习题２

1. 用 X 表示掷两次出现的点数之和，则分布律：

X	2	3	4	5	6	7	8	9	10	11	12
	(1, 1)	(1, 2) (2, 1)	(1, 3) (2, 2) (3, 1)	(1, 4) (2, 3) (3, 2) (4, 1)	(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)	(1, 6) (2, 5) (3, 4) (4, 3) (5, 2) (6, 1)	(2, 6) (3, 5) (4, 4) (5, 3) (6, 2)	(3, 6) (4, 5) (5, 4) (6, 3)	(4, 6) (5, 5) (6, 4)	(5, 6) (6, 5)	(6, 6)
p	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

分布函数为：$F(x)=\begin{cases}0, x<2\\ 1/36, 2\le x<3\\ 3/36, 3\le x<4\\ 6/36, 4\le x<5\\ \vdots\\ 1, x\ge 12\end{cases}$

2.

记$X=$“取出的次品数”$, i=0,1,2.$则

(1) 法 1：超几何分布，不放回取 3 次可按组合 作

$$P\{X=k\}=\frac{C_2^k C_{15-2}^{3-k}}{C_{15}^3}(k=0,1,2)$$

法 2：(按排列作)

$$P\{X=0\}=\frac{P_{13}^3}{P_{15}^3}=\frac{13\times12\times11}{15\times14\times13}=\frac{22}{35};$$

$$P\{X=1\}=\frac{C_3^1\times P_2^1P_{13}^1P_{12}^1}{P_{15}^3}=\frac{3\times2\times13\times12}{15\times14\times13}=\frac{12}{35};$$

$$P\{X=2\}=\frac{C_3^2\times P_2^1P_1^1P_{13}^1}{P_{15}^3}=\frac{3\times2\times13}{15\times14\times13}=\frac{1}{35}.$$

X	0	1	2
P	$\frac{22}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

（3）分布函数：

$x<0$: $F(x)=P\{X\le x\}=0$

$0\le x<1$: $F(x)=P\{X<0\}+P\{X=0\}$

$+P\{0<X\le x\}=0+22/35+0$

$1\le x<2$: $F(x)=P\{X=0\}+P\{X=1\}=34/35$

$2\le x$: $F(x)=P\{X=0\}+P\{X=1\}+P\{X=2\}=1$

$$\therefore\quad F(x)=\begin{cases}0 & x<0\\ 22/35 & 0\le x<1\\ 34/35 & 1\le x<2\\ 1 & 2\le x\end{cases}$$

(2) $P\{X\le \frac{1}{2}\}=F(\frac{1}{2})=\frac{22}{35}$;

$P\{1<X\le \frac{3}{2}\}=F(\frac{3}{2})-F(1)=\frac{34}{35}-\frac{34}{35}=0$;

$P\{1\le X\le \frac{3}{2}\}=P\{X=1\}+P\{1<X\le \frac{3}{2}\}=\frac{12}{35}+0=\frac{12}{35}$.

3. 二项分布：设 X 表示投中的次数，则

$P\{X=k\}=C_3^k 0.8^k\cdot 0.2^{3-k},\quad k=0,1,2,3.$

$P\{X=0\}=C_3^0 0.8^0\times(1-0.8)^3=0.008$

$P\{X=1\}=C_3^1 0.8^1\times(1-0.8)^2=0.096$

$P\{X=2\}=C_3^2 0.8^2\times(1-0.8)=0.384$

$P\{X=3\}=C_3^3 0.8^3\times(1-0.8)^0=0.512$

X	0	1	2	3
P	0.008	0.096	0.384	0.512

$P\{X\ge 2\}=P\{X=2\}+P\{X=3\}=0.896$

4.

$\because P\{X=k\}=C_2^k p^k(1-p)^{2-k}\ (k=0,1,2)$

$P\{X\ge 1\}=1-P\{X=0\}$

$\therefore 1-C_2^0 p^0(1-p)^2=\frac{5}{9}\Rightarrow p=\frac{1}{3}$

$\therefore P\{Y\ge 1\}=1-P\{Y=0\}=1-C_3^0 p^0(1-p)^3=1-(\frac{2}{3})^3=\frac{19}{27}$

5. 几何分布：

$P\{X=k\}=(1-p)^{k-1}p,\quad k=1,2,\cdots.$

6.

(1) $P\{X=k\}=\dfrac{\lambda^k e^{-\lambda}}{k!}$， 由$P\{X=0\}=\dfrac{1}{3}$得

$$\frac{1}{3}=\frac{e^{-\lambda}}{0!}\quad\Rightarrow\quad \lambda=\ln 3.$$

分布律: $P\{X=k\}=\dfrac{(\ln 3)^k e^{-\ln 3}}{k!}=\dfrac{\ln^k 3}{3\cdot k!},\quad k=0,1,2,\cdots.$

(2) $P\{X>1\}=1-P\{X\le 1\}=1-P\{X=0\}-P\{X=1\}$

$$=1-\frac{1}{3}-\frac{\ln 3}{3}=\frac{1}{3}(2-\ln 3)$$

7. 则 X 表“发生事故的次数”，则

$$X\sim B(1000,0.0001).$$

$\because \lambda=1000\times 0.0001=0.1,$

$\therefore P\{X\ge 2\}=1-P\{X<2\}=1-P\{X=0\}-P\{X=1\}$

$$\approx 1-\frac{0.1^0 e^{-0.1}}{0!}-\frac{0.1e^{-0.1}}{1!}\approx 0.00468$$

8. $p(k)=P\{X=k\}=\dfrac{\lambda^k e^{-\lambda}}{k!},k=1,2,\cdots$

$$\frac{P\{X=k\}}{P\{X=k-1\}}=\frac{\lambda^k e^{-\lambda}}{k!}\frac{(k-1)!}{\lambda^{k-1}e^{-\lambda}}=\frac{\lambda}{k}$$

可见：当$\dfrac{\lambda}{k}>1$时， $P\{X=k\}\uparrow$；当$\dfrac{\lambda}{k}<1$时， $P\{X=k\}\downarrow$.

取最大的情况是$k=\begin{cases}\lambda \text{ or } \lambda-1, & \lambda\in Z^+\\ [\lambda], & others\end{cases}$.

9.

X	−2	−1	0	1	3
P	1/5	1/6	1/5	1/15	11/30
2X+5	−5	3	5	7	11
X^2	4	1	0	1	9

2X+5	−5	3	5	7	11
P	1/5	1/6	1/5	1/15	11/30

X^2	4	1	0	9
P	1/5	7/30	1/5	11/30

10.

X	0	$\pi/2$	π
P	1/4	1/2	1/4
Y=2X/3	2	$\pi/3+2$	$2\pi/3+2$
cosX	1	0	−1

11.

$$(1)\ \because \begin{cases} F(-\infty)=\lim\limits_{x\to-\infty}F(x)=0 \\ F(+\infty)=\lim\limits_{x\to+\infty}F(x)=1 \end{cases},\quad \therefore \begin{cases} A+B(-\frac{\pi}{2})=0 \\ A+B\frac{\pi}{2}=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{\pi} \end{cases}.$$

$$\therefore\ F(x)=\frac{1}{2}+\frac{1}{\pi}\arctan x \quad -\infty<x<+\infty.$$

$$(2)P\{-1<X<1\}=P\{-1<X\le 1\}\ (\because X\text{为连续型随机变量},\therefore P\{X=1\}=0)$$

$$=F(1)-F(-1)=\frac{1}{\pi}\arctan 1-\frac{1}{\pi}\arctan(-1)=\frac{1}{2}.$$

$$(3)f(x)=F'(x)=\frac{1}{\pi}\frac{1}{1+x^2}.$$

12.

$(1) P\{X=a\}=F(a)-F(a-0)$
$(2) P\{X<a\}=P\{X\le a\}-P\{X=a\}=F(a-0)$
$(3) P\{X>a\}=1-P\{X\le a\}=1-F(a)$
$(4) P\{X\ge a\}=1-P\{X<a\}=1-F(a-0)$

13. $R.V.X$ 的概率密度为 $f(x)=\begin{cases}\frac{1}{2}e^x & x\le 0\\ \frac{1}{4} & 0<x\le 2\\ 0 & x>2\end{cases}$，求分布函数.

(1) 当 $x\le 0$ 时，$F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{x}\frac{1}{2}e^t dt=\frac{1}{2}e^x$.

(2) 当 $0<x\le 2$ 时，

$$F(x)=\int_{-\infty}^{0} f(t)dt+\int_{0}^{x} f(t)dt=F(0)+\int_{0}^{x}\frac{1}{4}dx$$
$$=\frac{1}{2}e^0+\frac{1}{4}x=\frac{1}{2}+\frac{x}{4}.$$

(3) 当 $x>2$ 时，

$$F(x)=\int_{-\infty}^{2} f(t)dt+\int_{2}^{x} f(t)dt=F(2)+\int_{2}^{x}0dt$$
$$=\frac{1}{2}+\frac{2}{4}+0=1.$$

$$\therefore\ F(x)=\begin{cases}\frac{1}{2}e^x & x\le 0\\ \frac{1}{2}+\frac{x}{4} & 0<x\le 2\\ 1 & x>2\end{cases}.$$

14. 连续型 $R.V.X$, 属无放回抽样问题. 5 重贝努利概型.

(1) 由 $1=F(+\infty)=\int_{-\infty}^{+\infty}f(x)dx=\int_{0}^{3}Axe^{-x^2}dx=-\frac{A}{2}e^{-x^2}\Big|_0^3=-\frac{A}{2}(e^{-9}-1)$.

得 $A=\frac{2}{1-e^{-9}}$. $\therefore f(x)=\begin{cases}\frac{2x}{1-e^{-9}}e^{-x^2} & 0<x<3\\ 0 & \text{其它}\end{cases}$.

$$(2)P\{0\le X\le 2\}=\int_0^2 f(x)dx=\int_0^2\frac{2x}{1-e^{-9}}e^{-x^2}dx=-\frac{1}{1-e^{-9}}e^{-x^2}\Big|_0^2=\frac{1-e^{-4}}{1-e^{-9}}.$$

记A_i =“第i发子弹落在距靶心2cm的圆域上”, $i=1,2,3,4,5$, 则5发均落在圆上的概率

$$P(A_1A_2A_3A_4A_5)=(P(A_1))^5=(P\{0\le X\le 2\})^5=\left(\frac{1-e^{-4}}{1-e^{-9}}\right)^5.$$

注：(1)也可先求出连续型随机变量X的分布函数确定$P\{0\le X\le 2\}=F(2)-F(0-0)$;

(2)视该批子弹数量很大为不放回抽样问题

$P\{X\le 2\}=P\{0\le X\le 2\}$

15.

(1) $1=\int_{-1}^{1}\frac{A}{\sqrt{1-x^2}}dx=A\pi\Rightarrow A=\frac{1}{\pi}$

$$f(x)=\begin{cases}\dfrac{1}{\pi\sqrt{1-x^2}} & |x|<1\\ 0 & \text{其它}\end{cases}.$$

(2) $P\{-\frac{1}{2}<x<\frac{1}{2}\}=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\pi\sqrt{1-x^2}}dx=\left[\frac{1}{\pi}\arcsin x\right]_{-\frac{1}{2}}^{\frac{1}{2}}=\frac{1}{3}$

(3) $F(x)=\begin{cases}0 & x\le -1\\ \dfrac{1}{\pi}\arcsin x+\dfrac{1}{2} & |x|<1.\\ 1 & x\ge 1\end{cases}$

$P\{-\frac{1}{2}<x<\frac{1}{2}\}=F(\frac{1}{2})-F(-\frac{1}{2})=\frac{1}{3}$

16. $f(t)=\begin{cases}\dfrac{1}{T} & 0<t<T\\ 0 & \text{others}\end{cases}$

(1) $P\{t_0<t<t_1\}=\int_{t_0}^{t_1}\frac{1}{T}dt=\frac{t_1-t_0}{T}$

$$(2)P\{\{t_0<t<t_1\}\mid\{t\ge t_0\}\}=\frac{P\{\{t_0\le t\le t_1\}\{t>t_0\}\}}{P\{t\ge t_0\}}=\frac{P\{t_0<t<t_1\}}{P\{t\ge t_0\}}$$

$$=\frac{\int_{t_0}^{t_1}f(t)dt}{\int_{t_0}^{T}f(t)dt+\int_{T}^{+\infty}f(t)dt}=\frac{(t_1-t_0)/T}{(T-t_0)/T}.$$

17.

$$P\{2 < X \le 5\} = \Phi(\frac{5-3}{2}) - \Phi(\frac{2-3}{2}) = \Phi(1) - \Phi(-\frac{1}{2}) = \Phi(1) - (1 - \Phi(\frac{1}{2}))$$

$$= \Phi(1) - 1 + \Phi(\frac{1}{2}) = 0.8413 - 1 + 0.6915 \cancel{-1} = 0.5328.$$

$$P\{-4 < X \le 10\} = \Phi(\frac{10-3}{2}) - \Phi(\frac{4-3}{2}) = \Phi(0.35) - \Phi(-0.35)$$

×2

$$= 2\Phi(0.35) - 1 = 0.9998 - 1 = 0.9996$$

$$P\{X > 3\} = 1 - F\{X \le 3\} = 1 - \Phi(\frac{3-3}{2}) = 1 - \Phi(0) = 1 - 0.5 = 0.5.$$

设$P\{X < C\} = P\{X \ge C\}$得

$$\Phi(\frac{C-3}{2}) = 1 - \Phi(\frac{C-3}{2}) \Rightarrow \Phi(\frac{C-3}{2}) = 0.5 \Rightarrow C = \mu = 3.$$

18.

$X \sim N(108, 3^2)$,

(1) $\because 0.9 = P\{X < a\} = \Phi(\frac{a-108}{3})$,

$\therefore$ 查表得 $\frac{a-108}{3} = 1.28 \Rightarrow a = 111.84.$

(2) $P\{|X-a| > a\} = 0.01$, 即$P\{|X-a| \le a\} = 0.99$

$\therefore \quad P\{0 \le X \le 2a\} = 0.99$ 或 $F(2a) - F(0) = 0.99$

$\therefore \quad \Phi(\frac{2a-108}{3}) - \Phi(\frac{0-108}{3}) = 0.99$

$\therefore \quad \Phi(\frac{2a-108}{3}) - 0 \approx 0.99 \quad \Rightarrow \quad \frac{2a-108}{3} \approx 2.33 \quad (\because \Phi(2.33) \approx 0.9901)$

$\therefore \quad a = 57.5.$

19.

$X \sim N(160, \sigma^2)$

由 $P\{120 < X \le 200\} = P(\frac{120-160}{\sigma} < X \le \frac{200-160}{\sigma}) = 2\Phi(\frac{40}{\sigma}) - 1 \ge 0.8,$

$\therefore \Phi(\frac{40}{\sigma}) \ge 0.9$, 查表得 $\frac{40}{\sigma} \ge 1.28 \Rightarrow \sigma \le \frac{40}{1.28} = 31.25.$

20.

$$X \sim U(0,1), \quad f_X(x)=\begin{cases}1 & 0<x<1\\ 0 & \text{其它}\end{cases},$$

（1）法 1　定义法：

$$F_Y(y)=P\{Y\le y\}=P\{e^X\le y\}=\begin{cases}P\{X\le \ln y\} & y>0\\ 0 & y\le 0\end{cases}$$

$$=\begin{cases}F_X(\ln y) & y>0\\ 0 & y\le 0\end{cases}$$

$$\therefore \quad f_Y(y)=F_Y'(y)=\begin{cases}\frac{1}{y}f_X(\ln y) & y>0\\ 0 & y\le 0\end{cases}=\begin{cases}\frac{1}{y}\cdot 1 & 1<y<e\\ 0 & \text{其它}\end{cases}$$

法 2　可用定理直接求．值域为 $y>0$，且

$$g'(x)=e^x>0,\quad x=h(y)=\ln y,$$

$$f_Y(y)=\begin{cases}f_X(\ln y)\left|\frac{1}{y}\right| & y>0\\ 0 & y\le 0\end{cases}=\begin{cases}1\cdot\frac{1}{y} & 0<\ln y<1\\ 0 & \text{others}\end{cases}.$$

（2）理解为 $X\le 0$ 为不可能事件，此时 $Y=\varnothing$；

$$F_Y(y)=P\{-2\ln X\le y\}=P\{X\ge e^{-\frac{y}{2}}\}$$

$$=1-P\{X<e^{-\frac{y}{2}}\}=1-F_X(e^{-\frac{y}{2}})$$

$$f_Y(y)=F_Y'(y)=-F_X'(e^{-\frac{y}{2}})\cdot e^{-\frac{y}{2}}(-\frac{1}{2})=\frac{1}{2}e^{-\frac{y}{2}}f_X'(e^{-\frac{y}{2}})$$

$$=\begin{cases}\frac{1}{2}e^{-\frac{y}{2}}\cdot 1 & 0<e^{-\frac{y}{2}}<1\\ 0 & y\le 0\end{cases}=\begin{cases}\frac{1}{2}e^{-\frac{y}{2}} & y>0\\ 0 & y\le 0\end{cases}$$

法 2　可用定理直接求．理解为 $X\le 0$ 为不可能事件，此时 $Y=\varnothing$；值域为 $y\in(-\infty,+\infty)$

$$g'(x)=\frac{-2}{x}<0(x>0),\quad x=h(y)=e^{-\frac{y}{2}}\ (-\infty<y<+\infty),$$

$$f_Y(y)=f_X(e^{-\frac{y}{2}})\left|e^{-\frac{y}{2}}(-\frac{1}{2})\right|=\begin{cases}1\cdot e^{-\frac{y}{2}}\cdot\frac{1}{2} & 0<e^{-\frac{y}{2}}<1\\ 0 & \text{others}\end{cases}$$

21.

$$X \sim N(0,1) \quad \varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

（1）法 1　定义法：

$$F_Y(y)=P\{Y\le y\}=P\{e^X\le y\}=\begin{cases}P\{X\le \ln y\} & y>0\\ 0 & y\le 0\end{cases}=\begin{cases}\Phi(\ln y) & y>0\\ 0 & y\le 0\end{cases}$$

$$f_Y(y)=F_Y'(y)=\begin{cases}\frac{1}{y}\Phi'(\ln y) & y>0\\ 0 & y\le 0\end{cases}=\begin{cases}\frac{1}{y}\varphi(\ln y) & y>0\\ 0 & y\le 0\end{cases}$$

$$=\begin{cases}\frac{1}{\sqrt{2\pi}\ y}e^{-\frac{\ln^2 y}{2}} & y>0\\ 0 & y\le 0\end{cases}$$

法 2　定理法：

$$g'(x)=e^x>0,\quad x=h(y)=\ln y(y>0).$$

$$f_Y(y)=\begin{cases}\varphi(\ln y)\left|\frac{1}{y}\right|, & y>0\\ 0 & y\le 0\end{cases}$$

$$(2)\ F_Y(y)=P\{2X^2+1\le y\}=P\{X^2\le\frac{y-1}{2}\}=\begin{cases}P\left\{-\frac{\sqrt{y-1}}{\sqrt{2}}\le X\le\frac{\sqrt{y-1}}{\sqrt{2}}\right\} & y>1\\ 0 & y\le 1\end{cases}$$

$$=\begin{cases}\Phi\left(\frac{\sqrt{y-1}}{\sqrt{2}}\right)-\Phi\left(-\frac{\sqrt{y-1}}{\sqrt{2}}\right) & y>1\\ 0 & y\le 1\end{cases}=\begin{cases}2\Phi\left(\frac{\sqrt{y-1}}{\sqrt{2}}\right)-1 & y>1\\ 0 & y\le 1\end{cases}$$

$$f_Y(y)=F_Y'(y)=\begin{cases}2\Phi'\left(\frac{\sqrt{y-1}}{\sqrt{2}}\right)\cdot\frac{1}{\sqrt{2}}\frac{1}{2\sqrt{y-1}} & y>1\\ 0 & y\le 1\end{cases}=\begin{cases}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{y-1}}\varphi\left(\frac{\sqrt{y-1}}{\sqrt{2}}\right) & y>1\\ 0 & y\le 1\end{cases}$$

$$=\begin{cases}\frac{1}{2\sqrt{\pi(y-1)}}e^{-\frac{y-1}{4}} & y>1\\ 0 & y\le 1\end{cases}.$$

$$(3)\ F_Y(y)=P\{Y\le y\}=P\{|X|\le y\}=\begin{cases}P\{-y\le X\le y\} & y>0\\ 0 & y\le 0\end{cases}=\begin{cases}2\Phi(y)-1 & y>0\\ 0 & y\le 0\end{cases}$$

$$f_Y(y)=F_Y'(y)=\begin{cases}2\Phi'(y) & y>0\\ 0 & y\le 0\end{cases}=\begin{cases}\sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2}} & y>0\\ 0 & y\le 0\end{cases}.$$

注意： 题(2)中，$P\{X^2 \le \frac{y-1}{2}\} \ne P\{-\frac{\sqrt{y-1}}{\sqrt{2}} \le X \text{ or } X \le \frac{\sqrt{y-1}}{\sqrt{2}}\}$

题(3)中，$P\{-y \le X \le y\} \ne P\{-y \le X \text{ or } Y \le y\}$.

22.

$$f_X(x) = \begin{cases} \frac{2x}{\pi^2} & 0 < x < \pi \\ 0 & \text{其它} \end{cases} \quad F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{\pi^2} & 0 < x < \pi \\ 1 & x \ge \pi \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ P\{\sin X \le y\} & 0 < y < 1 \quad (\text{注意：密度只在}X\text{在}(0,\pi)\text{上不为零}) \\ 1 & y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & y \le 0 \\ P\{0 \le X \le \arcsin y\} + P\{\pi - \arcsin y \le X \le \pi\} & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & y \le 0 \\ F_X(\arcsin y) - F_X(0) + F_X(\pi) - F_X(\pi - \arcsin y) & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & y \le 0 \\ F_X(\arcsin y) + 1 - F_X(\pi - \arcsin y) & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}$$

$$\therefore f_Y(y) = F_Y'(y) = \begin{cases} f_X(\arcsin y)\frac{1}{\sqrt{1-y^2}} + f_X(\pi - \arcsin y)\frac{1}{\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{others} \end{cases}$$

$$= \begin{cases} \frac{2\arcsin y + 2(\pi - \arcsin y)}{\pi^2\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{其它} \end{cases} = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{其它} \end{cases}.$$