4. 哈尔变换可以用矩阵的形式表示为：

$$T = HFH^T$$

其中，F是一个$N \times N$的图像矩阵，H是$N \times N$变换矩阵，T是$N \times N$变换结果。对于哈尔变换，变换矩阵H包含基函数$h_k(z)$，它们定义在连续闭区间$z \in [0,1], k = 0,1,2 \cdots N-1$，其中$N = 2^n$。为了生成矩阵，定义整数k，即$k = 2^p + q - 1$（这里$0 \le p \le n-1$，当 p=0 时 q=0，或 1；当$p \ne 0$时，$1 \le q \le 2^p$）。可得哈尔基函数为：

$$h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}, z \in [0,1]$$

且$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}}\begin{cases} 2^{\frac{p}{2}}, (q-1)/2^p \le z < (q-0.5)/2^p \\ -2^{\frac{p}{2}}, (q-0.5)/2^p \le z < q/2^p \\ 0, 其它，z \in [0,1] \end{cases}$

N×N 哈尔变换矩阵的第i行包含了元素$h_i(z)$，其中$z = \frac{0}{N}, \frac{1}{N}, \cdots (N-1)/N$。计算当$N = 16$时的$H_{16}$矩阵。

解:当 N=16 时，k=0,1,2,3⋯15。而此时 p 和 q 相应的取值如下表所示：

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	0	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
q	0	1	1	2	1	2	3	4	1	2	2	4	5	6	7	8

由课上所讲的求解变换矩阵 H 的规律，可得：

$$H_{16} = \frac{1}{4}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\
2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2}
\end{bmatrix}$$