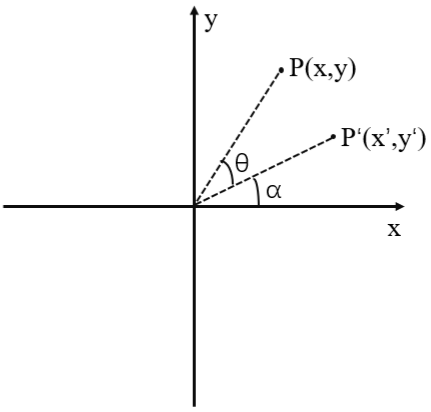
3. 完成课本数字图像处理第二版 116 页，习题 3.25，即拉普拉斯算子具有理论上的旋转不变性。

题目：证明如式(3.7.1)所示的拉普拉斯变换是各向同性的(旋转不变)。需要下列轴旋转 θ 角的坐标方程：

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

其中(x,y)为非旋转坐标，而(x',y')为旋转坐标

解：如下图所示，P 点绕原点旋转 θ 角得到 P'：



式(3.7.1)为：$\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$，而我们要证明的是 $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 f}{\partial x'^2} + \dfrac{\partial^2 f}{\partial y'^2}$

由题中所给出的关系式，我们显然可以得到：

$$\frac{\partial x}{\partial x'} = \cos\theta \quad \frac{\partial x}{\partial y'} = -\sin\theta$$
$$\frac{\partial y}{\partial x'} = \sin\theta \quad \frac{\partial y}{\partial y'} = \cos\theta$$

那么二元图像函数 f 对 x'的一阶偏导数为:

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'}$$

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

二元图像函数 f 对 y'的一阶偏导数为:

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'}$$

$$\frac{\partial f}{\partial y'}=-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta$$

那么 f 对 x’的二阶偏导数为:

$$\begin{aligned}
\frac{\partial f}{\partial x'^2}&=\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'})\\
&=\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta)\\
&=\cos\theta\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x})+\sin\theta\frac{\partial}{\partial x'}(\frac{\partial f}{\partial y})\\
&=\cos\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\frac{\partial x}{\partial x'}+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\frac{\partial y}{\partial x'})+\sin\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\frac{\partial x}{\partial x'}+\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\frac{\partial y}{\partial x'})\\
&=\cos\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\cos\theta+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\sin\theta)+\sin\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\cos\theta+\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\sin\theta)\\
&=\cos\theta^2\frac{\partial^2 f}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial f}{\partial x\partial y}+\sin\theta^2\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

那么 f 对 y’的二阶偏导数为:

$$\begin{aligned}
\frac{\partial f}{\partial y'^2}&=\frac{\partial}{\partial y'}(\frac{\partial f}{\partial y'})\\
&=\frac{\partial}{\partial y'}(-\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta)\\
&=-\sin\theta\frac{\partial}{\partial y'}(\frac{\partial f}{\partial x})+\cos\theta\frac{\partial}{\partial y'}(\frac{\partial f}{\partial y})\\
&=-\sin\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\frac{\partial x}{\partial y'}+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\frac{\partial y}{\partial y'})+\cos\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\frac{\partial x}{\partial y'}+\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\frac{\partial y}{\partial y'})\\
&=-\sin\theta(-\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\sin\theta+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\cos\theta)+\cos\theta(-\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin\theta+\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\cos\theta)\\
&=\sin\theta^2\frac{\partial^2 f}{\partial x^2}-2\cos\theta\sin\theta\frac{\partial f}{\partial x\partial y}+\cos\theta^2\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

将 f 对 x’的二阶偏导数和 f 对 y’的二阶偏导数相加可得到：

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}$$

故而可知，拉普拉斯算子具有旋转不变性。