# 习题 4

1. $E(X)=(-1)\times 0.2+0\times 0.3+1\times 0.5=0.3.$

$E(X^2)=(-1)^2\times 0.2+0^2\times 0.3+1^2\times 0.5=0.7$

$E(-2X^2+3)=-2E(X^2)+3=-2\times 0.7+3=1.6$

**2. 参习题 3 第 6 题：**

数 1 出现的概率是 1/10,它可被自己整除 X=1

数 2、3、5、7 出现的概率是 4/10,它可被 X=2 个整数整除

数 4、9 出现的概率是 2/10,它们可以被 X=3 个整数整除

数 6、8、10 出现的概率是 2/10,它们可以被 X=4 个整数整除

于是给出分布率：

| X | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P | 1/10 | 4/10 | 2/10 | 3/10 |

$$E(X)=1\times\frac{1}{10}+2\times\frac{4}{10}+3\times\frac{2}{10}+4\times\frac{3}{10}=\frac{27}{10}=2.7.$$

3. $E(X)=\int_{-\infty}^{+\infty}xf(x)dx=\int_{-\infty}^{+\infty}x\cdot\frac{1}{2}e^{-|x|}dx=\int_{-\infty}^{0}\frac{x}{2}e^{x}dx+\int_{0}^{+\infty}\frac{x}{2}e^{-x}dx=0$

$$E(X^2)=\int_{-\infty}^{+\infty}x^2f(x)dx=\int_{-\infty}^{+\infty}x^2\cdot\frac{1}{2}e^{-|x|}dx=\int_{0}^{+\infty}x^2e^{-x}dx=-\int_{0}^{+\infty}x^2de^{-x}$$

$$=-\left[0+2\int_{0}^{+\infty}xde^{-x}\right]=-2\left[xe^{-x}\Big|_0^{+\infty}-\int_{0}^{+\infty}e^{-x}dx\right]=-2\left[0+e^{-x}\Big|_0^{+\infty}\right]=2.$$

4.

10. $E(X)=\int_{-\infty}^{+\infty}xf(x)dx=\int_0^1 x\cdot xdx+\int_1^2 x\cdot(2-x)dx=\frac{x^3}{3}\Big|_0^1+\left[x^2-\frac{x^3}{3}\right]_1^2=1$

$$D(X)=(\int_{-\infty}^{+\infty}[x-E(X)]^2f(x)dx=)E(X^2)-[E(X)]^2=\int_{-\infty}^{+\infty}x^2f(x)dx-1^2$$

$$=\int_0^1 x^2\cdot xdx+\int_1^2 x^2\cdot(2-x)dx-1=\frac{x^4}{4}\Big|_0^1+\left[\frac{2}{3}x^3-\frac{x^4}{4}\right]_1^2-1=\frac{1}{6}.$$

5.

$$E(Y)=E(X_1)-2E(X_2)+3E(X_3)$$
$$=\frac{0+6}{2}-2\cdot 1+3\cdot\frac{1}{3}$$
$$=2.$$
$$D(Y)=D(X_1)+(-2)^2D(X_2)+3^2D(X_3)$$
$$=\frac{(6-0)^2}{12}+4\cdot 3+9\cdot\frac{1}{3^2}$$
$$=16.$$

$$E(Y^2)=D(Y)+(E(Y))^2=16+4=20.$$

12. 设(X,Y)的分布律为

| X \ Y | −1 | 1 | 2 |
|---|---|---|---|
| −1 | $\frac{5}{20}$ | $\frac{2}{20}$ | $\frac{6}{20}$ |
| 1 | $\frac{3}{20}$ | $\frac{3}{20}$ | $\frac{1}{20}$ |

求：(1) X＋Y；(2) X－Y 的期望和方差.

**6. 联合分布与边缘分布为**

| X \ Y | -1 | 1 | 2 | P{*X*=*i*} |
|---|---|---|---|---|
| -1 | 5/20 | 2/20 | 6/20 | 13/20 |
| 1 | 3/20 | 3/20 | 1/20 | 7/20 |
| P{*Y*=*j*} | 8/20 | 5/20 | 7/20 | |

$$E(X)=(-1)\times\frac{13}{20}+1\times\frac{7}{20}=-\frac{6}{20},$$

$$E(X^2)=(-1)^2\times\frac{13}{20}+1^2\times\frac{7}{20}=1,$$

$$E(Y)=(-1)\times\frac{8}{20}+1\times\frac{5}{20}+2\times\frac{7}{20}=\frac{11}{20},$$

$$E(Y^2)=(-1)^2\times\frac{8}{20}+1^2\times\frac{5}{20}+2^2\times\frac{7}{20}=\frac{41}{20}$$

(1) $E(X+Y)=E(X)+E(Y)=-\frac{6}{20}+\frac{11}{20}=\frac{5}{20}=\frac{1}{4}$.

因 $X$ 与 $Y$ 不独立，因此不能用 $D(X+Y)=D(X)+D(Y)$ 计算

$$D(X+Y)=E\{(X+Y)-[E(X+Y)]^2\}$$

$$=(-1-1-\frac{1}{4})^2\times\frac{5}{20}+(-1+1-\frac{1}{4})^2\times\frac{2}{20}+(-1+2-\frac{1}{4})^2\times\frac{6}{20}$$

$$+(1-1-\frac{1}{4})^2\times\frac{3}{20}+(1+1-\frac{1}{4})^2\times\frac{3}{20}+(1+2-\frac{1}{4})^2\times\frac{1}{20}$$

$$=\frac{1}{4^2\times 20}(81\times 5+2+9\times 6+3+49\times 3+121)=\frac{183}{80}$$

(也可先求 $E((X+Y)^2)$, 再求 $D(X+Y)$.)

(2) $E(X-Y)=E(X)-E(Y)=-\frac{11}{20}-\frac{6}{20}=-\frac{17}{20}.$

$$D(X-Y)=E\{[(X-Y)-E(X-Y)]^2\}$$

$$=(-1+1+\frac{17}{20})^2\times\frac{5}{20}+(-1-1+\frac{17}{20})^2\times\frac{2}{20}+(-1-2+\frac{17}{20})^2\times\frac{6}{20}$$

$$+(1+1+\frac{17}{20})^2\times\frac{3}{20}+(1-1+\frac{17}{20})^2\times\frac{3}{20}+(1-2+\frac{17}{20})^2\times\frac{1}{20}$$

$$=\frac{24220}{20^2\times20}=\frac{1211}{400}$$

**法 2：**

| P | 5/20 | 2/20 | 6/20 | 3/20 | 3/20 | 1/20 |
|---|---|---|---|---|---|---|
| (X,Y) | (-1,-1) | (-1,1) | (-1,2) | (1,-1) | (1,1) | (1,2) |
| X+Y | -2 | 0 | 1 | 0 | 2 | 3 |
| X-Y | 0 | -2 | -3 | 2 | 0 | -1 |

(1)

$$E(X+Y)^2=4\cdot\frac{5}{20}+1\cdot\frac{6}{20}+4\cdot\frac{3}{20}+9\cdot\frac{1}{20}=\frac{47}{20}$$

$$D(X+Y)=E(X+Y)^2-(E(X+Y))^2=\frac{47}{20}-(\frac{1}{4})^2=\frac{183}{80}$$

(2)

$$E(X-Y)^2=4\cdot\frac{2}{20}+9\cdot\frac{6}{20}+4\cdot\frac{3}{20}+1\cdot\frac{1}{20}=\frac{75}{20}$$

$$D(X-Y)=E(X-Y)^2-(E(X-Y))^2=\frac{75}{20}-(-\frac{17}{20})^2=\frac{1211}{400}$$

7. $E(Z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}zf(x,y)dydx=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}z(x,y)f_X(x)f_Y(y)dydx$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sqrt{x^2+y^2}\cdot\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dydx=\int_0^{2\pi}\int_0^{+\infty}r\cdot\frac{1}{2\pi}e^{-\frac{r^2}{2}}rdrd\theta$$

$$=\frac{1}{2\pi}\cdot 2\pi\int_0^{+\infty}(-r)de^{-\frac{r^2}{2}}=-re^{-\frac{r^2}{2}}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-\frac{r^2}{2}}dr=\sqrt{2}\int_0^{+\infty}e^{-\frac{r^2}{2}}d\frac{r}{\sqrt{2}}=\sqrt{\frac{\pi}{2}}\hat{=}a.$$

$$D(Z)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[z-E(Z)]^2f(x,y)dydx=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\sqrt{x^2+y^2}-a)^2\cdot\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dydx$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}(r-a)^2e^{-\frac{r^2}{2}}rdrd\theta=\int_0^{+\infty}(r-a)^2e^{-\frac{r^2}{2}}rdr=-\int_0^{+\infty}(r-a)^2de^{-\frac{r^2}{2}}$$

$$=-(r-a)^2e^{-\frac{r^2}{2}}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-\frac{r^2}{2}}\cdot 2(r-a)dr=a^2+2\int_0^{+\infty}e^{-\frac{r^2}{2}}rdr-2a\int_0^{+\infty}e^{-\frac{r^2}{2}}dr$$

$$=a^2-2e^{-\frac{r^2}{2}}\Big|_0^{+\infty}-2a\sqrt{2}\int_0^{+\infty}e^{-\frac{r^2}{2}}d\frac{r}{\sqrt{2}}=a^2+2-2a\sqrt{2}\cdot\frac{\sqrt{\pi}}{2}=\frac{\pi}{2}+2-\pi=2-\frac{\pi}{2}.$$

8. 设$(X,Y)$的密度函数为

$$f(x,y)=\begin{cases}12y^2 & 0\le y\le x\le 1\\ 0 & 其它\end{cases}$$

求 $E(X),E(Y),E(XY),E(X^2+Y^2)$.

$$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\begin{cases}\int_0^x 12y^2dy, 0<x\le 1\\ 0,\quad 其它\end{cases}=\begin{cases}4x^3, 0<x\le 1\\ 0,\quad 其它\end{cases}$$

$$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\begin{cases}\int_y^1 12y^2dx, 0<y<1\\ 0,\quad 其它\end{cases}=\begin{cases}12y^2(1-y), 0<y<1\\ 0,\quad 其它\end{cases}$$

$$E(X)=\int_{-\infty}^{+\infty}xf_X(x)dx=\int_0^1 x\cdot 4x^3dx=\frac{4}{5}.$$

$$E(Y)=\int_{-\infty}^{+\infty}yf_Y(y)dy=\int_0^1 y\cdot 12y^2(1-y)dy=3y^4\Big|_0^1-\frac{12}{5}y^5\Big|_0^1=\frac{3}{5}.$$

$$E(XY)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf(x,y)dydx=\int_0^1\int_0^x xy\cdot 12y^2dydx=3\int_0^1 x^5dx=\frac{3}{6}=\frac{1}{2}.$$

$$E(X^2+Y^2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x^2+y^2)f(x,y)dydx=\int_0^1\int_0^x(x^2+y^2)\cdot 12y^2dydx$$

$$=3\int_0^1(4x^5+\frac{12}{5}x^5)dx=\frac{16}{15}.$$

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9. $1=F(+\infty,+\infty)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)dydx=\int_0^1\int_0^x kdydx=k\int_0^1 xdx=\frac{k}{2}$

(或由分布均匀知$k=\frac{1}{f\text{非零区域面积}}=\frac{1}{\frac{1}{2}\cdot 1\cdot 1}=2.$)

$$\therefore \qquad f(x,y)=\begin{cases}2, & 0<y<x, 0<x<1\\ 0, & \text{其它}\end{cases}$$

$$E(XY)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf(x,y)dydx=\int_0^1\int_0^x xy\cdot 2dydx=\frac{1}{4}.$$

10. $\boldsymbol{E(X)=\int_{-\infty}^{+\infty}xf(x)dx=\int_0^{+\infty}x\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx=-\int_0^{+\infty}xde^{-\frac{x^2}{2\sigma^2}}}$

$$\boldsymbol{=-xe^{-\frac{x^2}{2\sigma^2}}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-\frac{x^2}{2\sigma^2}}dx=0+\frac{\sqrt{2\pi}\sigma}{2}\cdot 2\int_0^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}dx}$$

$$\boldsymbol{=\sqrt{\frac{\pi}{2}}\sigma\cdot 1}$$

$$\boldsymbol{E(X^2)=\int_0^{+\infty}x^2\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx=-\int_0^{+\infty}x^2de^{-\frac{x^2}{2\sigma^2}}}$$

$$\boldsymbol{=-x^2e^{-\frac{x^2}{2\sigma^2}}\Big|_0^{+\infty}+2\int_0^{+\infty}xe^{-\frac{x^2}{2\sigma^2}}dx=-2\sigma^2e^{-\frac{x^2}{2\sigma^2}}\Big|_0^{+\infty}}$$

$$\boldsymbol{=2\sigma^2}$$

$$\boldsymbol{DX=E(X^2)-(EX)^2=2\sigma^2-\frac{\pi}{2}\sigma^2=\frac{4-\pi}{4}\sigma^2.}$$

11. $\boldsymbol{E(X)=\sum_{k=1}^{\infty}kP\{X=k\}=\sum_{k=1}^{\infty}kp(1-p)^{k-1}=-p\sum_{k=1}^{\infty}[(1-p)^k]'=-p\left(\sum_{k=1}^{\infty}(1-p)^k\right)'}$

$$\boldsymbol{=-p\left(\sum_{k=0}^{\infty}(1-p)^k-1\right)'=-p\left(\frac{1}{1-(1-p)}-1\right)'=\frac{1}{p}.}$$

$$\boldsymbol{D(X)=\sum_{k=1}^{\infty}(k-\frac{1}{p})^2\cdot p(1-p)^{k-1}\}=\sum_{k=1}^{\infty}k^2p(1-p)^{k-1}-2\sum_{k=1}^{\infty}k(1-p)^{k-1}+\frac{1}{p}\sum_{k=1}^{\infty}(1-p)^{k-1}}$$

$$\boldsymbol{=\sum_{k=1}^{\infty}k^2p(1-p)^{k-1}-\frac{2}{p}E(x)+\frac{1}{p}\cdot\frac{1}{1-(1-p)}=\sum_{k=1}^{\infty}k^2p(1-p)^{k-1}-\frac{1}{p^2}}$$

而 $\boldsymbol{\sum_{k=1}^{\infty}k^2p(1-p)^{k-1}=p\sum_{k=1}^{\infty}(k+1)k(1-p)^{k-1}-\sum_{k=1}^{\infty}kp(1-p)^{k-1}=p\sum_{k=1}^{\infty}[(1-p)^{k+1}]''-E(X)}$

$$= p[\sum_{k=1}^{\infty}(1-p)^{k+1}]'' - \frac{1}{p} = p[\frac{1}{1-(1-p)} - (1-p) - 1]'' - \frac{1}{p} = \frac{2}{p^2} - \frac{1}{p}$$

$$\therefore D(X) = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p}.$$

12. (1)若X为离散型随机变量,则

$$\varphi(k) = E(X-k)^2 = \sum_{i=1}^{\infty}(x_i - k)^2 p_i = \sum_{i=1}^{\infty} x_i^2 p_i - 2k\sum_{i=1}^{\infty} x_i p_i + k^2 \sum_{i=1}^{\infty} p_i$$

$$= \sum_{i=1}^{\infty} x_i^2 p_i - 2kE(X) + k^2 \cdot 1$$

令$\varphi'(k) = 0 - 2E(X) + 2k = 0$得$k = E(X)$,又$\varphi''(k) = 2 > 0$,

故$\varphi(k)$在$k = E(X)$取得最小值.

(2)若X为连续型随机变量,则

$$\varphi(k) = E(X-k)^2 = \int_{-\infty}^{+\infty}(x-k)^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 f(x)dx - 2k\int_{-\infty}^{+\infty} xf(x)dx + k^2 \int_{-\infty}^{+\infty} f(x)dx$$

$$= E(X^2) - 2kE(X) + k^2 \cdot 1$$

令$\varphi'(k) = 0 - 2E(X) + 2k = 0$得$k = E(X)$,又$\varphi''(k) = 2 > 0$,

故$\varphi(k)$在$k = E(X)$取得最小值.

13. (1)对概率密度为$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$的柯西分布,有

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{1}{\pi}\lim_{\substack{a\to-\infty \\ b\to+\infty}}\left[\ln(1+x^2)\right]_a^b$$

$$= \frac{1}{\pi}\lim_{\substack{a\to-\infty \\ b\to+\infty}}\left[\ln(1+b^2) - \ln(1+a^2)\right] = \frac{1}{\pi}\lim_{\substack{a\to-\infty \\ b\to+\infty}}\ln\frac{1+b^2}{1+a^2} \text{不存在}$$

(2)对概率分布为$P\{X = (-1)^{j+1}\frac{3^j}{j}\} = \frac{j}{3^j}$的离散型随机变量$X$,有

$$E(X) = \sum_{j=1}^{\infty} x_j p_j = \sum_{j=1}^{\infty}\left((-1)^{j+1}\frac{3^j}{j}\cdot\frac{j}{3^j}\right) = \sum_{j=1}^{\infty}(-1)^{j+1} \text{不存在.}$$

14. **因$E(X),D(X)$为常数,所以由均值的性质得**

$$E(X^*) = E(\frac{X-E(X)}{\sqrt{D(X)}}) = \frac{1}{\sqrt{D(X)}}E(X-E(X))$$

$$= \frac{1}{\sqrt{D(X)}}(E(X) - E(X)) = 0$$

$$D(X^*) = (\frac{1}{\sqrt{D(X)}})^2 D(X + (-E(X))) = \frac{1}{D(X)}D(X) = 1(\text{性质3的推论})$$

15. (1) $E(\overline{X})=E(\frac{1}{n}\sum_{i=1}^{n}X_i)=\frac{1}{n}E(\sum_{i=1}^{n}X_i)=\frac{1}{n}\sum_{i=1}^{n}E(X_i)=\frac{1}{n}\sum_{i=1}^{n}\mu=\mu.$

由$X_1,X_2,\cdots,X_n$相互独立,得

$$D(\overline{X})=(\frac{1}{n})^2D(\sum_{i=1}^{n}X_i)=\frac{1}{n^2}\sum_{i=1}^{n}D(X_i)=\frac{1}{n^2}\sum_{i=1}^{n}\sigma^2=\frac{\sigma^2}{n}.$$

(2) $E(S^2)=E(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2)=\frac{1}{n-1}E(\sum_{i=1}^{n}(X_i^2-2X_i\overline{X}+\overline{X}^2))$

$$=\frac{1}{n-1}E(\sum_{i=1}^{n}X_i^2-2n\overline{X}\cdot\overline{X}+n\overline{X}^2)=\frac{1}{n-1}\left[\sum_{i=1}^{n}E(X_i^2)-nE(\overline{X}^2)\right]$$

$$=\frac{1}{n-1}\left\{\sum_{i=1}^{n}[DX_i+(EX_i)^2]-n[D(\overline{X})+(E(\overline{X})^2]\right\}$$

$$=\frac{1}{n-1}[n\sigma^2+n\mu^2-n(\frac{\sigma^2}{n}+\mu^2)]=\frac{1}{n-1}(n-1)\sigma^2=\sigma^2.$$

16. 法1 $E(X)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xf(x,y)dydx=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}x\cdot\frac{1}{2}\sin(x+y)dydx$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}[\int_0^{\frac{\pi}{2}}xd\cos(x+y)]dx=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[x\cos(x+y)-\sin(x+y)\right]_0^{\frac{\pi}{2}}dx$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{\pi}{2}\cos(x+\frac{\pi}{2})-\sin(x+\frac{\pi}{2})+\sin x\right]dx$$

$$=-\frac{1}{2}\left[\frac{\pi}{2}\sin(x+\frac{\pi}{2})+\cos(x+\frac{\pi}{2})-\cos x\right]_0^{\frac{\pi}{2}}$$

$$=-\frac{1}{2}\left[-1-(\frac{\pi}{2}-1)\right]=\frac{\pi}{4}.$$

$E(X^2)=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}x^2\cdot\frac{1}{2}\sin(x+y)dydx$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}[\int_0^{\frac{\pi}{2}}x^2d\cos(x+y)]dx=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\left[x^2\cos(x+y)\right]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\cos(x+y)dy\right]dx$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{\pi^2}{4}\cos(x+\frac{\pi}{2})-2\left[x\sin(x+y)+\cos(x+y)\right]_0^{\frac{\pi}{2}}\right]dx$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{\pi^2}{4}\cos(x+\frac{\pi}{2})-2\left[\frac{\pi}{2}\sin(x+\frac{\pi}{2})+\cos(x+\frac{\pi}{2})-\cos x\right]\right]dx$$

$$=-\frac{1}{2}\left[\frac{\pi^2}{4}\sin(x+\frac{\pi}{2})+\pi\cos(x+\frac{\pi}{2})-2\sin(x+\frac{\pi}{2})+2\sin x\right]_0^{\frac{\pi}{2}}$$

$$=-\frac{1}{2}\left[-\pi+2-(\frac{\pi^2}{4}-2)\right]=\frac{\pi}{2}+\frac{\pi^2}{8}-2$$

16. $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\begin{cases}\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(x+y)dy,0\le x\le\frac{\pi}{2}\\ 0,\quad 其它\end{cases}=\begin{cases}\frac{1}{2}(\sin x+\cos x),0\le x\le\frac{\pi}{2}\\ 0,其它\end{cases}$

$$f_Y(y)=\begin{cases}\frac{1}{2}(\sin y+\cos y),0\le y\le\frac{\pi}{2}\\ 0,其它\end{cases}$$

$$E(Y)=E(X)=\int_{-\infty}^{+\infty}xf_X(x)dx=\int_0^{\frac{\pi}{2}}x\cdot\frac{1}{2}(\sin x+\cos x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}xd(-\cos x+\sin x)$$

$$=\frac{1}{2}[x(-\cos x+\sin x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(-\cos x+\sin x)dx]=\frac{1}{2}[\frac{\pi}{2}+[\sin x+\cos x]_0^{\frac{\pi}{2}}=\frac{\pi}{4}.$$

$$E(X^2)=\int_{-\infty}^{+\infty}x^2\cdot\frac{1}{2}(\sin x+\cos x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}x^2d(-\cos x+\sin x)$$

$$=\frac{1}{2}[x^2(-\cos x+\sin x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}2x(-\cos x+\sin x)dx]=\frac{1}{2}[\frac{\pi^2}{4}+2\int_0^{\frac{\pi}{2}}xd(\sin x+\cos x)]$$

$$=\frac{\pi^2}{8}+x(\sin x+\cos x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(\sin x+\cos x)dx=\frac{\pi^2}{8}+\frac{\pi}{2}+[\cos x-\sin x]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi^2}{8}+\frac{\pi}{2}-2$$

$$D(Y)=D(X)=E(X^2)-(E(X))^2=\frac{\pi^2}{8}+\frac{\pi}{2}-2-(\frac{\pi}{4})^2=\frac{\pi^2+8\pi-32}{16}.$$

17. **由习题册第10题(书第4题)知 $E(X)=1,\quad D(X)=\frac{1}{6}$. 由$Y\sim\pi(1)$知**

$$E(Y)=\frac{1}{\lambda}=1\text{ , }D(Y)=\frac{1}{\lambda^2}=1$$

**由$X$与$Y$相互独立得** $E(XY)=E(X)E(Y)=1$

$$E((XY)^2)=E(X^2)E(Y^2)=[D(X)+(E(X))^2][D(Y)+(E(Y))^2]=\frac{7}{6}\cdot2=\frac{7}{3}$$

$$D(XY)=E((XY)^2)-(E(XY))^2=\frac{7}{3}-1=\frac{4}{3}\qquad(书上答案为\frac{7}{6}=E(X^2)).$$

18. 利用书上 P237 答案中给的离散型随机变量的例子：
写出联合及边缘分布律如下

| Y / X | -1 | 0 | 1 | P{X=i} |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| -1 | 1/8 | 1/8 | 1/8 | 3/8 |
| 0 | 1/8 | 0 | 1/8 | 2/8 |
| 1 | 1/8 | 1/8 | 1/8 | 3/8 |
| P{Y=j} | 3/8 | 2/8 | 3/8 | |

由表可知$X$与$Y$不是相互独立的,即

$$P(X=x_i, Y=y_j\} \neq P(X=x_i\}\cdot P(Y=y_j\}$$

再由

$$E(XY)=\sum_{j=1}^{3}\sum_{i=1}^{3}x_i y_j p_{ij}=(-1)^2\times\frac{1}{8}+0+(-1)\times 1\times\frac{1}{8}+0+0+0+1\times(-1)\times\frac{1}{8}+1^2\times\frac{1}{8}=0$$

$$E(Y)=E(X)=(-1)\times\frac{3}{8}+0\times\frac{2}{8}+1\times\frac{3}{8}=0$$

$$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=0,$$

$\therefore\quad \rho_{XY}=0,\quad$ 故$X$与$Y$也不相关.

19. (1) $1=F(+\infty,+\infty)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)dydx=\int_0^1\int_0^1 k(x+y)dydx=k$

(2) $f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\begin{cases}\int_0^1(x+y)dy, & 0\le x\le 1\\ 0,\text{其它}\end{cases}=\begin{cases}x+1/2, & 0\le x\le 1\\ 0, & \text{其它}\end{cases},$

$$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\begin{cases}y+1/2, & 0\le y\le 1\\ 0, & \text{其它}\end{cases}.$$

$$E(Y)=E(X)=\int_{-\infty}^{+\infty}xf_X(x)dx=\int_0^1 x(x+\frac{1}{2})dx=\frac{7}{12}.$$

$$E(Y^2)=E(X^2)=\int_{-\infty}^{+\infty}x^2 f_X(x)dx=\int_0^1 x^2(x+\frac{1}{2})dx=\frac{5}{12}.$$

$$D(Y)=D(X)=E(X^2)-[E(X)]^2=\frac{5}{12}-\frac{49}{144}=\frac{11}{144}.$$

$$E(XY)=\int_0^1\int_0^1 xy(x+y)dydx=\int_0^1\left[x^2\frac{y^2}{2}+x\frac{y^3}{3}\right]_0^1 dx=\int_0^1(\frac{x^2}{2}+\frac{x}{3})dx=\frac{1}{3}.$$

$$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{1}{3}-\frac{49}{144}=-\frac{1}{144}.$$

$$\rho_{XY}=\frac{\mathrm{Cov}(X,Y)}{\sqrt{D(X)D(Y)}}=\frac{-1/144}{11/144}=-\frac{1}{11}.$$

$$D(X+Y)=D(X)+D(Y)+2\mathrm{Cov}(X,Y)=\frac{11}{144}+\frac{11}{144}+2\times(-\frac{1}{144})=\frac{5}{36}.$$

20. $X+Y=n,\quad X\sim B(n,\frac{1}{2}),\quad Y\sim B(n,\frac{1}{2}).$

$$Cov(X,Y)=E(X(n-X))-EX\cdot E(n-X)$$
$$=nEX-E(X^2)-nEX+(EX)^2$$
$$=-DX=-npq=-\frac{n}{4}$$
$$\rho_{XY}=\frac{-DX}{\sqrt{DX}\cdot\sqrt{DY}}=\frac{-DX}{\sqrt{DX}\cdot\sqrt{DX}}=-1$$

21. (1) $Y_i=X_i-\bar{X}=\frac{n-1}{n}X_i-\frac{n-1}{n}\sum_{\substack{k=1\\k\neq i}}^{n}X_k$

$$D(Y_i)=(\frac{n-1}{n})^2D(X_i)+\frac{1}{n^2}\sum_{\substack{k=1\\k\neq i}}^{n}D(X_k)=\frac{(n-1)^2}{n^2}\cdot1+\frac{1}{n^2}\sum_{\substack{k=1\\k\neq i}}^{n}1$$
$$=\frac{(n-1)^2}{n^2}+\frac{n-1}{n^2}=\frac{n-1}{n}$$

(2)

$$E(Y_1)=E(X_1)-E(\bar{X})=0-0=0=E(Y_n)$$
$$E(Y_1Y_n)=E((X_1-\bar{X})(X_n-\bar{X}))$$
$$=E(X_1X_n-X_1\bar{X}-X_n\bar{X}+\bar{X}^2)$$
$$=E(X_1X_n)-2E(X_1\bar{X})+E(\bar{X}^2)$$
$$=E(X_1)E(X_n)-\frac{2}{n}E(X_1^2+\sum_{k=2}^{n}X_1X_k)+D\bar{X}+(E\bar{X})^2$$
$$=0\cdot0-\frac{2}{n}[(DX_1+(EX_1)^2+\sum_{k=2}^{n}0]+\frac{1^2}{n}+0^2$$
$$=0\cdot0-\frac{2}{n}(1+0^2+0)+\frac{1}{n}=-\frac{1}{n}$$
$$\mathrm{Cov}(Y_1,Y_n)=E(Y_1Y_n)-E(Y_1)E(Y_n)=-\frac{1}{n}$$

22. �任意��$\lambda$有 $E[(\lambda X+Y)^2]\geq0$, 即

$$E(\lambda^2X^2+2\lambda XY+Y^2)\geq0$$

或 $E(X^2)\cdot\lambda^2+2E(XY)\cdot\lambda+E(Y^2)\geq0$

$\because$ $E(X^2)>0$

$\therefore$ $\Delta=(2E(XY))^2-4E(X^2)E(Y^2)\leq0$

即 $(E(XY))^2\leq E(X^2)E(Y^2)$.