

Bandits with Knapsacks*

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Abstract

Multi-armed bandit problems are the predominant theoretical model of exploration-exploitation trade-offs in learning, and they have countless applications ranging from medical trials, to communication networks, to Web search and advertising. In many of these application domains the learner may be constrained by one or more supply (or budget) limits, in addition to the customary limitation on the time horizon. The literature lacks a general model encompassing these sorts of problems. We introduce such a model, called “bandits with knapsacks”, that combines aspects of stochastic integer programming with online learning. A distinctive feature of our problem, in comparison to the existing regret-minimization literature, is that the optimal policy for a given latent distribution may significantly outperform the policy that plays the optimal fixed arm. Consequently, achieving sublinear regret in the bandits-with-knapsacks problem is significantly more challenging than in conventional bandit problems.

We present two algorithms whose reward is close to the information-theoretic optimum: one is based on a novel “balanced exploration” paradigm, while the other is a primal-dual algorithm that uses multiplicative updates. Further, we prove that the regret achieved by both algorithms is optimal up to polylogarithmic factors. We illustrate the generality of the problem by presenting applications in a number of different domains including electronic commerce, routing, and scheduling. As one example of a concrete application, we consider the problem of dynamic posted pricing with limited supply and obtain the first algorithm whose regret, with respect to the optimal dynamic policy, is sublinear in the supply.

1 Introduction

For more than fifty years, the multi-armed bandit problem (henceforth, *MAB*) has been the predominant theoretical model for sequential decision problems that embody the tension between exploration and exploitation, “the conflict between taking actions which yield immediate reward and taking actions whose benefit (e.g. acquiring information or preparing the ground) will come only later,” to quote Whittle’s apt summary [41]. Owing to the universal nature of this conflict, it is not surprising that MAB algorithms have found diverse applications ranging from medical trials, to communication networks, to Web search and advertising.

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A common feature in many of these application domains is the presence of one or more limited-supply resources that are consumed during the decision process. For example, scientists experimenting with alternative medical treatments may be limited not only by the number of patients participating in the study but also by the cost of materials used in the treatments. A website experimenting with displaying advertisements is constrained not only by the number of users who visit the site but by the advertisers’ budgets. A retailer engaging in price experimentation faces inventory limits along with a limited number of consumers. The literature on MAB problems lacks a general model that encompasses these sorts of decision problems with supply limits. Our paper contributes such a model, and it presents algorithms whose regret (normalized by the payoff of the optimal policy) converges to zero as the resource budget and the optimal payoff tend to infinity. In fact, we prove that this convergence takes place at the information-theoretically optimal rate.

Problem description: bandits with knapsacks. Our problem formulation, which we call the *bandits with knapsacks* problem (henceforth, BwK), is easy to state. A learner has a fixed set of potential actions, denoted by X and known as *arms*. (In our main results, X will be finite, but we will also consider extensions with an infinite set of arms, see Section 5 and Appendix 8.) Over a sequence of time steps, the learner chooses an arm and observes two things: a *reward* and a *resource consumption vector*. Rewards are scalar-valued whereas resource consumption vectors are d -dimensional. For each resource there is a pre-specified *budget* representing the maximum amount that may be consumed, in total. At the first time τ when the total consumption of some resource exceeds its budget, the process stops.¹ The objective is to maximize the total reward received before time τ .

The conventional MAB problem, with a finite time horizon T , naturally fits into this framework. There is a single resource called “time”, one unit of which is deterministically consumed in each decision period, and the budget is T . A more interesting example is the *dynamic pricing* problem faced by a retailer selling k items to a population of T unit-demand consumers who arrive sequentially. Modeling this as a BwK problem, the arms correspond to the possible prices which may be offered to a consumer. Resource consumption vectors express the number of items sold and consumers seen. Thus, if a price p is offered and accepted, the reward is p and the resource consumption is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If the offer is declined, the reward is 0 and the resource consumption is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Benchmark and regret. We will assume that the data for a fixed arm x in each time step (i.e. the reward and resource consumption vector) are i.i.d. samples from a fixed joint distribution on $[0, 1] \times [0, 1]^d$, called the *latent distribution* for arm x . The performance of an algorithm will be measured by its *regret*: the worst case, over all possible tuples of latent distributions, of the difference between the algorithm’s expected reward and the expected reward of the benchmark: an optimal policy given foreknowledge of the latent distribution. In a conventional MAB problem, the optimal policy given foreknowledge of the latent distribution is to play a fixed arm, namely the one with the highest expected reward. In the BwK problem, the optimal policy for a given distribution is more complex: the choice of optimal arm depends on the remaining supply of each resource. In fact, we doubt there is a polynomial-time algorithm to compute the optimal policy; similar problems in optimal control have long been known to be PSPACE-hard [35].

Nevertheless we are able to bound the regret of our algorithms with respect to the optimal policy, by showing that the reward of the optimal policy is closely approximated by that of the best *time-invariant mixture* of arms, i.e. a policy that samples in each period from a fixed probability distribution over arms

¹More generally we could model the budget constraint as a downward-closed polytope $\mathcal{P} \subset \mathbb{R}_+^d$ such that the process stops when the sum of resource consumption vectors is no longer in \mathcal{P} . However, our assumption that \mathcal{P} is a box constraint is virtually without loss of generality. If \mathcal{P} is instead specified by a system of inequalities $\{Ax \preceq b\}$, we can redefine the resource consumption vectors to be Ax instead of x and then the budget constraint is the box constraint defined by the vector b . The only potential downside of this transformation is that it increases the dimension of the resource vector space, when the constraint matrix A has more rows than columns. However, one of our algorithms has regret depending only logarithmically on d , so this increase typically has only a mild effect on regret.

regardless of the remaining resource supplies, and that maximizes expected reward subject to this constraint. The fact that a mixture of arms may be strictly superior to any fixed arm (see Appendix A) highlights a qualitative difference between the BwK problem and conventional MAB problems,² and it illustrates one reason why the former problem is significantly more difficult to solve.

Our results and techniques. In analyzing MAB algorithms one typically expresses the regret as a function of the time horizon, T . The regret guarantee is considered nontrivial if this function grows sublinearly as $T \rightarrow \infty$. In the BwK problem a regret guarantee of the form $o(T)$ may be unacceptably weak because supply limits prevent the optimal policy from achieving a reward close to T . An illustrative example is the dynamic pricing problem with supply $k \ll T$: the seller can only sell k items, each at a price of at most 1, so bounding the regret by any number greater than k is worthless.

We instead seek regret bounds that are sublinear in OPT, the expected reward of the optimal policy, or (at least) sublinear in M_{LP} , the maximal possible value of OPT given the budget constraints. To achieve sublinear regret, the algorithm must be able to explore each arm a significant number of times without exhausting its resource budget. Accordingly we also assume, for some $B \geq 1$, that the amount of any resource consumed in a single round is guaranteed to be no more than $1/B$ fraction of that resource’s budget, and we parameterize our regret bound by B . We present an algorithm (called PD-BwK) whose regret is sublinear in OPT as both OPT and B tend to infinity. More precisely, denoting the number of arms by m , our algorithm’s regret is

$$\tilde{O} \left(\sqrt{m \text{OPT}} + \text{OPT} \sqrt{m/B} \right). \quad (1)$$

In particular, if all budget constraints (including the time horizon) are scaled up by the factor of $\alpha > 1$, regret scales as $\sqrt{\alpha}$. The algorithm is computationally efficient, in a strong sense: the per-round running time is polynomial in the number of arms and the number of resources. We also present another algorithm (called BalanceBwK) whose regret bound is qualitatively similar; instead of depending on OPT it depends on M_{LP} . The two algorithms use quite different techniques.

Algorithm BalanceBwK explicitly optimizes over mixtures of arms, based on a very simple idea: balanced exploration inside confidence bounds. The design principle underlying confidence-bound based algorithms for stochastic MAB (including the famous UCB1 algorithm [4] and our algorithm PD-BwK) is generally, “Exploit as much as possible, but use confidence bounds that are wide enough to encourage some exploration.” Our algorithm’s design principle, in contrast, could be summarized as, “Explore as much as possible, but use confidence bounds that are narrow enough to eliminate obviously suboptimal alternatives.” In fact, BalanceBwK balances its exploration across arms, so that (essentially) *each arm* is explored as much as possible. More precisely, for each arm, there are designated rounds when, once the obviously suboptimal mixtures of arms are eliminated, the algorithm picks a mixture that approximately maximizes the probability of choosing this arm. Intriguingly, the algorithm matches the logarithmic regret bound of UCB1 for stochastic MAB (up to constant factors) and achieves qualitatively similar bounds to PD-BwK for BwK, despite being based on a design principle that is the polar opposite of those algorithms. We believe that the “balanced exploration” principle underlying BalanceBwK is a novel and important conceptual contribution to the theory of MAB algorithms that is likely to find other applications.

Algorithm PD-BwK is a primal-dual algorithm based on the multiplicative weights update method. It maintains a vector of “resource costs” that is adjusted using multiplicative updates. In every period it estimates each arm’s expected reward and expected resource consumption, using upper confidence bounds for the former and lower confidence bounds for the latter; then it plays the most “cost-effective” arm, namely the one with the highest ratio of estimated resource consumption to estimated resource cost, using the current

²Algorithms for explore-exploit learning problems do not typically improve over the performance of the best fixed action, with a few notable exceptions in [5, 10, 7, 1].

cost vector. Although confidence bounds and multiplicative updates are the bread and butter of online learning theory, we consider this way of combining the two techniques to be quite novel. In particular, previous multiplicative-update algorithms in online learning theory — such as the `EXP3` algorithm for MAB [5] or the weighted majority [30] and Hedge [18] algorithms for learning from expert advice — applied multiplicative updates to the probabilities of choosing different arms (or experts). Our application of multiplicative updates to the dual variables of the LP relaxation of BwK is conceptually quite a different usage of this technique.

Further, we provide a matching lower bound: we prove that regret (1) is optimal up to polylog factors. Specifically, we show that any algorithm for BwK must incur regret

$$\Omega \left(\min \left(\text{OPT}, \text{OPT} \sqrt{m/B} + \sqrt{m \text{OPT}} \right) \right), \quad (2)$$

in the worst-case over all instances of BwK with given (m, B, OPT) . We derive this lower bound using a simple example in which all arms have reward 1 and 0-1 consumption of a single resource, and one arm has slightly smaller expected resource consumption than the rest. To analyze this example, we apply the KL-divergence technique from the MAB lower bound in [5]. Some technical difficulties arise (compared to the derivation in [5]) because the arms are different in terms of the expected consumption rather than expected reward, and because we need to match the desired value for OPT .

Applications and special cases. Due to its generality, the BwK problem admits applications in diverse domains such as dynamic pricing, dynamic procurement, ad allocation, repeated auctions, network routing, and scheduling. These applications are discussed in Section 5; below we provide some highlights.

The BwK setting subsumes dynamic pricing with limited supply, as studied in [11, 26, 9, 6, 10].³ Specializing our regret bounds to this setting, we obtain the optimal regret $\tilde{O}(k^{2/3})$ with respect to the optimal policy, where k is the number of items. The prior work [6] achieved the same regret bound with respect to the best fixed price, which is a much weaker benchmark, and proved a matching lower bound. Further, our setting allows to incorporate a number of generalizations such as selling multiple products (with a limited supply of each), volume pricing, pricing bundles of goods, and profiling of buyers according to their types. In particular, we improve over the result in [10] for multiple products.

A “dual” problem to dynamic pricing is *dynamic procurement* [7, 37], where the algorithm is “dynamically buying” rather than “dynamically selling”. (The budget constraint now applies to the amount spent rather than the quantity of items sold, which is why the problems are not merely identical up to sign reversal.) This problem is particularly relevant to the emerging domain of *crowdsourcing*: the items “bought” then correspond to microtasks ordered on a crowdsourcing platform such as Amazon Turk. We obtain significant improvements for the basic version of dynamic procurement studied in [7, 37]. In particular, [7] achieves a constant-factor approximation to the optimum with a prohibitively large constant (at least in the tens of thousands), whereas our approximation factor $1 + \tilde{O}((T/\text{OPT})B^{-1/4})$ is typically much smaller. The regret bound in [37] is against the best-fixed-price benchmark, which may be much smaller than OPT (see Appendix A).⁴ Furthermore, the generality of BwK allows to incorporate generalizations such as multiple types of items/microtasks, and the presence of other, competing algorithms.

Further, BwK applies to the problem of dynamic ad allocation, in the context of pay-per-click advertising with unknown click probabilities. It allows to extend a standard (albeit idealized) model of ad allocation to incorporate advertisers’ budgets. In fact, BwK allows advertisers to specify multiple budget constraints on possibly overlapping subsets of ads.

³The Bayesian version of this problem has a rich literature in Operations Research, see [9] for an overview. The earlier papers [11, 26] on the regret-minimizing version focus on the special case of unlimited supply.

⁴The paper [37] is simultaneous with respect to our paper. One cannot directly compare our regret bound and theirs (benchmarks aside), because [37] does not derive a worst-case regret bound.

Discussion. If the BwK instance includes multiple, different budget constraints, our regret bounds are with respect to the *smallest* of these constraints. This, however, is inevitable for the worst-case regret bounds.

Algorithm `BalanceBwK` is *domain-aware*, in the sense that it explicitly optimizes over all latent distributions that are feasible for a given BwK domain (such as dynamic pricing with limited supply). We do not provide a computationally efficient implementation for this optimization. In fact, it is not clear what model of computation would be appropriate to characterize access to the domain knowledge. Efficient implementation of `BalanceBwK` may be possible for some specific BwK domains, although we do not pursue that direction in this paper. (Recall that `PD-BwK`, on the other hand, is very computationally efficient.)

It is worth noting that our regret bounds for `BalanceBwK` and `PD-BwK` are incomparable: while `PD-BwK` achieves a stronger general bound, `BalanceBwK` performs better on some important special cases by virtue of being domain-aware. Specifically, it leads to a stronger regret bound of $\tilde{O}(\sqrt{k})$ for dynamic pricing with supply k under a monotone hazard rate assumption. (We omit the details from this version.)

Open questions. While we prove that the regret bound for `PD-BwK` is optimal up to logarithmic factors, improved results may be possible for various special cases, especially ones involving discretization over a multi-dimensional action space.

The study of multi-armed bandit problems with large strategy sets has been a very fruitful line of investigation. It seems likely that some of the techniques introduced here could be wedded with the techniques from that literature. In particular, it would be intriguing to try combining our primal-dual algorithm `PD-BwK` with confidence-ellipsoid algorithms for stochastic linear optimization (e.g. see [14]), or enhancing the `BalanceBwK` algorithm with the technique of adaptively refined discretization, as in the zooming algorithm of [27].

It is tempting to ask about the *adversarial* version of BwK. However, achieving sublinear regret bounds for such version appears hopeless even for fixed-arm benchmark. In order to make progress in the positive direction, one may require a more subtle notion of benchmark, and perhaps also some restrictions on the power of the adversary.

Related work. The study of prior-free algorithms for stochastic MAB problems was initiated by [28, 4]. Subsequent work supplied algorithms for stochastic MAB problems in which the set of arms can be infinite and the payoff function is linear, concave, or Lipschitz-continuous; see a recent survey [12] for more background. Confidence bound techniques have been an integral part of this line of work, and they remain integral to ours.

As explained earlier, stochastic MAB problems constitute a very special case of bandits with knapsacks, in which there is only one type of resource and it is consumed deterministically at rate 1. Several papers have considered the natural generalization in which there is a single resource, with deterministic consumption, but different arms consume the resource at different rates. Guha and Munagala [20] gave a constant-factor approximation algorithm for the Bayesian case of this problem, which was later generalized by Gupta et al. [22] to settings in which the arms' reward processes need not be martingales. Tran-Thanh et al. [38, 39, 40] presented prior-free algorithms for this problem; the best such algorithm achieves a regret guarantee qualitatively similar to that of the UCB1 algorithm.

As discussed above, several recent papers studied dynamic pricing with limited supply [9, 6, 10] and dynamic procurement on a budget [7, 37] which, in hindsight, can be cast as special cases of BwK featuring a two-dimensional resource constraint. We significantly improve over their results. Another recent paper [13] considers a special case of adjusting a repeated auction (albeit without inventory constraints).

While BwK is primarily an online learning problem, it also has elements of a stochastic packing problem. The literature on prior-free algorithms for stochastic packing has flourished in recent years, starting with prior-free algorithms for the stochastic AdWords problem [15], and continuing with a series of papers extending these results from AdWords to more general stochastic packing integer programs while also

achieving stronger performance guarantees [2, 16, 17, 32]. A running theme of these papers (and also of the primal-dual algorithm in this paper) is the idea of estimating of an optimal dual vector from samples, then using this dual to guide subsequent primal decisions. Particularly relevant to our work is the algorithm of [16], in which the dual vector is adjusted using multiplicative updates, as we do in our algorithm. However, unlike the BwK problem, the stochastic packing problems considered in prior work are not learning problems: they are full information problems in which the costs and rewards of decisions in the past and present are fully known. (The only uncertainty is about the future.) As such, designing algorithms for BwK requires a substantial departure from past work on stochastic packing. Our primal-dual algorithm depends upon a hybrid of confidence-bound techniques from online learning and primal-dual techniques from the literature on solving packing LPs; combining them requires entirely new techniques for bounding the magnitude of the error terms that arise in the analysis. Moreover, our `BalanceBwK` algorithm manages to achieve strong regret guarantees without even computing a dual solution.

2 Preliminaries

BwK: problem formulation. There is a fixed and known, finite set of m *arms* (possible actions), denoted X . There are d resources being consumed. In each round t , an algorithm picks an arm $x_t \in X$, receives reward $r_t \in [0, 1]$, and consumes some amount $c_{t,i} \in [0, 1]$ of each resource i . The values r_t and $c_{t,i}$ are revealed to the algorithm after the round. There is a hard constraint $B_i \in \mathbb{R}_+$ on the consumption of each resource i ; we call it a *budget* for resource i . The algorithm stops at the earliest time τ when one or more budget constraint is violated; its total reward is equal to the sum of the rewards in all rounds strictly preceding τ . The goal of the algorithm is to maximize the expected total reward.

The vector $(r_t; c_{t,1}, c_{t,2}, \dots, c_{t,d}) \in [0, 1]^{d+1}$ is called the *outcome vector* for round t . We assume *stochastic outcomes*: if an algorithm picks arm x , the outcome vector is chosen independently from some fixed distribution π_x over $[0, 1]^{d+1}$. The distributions π_x , $x \in X$ are latent. The tuple $(\pi_x : x \in X)$ comprises all latent information in the problem instance. A particular BwK setting (such as “dynamic pricing with limited supply”) is defined by the set of all feasible tuples $(\pi_x : x \in X)$. This set, called the *BwK domain*, is known to the algorithm.

We will assume that there is a fixed *time horizon* T , known in advance to the algorithm, such that the process is guaranteed to stop after at most T rounds. One way of assuring this is to assume that there is a specific resource, say resource 1, such that every arm deterministically consumes B_1/T units whenever it is picked. We make this assumption henceforth. Without loss of generality, $B_i \leq T$ for every resource i .

For technical convenience, we assume there exists a *null arm*: an arm with 0 reward and 0 consumption. Equivalently, an algorithm is allowed to spend a unit of time without doing anything.

Benchmark. We compare the performance of our algorithms to the expected total reward of the optimal dynamic policy given all the latent information, which we denote by OPT . Note that OPT depends on the latent structure μ , and therefore is a latent quantity itself. Time-invariant policies — those which use the same distribution \mathcal{D} over arms in all rounds — will also be relevant to the analysis of one of our algorithms. Let $\text{REW}(\mathcal{D}, \mu)$ denote the expected total reward of the time-invariant policy that uses distribution \mathcal{D} .

Uniform budgets. We say that the budgets are *uniform* if $B_i = B$ for each resource i . Any BwK instance can be reduced to one with uniform budgets by dividing all consumption values for every resource i by B_i/B , where $B = \min_i B_i$. (That is tantamount to changing the units in which we measure consumption of resource i .) Our technical results are for BwK with uniform budgets. We will assume uniform budgets B from here on.

Useful notation. Let $\mu_x = \mathbb{E}[\pi_x] \in [0, 1]^{d+1}$ be the expected outcome vector for each arm x , and denote

$\mu = (\mu_x : x \in X)$. We call μ the *latent structure* of a problem instance. The BwK domain induces a set of feasible latent structures, which we denote $\mathcal{M}_{\text{feas}}$.

For notational convenience, we will write $\mu_x = (r(x, \mu); c_1(x, \mu), \dots, c_d(x, \mu))$. Also, we will write the expected consumption as a vector $c(x, \mu) = (c_1(x, \mu), \dots, c_d(x, \mu))$.

If \mathcal{D} is a distribution over arms, let $r(\mathcal{D}, \mu) = \sum_{x \in X} \mathcal{D}(x) r(x, \mu)$ and $c(\mathcal{D}, \mu) = \sum_{x \in X} \mathcal{D}(x) c(x, \mu)$ be, respectively, the expected reward and expected resource consumption in a single round if an arm is sampled from distribution \mathcal{D} .

When discussing our primal-dual algorithm, it will be useful to represent the latent values and the algorithm's decisions as matrices and vectors. For this purpose, we will number the arms as x_1, \dots, x_m and let $r \in \mathbb{R}^m$ denote the vector whose j^{th} component is $r(x_j, \mu)$. Similarly we will let $C \in \mathbb{R}^{d \times m}$ denote the matrix whose $(i, j)^{\text{th}}$ entry is $c_i(x_j, \mu)$.

2.1 High-probability events

We will use the following expression, which we call the *confidence radius*.

$$\text{rad}(\nu, N) = \sqrt{\frac{C_{\text{rad}} \nu}{N}} + \frac{C_{\text{rad}}}{N}. \quad (3)$$

Here $C_{\text{rad}} = \Theta(\log(dT|X|))$ is a parameter which we will fix later; we will keep it implicit in the notation. The meaning of Equation (3) and C_{rad} is explained by the following tail inequality from [27, 6].⁵

Theorem 2.1 ([27, 6]). *Consider some distribution with values in $[0, 1]$ and expectation ν . Let $\hat{\nu}$ be the average of N independent samples from this distribution. Then*

$$\Pr[|\nu - \hat{\nu}| \leq \text{rad}(\hat{\nu}, N) \leq 3 \text{rad}(\nu, N)] \geq 1 - e^{-\Omega(C_{\text{rad}})}, \quad \text{for each } C_{\text{rad}} > 0. \quad (4)$$

More generally, Equation (4) holds if $X_1, \dots, X_N \in [0, 1]$ are random variables, $\hat{\nu} = \frac{1}{N} \sum_{t=1}^N X_t$ is the sample average, and $\nu = \frac{1}{N} \sum_{t=1}^N \mathbb{E}[X_t | X_1, \dots, X_{t-1}]$.

If the expectation ν is a latent quantity, Equation (4) allows us to estimate ν by a high-confidence interval

$$\nu \in [\hat{\nu} - \text{rad}(\hat{\nu}, N), \hat{\nu} + \text{rad}(\hat{\nu}, N)], \quad (5)$$

whose endpoints are observable (known to the algorithm). For small ν , such estimate is much sharper than the one provided by Azuma-Hoeffding inequality.⁶ Our algorithm uses several estimates of this form. For brevity, we say that $\hat{\nu}$ is an *N -strong estimate* of ν if $\nu, \hat{\nu}$ satisfy Equation (5).

It is sometimes useful to argue about *any* ν which lies in the high-confidence interval (5), not just the latent $\nu = \mathbb{E}[\hat{\nu}]$. We use the following claim which is implicit in [27].

Claim 2.2 ([27]). *For any $\nu, \hat{\nu} \in [0, 1]$, Equation (5) implies that $\text{rad}(\hat{\nu}, N) \leq 3 \text{rad}(\nu, N)$.*

⁵Specifically, this follows from Lemma 4.9 in the full version of [27] (on arxiv.org), and Theorem 4.8 and Theorem 4.10 in the full version of [6] (on arxiv.org).

⁶Essentially, Azuma-Hoeffding inequality states that $|\nu - \hat{\nu}| \leq O(\sqrt{C_{\text{rad}}/N})$, whereas by Theorem 2.1 for small ν it holds with high probability that $\text{rad}(\hat{\nu}, N) \sim C_{\text{rad}}/N$.

3 LP-relaxation

The expected reward of the optimal policy, given foreknowledge of the distribution of outcome vectors, is typically difficult to characterize exactly. In fact, even for a time-invariant policy, it is difficult to give an exact expression for the expected reward due to the dependence of the reward on the random stopping time, τ , when the resource budget is exhausted. To approximate these quantities, we consider the fractional relaxation of BwK in which the stopping time (i.e., the total number of rounds) can be fractional, and the reward and resource consumption per unit time are deterministically equal to the corresponding expected values in the original instance of BwK.

The following LP (shown here along with its dual) constitutes our fractional relaxation of the optimal policy. We denote by OPT_{LP} the value of the linear program (P).

$$\begin{array}{ll}
 \max & r^\top \xi \\
 \text{s.t.} & C\xi \preceq B\mathbf{1} \\
 & \xi \succeq 0
 \end{array} \quad (\text{P})
 \qquad
 \begin{array}{ll}
 \min & B\mathbf{1}^\top \eta \\
 \text{s.t.} & C^\top \eta \succeq r \\
 & \eta \succeq 0
 \end{array} \quad (\text{D})$$

Lemma 3.1. OPT_{LP} is an upper bound on the value of the optimal dynamic policy.

Proof. One way to prove the lemma is to define ξ_j to be the expected number of times arm x_j is played by the optimal dynamic policy, and argue that ξ is primal-feasible and that $r^\top \xi$ is the expected reward of the optimal policy. We instead present a simple proof using the dual LP (D), since it introduces ideas that motivate the design of our primal-dual algorithm.

Let η^* denote an optimal solution to (D). By strong duality, $B\mathbf{1}^\top \eta^* = \text{OPT}_{\text{LP}}$. Interpret η_i^* as a unit cost for resource i . Dual feasibility implies that for each arm x_j , the expected cost of resources consumed when x_j is pulled exceeds the expected reward produced. Thus, if we let Z_t denote the sum of rewards gained in rounds $1, \dots, t$ plus the cost of the remaining resource endowment after round t , then the stochastic process Z_0, Z_1, \dots, Z_T is a supermartingale. Note that $Z_0 = B\mathbf{1}^\top \eta^*$ is the LP optimum, and $Z_{\tau-1}$ equals the algorithm's total payoff, plus the cost of the remaining (non-negative) resource supply at the start of round τ . By Doob's optional stopping theorem, $Z_0 \geq \mathbb{E}[Z_{\tau-1}]$ and the lemma is proved. \square

In a similar way, the expected total reward of time-invariant policy \mathcal{D} is bounded above by the solution to the following linear program in which t is the only LP variable:

$$\begin{array}{ll}
 \text{Maximise} & t r(\mathcal{D}, \mu) \\
 \text{subject to} & t c_i(\mathcal{D}, \mu) \leq B \quad \text{for each resource } i \\
 & t \geq 0.
 \end{array} \quad (6)$$

The solution to (6), which we call the *LP-value*, is

$$\text{LP}(\mathcal{D}, \mu) = r(\mathcal{D}, \mu) \min_i \left(\frac{B}{c_i(\mathcal{D}, \mu)} \right). \quad (7)$$

Observe that t is feasible for $\text{LP}(\mathcal{D}, \mu)$ if and only if $\xi = t\mathcal{D}$ is feasible for (P), and therefore

$$\text{OPT}_{\text{LP}} = \sup_{\mathcal{D}} \text{LP}(\mathcal{D}, \mu).$$

A distribution $D^* \in \arg\max_{\mathcal{D}} \text{LP}(\mathcal{D}, \mu)$ is called *LP-optimal* for latent structure μ . Any optimal solution ξ to (P) corresponds to an LP-optimal distribution $D^* = \xi / \|\xi\|_1$.

Claim 3.2. *For any latent structure μ , there exists a distribution \mathcal{D} over arms which is LP-optimal for μ and moreover satisfies the following three properties:*

- (a) $c_i(\mathcal{D}, \mu) \leq B/T$ for each resource i .
- (b) \mathcal{D} has a support of size at most $d + 1$.
- (c) If \mathcal{D} has a support of size at least 2 then for some resource i we have $c_i(\mathcal{D}, \mu) = B/T$.

(Such distribution \mathcal{D} will be called **LP-perfect** for μ .)

Proof. Fix the latent structure μ . It is a well-known fact that for any linear program there exists an optimal solution whose support has size that is exactly equal to the number of constraints that are tight for this solution. Take any such optimal solution ξ for linear program (P), and take the corresponding LP-optimal distribution $\mathcal{D} = \xi / \|\xi\|_1$. Since there are $d + 1$ constraints in (P), distribution \mathcal{D} has support of size at most $d + 1$. If it satisfies property (a), we are done. Note that such \mathcal{D} also satisfies property (c).

Suppose property (a) does not hold for \mathcal{D} . Then there exists a resource i such that $c_i(\mathcal{D}, \mu) > B/T$. It follows that $\sum_i \xi_i < T$. Therefore, at most d constraints in (P) are tight for ξ , which implies that the support of \mathcal{D} has size at most d .

Now, let us modify \mathcal{D} to obtain another LP-optimal distribution \mathcal{D}' which satisfies both (a) and (b-c). Namely, let $\mathcal{D}'(x) = \alpha \mathcal{D}(x)$ for each non-null arm x , for some $\alpha \in (0, 1)$ that is sufficiently low to ensure property (a) (with equality for some resource i), and place the remaining probability in \mathcal{D}' on the null arm. Then $\text{LP}(\mathcal{D}', \mu) = \text{LP}(\mathcal{D}, \mu)$, so \mathcal{D}' is LP-optimal; \mathcal{D}' satisfies properties (a,c) by design, and it satisfies property (b) because it adds at most one to the support of \mathcal{D} . \square

4 Two algorithms

Our main contribution is a pair of algorithms for solving the BwK problem: a “balanced exploration” algorithm (BalanceBwK) that prioritizes information acquisition, exploring each arm as frequently as possible given current confidence intervals, and a primal-dual algorithm (PD-BwK) that chooses arms to greedily maximize the estimated reward per unit of resource cost. The two algorithms are complementary. While their regret guarantees have the same worst-case dependence on the budget parameter B , the primal-dual algorithm achieves better worst-case dependence on d and OPT. On the other hand, the balanced exploration algorithm, being more flexible, is better able to take advantage of domain-specific side information to improve its regret guarantee in favorable instances. For example, when applied to the dynamic pricing problem with supply k and with monotone hazard rate distributions, BalanceBwK achieves $\tilde{O}(k^{1/2})$ regret whereas the primal-dual approach achieves $\tilde{O}(k^{2/3})$ regret, just as it does for worst-case distributions.

For continuity reasons, we have chosen to defer the analysis of the algorithms until later sections of the paper. Section 7 presents the analysis of PD-BwK, and Section 6 presents the analysis of BalanceBwK.

4.1 Balanced exploration: algorithm BalanceBwK

The design principle behind BalanceBwK is to explore as much as possible while avoiding obviously suboptimal strategies. On a high level, the algorithm is very simple. The goal is to converge on an LP-perfect distribution. The time is divided in phases of $|X|$ rounds each. In the beginning of each phase p , the algorithm prunes away all distributions \mathcal{D} over arms that with high confidence are not LP-perfect given the observations so far. The remaining distributions over arms are called *potentially perfect*. Throughout the phase, the algorithm chooses among the potentially perfect distributions. Specifically, for each arm x , the algorithm chooses a potentially perfect distribution $\mathcal{D}_{p,x}$ which approximately maximizes $\mathcal{D}_{p,x}(x)$, and “pulls” an arm sampled independently from this distribution. This choice of $\mathcal{D}_{p,x}$ is crucial; we call it the *balancing step*. The algorithm halts as soon as the time horizon is met, or any of the constraints is exhausted. See Algorithm 1 for the pseudocode.

Algorithm 1 BalanceBwK

- 1: **For** each phase $p = 0, 1, 2, \dots$ **do**
 - 2: Recompute the set Δ_p of potentially perfect distributions \mathcal{D} over arms.
 - 3: Over the next $|X|$ rounds, for each $x \in X$:
 - 4: pick any distribution $\mathcal{D} = \mathcal{D}_{p,x} \in \Delta_p$ such that $\mathcal{D}(x) \geq \frac{1}{2} \max_{\mathcal{D} \in \Delta_p} \mathcal{D}(x)$.
 - 5: choose an arm to “pull” as an independent sample from \mathcal{D} .
 - 6: **halt** if time horizon is met or one of the resources is exhausted.
-

We believe that BalanceBwK, like UCB1 [4], is a very general design principle and has the potential to be a meta-algorithm for solving stochastic online learning problems.

Theorem 4.1 (BwK: BalanceBwK). *Consider an instance of BwK with d resources, $m = |X|$ arms, and the smallest budget $B = \min_i B_i$. Let M_{LP} be the maximum of OPT_{LP} , for given time horizon and budgets, over all problem instances in a given BwK domain. Algorithm BalanceBwK achieves total expected regret*

$$\text{OPT} - \text{REW} \leq O(\log(dmT) \log(T/m)) \left(\sqrt{dm M_{\text{LP}}} + M_{\text{LP}} \left(\sqrt{\frac{dm}{B}} + \frac{dm}{B} \right) + dm \right). \quad (8)$$

Let us fill in the details in the specification of BalanceBwK. In the beginning of each phase p , the algorithm recomputes a “confidence interval” I_p for the latent structure μ , so that (informally) $\mu \in I_p$ with high probability. Then the algorithm determines which distributions \mathcal{D} over arms can potentially be LP-perfect given that $\mu \in I_p$. Specifically, let Δ_p be set of all distributions \mathcal{D} that are LP-perfect for some latent structure $\mu' \in I_p$; such distributions are called *potentially perfect* (for phase p).

It remains to define the confidence intervals I_p . For phase $p = 0$, the confidence interval I_0 is simply $\mathcal{M}_{\text{feas}}$, the set of all feasible latent structures. For each subsequent phase $p \geq 1$, the confidence interval I_p is defined as follows. For each arm x , consider all rounds before phase p in which this arm has been chosen. Let $N_p(x)$ be the number of such rounds, let $\hat{r}_p(x)$ be the time-averaged reward in these rounds, and let $\hat{c}_{p,i}(x)$ be the time-averaged consumption of resource i in these rounds. We use these averages to estimate $r(x, \mu)$ and $c_i(x, \mu)$ as follows:

$$|r(x, \mu) - \hat{r}_p(x)| \leq \text{rad}(\hat{r}_p(x), N_p(x)) \quad (9)$$

$$|c_i(x, \mu) - \hat{c}_{p,i}(x)| \leq \text{rad}(\hat{c}_{p,i}(x), N_p(x)) \quad \text{for each resource } i \quad (10)$$

The confidence interval I_p is the set of all latent structures $\mu' \in I_{p-1}$ that are consistent with these estimates. This completes the specification of BalanceBwK.

For each phase of BalanceBwK, the round in which an arm is sampled from distribution $\mathcal{D}_{p,x}$ will be called *designated* to arm x . We need to use approximate maximization to choose $\mathcal{D}_{p,x}$, rather than exact maximization, because an exact maximizer $\arg\max_{\mathcal{D} \in \Delta_p} \mathcal{D}(x)$ is not guaranteed to exist.

4.2 A primal-dual algorithm for BwK

This section develops an algorithm that solves the BwK problem using a very natural and intuitive idea: greedily select arms with the greatest estimated “bang per buck,” i.e. reward per unit of resource consumption. One of the main difficulties with this idea is that there is no such thing as a “unit of resource consumption”: there are d different resources, and it is unclear how to trade off consumption of one resource versus another. The proof of Lemma 3.1 gives some insight into how to quantify this trade-off: an optimal dual solution η^* can be interpreted as a vector of unit costs for resources, such that for every arm the expected

reward is less than or equal to the expected cost of resources consumed. Since our goal is to match the optimum value of the LP as closely as possible, we must minimize the shortfall between the expected reward of the arms we pull and their expected resource cost as measured by η^* . Thus, our algorithm will try to learn an optimal dual vector η^* in tandem with learning the latent structure μ .

Borrowing an idea from [3, 19, 36], we will use the multiplicative weights update method to learn the optimal dual vector. This method raises the cost of a resource exponentially as it is consumed, which ensures that heavily demanded resources become costly, and thereby promotes balanced resource consumption. Meanwhile, we still have to ensure (as in any multi-armed bandit problem) that our algorithm explores the different arms frequently enough to gain adequately accurate estimates of the latent structure. We do this by estimating rewards and resource consumption as optimistically as possible, i.e. using upper confidence bound (UCB) estimates for rewards and lower confidence bound (LCB) estimates for resource consumption. Although both of these techniques — multiplicative weights and confidence bounds — have both successfully applied in previous online learning algorithms, it is far from obvious that this particular hybrid of the two methods should be effective. In particular, the use of multiplicative updates on dual variables, rather than primal ones, distinguishes our algorithm from other bandit algorithms that use multiplicative weights (e.g. the Exp3 algorithm [5]) and brings it closer in spirit to the literature on stochastic packing algorithms (especially [16]).

The pseudocode for the algorithm is presented as Algorithm 2, which we call PD-BwK. When we refer to the UCB or LCB for a latent parameter (the reward of an arm, or the amount of some resource that it utilizes), these are computed as follows. Letting $\hat{\nu}$ denote the empirical average of the observations of that random variable⁷ and letting N denote the number of times the random variable has been observed, the lower confidence bound (LCB) and upper confidence bound (UCB) are the left and right endpoints, respectively, of the *confidence interval* $[0, 1] \cap [\hat{\nu} - \text{rad}(\hat{\nu}, N), \hat{\nu} + \text{rad}(\hat{\nu}, N)]$. The UCB or LCB for a vector or matrix are defined componentwise.

In step 8, the pseudocode asserts that the set $\text{argmin}_{z \in \Delta[X]} \{ \frac{y_t^\top L_t z}{u_t^\top z} \}$ contains at least one of the point-mass distributions $\{e_1, \dots, e_m\}$. This is true, because if $\rho = \min_{z \in \Delta[X]} \{ (y_t^\top L_t z) / (u_t^\top z) \}$ then the linear inequality $y_t^\top L_t z \leq \rho u_t^\top z$ holds at some point $z \in \Delta[X]$, and hence it holds at some extreme point, i.e. one of the point-mass distributions.

Algorithm 2 Algorithm PD-BwK

- 1: Set $\epsilon = \sqrt{\ln(d)/B}$.
 - 2: In the first m rounds, pull each arm once.
 - 3: $v_1 = \mathbf{1}$
 - 4: **for** $t = m + 1, \dots, \tau$ (*i.e., until resource budget exhausted*) **do**
 - 5: Compute UCB estimate for reward vector, u_t .
 - 6: Compute LCB estimate for resource consumption matrix, L_t .
 - 7: $y_t = v_t / (\mathbf{1}^\top v_t)$.
 - 8: Pull arm x_j such that point-mass distribution $z_t = e_j$ belongs to $\text{argmin}_{z \in \Delta[X]} \{ \frac{y_t^\top L_t z}{u_t^\top z} \}$.
 - 9: $v_{t+1} = \text{Diag}\{(1 + \epsilon)^{\mathbf{e}_j^\top L_t z_t}\} v_t$.
-

Another feature of our algorithm that deserves mention is a variant of the Garg-Könemann width reduction technique [19]. The ratio $(y_t^\top L_t z_t) / (u_t^\top z_t)$ that we optimize in step 8 may be unboundedly large, so in the multiplicative update in step 9 we rescale this value to $y_t^\top L_t z_t$, which is guaranteed to be at most 1. This rescaling is mirrored in the analysis of the algorithm, when we define the dual vector \bar{y} by averaging

⁷Note that we initialize the algorithm by pulling each arm once, so empirical averages are always well-defined.

the vectors y_t using the aforementioned scale factors. (Interestingly, unlike the Garg-Könemann algorithm which applies multiplicative updates to the dual vectors and weighted averaging to the primal ones, in our algorithm the multiplicative updates and weighted averaging are *both* applied to the dual vectors.) The following theorem expresses the regret guarantee for PD-BwK. The proof is in Section 7.

Theorem 4.2. *For any instance of the BwK problem, the regret of Algorithm PD-BwK satisfies*

$$\text{OPT} - \text{REW} \leq O\left(\sqrt{\log(dmT)}\right) \left(\sqrt{m \text{OPT}} + \text{OPT} \sqrt{\frac{m}{B}} + m\sqrt{\log(dmT)}\right). \quad (11)$$

Comparing this bound to Theorem 4.1, when $d = O(1)$ and $M_{\text{LP}} = O(\text{OPT})$ the two guarantees are the same up to logarithmic factors. The regret guarantee for the primal-dual algorithm scales better with the number of resources, d , and it can be vastly superior in cases where $\text{OPT} \ll M_{\text{LP}}$.

5 Applications

Owing to its generality, the BwK problem admits applications in many different domains. We describe some of these applications below. Instantiating specific regret bounds is straightforward for most of these applications, as soon as one precisely defines the setting.

BwK with discretization. In many applications of BwK the action space X is very large or infinite, so the algorithms developed in the previous sections are not immediately applicable. However, in these applications the action space has some structure that our algorithms can leverage. For example, in dynamic pricing (with one type of good) every possible action (arm) corresponds to a price – i.e., a number in some fixed interval.

To handle such applications, we use a simple approach called *uniform discretization*: we select a subset $S \subset X$ of arms which are, in some sense, uniformly spaced in X , and apply a BwK algorithm for this subset. In the dynamic pricing application, S can consist of all prices in the allowed interval that are of the form $\ell\epsilon$, $\ell \in \mathbb{N}$, for some fixed discretization parameter $\epsilon > 0$. We call this subset the *additive ϵ -mesh*. The granularity of discretization (i.e., the value of ϵ) is adjusted in advance so as to minimize regret.

This generic approach has been successfully used in past work on MAB on metric spaces (e.g., [24, 23, 27, 31]) and dynamic pricing (e.g., [26, 11, 9, 6]) to provide worst-case optimal regret bounds. To carry it through to the setting of BwK, we need to define an appropriate notion of discretization (which would generalize the additive ϵ -mesh for dynamic pricing), and argue that it does not cause too much damage. Compared to the usage of discretization in prior work, we need to deal with two technicalities. First, we need to argue about distributions over arms rather than individual arms. Second, in order to compare an arm with its “image” in the discretization, we need to consider the difference in the ratio of expected reward to expected consumption, rather than the difference in rewards.⁸ We flesh out the details in Section 8.

For dynamic pricing, we show that if we use the additive ϵ -mesh S_ϵ , and $R(S_\epsilon)$ is the regret on the problem instance restricted to S_ϵ , then regret on the original problem instance is at most $\epsilon dB + R(S_\epsilon)$, where d is the number of constraints. More generally, we prove this for any BwK domain, and any subset $S_\epsilon \subset X$ which satisfies some axioms for a given parameter ϵ ; we call such S_ϵ an *ϵ -discretization*. Once we define an ϵ -discretization S_ϵ for every $\epsilon > 0$, then in order to optimize the regret bound $\epsilon dB + R(S_\epsilon)$ up to constant factors it suffices to pick ϵ such that $\epsilon dB = R(S_\epsilon)$. We need to consider regret bounds $R(S_\epsilon)$ that are in terms of the maximal possible expected total reward M_{LP} rather than the (possibly smaller) OPT , since the value of OPT is not initially known to the algorithm.⁹

⁸Because of these technicalities, our approach to discretization does not immediately extend to some of the generalizations of dynamic pricing and dynamic procurement that we consider in the subsequent subsections.

⁹For problem instances with $\text{OPT} \ll M_{\text{LP}}$, additional information about the problem instance may help the algorithm: it may

5.1 Dynamic pricing with limited supply

Basic version: unit demands. In the basic version of the *dynamic pricing problem*, the algorithm is a seller which has k identical items for sale and is facing n agents (potential buyers) that are arriving sequentially. Each agent is interested in buying one item. The algorithm offers a take-it-or-leave-it price p_t to each arriving agent t . The agent has a fixed value $v_t \in [0, 1]$ for an item, which is *private*: not known to the algorithm. The agent buys an item if and only if $p_t \geq v_t$. We assume that each v_t is an independent sample from some fixed (but unknown) distribution, called the buyers' *demand distribution*. The goal of the algorithm is to maximize his expected revenue.

This problem has been studied in [26, 11, 9, 6]. The best result is an algorithm which achieves regret $\tilde{O}(k^{2/3})$ compared to the best fixed price [6];¹⁰ this regret rate is proved to be worst-case optimal. For demand distributions that satisfy a standard (but limiting) assumption of *regularity*, the best fixed price is essentially as good as the best dynamic policy.

Dynamic pricing is naturally interpreted as a BwK domain: arms correspond to prices, rounds correspond to arriving agents (so there is a time horizon of n), and there is a single type of resource: the k items. One can think of the agents as simply units of time, so that n is the time horizon. Let $S(p)$ be the probability of a sale at price p . The latent structure μ is determined by the function $S(\cdot)$: the expected reward is $r(p, \mu) = p S(p)$, and expected resource consumption is simply $c(x, \mu) = S(p)$. Since there are only k items for sale, it follows that $\text{OPT} \leq k$. (To analyze `BalanceBwK`, note that the LP-values are bounded above by k , so we can take $M_{\text{LP}} = k$.) Prices can be discretized using the ϵ -uniform mesh over $[0, 1]$, resulting in $\frac{1}{\epsilon}$ arms.

We recover the optimal $\tilde{O}(k^{2/3})$ regret result from [6] with respect to offering the same price to each buyer, and moreover extend this result to regret with respect to the optimal dynamic policy. We observe that the expected revenue of the optimal dynamic policy can be twice the expected revenue of the best fixed price (see Appendix A). The crucial technical advance compared to the prior work is that our algorithms strive to converge to an optimal *distribution* over prices, whereas the algorithms in prior work target the best fixed price. Technically, the result follows by applying Theorem 8.2 for the uniform ϵ -mesh, in conjunction with either of the two main algorithms, and optimizing the ϵ .

Extension: non-unit demands. Agents may be interested in buying more than one unit of the product. Accordingly, let us consider an extension where an algorithm can offer each agent multiple units. More specifically: in each round t , the algorithm offers up to λ_t units at a fixed price p_t per unit, where the pair (p_t, λ_t) is chosen by the algorithm. The t -th agent then chooses how many units to buy. Agents' valuations may be non-linear in k_t , the number of units they receive. Each agent chooses k_t that maximizes her utility. We restrict $\lambda_t \leq \Lambda$, where Λ is an a-priori chosen parameter.

Let us solve this generalization via BwK. We model it as a BwK domain with a single resource and action space $X = [0, 1] \times \{0, 1, \dots, \Lambda\}$. Note that for each arm $x = (p_t, \lambda_t)$ the expected reward is $r(x, \mu) = p_t \mathbb{E}[k_t]$, and the expected consumption is $c(x, \mu) = \mathbb{E}[k_t]$. As before, we can use the additive ϵ -mesh on $[0, 1]$ for discretization; this results in Λ/ϵ arms. Optimizing the ϵ , we obtain regret $\tilde{O}(k \Lambda)^{2/3}$.¹¹

It is also interesting to consider a more restricted version where $\lambda_t = \Lambda$. Then in each round t the algorithm only chooses the price p_t , so that the action space is X . A similar argument gives regret $\tilde{O}(k^{2/3} \Lambda^{1/3})$.

Other extensions. The generality of BwK allows to handle several other extensions. For these extensions,

effectively reduce the problem to a smaller BwK domain with a smaller value of M_{LP} , which would allow the algorithm to choose ϵ more efficiently so as to reduce the number of arms in the ϵ -discretization and hence reduce the resulting regret.

¹⁰The $\tilde{O}(\cdot)$ notation hides poly-log factors.

¹¹Formally, the definition of BwK requires that the per-round reward and the per-round resource consumption are at most 1. Therefore we scale down the rewards and the consumption by the factor of Λ ; effectively, the supply constraint is divided by Λ , and regret is multiplied by Λ . After the re-scaling, the LP-values are upper-bounded by k , so that we can take $M_{\text{LP}} = k$.

it appears more difficult to bound the “damage” due to discretization. We will assume that the action space is restricted to a specific finite subset $S \subset X$ that is given to the algorithm, such as the additive ϵ -mesh for some specific ϵ . We bound the S -regret: regret with respect to the value of OPT_{LP} on the restricted action space. Such regret bounds depend on $|S|$.

- *Multiple products.* The algorithm has d products for sale, with k_i units of each product i . (To simplify regret bounds, let us assume $k_i = k$.) In each round t , an agent arrives. This agent is interested in buying only one unit of each product. The agent is characterized by the vector of values $(v_{t,1}, \dots, v_{t,d}) \in [0, 1]^d$, one for each product i . This vector is private: not known to the algorithm. We assume that it is an independent sample from a fixed but unknown distribution over $[0, 1]^d$; note that arbitrary correlations between values of different products are allowed. The algorithm offers a vector of prices $(p_{t,1}, \dots, p_{t,d}) \in [0, 1]^d$, one for each product i . The agent buys one unit of each product i such that $p_{t,i} \geq v_{t,i}$.

This problem corresponds to a BwK domain with d resources, one for each product. Arms are price vectors, so that $X = [0, 1]^d$. We assume that the algorithm is given a restricted action space $S \subset X$, such as an additive ϵ -mesh for some specific ϵ . We obtain S -regret $\tilde{O}(d\sqrt{k|S|})$.¹²

Similarly, we can handle a version where the t -th agent buys only one item; that is, she buys one unit of type $i^* = \text{argmax}_i v_i - p_i$, if and only if $v_{i^*} \geq p_{i^*}$. We obtain S -regret $\tilde{O}(\sqrt{dk|S|})$.

- *Network revenue management.* More generally, an algorithm may have d products for sale which may be produced on demand from limited *primitive resources*, so that each unit of each product i consumes a fixed and known amount $c_{ij} \in [0, 1]$ of each primitive resource j . This generalization is known as network revenue management problem (see [10] and references therein). All other details are the same as above; for simplicity, let us focus on a version in which each agent buys at most one item.

This corresponds to a BwK domain with the same action space as above: $X = [0, 1]^d$. For any given restricted action space $S \subset X$ with m arms, we obtain S -regret given by Equation (1). In particular, if all resource constraints (including the time horizon) are scaled up by factor γ , regret scales as $\sqrt{\gamma}$. This improves over the main result in [10], where (essentially) regret is stated in terms of γ and scales as $\gamma^{2/3}$.

- *Bundling and volume pricing.* When selling to agents with non-unit demands, an algorithm may use discounts and/or surcharges for buying multiple units of a product (the latter may make sense for high-valued products such as tickets to events at the Olympics). More generally, an algorithm can may use discounts and/or surcharges for some *bundles* of products, where each bundle can include multiple units of multiple products (e.g.: two beers and one snack). In full generality, there is a collection \mathcal{F} of allowed bundles. In each round an algorithm offers a *menu* of options which consists of a price for every allowed bundle in \mathcal{F} . Each buyer has a private valuation for every bundle; for each product, she chooses the bundle which maximizes her utility. The vector of private valuations over bundles comes from a fixed but unknown distribution.

This corresponds to a BwK domain where arms correspond to the feasible menus. In other words, each arm is a price vector over all possible bundles in \mathcal{F} , so that the (full) action space is $X = [0, 1]^{\mathcal{F}}$. We assume that the algorithm is given a restricted action space $S \subset X$. If each feasible bundle contains at most ℓ units, we obtain S -regret $\tilde{O}(\sqrt{d\ell k|S|})$.

We can reduce the “dimensionality” of the action space by restricting the bundle pricing. For example, the algorithm can offer a volume discounts of $x\%$ compared to the price of a single item, where x depends only on the number of items in the bundle, but not on the specific products in this bundle and not on the prices of a single unit of these products.

- *Buyer targeting.* Suppose there are ℓ different types of buyers (say *men* and *women*), and the demand

¹²To ensure that the maximal per-round reward is at most 1, as required by BwK, we re-scale all rewards down by the factor of d ; effectively, regret is increased by the factor of d . After the re-scaling, $\text{OPT} \leq M_{\text{LP}} \leq k$.

distribution of a buyer depends on her type. The buyer type is modeled as a sample from a fixed but unknown distribution. In each round the seller observes the type of the current buyer (e.g., using a *cookie* or a user profile), and can choose the price depending on this type.

This can be modeled as a BwK domain where arms correspond to functions from buyer types to prices. For example, with ℓ buyer types and a single product, the (full) action space is $X = [0, 1]^\ell$. Assuming we are given a restricted action space $S \subset X$, we obtain S -regret $\tilde{O}(\sqrt{k|S|})$.

Our regret guarantees for the above extensions are with respect to the optimal dynamic policy on the restricted action space. It is worth emphasizing that the benchmark equivalence result (the best fixed action is almost as good as the best dynamic policy, for regular demand distributions) applies only to the basic version of dynamic pricing. For example, no such result is known for multiple types of goods, even if the demand distribution for each individual type is regular. (See Appendix A for a related discussion.)

5.2 Dynamic procurement and crowdsourcing

Basic version: unit supply. A “dual” problem to dynamic pricing is *dynamic procurement*, where the algorithm is buying rather than selling. The algorithm has a budget B to spend, and is facing n agents (potential sellers) that are arriving sequentially. Each seller is interested in selling one item. Each seller’s value for an item is an independent sample from some fixed (but unknown) distribution with support $[0, 1]$. The algorithm offers a take-it-or-leave-it price to each arriving agent. The goal is to maximize the number of items bought. This problem has been studied in [7]; they achieve a multiplicative constant-factor approximation with respect to the optimal dynamic policy.

Application to crowdsourcing. The problem is particularly relevant to the emerging domain of *crowdsourcing*, where agents correspond to the (relatively inexpensive) workers on a crowdsourcing platform such as Amazon Mechanical Turk, and “items” bought/sold correspond to simple jobs (“microtasks”) that can be performed by these workers. The algorithm corresponds to the “requester”: an entity that submits jobs and benefits from them being completed. The (basic) dynamic procurement model captures an important issue in crowdsourcing that a requester interacts with multiple users with unknown values-per-item, and can adjust its behavior (such as the posted price) over time as it learns the distribution of users. While this basic model ignores some realistic features of crowdsourcing environments, some of these limitations are addressed by the generalizations which we present below.

Dynamic procurement via BwK. Let us cast dynamic procurement as a BwK domain. As in dynamic pricing, agents correspond to time rounds, and n is the time horizon. The only resource is money. Arms correspond to prices: $X = [0, 1]$. Letting $S(p)$ be the probability of a sale at price p , the expected reward (items bought) is $r(p, \mu) = S(p)$, and the expected resource consumption is $c(p, \mu) = pS(p)$. The best a-priori upper bound on the LP-value is $M_{LP} = n$.

We find that arbitrarily small prices are not amenable to discretization. Instead, we focus on prices $p \geq p_0$, where $p_0 \in (0, 1)$ is a parameter to be adjusted, and construct an ϵ -discretization on the set $[p_0, 1]$. We find that the additive δ -mesh (for a suitably chosen δ) is not the most efficient way to construct an ϵ -discretization in this domain. Instead, we use a mesh of the form $\{\frac{1}{1+j\epsilon} : j \in \mathbb{N}\}$ (we call it the *hyperbolic ϵ -mesh*). Then we obtain an ϵ -discretization with significantly fewer arms. Optimizing the parameters p_0 and ϵ , we obtain regret $\tilde{O}(n/B^{1/4})$; see Section 8.1 for more details.

Our result is meaningful, and improves over the constant-factor approximation in [7], as long as OPT is not too small compared to $n/B^{1/4}$. Recall that our result is with respect to the optimal dynamic policy. We observe that in this domain the optimal dynamic policy can be vastly superior compared to the best fixed price (see Appendix A for a specific example).

Extension: non-unit supply. As in dynamic pricing, we consider an extension where each agent may be

interested in more than one item. For example, a worker may be interested in performing several jobs. In each round t , the algorithm offers to buy up to λ_t units at a fixed price p_t per unit, where the pair (p_t, λ_t) is chosen by the algorithm. The t -th agent then chooses how many units to sell. Agents' valuations may be non-linear in k_t , the number of units they sell. Each agent chooses k_t that maximizes her utility. We restrict $\lambda_t \leq \Lambda$, where Λ is an a-priori chosen parameter.

We model it as a BwK domain with a single resource (money) and action space $X = [0, 1] \times \{0, 1, \dots, \Lambda\}$. Note that for each arm $x = (p_t, \lambda_t)$ the expected reward is $r(x, \mu) = \mathbb{E}[k_t]$, and the expected consumption is $c(x, \mu) = p_t \mathbb{E}[k_t]$. As in the basic dynamic procurement problem, we focus on prices $p \geq p_0$ and use the hyperbolic ϵ -mesh on $[p_0, 1]$ for discretization, for some parameters $p_0, \epsilon \in (0, 1)$. Optimizing the ϵ , we obtain regret $\tilde{O}(\Lambda^{3/2} n / B^{1/4})$; see Section 8.1 for more details.

In a more restricted version with $\lambda_t = \Lambda$, we obtain regret $\tilde{O}(\Lambda^{5/4} n / B^{1/4})$.

Other extensions. Further, we can model and handle a number of other extensions. As in “other extensions” for dynamic pricing, will assume that the action space is restricted to a specific finite subset $S \subset X$ that is given to the algorithm, and bound the S -regret – regret with respect to the restricted action space.

- *Multiple types of jobs.* There are ℓ types of jobs requested on the crowdsourcing platform. Each agent t has a private cost $v_{t,i} \in [0, 1]$ for each type i ; the vector of private costs comes from a fixed but unknown distribution (i.e., arbitrary correlations are allowed). The algorithm derives utility $u_i \in [0, 1]$ from each job of type i . In each round t , the algorithm offers a vector of prices $(p_{t,1}, \dots, p_{t,\ell})$, where $p_{t,i}$ is the price for one job of type i . For each type i , the agent performs one job of this type if and only if $p_{t,i} \geq v_{t,i}$, and receives payment $p_{t,i}$ from the algorithm.

Here arms correspond to the ℓ -dimensional vectors of prices, so that the (full) action space is $X = [0, 1]^\ell$. Given the restricted action space $S \subset X$, we obtain S -regret $\tilde{O}(\ell)(\sqrt{n|S|} + n\sqrt{\ell|S|/B})$.

- *Additional features.* We can also model more complicated “menus” so that each agent can perform several jobs of the same type. Then in each round, for each type i , the algorithm specifies the maximal offered number of jobs of this type and the price per one such job.

We can also incorporate constraints on the maximal number of jobs of each type that is needed by the requester, and/or the maximal amount of money spend on each type.

- *Competitive environment.* There may be other requesters in the system, each offering its own vector of prices in each round. (This is a realistic scenario in crowdsourcing, for example.) Each seller / worker chooses the requester and the price that maximize her utility. One standard way to model such competitive environment is to assume that the “best offer” from the competitors is a vector of prices which comes from a fixed but unknown distribution. This can be modeled as a BwK instance with a different distribution over outcomes which reflects the combined effects of the demand distribution of agents and the “best offer” distribution of the environment.

5.3 Other applications to Electronic Markets

Ad allocation with unknown click probabilities. Consider *pay-per-click* (PPC) advertising on the web (in particular, this is a prevalent model in sponsored search auctions). The central premise in PPC advertising is that an advertiser derives value from her ad only when the user clicks on this ad. The ad platform allocates ads to users that arrive over time.

Consider the following simple (albeit highly idealized) model for PPC ad allocation. Users arrive over time, and the ad platform needs to allocate an ad to each arriving user. There is a set X of available ads. Each ad x is characterized by the payment-per-click π_x and click probability μ_x ; the former quantity is known to the algorithm, whereas the latter is not. If an ad x is chosen, it is clicked on with probability μ_x ,

in which case payment π_x is received. The goal is to maximize the total payment. This setting, and various extensions thereof that incorporate user/webpage context, has received a considerable attention in the past several years (starting with [33, 34, 29]). In fact, the connection to PPC advertising has been one of the main motivations for the recent surge of interest in MAB.

We enrich the above setting by incorporating advertisers' *budgets*. In the most basic version, for each ad x there is a budget B_x — the maximal amount of money that can be spent on this ad. More generally, an advertiser can have an ad campaign which consists of a subset S of ads, so that there is a per-campaign budget S . Even more generally, an advertiser can have a more complicated budget structure: a family of overlapping subsets $S \subset X$ and a separate budget B_S for each S . For example, BestBuy can have a total budget for the ad campaign, and also separate budgets for ads about TVs and ads about computers. Finally, in addition to budgets (i.e., constraints on the number of times ads are clicked), an advertiser may wish to have similar constraints on the number of times ads are shown. BwK allows us to express all these constraints.

Adjusting a repeated auction. An auction is held in every round, with a fresh set of participants. The number of participants and a vector of their types come from a fixed but unknown distribution. The auction is *adjustable*: it has some parameter that can be adjusted by the auctioneer. For example, [13] studies a repeated second price auction with adjustable reserve price. The auctioneer has a limited inventory and wishes to optimize revenue.

The prior work [13] considers the case of a single type of good and unlimited inventory. They design an algorithm with $O(\sqrt{n \log n})$ regret, where n is the number of rounds; this regret is proved optimal.

BwK allows us to model limited inventory. For example, for the setting in [13] our algorithm achieves $O(k \log n)^{2/3}$ regret, where k is the number of items. Moreover, we can incorporate any auction design with adjustable parameter that admits discretization (as defined in Section 8). Further, we can handle multiple types of goods; such extension would correspond to multiple resource constraints in BwK.

Repeated bidding. A bidder participates in a repeated auction, such as a sponsored search auction. In each round t , the bidder can adjust her bid b_t based on the past performance. The outcome for this bidder is a vector (p_t, u_t) , where p_t is the payment and u_t is the utility received. We assume that this vector comes from a fixed but unknown distribution. The bidder has a fixed budget.

We model this as a BwK problem where arms correspond to the possible bids, and the single resource is money. Note that (the basic version of) dynamic procurement corresponds to this setting with two possible outcome vectors (p_t, u_t) : $(0, 0)$ and $(b_t, 1)$.

The BwK setting also allows to incorporate more complicated constraints. For example, an action can result in several different types of outcomes that are useful for the bidder (e.g., an ad shown to a *male* or an ad shown to a *female*), but the bidder is only interested in a limited quantity of each outcome.

5.4 Application to network routing and scheduling

In addition to applications to Electronic Markets, we describe two applications to network routing and scheduling. In both applications an algorithm choose between different feasible policies to handle arriving “service requests”, such as connection requests in network routing and jobs in scheduling.

Adjusting a routing protocol. Consider the following stylized application to routing in a communication network. Connection requests arrive one by one. A connection request consists of a pair of terminals; assume the pair comes from a fixed but unknown distribution. The system needs to choose a *routing protocol* for each connection, out of several possible routing protocols. The routing protocol defines a path that connects the terminals; abstractly, each protocol is simply a mapping from terminal pairs to paths. Once the path is chosen, a connection between the terminals is established. Connections persist for a significant amount of time. Each connection uses some amount of bandwidth. For simplicity, we can assume that this amount

is fixed over time for every connection, and comes from a fixed but unknown distribution (although even a deterministic version is interesting). Each edge in the network (or perhaps each node) has a limited capacity: the total bandwidth of all connections that pass through this edge or node cannot exceed some value. A connection which violates any capacity constraint is terminated. The goal is to satisfy a maximal number of connections.

We model this problem as BwK as follows: arms correspond to the feasible routing protocols, each edge/node is a limited resource, each satisfied connection is a unit reward.

Further, if the time horizon is partitioned in epochs, we can model different bandwidth utilization in each phase; then a resource in BwK is a pair (edge, epoch).

Adjusting a scheduling policy. An application with a similar flavor arises in the domain of scheduling long-running jobs to machines. Suppose jobs arrive over time. Each job must be assigned to one of the machines (or dropped); once assigned, a job stays in the system forever (or for some number of “epochs”), and consumes some resources. Jobs have multiple “types” that can be observed by the scheduler. For each type, the resource utilization comes from a fixed but unknown distribution. Note that there may be multiple resources being consumed on each machine: for example, jobs in a datacenter can consume CPU, RAM, disk space, and network bandwidth. Each satisfied job of type i brings utility u_i . The goal of the scheduler is to maximize utility given the constrained resources.

The mapping of this setting to BwK is straightforward. The only slightly subtle point is how to define the arms: in BwK terms, arms correspond to all possible mappings from job types to machines.

One can also consider an alternative formulation where there are several allowed scheduling policies (mappings from types and current resource utilizations to machines), and in every round the scheduler can choose to use one of these policies. Then the arms in BwK correspond to the allowed policies.

6 Analysis of BalanceBwK (proof of Theorem 4.1)

Let us summarize various properties of the confidence radius which we use throughout the analysis.

Claim 6.1. *The confidence radius $\text{rad}(\nu, N)$, defined in Equation (3), satisfies the following properties:*

- (a) *monotonicity: $\text{rad}(\nu, N)$ is non-decreasing in ν and non-increasing in N .*
- (b) *concavity: $\text{rad}(\nu, N)$ is concave in ν , for any fixed N .*
- (c) *$\max(0, \nu - \text{rad}(\nu, N))$ is non-decreasing in ν .*
- (d) *$\nu - \text{rad}(\nu, N) \geq \frac{1}{4}\nu$ whenever $4\frac{C_{\text{rad}}}{N} \leq \nu \leq 1$.*
- (e) *$\text{rad}(\nu, N) \leq 3\frac{C_{\text{rad}}}{N}$ whenever $\nu \leq 4\frac{C_{\text{rad}}}{N}$.*
- (f) *$\text{rad}(\nu, \alpha N) = \frac{1}{\alpha} \text{rad}(\alpha\nu, N)$, for any $\alpha \in (0, 1]$.*
- (g) *$\frac{1}{N} \sum_{\ell=1}^N \text{rad}(\nu, \ell) \leq O(\log N) \text{rad}(\nu, N)$.*

6.1 Deterministic properties of BalanceBwK

First, we show that any two latent structures in the confidence interval I_p correspond to similar consumptions and rewards, for each arm x . This follows deterministically from the specification of I_p .

Claim 6.2. *Fix any phase p , any two latent structures $\mu', \mu'' \in I_p$, an arm x , and a resource i . Then*

$$|c_i(x, \mu') - c_i(x, \mu'')| \leq 6 \text{rad}(c_i(x, \mu'), N_p(x)) \quad (12)$$

$$|r(x, \mu') - r(x, \mu'')| \leq 6 \text{rad}(r(x, \mu'), N_p(x)). \quad (13)$$

Proof. We prove Equation (12); Equation (13) is proved similarly.

Let $N = N_p(x)$. By specification of `BalanceBwK`, any $\mu' \in I_p$ is consistent with estimate (10):

$$|c_i(x, \mu') - \widehat{c}_{p,i}(x)| \leq \text{rad}(\widehat{c}_{p,i}(x), N).$$

It follows that

$$|c_i(x, \mu') - c_i(x, \mu'')| \leq 2 \text{rad}(\widehat{c}_{p,i}(x), N).$$

Finally, we observe that by Claim 2.2,

$$\text{rad}(\widehat{c}_{p,i}(x), N) \leq 3 \text{rad}(c_i(x, \mu'), N). \quad \square$$

For each phase p and arm x , let $\bar{\mathcal{D}}_{p,x} = \frac{1}{p} \sum_{q < p} \mathcal{D}_{q,x}(x)$ be the average of probabilities for arm x among the distributions in the preceding phases that are designated to arm x . Because of the balancing step in `BalanceBwK`, we can compare this quantity to $\mathcal{D}(x)$, for any $\mathcal{D} \in \Delta_p$. (Here we also use the fact that the confidence intervals I_p are non-increasing from one phase to another.)

Claim 6.3. $\bar{\mathcal{D}}_{p,x} \geq \frac{1}{2} \mathcal{D}(x)$ for each phase p , each arm x and any distribution $\mathcal{D} \in \Delta_p$.

Proof. Fix arm x . Recall that $\bar{\mathcal{D}}_{p,x} = \frac{1}{p} \sum_{q < p} \mathcal{D}_{q,x}(x)$, where $\mathcal{D}_{q,x}$ is the distribution chosen in the round in phase q that is designated to arm x . Fix any phase $q < p$. Because of the balancing step, $\mathcal{D}_{q,x}(x) \geq \frac{1}{2} \mathcal{D}'(x)$ for any distribution $\mathcal{D}' \in \Delta_q$. Since the confidence intervals I_q are non-increasing from one phase to another, we have $I_p \subset I_q$ for any $q \leq p$, which implies that $\Delta_p \subset \Delta_q$. Consequently, $\mathcal{D}_{q,x}(x) \geq \frac{1}{2} \mathcal{D}(x)$ for each $q < p$, and the claim follows. \square

6.2 High-probability events

We keep track of several quantities: the averages $\widehat{r}_p(x)$ and $\widehat{c}_{p,i}(x)$ defined in Section 4.1, as well as several other quantities that we define below.

Fix phase p . For each arm x , consider all rounds in phases $q < p$ that are designated to arm x . Let $n_p(x)$ denote the number of times arm x has been chosen in these rounds. Let $\widehat{\mathcal{D}}_{p,x} = n_t(x)/p$ be the corresponding empirical probability of choosing x . We compare this to $\bar{\mathcal{D}}_{p,x}$.

Further, consider all rounds in phases $q < p$. There are $N = p|X|$ such rounds. The average distribution chosen by the algorithm in these rounds is $\bar{\mathcal{D}}_p = \frac{1}{N} \sum_{q < p, x \in X} \mathcal{D}_{q,x}$. We are interested in the corresponding quantities $r(\bar{\mathcal{D}}_p, \mu)$ and $c_i(\bar{\mathcal{D}}_p, \mu)$. We compare these quantities to $\widehat{r}_p = \frac{1}{N} \sum_{t=1}^N r_t$ and $\widehat{c}_{p,i} = \frac{1}{N} \sum_{t=1}^N c_{t,i}$, the average reward and the average resource- i consumption in phases $q < p$.

We consider several high-probability events which follow from applying Theorem 2.1 to the various quantities defined above. All these events have a common shape: $\widehat{\nu}$ is an N -strong estimate for ν , for some quantities $\nu, \widehat{\nu} \in [0, 1]$ and $N \in \mathbb{N}$.

Lemma 6.4. For each phase p , arm x , and resource i , with probability $e^{-\Omega(C_{\text{rad}})}$ it holds that:

- (a) $\widehat{r}_p(x)$ is an $N_p(x)$ -strong estimator for $r(x, \mu)$, and $\widehat{c}_{p,i}(x)$ is an $N_p(x)$ -strong estimator for $c_i(x, \mu)$.
- (b) $\bar{\mathcal{D}}_{p,x}$ is an p -strong estimator for $\widehat{\mathcal{D}}_{p,x}$.
- (c) $r(\bar{\mathcal{D}}_p, \mu)$ is an $(p|X|)$ -strong estimator for \widehat{r}_p , and $c_i(\bar{\mathcal{D}}_p, \mu)$ is an $(p|X|)$ -strong estimator for $\widehat{c}_{p,i}$.

6.3 Clean execution of BalanceBwK

A *clean execution* of the algorithm is one in which all events in Lemma 6.4 hold. It is convenient to focus on a clean execution. In the rest of the analysis, we assume clean execution without further notice. Also, we fix an arbitrary phase p in such execution.

Since a clean execution satisfies the event in Claim 6.4(a), it immediately follows that:

Claim 6.5. *The confidence interval I_p contains the (actual) latent structure μ . Therefore, $\mathcal{D}^* \in \Delta_p$ for any distribution \mathcal{D}^* that is LP-perfect for μ .*

Claim 6.6. *Fix any latent structures $\mu', \mu'' \in I_p$ and any distribution $\mathcal{D} \in \Delta_p$. Then for each resource i ,*

$$|c_i(\mathcal{D}, \mu') - c_i(\mathcal{D}, \mu'')| \leq O(1) \text{rad}(c_i(\mathcal{D}, \mu'), p/d) \quad (14)$$

$$|r(\mathcal{D}, \mu') - r(\mathcal{D}, \mu'')| \leq O(1) \text{rad}(r(\mathcal{D}, \mu'), p/d). \quad (15)$$

Proof. We prove Equation (14); Equation (15) is proved similarly. Let us first prove the following:

$$\forall x \in X, \quad \mathcal{D}(x) |c_i(x, \mu') - c_i(x, \mu'')| \leq O(1) \text{rad}(\mathcal{D}(x) c_i(x, \mu'), p). \quad (16)$$

Intuitively, in order to argue that we have good estimates on quantities related to arm x , it helps to prove that this arm has been chosen sufficiently often. Using the definition of clean execution and Claim 6.3, we accomplish this as follows:

$$\begin{aligned} \frac{1}{p} N_p(x) &\geq \frac{1}{p} n_p(x) = \widehat{\mathcal{D}}_{p,x} \\ &\geq \bar{\mathcal{D}}_{p,x} - \text{rad}(\bar{\mathcal{D}}_{p,x}, p) && \text{(by clean execution)} \\ &\geq \frac{1}{2} \mathcal{D}(x) - \text{rad}(\frac{1}{2} \mathcal{D}(x), p) && \text{(by Claim 6.3 and Claim 6.1(c)).} \end{aligned}$$

Consider two cases depending on $\mathcal{D}(x)$. For the first case, assume $\mathcal{D}(x) \geq 8 \frac{C_{\text{rad}}}{p}$. Using Claim 6.1(d) and the previous equation, it follows that $N_p(x) \geq \frac{1}{8} p \mathcal{D}(x)$. Therefore:

$$\begin{aligned} \mathcal{D}(x) |c_i(x, \mu') - c_i(x, \mu'')| &\leq 6 \mathcal{D}(x) \text{rad}(c_i(x, \mu'), N_p(x)) && \text{(by Claim 6.2)} \\ &\leq 6 \mathcal{D}(x) \text{rad}(c_i(x, \mu'), \frac{1}{8} p \mathcal{D}(x)) && \text{(by monotonicity of rad)} \\ &= 48 \text{rad}(\mathcal{D}(x) c_i(x, \mu'), p) && \text{(by Claim 6.1(f)).} \end{aligned}$$

The second case is that $\mathcal{D}(x) < 8 \frac{C_{\text{rad}}}{p}$. Then Equation (16) follows simply because $\frac{C_{\text{rad}}}{p} \leq \text{rad}(\cdot, p)$.

We have proved Equation (16). We complete the proof of Equation (14) using concavity of $\text{rad}(\cdot, p)$ and the fact that, by the specification of BalanceBwK, \mathcal{D} has support of size at most $d + 1$.

$$\begin{aligned} |c_i(\mathcal{D}, \mu') - c_i(\mathcal{D}, \mu'')| &\leq \sum_{x \in X} \mathcal{D}(x) |c_i(x, \mu') - c_i(x, \mu'')| \\ &\leq \sum_{x \in X, \mathcal{D}(x) > 0} O(1) \text{rad}(\mathcal{D}(x) c_i(x, \mu'), p) \\ &\leq O(d) \text{rad}\left(\frac{1}{d+1} \sum_{x \in X} \mathcal{D}(x) c_i(x, \mu'), p\right) \\ &= O(d) \text{rad}\left(\frac{1}{d+1} c_i(\mathcal{D}, \mu'), p\right) \\ &\leq O(1) \text{rad}\left(c_i(\mathcal{D}, \mu'), \frac{p}{d}\right) \quad \text{(by Claim 6.1(f)).} \quad \square \end{aligned}$$

Claim 6.7. *Fix any latent structures $\mu', \mu'' \in I_p$ and any distribution $\mathcal{D} \in \Delta_p$. Then*

$$|\text{LP}(\mathcal{D}, \mu') - \text{LP}(\mathcal{D}, \mu'')| \leq O(T) \text{rad}\left(M_{\text{LP}}/T, \frac{p}{d}\right) + O(M_{\text{LP}} \frac{T}{B}) \text{rad}\left(\frac{B}{T}, \frac{p}{d}\right). \quad (17)$$

Proof. Since $\mathcal{D} \in \Delta_p$, it is LP-perfect for some latent structure μ . Then $\text{LP}(\mathcal{D}, \mu) = T r(\mathcal{D}, \mu)$. Therefore:

$$\begin{aligned} \text{LP}(\mathcal{D}, \mu') - \text{LP}(\mathcal{D}, \mu) &\leq T (r(\mathcal{D}, \mu') - r(\mathcal{D}, \mu)) \\ &\leq O(T) \text{rad} \left(r(\mathcal{D}, \mu), \frac{B}{d} \right) \end{aligned} \quad (\text{by Claim 6.6}). \quad (18)$$

We need a little more work to bound the difference in the LP values in the other direction.

Consider $t_0 = \text{LP}(\mathcal{D}, \mu')/r(\mathcal{D}, \mu')$; this is the optimal value of variable t in linear program (6). Let us obtain a lower bound on this quantity. Assume $t_0 < T$. Then one of the budget constraints in (6) must be tight, i.e. $t_0 c_i(\mathcal{D}, \mu') = B$ for some resource i .

$$\begin{aligned} c_i(\mathcal{D}, \mu') &\leq c_i(\mathcal{D}, \mu) + O(1) \text{rad} \left(c_i(\mathcal{D}, \mu), \frac{B}{d} \right) \\ &\leq \frac{B}{T} + O(1) \text{rad} \left(\frac{B}{T}, \frac{B}{d} \right) \end{aligned} \quad (\text{by Claim 6.6})$$

Let $\Psi = \text{rad} \left(\frac{B}{T}, \frac{B}{d} \right)$. It follows that $t_0 = B/c_i(\mathcal{D}, \mu') \geq T(1 - O(\frac{T}{B} \Psi))$. Therefore:

$$\begin{aligned} \text{LP}(\mathcal{D}, \mu) - \text{LP}(\mathcal{D}, \mu') &= T r(\mathcal{D}, \mu) - t_0 r(\mathcal{D}, \mu') \\ &\leq T r(\mathcal{D}, \mu) - [T(1 - O(\frac{T}{B} \Psi))] r(\mathcal{D}, \mu') \\ &\leq T [r(\mathcal{D}, \mu) - r(\mathcal{D}, \mu')] + O(\frac{T}{B} \Psi) T r(\mathcal{D}, \mu) \\ &\leq O(T) \text{rad} \left(r(\mathcal{D}, \mu), \frac{B}{d} \right) + O(\frac{T}{B} \Psi) T r(\mathcal{D}, \mu) \end{aligned} \quad (\text{by Claim 6.6}).$$

Using Equation (18) and noting that $r(\mathcal{D}, \mu) = \text{LP}(\mathcal{D}, \mu)/T \leq M_{\text{LP}}/T$, we conclude that

$$|\text{LP}(\mathcal{D}, \mu) - \text{LP}(\mathcal{D}, \mu')| \leq O(T) \text{rad} \left(M_{\text{LP}}/T, \frac{B}{d} \right) + O(M_{\text{LP}} \frac{T}{B}) \text{rad} \left(\frac{B}{T}, \frac{B}{d} \right).$$

We obtain the same upper bound on $|\text{LP}(\mathcal{D}, \mu) - \text{LP}(\mathcal{D}, \mu'')|$, and the claim follows. \square

We will carry the upper bound in Equation (17) through the rest of the analysis. To simplify the equations, we will denote it Φ_p .

Claim 6.8. Let Φ_p denote the right-hand side of Equation (17).

(a) Fix any latent structure $\mu^* \in I_p$, and any distributions $\mathcal{D}', \mathcal{D}'' \in \Delta_p$. Then

$$|\text{LP}(\mathcal{D}', \mu^*) - \text{LP}(\mathcal{D}'', \mu^*)| \leq 2\Phi_p.$$

(b) $\text{LP}(\mathcal{D}_{p,x}, \mu) \geq \text{OPT}_{\text{LP}} - 2\Phi_p$, for each distribution $\mathcal{D}_{p,x}$ chosen by the algorithm in phase p .

Proof. (a). Since $\mathcal{D}', \mathcal{D}'' \in \Delta_p$, it holds that \mathcal{D}' and \mathcal{D}'' are LP-perfect for some latent structures μ' and μ'' . Further, pick a distribution \mathcal{D}^* that is LP-perfect for μ^* . Then:

$$\begin{aligned} \text{LP}(\mathcal{D}', \mu^*) &\geq \text{LP}(\mathcal{D}', \mu') - \Phi_p && (\text{by Lemma 6.7 with } \mathcal{D} = \mathcal{D}') \\ &\geq \text{LP}(\mathcal{D}^*, \mu') - \Phi_p \\ &\geq \text{LP}(\mathcal{D}^*, \mu^*) - 2\Phi_p && (\text{by Lemma 6.7 with } \mathcal{D} = \mathcal{D}^*) \\ &\geq \text{LP}(\mathcal{D}'', \mu^*) - 2\Phi_p. \end{aligned} \quad \square$$

(b). Follows from part (a) because $\mathcal{D}_{p,x} \in \Delta_p$ by the specification of BalanceBwK, and $\text{OPT}_{\text{LP}} = \text{LP}(\mathcal{D}^*, \mu)$ for some distribution $\mathcal{D}^* \in \Delta_p$.

The following corollary lower-bounds the average reward; once we have it, it essentially remains to lower-bound the stopping time of the algorithm.

Corollary 6.9. $\hat{r}_p \geq \frac{1}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p \log p))$, where Φ_p denotes the right-hand side of Equation (17).

Proof. By Claim 6.8(b), for each distribution $\mathcal{D}_{q,x}$ chosen by the algorithm in phase $q < p$ it holds that

$$r(\mathcal{D}_{q,x}, \mu) \geq \frac{1}{T} \text{LP}(\mathcal{D}_{q,x}, \mu) \geq \frac{1}{T} (\text{OPT}_{\text{LP}} - 2\Phi_q).$$

Averaging the above equation over all rounds in phases $q < p$, we obtain

$$\begin{aligned} r(\bar{\mathcal{D}}_p, \mu) &\geq \frac{1}{T} \left(\text{OPT}_{\text{LP}} - \frac{2}{p} \sum_{q < p} \Phi_q \right) \\ &\geq \frac{1}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p \log p)). \end{aligned}$$

For the last inequality, we used Claim 6.1(fg) to average the confidence radii in Φ_q .

Using the high-probability event in Claim 6.4(c):

$$\hat{r}_p \geq r(\bar{\mathcal{D}}_p, \mu) - \text{rad}(r(\bar{\mathcal{D}}_p, \mu), p|X|).$$

Now using the monotonicity of $\nu - \text{rad}(\nu, N)$ (Claim 6.1(c)) we obtain

$$\begin{aligned} \hat{r}_p &\geq \frac{1}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p)) - \text{rad} \left(\frac{1}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p)), p|X| \right) \\ &\geq \frac{1}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p)) - \text{rad} \left(M_{\text{LP}}/T, p|X| \right) \\ &\geq \frac{1}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p)). \end{aligned}$$

For the last equation, we use the fact that $\Phi_p/T \geq \Omega(\text{rad}(M_{\text{LP}}/T, \frac{p}{d})) \geq \Omega(\text{rad}(M_{\text{LP}}/T, p|X|))$. \square

The following two claims help us to lower-bound the stopping time of the algorithm.

Claim 6.10. $c_i(\mathcal{D}_{p,x}, \mu) \leq \frac{B}{T} + O(1) \text{rad} \left(\frac{B}{T}, \frac{p}{d} \right)$ for each resource i .

Proof. By the algorithm's specification, $\mathcal{D}_{p,x} \in \Delta_p$, and moreover there exists a latent structure $\mu' \in I_p$ such that $\mathcal{D}_{p,x}$ is LP-perfect for μ' . Apply Claim 6.6, noting that $c_i(\mathcal{D}_{p,x}, \mu') \leq \frac{B}{T}$ by LP-perfectness. \square

Corollary 6.11. $\hat{c}_{p,i} \leq \frac{B}{T} + O(\log p) \text{rad} \left(\frac{B}{T}, \frac{p}{d} \right)$ for each resource i .

Proof. Using a property of the clean execution, namely the event in Claim 6.4(c), we have

$$\hat{c}_{p,i} \leq c_i(\bar{\mathcal{D}}, \mu) + \text{rad} \left(c_i(\bar{\mathcal{D}}, \mu), p \right). \quad (19)$$

Consider all rounds preceding phase p .

$$\begin{aligned} c_i(\bar{\mathcal{D}}_p, \mu) &= \frac{1}{p|X|} \sum_{q < p, x \in X} c_i(\mathcal{D}_{q,x}, \mu) \\ &\leq \frac{B}{T} + \frac{O(1)}{p|X|} \sum_{q < p, x \in X} \text{rad} \left(\frac{B}{T}, \frac{p}{d} \right) \quad (\text{by Claim 6.10}) \\ &\leq \frac{B}{T} + O(\log p) \text{rad} \left(\frac{B}{T}, \frac{p}{d} \right) \quad (\text{by Claim 6.1(fg)}). \end{aligned} \quad (20)$$

For the last inequality, we used Claim 6.1(fg) to average the confidence radii.

Using the upper bound on $c_i(\bar{\mathcal{D}}, \mu)$ that we derived above,

$$\text{rad} \left(c_i(\bar{\mathcal{D}}, \mu), \frac{p}{d} \right) \leq O(\log p) \text{rad} \left(\frac{B}{T} + \text{rad} \left(\frac{B}{T}, \frac{p}{d} \right), \frac{p}{d} \right).$$

Using a general property of the confidence radius that

$$\text{rad}(\nu + \text{rad}(\nu, N), N) \leq O(\text{rad}(\nu, N)),$$

we conclude that

$$\text{rad}(c_i(\bar{\mathcal{D}}, \mu), \frac{p}{d}) \leq O(\log p) \text{rad}(\frac{B}{T}, \frac{p}{d}). \quad (21)$$

We obtain the claim by plugging the upper bounds (20) and (21) into (19). \square

We are ready to put the pieces together and derive the performance guarantee for a clean execution of BalanceBwK.

Lemma 6.12. *Consider a clean execution of BalanceBwK. Letting Φ_p denote the right-hand side of Equation (17), the total reward of the algorithm is $\text{REW} \geq \text{OPT}_{\text{LP}} - O(\Phi_{T/|X|} \log(T/|X|))$.*

Proof. Let p be the last phase in the execution of the algorithm, and let T_0 be the stopping time. Letting $m = |X|$, note that $pm < T_0 \leq (p+1)m$.

We can use Corollary 6.9 to bound REW from below:

$$\text{REW} = T_0 \hat{r}_{p+1} > pm \hat{r}_{p+1} \geq \frac{pm}{T} (\text{OPT}_{\text{LP}} - O(\Phi_p \log p)). \quad (22)$$

Let us bound $\frac{pm}{T}$ from below. The algorithm stops either when it runs out of time or if it runs out of resources during phase p . In the former case, $p = \lfloor T/m \rfloor$. In the latter case, $B = T_0 \hat{c}_{p+1,i}$ for some resource i , so $B \leq m(p+1) \hat{c}_{p+1,i}$. Using Corollary 6.11, we obtain the following lower bound on p :

$$\frac{pm}{T} \geq 1 - O\left(\frac{pm \log p}{B}\right) \text{rad}\left(\frac{B}{T}, \frac{p}{d}\right).$$

Plugging this into Equation (22), we conclude:

$$\begin{aligned} \text{REW} &\geq \text{OPT}_{\text{LP}} - O\left(\frac{pm \log p}{B}\right) \text{rad}\left(\frac{B}{T}, \frac{p}{d}\right) \text{OPT}_{\text{LP}} - O\left(\frac{pm \log p}{T}\right) \Phi_p \\ &\geq \text{OPT}_{\text{LP}} - O\left(\frac{pm \log p}{T}\right) \left(\Phi_p + \frac{T}{B} \text{rad}\left(\frac{B}{T}, \frac{p}{d}\right) \text{OPT}_{\text{LP}}\right) \\ &\geq \text{OPT}_{\text{LP}} - \frac{pm \log p}{T} O(\Phi_p). \end{aligned}$$

To complete the proof, we observe that $(p \Phi_p \log p)$ is increasing in p (by definition of Φ_p), and plug in a trivial upper bound $p \leq T/m$. \square

Using Claim 6.1(c), we can write

$$\Phi_{T/|X|} = O(1) \left(\text{rad}(M_{\text{LP}}, \frac{1}{d|X|}) + \frac{M_{\text{LP}}}{B} \text{rad}(B, \frac{1}{d|X|}) \right).$$

Rearranging the term, we obtain the bound claimed in Theorem 4.1.

7 Analysis of PD-BwK (Proof of Theorem 4.2)

In this section we present the analysis of the PD-BwK algorithm. We begin by recalling the Hedge algorithm from online learning theory, and its performance guarantee. Next, we present a simplified analysis of PD-BwK in a toy model in which the outcome vectors are deterministic. (This toy model of the BwK problem is uninteresting as a learning problem since the latent structure does not need to be learned, and the problem reduces to solving a linear program. We analyze PD-BwK first in this context because the analysis is simple, yet its main ideas carry over to the general case which is more complicated.)

7.1 The Hedge algorithm

In this section we present the Hedge algorithm, also known as the multiplicative weights algorithm. It is an online algorithm for maintaining a d -dimensional probability vector y while observing a sequence of d -dimensional payoff vectors π_1, \dots, π_τ . The algorithm is initialized with a parameter $\epsilon \in (0, 1)$.

Algorithm 3 The algorithm Hedge(ϵ)

- 1: $v_1 = \mathbf{1}$
 - 2: **for** $t = 1, 2, \dots, \tau$ **do**
 - 3: $y_t = v_t / (\mathbf{1}^\top v_t)$.
 - 4: $v_{t+1} = \text{Diag}\{(1 + \epsilon)^{\pi_{ti}}\} v_t$.
-

The Hedge algorithm is due to Freund and Schapire [18]. The version presented above, along with the following performance guarantee, are adapted from [25].

Proposition 7.1. *For any $0 < \epsilon < 1$ and any sequence of payoff vectors $\pi_1, \dots, \pi_\tau \in [0, 1]^d$, we have*

$$\forall y \in \Delta[d] \quad \sum_{t=1}^{\tau} y_t^\top \pi_t \geq (1 - \epsilon) \sum_{t=1}^{\tau} y^\top \pi_t - \frac{\ln d}{\epsilon}.$$

A self-contained proof of the proposition, for the reader's convenience, appears in Appendix B.

7.2 Warm-up: The deterministic case

To present the application of Hedge to BwK in its purest form, we first present an algorithm for the case in which the rewards of the various arms are deterministically equal to the components of a vector $r \in \mathbb{R}^m$, and the resource consumption vectors are deterministically equal to the columns of a matrix $C \in \mathbb{R}^{d \times m}$.

Algorithm 4 Algorithm PD-BwK, adapted for deterministic outcomes

- 1: In the first m rounds, pull each arm once.
 - 2: Let r, C denote the reward vector and resource consumption matrix revealed in Step 1.
 - 3: $v_1 = \mathbf{1}$
 - 4: $t = 1$
 - 5: **for** $t = m + 1, \dots, \tau$ (i.e., until resource budget exhausted) **do**
 - 6: $y_t = v_t / (\mathbf{1}^\top v_t)$.
 - 7: Pull arm x_j such that point-mass distribution $z_t = \mathbf{e}_j$ belongs to $\text{argmin}_{z \in \Delta[X]} \{ \frac{y_t^\top C z}{r^\top z} \}$.
 - 8: $v_{t+1} = \text{Diag}\{(1 + \epsilon)^{\mathbf{e}_j^\top C z_t}\} v_t$.
 - 9: $t = t + 1$.
-

This algorithm is an instance of the multiplicative-weights update method for solving packing linear programs. Interpreting it through the lens of online learning, as in the survey [3], it is updating a vector y using the Hedge algorithm. The payoff vector in round t is given by Cx_t . Note that these payoffs belong to $[0, 1]$, by our assumption that C has $[0, 1]$ entries.

Now let ξ^* denote an optimal solution of the primal linear program (P) from Section 3, and let $\text{OPT}_{\text{LP}} = r^\top \xi^*$ denote the optimal value of that LP. Let $\text{REW} = \sum_{t=m+1}^{\tau-1} r^\top z_t$ denote the total payoff obtained by the algorithm after its start-up phase (the first m rounds). By the stopping condition for the algorithm, we know there is a vector $y \in \Delta[\mathcal{X}]$ such that $y^\top C (\sum_{t=1}^{\tau} z_t) \geq B$ so fix one such y . Note that the algorithm's

start-up phase consumes at most m units of any resource, and the final round τ consumes at most one unit more, so

$$y^\top C \left(\sum_{m < t < \tau} z_t \right) \geq B - m - 1. \quad (23)$$

Finally let

$$\bar{y} = \frac{1}{\text{REW}} \sum_{m < t < \tau} (r^\top z_t) y_t.$$

Now we have

$$\begin{aligned} B &\geq \bar{y}^\top C \xi^* = \frac{1}{\text{REW}} \sum_{m < t < \tau} (r^\top z_t) (y_t^\top C \xi^*) \\ &\geq \frac{1}{\text{REW}} \sum_{m < t < \tau} (r^\top \xi^*) (y_t^\top C z_t) \\ &\geq \frac{\text{OPT}_{\text{LP}}}{\text{REW}} \left[(1 - \epsilon) \sum_{m < t < \tau} y_t^\top C z_t - \frac{\ln d}{\epsilon} \right] \\ &\geq \frac{\text{OPT}_{\text{LP}}}{\text{REW}} \left[B - \epsilon B - m - 1 - \frac{\ln d}{\epsilon} \right]. \end{aligned}$$

The first inequality is by primal feasibility of ξ^* , the second is by the definition of z_t , the third is the performance guarantee of Hedge, the fourth is by (23). Putting these inequalities together we have derived

$$\frac{\text{REW}}{\text{OPT}_{\text{LP}}} \geq 1 - \epsilon - \frac{m + 1}{B} - \frac{\ln d}{\epsilon B}.$$

Setting $\epsilon = \sqrt{\frac{\ln d}{B}}$ we find that the algorithm's regret is bounded above by $\text{OPT}_{\text{LP}} \cdot O\left(\sqrt{\frac{\ln d}{B}} + \frac{m}{B}\right)$.

7.3 Analysis modulo error terms

We now commence the analysis of Algorithm PD-BwK. In this section we show how to reduce the problem of bounding the algorithm's regret to a problem of estimating two error terms that reflect the difference between the algorithm's confidence-bound estimates of its own reward and resource consumption with the empirical values of these random variables. The analysis of those error terms will be the subject of Section 7.4.

By Theorem 2.1 and our choice of C_{rad} , it holds with probability at least $1 - T^{-1}$ that the confidence interval for every latent parameter, in every round of execution, contains the true value of that latent parameter. We call this high-probability event a *clean execution* of PD-BwK. Our regret guarantee will hold deterministically assuming that a clean execution takes place. The regret can be at most T when a clean execution does not take place, and since this event has probability at most T^{-1} it contributes only $O(1)$ to the regret. We will henceforth assume a clean execution of PD-BwK.

Claim 7.2. *In a clean execution of Algorithm PD-BwK, the algorithm's total reward satisfies the bound*

$$\text{REW} \geq \text{OPT}_{\text{LP}} - \left[2\text{OPT}_{\text{LP}} \left(\sqrt{\frac{\ln d}{B}} + \frac{m + 1}{B} \right) + m + 1 \right] - \frac{\text{OPT}_{\text{LP}}}{B} \left\| \sum_{m < t < \tau} E_t z_t \right\|_\infty - \left| \sum_{m < t < \tau} \delta_t^\top z_t \right|, \quad (24)$$

where E_t and δ_t are defined by

$$E_t = C_t - L_t, \quad \delta_t = u_t - r_t. \quad (25)$$

Proof. The claim is proven by mimicking the analysis of Algorithm 4 in the preceding section, incorporating error terms that reflect the differences between observable values and latent ones. As before, let ξ^* denote an optimal solution of the primal linear program (P), and let $\text{OPT}_{\text{LP}} = r^\top \xi^*$ denote the optimal value of that LP. Let $\text{REW}_{\text{UCB}} = \sum_{m < t < \tau} u_t^\top z_t$ denote the total payoff the algorithm would have obtained, after its start-up phase, if the actual payoff at time t were replaced with the upper confidence bound. By the stopping condition for the algorithm, we know there is a vector $y \in \Delta[\mathcal{R}]$ such that $y^\top (\sum_{t=1}^\tau C_t z_t) \geq B$, so fix one such y . As before,

$$y^\top \left(\sum_{m < t < \tau} C_t z_t \right) \geq B - m - 1. \quad (26)$$

Finally let

$$\bar{y} = \frac{1}{\text{REW}_{\text{UCB}}} \sum_{m < t < \tau} (u_t^\top z_t) y_t.$$

Assuming a clean execution, we have

$$\begin{aligned}
B &\geq \bar{y}^\top C \xi^* && (\xi^* \text{ is primal feasible}) \\
&= \frac{1}{\text{REW}_{\text{UCB}}} \sum_{m < t < \tau} (u_t^\top z_t) (y_t^\top C \xi^*) \\
&\geq \frac{1}{\text{REW}_{\text{UCB}}} \sum_{m < t < \tau} (u_t^\top z_t) (y_t^\top L_t \xi^*) && (\text{clean execution}) \\
&\geq \frac{1}{\text{REW}_{\text{UCB}}} \sum_{m < t < \tau} (u_t^\top \xi^*) (y_t^\top L_t z_t) && (\text{definition of } z_t) \\
&\geq \frac{1}{\text{REW}_{\text{UCB}}} \sum_{m < t < \tau} (r^\top \xi^*) (y_t^\top L_t z_t) && (\text{clean execution}) \\
&\geq \frac{\text{OPT}_{\text{LP}}}{\text{REW}_{\text{UCB}}} \left[(1 - \epsilon) y^\top \left(\sum_{m < t < \tau} L_t z_t \right) - \frac{\ln d}{\epsilon} \right] && (\text{Hedge guarantee}) \\
&= \frac{\text{OPT}_{\text{LP}}}{\text{REW}_{\text{UCB}}} \left[(1 - \epsilon) y^\top \left(\sum_{m < t < \tau} C_t z_t \right) - (1 - \epsilon) y^\top \left(\sum_{m < t < \tau} E_t z_t \right) - \frac{\ln d}{\epsilon} \right] \\
&\geq \frac{\text{OPT}_{\text{LP}}}{\text{REW}_{\text{UCB}}} \left[(1 - \epsilon) B - m - 1 - (1 - \epsilon) y^\top \left(\sum_{m < t < \tau} E_t z_t \right) - \frac{\ln d}{\epsilon} \right] && (\text{definition of } y; \text{ see eq. (26)}) \\
\text{REW}_{\text{UCB}} &\geq \text{OPT}_{\text{LP}} \left[1 - \epsilon - \frac{m+1}{B} - \frac{1}{B} \left\| \sum_{m < t < \tau} E_t z_t \right\|_\infty - \frac{\ln d}{\epsilon B} \right]. \quad (27)
\end{aligned}$$

The algorithm's actual payoff, $\text{REW} = \sum_{t=1}^\tau r_t^\top z_t$, satisfies the inequality

$$\text{REW} \geq \text{REW}_{\text{UCB}} - \sum_{m < t < \tau} (u_t - r_t)^\top z_t = \text{REW}_{\text{UCB}} - \sum_{m < t < \tau} \delta_t^\top z_t.$$

Combining this inequality with (27), we obtain the bound (24), as claimed. \square

7.4 Error analysis

In this section we complete the proof of Theorem 4.2 by deriving upper bounds on the terms $\left\| \sum_{m < t < \tau} E_t z_t \right\|_\infty$ and $\left| \sum_{m < t < \tau} \delta_t^\top z_t \right|$ appearing on the right side of (24). Both bounds are special cases of a more general

statement, presented as Lemma 7.4 below. Before stating the lemma, we need to establish a simple fact about confidence radii.

Lemma 7.3. *For any two vectors $a, N \in \mathbb{R}_+^m$, we have*

$$\sum_{j=1}^m \text{rad}(a_j, N_j) N_j \leq \sqrt{C_{\text{rad}} m (a^\top N)} + C_{\text{rad}} m. \quad (28)$$

Proof. The definition of $\text{rad}(\cdot, \cdot)$ implies that $\text{rad}(a_j, N_j) N_j \leq \sqrt{C_{\text{rad}} a_j N_j} + C_{\text{rad}}$. Summing these inequalities and applying Cauchy-Schwarz,

$$\sum_{j=1}^m \text{rad}(a_j, N_j) N_j \leq \sum_{j=1}^m \sqrt{C_{\text{rad}} a_j N_j} + C_{\text{rad}} m \leq \sqrt{m} \cdot \sqrt{\sum_{j \in X} C_{\text{rad}} a_j N_j} + C_{\text{rad}} m,$$

and the lemma follows by rewriting the expression on the right side. \square

The next lemma requires some notation and definitions. Suppose that a_1, \dots, a_τ is a sequence of vectors in $[0, 1]^m$ and that z_1, \dots, z_τ is a sequence of vectors in $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$. Assume that the latter sequence begins with a permutation of the elements of $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ so that, for every $s \geq m$, the diagonal matrix $\sum_{t=1}^s z_t z_t^\top$ is invertible. The vector

$$\bar{a}_s = \left(\sum_{t=1}^s z_t z_t^\top \right)^{-1} \left(\sum_{t=1}^s z_t z_t^\top a_t \right)$$

can be interpreted as the vector of empirical averages: the entry $\bar{a}_{s,j}$ is equal to the average of $a_{t,j}$ over all times $t \leq s$ such that $z_t = \mathbf{e}_j$. Let $\mathbf{N}_s = \sum_{t=1}^s z_t$ denote the vector whose entry $\mathbf{N}_{s,j}$ indicates the number of times \mathbf{e}_j occurs in the sequence z_1, \dots, z_s . A useful identity is

$$\bar{a}_s^\top \mathbf{N}_t = \sum_{s=1}^t a_s^\top z_s.$$

Lemma 7.4. *Suppose we are given sequences of vectors a_1, \dots, a_τ and z_1, \dots, z_τ as above. Suppose we are additionally a sequence of vectors b_1, \dots, b_τ in $[0, 1]^m$ and another vector a_0 such that for $m < t < \tau$ and $j \in X$,*

$$|b_{t,j} - a_{0,j}| \leq 2 \text{rad}(\bar{a}_{t,j}, N_{t,j}) \leq 6 \text{rad}(a_{0,j}, N_{t,j}).$$

Let $s = \tau - 1$ and suppose that for all $j \in X$, $|\bar{a}_{s,j} - a_{0,j}| \leq \text{rad}(\bar{a}_{s,j}, N_{s,j})$. Then

$$\left| \sum_{m < t < \tau} (b_t - a_t)^\top z_t \right| \leq O \left(\sqrt{C_{\text{rad}} m A} + C_{\text{rad}} m \right), \quad (29)$$

where $A = (\bar{a}_{\tau-1})^\top \mathbf{N}_{\tau-1} = \sum_{t=1}^{\tau-1} a_t^\top z_t$.

Proof. The proof decomposes the left side of (29) as a sum of three terms,

$$\sum_{m < t < \tau} (b_t - a_t)^\top z_t = \sum_{t=1}^m (a_t - b_t)^\top z_t + \sum_{t=1}^{\tau-1} (b_t - a_0)^\top z_t + \sum_{t=1}^{\tau-1} (a_0 - a_t)^\top z_t, \quad (30)$$

then bounds the three terms separately. The first sum is clearly bounded above by m . We next work on bounding the third sum. Let $s = \tau - 1$.

$$\begin{aligned} \sum_{t=1}^s (a_0 - a_t)^\top z_t &= a_0^\top \mathbf{N}_t - \sum_{t=1}^s a_t^\top z_t = (a_0 - \bar{a}_s)^\top \mathbf{N}_s \\ \left| \sum_{t=1}^s (a_0 - a_t)^\top z_t \right| &= |(a_0 - \bar{a}_s)^\top \mathbf{N}_s| \leq \sum_{j \in X} \text{rad}(\bar{a}_{s,j}, N_{s,j}) N_{s,j} \\ &\leq \sqrt{C_{\text{rad}} m A} + C_{\text{rad}} m, \end{aligned}$$

where the last line follows from Lemma 7.3.

Finally we bound the middle sum in (30).

$$\begin{aligned} \left| \sum_{t=1}^s (b_t - a_0)^\top z_t \right| &\leq 6 \sum_{t=1}^s \sum_{j \in X} \text{rad}(a_{0,j}, N_{t,j}) z_{t,j} \\ &= 6 \sum_{j \in X} \sum_{\ell=1}^{N_{s,j}} \text{rad}(a_{0,j}, \ell) \\ &= O \left(\sum_{j \in X} \text{rad}(a_{0,j}, N_{s,j}) N_{s,j} \right) \\ &\leq O \left(\sqrt{C_{\text{rad}} n a_0^\top \mathbf{N}_s} + C_{\text{rad}} m \right). \end{aligned}$$

We would like to replace the expression $a_0^\top \mathbf{N}_s$ on the last line with the expression $\bar{a}_s^\top \mathbf{N}_s = A$. To do so, recall from earlier in this proof that $|(a_0 - \bar{a}_s)^\top \mathbf{N}_s| \leq \sqrt{C_{\text{rad}} m A} + C_{\text{rad}} m$, and now apply the following calculation:

$$\begin{aligned} a_0^\top \mathbf{N}_s &\leq \bar{a}_s^\top \mathbf{N}_s + \sqrt{C_{\text{rad}} m A} + C_{\text{rad}} m = A + \sqrt{C_{\text{rad}} m A} + C_{\text{rad}} m \leq \left(\sqrt{A} + \sqrt{C_{\text{rad}} m} \right)^2 \\ \sqrt{C_{\text{rad}} m a_0^\top \mathbf{N}_s} &\leq \sqrt{C_{\text{rad}} m} \left(\sqrt{A} + \sqrt{C_{\text{rad}} m} \right) = \sqrt{C_{\text{rad}} m A} + C_{\text{rad}} m. \end{aligned}$$

Summing up the upper bounds for the three terms on the right side of (30), we obtain (29). \square

Corollary 7.5. *In a clean execution of PD-BwK,*

$$\left| \sum_{m < t < \tau} \delta_t z_t \right| \leq O \left(\sqrt{C_{\text{rad}} m \text{REW}} + C_{\text{rad}} m \right)$$

and

$$\left\| \sum_{m < t < \tau} E_t z_t \right\|_\infty \leq O \left(\sqrt{C_{\text{rad}} m B} + C_{\text{rad}} m \right).$$

Proof. The first inequality is obtained by applying Lemma 7.4 with vector sequences $a_t = r_t$ and $b_t = u_t$. The second one is obtained by applying the same lemma with vector sequences $a_t = \mathbf{e}_i^\top C_t$ and $b_t = \mathbf{e}_i^\top L_t$, for each resource i . \square

Proof of Theorem 4.2: The theorem asserts that

$$\text{REW} \geq \text{OPT} - O\left(\sqrt{m \log(dmT)} \left(\frac{\text{OPT}}{\sqrt{B}} + \sqrt{\text{OPT}} + \sqrt{m \log(dmT)}\right)\right).$$

Denote the right-hand side by $f(\text{OPT})$. The function $\max\{f(x), 0\}$ is a non-decreasing function of x , so to establish the theorem we will prove that $\text{REW} \geq f(\text{OPT}_{\text{LP}}) \geq f(\text{OPT})$, where the latter inequality is an immediate consequence of the fact that $\text{OPT}_{\text{LP}} \geq \text{OPT}$ (Lemma 3.1).

To prove $\text{REW} \geq f(\text{OPT}_{\text{LP}})$, we begin by recalling inequality (24) and observing that

$$2\text{OPT}_{\text{LP}} \left(\sqrt{\frac{\ln d}{B}} + \frac{m+1}{B} \right) = O\left(\sqrt{m \log(dmT)} \frac{\text{OPT}_{\text{LP}}}{\sqrt{B}}\right)$$

by our assumption that $m \leq B$. The term $m+1$ on the right side of Equation (24) is bounded above by $m \log(dmT)$. Finally, using Corollary 7.5 we see that the sum of the final two terms on the right side of (24) is bounded by $O\left(\sqrt{C_{\text{rad}} m} \left(\frac{\text{OPT}_{\text{LP}}}{\sqrt{B}} + \sqrt{\text{OPT}_{\text{LP}}} + \sqrt{C_{\text{rad}} m}\right)\right)$. The theorem follows by plugging in $C_{\text{rad}} = \Theta(\log(dmT))$. \square

8 BwK with discretization

In this section we flesh out our approach to BwK with uniform discretization, as discussed early in Section 5. The technical contribution here is a definition of the appropriate structure, and proof that discretization does not do much damage.

Definition 8.1. We say that arm x ϵ -covers arm y if the following two properties are satisfied for each resource i and every latent structure μ in a given BwK domain:

- (i) $r(x, \mu)/c_i(x, \mu) \geq r(y, \mu)/c_i(y, \mu) - \epsilon$.
- (ii) $c_i(x, \mu) \geq c_i(y, \mu)$.

A subset $S \subset X$ of arms is called an ϵ -discretization of X if each arm in X is ϵ -covered by some arm in S .

Note that we consider the difference in the ratio of expected reward to expected consumption, whereas for MAB in metric spaces it suffices to consider the difference in expected rewards. For a typical application of uniform discretization it holds that $X \subset \mathbb{R}^\ell$ for some $\ell \in \mathbb{N}$, and a suitably chosen mesh on X is an ϵ -discretization. There are several types of meshes one could consider, depending on the particular BwK domain: *additive δ -mesh* $\delta \mathbb{N}^\ell$, *multiplicative δ -mesh* $\{(1 - \delta)^j : j \in \mathbb{N}\}^\ell$, and *hyperbolic δ -mesh* $\{\frac{1}{1+j\delta} : j \in \mathbb{N}\}^\ell$. For a particular mesh $M \subset \mathbb{R}^\ell$, the “mesh on X ” is simply $X \cup M$.

Once we construct an ϵ -discretization S of X , we can run a BwK algorithm on S and obtain expected total reward competitive with $\text{OPT}_{\text{LP}}(S)$, the value of OPT_{LP} if the action space is restricted to S . This makes sense if $\text{OPT}_{\text{LP}}(S)$ is not much worse than $\text{OPT}_{\text{LP}}(X)$.

Theorem 8.2 (BwK: discretization). *Fix a BwK domain with action space X . Consider an algorithm ALG which achieves expected total reward $\text{REW} \geq \text{OPT}_{\text{LP}}(S) - R(S)$ if applied to the restricted action space $S \subset X$, for any given $S \subset X$. If S is an ϵ -discretization of X , for some $\epsilon \geq 0$, then*

$$\text{REW} \geq \text{OPT}_{\text{LP}}(X) - \epsilon d B - R(S).$$

The technical content in Theorem 8.2 is that $\text{OPT}_{\text{LP}}(S)$ is not much worse than $\text{OPT}_{\text{LP}}(X)$. Such result is typically trivial in MAB settings where the benchmark is the best fixed arm, but requires some work in our setting because we need to consider distributions over arms.

Lemma 8.3. *Consider an instance of BwK with action space X . Let $S \subset X$ be an ϵ -discretization of X , for some $\epsilon \geq 0$. Then $\text{OPT}_{\text{LP}}(S) \geq \text{OPT}_{\text{LP}}(X) - \epsilon d B$*

Proof. Let \mathcal{D} be the distribution over arms in X which maximizes $\text{LP}(\mathcal{D}, \mu)$. We use \mathcal{D} to construct a distribution \mathcal{D}_S over S which is nearly as good.

Let μ be the (actual) latent structure. For brevity, we will suppress μ from the notation: $c_i(x) = c_i(x, \mu)$ and $c_i(\mathcal{D}) = c_i(\mathcal{D}, \mu)$ for arms x , distributions \mathcal{D} and resources i . Similarly, we will write $r(x) = r(x, \mu)$ and $r(\mathcal{D}) = r(\mathcal{D}, \mu)$.

We define \mathcal{D}_S as follows. Since S is an ϵ -discretization of X , there exists a family of subsets $(\text{cov}(x) \subset X : x \in X)$ so that each arm x ϵ -covers all arms in $\text{cov}(x)$, the subsets are disjoint, and their union is X . Fix one such family of subsets, and define

$$\mathcal{D}_S(x) = \sum_{y \in \text{cov}(x)} \mathcal{D}(y) \min_i \frac{c_i(y)}{c_i(x)}, \quad x \in S.$$

Note that $\sum_{x \in S} \mathcal{D}_S(x) \leq 1$ by Definition 8.1(ii). With the remaining probability, the null arm is chosen (i.e., the algorithm skips a given round).

To argue that $\text{LP}(\mathcal{D}_S, \mu)$ is large, we upper-bound the resource consumption $c_i(\mathcal{D}_S)$, for each resource i , and lower-bound the reward $r(\mathcal{D}_S)$.

$$\begin{aligned} c_i(\mathcal{D}_S) &= \sum_{x \in S} c_i(x) \mathcal{D}_S(x) \\ &\leq \sum_{x \in S} c_i(x) \sum_{y \in \text{cov}(x)} \mathcal{D}(y) \frac{c_i(y)}{c_i(x)} = \sum_{x \in S} \sum_{y \in \text{cov}(x)} \mathcal{D}(y) c_i(y) = \sum_{y \in X} \mathcal{D}(y) c_i(y) \\ &= c_i(\mathcal{D}) \end{aligned} \tag{31}$$

(Note that the above argument did not use the property (i) in Definition 8.1.)

$$\begin{aligned} r(\mathcal{D}_S) &= \sum_{x \in S} r(x) \mathcal{D}_S(x) \\ &= \sum_{x \in S} r(x) \sum_{y \in \text{cov}(x)} \mathcal{D}(y) \min_i \frac{c_i(y)}{c_i(x)} \\ &= \sum_{x \in S} \sum_{y \in \text{cov}(x)} \mathcal{D}(y) \min_i \frac{c_i(y) r(x)}{c_i(x)} \\ &\geq \sum_{x \in S} \sum_{y \in \text{cov}(x)} \mathcal{D}(y) \min_i r(y) - \epsilon c_i(y) && \text{(by Definition 8.1(i))} \\ &= \sum_{y \in X} \mathcal{D}(y) \min_i r(y) - \epsilon c_i(y) \\ &\geq \sum_{y \in X} \mathcal{D}(y) (r(y) - \epsilon \sum_i c_i(y)) \\ &= r(\mathcal{D}) - \epsilon \sum_i c_i(\mathcal{D}). \end{aligned} \tag{32}$$

Let $\tau(\mathcal{D}) = \min_i \frac{B}{c_i(\mathcal{D})}$ be the stopping time in the linear relaxation, so that $\text{LP}(\mathcal{D}, \mu) = \tau(\mathcal{D}) r(\mathcal{D})$. By

Equation (31) we have $\tau(\mathcal{D}_S) \geq \tau(\mathcal{D})$. We are ready for the final computation:

$$\begin{aligned}
\text{LP}(\mathcal{D}_S, \mu) &= \tau(\mathcal{D}_S) r(\mathcal{D}_S) \\
&\geq \tau(\mathcal{D}) r(\mathcal{D}_S) \\
&\geq \tau(\mathcal{D}) (r(\mathcal{D}) - \epsilon \sum_i c_i(\mathcal{D})) && \text{(by Equation (32))} \\
&\geq r(\mathcal{D}) \tau(\mathcal{D}) - \epsilon \tau(\mathcal{D}) \sum_i c_i(\mathcal{D}) \\
&\geq \text{LP}(\mathcal{D}, \mu) - \epsilon d B. && \square
\end{aligned}$$

8.1 Discretization for dynamic procurement

A little more work is needed to apply uniform discretization for dynamic procurement. Consider dynamic procurement with non-unit supply, as defined in Section 5. Throughout this subsection, assume n agents, budget B , and ceiling Λ . Recall that in each round t the algorithm makes an offer of the form (p_t, λ_t) , where $p_t \in [0, 1]$ is the posted price per unit, and $\lambda_t \leq \Lambda$ is the maximal number of units offered for sale in this round. The action space here is $X = [0, 1] \times \{1, \dots, \Lambda\}$, the set of all possible (p, λ) pairs.

For each arm $x = (p, \lambda)$, let $S(x)$ be the expected number of items sold. Recall that expected consumption is $c(x) = p S(x)$, and expected reward is $S(x)$. It follows that $\frac{r(x)}{c(x)} = \frac{1}{p}$. By Definition 8.1 price p ϵ -covers price $q < p$ if and only if $\frac{1}{p} \geq \frac{1}{q} - \epsilon$.

It is easy to see that the hyperbolic ϵ -mesh S on $[0, 1]$ is an ϵ -discretization: namely, each arm (q, λ) is ϵ -covered by (p, λ) , where p is the smallest price in S such that $p \geq q$. Unfortunately, such S has infinitely many points. In fact, it is easy to see that any ϵ -discretization on X must be infinite (even for $\Lambda = 1$). To obtain a finite ϵ -discretization, we only consider prices $p \geq p_0$, for some parameter p_0 to be tuned later. We argue that this restriction is not too damaging:

Claim 8.4. *Consider dynamic procurement with non-unit supply. Then for any $p_0 \in (0, 1)$ it holds that*

$$\text{OPT}_{\text{LP}}(\{0\} \cup [p_0, 1]) \geq \text{OPT}_{\text{LP}} - \Lambda^2 p_0 n^2 / B.$$

Proof. When $p_0 > B/(n\Lambda)$ the bound is trivial and for the rest of the proof we assume that $p_0 \leq B/(n\Lambda)$. Let \mathcal{D} be the distribution over arms which maximizes $\text{LP}(\mathcal{D}, \mu)$ to get OPT_{LP} . By Claim 3.2, we can further assume that \mathcal{D} is LP-perfect. Thus, \mathcal{D} has a support set of size at most 2, say arms $x_i = (p_i, \lambda_i)$, $i = 1, 2$, where p_i is the posted price and λ_i is the maximal allowed number of items. W.l.o.g. assume $p_1 \leq p_2$. Note that p_1 can be 0, which would correspond to the “null arm”. Moreover, letting s_i denote the expected number of items bought for arm (p_i, λ_i) , then by LP-perfectness the expected consumption is

$$c(\mathcal{D}, \mu) = \sum_{i=1}^2 \mathcal{D}(x_i) p_i s_i \leq B/n. \quad (33)$$

(Formally, to apply Claim 3.2 one needs to divide the rewards and consumptions (and hence the budget) by Λ , so that consumptions and rewards in each round are at most 1.)

Let us define a distribution \mathcal{D}' which has support in $\{0\} \cup [p_0, 1]$.

- If $p_1 \geq p_0$ then define $\mathcal{D}' = \mathcal{D}$ and we get $\text{OPT}_{\text{LP}}(\{0\} \cup [p_0, 1]) \geq \text{OPT}_{\text{LP}}$.
- From here on, assume $p_1 < p_0$. If \mathcal{D} is a distribution of support 1 (i.e $\mathcal{D}'(x_2) = 0$) then define $\mathcal{D}'(B/(n\Lambda), \lambda_1) = 1$ and we get $\text{OPT}_{\text{LP}}(\{0\} \cup [p_0, 1]) \geq \text{OPT}_{\text{LP}}$.

- If \mathcal{D} is a distribution of support 2 then by LP-perfectness Equation (33) is satisfied with equality. Therefore $p_2 \geq B/(ns_2) \geq B/(n\Lambda)$. Define

$$\begin{aligned}\mathcal{D}'(p_0, \lambda_1) &= \mathcal{D}(p_1, \lambda_1) \\ \mathcal{D}'(p_2, \lambda_2) &= \max(0, \mathcal{D}(p_2, \lambda_2) - \frac{p_0\Lambda}{p_2}) \\ \mathcal{D}'(0) &= 1 - \mathcal{D}'(p_0) - \mathcal{D}'(p_2).\end{aligned}$$

Then \mathcal{D}' forms a feasible solution to the linear program (6), with support in $\{0\} \cup [p_0, 1]$, and with value $\text{LP}(\mathcal{D}', \mu) \geq \text{OPT}_{\text{LP}} - \Lambda p_0 n / p_2 \geq \text{OPT}_{\text{LP}} - \Lambda^2 p_0 n^2 / B$.

□

Let ALG be a BwK algorithm which achieves regret bound (1). Suppose ALG is applied to a discretized instance of dynamic procurement, with m arms, n agents and budget B . For unit supply ($\Lambda = 1$), ALG has regret $R(m, n, B) = \tilde{O}(\sqrt{mn} + n\sqrt{m/B})$. For non-unit supply, in order to apply ALG we need to normalize the problem instance so that per-round rewards and consumptions are at most 1. That is, we need to divide all rewards, consumptions, and the budget, by the factor of Λ . Thus we obtain regret $R(m, n, B/\Lambda)$ on the normalized instance, and therefore regret $\Lambda R(m, n, B/\Lambda)$ on the original problem instance. Using Lemma 8.3 and Claim 8.4 (and normalizing / re-normalizing) we obtain the following regret bound:

Corollary 8.5. *Consider dynamic procurement with non-unit supply. Fix $\epsilon, p_0 \in (0, 1)$, and let S be the hyperbolic ϵ -mesh on $[p_0, 1]$. Then running ALG on arms $S \times \{1, \dots, \Lambda\}$ achieves regret*

$$\Lambda R(\Lambda|S|, n, B/\Lambda) + \epsilon B + p_0 \Lambda^2 n^2 / B.$$

Optimizing the parameters ϵ, p_0 in Corollary 8.5, we obtain the final regret bound.

Theorem 8.6. *Consider dynamic procurement with non-unit supply. Assume n agents, budget B and ceiling Λ . Let ALG be a BwK algorithm which achieves regret bound (1). Applying ALG with a suitably chosen discretization yields regret $\tilde{O}(\Lambda^{3/2} n / B^{1/4})$.*

9 Lower Bound

We prove that regret (1) obtained by algorithm PD-BwK is optimal up to polylog factors.

Theorem 9.1. *Informally, any algorithm for BwK must incur regret*

$$\Omega \left(\min \left(\text{OPT}, \text{OPT} \sqrt{\frac{m}{B}} + \sqrt{m \text{OPT}} \right) \right), \quad (34)$$

where $m = |X|$ is the number of arms and $B = \min_i B_i$ is the smallest budget.

More precisely, fix any $m \geq 2$, $d \geq 1$, $\text{OPT} \geq m$, and $(B_1, \dots, B_d) \in [2, \infty)$. Let \mathcal{F} be the family of all BwK problem instances with m arms, d resources, budgets (B_1, \dots, B_d) and optimal reward OPT . Then any algorithm for BwK must incur regret (34) in the worst case over \mathcal{F} .

We treat the two summands in Equation (34) separately:

Claim 9.2. *Consider the family \mathcal{F} from Theorem 9.1, and let ALG be some algorithm for BwK.*

- ALG incurs regret $\Omega \left(\min \left(\text{OPT}, \sqrt{m \text{OPT}} \right) \right)$ in the worst case over \mathcal{F} .
- ALG incurs regret $\Omega \left(\min \left(\text{OPT}, \text{OPT} \sqrt{\frac{m}{B}} \right) \right)$ in the worst case over \mathcal{F} .

Theorem 9.1 follows from Claim 9.2(ab). For part (a), we use a standard lower-bounding example for MAB. For part (b), we construct a new example, specific to BwK, and analyze it using KL-divergence.

Proof of Claim 9.2(a). Fix $m \geq 2$ and $0\text{OPT} \geq m$. Let \mathcal{F}_0 be the family of all MAB problem instances with m arms and time horizon $T = \lfloor 2\text{OPT} \rfloor$, where the “best arm” has expected reward $\mu^* = T/\text{OPT}$ and all other arms have reward $\mu^* - \epsilon$ with $\epsilon = \frac{1}{4}\sqrt{m/T}$. Note that $\mu^* \in [\frac{1}{2}, \frac{3}{4}]$ and $\epsilon \leq \frac{1}{4}$. It is well-known [5] that any MAB algorithm incurs regret $\Omega(\sqrt{m\text{OPT}})$ in the worst case over \mathcal{F}_0 .

To ensure that $\mathcal{F}_0 \subset \mathcal{F}$, let us treat each MAB instance in \mathcal{F}_0 as a BwK instance with d resources, budgets (B_1, \dots, B_d) , and no resource consumption. \square

9.1 The new lower-bounding example: proof of Claim 9.2(b)

Our lower-bounding example is very simple. There are m arms. Each arm gives reward 1 deterministically. There is a single resource with budget B .¹³ The resource consumption, for each arm and each round, is either 0 or 1. The expected resource consumption is $p - \epsilon$ for the “best arm” and p for all other arms, where $0 < \epsilon < p < 1$. There is time horizon $T < \infty$. Let $\mathcal{F}_{(p,\epsilon)}$ denote the family of all such problem instances, for fixed parameters (p, ϵ) . We analyze this family in the rest of this section.

We rely on the following fact about stopping times of random sums. For the sake of completeness, we provide a proof in Appendix C.

Fact 9.3. *Let S_t be the sum of t i.i.d. 0-1 variables with expectation q . Let τ^* be the first time this sum reaches a given number $B \in \mathbb{N}$. Then $\mathbb{E}[\tau^*] = B/q$. Moreover, for each $T > \mathbb{E}[\tau^*]$ it holds that*

$$\sum_{t > T} \Pr[\tau^* \geq t] \leq \mathbb{E}[\tau^*]^2/T.$$

Infinite time horizon. It is convenient to consider the family of problem instances which is the same as $\mathcal{F}_{(p,\epsilon)}$ except that it has the *infinite* time horizon; denote it $\mathcal{F}_{(p,\epsilon)}^\infty$. We will first prove the desired lower bound for this family, then extend it to $\mathcal{F}_{(p,\epsilon)}$.

The two crucial quantities that describe algorithm’s performance on an instance in $\mathcal{F}_{(p,\epsilon)}^\infty$ is the stopping time and the total number of plays of the best arm. (Note that the total reward is equal to the stopping time minus 1.) The following claim connects these two quantities.

Claim 9.4 (Stopping time). *Fix an algorithm ALG for BwK and a problem instance in $\mathcal{F}_{(p,\epsilon)}^\infty$. Consider an execution of ALG on this problem instance. Let τ be the stopping time of ALG. For each round t , let N_t be the number of rounds $s \leq t$ in which the best arm is selected. Then*

$$p\mathbb{E}[\tau] - \epsilon\mathbb{E}[N_\tau] = \lfloor B + 1 \rfloor.$$

Proof. Let C_t be the total resource consumption after round t . Note that $\mathbb{E}[C_t] = pt - \epsilon N_t$. We claim that

$$\mathbb{E}[C_\tau] = \mathbb{E}[p\tau - \epsilon N_\tau]. \tag{35}$$

Indeed, let $Z_t = C_t - (pt - \epsilon N_t)$. It is easy to see that Z_t is a martingale with bounded increments, and moreover that $\Pr[\tau < \infty] = 1$. Therefore the Optional Stopping Theorem applies to Z_t and τ , so that $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0] = 0$. Therefore we obtain Equation (35).

To complete the proof, it remains to show that $C_\tau = \lfloor B + 1 \rfloor$. Recall that ALG stops if and only if $C_t > B$. Since resource consumption in any round is either 0 or 1, it follows that $C_\tau = \lfloor B + 1 \rfloor$. \square

¹³More formally, other resources in the setting of Theorem 9.1 are not consumed. For simplicity, we leave them out.

Corollary 9.5. *Consider the setting in Claim 9.4. Then:*

- (a) *If ALG always chooses the best arm then $\mathbb{E}[\tau] = \lfloor B + 1 \rfloor / (p - \epsilon)$.*
- (b) *$\text{OPT} = \lfloor B + 1 \rfloor / (p - \epsilon) - 1$ for any problem instance in $\mathcal{F}_{(p, \epsilon)}^\infty$.*
- (c) *$p \mathbb{E}[\tau] - \epsilon \mathbb{E}[N_\tau] = (p - \epsilon) (1 + \text{OPT})$.*

Proof. For part(b), note that we have $\mathbb{E}[\tau] \leq \lfloor B + 1 \rfloor / (p - \epsilon)$, so $\text{OPT} \leq \lfloor B + 1 \rfloor / (p - \epsilon) - 1$. By part (a), the equality is achieved by the policy that always selects the best arm. \square

The heart of the proof is a KL-divergence argument which bounds the number of plays of the best arm. This argument is encapsulated in the following claim, whose proof is deferred to Section 9.3.

Lemma 9.6 (best arm). *Assume $p \leq \frac{1}{2}$ and $\frac{\epsilon}{p} \leq \frac{1}{16} \sqrt{\frac{m}{B}}$. Then for any BwK algorithm there exists a problem instance in $\mathcal{F}_{(p, \epsilon)}^\infty$ such that the best arm is chosen at most $\frac{3}{4} \text{OPT}$ times in expectation.*

Armed with this bound and Corollary 9.5(c), it is easy to lower-bound regret over $\mathcal{F}_{(p, \epsilon)}^\infty$.

Claim 9.7 (regret). *If $p \leq \frac{1}{2}$ and $\frac{\epsilon}{p} \leq \frac{1}{16} \sqrt{\frac{m}{B}}$ then any BwK algorithm incurs regret $\frac{\epsilon}{4p} \text{OPT}$ over $\mathcal{F}_{(p, \epsilon)}^\infty$.*

Proof. Fix any algorithm ALG for BwK. Consider the problem instance whose existence is guaranteed by Lemma 9.6. Let τ be the stopping time of ALG, and let N_t be the number of rounds $s \leq t$ in which the best arm is selected. By Lemma 9.6 we have $\mathbb{E}[N_\tau] \leq \frac{3}{4} \text{OPT}$. Plugging this into Corollary 9.5(c) and rearranging the terms, we obtain $\mathbb{E}[\tau] \leq (1 + \text{OPT})(1 - \frac{\epsilon}{4p})$. Therefore, regret of ALG is $\text{OPT} - (\mathbb{E}[\tau] - 1) \geq \frac{\epsilon}{4p} \text{OPT}$. \square

Thus, we have proved the lower bound for the infinite time horizon.

Finite time horizon. Let us “translate” a regret bound for $\mathcal{F}_{(p, \epsilon)}^\infty$ into a regret bound for $\mathcal{F}_{(p, \epsilon)}$.

We will need a more nuanced notation for OPT. Consider the family of problem instances in $\mathcal{F}_{(p, \epsilon)} \cup \mathcal{F}_{(p, \epsilon)}^\infty$ with a particular time horizon $T \leq \infty$. Let $\text{OPT}_{(p, \epsilon, T)}$ be the optimal expected total reward for this family (by symmetry, this quantity does not depend on which arm is the best arm). We will write $\text{OPT}_T = \text{OPT}_{(p, \epsilon, T)}$ when parameters (p, ϵ) are clear from the context.

Claim 9.8. *For any fixed (p, ϵ) and any $T > \text{OPT}_\infty$ it holds that $\text{OPT}_T \geq \text{OPT}_\infty - \text{OPT}_\infty^2 / T$.*

Proof. Let τ^* be the stopping time of a policy that always plays the best arm on a problem instance in $\mathcal{F}_{(p, \epsilon)}^\infty$.

$$\begin{aligned} \text{OPT}_\infty - \text{OPT}_T &= \mathbb{E}[\tau^*] - \mathbb{E}[\min(\tau^*, T)] \\ &= \sum_{t > T} (t - T) \Pr[\tau^* = t] \\ &= \sum_{t > T} \Pr[\tau^* \geq t] \\ &\leq \mathbb{E}[\tau^*] / T = \text{OPT}_\infty / T. \end{aligned}$$

The inequality is due to Fact 9.3. \square

Claim 9.9. *Fix (p, ϵ) and fix algorithm ALG. Let REG_T be the regret of ALG over the problem instances in $\mathcal{F}_{(p, \epsilon)} \cup \mathcal{F}_{(p, \epsilon)}^\infty$ with a given time horizon $T \leq \infty$. Then $\text{REG}_T \geq \text{REG}_\infty - \text{OPT}_\infty^2 / T$.*

Proof. For each problem instance $\mathcal{I} \in \mathcal{F}_{(p, \epsilon)}^\infty$, let $\text{REW}_T(\mathcal{I})$ be the expected total reward of ALG on \mathcal{I} , if the time horizon is $T \leq \infty$. Clearly, $\text{REW}_\infty(\mathcal{I}) \geq \text{REW}_T(\mathcal{I})$. Therefore, using Claim 9.8, we have:

$$\begin{aligned} \text{REG}_T &= \text{OPT}_T - \inf_{\mathcal{I}} \text{REW}_T(\mathcal{I}) \\ &\geq \text{OPT}_T - \inf_{\mathcal{I}} \text{REW}_\infty(\mathcal{I}) \\ &= \text{REG}_\infty + \text{OPT}_T - \text{OPT}_\infty \\ &\geq \text{REG}_\infty - \text{OPT}_\infty^2 / T. \end{aligned} \quad \square$$

Lemma 9.10 (regret: finite time horizon). *Fix $p \leq \frac{1}{2}$ and $\epsilon = \frac{p}{16} \sqrt{\frac{\min(m, B)}{B}}$. Then for any time horizon $T > 2 \text{OPT}_\infty$ and any BwK algorithm ALG there exists a problem instance in $\mathcal{F}_{(p, \epsilon)}$ with time horizon T for which ALG incurs regret $\Omega(\text{OPT}_T) \sqrt{\frac{\min(m, B)}{B}}$.*

Proof. By Claim 9.7, ALG incurs regret $\Omega(\frac{\epsilon}{p} \text{OPT}_\infty)$ for some problem instance in $\mathcal{F}_{(p, \epsilon)}^\infty$. By Claim 9.9, ALG incurs regret $\Omega(\frac{\epsilon}{p} \text{OPT}_\infty)$ for the same problem instance in $\mathcal{F}_{(p, \epsilon)}$ with time horizon T . Finally, it is clear that $\text{OPT}_\infty \geq \text{OPT}_T$. \square

Let us complete the proof of Claim 9.2(b). Recall that Claim 9.2(b) specifies the values for (m, B, OPT) that our problem instance must have. Since we have already proved Claim 9.2(a) and $\text{OPT} \sqrt{\frac{m}{B}} \leq O(\sqrt{m \text{OPT}})$ for $\text{OPT} < 3B$, it suffices to assume $\text{OPT} \geq 3B$.

Let us set ϵ as prescribed by Lemma 9.10. Then we obtain the desired regret for any parameter $p \leq \frac{1}{2}$ and any time horizon $T > 2 \text{OPT}_{(p, \epsilon, \infty)}$. Recall that

$$\text{OPT}_{(p, \epsilon, \infty)} = \frac{\Gamma}{p}, \quad \text{where } \Gamma = \lfloor B + 1 \rfloor / \left(1 - \frac{1}{16} \sqrt{\frac{\min(m, B)}{B}} \right).$$

Thus, it remains to pick such p and T so that $\text{OPT}_{(p, \epsilon, T)} = \text{OPT}$.

Let $f(p, T) = \text{OPT}_{(p, \epsilon, T)}$. By Claim 9.8, for any $p \leq \frac{1}{2}$ and $T > 2 \Gamma / p$ we have

$$\frac{\Gamma}{p} \geq f(p, T) \geq \frac{\Gamma}{2p}. \quad (36)$$

Since $\text{OPT} \geq 3B$, there exists $p_0 \leq \frac{1}{2}$ such that $\Gamma / p_0 \leq \text{OPT} \leq 2 \Gamma / p_0$. Let $T = 8 \Gamma / p_0$. Then Equation (36) holds for all $p \in [p_0/4, \frac{1}{2}]$. In particular,

$$f(p_0, T) \leq \Gamma / p_0 \leq \text{OPT} \leq 2 \Gamma / p_0 \leq f(p_0/4, T).$$

Since $f(p, T)$ is continuous in p , there exists $p \in [p_0/4, \frac{1}{2}]$ such that $f(p, T) = \text{OPT}$.

This completes the proof of Claim 9.2(b) and Theorem 9.1.

9.2 Background on KL-divergence (for the proof of Lemma 9.6)

The proof of Lemma 9.6 relies on the concept of KL-divergence. Let us provide some background to make on KL-divergence to make this proof self-contained. We use a somewhat non-standard notation that is tailored to the needs of our analysis.

The *KL-divergence* (a.k.a. *relative entropy*) is defined as follows. Consider two distributions μ, ν on the same finite universe Ω . Assume $\mu \ll \nu$ (in words, μ is *absolutely continuous* with respect to ν), meaning that $\nu(w) = 0 \Rightarrow \mu(w) = 0$ for all $w \in \Omega$. Then KL-divergence of μ given ν is

$$\text{KL}(\mu \parallel \nu) \triangleq \mathbb{E}_{w \sim (\Omega, \mu)} \log \left(\frac{\mu(w)}{\nu(w)} \right) = \sum_{w \in \Omega} \log \left(\frac{\mu(w)}{\nu(w)} \right) \mu(w).$$

In this formula we adopt a convention that $\frac{0}{0} = 1$. We will use the fact that

$$\text{KL}(\mu \parallel \nu) \geq \frac{1}{2} \|\mu - \nu\|_1^2. \quad (37)$$

Henceforth, let μ, ν be distributions on the universe Ω^∞ , where Ω is a finite set. For $\vec{w} = (w_1, w_2, \dots) \in \Omega^\infty$ and $t \in \mathbb{N}$, let us use the notation $\vec{w}_t = (w_1, \dots, w_t) \in \Omega^t$. Let μ_t be a restriction of μ to Ω^t : that is, a distribution on Ω^t given by

$$\mu_t(\vec{w}_t) \triangleq \mu(\{\vec{u} \in \Omega^\infty : \vec{u}_t = \vec{w}_t\}).$$

The *next-round conditional distribution* of μ given \vec{w}_t , $t < T$ is defined by

$$\mu(w_{t+1} \mid \vec{w}_t) \triangleq \frac{\mu_{t+1}(\vec{w}_{t+1})}{\mu_t(\vec{w}_t)}.$$

Note that $\mu(\cdot \mid \vec{w}_t)$ is a distribution on Ω for every fixed \vec{w}_t .

The *conditional KL-divergence* at round $t + 1$ is defined as

$$\text{KL}_{t+1}(\mu \parallel \nu) \triangleq \mathbb{E}_{\vec{w}_t \sim (\Omega^t, \mu_t)} \text{KL}(\mu(\cdot \mid \vec{w}_t) \parallel \nu(\cdot \mid \vec{w}_t)).$$

In words, this is the KL-divergence between the next-round conditional distributions $\mu(\cdot \mid \vec{w}_t)$ and $\nu(\cdot \mid \vec{w}_t)$, in expectation over the random choice of \vec{w}_t according to distribution μ_t .

We will use the following fact, known as the *chain rule* for KL-divergence:

$$\text{KL}(\mu_T \parallel \nu_T) = \sum_{t=1}^T \text{KL}_t(\mu \parallel \nu), \quad \text{for each } T \in \mathbb{N}. \quad (38)$$

Here for notational convenience we define $\text{KL}_1(\mu \parallel \nu) \triangleq \text{KL}(\mu_1 \parallel \nu_1)$.

9.3 The KL-divergence argument: proof of Lemma 9.6

Fix some BwK algorithm ALG and fix parameters (p, ϵ) . Let \mathcal{I}_x be the problem instance in $\mathcal{F}_{(p, \epsilon)}^\infty$ in which the best arm is x . For the analysis, we also consider an instance \mathcal{I}_0 which coincides with \mathcal{I}_x but has no best arm: that is, all arms have expected resource consumption p . Let $\tau(\mathcal{I})$ be the stopping time of ALG for a given problem instance \mathcal{I} , and let $N_x(\mathcal{I})$ be the expected number of times a given arm x is chosen by ALG on this problem instance.

Consider problem instance \mathcal{I}_0 . Since all arms are the same, we can apply Corollary 9.5(a) (suitably modified to the non-best arm) and obtain $\mathbb{E}[\tau(\mathcal{I}_0)] = \lfloor B + 1 \rfloor / p$. We focus on an arm x with the smallest $N_x(\mathcal{I}_0)$. For this arm it holds that

$$N_x(\mathcal{I}_0) \leq \frac{1}{m} \sum_{x \in X} N_x(\mathcal{I}_0) = \frac{1}{m} \mathbb{E}[\tau(\mathcal{I}_0)] \leq \frac{\lfloor B + 1 \rfloor}{pm}. \quad (39)$$

In what follows, we use this inequality to upper-bound $N_x(\mathcal{I}_x)$. Informally, if arm x is not played sufficiently often in \mathcal{I}_0 , ALG cannot tell apart \mathcal{I}_0 and \mathcal{I}_x .

The *transcript* of ALG on a given problem instance \mathcal{I} is a sequence of pairs $\{(x_t, c_t)\}_{t \in \mathbb{N}}$, where for each round $t \leq \tau(\mathcal{I})$ it holds that x_t is the arm chosen by ALG in round t , and c_t is the realized resource consumption in this round. For all $t > \tau(\mathcal{I})$, we define $(x_t, c_t) = (\text{null}, 0)$. To map this to the setup in Section 9.2, denote $\Omega = (X \cup \{\text{null}\}) \times \{0, 1\}$. Then the set of all possible transcripts is a subset of Ω^∞ .

Every given problem instance \mathcal{I} induces a distribution over Ω^∞ . Let μ, ν be the distributions over Ω^∞ that are induced by \mathcal{I}_0 and \mathcal{I}_x , respectively. We will use the following shorthand:

$$\text{diff}[T_0, T_*] \triangleq \sum_{t=T_0}^{T_*} \nu(x_t = x) - \mu(x_t = x), \quad \text{where } 1 \leq T_0 \leq T_* \leq \infty.$$

For any $T \in \mathbb{N}$ (which we will fix later), we can write

$$N_x(\mathcal{I}_x) - N_x(\mathcal{I}_0) = \text{diff}[1, \infty] = \text{diff}[1, T] + \text{diff}[T + 1, \infty]. \quad (40)$$

We will bound $\text{diff}[1, T]$ and $\text{diff}[T + 1, \infty]$ separately.

Upper bound on $\text{diff}[1, T]$. This is where we use KL-divergence. Namely, by Equation (37) we have

$$\text{diff}[1, T] \leq \frac{T}{2} \|\mu_T - \nu_T\|_1 \leq T \sqrt{\frac{1}{2} \text{KL}(\mu_T \| \nu_T)}. \quad (41)$$

Now, by the chain rule (Equation (38)), we can focus on upper-bounding the conditional KL-divergence $\text{KL}_t(\mu \| \nu)$ at each round $t \leq T$.

Claim 9.11. *For each round $t \leq T$ it holds that*

$$\text{KL}_t(\mu \| \nu) = \mu(x_t = x) \left(p \log\left(\frac{p}{p-\epsilon}\right) + (1-p) \log\left(\frac{1-p}{1-p+\epsilon}\right) \right). \quad (42)$$

Proof. The main difficulty here is to carefully “unwrap” the definition of $\text{KL}_t(\mu \| \nu)$.

Fix $t \leq T$ and let $\vec{w}_t \in \Omega^t$ be the partial transcript up to and including round t . For each arm y , let $f(y|\vec{w}_t)$ be the probability that ALG chooses arm y in round t , given the partial transcript \vec{w}_t . Let $c(y|\mathcal{I})$ be the expected resource consumption for arm y under a problem instance \mathcal{I} . The transcript for round $t+1$ is a pair $w_{t+1} = (x_{t+1}, c_{t+1})$, where x_{t+1} is the arm chosen by ALG in round $t+1$, and $c_{t+1} \in \{0, 1\}$ is the resource consumption in that round. Therefore if $c_{t+1} = 1$ then

$$\begin{aligned} \mu(w_{t+1} | \vec{w}_t) &= f(x_{t+1} | \vec{w}_t) c(x_{t+1} | \mathcal{I}_0) = f(x_{t+1} | \vec{w}_t) p, \\ \nu(w_{t+1} | \vec{w}_t) &= f(x_{t+1} | \vec{w}_t) c(x_{t+1} | \mathcal{I}_x) = f(x_{t+1} | \vec{w}_t) (p - \epsilon \mathbf{1}_{\{x_{t+1}=x\}}). \end{aligned}$$

Similarly, if $c_{t+1} = 0$ then

$$\begin{aligned} \mu(w_{t+1} | \vec{w}_t) &= f(x_{t+1} | \vec{w}_t) (1 - c(x_{t+1} | \mathcal{I}_0)) = f(x_{t+1} | \vec{w}_t) (1 - p), \\ \nu(w_{t+1} | \vec{w}_t) &= f(x_{t+1} | \vec{w}_t) (1 - c(x_{t+1} | \mathcal{I}_x)) = f(x_{t+1} | \vec{w}_t) (1 - p + \epsilon \mathbf{1}_{\{x_{t+1}=x\}}). \end{aligned}$$

It follows that

$$\log \frac{\mu(w_{t+1} | \vec{w}_t)}{\nu(w_{t+1} | \vec{w}_t)} = \mathbf{1}_{\{x_t=x\}} \left(\log\left(\frac{p}{p-\epsilon}\right) \mathbf{1}_{\{c_{t+1}=1\}} + \log\left(\frac{1-p}{1-p+\epsilon}\right) \mathbf{1}_{\{c_{t+1}=0\}} \right).$$

Taking expectations over $w_{t+1} = (x_t, c_t) \sim \mu(\cdot | \vec{w}_t)$, we obtain

$$\text{KL}(\mu(\cdot | \vec{w}_t) \| \nu(\cdot | \vec{w}_t)) = f(x | \vec{w}_t) \left(p \log\left(\frac{p}{p-\epsilon}\right) + (1-p) \log\left(\frac{1-p}{1-p+\epsilon}\right) \right).$$

Taking expectations over $\vec{w}_t \sim \mu_t$, we obtain the conditional KL-divergence $\text{KL}_t(\mu \| \nu)$. Equation (42) follows because

$$\mathbb{E}_{\vec{w}_t \sim \mu_t} f(x | \vec{w}_t) = \mu(x_t = x). \quad \square$$

We will use the following fact about logarithms, which is proved using standard quadratic approximations for the logarithm. The proof is in Appendix C.

Fact 9.12. *Assume $\frac{\epsilon}{p} \leq \frac{1}{2}$ and $p \leq \frac{1}{2}$. Then*

$$p \log\left(\frac{p}{p-\epsilon}\right) + (1-p) \log\left(\frac{1-p}{1-p+\epsilon}\right) \leq \frac{2\epsilon^2}{p}.$$

Now we can put everything together and derive an upper bound on $\text{diff}[1, T]$.

Claim 9.13. *Assume $\frac{\epsilon}{p} \leq \frac{1}{2}$ and $p \leq \frac{1}{2}$. Then $\text{diff}[1, T] \leq T \frac{\epsilon}{p} \sqrt{\frac{B+1}{m}}$.*

Proof. By Claim 9.11 and Fact 9.12, for each round $t \leq T$ we have

$$\text{KL}_t(\mu \parallel \nu) \leq \frac{2\epsilon^2}{p} \mu(x_t = x).$$

By the chain rule (Equation (38)), we have

$$\text{KL}(\mu_T \parallel \nu_T) \leq \frac{2\epsilon^2}{p} \sum_{t=1}^T \mu(x_t = x) \leq \frac{2\epsilon^2}{p} N_x(\mathcal{I}_0) \leq 2 \frac{B+1}{m} \left(\frac{\epsilon}{p}\right)^2.$$

The last inequality is the place where we use our choice of x , as expressed by Equation (39).

Plugging this back into Equation (41), we obtain $\text{diff}[1, T] \leq T \frac{\epsilon}{p} \sqrt{\frac{B+1}{m}}$. \square

Upper bound on $\text{diff}[T, \infty]$. Consider the problem instance \mathcal{I}_x , and consider the policy that always chooses the best arm. Let ν^* be the corresponding distribution over transcripts Ω^∞ , and let τ be the corresponding stopping time. Note that $\nu^*(x_t = x)$ if and only if $\tau > t$. Therefore:

$$\text{diff}[T, \infty] \leq \sum_{t=T}^{\infty} \nu(x_t = x) \leq \sum_{t=T}^{\infty} \nu^*(x_t = x) = \sum_{t=T}^{\infty} \nu^*(\tau > t) \leq \text{OPT}^2/T.$$

The second inequality can be proved using a simple “coupling argument”. The last inequality follows from Fact 9.3, observing that $\mathbb{E}[\tau] = \text{OPT}$.

Putting the pieces together. Assume $p \leq \frac{1}{2}$ and $\frac{\epsilon}{p} \leq \frac{1}{2}$. Denote $\gamma = \frac{\epsilon}{p} \sqrt{\frac{B+1}{m}}$. Using the upper bounds on $\text{diff}[1, T]$ and $\text{diff}[T+1, \infty]$ and plugging them into Equation (40), we obtain

$$N_x(\mathcal{I}_x) - N_x(\mathcal{I}_0) \leq \gamma T + \text{OPT}^2/T \leq \text{OPT} \sqrt{\gamma}$$

for $T = \text{OPT}/\sqrt{\gamma}$. Recall that $N_x(\mathcal{I}_0) < \text{OPT}/m$. Thus, we obtain

$$N_x(\mathcal{I}_x) \leq \left(\frac{1}{m} + \sqrt{\gamma}\right) \text{OPT}.$$

Recall that we need to conclude that $N_x(\mathcal{I}_x) \leq \frac{3}{4} \text{OPT}$. For that, it suffices to have $\gamma \leq \frac{1}{16}$.

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Appendix A: Optimal dynamic policy beats the best fixed arm

Let us provide some examples of BwK problem instances in which the best dynamic policy (in fact, the best fixed distribution over arms) beats the best fixed arm. We start with a very simple (but perhaps contrived) example of this phenomenon. Then we provide more realistic examples that arise in the domains of dynamic pricing and dynamic procurement.

Simple example. There are d arms; pulling arm i deterministically produces a reward of 1, consumes one unit of resource i , and does not consume any other resources. We are given an initial endowment of B units of each resource. Any policy that plays a fixed arm i in each period is limited to a total reward of B before running out of its budget of resource i . A policy that always plays a uniformly random arm achieves an expected reward of nearly $d \cdot B$ before its resource budget runs out.

Dynamic pricing. Consider the basic setting of “dynamic pricing with limited supply”: in each round a potential buyer arrives, and the seller offers him one item at a price; there are k items and $n > k$ potential buyers. One can easily construct distributions for which offering a mixture of two prices is strictly superior to offering any fixed price. In fact this situation arises whenever the “revenue curve” (the mapping from prices to expected revenue) is non-concave and its value at the quantile k/n lies below its concave hull.

Let us provide a specific example. Fix any constant $\delta > 0$, and let $\epsilon = k^{\delta-1/2}$. Each buyer has the following two-point demand distribution: the buyer’s value for item is $v = 1$ with probability $\frac{1}{n} k^{1/2+\delta}$, and $v = \epsilon$ with the remaining probability.

To analyze this example, let $\text{REW}(\mathcal{D})$ be the expected total reward (i.e., the expected total revenue) from using a fixed distribution \mathcal{D} over prices in each round; let $\text{REW}(p)$ be the same quantity when \mathcal{D} deterministically picks a given price p .

- Clearly, if one offers a fixed price in all rounds, it only makes sense to offer prices $p = \epsilon$ and $p = 1$. It is easy to see that $\text{REW}(\epsilon) = k \cdot \epsilon \leq k^{1/2+\delta}$, and $\text{REW}(1) = 1 \cdot \Pr[\text{sale at price } 1] \leq k^{1/2+\delta}$.

- Now consider a distribution \mathcal{D} which picks price ϵ with probability $\frac{k-k^{1/2+\delta}}{n}$, and picks price 1 with the remaining probability. It is easy to show that $\text{REW}(\mathcal{D}) \geq (2 - o(1)) k^{1/2+\delta}$.

So, $\text{REW}(\mathcal{D})$ is essentially twice as large compared to the total expected revenue of the best fixed arm.

Dynamic procurement. A similar example can be constructed in the domain of dynamic procurement. Consider the basic setting thereof: in each round a potential seller arrives, and the buyer offers to buy one item at a price; there are T sellers and the buyer is constrained to spend at most budget B . The buyer has no value for left-over budget and each seller's value for the item is drawn i.i.d from an unknown distribution. Then a mixture of two prices is strictly superior to offering any fixed price whenever the “sales curve” (the mapping from prices to probability of selling) is non-concave and its value at the quantile B/T lies below its concave hull.

Let us provide a specific example. Fix any constant $\delta > 0$, and let $\epsilon = B^{1/2+\delta}$. Each seller has the following two-point demand distribution: the seller's value for item is $v = 1$ with probability $\frac{B}{T}$, and $v = 0$ with the remaining probability. We use the notation $\text{REW}(\mathcal{D})$ and $\text{REW}(p)$ as defined above.

- Clearly, if one offers a fixed price in all rounds, it only makes sense to offer prices $p = 0$ and $p = 1$. It is easy to see that $\text{REW}(0) \leq T \cdot \mathbb{E}[\text{Probability of selling at price } 0] = B$, and $\text{REW}(1) = B$.
- Now consider a distribution \mathcal{D} which picks price 0 with probability $1 - \frac{B-\epsilon}{T}$, and picks price 1 with the remaining probability. It is easy to show that $\text{REW}(\mathcal{D}) \geq (2 - o(1)) B$.

Again, $\text{REW}(\mathcal{D})$ is essentially twice as large compared to the total expected sales of the best fixed arm.

Appendix B: Analysis of the Hedge Algorithm

In this section we analyze the Hedge algorithm, also known as the multiplicative weights algorithm. It is an online algorithm for maintaining a d -dimensional probability vector y while observing a sequence of d -dimensional payoff vectors π_1, \dots, π_τ . The algorithm is initialized with a parameter $\epsilon \in (0, 1)$.

Algorithm 5 The algorithm Hedge(ϵ)

- 1: $v_1 = \mathbf{1}$
 - 2: **for** $t = 1, 2, \dots, \tau$ **do**
 - 3: $y_t = v_t / (\mathbf{1}^\top v_t)$.
 - 4: $v_{t+1} = \text{Diag}\{(1 + \epsilon)^{\pi_{ti}}\} v_t$.
-

The performance guarantee of the algorithm is expressed by the following proposition.

Proposition B.1. *For any $0 < \epsilon < 1$ and any sequence of payoff vectors $\pi_1, \dots, \pi_\tau \in [0, 1]^d$, we have*

$$\forall y \in \Delta[d] \quad \sum_{t=1}^{\tau} y_t^\top \pi_t \geq (1 - \epsilon) \sum_{t=1}^{\tau} y^\top \pi_t - \frac{\ln d}{\epsilon}.$$

Proof. The analysis uses the potential function $\Phi_t = \mathbf{1}^\top v_t$. We have

$$\begin{aligned}
\Phi_{t+1} &= \mathbf{1}^\top \text{Diag}\{(1 + \epsilon)^{\pi_{ti}}\} v_t \\
&= \sum_{i=1}^d (1 + \epsilon)^{\pi_{ti}} v_{ti} \\
&\leq \sum_{i=1}^d (1 + \epsilon \pi_{ti}) v_{ti} \\
&= \Phi_t (1 + \epsilon y_t^\top \pi_t) \\
\ln(\Phi_{t+1}) &\leq \ln(\Phi_t) + \ln(1 + \epsilon y_t^\top \pi_t) \leq \ln(\Phi_t) + \epsilon y_t^\top \pi_t.
\end{aligned}$$

On the third line, we have used the inequality $(1 + \epsilon)^x \leq 1 + \epsilon x$ which is valid for $0 \leq x \leq 1$. Now, summing over $t = 1, \dots, \tau$ we obtain

$$\sum_{t=1}^{\tau} y_t^\top \pi_t \geq \frac{1}{\epsilon} (\ln \Phi_{\tau+1} - \ln \Phi_1) = \frac{1}{\epsilon} \ln \Phi_{\tau+1} - \frac{\ln d}{\epsilon}.$$

The maximum of $y^\top (\sum_{t=1}^{\tau} \pi_t)$ over $y \in \Delta[d]$ must be attained at one of the extreme points of $\Delta[d]$, which are simply the standard basis vectors of \mathbb{R}^d . Say that the maximum is attained at e_i . Then we have

$$\begin{aligned}
\Phi_{\tau+1} &= \mathbf{1}^\top v_{\tau+1} \geq v_{\tau+1,i} = (1 + \epsilon)^{\pi_{1i} + \dots + \pi_{\tau i}} \\
\ln \Phi_{\tau+1} &\geq \ln(1 + \epsilon) \sum_{t=1}^{\tau} \pi_{ti} \\
\sum_{t=1}^{\tau} y_t^\top \pi_t &\geq \frac{\ln(1 + \epsilon)}{\epsilon} \sum_{t=1}^{\tau} \pi_{ti} - \frac{\ln d}{\epsilon} \\
&\geq (1 - \epsilon) \sum_{t=1}^{\tau} y^\top \pi_t - \frac{\ln d}{\epsilon}.
\end{aligned}$$

The last line follows from two observations. First, our choice of i ensures that $\sum_{t=1}^{\tau} \pi_{ti} \geq \sum_{t=1}^{\tau} y^\top \pi_t$ for every $y \in \Delta[d]$. Second, the inequality $\ln(1 + \epsilon) > \epsilon - \epsilon^2$ holds for every $\epsilon > 0$. In fact,

$$\begin{aligned}
-\ln(1 + \epsilon) &= \ln\left(\frac{1}{1 + \epsilon}\right) = \ln\left(1 - \frac{\epsilon}{1 + \epsilon}\right) < -\frac{\epsilon}{1 + \epsilon} \\
\ln(1 + \epsilon) &> \frac{\epsilon}{1 + \epsilon} > \frac{\epsilon(1 - \epsilon^2)}{1 + \epsilon} = \epsilon - \epsilon^2.
\end{aligned}$$

□

Appendix C: Facts for the proof of the lower bound

We restate and prove two facts that we used in the proof of the lower bound in Section 9.

Fact (Fact 9.3, restated). *Let S_t be the sum of t i.i.d. 0-1 variables with expectation q . Let τ be the first time this sum reaches a given number $B \in \mathbb{N}$. Then $\mathbb{E}[\tau] = B/q$. Moreover, for each $T > \mathbb{E}[\tau]$ it holds that*

$$\sum_{t > T} \Pr[\tau \geq t] \leq \mathbb{E}[\tau]^2 / T. \tag{43}$$

Proof. $\mathbb{E}[\tau] = B/q$ follows from the martingale argument presented in the proof of Claim 9.4. Formally, take $q = p - \epsilon$ and $N_\tau = \tau$.

Assume $T > \mathbb{E}[\tau]$. The proof of Equation (43) uses two properties, one being that a geometric random variable is memoryless and other being Markov's inequality. Let us first bound the random variable $\tau - T$ conditional on the event that $\tau > T$.

$$\begin{aligned}\mathbb{E}[\tau - T | \tau > T] &= \sum_{t=1}^B \Pr[S_T = t] \mathbb{E}[\tau - T | \tau > T, S_T = t] \\ &= \sum_{t=1}^B \Pr[S_T = t] \mathbb{E}[\tau - T | S_T = t] \\ &\leq \sum_{t=1}^B \Pr[S_T = t] \mathbb{E}[\tau - T | S_T = 0] \\ &\leq \mathbb{E}[\tau - T | S_T = 0] = \mathbb{E}[\tau].\end{aligned}$$

By Markov's inequality we have $\Pr[\tau \geq T] \leq \frac{\mathbb{E}[\tau]}{T}$. Combining the two inequalities, we have

$$\begin{aligned}\sum_{t>T} \Pr[\tau \geq t] &= \sum_{t>T} \Pr[\tau \geq T] \Pr[\tau \geq t | \tau > T] \\ &= \Pr[\tau \geq T] \mathbb{E}[\tau - T | \tau > T] \\ &\leq \mathbb{E}[\tau]^2 / T.\end{aligned}$$

□

Fact (Fact 9.12, restated). Assume $\frac{\epsilon}{p} \leq \frac{1}{2}$ and $p \leq \frac{1}{2}$. Then

$$p \log \left(\frac{p}{p - \epsilon} \right) + (1 - p) \log \left(\frac{1 - p}{1 - p + \epsilon} \right) \leq \frac{2\epsilon^2}{p}.$$

Proof. To prove the inequality we use the following standard inequalities:

$$\begin{aligned}\log(1 + x) &\geq x - x^2/2 & \forall x \in [0, 1] \\ \log(1 - x) &\geq -x - x^2 & \forall x \in [0, \frac{1}{2}].\end{aligned}$$

It follows that:

$$\begin{aligned}p \log \left(\frac{p}{p - \epsilon} \right) + (1 - p) \log \left(\frac{1 - p}{1 - p + \epsilon} \right) &= -p \log \left(1 - \frac{\epsilon}{p} \right) - (1 - p) \log \left(1 + \frac{\epsilon}{1 - p} \right) \\ &\leq p \left(\frac{\epsilon}{p} + \frac{\epsilon^2}{p^2} \right) + (1 - p) \left(-\frac{\epsilon}{1 - p} + \frac{\epsilon^2}{(1 - p)^2} \right) \\ &= \frac{\epsilon^2}{p} + \frac{\epsilon^2}{1 - p} \leq \frac{2\epsilon^2}{p}\end{aligned}$$

□