第六节　微分法在几何中的应用

复习：平面曲线的切线与法线

1. 空间直线方程

一般式 $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$

对称式 $\dfrac{x - x_0}{m} = \dfrac{y - y_0}{n} = \dfrac{z - z_0}{p}$

参数式 $\begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$ $(m^2 + n^2 + p^2 \neq 0)$

2.平面基本方程:

一般式 $Ax + By + Cz + D = 0 \quad (A^2 + B^2 + C^2 \neq 0)$

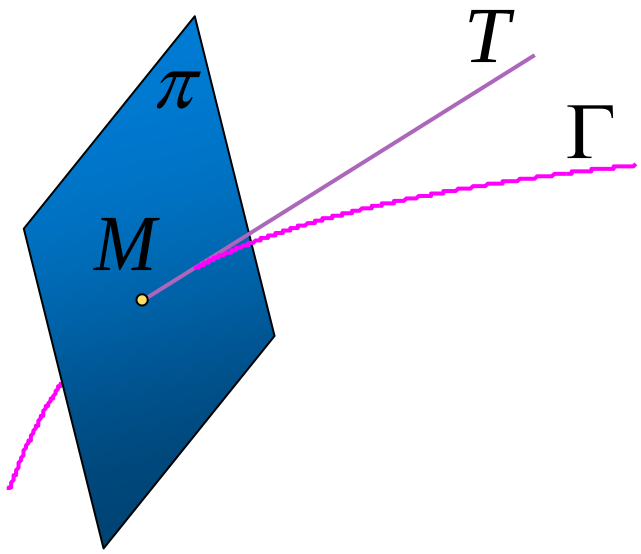
点法式 $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

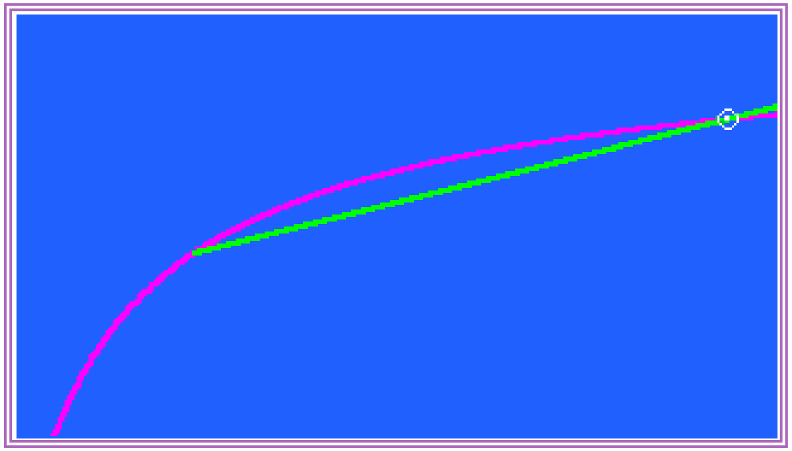
截距式 $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1 \quad (abc \neq 0)$

三点式
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

6.1 空间曲线的切线与法平面

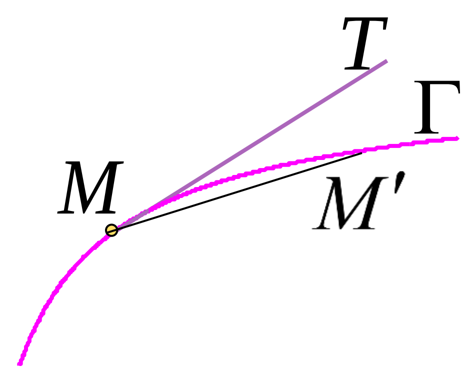
空间光滑曲线在点 M 处的切线为此点处割线的极限位置. 过点 M 与切线垂直的平面称为曲线在该点的法平面.





点击图中任意点动画开始或暂停

1. 曲线方程为参数方程的情况



$$\Gamma: \quad x=\varphi(t),\ y=\psi(t),\ z=\omega(t)$$

设 $t=t_0$ 对应 $M(x_0, y_0, z_0)$

$t=t_0+\Delta t$ 对应 $M'(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)$

割线 MM' 的方程：

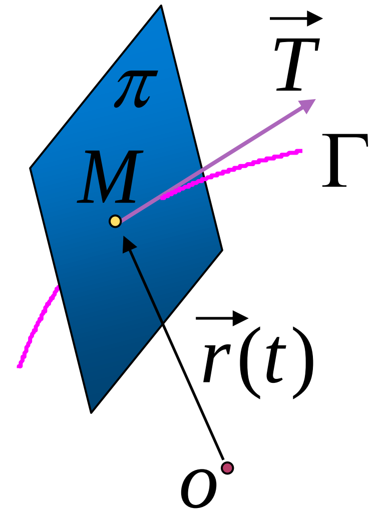
$$\frac{x-x_0}{\Delta x}=\frac{y-y_0}{\Delta y}=\frac{z-z_0}{\Delta z}$$

上述方程之分母同除以 Δt，令 $\Delta t \to 0$，得

切线方程 $$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{\omega'(t_0)}$$

此处要求$\varphi'(t_0), \psi'(t_0), \omega'(t_0)$不全为0，

如个别为0，则理解为分子为 0 .



切线的方向向量：

$$\vec{T} = (\varphi'(t_0), \psi'(t_0), \omega'(t_0))$$

称为曲线上点M处的切向量.

$\vec{T}$ 也是法平面的法向量，因此得法平面方程

$$\varphi'(t_0)(x - x_0) + \psi'(t_0)(y - y_0) + \omega'(t_0)(z - z_0) = 0$$

说明：若引进向量值函数 $\vec{r}(t) = (\varphi(t), \psi(t), \omega(t))$，则 Γ 为 $\vec{r}(t)$ 的矢端曲线，而在 t_0 处的导向量

$$\vec{r}'(t_0) = (\varphi'(t_0), \psi'(t_0), \omega'(t_0))$$

就是该点的切向量.

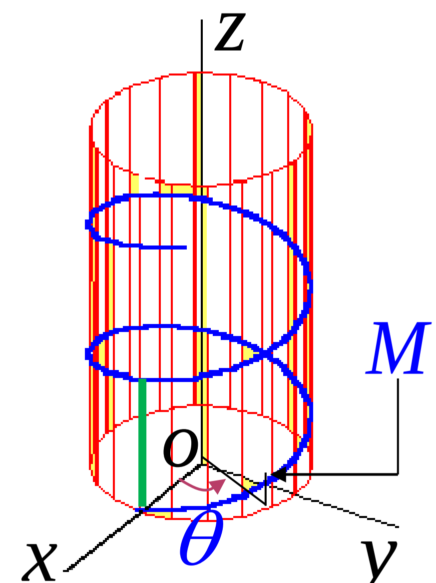
例1. 求圆柱螺旋线 $x = R\cos\varphi,\ y = R\sin\varphi,\ z = k\varphi$ 在 $\varphi = \frac{\pi}{2}$ 对应点处的切线方程和法平面方程.

解: 由于 $x' = -R\sin\varphi,\ y' = R\cos\varphi,\ z' = k$, 当 $\varphi = \frac{\pi}{2}$ 时,

对应的切向量为 $\overrightarrow{T} = (-R, 0, k)$, 对应点 $M_0(0, R, \frac{\pi}{2}k)$

切线方程 $$\frac{x}{-R} = \frac{y-R}{0} = \frac{z-\frac{\pi}{2}k}{k}$$

即 $$\begin{cases} kx + Rz - \frac{\pi}{2}Rk = 0 \\ y - R = 0 \end{cases}$$



法平面方程 $$-Rx + k(z - \frac{\pi}{2}k) = 0$$

即 $$Rx - kz + \frac{\pi}{2}k^2 = 0$$

2. 曲线为一般式的情况

光滑曲线 $\Gamma:\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

$$\longrightarrow \begin{cases} x = x \\ y=\phi(x) \\ z=\psi(x) \end{cases} \Rightarrow \begin{cases} x'=1 \\ \dfrac{dy}{dx}=\phi'(x) \\ \dfrac{dz}{dx}=\psi'(x) \end{cases}$$

曲线上一点 $M(x_0,y_0,z_0)$ 处的切向量为

$$\vec{T}=\{1,\varphi'(x_0),\psi'(x_0)\}=\left\{1,\frac{dy}{dx},\frac{dz}{dx}\right\}$$

2. 曲线为一般式的情况

光滑曲线 $\Gamma:\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ $\qquad \vec{T}=\left\{1,\dfrac{dy}{dx},\dfrac{dz}{dx}\right\}$

$$\begin{cases} F(x,y(x),z(x))=0 \\ G(x,y(x),z(x))=0 \end{cases} \Longrightarrow \boxed{\begin{cases} F_x'+F_y'\cdot\dfrac{dy}{dx}+F_z'\cdot\dfrac{dz}{dx}=0 \\ G_x'+G_y'\cdot\dfrac{dy}{dx}+G_z'\cdot\dfrac{dz}{dx}=0 \end{cases}}$$

$$\frac{dy}{dx}=-\frac{1}{J}\begin{vmatrix} F_x' & F_z' \\ G_x' & G_z' \end{vmatrix};\quad \frac{dz}{dx}=-\frac{1}{J}\begin{vmatrix} F_y' & F_x' \\ G_y' & G_x' \end{vmatrix}$$

$$\vec{T}=\left\{1,\frac{dy}{dx},\frac{dz}{dx}\right\}=\left\{J,-\begin{vmatrix} F_x' & F_z' \\ G_x' & G_z' \end{vmatrix},-\begin{vmatrix} F_y' & F_x' \\ G_y' & G_x' \end{vmatrix}\right\}$$

y—z
z—x
x—y

$$=\left\{\begin{vmatrix} F_y' & F_z' \\ G_y' & G_z' \end{vmatrix},\begin{vmatrix} F_z' & F_x' \\ G_z' & G_x' \end{vmatrix},\begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix}\right\}$$

2. 曲线为一般式的情况

光滑曲线 $\Gamma:\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

$$\vec{T}=\left\{1,\frac{dy}{dx},\frac{dz}{dx}\right\}$$

$$\boxed{\vec{T}=\left\{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix},\begin{vmatrix} F'_z & F'_x \\ G'_z & G'_x \end{vmatrix},\begin{vmatrix} F'_x & F'_y \\ G'_x & G'_y \end{vmatrix}\right\}}$$

$$\boxed{\begin{cases} F'_x+F'_y\cdot\dfrac{dy}{dx}+F'_z\cdot\dfrac{dz}{dx}=0 \\ G'_x+G'_y\cdot\dfrac{dy}{dx}+G'_z\cdot\dfrac{dz}{dx}=0 \end{cases}}$$

$$\begin{cases} \left(1,\dfrac{dy}{dx},\dfrac{dz}{dx}\right)\cdot\left(F'_x,F'_y,F'_z\right)=0 \\ \left(1,\dfrac{dy}{dx},\dfrac{dz}{dx}\right)\cdot\left(G'_x,G'_y,G'_z\right)=0 \end{cases}$$

$$\left\{1,\frac{dy}{dx},\frac{dz}{dx}\right\}=\begin{vmatrix} i & j & k \\ F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{vmatrix}=\left\{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix},\begin{vmatrix} F'_z & F'_x \\ G'_z & G'_x \end{vmatrix},\begin{vmatrix} F'_x & F'_y \\ G'_x & G'_y \end{vmatrix}\right\}$$

或 $\vec{T} = \left\{ \left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_M, \left.\dfrac{\partial(F,G)}{\partial(z,x)}\right|_M, \left.\dfrac{\partial(F,G)}{\partial(x,y)}\right|_M \right\}$

则在点 $M(x_0, y_0, z_0)$ 有

切线方程
$$\frac{x - x_0}{\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_M} = \frac{y - y_0}{\left.\dfrac{\partial(F,G)}{\partial(z,x)}\right|_M} = \frac{z - z_0}{\left.\dfrac{\partial(F,G)}{\partial(x,y)}\right|_M}$$

法平面方程
$$\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_M (x - x_0) + \left.\frac{\partial(F,G)}{\partial(z,x)}\right|_M (y - y_0) + \left.\frac{\partial(F,G)}{\partial(x,y)}\right|_M (z - z_0) = 0$$

法平面方程:

$$\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_M (x-x_0)+\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_M (y-y_0)$$
$$+\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_M (z-z_0)=0$$

也可表为

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ F_x(M) & F_y(M) & F_z(M) \\ G_x(M) & G_y(M) & G_z(M) \end{vmatrix}=0$$

例2. 求曲线 $x^2+y^2+z^2=6,\ x+y+z=0$ 在点 $M(1,-2,1)$ 处的切线方程与法平面方程.

解法1 令 $F=x^2+y^2+z^2,\ G=x+y+z$, 则

$$F_x(M)=2,\quad F_y(M)=-4,\quad F_z(M)=2;$$

$$G_x(M)=1,\quad G_y(M)=1,\quad G_z(M)=1$$

切向量 $\vec{T}=(-6,\ 0,\ 6)$

切线方程 $\dfrac{x-1}{-6}=\dfrac{y+2}{0}=\dfrac{z-1}{6}$ 即 $\begin{cases} x+z-2=0 \\ y+2=0 \end{cases}$

法平面方程 $-6\cdot(x-1)+0\cdot(y+2)+6\cdot(z-1)=0$

即 $x-z=0$

例2. 求曲线 $x^2+y^2+z^2=6,\ x+y+z=0$ 在点 $M(1,-2,1)$ 处的切线方程与法平面方程.

解法2. 方程组两边对 x 求导, 得 $\begin{cases} y\dfrac{\mathrm{d}y}{\mathrm{d}x}+z\dfrac{\mathrm{d}z}{\mathrm{d}x}=-x \\ \dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{\mathrm{d}z}{\mathrm{d}x}=-1 \end{cases}$

解得 $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\begin{vmatrix}-x & z\\ -1 & 1\end{vmatrix}}{\begin{vmatrix}y & z\\ 1 & 1\end{vmatrix}}=\frac{z-x}{y-z},\qquad \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\begin{vmatrix}y & -x\\ 1 & -1\end{vmatrix}}{\begin{vmatrix}y & z\\ 1 & 1\end{vmatrix}}=\frac{x-y}{y-z}$$

曲线在点 $M(1,-2,1)$ 处有:

切向量 $$\vec{T}=\left(1,\ \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_M,\ \frac{\mathrm{d}z}{\mathrm{d}x}\bigg|_M\right)=(1,0,-1)$$

点 $M\,(1,-2,1)$ 处的切向量

$$\vec{T}=(1,0,-1)$$

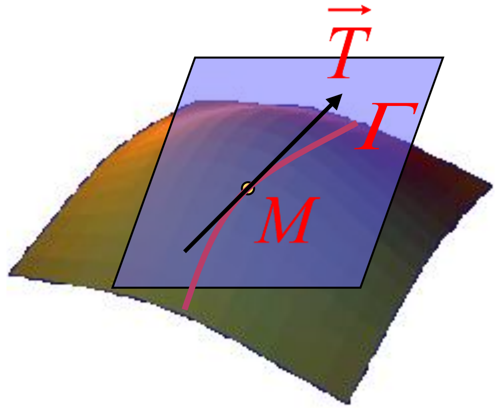
切线方程 $$\frac{x-1}{1}=\frac{y+2}{0}=\frac{z-1}{-1}$$

即 $$\begin{cases} x+z-2=0 \\ y+2=0 \end{cases}$$

法平面方程 $$1\cdot(x-1)+0\cdot(y+2)+(-1)\cdot(z-1)=0$$

即 $$x-z=0$$

6.2 曲面的切平面与法线



设有光滑曲面 $\Sigma: F(x,y,z)=0$

通过其上定点 $M(x_0,y_0,z_0)$ **($t = t_0$)**

任意引一条光滑曲线 $\Gamma: x=\varphi(t),\ y=\psi(t),\ z=\omega(t)$,

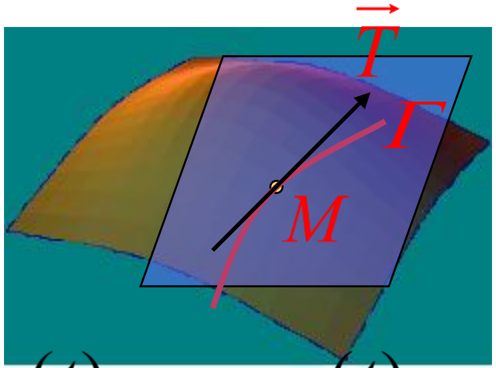
曲线上所有点在 Σ 上, 满足曲面方程

$$F(\varphi(t), \psi(t), \omega(t)) \equiv 0$$

两边对 t 求导, 代入 t_0

得

$$\boxed{\begin{aligned} F_x(x_0,y_0,z_0)\,\varphi'(t_0) + F_y(x_0,y_0,z_0)\,\psi'(t_0) \\ + F_z(x_0,y_0,z_0)\,\omega'(t_0) = 0 \end{aligned}}$$



设有光滑曲面 $\Sigma: F(x,y,z)=0$

通过其上定点 $M(x_0,y_0,z_0)$ **$(t=t_0)$**

任意引一条光滑曲线 $\Gamma: x=\varphi(t),\ y=\psi(t),\ z=\omega(t)$,

$$\boxed{\begin{aligned}F_x(x_0,y_0,z_0)\,\varphi'(t_0)+F_y(x_0,y_0,z_0)\,\psi'(t_0)&\\+F_z(x_0,y_0,z_0)\,\omega'(t_0)&=0\end{aligned}}$$

令 $\vec{T}=(\varphi'(t_0),\psi'(t_0),\omega'(t_0))$

$\vec{n}=(F_x'(x_0,y_0,z_0),F_y'(x_0,y_0,z_0),F_z'(x_0,y_0,z_0))$

切向量 $\vec{T}\perp\vec{n}$

由曲线 Γ 的任意性知这些切线都在以

$\vec{n}$ 为法向量的平面上，从而切平面存在．

曲面Σ在点M的法向量　　$\Sigma: F(x, y, z)=0$

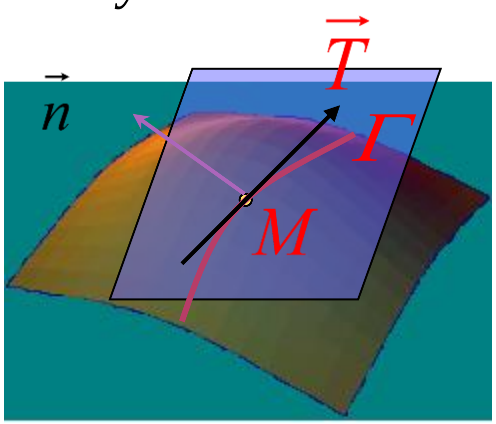
$$\vec{n}=(F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))$$

切平面方程

$$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0)=0$$

法线方程

$$\frac{x-x_0}{F_x(x_0, y_0, z_0)}=\frac{y-y_0}{F_y(x_0, y_0, z_0)}=\frac{z-z_0}{F_z(x_0, y_0, z_0)}$$



特别，当光滑曲面Σ的方程为显式$z=f(x,y)$时，令

$$F(x,y,z)=z-f(x,y)$$

则在点(x,y,z)，$F_x=-f_x$，　$F_y=-f_y$，　$F_z=1$

曲面$z=f(x,y)$上点$M_0(x_0,y_0,z_0)$处的

法向量为 $\vec{n}=(-f_x(x_0,y_0),-f_y(x_0,y_0),1)$

切平面方程

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

法线方程

$$\frac{x-x_0}{-f_x(x_0,y_0)}=\frac{y-y_0}{-f_y(x_0,y_0)}=\frac{z-z_0}{1}$$

例3. **求球面** $x^2+2y^2+3z^2=36$ **在点(1 , 2 , 3) 处的切平面及法线方程.**

解: 令 $F(x,y,z)=x^2+2y^2+3z^2-36$

法向量 $\qquad \vec{n}=(2x,4y,6z)$

$$\vec{n}\,\Big|_{(1,2,3)}=(2,8,18)$$

所以球面在点 (1 , 2 , 3) 处有:

切平面方程 $\qquad 2(x-1)+8(y-2)+18(z-3)=0$

即 $\qquad x+4y+9z-36=0$

法线方程 $\qquad \dfrac{x-1}{1}=\dfrac{y-2}{4}=\dfrac{z-3}{9}$

例4 求旋转抛物面 $z = x^2 + y^2 - 1$在点(2, 1, 4)处的切平面及法线方程.

解 $f(x, y) = x^2 + y^2 - 1$, $\vec{n}\big|_{(2,1,4)} = \{2x, 2y, -1\}\big|_{(2,1,4)} = \{4, 2, -1\}$,

切平面方程为 $4(x-2) + 2(y-1) - (z-4) = 0$, 即 $4x + 2y - z - 6 = 0$

法线方程为 $\dfrac{x-2}{4} = \dfrac{y-1}{2} = \dfrac{z-4}{-1}$.

例5 求曲面 $z - e^z + 2xy = 3$在点(1, 2, 0)处的切平面及法线方程.

解 令 $F(x, y, z) = z - e^z + 2xy - 3$,

$F'_x\big|_{(1,2,0)} = 2y\big|_{(1,2,0)} = 4$, $F'_y\big|_{(1,2,0)} = 2x\big|_{(1,2,0)} = 2$, $F'_z\big|_{(1,2,0)} = 1 - e^z\big|_{(1,2,0)} = 0$,

切平面方程 $4(x-1) + 2(y-2) + 0\cdot(z-0) = 0$, 即 $2x + y - 4 = 0$

法线方程 $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-0}{0}$.

例6 求曲面 $x^2+2y^2+3z^2=21$ 平行于平面 $x+4y+6z=0$ 的各切平面方程.

解 设 (x_0,y_0,z_0) 为曲面上的切点，

切平面方程为 $2x_0(x-x_0)+4y_0(y-y_0)+6z_0(z-z_0)=0$

依题意，切平面方程平行于已知平面，得 $\dfrac{2x_0}{1}=\dfrac{4y_0}{4}=\dfrac{6z_0}{6}$，$\Rightarrow$ $2x_0=y_0=z_0$.

因为 (x_0,y_0,z_0) 为曲面上的切点，满足方程 $\therefore\ x_0=\pm1$,

所求切点为 （1,2,2），（-1,-2,-2）

切平面方程(1) $2(x-1)+8(y-2)+12(z-2)=0\Rightarrow\ x+4y+6z=21$；

切平面方程(2) $-2(x+1)-8(y+2)-12(z+2)=0\Rightarrow\ x+4y+6z=-21$.

内容小结

1. 空间曲线的切线与法平面

1) 参数式情况. 空间光滑曲线 $\Gamma:\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=\omega(t) \end{cases}$

切向量 $\vec{T}=(\varphi'(t_0),\psi'(t_0),\omega'(t_0))$

切线方程 $$\frac{x-x_0}{\varphi'(t_0)}=\frac{y-y_0}{\psi'(t_0)}=\frac{z-z_0}{\omega'(t_0)}$$

法平面方程

$$\varphi'(t_0)(x-x_0)+\psi'(t_0)(y-y_0)+\omega'(t_0)(z-z_0)=0$$

2) 一般式情况．空间光滑曲线 $\Gamma:\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$

切向量 $\vec{T}=\left(\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_M,\ \left.\dfrac{\partial(F,G)}{\partial(z,x)}\right|_M,\ \left.\dfrac{\partial(F,G)}{\partial(x,y)}\right|_M\right)$

切线方程
$$\frac{x-x_0}{\left.\dfrac{\partial(F,G)}{\partial(y,z)}\right|_M}=\frac{y-y_0}{\left.\dfrac{\partial(F,G)}{\partial(z,x)}\right|_M}=\frac{z-z_0}{\left.\dfrac{\partial(F,G)}{\partial(x,y)}\right|_M}$$

法平面方程
$$\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_M(x-x_0)+\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_M(y-y_0)+\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_M(z-z_0)=0$$

2. 曲面的切平面与法线

1) 隐式情况 . 空间光滑曲面 $\Sigma: F(x, y, z) = 0$

曲面 Σ **在点** $M(x_0, y_0, z_0)$ **的法向量**

$$\vec{n} = (F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))$$

切平面方程

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

法线方程

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

2) 显式情况. 空间光滑曲面 $\Sigma: z = f(x, y)$

法向量 $\qquad \vec{n} = (-f_x, -f_y, 1)$

法线的方向余弦

$$\cos\alpha = \frac{-f_x}{\sqrt{1+f_x^2+f_y^2}}, \quad \cos\beta = \frac{-f_y}{\sqrt{1+f_x^2+f_y^2}},$$

$$\cos\gamma = \frac{1}{\sqrt{1+f_x^2+f_y^2}}$$

切平面方程

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

法线方程 $\qquad \dfrac{x - x_0}{f_x(x_0, y_0)} = \dfrac{y - y_0}{f_y(x_0, y_0)} = \dfrac{z - z_0}{-1}$