第四节 多元复合函数的求导法则

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

微分法则 $\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$

4.1 多元复合函数求导的链式法则

4.2 多元复合函数的全微分

1) 中间变量为一元函数 $z=f(u,v),\quad \begin{cases} u=\phi(t) \\ v=\psi(t) \end{cases}$

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$

$$=f_1'\phi'+f_2'\psi'$$

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ u \arrow[dr] & & v \arrow[dl] \\ & t & \end{tikzcd}$$

例 $z=e^u\sin v,\quad \begin{cases} u=t^2 \\ v=2t \end{cases}$，求 $\dfrac{dz}{dt}$.

法一： $z=e^{t^2}\sin(2t)$

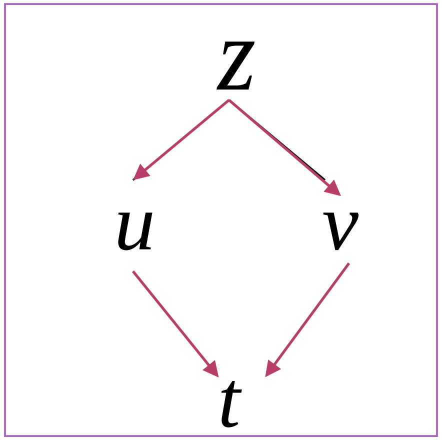
$$\Rightarrow\frac{dz}{dt}=\left(e^{t^2}\cdot 2t\right)\cdot\sin(2t)+e^{t^2}\cdot\left(\cos(2t)\cdot 2\right)$$

$$=2te^{t^2}\sin(2t)+2e^{t^2}\cos(2t)$$

1) 中间变量为一元函数 $z=f(u,v)$, $\begin{cases} u=\phi(t) \\ v=\psi(t) \end{cases}$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$=f_1'\phi'+f_2'\psi'$$



例 $z=e^u\sin v$, $\begin{cases} u=t^2 \\ v=2t \end{cases}$, 求 $\dfrac{dz}{dt}$.

法二： $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial f}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial f}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

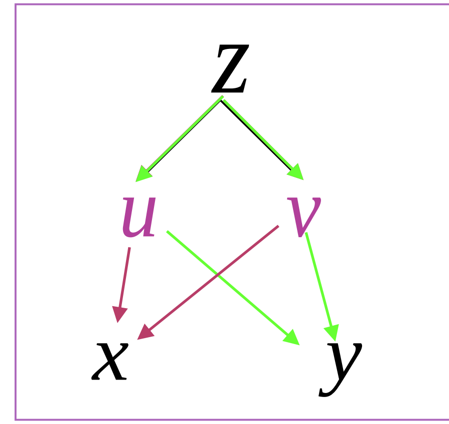
$$=e^u\sin v\cdot 2t+e^u\cos v\cdot 2$$

$$=2te^{t^2}\sin(2t)+2e^{t^2}\cos(2t)$$

2) 中间变量是多元函数 $z = f(u, v)$, $\begin{cases} u = \phi(x, y) \\ v = \psi(x, y) \end{cases}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= f_1' \varphi_1' + f_2' \psi_1'$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = f_1' \varphi_2' + f_2' \psi_2'$$

例 $z = e^{u+2v}$, $\begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases}$, 求 $\dfrac{\partial z}{\partial x}$

法一：

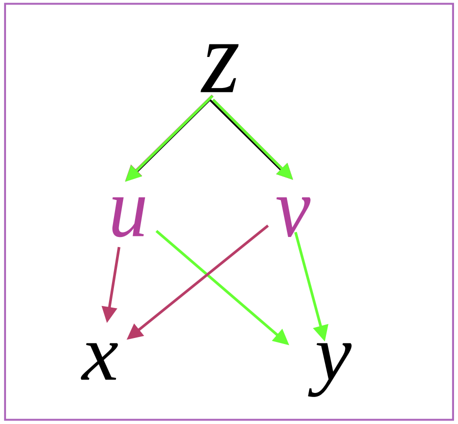
$$z = e^{xy+2\frac{y}{x}} \Rightarrow \frac{\partial z}{\partial x} = e^{xy+2\frac{y}{x}} \cdot \left(y - 2\frac{y}{x^2} \right)$$

2) 中间变量是多元函数 $z=f(u,v)$, $\begin{cases} u=\phi(x,y) \\ v=\psi(x,y) \end{cases}$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=f_1'\varphi_1'+f_2'\psi_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=f_1'\varphi_2'+f_2'\psi_2'$$



例 $z=e^{u+2v}$, $\begin{cases} u=xy \\ v=\dfrac{y}{x} \end{cases}$, 求 $\dfrac{\partial z}{\partial x}$

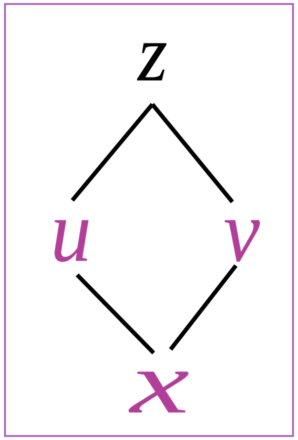
法二: $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=(e^{u+2v}\cdot 1)\cdot y+(e^{u+2v}\cdot 2)\cdot\left(-\frac{y}{x^2}\right)=e^{xy+2\frac{y}{x}}\cdot\left(y-2\frac{y}{x^2}\right)$$

3.1　多元复合函数求导的链式法则

定理1. 若函数 $u=\phi(x), v=\psi(x)$ 在点 x 可导, $z=f(u,v)$ 在点 (u,v) 处偏导连续, 则复合函数 $z=f(\phi(x),\psi(x))$ 在点 x 可导, 且有链式法则

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial f}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial f}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}$$



证: 设 x 取增量 Δx, 则相应中间变量有增量 Δu, Δv,

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\quad\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta x}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta x}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta x}+\frac{o(\rho)}{\Delta x}\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}=\frac{\partial z}{\partial u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\frac{\partial z}{\partial v}\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}+\lim_{\Delta x\to 0}\cdot\frac{o(\rho)}{\Delta x}$$

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}x}}$$

（全导数公式）

口诀：分段用乘，分叉用加，单路全导，叉路偏导

注:1 $z = f(u,v),\ \begin{cases} u = \\ v = \end{cases}$

$$\frac{\partial f}{\partial u} \overset{\Delta}{=} f_1' = f_u' = f_1'(u,v)$$

$$\frac{\partial f}{\partial v} \overset{\Delta}{=} f_2' = f_v' = f_2'(u,v)$$

$$\frac{\partial^2 f}{\partial u^2} \overset{\Delta}{=} f_{11}'' \qquad \frac{\partial^2 f}{\partial u \partial v} = f_{12}''$$

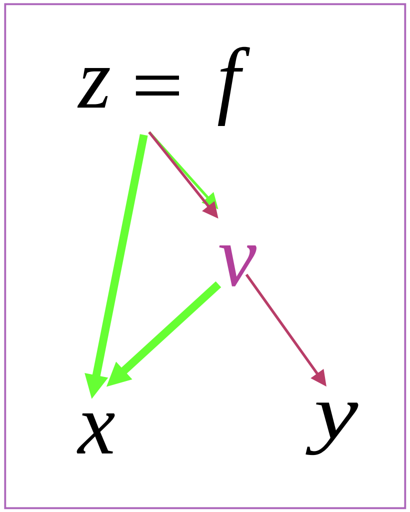
$$\frac{\partial^2 f}{\partial v^2} \overset{\Delta}{=} f_{22}'' \qquad \frac{\partial^2 f}{\partial v \partial u} = f_{21}''$$

注:2 $\dfrac{\partial z}{\partial x}$ 与 $\dfrac{\partial f}{\partial x}$

$z=f(x,v),\ v=\psi(x,y)$ 当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}}=\boxed{\frac{\partial f}{\partial x}}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'+f_2'\psi_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=f_2'\psi_2'$$



注意: 这里 $\dfrac{\partial z}{\partial x}$ 与 $\dfrac{\partial f}{\partial x}$ 不同,

$\dfrac{\partial z}{\partial x}$ 表示固定 y 对 x 求导, $\dfrac{\partial f}{\partial x}$ 表示固定 v 对 x 求导

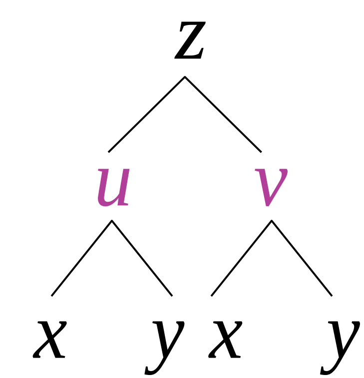
定理2 如果$u=u(x,y)$及$v=v(x,y)$在点(x,y)对x和y的偏导数都存在，且函数$z=f(u,v)$在对应点(u,v)可微，则复合函数 $z=f(u(x,y),v(x,y))$在点(x,y)的两个偏导数都存在，且

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

称为复合函数求导的**链式法则**.

例1. 设 $z = e^u \sin v$，$u = xy$，$v = x + y$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

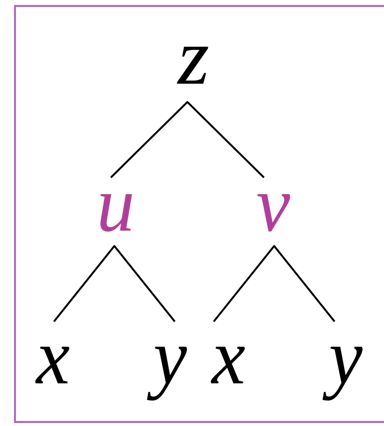


例1. 设 $z = e^u \sin v$, $u = xy$, $v = x + y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解: $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$

$$= e^{xy}[y \cdot \sin(x+y) + \cos(x+y)]$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^{xy}[x \cdot \sin(x+y) + \cos(x+y)]$$

例2. $u = f(x, y, z) = e^{x^2+y^2+z^2}$, $z = x^2 \sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

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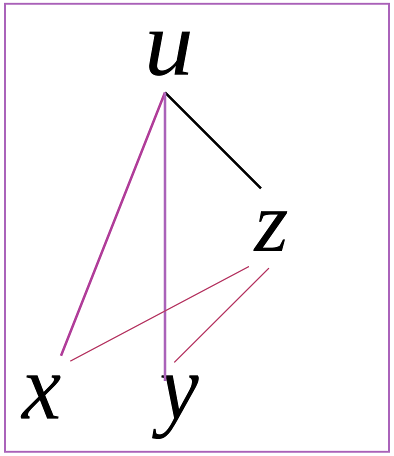


解: $$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= 2xe^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$= 2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$= 2ye^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2}\cdot x^2\cos y$$

$$= 2(y + x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$$

例3. $z=f\left(t^2,\sin 2t\right)$，求 $\dfrac{d^2z}{dt^2}$

例3. $z = f\left(t^2, \sin 2t\right)$，求 $\dfrac{d^2 z}{dt^2}$

解： $$\frac{dz}{dt} = f_1' \cdot 2t + f_2' \cdot \cos(2t) \cdot 2$$

$$\frac{d^2 z}{dt^2} = \left(f_{11}'' \cdot 2t + f_{12}'' \cdot \cos(2t) \cdot 2\right) \cdot 2t + f_1 \cdot 2$$

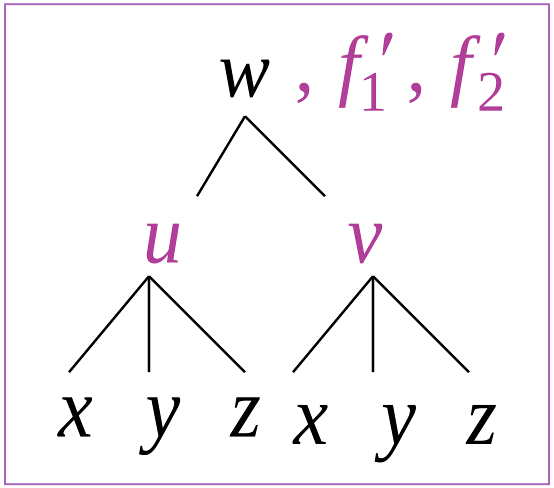
$$+ \left(f_{21}'' \cdot 2t + f_{22}'' \cdot \cos(2t) \cdot 2\right) \cdot 2\cos(2t)$$

$$+ f_2' \cdot \left[-\sin(2t)\right] \cdot 4$$

例4. 设 $w = f(x+y+z, xyz)$, f 具有二阶连续偏导数,

求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.

例4. 设 $w=f(x+y+z,xyz)$, f 具有二阶连续偏导数, 求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.



解:

$$\frac{\partial w}{\partial x}=f_1'\cdot 1+f_2'\cdot yz$$

$$=f_1'(x+y+z,xyz)$$

$$+yz\,f_2'(x+y+z,xyz)$$

$$\frac{\partial^2 w}{\partial x \partial z}=f_{11}''\cdot 1+f_{12}''\cdot xy+y\,f_2'+yz[\,f_{21}''\cdot 1+f_{22}''\cdot xy]$$

$$=f_{11}''+y(x+z)\,f_{12}''+xy^2z\,f_{22}''+y\,f_2'$$

4.2　多元复合函数的全微分

设函数 $z=f(u,v),\ u=\varphi(x,y),\ v=\psi(x,y)$ 都可微，

则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$
\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y \\
&=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y \\
&=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\right) \\
&=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v
\end{aligned}
$$

可见无论 u, v 是自变量还是中间变量，其全微分表达形式都一样，这性质叫做全微分形式不变性.

例 6. $z = e^u \sin v$，$u = xy$ ，$v = x + y$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

例 6. $z = e^u \sin v$, $u = xy$, $v = x + y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解: $\mathrm{d}z = \mathrm{d}(e^u \sin v) = \mathrm{d}(e^u) \cdot \sin v + \mathrm{d}(\sin v) \cdot e^u$

$= e^u \sin v \,\mathrm{d}u + e^u \cos v \,\mathrm{d}v$

$= e^{xy}[\sin(x+y)\,\mathrm{d}(xy) + \cos(x+y)\,\mathrm{d}(x+y)]$

$= e^{xy}[\sin(x+y)(y\mathrm{d}x + x\mathrm{d}y) + \cos(x+y)(\mathrm{d}x + \mathrm{d}y)]$

$= e^{xy}[y\sin(x+y) + \cos(x+y)]\mathrm{d}x$

$+ e^{xy}[x\sin(x+y) + \cos(x+y)]\mathrm{d}y$

所以 $\dfrac{\partial z}{\partial x} = e^{xy}[y \cdot \sin(x+y) + \cos(x+y)]$

$\dfrac{\partial z}{\partial y} = e^{xy}[x \cdot \sin(x+y) + \cos(x+y)]$

内容小结

1. 复合函数求导的链式法则

例如， $u = f(x, y, v), v = \varphi(x, y),$

$$\frac{\partial u}{\partial x} = f_1' + f_3' \cdot \varphi_1' ; \quad \frac{\partial u}{\partial y} = f_2' + f_3' \cdot \varphi_2'$$

练习1. 设 $z = f(x+y, xy, x^2)$ 有二阶连续偏导数，求 $\dfrac{\partial^2 z}{\partial x \partial y}$

内容小结

2. 全微分形式不变性

对 $z=f(u,v)$, **不论 $\boldsymbol{u}$, $\boldsymbol{v}$ 是自变量还是因变量,**

$$\mathrm{d}z = f_u(u,v)\,\mathrm{d}u + f_v(u,v)\,\mathrm{d}v$$

附加题

1. 已知 $f(x,y)\big|_{y=x^2}=1$，$f_1'(x,y)\big|_{y=x^2}=2x$，求 $f_2'(x,y)\big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导，得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$