_t(C) + \mathrm{rad}_t(D) - 2\,\mathrm{rad}_t(C_+ \cap D_-) - 2\,\mathrm{rad}_t(C_- \cap D_+).$$

In addition, if $\neg D \prec C$*, then*

$$\mathrm{rad}_t(C \ominus D) = \mathrm{rad}_t(C) + \mathrm{rad}_t(D) - 2\,\mathrm{rad}_t(C_+ \cap D_+) - 2\,\mathrm{rad}_t(C_- \cap D_-).$$

Proof. We prove the first part of the lemma. The second part follows from the first part and the definition of $\neg D$.

By definition, we have $C \oplus D = ((C_+ \cup D_+) \backslash (C_- \cup D_-), (C_- \cup D_-) \backslash (C_+ \cup D_+))$. Hence, we have

$$\begin{aligned}\mathrm{rad}_t((C_+ \cup D_+) \backslash (C_- \cup D_-)) &= \mathrm{rad}_t(C_+ \cup D_+) - \mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)) \qquad (7)\\ &= \mathrm{rad}_t(C_+) + \mathrm{rad}_t(D_+) - \mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)), \qquad (8)\end{aligned}$$

where the second equality holds due to $C_+ \cap D_+ = \emptyset$ by the definition of $D \prec C$.

Similarly, we have

$$\mathrm{rad}_t((C_- \cup D_-) \backslash (C_+ \cup D_+)) = \mathrm{rad}_t(C_-) + \mathrm{rad}_t(D_-) - \mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)).$$

Combine both equalities, we have

$$\mathrm{rad}_t(C \oplus D) = \mathrm{rad}_t((C_+ \cup D_+) \backslash (C_- \cup D_)) + \mathrm{rad}_t((C_- \cup D_-) \backslash (C_- \cap D_+)) \tag{9}$$

$$= \mathrm{rad}_t(C_+) + \mathrm{rad}_t(D_+) + \mathrm{rad}_t(C_-) + \mathrm{rad}_t(D_-) - 2\,\mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)) \quad (10)$$
$$= \mathrm{rad}_t(C) + \mathrm{rad}_t(D) - 2\,\mathrm{rad}_t((C_+ \cup D_+) \cap (C_- \cup D_-)). \quad (11)$$

□

2 Algorithm and Main Results

2.1 Algorithm

1. Input Parameter: $\delta \in (0,1)$.
2. For $t = 1, \ldots,$
3. Maintain $\bar{w}_t$ and rad_t.
4. Let $M_t = \arg\max_{M \in \mathcal{M}} \bar{w}_t(M)$.
5. Let $D = \arg\max_{C \in \mathsf{diff}[n], C \prec M_t} w_t^+(C)$.
6. If $w_t^+(D) \leq 0$. Then stop and return M_t.
7. Otherwise, find $p_t = \arg\min_{e \in D} \mathrm{rad}_t(e)$.
8. Play p_t and observe outcome x_t.
9. Go back to step 2.

The step 5 of above procedure can be implemented by:

1. Let $M_t^+ = \arg\max_{M \in \mathcal{M}} \tilde{w}_t(M)$, where $\tilde{w}_t$ is a weight function defined by:
   (a) $\forall e \in M_t$, $\tilde{w}_t(e) = \bar{w}_t(e) - \mathrm{rad}_t(e)$.
   (b) $\forall e \notin M_t$, $\tilde{w}_t(e) = \bar{w}_t(e) + \mathrm{rad}_t(e)$.
2. $D = M_t^+ \ominus M_t$

2.2 Main result

Definition 17 (Optimal diff-sets)**.** *Given a diff-set class $\mathcal{B}$ and the optimal set M_*. We define $\mathcal{B}_{\mathsf{opt}}$ as a subset of $\mathcal{B}$, and for all $b \in \mathcal{B}$, $b \in \mathcal{B}_{\mathsf{opt}}$ if and only if, there exists $M \neq M_*$ and $M_* \ominus M$ can be decomposed as $b, b_1, \ldots, b_k$ on $\mathcal{B}$.*

Definition 18 (Hardness Δ_e of base arm e)**.** *For each $e \in [n]$, we define its hardness Δ_e as follows*

$$\Delta_e = \min_{b \in \mathcal{B}_{\mathsf{opt}}, e \in b} \frac{1}{\mathrm{rank}(\mathcal{B})} w(b).$$

Definition 19 (Sufficient exploration)**.** *For all $t > 0$, we define $E_t^3 \subseteq [n]$, such that, for all $e \in [n]$ $e \in E_t^3$ if and only if $\mathrm{rad}_t(e) < \frac{1}{3}\Delta_e$.*

Corollary 2. *For all $t > 0$ and $e \in [n]$*

$$n_t(e) \geq O(\frac{1}{\Delta_e^2} \log(\Delta_e n/\delta)) \implies e \in E_t^3.$$

Theorem 1. *With probability at least $1-\delta$, the algorithm returns M_*, and the number of samples used by the algorithm are at most*

$$\sum_{e \in [n]} \Delta_e^{-2} \log(\Delta_e n/\delta).$$

3 Proof of Main Results

Unless specified, we shall assume the random event ξ (defined in Lemma 5) holds in all the following proofs.

Lemma 8. *For any $t > 0$, if the algorithm terminates on round t, then $M_t = M_*$.*

Proof. Suppose $M_t \neq M_*$. Then $w(M_*) > w(M_t)$. Then, there exists $b \in \mathcal{B}$ such that $b \prec M_t$ and $w(b) > 0$. On the other hand, by Corollary 1, we have $w_t^+(b) > w(b)$. Hence $w_t^+(b) > 0$. This contradicts to the stopping condition of our algorithm. □

Lemma 9. *For any $t > 0$. If $e \in E_t^3$, then $p_t \neq e$.*

Proof. Suppose that $p_t = e$. Let $D = M_t^+ \ominus M_t$. Let $c, c_1, \ldots, c_k$ be decomposition of D on $\mathcal{B}$. And since $\mathcal{B}$ is a diff-set class, such decomposition exists. Assume, without loss of generality, that $e \in c$.

By Lemma Y, we know that

$$D_+ = c_+ \cup c_1^+ \cup \ldots \cup c_k^+ \quad \text{and} \quad D_- = c_- \cup c_1^- \cup \ldots c_k^-. \tag{12}$$

We also denote $K = \mathrm{rank}(\mathcal{B})$.

Case (1). Suppose that $c \in \mathcal{B}_{\mathsf{opt}}$. Then $w(c) > 0$. Since $e \in E_t^3$, we have $\mathrm{rad}_t(e) \leq \frac{1}{3}\Delta_e \leq \frac{1}{3K}w(c)$. In addition, $\forall g \in c_t, g \neq e$, $\mathrm{rad}_t(g) \leq \mathrm{rad}_t(e) \leq \frac{1}{3K}w(c)$. Hence, $\mathrm{rad}_t(c) = \sum_{g \in c_t} \mathrm{rad}_t(g) \leq \frac{|c_t|}{3K}w(c) \leq \frac{1}{3}w(c)$.

Hence, $\bar{w}_t(c) \geq w(c) - \mathrm{rad}_t(c) \geq \frac{2}{3}w(c) > 0$. This means that $\bar{w}_t(M_t \oplus c) = \bar{w}_t(M_t) + \bar{w}_t(c) > \bar{w}_t(M_t)$. Therefore, $M_t \neq \max_{M \in \mathcal{M}} \bar{w}_t(M)$. This contradicts to the definition of M_t.

Case (2). Suppose that $c_t \notin \mathcal{B}_{\mathsf{opt}}$. Then, one of the following mutually exclusive cases must hold.

Case (2.1). $(e \in M_* \wedge e \in c_+)$ or $(e \notin M_* \wedge e \in c_-)$.

Let the decomposition of $M_* \ominus (M_t \oplus D \ominus c)$ on $\mathcal{B}$ be $b, b_1, \ldots, b_l$, which exists due to $\mathcal{B}$ is a diff-set class. Assume wlog that $e \in b$. We write $b = (b_+, b_-)$. It is easy to see that $b \in \mathcal{B}_{\mathsf{opt}}$.

Define $\tilde{D} = (M_t \oplus D \ominus c) \ominus M_t$ and $D' = (M_t \oplus \tilde{D} \oplus b) \ominus M_t$. By Lemma 2, we know that $\tilde{D} = D \ominus c$ and $D' = \tilde{D} \oplus b$. We also write $\tilde{D} = (\tilde{D}_+, \tilde{D}_-)$ and $D' = (D'_+, D'_-)$. By definition, we have

$$\begin{aligned}
\tilde{D}_+ &= (D_+ \cup c_-) \backslash (D_- \cup c_+) \\
&= (D_+ \cup c_- \backslash D_-) \cap (D_+ \cup c_- \backslash c_+) \\
&= D_+ \cap (D_+ \backslash c_-) \\
&= D_+ \backslash c_+.
\end{aligned}$$

By the same method, we are able to show that $\tilde{D}_- = D_- \backslash c_-$. Therefore we have

$$\tilde{D}_+ \subseteq D_+ \quad \text{and} \quad \tilde{D}_- \subseteq D_-. \tag{13}$$

First, we show that $\mathrm{rad}_t(c) \leq \frac{1}{3}w(b)$. Since $e \in E_t^3$, $e \in b$ and $b \in \mathcal{B}_{\mathsf{opt}}$, we have $\mathrm{rad}_t(e) \leq \frac{1}{3}\Delta_e \leq \frac{1}{3K}w(b)$. In addition, $\forall g \in c, g \neq e$, $\mathrm{rad}_t(g) \leq \mathrm{rad}_t(e) \leq \frac{1}{3K}w(b)$. Hence,

$$\begin{aligned}
\mathrm{rad}_t(c) &= \sum_{g \in c} \mathrm{rad}_t(g) \\
&\leq \frac{|c|}{3K}w(b) \\
&\leq \frac{1}{3}w(b).
\end{aligned} \tag{14}$$

Now, we show that $\mathrm{rad}_t(\tilde{D}_+ \cap b_-) + \mathrm{rad}_t(\tilde{D}_- \cap b_+) + \leq \frac{1}{3}w(b)$. Since Eq. (13), we have $\forall g \in (\tilde{D}_+ \cap b_-) \cup$

$(\tilde{D}_- \cap b_+), g \neq e$, $\mathrm{rad}_t(g) \leq \mathrm{rad}_t(e) \leq \frac{1}{3K} w(b)$. Note that $|\tilde{D}_+ \cap b_-| + |\tilde{D}_- \cap b_+| \leq |b_+| + |b_-| \leq K$. Hence,

$$\begin{aligned}
\mathrm{rad}_t(\tilde{D}_+ \cap b_-) + \mathrm{rad}_t(\tilde{D}_- \cap b_+) &= \sum_{g \in (\tilde{D}_+ \cap b_-) \cup (\tilde{D}_- \cap b_+)} \mathrm{rad}_t(g) \\
&\leq \frac{K}{3K} w(b) \\
&\leq \frac{1}{3} w(b). \qquad (15)
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mathrm{rad}_t(D') - \mathrm{rad}_t(D) &= \mathrm{rad}_t(\tilde{D} \oplus b) - \mathrm{rad}_t(D) && (16)\\
&= \mathrm{rad}_t(\tilde{D}) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) && (17)\\
&= \mathrm{rad}_t(D \ominus c) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) && (18)\\
&= \mathrm{rad}_t(D) + \mathrm{rad}_t(c) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(D_+ \cap c_+) - 2\,\mathrm{rad}_t(D_- \cap c_-) \\
&\quad - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) && (19)\\
&= \mathrm{rad}_t(D) + \mathrm{rad}_t(c) + \mathrm{rad}_t(b) - 2\,\mathrm{rad}_t(c_+) - 2\,\mathrm{rad}_t(c_-) \\
&\quad - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) - \mathrm{rad}_t(D) && (20)\\
&= \mathrm{rad}_t(b) - \mathrm{rad}_t(c) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+), && (21)
\end{aligned}$$

where Eq. (17) and Eq. (19) follow from Lemma 7, and Eq. (20) follows from Eq. (12).

By the definition of D, we have that $w_t^+(D) \geq w_t^+(D')$. This means that

$$\begin{aligned}
\bar{w}_t(D) + \mathrm{rad}_t(D) &\geq \bar{w}_t(D') + \mathrm{rad}_t(D') && (22)\\
&= \bar{w}_t(D) - \bar{w}_t(c) + \bar{w}_t(b) + \mathrm{rad}_t(D'). && (23)
\end{aligned}$$

By regrouping the above inequality, we have

$$\begin{aligned}
\bar{w}_t(c) &\geq \bar{w}_t(b) + \mathrm{rad}_t(D') - \mathrm{rad}_t(D) && (24)\\
&= \bar{w}_t(b) + \mathrm{rad}_t(b) - \mathrm{rad}_t(c) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) && (25)\\
&\geq w(b) - \mathrm{rad}_t(c) - 2\,\mathrm{rad}_t(\tilde{D}_+ \cap b_-) - 2\,\mathrm{rad}_t(\tilde{D}_- \cap b_+) && (26)\\
&> w(b) - \frac{1}{3} w(b) - \frac{2}{3} w(b) && (27)\\
&= 0, && (28)
\end{aligned}$$

where Eq. (27) follows from Eq. (14) and Eq. (15).

This contradicts to the definition of M_t.

Case (2.2). $(e \in M_* \wedge e \in c_-)$ or $(e \notin M_* \wedge e \in c_+)$.

Let the decomposition of $M_* \ominus (M_t \oplus D)$ on $\mathcal{B}$ be $b, b_1, \ldots, b_l$. Assume wlog that $e \in b$. We write that $b = (b_+, b_-)$. Note that $b \in \mathcal{B}_{\mathsf{opt}}$ and hence $w(b) > 0$.

Define $D' = (M_t \oplus D \oplus b) \ominus M_t$. By Lemma 2, we know that $D' = D \oplus b$.

First, we show that $|D \backslash D'| \leq |b|$. Let $C = D \backslash D'$ and write $C = (C_+, C_-)$. We can bound $|C_+|$ as follows.

$$\begin{aligned}
C_+ &= D_+ \backslash D'_+ \\
&= D_+ \backslash \left((D_+ \cup b_+) \backslash (D_- \cup b_-) \right) \\
&= (D_+ \cap (D_- \cup b_-)) \cup (D_+ \backslash (D_+ \cup b_+)) \\
&= D_+ \cap b_-.
\end{aligned}$$

Hence, we have $|C_+| \leq |b_-|$. Then, we move to bounding $|C_-|$

$$\begin{aligned} C_- &= D_- \backslash D'_- \\ &= D_- \backslash \left((D_- \cup b_-) \backslash (D_+ \cup b_+) \right) \\ &= (D_- \cap (D_+ \cup b_+)) \cup (D_- \backslash (D_- \cup b_-)) \\ &= D_- \cap b_+. \end{aligned}$$

Thus $|C_-| \leq |b_+|$ and we proved that $|D \backslash D'| \leq |b|$.

Next, we show that $\mathrm{rad}_t(D \backslash D') \leq \frac{1}{3} w(b)$. Since $e \in E_t^3$, $e \in b$ and $b \in \mathcal{B}_{\mathsf{opt}}$, we have $\mathrm{rad}_t(e) \leq \frac{1}{3} \Delta_e \leq \frac{1}{3K} w(b)$. In addition, $\forall g \in (D \backslash D'), g \neq e$, $\mathrm{rad}_t(g) \leq \mathrm{rad}_t(e) \leq \frac{1}{3K} w(b)$. Note that $|D \backslash D'| \leq |b| \leq K$. Hence, $\mathrm{rad}_t(D \backslash D') = \sum_{g \in (D \backslash D')} \mathrm{rad}_t(g) \leq \frac{K}{3K} w(b) \leq \frac{1}{3} w(b)$.

We also note that

$$\begin{aligned} w(D' \backslash D) - w(D \backslash D') &= w(D' \backslash D) + w(D' \cap D) - w(D \cap D') - w(D \backslash D') && (29) \\ &= w(D') - w(D) && (30) \\ &= w(b), && (31) \end{aligned}$$

where we have repeatedly applied Lemma 4.

Then, we show that $w_t^+(D') > w_t^+(D)$.

$$\begin{aligned} w_t^+(D') - w_t^+(D) &= \bar{w}_t(D') - \bar{w}_t(D) + \mathrm{rad}_t(D') - \mathrm{rad}_t(D) && (32) \\ &= \bar{w}_t(D' \backslash D) - \bar{w}_t(D \backslash D') + \mathrm{rad}_t(D' \backslash D) - \mathrm{rad}_t(D \backslash D') && (33) \\ &\geq w(D' \backslash D) - w(D \backslash D') - 2\,\mathrm{rad}_t(D \backslash D') && (34) \\ &= w(b) - 2\,\mathrm{rad}_t(D \backslash D') && (35) \\ &> w(b) - \frac{2}{3} w(b) && (36) \\ &= \frac{1}{3} w(b) > 0, && (37) \end{aligned}$$

where Eq. (33) follows from Lemma 6 and Eq. (34) follows from the fact that $\bar{w}_t(D' \backslash D) + \mathrm{rad}_t(D' \backslash D) \geq w(D' \backslash D)$ and that $\bar{w}_t(D \backslash D') + \mathrm{rad}_t(D \backslash D') \geq w(D \backslash D')$, under the random event ξ.

This contradicts to the fact that D is chosen on round t. ☐

4 Lower bounds

Definition 20 (Hardness of arm)**.** *Given $\mathcal{M}$, M_* and w. For any $e \in [n]$, we define its hardness Δ_e as follows*

$$\Delta_e = \begin{cases} \min_{M: e \in M, M \in \mathcal{M}} w(M_*) - w(M) & \text{if } e \notin M_*, \\ \min_{M: e \notin M, M \in \mathcal{M}} w(M_*) - w(M) & \text{if } e \in M_*. \end{cases}$$

Lemma 10.

$$\Delta_e = \min_{b: e \in b, b \in \mathcal{B}_{\mathsf{opt}}} w(b).$$

Theorem 2.

$$\mathbb{E}[T] \geq \Omega\left(\sum_e \Delta_e^{-2} \log(1/\delta) \right).$$

Lemma 11.

$$\mathbb{E}[T_e] \geq c_1 \Delta_e^{-2} \log(1/\delta).$$

Proof. Given $\vec{p} \in [0,1]^n$.

Consider two hypothesis H_0 and H_1. Under hypothesis H_0, the true bias of each arm (coin) is

$$H_0 : q_l = p_l, \quad \text{for all } l \in [n].$$

And under hypothesis H_1, the true bias of each arm is

$$H_1 : q_e = \begin{cases} p_e - 2\Delta_e & e \in M_* \\ p_e + 2\Delta_e & e \notin M_* \end{cases}, \quad q_l = p_l \quad \text{for all } l \neq e.$$

Define M_e be the "next-to-optimal" set as follows

$$M_e = \begin{cases} \arg\max_{M \in \mathcal{M}: e \in M} w(M) & e \notin M_*, \\ \arg\max_{M \in \mathcal{M}: e \notin M} w(M) & e \in M_*. \end{cases}$$

By definition, we know that $w(M_*) - w(M_e) = \Delta_e$.

Let w_0, w_1 be the weighting functions under H_0, H_1 respectively. Notice that $w_0(M_*) - w_0(M_e) = \Delta_e > 0$. On the other hand, $w_1(M_*) - w_1(M_e) = -\Delta < 0$. This means that under H_1, M_* is not the optimal set.

For $l \in \{0,1\}$, we use $\mathbb{E}_l$ and $\Pr_l$ to denote the expectation and probability, respectively, under the hypothesis H_l.

Define $\theta = 8\delta$. Let

$$t_e^* = \frac{1}{c\Delta_e^2} \log\left(\frac{1}{\theta}\right),$$

where c is a constant whose value will be determined later.

Recall that T_e denotes the total number of samples of arm e. Define the event $\mathcal{A} = \{T_e \leq 4t_e^*\}$.

First, we show that $\Pr_0[\mathcal{A}] \geq 3/4$. This can be proved by Markov inequality as follows.

$$\begin{aligned} \Pr_0[T_e > 4t_e^*] &\leq \frac{\mathbb{E}_0[T_e]}{4t_e^*} \\ &= \frac{t_e^*}{4t_e^*} = \frac{1}{4}. \end{aligned}$$

We define $K_t(e)$ as the sum of outcomes of arm e up to round t, i.e.

$$K_t(e) = \sum_{i \in [t]: p_i = e} x_i.$$

Next, we define the event

$$\mathcal{C} = \left\{ \max_{1 \leq t \leq 4t_e^*} |K_t(e) - p_e t| < \sqrt{t_e^* \log(1/\theta)} \right\}.$$

We now show that $\Pr_0[\mathcal{C}] \geq 3/4$. First, notice that $K_t(e) - p_e t$ is a martingale under H_0. Then, by Kolmogorov's inequality, we have

$$\begin{aligned} \Pr_0\left[\max_{1 \leq t \leq 4t_e^*} |K_t(e) - p_e t| \geq \sqrt{t_e^* \log(1/\theta)}\right] &\leq \frac{\mathbb{E}_0[(K_{4t_e^*}(e) - 4p_e t_e^*)^2]}{t_e^* \log(1/\theta)} \\ &= \frac{4p_e(1-p_e)t_e^*}{t_e^* \log(1/\theta)} \\ &< \frac{1}{4}, \end{aligned}$$

where the second inequality follows from the fact that $\mathbb{E}_0[(K_{4t_e^*}(e) - 4p_e t_e^*)^2] = 4p_e(1-p_e)t_e^*$; the last inequality follows since $\theta < e^{-4}$ and $4p_e(1-p_e) \leq 1$.

Then, we define the event $\mathcal{B}$ as the event that the algorithm eventually returns M_*, i.e.

$$\mathcal{B} = \{O = M_*\}.$$

Since the probability of error of the algorithm is smaller than $\delta < 1/4$, we have $\Pr_0[\mathcal{B}] \geq 3/4$. Define $\mathcal{S}$ be $\mathcal{S} = \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$. Then, by union bound, we have $\Pr_0[\mathcal{S}] \geq 1/4$.

Now, we show that if $\mathbb{E}_0[T_e] \leq t_e^*$, then $\Pr_1[\mathcal{B}] \geq \delta$. Let W be the history of the sampling process until the algorithm stops (including the sequence of arms chosen at each time and the sequence of observed outcomes). Define the likelihood function L_l as

$$L_l(w) = \Pr_l[W = w].$$

Let K be the shorthand of $K_e(T_e)$ and T stands for T_e. Now consider the ratio $L_1(W)/L_0(W)$, we have

$$\begin{aligned}
\frac{L_1(W)}{L_0(W)} &= \frac{(p_e + 2\Delta_e)^K (1 - p_e - 2\Delta_e)^{T-K}}{p_e^K (1-p_e)^{T-K}} \\
&= \left(1 + \frac{2\Delta_e}{p_e}\right)^K \left(1 - \frac{2\Delta_e}{1-p_e}\right)^{T-K} \\
&= \left(1 + \frac{2\Delta_e}{p_e}\right)^K \left(1 - \frac{2\Delta_e}{p_e}\right)^K \left(1 - \frac{2\Delta_e}{p_e}\right)^{-K} \left(1 - \frac{2\Delta_e}{1-p_e}\right)^{T-K} \\
&= \left(1 - \left(\frac{2\Delta_e}{p_e}\right)^2\right)^K \left(1 - \frac{2\Delta_e}{p_e}\right)^{-K} \left(1 - \frac{2\Delta_e}{1-p_e}\right)^{T-K} \\
&= \left(1 - \left(\frac{2\Delta_e}{p_e}\right)^2\right)^K \left(1 - \frac{2\Delta_e}{p_e}\right)^{-K} \left(1 - \frac{2\Delta_e}{1-p_e}\right)^{K(1-p_e)/p_e} \left(1 - \frac{2\Delta_e}{1-p_e}\right)^{(p_e T_e - K)/p_e}. \qquad (38)
\end{aligned}$$

Now, we assume that the event $\mathcal{S}$ occurred and bound each individual terms on the right-hand side of Eq. (38) under this assumption. First, we have

$$\begin{aligned}
\left(1 - \left(\frac{2\Delta_e}{p_e}\right)^2\right)^K &\geq \left(1 - \left(\frac{2\Delta_e}{p_e}\right)^2\right)^{4t_e^*} && (39) \\
&= \left(1 - \left(\frac{2\Delta_e}{p_e}\right)^2\right)^{(4/c\Delta_l^2)\log(1/\theta)} && (40) \\
&\geq \exp\left(-d\frac{4}{c}\left(\frac{1}{p_e}\right)^2 \log(1/\theta)\right) && (41) \\
&= \theta^{4d/p_e^2 c}, && (42)
\end{aligned}$$

where Eq. (39) follows from the fact that $K \leq T_e$ and the assumption that the event $\mathcal{A}$ occurred, i.e. $T_e \leq 4t_e^*$; Eq. (40) uses the definition of t_e; Eq. (41) follows from Lemma X.

Next, we bound the second and third term on the right-hand side of Eq. (38). Notice that $(1-x)^b \geq 1-bx$ for any $b \geq 1$ and $x \in [0,1]$. Therefore,

$$\left(1 - \frac{\Delta_e}{1-p_e}\right)^{(1-p_e)/p_e} \geq \left(1 - \frac{\Delta_e}{p_e}\right).$$

Plugging in the above inequality into the product of the second and third terms, we have

$$\left(1-\frac{2\Delta_e}{p_e}\right)^{-K}\left(1-\frac{2\Delta_e}{1-p_e}\right)^{K(1-p_e)/p_e} \geq \left(1-\frac{2\Delta_e}{p_e}\right)^{-K}\left(1-\frac{\Delta_e}{p_e}\right)^{K} = 1. \tag{43}$$

Finally, we bound the last term of Eq. (38). We have

$$\begin{aligned}
\left(1-\frac{\Delta_e}{1-p_e}\right)^{(p_eT_e-K)/p_e} &\geq \left(1-\frac{\Delta_e}{1-p_e}\right)^{(1/p_e)\sqrt{t_e^*\log(1/\theta)}} && (44)\\
&= \left(1-\frac{\Delta_e}{1-p_e}\right)^{(1/p_e\sqrt{c}\Delta_e)\log(1/\theta)} && (45)\\
&\geq \exp\left(-\frac{d}{\sqrt{c}p_e\Delta_e}\frac{\Delta_e}{1-p_e}\log(1/\theta)\right) && (46)\\
&= \exp\left(-\frac{d}{\sqrt{c}p_e(1-p_e)}\log(1/\theta)\right) \\
&\geq \exp\left(-\frac{2d}{\sqrt{c}p_e}\log(1/\theta)\right) && (47)\\
&= \theta^{2d/(p_e\sqrt{c})}, && (48)
\end{aligned}$$

where Eq. (44) follows from the assumption that the event $\mathcal{C}$ occurred; Eq. (45) follows from the definition of t_e^*; Eq. (46) follows from Lemma X; Eq. (47) follows from the assumption that $p_e \leq 1/2$.

Combining Eq. (42), Eq. (43) and Eq. (48), we can finally bound the right-hand side of Eq. (38) as follows

$$\begin{aligned}
\frac{L_1(W)}{L_0(W)} &\geq \theta^{4d/p_e^2c+2d/(p_e\sqrt{c})} && (49)\\
&\geq \theta = 8\delta. && (50)
\end{aligned}$$

where c is a sufficiently large constant such that $\theta^{4d/p_e^2c+2d/(p_e\sqrt{c})} > \theta$.

Define 1_S as the indicator variable of event $\mathcal{S}$, i.e. $1_S = 1$ if and only if $\mathcal{S}$ occurs and otherwise $1_S = 0$. Then, we have

$$\frac{L_1(W)}{L_0(W)}1_S \geq 8\delta 1_S.$$

Then, we have

$$\begin{aligned}
\Pr_1[\mathcal{B}] \geq \Pr_1[\mathcal{S}] &= \mathbb{E}_1[1_S]\\
&= \mathbb{E}_0\left[\frac{L_1(W)}{L_0(W)}1_S\right]\\
&\geq 8\delta\mathbb{E}_0[1_S]\\
&= 8\delta\Pr_0[\mathcal{S}] > \delta.
\end{aligned}$$

Now we have proved that, if $\mathbb{E}_0[T_e] \leq t_e^*$, then $\Pr_1[\mathcal{B}] > \delta$. This means that, if $\mathbb{E}_0[T_e] \leq t_e^*$, the algorithm will choose M_* as the output with probability at least δ, under hypothesis H_1. However, under H_1, we have shown that M_* is not the optimal set since $w_1(M_e) > w_1(M_*)$. Therefore, the algorithm has a probability of error larger than δ under H_1. Contradiction. □