第五节 隐函数（组）求导

(1) 隐函数存在定理

(2) 隐函数的导数和偏导数.

方程（组）确定的隐函数（组）

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \longrightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$$

$$F(x,y,z) \longrightarrow z=f(x,y)$$

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

5.1 一个方程所确定的隐函数及其导数

定理1 设函数 $F(x,y)$ 在点 $P(x_0, y_0)$ 的某一邻域内满足

一个

自变量

① 具有连续的偏导数 $F_x'(x,y)$, $F_y'(x,y)$,

② $F(x_0, y_0)=0$;

③ $F_y'(x_0, y_0)\neq 0$

则方程 $F(x,y)=0$ 在点P 的某邻域内可唯一确定一个单值连续函数 $y=f(x)$，满足条件 $y_0=f(x_0)$，并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$ (隐函数求导公式)

注 若③换成$F_x(x_0, y_0)\neq 0$, 则确定隐函数$x=x(y)$, 在点(x_0, y_0)可导, 且

$$\frac{\mathrm{d}x}{\mathrm{d}y}=-\frac{F_y}{F_x}.$$

例1 设$x^2+y^2=1$, 求 $\dfrac{dy}{dx} \cdot \dfrac{d^2y}{dx^2}$

例1 设$x^2+y^2=1$, 求 $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$

解: $\longrightarrow$ $y = f(x)$

(1)法一 $2x + 2y \cdot \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

法二 $F(x, y)=x^2+y^2-1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{2x}{2y} = -\dfrac{x}{y}$

(2)

$$\frac{d^2y}{dx^2} = -\frac{y - x \cdot \dfrac{dy}{dx}}{y^2} = -\frac{x^2+y^2}{y^3} = -\frac{1}{y^3}$$

定理2 若函数 $F(x,y,z)$ 满足:

一个方程
二个自变量

① 在点 $P(x_0,y_0,z_0)$ 的某邻域内具有**连续偏导数,**

② $F(x_0,y_0,z_0)=0$

③ $F_z(x_0,y_0,z_0)\neq 0$

则方程 $F(x,y,z)=0$ 在点 P 的某一邻域内可唯一确定一个单值连续函数 $z=f(x,y)$, 满足 $z_0=f(x_0,y_0)$,

并有连续偏导数 $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$, $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}$

例2 设$e^{x+y+z} - xyz = e$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

例2 设$e^{x+y+z}-xyz=e$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$

解： $\Rightarrow z=f(x,y)$

法一

$$e^{x+y+z}\cdot\left(1+\frac{\partial z}{\partial x}\right)-y\left(z+x\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\frac{\partial z}{\partial x}=\frac{yz-e^{x+y+z}}{e^{x+y+z}-xy};\qquad \frac{\partial z}{\partial y}=\frac{xz-e^{x+y+z}}{e^{x+y+z}-xy}$$

法二 $F(x,y,z)=e^{x+y+z}-xyz-e.$ $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$

$$\frac{\partial z}{\partial x}=\frac{e^{x+y+z}-yz}{xy-e^{x+y+z}};\qquad \frac{\partial z}{\partial y}=\frac{e^{x+y+z}-xz}{xy-e^{x+y+z}};$$

例3 设$x^2+y^2+z^2-4z=0$, 求 $\dfrac{\partial^2 z}{\partial x^2}$

例3 设$x^2+y^2+z^2-4z=0$, 求 $\frac{\partial^2 z}{\partial x^2}$

解: $\Longrightarrow$ $z=f(x,y)$

$$2x+2z\cdot\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{(2-z)-x\cdot\left(-\frac{\partial z}{\partial x}\right)}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

5.2 方程组所确定的隐函数组及其导数

隐函数存在定理还可以推广到方程组的情形.通常,两个方程可确定两个隐函数,即

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \longrightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$$

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

定义 由 F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F、G 的**雅可比(Jacobi)**行列式.

$$\frac{\partial(F,G,H)}{\partial(x,y,z)}=\begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}$$

定理3. 设函数 $F(x,y,z)$，$G(x,y,z)$，满足:

二个方程
一个自变量

① 在点$M_0(x_0,y_0,z_0)$,的某邻域内具有连续偏导数;

② $F(x_0,y_0,z_0)=0$，$G(x_0,y_0,z_0)=0$;

③ $J\Big|_{M_0}=\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_{M_0}\neq 0$

则方程组 $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ 在点$\mathbf{M_0}$的某一邻域内可唯一确定一组满足条件$y_0=y(x_0), z_0=z(x_0)$的单值连续函数

$y=y(x), z=z(x)$, 且有连续导数公式:

$$\frac{dy}{dx} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)} = -\frac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}$$

$$\frac{dz}{dx} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,x)} = -\frac{\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}$$

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \qquad \begin{cases} y=y(x) \\ z=z(x) \end{cases}$$

$$\begin{cases} F(x,y(x),z(x))=0 \\ G(x,y(x),z(x))=0 \end{cases} \Rightarrow \begin{cases} F'_x+F'_y\cdot\dfrac{dy}{dx}+F'_z\cdot\dfrac{dz}{dx}=0 \\ G'_x+G'_y\cdot\dfrac{dy}{dx}+G'_z\cdot\dfrac{dz}{dx}=0 \end{cases}$$

$$\Rightarrow \begin{cases} F'_y\cdot\dfrac{dy}{dx}+F'_z\cdot\dfrac{dz}{dx}=-F'_x \\ G'_y\cdot\dfrac{dy}{dx}+G'_z\cdot\dfrac{dz}{dx}=-G'_x \end{cases} \Rightarrow \begin{cases} \dfrac{dy}{dx}= \\ \dfrac{dz}{dx}= \end{cases}$$

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \longrightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$$

$$\begin{cases} F(x,y(x),z(x))=0 \\ G(x,y(x),z(x))=0 \end{cases} \Rightarrow \begin{cases} F'_x+F'_y\cdot\dfrac{dy}{dx}+F'_z\cdot\dfrac{dz}{dx}=0 \\ G'_x+G'_y\cdot\dfrac{dy}{dx}+G'_z\cdot\dfrac{dz}{dx}=0 \end{cases}$$

$$\Rightarrow \begin{cases} F'_y\cdot\dfrac{dy}{dx}+F'_z\cdot\dfrac{dz}{dx}=-F'_x \\ G'_y\cdot\dfrac{dy}{dx}+G'_z\cdot\dfrac{dz}{dx}=-G'_x \end{cases} \Rightarrow \begin{cases} \dfrac{dy}{dx}=\dfrac{\begin{vmatrix} -F'_x & F'_z \\ -G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\dfrac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} \\ \dfrac{dz}{dx}=\dfrac{\begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\dfrac{\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} \end{cases}$$

例4 $\begin{cases} x+2y-3z=0 \\ x^2+y^2+z^2-21=0 \end{cases}$ 求 $\dfrac{dz}{dx}$

例4 $\begin{cases} x+2y-3z=0 \\ x^2+y^2+z^2-21=0 \end{cases}$ 求 $\dfrac{dz}{dx}$

解： $\Rightarrow \begin{cases} y=y(x) \\ z=z(x) \end{cases}$

法一

法二 $\dfrac{dz}{dx} = -\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(y,x)}$

$$J=\frac{\partial(F,G)}{\partial(y,z)}=\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}=\begin{vmatrix} 2 & -3 \\ 2y & 2z \end{vmatrix}=4z+6y$$

$$\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}=\begin{vmatrix} 2 & 1 \\ 2y & 2x \end{vmatrix}=4x-2y$$

$$\left.\right\} \frac{dz}{dx}=-\frac{4x-2y}{4z+6y}=\frac{-2x+y}{2z+3y}$$

例5 设$u=F(x, y, z)$, $z=f(x, y)$, $y=\sin x$, 求 $\dfrac{du}{dx}$

解：

$$\left\{\begin{array}{l} u=F(x, y, z) \\ z=f(x, y) \\ y=\sin x \end{array}\right. \longrightarrow \left\{\begin{array}{l} u=F[x, \sin x, f(x, \sin x)] \\ z=f(x, \sin x) \\ y=\sin x \end{array}\right.$$

$$\left\{\begin{array}{l} \dfrac{du}{dx} = F'_x + F'_y \cdot \dfrac{dy}{dx} + F'_z \cdot \dfrac{dz}{dx} \\ \dfrac{dz}{dx} = f'_x + f'_y \cdot \dfrac{dy}{dx} \\ \dfrac{dy}{dx} = \cos x \end{array}\right.$$

$$\frac{du}{dx} = F'_x + F'_y \cdot \cos x + F'_z \cdot (f'_x + f'_y \cdot \cos x)$$

定理4. 设函数 $F(x,y,u,v), G(x,y,u,v)$ 满足:

① 在点 $M_0(x_0, y_0, u_0, v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0, y_0, u_0, v_0) = 0,\ G(x_0, y_0, u_0, v_0) = 0;$

③ $J\Big|_{M_0} = \dfrac{\partial(F,G)}{\partial(u,v)}\Big|_{M_0} \neq 0$

二个方程

二个自变量

则方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 在点 $M_0(x_0, y_0, u_0, v_0)$ 的某一邻域内可唯一确定一组满足条件 $u_0 = u(x_0, y_0)$, $v_0 = v(x_0, y_0)$ 的单值连续函数 $u = u(x,y), v = v(x,y)$, 且有连续偏导数:

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{\begin{vmatrix}F'_x & F'_v\\ G'_x & G'_v\end{vmatrix}}{\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}},\qquad \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{\begin{vmatrix}F'_u & F'_x\\ G'_u & G'_x\end{vmatrix}}{\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}},$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{\begin{vmatrix}F'_{\mathbf{y}} & F'_v\\ G'_y & G'_v\end{vmatrix}}{\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}},\qquad \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{\begin{vmatrix}F'_u & F'_y\\ G'_u & G'_y\end{vmatrix}}{\begin{vmatrix}F'_u & F'_v\\ G'_u & G'_v\end{vmatrix}}.$$

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

$$\begin{cases} F(x,y,u(x,y),v(x,y))=0 \\ G(x,y,u(x,y),v(x,y))=0 \end{cases} \Rightarrow \begin{cases} F'_x+F'_u\cdot\dfrac{\partial u}{\partial x}+F'_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G'_x+G'_u\cdot\dfrac{\partial u}{\partial x}+G'_z\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$$

$$\Rightarrow \begin{cases} F'_u\cdot\dfrac{\partial u}{\partial x}+F'_v\cdot\dfrac{\partial v}{\partial x}=-F'_x \\ G'_u\cdot\dfrac{\partial u}{\partial x}+G'_z\cdot\dfrac{\partial v}{\partial x}=-G'_x \end{cases} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\begin{vmatrix} -F'_x & F'_v \\ -G'_x & G'_v \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}=-\dfrac{\begin{vmatrix} F'_x & F'_v \\ G'_x & G'_v \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}} \\ \dfrac{\partial v}{\partial x}=\dfrac{\begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}=-\dfrac{\begin{vmatrix} F'_u & F'_x \\ G'_u & G'_x \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}} \end{cases}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}$$

例6 $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

$$\boxed{\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}}$$

例6 $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

$$\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

法一

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}$$

$$D = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$$

$$D_1 = \begin{vmatrix} -u & -y \\ -v & x \end{vmatrix} = -xu - vy$$

$$D_2 = \begin{vmatrix} x & -u \\ y & -v \end{vmatrix} = -xv + uy$$

$$\begin{cases} x\dfrac{\partial u}{\partial y} - y\dfrac{\partial v}{\partial y} = v \\ y\dfrac{\partial u}{\partial y} + x\dfrac{\partial v}{\partial y} = -u \end{cases}$$

$$D = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$$

$$D_1 = \begin{vmatrix} v & -y \\ -u & x \end{vmatrix} = vx - yu$$

$$D_2 = \begin{vmatrix} x & v \\ y & -u \end{vmatrix} = -xu - vy$$

例6 $\begin{cases} xu - yv=0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

法二 $\begin{cases} F(x,y,u,v)=xu - yv=0 \\ G(x,y,u,v)=yu + xv - 1 = 0 \end{cases}$

$$\boxed{\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}}$$

$$\boxed{J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2}$$

$$\boxed{\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)} = -\frac{\begin{vmatrix} u & -y \\ v & x \end{vmatrix}}{x^2+y^2} = \frac{-ux - yv}{x^2+y^2}}$$

例6 $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

法二

$$\begin{cases} F(x,y,u,v) = xu - yv = 0 \\ G(x,y,u,v) = yu + xv - 1 = 0 \end{cases}$$

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$$J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)} = -\frac{\begin{vmatrix} -v & -y \\ u & x \end{vmatrix}}{x^2+y^2} = \frac{vx - yu}{x^2+y^2}$$

例6 $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

法二

$$\begin{cases} F(x,y,u,v) = xu - yv = 0 \\ G(x,y,u,v) = yu + xv - 1 = 0 \end{cases}$$

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$$J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\ x)} = -\frac{\begin{vmatrix} x & u \\ y & v \end{vmatrix}}{x^2 + y^2} = -\frac{vx - yu}{x^2 + y^2}$$

例6 $\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$

法二

$$\begin{cases} F(x,y,u,v) = xu - yv = 0 \\ G(x,y,u,v) = yu + xv - 1 = 0 \end{cases}$$

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

$$J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)} = -\frac{\begin{vmatrix} x & -v \\ y & u \end{vmatrix}}{x^2+y^2} = -\frac{ux+yv}{x^2+y^2}$$

5.3 一阶全微分形式不变性的应用

利用一阶全微分形式不变性求偏导数

1. 将所有变量视为自变量，等式两端求全微分.

2. 同样适用于方程组确定的隐函数求偏导数.

3. 求(偏)导方法的选择.

例6 $\begin{cases} xu - yv=0 \\ yu + xv = 1 \end{cases}$ 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ $\boxed{\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}}$

法三

$$\begin{cases} xdu+udx - ydv-vdy=0 \\ ydu + udy + xdv + vdx = 0 \end{cases} \Rightarrow \begin{cases} xdu- ydv=-udx+vdy \\ ydu + xdv = -udy-vdx \end{cases}$$

$$\boxed{D= \begin{vmatrix} x & -y \\ y & x \end{vmatrix} =x^2 + y^2}$$

$$D_1 = \begin{vmatrix} -udx+vdy & -y \\ -udy-vdx & x \end{vmatrix} \quad D_2 = \begin{vmatrix} x & -udx+vdy \\ y & -udy-vdx \end{vmatrix} \Rightarrow \begin{cases} dv = \\ du = \end{cases}$$

例7. 设由方程组 $\begin{cases} x = -u^2 + v + z \\ y = u + vz \end{cases}$ 确定了隐函数

$u = u(x, y, z)$, $v = v(x, y, z)$, 求它们的偏导数.

例7. 设由方程组 $\begin{cases} x = -u^2 + v + z \\ y = u + vz \end{cases}$ **确定了隐函数**

$u = u(x, y, z), v = v(x, y, z)$, **求它们的偏导数.**

$$\begin{cases} dx = -2u\,du + dv + dz \\ dy = du + z\,dv + v\,dz \end{cases} \Rightarrow \begin{cases} du = \\ dv = \end{cases}$$

$$\begin{cases} 2u\,du - dv = dz - dx \\ du + z\,dv = -v\,dz + dy \end{cases} \Rightarrow D = \begin{vmatrix} 2u & -1 \\ 1 & z \end{vmatrix} = 2uz + 1$$

$$\begin{cases} du = \dfrac{\begin{vmatrix} dz - dx & -1 \\ -v\,dz + dy & z \end{vmatrix}}{2uz+1} = \dfrac{-z\,dx + dy + (z - v)dz}{2uz+1} \\ dv = \dfrac{\begin{vmatrix} 2u & dz - dx \\ 1 & -v\,dz + dy \end{vmatrix}}{2uz+1} = \dfrac{dx + 2u\,dy - (2uv+1)dz}{2uz+1} \end{cases}$$

例7. 设由方程组 $\begin{cases} x = -u^2 + v + z \\ y = u + vz \end{cases}$ 确定了隐函数

$u = u(x, y, z), v = v(x, y, z)$, 求它们的偏导数.

$$\begin{cases} du = \dfrac{-zdx + dy + (z-v)dz}{2uz+1} = \dfrac{-z}{2uz+1}dx + \dfrac{1}{2uz+1}dy + \dfrac{(z-v)}{2uz+1}dz \\ dv = \dfrac{dx + 2udy - (2uv+1)dz}{2uz+1} = \dfrac{1}{2uz+1}dx + \dfrac{2u}{2uz+1}dy - \dfrac{2uv+1}{2uz+1}dz \end{cases}$$

$$\frac{\partial u}{\partial x} = \frac{-z}{2uz+1} \qquad \frac{\partial u}{\partial y} = \frac{1}{2uz+1} \qquad \frac{\partial u}{\partial z} = \frac{(z-v)}{2uz+1}dz$$

$$\frac{\partial v}{\partial x} = \frac{1}{2uz+1} \qquad \frac{\partial v}{\partial y} = \frac{2u}{2uz+1}dy \qquad \frac{\partial v}{\partial z} = -\frac{2uv+1}{2uz+1}dz$$

内容小结

隐函数(组)求导方法

由方程（组）个数及未知量个数确定几个几元

方法1. 利用复合函数求导法则直接计算；

方法2. 利用微分形式不变性；

方法3. 代公式

练习1. 设$F(x,y)$具有连续偏导数,已知方程 $F(\frac{x}{z},\frac{y}{z})=0$,求 $\mathrm{d}z$.

解法1 利用偏导数公式. 设 $z=f(x,y)$ 是由方程 $F(\frac{x}{z},\frac{y}{z})=0$ 确定的隐函数,则

$$\frac{\partial z}{\partial x}=-\frac{F_1'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_1'}{xF_1'+yF_2'}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_2'}{xF_1'+yF_2'}$$

故 $$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

练习1. 设$F(x,y)$具有连续偏导数,已知方程 $F(\frac{x}{z},\frac{y}{z})=0$,求 $\mathrm{d}z$.

解法2 微分法. 对方程两边求微分:

$$F(\frac{x}{z},\frac{y}{z})=0 \qquad F_1'\cdot \mathrm{d}(\frac{x}{z})+F_2'\cdot \mathrm{d}(\frac{y}{z})=0$$

$$F_1'\cdot(\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2})+F_2'\cdot(\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2})=0$$

$$\frac{xF_1'+yF_2'}{z^2}\mathrm{d}z=\frac{F_1'\mathrm{d}x+F_2'\mathrm{d}y}{z}$$

$$\mathrm{d}z=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

练习2. 设 $z=f(x+y+z,xyz)$，求 $\frac{\partial z}{\partial x}, \frac{\partial x}{\partial z}, \frac{\partial x}{\partial y}$.

- $$\frac{\partial z}{\partial x}=f_1'\cdot(1+\frac{\partial z}{\partial x})+f_2'\cdot(yz+xy\frac{\partial z}{\partial x})$$

$$\Longrightarrow \frac{\partial z}{\partial x}=\frac{f_1'+yz f_2'}{1-f_1'-xy f_2'}$$

- $$1=f_1'\cdot(\frac{\partial x}{\partial z}+1)+f_2'\cdot(yz\frac{\partial x}{\partial z}+xy)$$

$$\Longrightarrow \frac{\partial x}{\partial z}=\frac{1-f_1'-xy f_2'}{f_1'+yz f_2'}$$

- $$0=f_1'\cdot(\frac{\partial x}{\partial y}+1)+f_2'\cdot(yz\frac{\partial x}{\partial y}+xz)$$

$$\Longrightarrow \frac{\partial x}{\partial y}=-\frac{f_1'+xz f_2'}{f_1'+yz f_2'}$$

解法2. 利用全微分形式不变性同时求出各偏导数.

$$z = f(x+y+z, xyz)$$

$$\mathrm{d}z = f_1'\cdot(\mathrm{d}x+\mathrm{d}y+\mathrm{d}z) + f_2'\cdot(yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z)$$

解出 $\mathrm{d}x$:

$$\mathrm{d}x = \frac{-(f_1' + xz f_2')\mathrm{d}y + (1 - f_1' - xy f_2')\mathrm{d}z}{f_1' + yz f_2'}$$

由 $\mathrm{d}y$, $\mathrm{d}z$ 的系数即可得 $\dfrac{\partial x}{\partial y}, \dfrac{\partial x}{\partial z}$.

练习3. 设 $y=y(x), z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数，求 $\frac{\mathrm{d}z}{\mathrm{d}x}$.

解法1 分别在各方程两端对 x 求导，得

$$\begin{cases} z' = f + x\cdot f'\cdot(1+y') \\ F_x + F_y\cdot y' + F_z\cdot z' = 0 \end{cases} \Longrightarrow \begin{cases} -xf'\cdot y' + z' = f + xf' \\ F_y\cdot y' + F_z\cdot z' = -F_x \end{cases}$$

$$\therefore \quad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\begin{vmatrix} -xf' & f+xf' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -xf' & 1 \\ F_y & F_z \end{vmatrix}} = \frac{(f+xf')F_y - xf'\cdot F_x}{F_y + xf'\cdot F_z}$$

$$(F_y + xf'\cdot F_z \neq 0)$$

解法2 微分法.

$$z = xf(x+y),\ F(x,y,z)=0$$

对各方程两边分别求微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + xf'\cdot(\mathrm{d}x+\mathrm{d}y) \\ F_1'\mathrm{d}x + F_2'\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

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$$\begin{cases} (f+xf')\,\mathrm{d}x + x\,f'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1'\,\mathrm{d}x + F_2'\,\mathrm{d}y + F_3'\mathrm{d}z = 0 \end{cases}$$

消去$\mathrm{d}\,y$可得$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.