习题 3

1. (因为不知X是否连续型$R.V.$,所以有)

(1) $P\{a \le X \le b, Y \le y\} = P\{X \le b, Y \le y\} - P\{X < a, Y \le y\}$
$= F(b, y) - F(a-0, y).$

(2) $P\{X = a, Y \le y\} = F(a, y) - F(a-0, y).$

(3) $P\{a < X \le b\} = P\{a < X \le b, Y < +\infty\}$
$= P\{X \le b, Y < +\infty\} - P\{X \le a, Y < +\infty\}$
$= F(b, +\infty) - F(a, +\infty).$

(4) $P\{c \le X \le d\} = P\{X \le d, Y < +\infty\} - P\{X < c, Y < +\infty\}$
$= F(d, +\infty) - F(c-0, +\infty).$

2. (1) $P\{X+Y>2\} = 1 - P\{X+Y \le 2\} = 1 - P\{X=1, Y=1\}$
$= 1 - 1/8 = 7/8.$

(2) $P\{X/Y > 1\} = P\{X=2, Y=1\} = 1/8.$

(3) $P\{XY \le 3\} = 1 - P\{XY > 3\} = 1 - P\{X=2, Y=2\}$
$= 1 - 1/4 = 3/4.$

(4) $P\{X=Y\} = P\{X=1, Y=1\} + P\{X=2, Y=2\}$
$= 1/8 + 1/4 = 3/8.$

或由下表求得如上结果:

p	1/8	1/2	1/8	1/4
(X,Y)	(1, 1)	(1, 2)	(2, 1)	(2, 2)
X+Y	2	3	3	4
X/Y	1	1/2	2	1
XY	1	2	2	4
X=Y	1			2

3.

X \ Y	1	2	3	$p_{i\cdot} = P\{X = x_i\}$
1	1/6	1/9	1/18	1/3
2	1/3	α	β	1/3+α+β
$p_{\cdot j} = P\{Y = y_i\}$	1/2	1/9+α	1/18+β	*1*

由X与Y独立$\Leftrightarrow p_{ij}=p_{1\bullet}\cdot p_{\bullet 2}$，

选用 $\begin{cases} p_{12}=p_{1\bullet}\cdot p_{\bullet 2} \\ p_{13}=p_{1\bullet}\cdot p_{\bullet 3}\end{cases} \Rightarrow \begin{cases} \frac{1}{9}=\frac{1}{3}\cdot(\frac{1}{9}+\alpha) \\ \frac{1}{18}=\frac{1}{3}\cdot(\frac{1}{18}+\beta)\end{cases} \Rightarrow \begin{cases} \alpha=\frac{2}{9} \\ \beta=\frac{1}{9}\end{cases}$

验知这个结果对其它也正确.

4. 把一枚均匀硬币抛掷三次，设 X 为三次抛掷中正面出现的次数，而 Y 为正面出现次数与反面出现次数之差的绝对值，求(X,Y)的概率分布.

解 （X, Y）的可能取值为：

(0,3)——0 次正, 3 次反：$P\{X=0,Y=3\}=(\frac{1}{2})^3=\frac{1}{8}$

(1,1)——1 次正, 2 次反: $P\{X=1, Y=1\}=C_3^1\frac{1}{2}(\frac{1}{2})^2=\frac{3}{8}$

(2,1)——2 次正, 1 次反: $P\{X=2, Y=1\}=C_3^2(\frac{1}{2})^2\frac{1}{2}=\frac{3}{8}$

(3,3)——3 次正, 0 次反：$P\{X=3,Y=0\}=(\frac{1}{2})^3=\frac{1}{8}$

联合分布与边缘分布（X 与 Y 不独立）：

Y / X	1	3	$p_{i\cdot}=P\{X=x_i\}$
0	0	1/8	1/8
1	3/8	0	3/8
2	3/8	0	3/8
3	0	1/8	1/8
$p_{\cdot j}=P\{Y=y_j\}$	6/8	2/8	1

5. $P\{X=0,Y=0\}=0,\quad P\{X=0,Y=1\}=0,\quad P\{X=0,Y=2\}=\frac{C_2^2C_2^2}{C_7^4}=\frac{1}{35}$

$P\{X=1,Y=0\}=0,\ P\{X=1,Y=1\}=\frac{C_3^1C_2^1C_2^2}{C_7^4}=\frac{6}{35},\ P\{X=1,Y=2\}=\frac{C_3^1C_2^2C_2^1}{C_7^4}=\frac{6}{35}$

$P\{X=2,Y=0\}=\frac{C_3^2C_2^2}{C_7^4}=\frac{3}{35},\ P\{X=2,Y=1\}=\frac{C_3^2C_2^1C_2^1}{C_7^4}=\frac{12}{35},\ P\{X=2,Y=2\}=\frac{C_3^2C_2^2}{C_7^4}=\frac{3}{35}$

$P\{X=3,Y=0\}=\frac{C_3^3C_2^1}{C_7^4}=\frac{2}{35},\quad P\{X=3,Y=1\}=\frac{C_3^3C_2^1}{C_7^4}=\frac{2}{35},\quad P\{X=3,Y=2\}=0$

Y \ X	0	1	2	$p_{i\cdot}=P\{X=x_i\}$
0	0	0	1/35	1/35
1	0	6/35	6/35	12/35
2	3/35	12/35	3/35	18/35
3	2/35	2/35	0	4/35
$p_{\cdot j}=P\{Y=y_i\}$	5/35	20/35	10/35	*1*

6. 由 1~10 的因数分解：

1	2	3	4	5	6	7	8	9	10
1×1	1×2	1×3	1×4	1×5	1×6	1×7	1×8	1×9	1×10
			2×2		2×3		2×4	3×3	2×5
							2×2×2		

可知:

数 1 出现的概率是 1/10,它可被自己整除 d=1,但不能被素数整除 F=0.

数 2、3、5、7 出现的概率是 4/10,它可被 d=2 个整数整除,可以被 F=1 个素数整除.

数 4、9 出现的概率是 2/10,它们可以被 d=3 个整数整除,且能被 F=1 个素数整除.

数 6、8、10 出现的概率是 2/10,它们可以被 d=4 个整数整除,且能被 F=2 个素数整除.

于是给出分联合分布率：

d \ F	1	2	3	4
0	1/10	0	0	0
1	0	4/10	2/10	1/10
2	0	0	0	2/10

7.（1）边缘分布律:

X	51	52	53	54	55
P	0.18	0.15	0.35	0.12	0.2

Y	51	52	53	54	55
P	0.28	0.28	0.22	0.09	0.13

$$(2)\, p_i = P\{X=x_i \mid Y=51\} = \frac{P\{X=x_i, Y=51\}}{P\{Y=51\}} = \frac{P\{X=x_i, Y=51\}}{0.28},$$

9月订单数的分布律 :

X	51	52	53	54	55
p	6/28	7/28	5/28	5/28	5/28

8. $T8$. (1)边缘分布律：

$$p_{n\cdot}=P\{X=n\}=\sum_{m=0}^{n}P\{X=n,Y=m\}=\sum_{m=0}^{n}\frac{e^{-14}\cdot 7.14^m\cdot 6.86^{n-m}}{m!(n-m)!}$$

$$=\frac{e^{-14}}{n!}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}7.14^m\cdot 6.86^{n-m}=\frac{e^{-14}}{n!}(7.14+6.86)^n=\frac{14^n}{n!}e^{-14}\quad n=0,1,2,\cdots.$$

$$p_{\cdot m}=P\{Y=m\}=\sum_{n=m}^{\infty}P\{X=n,Y=m\}=\sum_{n=m}^{\infty}\frac{e^{-14}\cdot 7.14^m\cdot 6.86^{n-m}}{m!(n-m)!}=\frac{7.14^m\cdot e^{-14}}{m!}\sum_{n=m}^{\infty}\frac{6.86^{n-m}}{(n-m)!}$$

$$=\frac{7.14^m\cdot e^{-14}}{m!}e^{6.86}=\frac{7.14^m}{m!}e^{-7.14}\quad m=0,1,2,\cdots.$$

$$(2)P\{X=n\mid Y=m\}=\frac{P\{X=n,Y=m\}}{P\{Y=m\}}=\frac{\dfrac{e^{-14}\cdot 7.14^m\cdot 6.86^{n-m}}{m!(n-m)!}}{\dfrac{7.14^m}{m!}e^{-7.14}}=\frac{6.86^{n-m}}{(n-m)!}e^{-6.86},$$

$$n=m,m+1,\cdots;\quad m=0,1,2,\cdots$$

$$P\{Y=m\mid X=n\}=\frac{P\{X=n,Y=m\}}{P\{X=n\}}=\frac{\dfrac{e^{-14}\cdot 7.14^m\cdot 6.86^{n-m}}{m!(n-m)!}}{\dfrac{14^n}{n!}e^{-14}}=\frac{n!}{m!(n-m)!}\left(\frac{7.14}{14}\right)^m\cdot\left(\frac{6.86}{14}\right)^{n-m}$$

$$=C_n^m 0.51^m 0.49^{n-m},\qquad m=0,1,2,\cdots,n;\quad n=0,1,2,\cdots.$$

$$(3)P\{Y=m\mid X=20\}=C_{20}^m 0.51^m 0.49^{20-m},m=0,1,2,\cdots,20.$$

9. $$P\{Z=i\}=P\{X+Y=i\}=\sum_{k=0}^{i}P\{X=k,Y=i-k\}=\sum_{k=0}^{i}P\{X=k\}P\{Y=i-k\}$$

$$=\sum_{k=0}^{i}p(k)q(i-k)\quad i=0,1,2,\cdots.$$

10. (1)由

$$\begin{cases} 1 = F(+\infty,+\infty) = A(B+\dfrac{\pi}{2})(C+\dfrac{\pi}{2}) \\ 0 = F(-\infty, y) = A(B-\dfrac{\pi}{2})(C+\arctan\dfrac{y}{3}) \\ 0 = F(x,-\infty) = A(B+\arctan\dfrac{x}{2})(C-\dfrac{\pi}{2}) \end{cases}$$

解得 $A=\dfrac{1}{\pi^2},\quad B=\dfrac{\pi}{2},\quad C=\dfrac{\pi}{2},$

$$\therefore F(x,y)=\frac{1}{\pi^2}(\frac{\pi}{2}+\arctan\frac{x}{2})(\frac{\pi}{2}+\arctan\frac{y}{3});$$

(2)边缘分布函数

$$F_X(x)=F(x,+\infty)=\frac{1}{\pi}(\frac{\pi}{2}+\arctan\frac{x}{2})\quad -\infty<x<+\infty$$

$$F_Y(y)=F(+\infty,y)=\frac{1}{\pi}(\frac{\pi}{2}+\arctan\frac{y}{3})\quad -\infty<y<+\infty$$

(3)边缘分布密度为:

$$f_X(x)=F_X'(x)=\frac{1}{\pi}\frac{1}{1+(x/2)^2}\frac{1}{2}=\frac{2}{\pi}\frac{1}{4+x^2}$$

$$f_Y(y)=F_Y'(y)=\frac{1}{\pi}\frac{1}{1+(y/3)^2}\frac{1}{3}=\frac{3}{\pi}\frac{1}{9+y^2}$$

注:由联合分布知X与Y是相互独立的.

11. (1)$F_X(x)=F(x,+\infty)=\begin{cases}1-e^{-0.01x}, & x>0\\ 0, & x\le 0\end{cases}.$

$$F_Y(y)=F(+\infty,y)=\begin{cases}1-e^{-0.01y}, & y>0\\ 0, & y\le 0\end{cases}.$$

(2)并联电路中电子元件都损还时才停止工作,所以工作120小时以上的概率为

$$P\{\max\{X,Y\}\ge 120\}=1-P\{\max\{X,Y\}<120\}$$
$$=1-P\{X<120,Y<120\}\ =1-F(120,120)$$
$$=1-(1-e^{-0.01\times120}-e^{-0.01\times120}+e^{-0.01\times(120+120)})$$
$$=2e^{-1.2}-e^{-2.4}=0.5116704705.$$

注：由分布与边缘分布的关系看出X与Y独立，故

$$P\{\max\{X,Y\}\geq 120\}=1-P\{\max\{X,Y\}<120\}$$
$$=1-P\{X<120,Y<120\}\ =1-P\{X<120\}P\{Y<120\}$$
$$=1-F_X(120)F_Y(120)=1-(1-e^{-0.01\times 120})(1-e^{-0.01\times 120})$$
$$=1-(1-e^{-1.2})^2=2e^{-1.2}-e^{-2.4}=0.511\,6704705.$$

*(3)当为串联时,电子部件能正常工作120小时以上的概率为

$$P\{\min\{X,Y\}\geq 120\}=P\{X>120,Y>120\}\ (\text{利用两部分独立})$$
$$=P\{X>120\}P\{Y>120\}$$
$$=(1-P\{X<120\})(1-P\{Y<120\})$$
$$=(1-F_X(120))(1-F_Y(120))=e^{-0.01\times 120}\cdot e^{-0.01\times 120}(\text{利用边缘分布函数})$$
$$=e^{-2.4}=0.090717953\,29.$$

12. $1=F(+\infty,+\infty)=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}f(x,y)dy=\int_0^{\pi}dx\int_c^{\pi/2}\sin x\cos ydy=2(1-\sin c)$

再由$f(x,y)\geq 0$知$\sin x\cos y\geq 0$, 因$[0,\pi]$上$\sin x\geq 0$, 所以在$c\leq y\leq\frac{\pi}{2}$上$\cos y\geq 0$,

故$-\frac{\pi}{2}\leq c\leq\frac{\pi}{2}$,再由上面的结果知 $c=\frac{\pi}{6}$.

13. (1)$P\{0<X<\frac{1}{2},\frac{1}{4}<Y<1\}=\int_0^{1/2}dx\int_{1/4}^{1}f(x,y)dy=\int_0^{1/2}dx\int_{1/4}^{1}4xydy$

$$=4\int_0^{1/2}xdx\cdot\int_{1/4}^{1}ydy=\frac{15}{64}$$

(2)$P\{X=Y\}=0$−−即沿着$y=x$直线上的概率, 因$y=x$上的面积为零而为零

(3)位于概率密度不为零的 矩形域内且在$y=x$左上方部分上的概率即 为所求.

$$P\{X<Y\}=\int_0^1dx\int_x^1f(x,y)dy=\int_0^1dx\int_x^1 4xydy=\frac{1}{2}.$$

(4)边缘分布密度为：

$$f_X(x)=\begin{cases}\int_0^1 4xydy & 0<x\leq 1\\ 0 & others\end{cases}=\begin{cases}2x & 0<x\leq 1\\ 0 & \text{others}\end{cases}$$

$$f_Y(y)=\begin{cases}2y & 0<y\leq 1\\ 0 & \text{others}\end{cases}$$

注：由联合分布知X与Y是相互独立的.

$$F(x,y)=\int_{-\infty}^{x}dx\int_{-\infty}^{y}\frac{A}{\pi^2}\frac{1}{16+x^2}\frac{1}{25+y^2}dy=\frac{A}{\pi^2}\int_{-\infty}^{x}\frac{1}{16+x^2}dx\cdot\int_{-\infty}^{y}\frac{1}{25+y^2}dy$$

$$=\frac{A}{\pi^2}\frac{1}{4}\left[\arctan\frac{x}{4}\right]_{-\infty}^{x}\cdot\frac{1}{5}\left[\arctan\frac{y}{5}\right]_{-\infty}^{y}$$

$$=\frac{A}{20\pi^2}(\arctan\frac{x}{4}+\frac{\pi}{2})(\arctan\frac{y}{5}+\frac{\pi}{2})$$

由 $1=F(+\infty,+\infty)=\frac{A}{20\pi^2}(\frac{\pi}{2}+\frac{\pi}{2})(\frac{\pi}{2}+\frac{\pi}{2})=\frac{A}{20}$,

得$A=20$. 故分布函数

$$F(x,y)=\frac{1}{\pi^2}(\frac{\pi}{2}+\arctan\frac{x}{4})(\frac{\pi}{2}+\arctan\frac{y}{5})$$

15. 设甲,乙到达时间分别为X, Y, 则X与Y相互 独立, 概率密度为

$$f_X(x)=\begin{cases}\frac{1}{24} & 0\le x\le 24\\ 0 & \text{others}\end{cases},\quad f_Y(x)=\begin{cases}\frac{1}{24} & 0\le y\le 24\\ 0 & \text{others}\end{cases}$$

$$f(x,y)=f_X(x)f_Y(y)=\begin{cases}\frac{1}{24}\cdot\frac{1}{24} & 0\le x,y\le 24\\ 0 & \text{others}\end{cases}.$$

(1)$P\{X<Y\}=\int_0^{24}dx\int_x^{24}f(x,y)dy=\int_0^{24}dx\int_x^{24}\frac{1}{24^2}dy$

$$=\frac{1}{24^2}\int_0^{24}(24-x)dx=\frac{1}{2}.$$

(2)两船相遇在区域 $D:\{0\le Y-X<2\}\cup\{0\le X-Y<1\}$,

故相遇的概率为

$$P\{(X,Y)\in D\}=\iint_D f(x,y)dxdy=\frac{1}{24^2}\iint_D dxdy$$

$$=\frac{1}{24^2}(24^2-\frac{1}{2}\cdot 22\cdot 22-\frac{1}{2}\cdot 23\cdot 23)$$

$$=\frac{139}{1152}\approx 0.1207.$$

注：对于均匀分布直接用面积计算更简单：

(1)$P\{X<Y\}=\dfrac{\frac{1}{2}\times 24\times 24}{24\times 24}=\frac{1}{2}$;

(2)$P\{(X,Y)\in D\}=\frac{1}{24^2}(24^2-\frac{1}{2}\cdot 22\cdot 22-\frac{1}{2}\cdot 23\cdot 23)$

$$=\frac{1}{24^2}[\frac{1}{2}(24^2-22^2)+\frac{1}{2}(24^2-23^2)]=\frac{139}{1152}.$$

16. 设甲, 乙到达时间分别为X,Y,则X与Y相互独立, 概率密度为

$$f(x,y)=f_X(x)f_Y(y)=\frac{1}{60}\cdot\frac{1}{60}=\frac{1}{60^2}.$$

(1)以分钟计算, 两人同乘一辆车应在以下时间段到达:

$$D_1:\quad \{60\le X,Y<75\}\cup\{75\le X,Y<90\}\cup\{90\le X,Y<105\}$$
$$\cup\{105\le X,Y<120\}$$

$$P\{(X,Y)\in D_1\}=\iint_D f(x,y)dxdy=\frac{1}{60^2}\iint_D dxdy=\frac{1}{60^2}\times 4\times 15^2=\frac{1}{4}.$$

(2) $D_2:\quad \{60\le X<75,60\le Y<90\}\cup\{75\le X<90,60\le Y<105\}$

$$\cup\{90\le X<105,75\le Y<120\}\cup\{105\le X<120,90\le Y<120\}$$

$$P\{(X,Y)\in D_2\}=\iint_D f(x,y)dxdy=\frac{1}{60^2}\iint_D dxdy=\frac{1}{60^2}\times 10\times 15^2=\frac{10}{16}.$$

17. 由P51例3.10,有

$$f(x,y)=\frac{1}{2\pi\cdot\sqrt{10}\cdot\sqrt{10}\cdot\sqrt{1-0^2}}e^{-\frac{1}{2(1-0)^2}(\frac{x^2}{10}+\frac{y^2}{10})}=\frac{1}{20\pi}e^{-\frac{x^2+y^2}{20}}.$$

$$P\{X<Y\}=\int_{\pi/4}^{5\pi/4}\left[\int_0^{+\infty}f(r\cos\theta,r\cos\theta)rdr\right]d\theta=\int_{\pi/4}^{5\pi/4}\left[\int_0^{+\infty}\frac{1}{20\pi}e^{-\frac{r^2}{20}}rdr\right]d\theta$$

$$=\int_{\pi/4}^{5\pi/4}d\theta\cdot\frac{-1}{2\pi}\int_0^{+\infty}e^{-\frac{r^2}{20}}d(-\frac{r^2}{20})=\pi\cdot\frac{-1}{2\pi}e^{-\frac{r^2}{20}}\Big|_0^{+\infty}=\frac{1}{2}$$

18. $$P\{X^2+Y^2<r\}=\iint_{x^2+y^2\le r}\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy=\int_0^{2\pi}\left[\int_0^{\sqrt{r}}\frac{1}{2\pi}e^{-\frac{\rho^2}{2}}\rho d\rho\right]d\theta$$

$$=\int_0^{2\pi}d\theta\cdot\frac{-1}{2\pi}e^{-\frac{\rho^2}{2}}\Big|_0^{\sqrt{r}}=2\pi\cdot\frac{-1}{2\pi}(e^{-\frac{r}{2}}-1)=1-e^{-\frac{r}{2}}.$$

19. $$f\{x,y)=\begin{cases}1/2, & |x|+|y|<1\\ 0,\text{其它}\end{cases}$$

$$f_X(x)=\int_{-\infty}^{+\infty}f\{x,y)dy=\begin{cases}\int_{-x-1}^{x+1}1/2dy, & -1\le x<0\\ \int_{x-1}^{-x+1}1/2dy, & 0\le x<1\\ 0,\text{其它}\end{cases}=\begin{cases}x+1, & -1\le x<0\\ 1-x, & 0\le x<1\\ 0,\text{其它}\end{cases}$$

$$f_Y(y)=\begin{cases}y+1, & -1\le y<0\\ 1-y, & 0\le y<1\\ 0,\text{其它}\end{cases}.$$

$$f_X(x)f_Y(y)=\begin{cases}(x+1)(y+1), & -1\le x<0,-1\le y<0\\(x+1)(1-y), & -1\le x<0,0\le y<1\\(1-x)(y+1) & 0\le x<1,-1\le y<0\\(1-x)(1-y) & 0\le x<1,0\le y<1\\0 & \text{其它}\end{cases}$$

显然$f\{x,y)\ne f_X(x)f_Y(y)$,故X与Y不是相互独立的.

20. $$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\begin{cases}\int_0^3\frac{1}{3}\sin xdy, & 0\le x\le\frac{\pi}{2}\\0, & \text{其它}\end{cases}=\begin{cases}\sin x, & 0\le x\le\frac{\pi}{2}\\0, & \text{其它}\end{cases}$$

$$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\begin{cases}\int_0^{\frac{\pi}{2}}\frac{1}{3}\sin xdx, & 0\le y\le 3\\0, & \text{其它}\end{cases}=\begin{cases}\frac{1}{3}, & 0\le y\le 3\\0, & \text{其它}\end{cases}$$

当$0<x<\frac{\pi}{2}$时, $$f_{Y|X}(y\mid x)=\begin{cases}\frac{f(x,y)}{f_X(x)}, & 0\le y\le 3\\0, & \text{其它}\end{cases}=\begin{cases}\frac{1}{3}, & 0\le y\le 3\\0, & \text{其它}\end{cases}$$

当$0<y<3$时, $$f_{X|Y}(x\mid y)=\begin{cases}\frac{f(x,y)}{f_Y(y)}, & 0\le x\le\frac{\pi}{2}\\0, & \text{其它}\end{cases}=\begin{cases}\sin x, & 0\le x\le\frac{\pi}{2},\\0, & \text{其它}\end{cases}$$

21. $$F_Z(z)=P\{Z\le z\}=P\{X+Y\le z\}=\iint\limits_{x+y\le z}f(x,y)dxdy$$

当$z\le 0$时, $$F_Z(z)=\iint\limits_{x+y\le z}0dxdy=0.$$

当$0<z\le 1$时,

$$F_Z(z)=\int_0^{\frac{z}{2}}dx\int_x^{z-x}2(x+y)dy=\int_0^{\frac{z}{2}}(x+y)^2\Big|_x^{z-x}dx=\int_0^{\frac{z}{2}}(z^2-4x^2)dx=\frac{1}{3}z^3.$$

当$1<z<2$时,

$$F_Z(z)=\int_0^{\frac{z}{2}}dy\int_0^y 2(x+y)dx+\int_{\frac{z}{2}}^1 dy\int_0^{z-y}2(x+y)dx$$

$$=\int_0^{\frac{z}{2}}(x+y)^2\Big|_0^y dy+\int_{\frac{z}{2}}^1(x+y)^2\Big|_0^{z-y}dy=\int_0^{\frac{z}{2}}3y^2dy+\int_{\frac{z}{2}}^1(z^2-y^2)dy$$

$$=(\frac{z^3}{8}-0)+z^2(1-\frac{z}{2})-(\frac{1}{3}-\frac{1}{24}z^3)=z^2-\frac{z^3}{3}-\frac{1}{3}$$

当$z \ge 2$时， $F_Z(z) = 1.$

$$\therefore \quad f_Z(z) = F_Z'(z) = \begin{cases} 0, & 其它 \\ z^2, & 0 \le z < 1 \\ 2z - z^2, & 1 \le z \le 2 \end{cases}$$

法2　套用$P56$的$X+Y$的分布公式求： $f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)dx.$

$$f(x, z-x)\text{非零范围}: 0 \le x \le z - x \le 1 \Rightarrow \begin{cases} 0 \le x \\ x \le z - x \le 1 \end{cases} \Rightarrow \begin{cases} 0 \le x \\ 2x \le z \le 1 + x \end{cases}$$

当$z < 0$时，$f_Z(z) = \int_{-\infty}^{+\infty} 0dx = 0;$

当$0 < z \le 1$时, $f_Z(z) = \int_0^{z/2} 2[x + (z-x)]dx = \int_0^{z/2} 2zdx = 2z \cdot z/2 = z^2;$

当$1 < z < 2$时, $f_Z(z) = \int_{z-1}^{z/2} 2[x + (z-x)]dx = 2z \cdot [\frac{z}{2} - (z-1)] = 2z - z^2.$

22.

$$f_Z(z) = \int_{-\infty}^{+\infty} |y| f(yz, y)dy = \int_{-\infty}^{+\infty} |y| f_X(yz) f_Y(y)dy$$

$$= \int_{-\infty}^{+\infty} |y| \frac{1}{\sqrt{2\pi}} e^{-\frac{(yz)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |y| e^{-\frac{z^2+1}{2}y^2} dy$$

$$= \frac{2}{2\pi} \int_0^{+\infty} y e^{-\frac{z^2+1}{2}y^2} dy = \frac{1}{\pi} \frac{-1}{z^2+1} e^{-\frac{z^2+1}{2}y^2} \Bigg|_0^{+\infty} = \frac{1}{\pi(z^2+1)}$$

23.　法1　$\because f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = f_X(x) f_Y(y),$

$\therefore X \sim N(0,1), Y \sim N(0,1),$

$\therefore X + Y \sim N(0,2),$

$$\therefore P\{-\sqrt{2} < X + Y < 2\sqrt{2}\} = \Phi(\frac{2\sqrt{2} - 0}{\sqrt{2}}) - \Phi(\frac{-\sqrt{2} - 0}{\sqrt{2}}) = \Phi(2) - \Phi(-1)$$

$$= \Phi(2) - [1 - \Phi(1)] = \Phi(2) + \Phi(1) - 1$$

$$= 0.9772 + 0.8413 - 1 = 0.8185.$$

法2 $P\{-\sqrt{2}<X+Y<2\sqrt{2}\}=\iint\limits_D f(x,y)dxdy=\int_{-\infty}^{+\infty}dy\int_{-\sqrt{2}-y}^{2\sqrt{2}-y}\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dx$

$$\left(\text{令}\begin{cases}u=x+y\\ v=y\end{cases}\Rightarrow\begin{cases}x=u-v\\ y=v\end{cases},J=1,\ \text{而}\ D':-\infty<v<+\infty,-\sqrt{2}\le u\le 2\sqrt{2}\right)$$

$$=\int_{-\sqrt{2}}^{2\sqrt{2}}\int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{-\frac{(u-v)^2+v^2}{2}}|J|dvdu=\int_{-\sqrt{2}}^{2\sqrt{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{4}}\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\sqrt{2}v+\frac{u}{\sqrt{2}})^2}{2}}d(\sqrt{2}v+\frac{u}{\sqrt{2}})du$$

$$=\int_{-\sqrt{2}}^{2\sqrt{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{4}}\frac{1}{\sqrt{2}}\cdot 1du=\int_{-\sqrt{2}}^{2\sqrt{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{4}}d\frac{u}{\sqrt{2}}\quad(\text{令}w=\frac{u}{\sqrt{2}})$$

$$=\int_{-1}^{2}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw=\Phi(2)-\Phi(-1)=\Phi(2)+\Phi(1)-1$$

$=0.9772+0.8413-1$

$=0.8185$.(书上答案为0.81855,可能是多录入一位)

24. (1) $1=\int_2^4 dy\int_0^2 k(6-x-y)dx=-\frac{k}{2}\int_2^4(6-x-y)^2\Big|_0^2dy=8k$

(2) $P\{X<1,Y<3\}=\int_2^3 dy\int_0^1\frac{1}{8}(6-x-y)dx$

$=\frac{1}{8}\int_2^3(6-x-y)^2\Big|_0^1dy=\frac{3}{8}.$

(3) $P\{X<1.5\}=P\{X<1.5,Y<+\infty\}$

$=\int_{-\infty}^{1.5}dx\int_{-\infty}^{+\infty}f(x,y)dy=\int_0^{1.5}dx\int_2^4\frac{1}{8}(6-x-y)dy=\frac{27}{32}.$

(4) $P\{X+Y\le 4\}=\int_2^4 dy\int_0^{4-y}\frac{1}{8}(6-x-y)dx=\frac{2}{3}.$

25. 法1(公式法) $f_Z(z)=\int_{-\infty}^{+\infty}f(x,z-x)dx=\int_{-\infty}^{+\infty}f_X(x)f_Y(z-x)dx$

$$f_X(x)f_Y(z-x)>0\Rightarrow\begin{cases}0<x<1\\ z-x>0\end{cases}$$

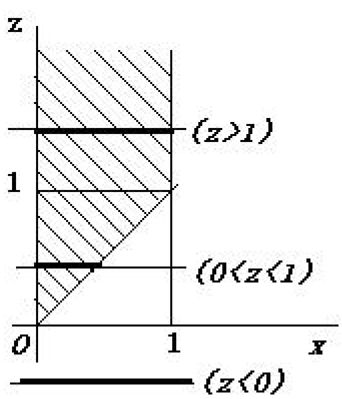
(1)当$z\le 0$时, $f_Z(z)=\int_{-\infty}^{+\infty}f_X(x)f_Y(z-x)dx=0$

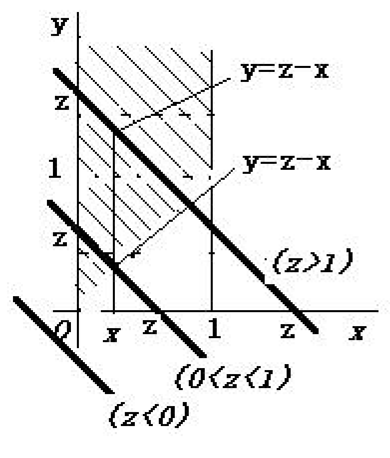
(2)当$0<z<1$时, $f_Z(z)=\int_0^z 1\cdot e^{-(z-x)}dx=e^{-z}(e^z-1)=1-e^{-z}$

(3)当$z\ge 1$时, $f_Z(z)=\int_0^1 1\cdot e^{-(z-x)}dx=e^{-z}(e-1)$

综上，有

$$f_Z(z)=\begin{cases}0 & z\le 0\\ 1-e^{-z} & 0<z<1.\\ e^{-z}(e-1) & z\ge 1\end{cases}$$





法2(直接法) $F_Z(z)=P\{X+Y\le z\}=\int_{-\infty}^{+\infty}\int_{-\infty}^{z-x} f_X(x)f_Y(y)dydx$

(1)当$z\le 0$时, $F_Z(z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{z-x} 0dydx=0$, $f_Z(z)=F_Z'(z)=0$

(2)当$0<z<1$时，$F_Z(z)=\int_0^z\int_0^{z-x} 1\cdot e^{-y}dydx=\int_0^z[-e^{-y}]_0^{z-x}dx$

$$=\int_0^z(-e^{x-z}+1)dx=[-e^{x-z}+x]_0^z=-1+z+e^{-z}$$

$f_Z(z)=1-e^{-z}$

(3)当$z\ge 1$时，$F_Z(z)=\int_0^1\int_0^{z-x} 1\cdot e^{-y}dydx=\int_0^1[-e^{-y}]_0^{z-x}dx$

$$=[-e^{x-z}+x]_0^1=-e^{1-z}+1+e^{-z}$$

$f_Z(z)=e^{1-z}-e^{-z}=e^{-z}(e-1)$

综上，有

$$f_Z(z)=\begin{cases}0 & z\le 0\\ 1-e^{-z} & 0<z<1.\\ e^{-z}(e-1) & z\ge 1\end{cases}$$

26. (1) $\because$ X与Y独立, $\therefore$ 当$y>0$时,

$$f_{X|Y}(x\mid y)=\frac{f(x,y)}{f_Y(y)}=\frac{f_X(x)f_Y(y)}{f_Y(y)}=f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x>0\\ 0 & x\le 0\end{cases}.$$

(2) $P\{Z=1\}=P\{X\le Y\}=\int_0^{+\infty}dy\int_0^y f(x,y)dxdy=\int_0^{+\infty}dy\int_0^y f_X(x)f_Y(y)dxdy$

$$=\int_0^{+\infty}\mu e^{-\mu y}dy\int_0^y\lambda e^{-\lambda x}dx=\int_0^{+\infty}\mu e^{-\mu y}(1-e^{-\lambda y})dy$$

$$=\mu\int_0^{+\infty}(e^{-\mu y}-e^{-(\mu+\lambda)y})dy=\mu\left[-\frac{1}{\mu}e^{-\mu y}+\frac{1}{\mu+\lambda}e^{-(\mu+\lambda)y}\right]_0^{+\infty}$$

$$=1-\frac{\mu}{\mu+\lambda}=\frac{\lambda}{\mu+\lambda}.$$

$$P\{Z=0\}=1-P\{Z=1\}=\frac{\mu}{\mu+\lambda}.$$

27. 由方程组解得 $Y=\frac{1}{2}X+\frac{1}{2},\quad Z=\frac{3}{2}X+\frac{1}{2}$

$$\therefore\quad P\{0\le Y\le 1\}=P\{0\le \frac{1}{2}X+\frac{1}{2}\le 1\}=P\{-1\le X\le 1\}$$

$$=P\{-1\le X<0\}+P\{0\le X\le 1\}$$

$$=0+1(\because X\text{是区间}(0,1)\text{上的均匀分布})$$

$$\therefore\quad P\{0\le Z\le 1\}=P\{0\le \frac{3}{2}X+\frac{1}{2}\le 1\}=P\{-\frac{1}{3}\le X\le \frac{1}{3}\}$$

$$=P\{-\frac{1}{3}\le X<0\}+P\{0\le X\le \frac{1}{3}\}$$

$$=0+\frac{1}{3}(\because X\text{是区间}[0,1]\text{上的均匀分布})$$

28. 当$w\le 0$时, $F_W(w)=P\{\sqrt{X^2+Y^2+Z^2}<w\}=0.$

当$w\le 0$时, $F_W(w)=P\{\sqrt{X^2+Y^2+Z^2}<w\}=\iiint\limits_{x^2+y^2+z^2\le w^2} f_X(x)f_Y(y)f_Z(z)dv$

$$=\frac{1}{2\pi\sqrt{2\pi}}\iiint\limits_{x^2+y^2+z^2} e^{-\frac{x^2+y^2+z^2}{2}}dv=\frac{1}{2\pi\sqrt{2\pi}}\int_0^{2\pi}\int_0^{\pi}\int_0^{w} e^{-\frac{r^2}{2}}r^2\sin\varphi drd\varphi d\theta$$

$$=\frac{1}{2\pi\sqrt{2\pi}}\cdot 2\pi\cdot 2\cdot\int_0^{w} e^{-\frac{r^2}{2}}r^2dr=\sqrt{\frac{2}{\pi}}\int_0^{w} e^{-\frac{r^2}{2}}r^2dr$$

$$\therefore\quad f_W(w)=F_W'(w)=\begin{cases}\sqrt{\frac{2}{\pi}}e^{-\frac{w^2}{2}}w^2, & w>0\\ 0, & w\le 0\end{cases}.$$

Book： Chapter3 习题参考答案

2. $\because X,Y$独立$),\therefore P\{Y=y_j \mid X=1\}=P\{Y=y_j\}$(如原题中右表)

3. $1=\dfrac{1}{2C}+\dfrac{1}{C}+\dfrac{1}{4C}+\dfrac{5}{4C}\Rightarrow 1=\dfrac{2+4+1+5}{4C}\Rightarrow C=3.$

20. $f\{x,y)=\begin{cases}1, & (x,y)\in D\\ 0,\text{其它}\end{cases}$

$$f_X(x)=\int_{-\infty}^{+\infty}f\{x,y)dy=\begin{cases}\int_0^x 1dy, & 0<x<1\\ \int_{x-1}^1 1dy\text{，} & 1\le x<2\\ 0,\text{其它}\end{cases}=\begin{cases}x, & 0<x<1\\ 2-x\text{，} & 1\le x<2\\ 0,\text{其它}\end{cases}$$

$$\text{当}0<x<1\text{时，}\quad f_{Y|X}(y\mid x)=\begin{cases}\dfrac{f\{x,y)}{f_X(x)}, & 0<y<x\\ 0, & \text{其它}\end{cases}=\begin{cases}1/x\text{，} & 0<y<x,\\ 0, & \text{其它}\end{cases}$$

$$\text{当}1\le x<2\text{时，}\quad f_{Y|X}(y\mid x)=\begin{cases}\dfrac{f\{x,y)}{f_X(x)}, & x-1<y<1\\ 0, & \text{其它}\end{cases}=\begin{cases}1/(2-x)\text{，} & x-1<y<1\\ 0, & \text{其它}\end{cases}$$

$$P(0<Y<0.5\mid X=0.5\}=\int_0^{0.5}f_{Y|X}(y\mid 0.5)dy=\int_0^{0.5}\frac{1}{0.5}dy=2\times(0.5-0)=1.$$

$$P(0<Y<0.5\mid X=1.2\}=\int_0^{0.2}f_{Y|X}(y\mid 1.2)dy+\int_{0.2}^{0.5}f_{Y|X}(y\mid 1.2)dy$$

$$=\int_0^{0.2}0dy+\int_{0.2}^{0.5}\frac{1}{2-1.2}dy=0+\frac{1}{0.8}0.3=0.375$$

(书上答案为0.625?)..

23. $F_Z(z)=P\{Z\le z\}=P\{X/Y\le z\}=\int_0^{+\infty}dy\int_{-\infty}^{yz}f(x,y)dx+\int_{-\infty}^{0}dy\int_{yz}^{+\infty}f(x,y)dx$

$$f_Z(z)=\int_0^{+\infty}f(yz,y)ydy+\int_{-\infty}^{0}(-f(yz,y)y)dy=\int_{-\infty}^{+\infty}f(yz,y)|y|dy$$

$$=\int_{-\infty}^{+\infty}f_X(yz)f_Y(y)|y|dy=\int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{-\frac{y^2z^2+y^2}{2}}|y|dy$$

$$=2\int_0^{+\infty}\frac{1}{2\pi}e^{-\frac{y^2z^2+y^2}{2}}ydy=\frac{1}{\pi}\int_0^{+\infty}e^{-\frac{z^2+1}{2}y^2}\frac{-1}{z^2+1}d(-\frac{z^2+1}{2}y^2)$$

$$=\frac{-1}{\pi(z^2+1)}e^{-\frac{z^2+1}{2}y^2}\Bigg|_0^{+\infty}=\frac{1}{\pi(z^2+1)}\qquad --\text{柯西分布的概率密度}$$

25. $$f_X(x)=\begin{cases}\int_{-\infty}^{+\infty}f(x,y)dy, & 0<x<1\\ 0, & \text{其它}\end{cases}=\begin{cases}\int_{-x}^{x}1\,dy, & 0<x<1\\ 0, & \text{其它}\end{cases}=\begin{cases}2x, & 0<x<1\\ 0, & \text{其它}\end{cases}$$

$$f_Y(y)=\begin{cases}\int_{-\infty}^{+\infty}f(x,y)dx, & -1<y<1\\ 0, & \text{其它}\end{cases}=\begin{cases}\int_{-y}^{1}1\,dx, & -1<y<0\\ \int_{y}^{1}1\,dx, & 0\le y<1\\ 0, & \text{其它}\end{cases}=\begin{cases}1+y, & -1<y<0\\ 1-y, & 0\le y<1\\ 0, & \text{其它}\end{cases}$$

$$=\begin{cases}1-|y|, & |y|<1\\ 0, & \text{其它}\end{cases}$$

当$0<x<1$时， $$f_{Y|X}(y|x)=\begin{cases}\dfrac{f(x,y)}{f_X(x)}, & |y|<x,\\ 0, & \text{其它}\end{cases}=\begin{cases}\dfrac{1}{2x}, & |y|<x\\ 0, & \text{其它}\end{cases}$$

当$-1<y<1$时， $$f_{X|Y}(x|y)=\begin{cases}\dfrac{f(x,y)}{f_Y(y)}, & |y|<x<1\\ 0, & \text{其它}\end{cases}=\begin{cases}\dfrac{1}{1-|y|}, & |y|<x<1\\ 0, & \text{其它}\end{cases}$$

注：在x或y其他范围内条件概率无 意义.