自抗扰控制系统稳定性的基本理论结果

薛文超

中国科学院数学与系统科学研究院

wenchaoxue@amss.ac.cn

2020年12月

ADRC as a powerful solution has been explored in many domain of control engineering:

- 电机控制 **(speed control of induction motor and permanent-magnet synchronous, noncircular turning process, etc)**
- 飞行控制 **(attitude control of space ship, high-speed vehicles, UAV, morphing aircraft, etc)**
- 机器人 **（force control, uncalibrated hand-eye coordination, AUV, etc）**
- 热工过程 **(unstable heat conduction systems, boiler-turbine-generator systems, ALSTOM gasifier, fractional-order system, etc)**
- 电力电子装置 **(DC-to DC power converters, rectifiers, inverters, HVDC SMC systems, etc)**
- 行器
- 电力系统、发动机、汽车、内燃机。。。

ADRC in U.S.: Milestones

- ***1997: made the 1st successful ADRC hardware test on a servo mechanism***
- ***2008: $1M venture capital, grew by $5M in 2012.***
- ***2010: 1st factory implementation, 10 Parker extrusion lines (挤压机生产线) at a Parker Hannifin Extrusion Plant in North America***

  (cpk: from 2.3 to >8; avg. 节能 57%)

- ***2011: implemented ADRC in several high energy particle accelerators (高能粒子加速器) in the National Superconducting Cyclotron （超导回旋加速器） Lab in the U.S.***

- ***2011: Texas Instrument adopts ADRC; 3 patents granted.***
- ***2013: Texas Instrument released the ADRC based motion control chips （德州仪器的运动控制芯片）***

从学术到工业的跨越

本次课程目的：通过理论分析了解
自抗扰控制系统的主要特点

自抗扰的发展历程：理论研究是远远落后于应用

致力于处理大范围的不确定性，导致理论分析较困难

不确定系统: 非线性，时变，多输入多输出

扰动信号: 不连续

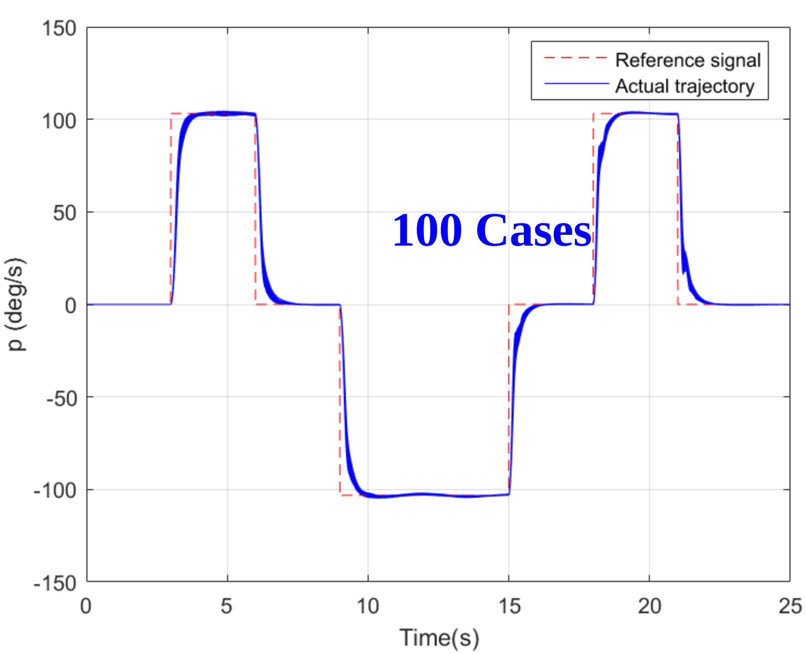
状态反馈: 非线性结构

实现闭环系统的预期动态，缺少分析方法

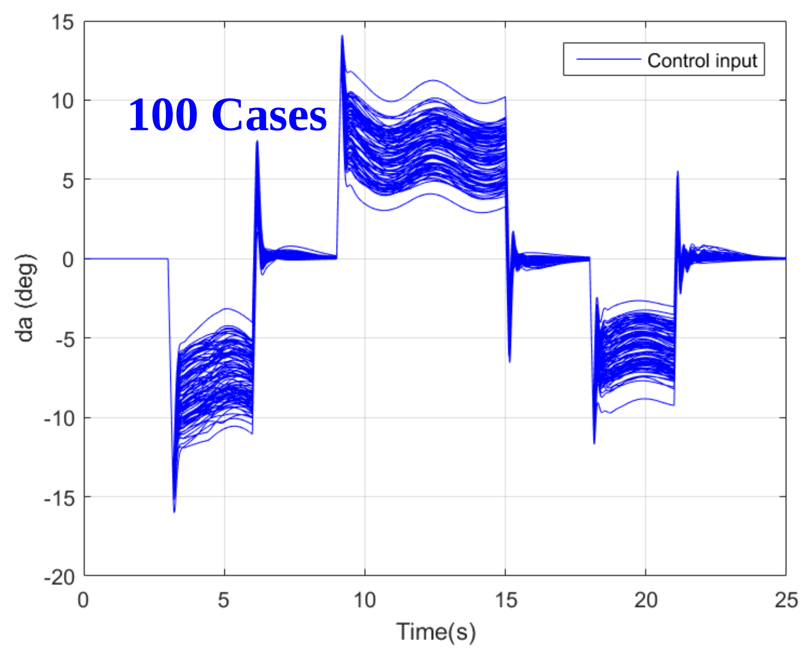
自抗扰控制系统的特点：
在不确定性保证了闭环系统输出达到期望的瞬态

Simulation results of flight control: roll rate tracking under ±40% random derivations of parameters

闭环系统的角速率响应



控制输入



$$\dot{p} = b(t)u + F(p,u,t)$$

p ：角速率

$b(t)$: 控制输入增益

F : 总扰动

u : 控制输入

线性ESO的估计误差分析

不确定系统：
$$\begin{cases} \dot{X}_1 = X_2 \\ \cdots \\ \dot{X}_{n-1} = X_n \\ \dot{X}_n = X_{n+1} + \bar{B}(t)U(t) \end{cases}$$

线性ESO：
$$\begin{cases} \dot{\hat{X}}_1 = \hat{X}_2 + l_1(\hat{X}_1 - Y) \\ \cdots \\ \dot{\hat{X}}_n = \hat{X}_{n+1} + l_n(\hat{X}_1 - Y) + \bar{B}(t)U(t) \\ \dot{\hat{X}}_{n+1} = l_{n+1}(\hat{X}_1 - Y) \end{cases}$$

总扰动： $X_{n+1} = F(X,t) + d(t) + \left(B(X,t) - \bar{B}(t)\right)U(t)$

ESO的带宽：ω_e $\quad s^{n+1} + \sum_{i=1}^{n+1} l_i s^{n+1-i} = (s+\omega_e)^{n+1}$

经典的理论分析结果：

[Q Zheng, L Gao, Z Gao, 2007CDC] **Assuming that $\dot{X}_{n+1}(.)$ is a bounded function, then there exists a finite $T_1 > 0$ such that**

$$\left|X_i - \hat{X}_i\right| \le O\left(\frac{1}{\omega_e^k}\right) \qquad \forall t \ge T_1 > 0,\ \omega_e > 0,\ i = 1, 2, \ldots, n+1$$

线性ESO的估计误差分析：定理1

定理1：假设总扰动的导数$\dot{X}_{n+1}$有界，则

$$\lim_{t\to\infty}\left|X_i(\mathrm{t})-\hat{X}_i(\mathrm{t})\right|\le O\left(\frac{1}{\omega_e^k}\right)\qquad k=n+2-i,\qquad \omega_e>0,\qquad i=1,2,\ldots,n+1.$$

证明：首先定义ESO的估计误差 $\hat{E}\triangleq\begin{bmatrix}\hat{E}_1\\ \hat{E}_2\\ \vdots\\ \hat{E}_{n+1}\end{bmatrix}=\begin{bmatrix}X_1-\hat{X}_1\\ X_2-\hat{X}_2\\ \vdots\\ X_{n+1}-\hat{X}_{n+1}\end{bmatrix}$

进而得到ESO估计误差的动态方程 $\begin{cases}\dot{\hat{E}}_1=\hat{E}_2-l_1\hat{E}_1\\ \cdots\\ \dot{\hat{E}}_n=\hat{E}_{n+1}-l_n\hat{E}_1\\ \dot{\hat{E}}_{n+1}=\dot{X}_{n+1}-l_{n+1}\hat{E}_1\end{cases}$

其中： $\begin{cases}s^{n+1}+\sum\limits_{i=1}^{n+1}l_is^{n+1-i}=(s+\omega_e)^{n+1}\\ or\\ l_i=C_{n+1}^i\omega_e{}^i\end{cases}$

线性ESO的估计误差分析：定理1（续）

再引入新状态及等价变换： $\xi \triangleq \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n+1} \end{bmatrix} = \begin{bmatrix} \omega_e^n & 0 & \cdots & 0 \\ 0 & \omega_e^{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \vdots \\ \hat{E}_{n+1} \end{bmatrix}$

即有： $\xi_i = \omega_e^{n+1-i} \hat{E}_i, \dot{\xi} = \omega_e^{n+1-i} \dot{\hat{E}},$

进而得到新状态的动态方程：

$$\dot{\xi} = \begin{bmatrix} \omega_e^n & 0 & \cdots & 0 \\ 0 & \omega_e^{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \dot{\hat{E}}$$

$$= \begin{bmatrix} \omega_e^n & 0 & \cdots & 0 \\ 0 & \omega_e^{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \left(\begin{bmatrix} l_1 & 1 & \cdots & 0 \\ l_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n+1} & 0 & \cdots & 0 \end{bmatrix} \hat{E} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dot{X}_{n+1} \end{bmatrix} \right)$$

$$= \omega_e \begin{bmatrix} -C_{n+1}^1 & 1 & \cdots & 0 \\ -C_{n+1}^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -C_{n+1}^{n+1} & 0 & \cdots & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dot{X}_{n+1} \end{bmatrix}$$

线性ESO的估计误差分析：定理1（续）

$$\dot{\xi}=\omega_e\begin{bmatrix}-C_{n+1}^1 & 1 & 0 & \cdots & 0\\ -C_{n+2}^2 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -C_{n+1}^n & 0 & 0 & \cdots & 1\\ -C_{n+1}^{n+1} & 0 & 0 & \cdots & 0\end{bmatrix}\xi+\begin{bmatrix}0\\0\\\vdots\\0\\\dot{X}_{n+1}\end{bmatrix}$$

因为矩阵 $\begin{bmatrix}-C_{n+1}^1 & 1 & 0 & \cdots & 0\\ -C_{n+2}^2 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -C_{n+1}^n & 0 & 0 & \cdots & 1\\ -C_{n+1}^{n+1} & 0 & 0 & \cdots & 0\end{bmatrix}$ 为赫尔维兹矩阵，则存在一个正定矩阵 P 使得：

$$\begin{bmatrix}-C_{n+1}^1 & 1 & 0 & \cdots & 0\\ -C_{n+2}^2 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -C_{n+1}^n & 0 & 0 & \cdots & 1\\ -C_{n+1}^{n+1} & 0 & 0 & \cdots & 0\end{bmatrix}^T P+P\begin{bmatrix}-C_{n+1}^1 & 1 & 0 & \cdots & 0\\ -C_{n+2}^2 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -C_{n+1}^n & 0 & 0 & \cdots & 1\\ -C_{n+1}^{n+1} & 0 & 0 & \cdots & 0\end{bmatrix}=-I$$

定义矩阵 $V(\xi)\triangleq\xi^T P\xi$ ，则有

$$\frac{dV(\xi)}{dt}=-\omega_e\xi^T\xi+2\xi^T P\begin{bmatrix}0\\0\\\vdots\\\dot{X}_{n+1}\end{bmatrix}$$

线性ESO的估计误差分析：定理1（续）

定义矩阵 $V(\xi)\triangleq\xi^T P\xi$ ，令 c_1,c_2 分别为矩阵 P 的最小和最大特征根，则有

$$\frac{dV(\xi)}{dt}=-\omega_e\xi^T\xi+2\xi^T P\begin{bmatrix}0\\0\\\vdots\\\dot{X}_{n+1}\end{bmatrix}$$

$$\frac{dV(\xi)}{dt}\le-\omega_e\frac{V(\xi)}{2c_2}+2\|\xi\|c_2\left\|\dot{X}_{n+1}\right\|\le-\omega_e\frac{V(\xi)}{2c_2}+2\frac{\sqrt{V(\xi)}}{\sqrt{c_1}}c_2\left\|\dot{X}_{n+1}\right\|$$

$$\frac{d\sqrt{V(\xi)}}{dt}\le-\omega_e\frac{\sqrt{V(\xi)}}{2c_2}+\frac{c_2}{\sqrt{c_1}}\left\|\dot{X}_{n+1}\right\|$$

线性ESO的估计误差分析：定理1（续）

$$\frac{d\sqrt{V(\xi)}}{dt} \le -\omega_e \frac{\sqrt{V(\xi)}}{2c_2} + \frac{c_2}{\sqrt{c_1}} \left\| \dot{X}_{n+1} \right\|$$

又由于 $\dot{X}_{n+1}$ 有界，则存在一个常数 M，使得

$$\left\| \dot{X}_{n+1} \right\| \le M, \qquad \forall t \ge t_0.$$

有

$$\frac{d\sqrt{V(\xi)}}{dt} \le -\omega_e \frac{\sqrt{V(\xi)}}{2c_2} + \frac{c_2}{\sqrt{c_1}} M, \qquad \forall t \ge t_0$$

由Gronwall–Bellman不等式有

$$\sqrt{V(\xi)} \le e^{-\frac{\omega_e}{2c_2}(t-t_0)} \sqrt{V(\xi(t_0))} + \frac{c_2}{\sqrt{c_1}} M \int_{t_0}^{t} e^{-\frac{\omega_e}{2c_2}(t-s)} ds, \qquad \forall t \ge t_0$$

线性ESO的估计误差分析：定理1（续）

由矩阵$V(\xi) \triangleq \xi^T P \xi$ 定义，

则有

$$\|\xi\| \le \sqrt{\frac{V(\xi)}{c_1}} \le e^{-\frac{\omega_e}{2c_2}(t-t_0)} \sqrt{\frac{V(\xi(t_0))}{c_1}} + \frac{c_2}{c_1} M \int_{t_0}^{t} e^{-\frac{\omega_e}{2c_2}(t-s)} ds, \qquad \forall t \ge t_0$$

又由于 $\xi_i = \omega_e^{n+1-i} \hat{E}_i, \hat{E}_i = \frac{1}{\omega_e^{n+1-i}} \xi_i$，则有

$$\begin{aligned}
\left|\hat{E}_i(t)\right| &= \frac{1}{\omega_e^{n+1-i}} \left|\xi_i(t)\right| \\
&\le \frac{1}{\omega_e^{n+1-i}} \left\|\xi(t)\right\| \\
&\le \frac{1}{\omega_e^{n+1-i}} \left[e^{-\frac{\omega_e}{2c_2}(t-t_0)} \sqrt{\frac{V(\xi(t_0))}{c_1}} + \frac{c_2}{c_1} M \int_{t_0}^{t} e^{-\frac{\omega_e}{2c_2}(t-s)} ds \right].
\end{aligned}$$

线性ESO的估计误差分析：定理1（续）

又由于

$$
\begin{aligned}
\lim_{t\to\infty}\left|\hat{E}_i(t)\right| &\le \lim_{t\to\infty}\frac{1}{\omega_e^{n+1-i}}\left[e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{\frac{V(\xi(t_0))}{c_1}}+\frac{c_2}{c_1}M\int_{t_0}^{t}e^{-\frac{\omega_e}{2c_2}(t-s)}ds\right]\\
&\le \lim_{t\to\infty}\frac{1}{\omega_e^{n+1-i}}\left[e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{\frac{V(\xi(t_0))}{c_1}}+\frac{c_2}{c_1}M\int_{t_0}^{t}e^{-\frac{\omega_e}{2c_2}(t-s)}ds\right]\\
&\le \lim_{t\to\infty}\frac{1}{\omega_e^{n+1-i}}\left[e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{\frac{V(\xi(t_0))}{c_1}}+\frac{c_2}{c_1}M\frac{2c_2}{\omega_e}\left(1-e^{-\frac{\omega_e}{2c_2}(t-t_0)}\right)\right]\\
&\le \frac{1}{\omega_e^{n+1-i}}\frac{c_2}{c_1}M\frac{2c_2}{\omega_e}.
\end{aligned}
$$

所以，$\left|X_i-\hat{X}_i\right|\le O\left(\frac{1}{\omega_e^k}\right)\quad k=n+2-i, \forall t\ge T_1>t_0, \omega_e>0,\ i=1,2,\dots,n+1$ 证毕。

线性ESO的估计误差分析：定理2

定理2：假设总扰动的导数，即$\dot{X}_{n+1}(.)$，满足$\lim_{t\to\infty}\dot{X}_{n+1}(.)=0$，则：

$$\lim_{t\to\infty}\left|X_i-\hat{X}_i\right|=0 \qquad i=1,2,\ldots,n+1$$

证明：同上定义矩阵$V(\xi)\triangleq\xi^T P\xi$, 有

$$\frac{dV(\xi)}{dt}=-\omega_e\xi^T\xi+2\xi^T P\begin{bmatrix}0\\0\\\vdots\\\dot{X}_{n+1}\end{bmatrix}$$

$$\frac{dV(\xi)}{dt}\le-\omega_e\frac{V(\xi)}{2c_2}+2\|\xi\|c_2\left\|\dot{X}_{n+1}\right\|\le-\omega_e\frac{V(\xi)}{2c_2}+2\frac{\sqrt{V(\xi)}}{\sqrt{c_1}}c_2\left\|\dot{X}_{n+1}\right\|$$

$$\frac{d\sqrt{V(\xi)}}{dt}\le-\omega_e\frac{\sqrt{V(\xi)}}{2c_2}+\frac{c_2}{\sqrt{c_1}}\left\|\dot{X}_{n+1}\right\|$$

由Gronwall–Bellman不等式有

$$\sqrt{V(\xi)}\le e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))}+\frac{c_2}{\sqrt{c_1}}\int_{t_0}^{t}e^{-\frac{\omega_e}{c_2}(t-s)}\left\|\dot{X}_{n+1}(s)\right\|ds$$

线性ESO的估计误差分析：定理2（续）

又由于 $\lim_{t\to\infty}\dot{X}_{n+1}=0$, 则有

$$\lim_{t\to\infty}\varepsilon(t)=0 \qquad \varepsilon(t)\triangleq\sup_{s\in\left[\frac{t}{2},\infty\right)}\left\|\dot{X}_{n+1}(s)\right\|$$

因此有

$$\sqrt{V(\xi)}\le e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))}+\frac{c_2}{\sqrt{c_1}}\left(\int_{t_0}^{t/2}+\int_{t/2}^{t}\right)e^{-\frac{\omega_e}{2c_2}(t-s)}\left\|\dot{X}_{n+1}(s)\right\|ds$$

$$\le e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))}+\frac{c_2}{\sqrt{c_1}}\int_{t_0}^{t/2}e^{-\frac{\omega_e}{2c_2}(t-s)}\left\|\dot{X}_{n+1}(s)\right\|ds+\varepsilon(t)\frac{c_2}{\sqrt{c_1}}\int_{t/2}^{t}e^{-\frac{\omega_e}{2c_2}(t-s)}ds$$

$$\le e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))}+\frac{c_2}{\sqrt{c_1}}M\int_{t_0}^{t/2}e^{-\frac{\omega_e}{2c_2}(t-s)}ds+\varepsilon(t)\frac{c_2}{\sqrt{c_1}}\int_{t/2}^{t}e^{-\frac{\omega_e}{2c_2}(t-s)}ds$$

$$\le e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))}+\frac{c_2}{\sqrt{c_1}}M\frac{2c_2}{\omega_e}\left(e^{-\frac{\omega_e}{2c_2}\frac{t}{2}}-e^{-\frac{\omega_e}{2c_2}(t-t_0)}\right)+\varepsilon(t)\frac{c_2}{\sqrt{c_1}}\frac{2c_2}{\omega_e}\left(1-e^{-\frac{\omega_e}{2c_2}\frac{t}{2}}\right).$$

线性ESO的估计误差分析：定理2（续）

进一步可以得到：

$$
\begin{aligned}
\lim_{t\to\infty}\sqrt{V(\xi)} \le & \lim_{t\to\infty} e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))} \\
& +\lim_{t\to\infty}\frac{c_2}{\sqrt{c_1}}M\frac{2c_2}{\omega_e}\left(e^{-\frac{\omega_e}{2c_2}\frac{t}{2}}-e^{-\frac{\omega_e}{2c_2}(t-t_0)}\right) \\
& +\lim_{t\to\infty}\varepsilon(t)\frac{c_2}{\sqrt{c_1}}\frac{2c_2}{\omega_e}\lim_{t\to\infty}\left(1-e^{-\frac{\omega_e}{2c_2}\frac{t}{2}}\right) \\
= & \ 0
\end{aligned}
$$

线性ESO的估计误差分析：定理3

定理3：假设总扰动的导数有界，则

$$\left|X_i(\mathrm{t})-\hat{X}_i(\mathrm{t})\right| \le O\left(\frac{1}{\omega_e^k}\right) \qquad \forall t \ge T \triangleq t_0 + 2(n+1)c_2 \max\left\{\frac{\ln \omega_e}{\omega_e}, 0\right\}, \omega_e > 0,\ i=1,2,\dots,n+1$$

证明：同上，定义矩阵$V(\xi) \triangleq \xi^T P \xi$**,有**

$$\frac{dV(\xi)}{dt} = -\omega_e \xi^T \xi + 2\xi^T P \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dot{X}_{n+1} \end{bmatrix}$$

$$\frac{dV(\xi)}{dt} \le -\omega_e \frac{V(\xi)}{2c_2} + 2\|\xi\| c_2 \left\|\dot{X}_{n+1}\right\| \le -\omega_e \frac{V(\xi)}{2c_2} + 2\frac{\sqrt{V(\xi)}}{\sqrt{c_1}} c_2 \left\|\dot{X}_{n+1}\right\|$$

$$\frac{d\sqrt{V(\xi)}}{dt} \le -\omega_e \frac{\sqrt{V(\xi)}}{2c_2} + \frac{c_2}{\sqrt{c_1}} \left\|\dot{X}_{n+1}\right\|$$

线性ESO的估计误差分析：定理3（续）

又由于 $\dot{X}_{n+1}$ 有界，则存在一个常数 M, 使得

$$\left\|\dot{X}_{n+1}\right\| \le M, \qquad \forall t \ge t_0.$$

因此有

$$\frac{d\sqrt{V(\xi)}}{dt} \le -\omega_e \frac{\sqrt{V(\xi)}}{2c_2} + \frac{c_2}{\sqrt{c_1}} M$$

由Gronwall–Bellman不等式有

$$\sqrt{V(\xi)} \le e^{-\frac{\omega_e}{2c_2}(t-t_0)} \sqrt{V(\xi(t_0))} + \frac{c_2}{\sqrt{c_1}} M \int_{t_0}^{t} e^{-\frac{\omega_e}{c_2}(t-s)} ds$$

设初始估计误差满足：

$$\left|\hat{E}(t_0)\right| < \rho_0$$

则有

$$\left\|\xi(t_0)\right\| \le \bar{\mu}_e(\omega_e)\rho_0 \qquad \bar{\mu}_e(\omega_e) \triangleq \max\left\{1, \frac{1}{\omega_e^{n-1}}\right\}$$

线性ESO的估计误差分析：定理3（续）

设初始估计误差满足：

$$\left|\hat{E}(t_0)\right| < \rho_0$$

则有

$$\left\|\xi(t_0)\right\| \le \bar{\mu}_e(\omega_e)\rho_0 \qquad \bar{\mu}_e(\omega_e) \triangleq \max\left\{1, \omega_e^n\right\}$$

则对 $\sqrt{V(\xi)} \le e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))} + \frac{c_2}{\sqrt{c_1}} M \int_{t_0}^{t} e^{-\frac{\omega_e}{c_2}(t-s)} ds$ **的右端第1项得如下分析结果**

当 $\forall t \ge T \triangleq t_0 + 2(n+1)c_2 \max\left\{\frac{\ln \omega_e}{\omega_e}, 0\right\}, \ \omega_e > 0, \ i = 1, 2, \ldots, n+1$ **时**

$$e^{-\frac{\omega_e}{2c_2}(t-t_0)}\sqrt{V(\xi(t_0))} \le \frac{1}{\omega_e^{n+1}}\sqrt{c_2}\,\rho_0 \bar{\mu}_e(\omega_e) \le \max\left\{\frac{1}{\omega_e^{n+1}}, \frac{1}{\omega_e}\right\}\sqrt{c_2}\,\rho_0.$$

线性ESO的估计误差分析：定理3（续）

又由于$\xi_i = \omega_e^{n+1-i}\hat{E}_i, \hat{E}_i = \dfrac{1}{\omega_e^{n+1-i}}\xi_i$，则

当 $\forall t \ge T \triangleq t_0 + 2(n+1)c_2 \max\left\{\dfrac{\ln \omega_e}{\omega_e}, 0\right\}, \omega_e > 0,\ i = 1, 2, \ldots, n+1$ 时

$$\left|\hat{E}_i(t)\right| = \frac{1}{\omega_e^{n+1-i}}\left\|\xi_i(t)\right\|$$

$$\le \frac{1}{\omega_e^{n+1-i}}\left[e^{-\frac{\omega_e}{2c_2}t}\sqrt{\frac{V(\xi(t_0))}{c_1}} + \frac{c_2}{c_1}M\int_{t_0}^{t} e^{-\frac{\omega_e}{2c_2}(t-s)}ds\right]$$

$$\le \frac{1}{\omega_e^{n+1-i}}\left[\max\left\{\frac{1}{\omega_e^{n+1}}, \frac{1}{\omega_e}\right\}\sqrt{c_2}\rho_0 + \frac{c_2}{c_1}M\frac{2c_2}{\omega_e}\right]$$

因此 $\left|X_i - \hat{X}_i\right| \le O\left(\dfrac{1}{\omega_e^k}\right)$ $k = n+2-i, \forall t \ge T_1 > t_0, \omega_e > 0,\ i = 1, 2, \ldots, n+1$. 证毕。

线性自抗扰控制的跟踪误差分析：定理4

Uncertain Systems:
$$\begin{cases}\dot{X}_1 = X_2 \\ \cdots \\ \dot{X}_{n-1} = X_n \\ \dot{X}_n = X_{n+1} + \bar{B}(t)U(t)\end{cases}$$

总扰动: $X_{n+1} = F(X,t) + d(t) + \left(B(X,t) - \bar{B}(t)\right)U(t)$

Linear ESO:
$$\begin{cases}\dot{\hat{X}}_1 = \hat{X}_2 - l_1(\hat{X}_1 - Y) \\ \cdots \\ \dot{\hat{X}}_n = -l_n(\hat{X}_1 - Y) - \hat{X}_{n+1} + \bar{B}(t)U(t) \\ \dot{\hat{X}}_{n+1} = -l_{n+1}(\hat{X}_1 - Y)\end{cases}$$

ESO的带宽: ω_e $\quad s^{n+1} + \sum_{i=1}^{n+1} l_i s^{n+1-i} = (s+\omega_e)^{n+1}$

控制器设计： $U(\mathrm{t}) = \begin{cases}0, t < T_1 \\ \bar{B}^{-1}(-k_1\hat{X}_1 - k_2\hat{X}_2 \cdots - k_n\hat{X}_n - \hat{X}_{n+1}), t \geq T\end{cases}$

定理4：考虑上述闭环系统，存在 ω_e^*，使得 $\forall \omega_e \geq \omega_e^*$：

$$\left\| y(t) - y^*(t) \right\| \leq \gamma_1^* \frac{\max\{\ln(\rho_2\omega_e), \ln\omega_e, 1\}}{\omega_e}, \quad \forall t \geq t_0$$

→ **瞬态性能：接近预期轨迹**

其中 $(\gamma_1^*, \gamma_3^*, \gamma_2^*)$ 为正常数。

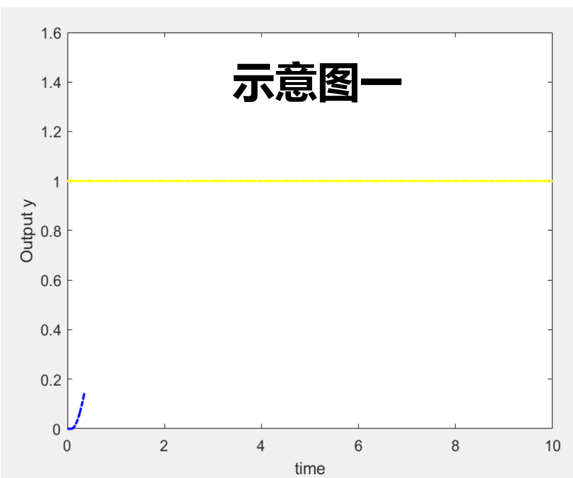
Wenchao Xue, and Yi Huang, Performance Analysis of 2-DOF Tracking Control for a Class of nonlinear Uncertain Systems with Discontinuous Disturbances, International Journal of Robust and nonlinear Control, 2018, 28:1456–1473.*

定理4：考虑上述闭环系统，存在 ω_e^*，使得 $\forall \omega_e \ge \omega_e^*$：

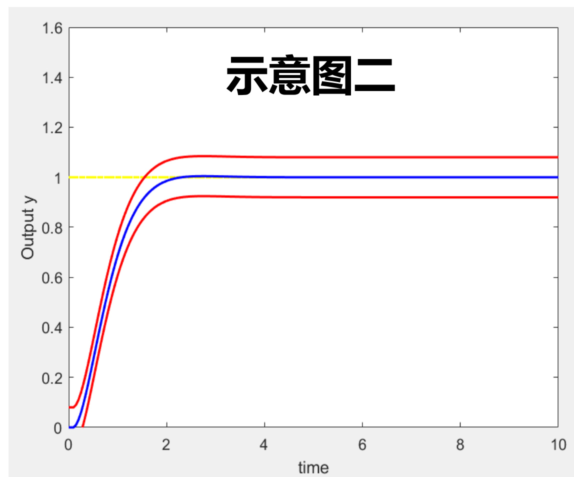
$$\left\| y(t) - y^*(t) \right\| \le \gamma_1^* \frac{\max\{\ln(\rho_2 \omega_e), \ln \omega_e, 1\}}{\omega_e}, \quad \forall t \ge t_0$$

→ **瞬态性能：接近预期轨迹**

其中 $(\gamma_1^*, \gamma_3^*, \gamma_2^*)$ 为正常数。



$y \to r \ (t \to \infty)$，
但超调、震荡…



输出曲线在理想轨线(蓝线)的小范围内(红线之间)波动

$$y^* = x_1^*$$

$$\begin{bmatrix} \dot{x}_1^* \\ \dot{x}_2^* \\ \vdots \\ \dot{x}_n^* \end{bmatrix} = A_k \left(\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} - \begin{bmatrix} r \\ r^{(1)} \\ \vdots \\ r^{(n-1)} \end{bmatrix} \right) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r^{(n)} \end{bmatrix}$$

$$A_k = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ k_{1,c} & k_{2,c} & \cdots & k_{n,c} \end{bmatrix}$$

Wenchao Xue, and Yi Huang, Performance Analysis of 2-DOF Tracking Control for a Class of nonlinear Uncertain Systems with Discontinuous Disturbances, International Journal of Robust and nonlinear Control, 2018, 28:1456–1473.*